LANNÉR DIAGRAMS AND COMBINATORIAL PROPERTIES OF COMPACT HYPERBOLIC COXETER POLYTOPES

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Abstract. In this paper we study $\times_0$-products of Lannér diagrams. We prove that every $\times_0$-product of at least four Lannér diagrams with at least one diagram of order $\geqslant 3$ is superhyperbolic. As a corollary, we obtain that known classifications exhaust all compact hyperbolic Coxeter polytopes that are combinatorially equivalent to products of simplices.

We also consider compact hyperbolic Coxeter polytopes whose every Lannér subdiagram has order 2. The second result of this paper slightly improves recent Burcroff's upper bound on the dimension of such polytopes to 12.

1. Introduction

A convex polytope is called a *Coxeter polytope* if its dihedral angles are all integer submultiples of π. Compact Coxeter polytopes in $\mathbb{S}^d$ and $\mathbb{E}^d$ were classified by Coxeter in [Cox34]. Vinberg initiated the study of such polytopes in $\mathbb{H}^d$ and proved in [Vin84] that there are no compact Coxeter polytopes in $\mathbb{H}^{\geqslant 30}$. Examples are known only in $\mathbb{H}^{\leqslant 8}$, the unique known example in $\mathbb{H}^8$ and both known examples in $\mathbb{H}^7$ are due to Bugaenko ([Bug92]).

Thus, there are two very hard long-standing open problems. The first one is the construction of new hyperbolic Coxeter polytopes, especially higher-dimensional ones. And the second one is the classification of such polytopes.

Generally speaking, there are two different approaches to both problems: classification of finite-volume Coxeter polytopes of some certain combinatorial types (see [Kap74, Ess96, Tum07, FT08, FT09, JT18, Bur22, MZ22a, MZ22b]) and the theory of arithmetic hyperbolic reflection groups (see [Vin72, Bel16, Bog17, BP18, Bog19, Bog20]). In particular, in the context of arithmetic and quasi-arithmetic reflection groups several authors constructed new Coxeter polytopes as faces or reflection centralizers of some higher dimensional polytopes (see [Bor87, All06, All13, BK21, BBKS21]).

This article is focused on the combinatorial approach, so let us give a brief summary of the results on the classification of compact hyperbolic Coxeter polytopes of certain combinatorial properties. A complete classification of Coxeter polytopes in $\mathbb{H}^2$ was obtained by Poincaré ([Poi82]). Andreev ([And70a, And70b]) described all Coxeter polytopes in $\mathbb{H}^3$. Compact Coxeter simplices were classified by Lannér ([Lan50]). Kaplinskaya ([Kap74]) used this classification to list all compact simplicial prisms. Esselmann ([Ess96]) used Gale diagrams to list the remain compact polytopes in $\mathbb{H}^d$ with $d+2$ facets. Tumarkin ([Tum07]) improved this technique and listed all compact polytopes in $\mathbb{H}^d$ with $d+3$ facets. All cubes were classified by Jacquemet and Tschantz ([JT18]). Very recently and independently, Burcroff

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([Bur22]) and Ma & Zheng ([MZ22a, MZ22b]) listed all compact Coxeter polytopes in $\mathbb{H}^d$ with $d+4$ facets for $d = 4, 5$.

The work is based on the author's bachelor thesis ([Ale21]) supervised by Nikolay Bogachev.

1.1. **Classification of compact Coxeter products of simplices.** First of all, we should provide some definitions. Each Coxeter polytope can be described by its Coxeter diagram. Such a diagram contains information about the angles and distances between every pair of facets. Coxeter diagrams of the compact simplices in hyperbolic spaces were listed by Lannér ([Lan50]) and are now called Lannér diagrams. They have an important property. Consider a compact hyperbolic Coxeter polytope and a minimal set of its facets with empty intersection. The subdiagram that corresponds to the set is a Lannér diagram.

The Lannér diagrams play an important role in many classifications as they are "unfriendly" to each other. These diagrams often form so-called superhyperbolic diagrams, which are not contained in any diagram of a hyperbolic Coxeter polytope. The first theorem provides a result of this type.

Denote by $\mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$ the set of all Coxeter diagrams generated[1] by pairwise disjoint Lannér diagrams of orders $k_1, \ldots, k_n$ and containing no other Lannér subdiagrams. Let us introduce the notation for some families of compact hyperbolic Coxeter polytopes:

- Simp^* for all products of simplices;
- Simp^k for all products of k simplices;
- Cubes for all cubes (not necessarily 3-dimensional).

Theorem A. *Let $n \geqslant 4$ and $2 \neq k_1 \geqslant \cdots \geqslant k_n = 2$. Every diagram contained in the set $\mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$ is superhyperbolic.*

As a simple corollary of this theorem, we obtain the following.

Theorem B. $\mathsf{Simp}^* = \mathsf{Simp}^1 \cup \mathsf{Simp}^2 \cup \mathsf{Simp}^3 \cup \mathsf{Cubes}$.

1.2. **Compact 3-free Coxeter polytopes.** Now let us consider the polytopes with diagram containing no Lannér subdiagrams of order $\geqslant 3$. These are exactly the polytopes with the following property: every set of facets with an empty intersection contains a pair of disjoint facets. Such polytopes are called 3-free polytopes. For example, cubes satisfy this property, so the Coxeter diagram of a cube does not contain a Lannér subdiagram of order $\geqslant 3$. Another example that satisfies this property is the family of compact right-angled polytopes in hyperbolic spaces (the reason is the structure of their diagrams). It is known that there are no Coxeter cubes in $\mathbb{H}^{\geqslant 6}$ ([JT18]) and that there are no compact right-angled polytopes in $\mathbb{H}^{\geqslant 5}$ ([PV05]). Recently Burcroff in [Bur22] used Vinberg's methods to estimate the dimension of such polytopes. We slightly improved this estimation.

Theorem C. *Every Coxeter diagram of a compact Coxeter polytope in $\mathbb{H}^{\geqslant 13}$ contains a Lannér diagram of order $\geqslant 3$.*

Acknowledgements. The author is grateful to Nikolay Bogachev for his supervision and remarks, and to Anna Felikson for maintaining her webpage on hyperbolic Coxeter polytopes.

[1] The exact definition is given at the beginning of Section 3.

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2. Preliminaries

2.1. **Abstract diagrams.** A *diagram* is a graph with positive real weights on the edges. The *order* $|S|$ of a diagram S is the number of vertices of the graph. A *subdiagram* of a diagram S is a diagram obtained from S by erasing some vertices. Consider a diagram S. A diagram *generated* by subdiagrams $S_1, \dots, S_k$ of S and vertices $v_1, \dots, v_l$ of S is a subdiagram $\langle S_1, \dots, S_k, v_1, \dots, v_l \rangle$ of S obtained from S by erasing every vertex v that is not contained in any S_i and is not equal to any v_j.

Let S be a diagram. Consider a symmetric matrix (g_{ij}) such that g_{ij} equals one if $i = j$, zero if $v_i v_j$ is not an edge of the diagram S, and $-w_{ij}$ if w_{ij} is the weight of the edge $v_i v_j$. Such a matrix $G(S) = (g_{ij})$ is called the *Gram matrix* of the diagram S.

We say that a diagram has some property if its Gram matrix has the same property (e.g., positive definiteness). A diagram has the same determinant and signature as its Gram matrix.

A diagram is said to be *elliptic* if it is positive definite, *parabolic* if it is positive semidefinite and not elliptic, and *hyperbolic* if it is indefinite with the negative inertia index equals one.

A *product* of diagrams S_1 and S_2 is a diagram whose vertex set is the disjoint union of the vertex sets of S_1 and S_2 and whose edge set is the union of the edge sets of S_1 and S_2 (informally speaking, we draw two diagrams side by side). The Gram matrix of such diagram is equal to $G(S_1) \oplus G(S_2)$ up to simultaneous permutation of rows and columns. A diagram is *connected* if it is not a product of some other non-empty diagrams.

Obviously, every elliptic diagram is a product of some connected elliptic diagrams. Every parabolic diagram is a product of some connected elliptic diagrams and some (at least one) connected parabolic diagrams.

Proposition 2.1. *A hyperbolic diagram does not contain a subdiagram that is a product of two hyperbolic diagrams.*

2.2. **Coxeter diagrams.** A diagram is called a *Coxeter diagram* if each of its weights is either $\geqslant 1$ or equal to $\cos\left(\frac{\pi}{m}\right)$ for some $m \geqslant 3$. Such diagrams are usually drawn as follows. If the weight of an edge $v_i v_j$ is greater than one, then a dashed edge is drawn connecting v_i and v_j. If the weight of an edge $v_i v_j$ is equal to one, then a bold edge is drawn. If the weight of an edge $v_i v_j$ is equal to $\cos(\frac{\pi}{m})$, then a $(m-2)$-fold edge or a simple edge with label m is drawn. We say that a vertex v is *joined* with a vertex u if they are joined by any edge other than a 2-labeled one.

Theorem 2.2 ([Cox34])**.** *Connected elliptic and parabolic diagrams are listed in Table 1 and Table 2.*

Corollary 2.3. *Every elliptic diagram contains no cycle. Every vertex of an elliptic diagram is joined with at most three other vertices.*

A hyperbolic Coxeter diagram S is called a *Lannér diagram* if any proper subdiagram of S is elliptic. All Lannér diagrams were classified by Lannér in [Lan50].

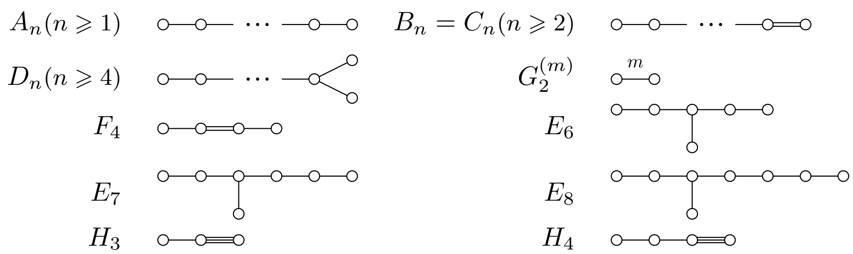


TABLE 1. Connected elliptic Coxeter diagrams

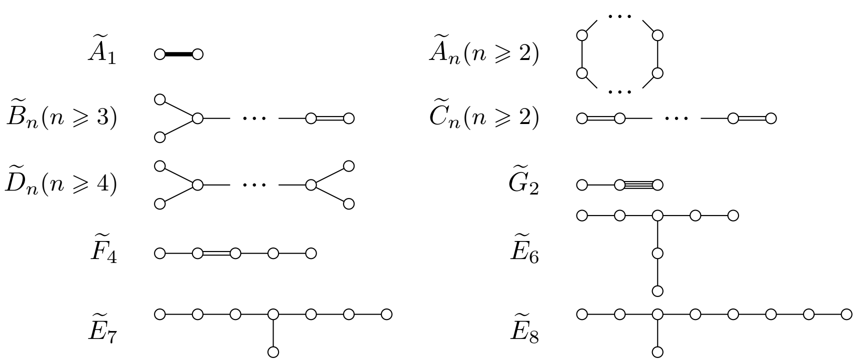


TABLE 2. Connected parabolic Coxeter diagrams

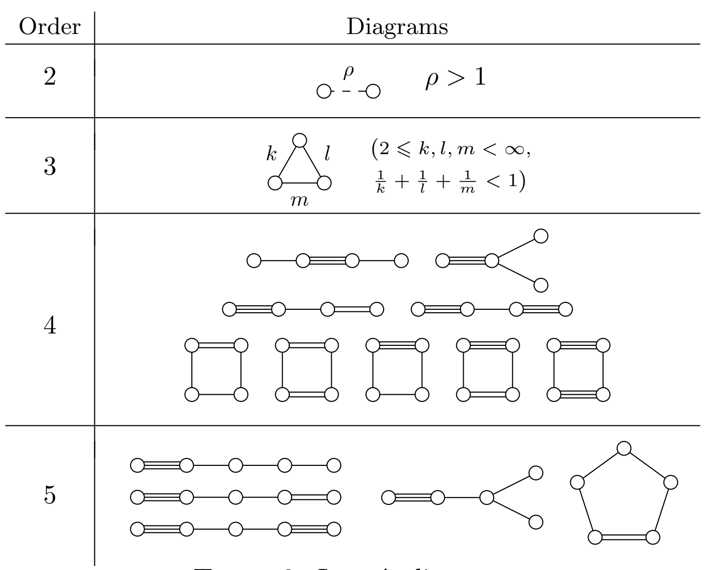


TABLE 3. Lannér diagrams

They are listed in Table 3. These diagrams correspond (in the sense defined further) to compact hyperbolic Coxeter simplices. Nevertheless, the importance of such diagrams can already be appreciated.

Proposition 2.4. *Every hyperbolic diagram contains either a parabolic or a Lannér subdiagram.*

2.3. **Hyperbolic polytopes.** *Minkowski space* $\mathbb{R}^{d,1}$ is a real vector space $\mathbb{R}^{d+1}$ equipped with the scalar product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_dy_d.$$

The upper sheet $\mathbb{H}^d = \{x \in \mathbb{R}^{d,1} \mid \langle x, x\rangle = -1 \;\&\; x_0 > 0\}$ of the hyperboloid is called a *hyperbolic space*. The projection of $\mathbb{H}^d$ onto the plane $x_0 = 1$ through the zero is the open ball. Its boundary is called the *boundary* $\partial\mathbb{H}^d$ of the hyperbolic space $\mathbb{H}^d$. Its points correspond to the isotropic vectors $\{x \mid \langle x, x\rangle = 0 \;\&\; x_0 > 0\}/\,\mathbb{R}_{>0}$. The union $\overline{\mathbb{H}^d} = \mathbb{H}^d \cup \partial\mathbb{H}^d$ is called the *compactification* of $\mathbb{H}^d$.

Every vector $e \in \mathbb{R}^{d,1}$ with $\langle e, e\rangle = 1$ defines a *hyperbolic hyperplane* $H^0_e = \mathbb{H}^d \cap \{x \mid \langle e, x\rangle = 0\}$ and a *hyperbolic half-space* $H^-_e = \mathbb{H}^d \cap \{x \mid \langle e, x\rangle \leqslant 0\}$. The hyperplane H^0_e is said to be a *supporting hyperplane* of the half-space H^-_e. By $\overline{H^0_e}$ we denote the closure of H^0_e in $\overline{\mathbb{H}^d}$. If $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1$ then $\langle e_1, e_2\rangle \geqslant 0$ and the following holds:

(1) if $\langle e_1, e_2\rangle < 1$ then the hyperplanes $H^0_{e_1}$ and $H^0_{e_2}$ intersect and the angle ϕ between them can be found from the equation $\cos\phi = \langle e_1, e_2\rangle$;
(2) if $\langle e_1, e_2\rangle = 1$ then the hyperplanes $H^0_{e_1}$ and $H^0_{e_2}$ do not intersect but its closures $\overline{H^0_{e_1}}$ and $\overline{H^0_{e_2}}$ share a unique point on the boundary $\partial\mathbb{H}^d$;
(3) if $\langle e_1, e_2\rangle > 1$ then the closures $\overline{H^0_{e_1}}$ and $\overline{H^0_{e_2}}$ do not intersect and the distance between $H^0_{e_1}$ and $H^0_{e_2}$ is equal to $\operatorname{arccosh}\langle e_1, e_2\rangle$.

Convex hyperbolic d-polytope is the intersection of finitely many closed half-spaces of $\mathbb{H}^d$ with non-empty interior. Consider a convex hyperbolic polytope P. We say that a vertex v of $\overline{P} \subset \overline{\mathbb{H}^d}$ is a *finite vertex* of P if $v \in \mathbb{H}^d$ and *ideal* if $v \in \partial\mathbb{H}^d$. The polytope P has finite volume if and only if it coincide with the convex hull of its vertices. The polytope P is compact if and only if its volume is finite and all of its vertices are finite. If the polytope has finite volume and all of its vertices are ideal then the polytope P is said to be *ideal*.

The closure $\overline{P} \subset \overline{\mathbb{H}^d}$ of a convex hyperbolic polytope $P \subset \mathbb{H}^d$ is combinatorially equivalent to a convex Euclidean polytope. The latter fact allows to use theorems on the structure of convex polytopes.

2.4. **Hyperbolic Coxeter polytopes.** Let $P \subset \mathbb{H}^d$ be a Coxeter polytope with facets $f_1, \dots, f_n$. The Coxeter diagram of the polytope P is a Coxeter diagram with vertices $v_1, \dots, v_n$. If the facets f_i and f_j intersect, then the weight of the edge v_iv_j is equal to the cosine of the dihedral angle between the facets. If the facets f_i and f_j are parallel, then the weight of the edge v_iv_j is equal to one. If the facets f_i and f_j diverge, then the weight of the edge v_iv_j is equal to the hyperbolic cosine of the distance between f_i and f_j.

Now let us list the essential results on combinatorics of compact hyperbolic Coxeter polytopes. Let P be a polytope. By $\mathcal{F}(P)$ we denote the partially ordered set of its faces. Let S be a Coxeter diagram. By $\mathcal{F}(S)$ we denote the dual (i.e., anti-isomorphic to the original) partially ordered set of its elliptic subdiagrams.

Proposition 2.5 ([Vin85, Theorem 3.1])**.** *Let $P \subset \mathbb{H}^d$ be a compact hyperbolic Coxeter polytope. Partially ordered sets $\mathcal{F}(S(P))$ and $\mathcal{F}(P)$ are isomorphic.*

Thus, the combinatorics of a compact polytope can be easily read according to its Coxeter diagram. A set of facets has a non-empty intersection if and only if the subdiagram generated by the corresponding vertices is elliptic.

Consider a compact hyperbolic Coxeter polytope. The structure of its Coxeter diagram is restricted by the propositions below.

Proposition 2.6 ([Vin85, Proposition 3.2]). *Let $P \subset \mathbb{H}^d$ be a compact hyperbolic Coxeter polytope. The Coxeter diagram $S(P)$ contains no parabolic subdiagrams.*

Proposition 2.7 ([Vin85, Proposition 4.2]). *A Coxeter diagram S is a Coxeter diagram of a compact hyperbolic Coxeter polytope if and only if the diagram is hyperbolic, contains no parabolic subdiagrams, and there is a polytope $P \subset \mathbb{E}^d$ such that $\mathcal{F}(P)$ and $\mathcal{F}(S)$ are isomorphic.*

The following statement is an easy corollary of the propositions above.

Corollary 2.8. *A polytope $P \subset \mathbb{H}^d$ is a compact simplex if and only if $S(P)$ is a Lannér diagram.*

Finally, the best known general estimation on dimension of a compact hyperbolic Coxeter polytope is the following.

Theorem 2.9 ([Vin84, Theorem 1]). *There are no compact Coxeter polytopes in $\mathbb{H}^{\geqslant 30}$.*

2.5. **Superhyperbolic diagrams.** A Coxeter diagram is said to be *superhyperbolic* if its negative inertia index is greater than one. A *local determinant* of a diagram S on its subdiagram T is

$$\det(S, T) = \frac{\det(S)}{\det(S \setminus T)}.$$

Usually we will mark the vertices of the subdiagram T with $\vee$.

We denote by $p(\gamma)$ the product of the edge weights of a cycle γ. The following proposition is very useful for computing determinants.

Proposition 2.10 ([Vin84, Proposition 11]). *A determinant of a Coxeter diagram S is equal to the sum of the products*

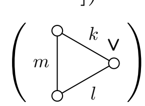
$$(-1)^k \cdot p(\gamma_1) \cdot \ldots \cdot p(\gamma_k)$$

over all sets $\{\gamma_1, \ldots, \gamma_k\}$ of positive length disjoint cycles.

Proposition 2.11 ([Vin84, Proposition 13]). *If a Coxeter diagram S is generated by two disjoint subdiagrams S_1 and S_2 joined by a unique edge $v_1 v_2$ of weight w, then*

$$\det(S, \langle v_1, v_2 \rangle) = \det(S_1, v_1) \cdot \det(S_2, v_2) - w^2.$$

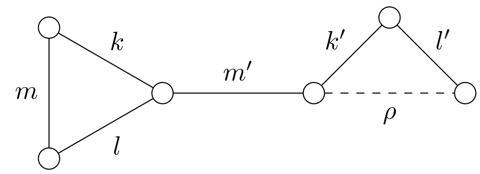
Proposition 2.12 ([Vin84, Table 2]).

det  $= -d(k, l, m),$

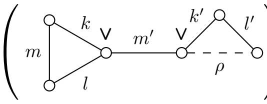
where

$$d(k, l, m) = \frac{\cos\left(\frac{\pi}{k}\right)^2 + \cos\left(\frac{\pi}{l}\right)^2 + 2\cos\left(\frac{\pi}{k}\right)\cos\left(\frac{\pi}{l}\right)\cos\left(\frac{\pi}{m}\right)}{\sin\left(\frac{\pi}{m}\right)^2} - 1.$$

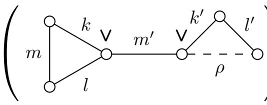
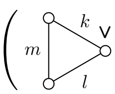
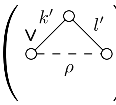
Now let us use these propositions to test the diagram below for hyperbolicity.

(1) 

This diagram contains an elliptic subdiagram of order 4 and a Lannér subdiagram of order 2. Therefore, its signature is either $(4,1,1)$ or $(5,1,0)$, or $(4,2,0)$. Hence, the diagram is hyperbolic if and only if

det () $\leqslant 0.$

But

det () $=$ det () $\cdot$ det () $- \cos\left(\frac{\pi}{m'}\right)^2$

$$= d(k,l,m)\frac{\cos\left(\frac{\pi}{l'}\right)^2 + \cos\left(\frac{\pi}{k'}\right)^2 + \rho^2 + 2\rho\cos\left(\frac{\pi}{k'}\right)\cos\left(\frac{\pi}{l'}\right) - 1}{\sin\left(\frac{\pi}{l'}\right)^2} - \cos\left(\frac{\pi}{m'}\right)^2.$$

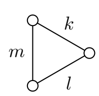
If $d(k,l,m) \neq 0$, then the last inequality is equivalent to the following:

$$\rho^2 + 2\rho\cos\left(\frac{\pi}{k'}\right)\cos\left(\frac{\pi}{l'}\right) + \cos\left(\frac{\pi}{l'}\right)^2 + \cos\left(\frac{\pi}{k'}\right)^2 - 1 - \frac{\sin\left(\frac{\pi}{l'}\right)^2\cos\left(\frac{\pi}{m'}\right)^2}{d(k,l,m)} \leqslant 0.$$

Consider the left part of this inequality as a quadratic function in ρ. One of the zeros of this function is not greater than 1. So there is a $\rho > 1$ satisfying the inequality if and only if for $\rho = 1$ the strict inequality holds, i.e.

$$D(k,l,m,k',l',m') = \left(\cos\left(\frac{\pi}{l'}\right) + \cos\left(\frac{\pi}{k'}\right)\right)^2 - \frac{\sin\left(\frac{\pi}{l'}\right)^2\cos\left(\frac{\pi}{m'}\right)^2}{d(k,l,m)} < 0.$$

This proves the following lemma.

Lemma 2.13. *Let*  *be a Lannér diagram. The Coxeter diagram (1) is superhyperbolic for any $\rho > 1$ if and only if*

$$D(k,l,m,k',l',m') \geqslant 0.$$

Remark 2.14. Direct calculations show that if $d(k,l,m) > 0$, then the function D is increasing in k,l,m,k',l', and decreasing in m'.

3. Proof of Theorems A and B

Let Σ_1 and Σ_2 be sets of Coxeter diagram. By $\Sigma_1 \times_k \Sigma_2$ we denote the set of all Coxeter diagrams S generated by subdiagrams $S_1 \in \Sigma_1$ and $S_2 \in \Sigma_2$ such that intersection $S_1 \cap S_2$ consists of k vertices and every Lannér or parabolic subdiagram is contained in either S_1 or S_2.

Denote by $\mathcal{L}_k$ the set of all Lannér diagrams of order k and by Δ_k the standard $(k-1)$-dimensional simplex. Consider a compact hyperbolic Coxeter polytope P. Suppose that $\mathcal{F}(P)$ and $\mathcal{F}(\Delta_{k_1} \times \cdots \times \Delta_{k_n})$ are isomorphic. Every face of $\Delta_{k_1} \times \cdots \times \Delta_{k_n}$ is equal to $f_1 \times \cdots \times f_n$ for some faces f_i of Δ_i. Therefore, the facets of $\Delta_{k_1} \times \cdots \times \Delta_{k_n}$ are equal to

$$f_j^i = \Delta_{k_1} \times \cdots \times \Delta_{k_{i-1}} \times f_j \times \Delta_{k_{i+1}} \times \cdots \times \Delta_{k_n}, \quad \text{where } f_j \text{ is a facet of } \Delta_{k_i}.$$

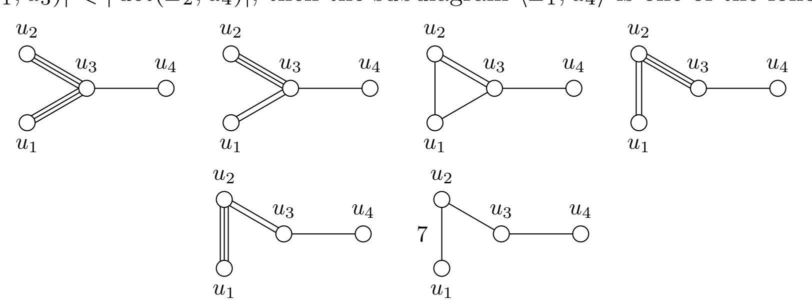
Let F be a set of the facets. The intersection $\bigcap_{f \in F} f$ is empty if and only if $\{f_1^i, \ldots, f_{k_i}^i\} \subseteq F$ for some $1 \leqslant i \leqslant n$. According to Proposition 2.4 and Proposition 2.7, $S(P) \in \mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$.

Without loss of generality, $k_1 \geqslant \cdots \geqslant k_n$. If $k_1 = \cdots = k_n = 2$, then P is a n-dimensional cube. If $k_n \neq 2$, then the diagram $S(P)$ contains no dashed edges. It is known that every such polytope is a product of at most two simplices (see [FT08, Theorem A]). Thus, Theorem B is a corollary of Theorem A. For the reader's convenience we present its statement again. The proof starts below.

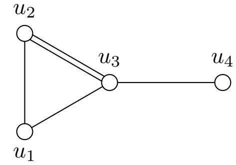
Theorem A. *Let $n \geqslant 4$ and $2 \neq k_1 \geqslant \cdots \geqslant k_n = 2$. Every diagram contained in the set $\mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$ is superhyperbolic.*

3.1. **Case $k_1 \geqslant 4$.** Let $S = \langle L_1, \ldots, L_n \rangle$ be a Coxeter diagram generated by disjoint Lannér diagrams $L_1, \ldots, L_n$ of orders $k_1, \ldots, k_n$. Let $v_2, \ldots, v_n$ be arbitrary vertices of the subdiagrams $L_2, \ldots, L_n$ respectively. The diagram $\langle L_1, v_2, v_3, v_4 \rangle$ does not contain any parabolic or Lannér subdiagrams other than L_1. But direct calculations show that each such diagram contains either a parabolic or a new Lannér subdiagram (see [Ess96, Lemma 4.2]). In this case, we say that no Lannér diagram of order 4 or 5 can be *expanded* with three vertices without forming a new Lannér or parabolic subdiagram. In other words, the sets $\mathcal{L}_4 \times_0 \{\circ\} \times_0 \{\circ\} \times_0 \{\circ\}$ and $\mathcal{L}_5 \times_0 \{\circ\} \times_0 \{\circ\} \times_0 \{\circ\}$ are empty ($\circ$ is a Coxeter diagram consisting of one vertex). Therefore, the required diagrams with $k_1 \geqslant 4$ do not exist.

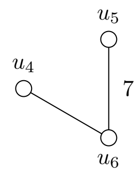
3.2. **Case $k_1 = k_2 = 3$.** Consider the Lannér subdiagrams $L_1 = \langle u_1, u_2, u_3 \rangle$ and $L_2 = \langle u_4, u_5, u_6 \rangle$ of order 3. It is shown in [Tum07, Lemma 4.10] that if $|\det(L_1, u_3)| \leqslant |\det(L_2, u_4)|$, then the subdiagram $\langle L_1, u_4 \rangle$ is one of the following.



The diagram L_1 can be expanded with two vertices, so the subdiagram $\langle L_1, u_4 \rangle$ is the following, or a new Lannér or parabolic subdiagram is forming.



So, $|\det(L_1, u_3)| = \frac{\sqrt{2}}{3}$. According to Proposition 2.11, $|\det(L_2, u_4)| \leqslant \frac{3}{4\sqrt{2}}$. The multiplicity of the edges u_4u_5 and u_4u_6 does not exceed one. There is the only Lannér diagram of order 3 with such properties, which is shown below.

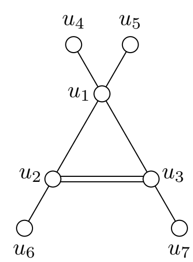


This diagram is not appropriate since it cannot be expanded with three vertices without forming a new Lannér or parabolic subdiagram.

3.3. **Case** $k_2 = 2$**.** A Lannér diagram of order 3 cannot be expanded with five vertices. Therefore, $n \leqslant 5$. Let us denote by $[u, v]$ the multiplicity of the edge connecting vertices u and v.

Lemma 3.1. *Under the conditions described above, $n \leqslant 4$.*

Proof. Suppose that $n = 5$. Denote the Lannér subdiagrams by $L_1 = \langle u_1, u_2, u_3\rangle$, $L_2 = \langle u_4, u_8\rangle$, $L_3 = \langle u_5, u_9\rangle$, $L_4 = \langle u_6, u_{10}\rangle$, and $L_5 = \langle u_7, u_{11}\rangle$. Without loss of generality, the vertices u_4, u_5, u_6, and u_7 are joined to the subdiagram L_1. The only subdiagram $\langle L_1, u_4, u_5, u_6, u_7\rangle$ that satisfies these properties is shown below.



It is easy to check that

$$[u_6, u_4] = [u_6, u_5] = [u_6, u_8] = [u_6, u_9] =$$
$$[u_7, u_4] = [u_7, u_5] = [u_7, u_8] = [u_7, u_9] = 0.$$

This implies that the vertices u_{10} and u_{11} are joined to the subdiagrams $\langle u_4, u_8\rangle$ and $\langle u_5, u_9\rangle$. There are two cases:

(1) Let $[u_{10}, u_{11}] \geqslant 1$. Without loss of generality, we may assume that
$$[u_{10}, u_8] = [u_{10}, u_5] = [u_{11}, u_4] = [u_{11}, u_9] \geqslant 1$$
and
$$[u_{11}, u_8] = [u_{11}, u_5] = [u_{10}, u_4] = [u_{10}, u_9] = 0.$$
Then
$$[u_4, u_5] = [u_4, u_9] = [u_8, u_5] = [u_8, u_9] = 0$$
and the subdiagram $\langle L_2, L_3\rangle$ is not connected.

(2) Let $[u_{10}, u_{11}] = 0$. Then, without loss of generality, we may assume that $[u_6, u_{11}] = 1$. In this case the subdiagram L_5 can be joined with L_2 and L_3 only if
$$[u_{11}, u_8] = [u_{11}, u_9] \geqslant 1.$$

Then

$$[u_4, u_5] = [u_4, u_9] = [u_8, u_5] = [u_8, u_9] = 0$$

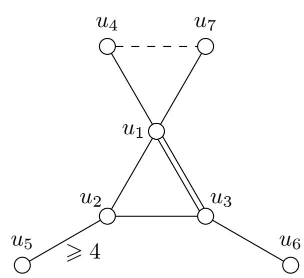
and the subdiagram $\langle L_2, L_3 \rangle$ is not connected.

□

Thus, only the products of a triangle and a 3-dimensional cube left. Denote the Lannér subdiagrams by $L_1 = \langle u_1, u_2, u_3 \rangle$, $L_2 = \langle u_4, u_7 \rangle$, $L_3 = \langle u_5, u_8 \rangle$, and $L_4 = \langle u_6, u_9 \rangle$. We suppose that the subdiagrams $\langle L_1, u_4 \rangle$, $\langle L_1, u_5 \rangle$, and $\langle L_1, u_6 \rangle$ are connected. If the subdiagram $\langle L_1, u_4, u_5, u_6 \rangle$ contains the only Lannér subdiagram, then all edges of the subdiagram L_1 have a positive multiplicity. This means that any vertex of the subdiagrams L_2, L_3, and L_4 is joined to L_1 by at most one edge. Denote the multiplicity of such an edge by $[u, L_1]$. If $[u_7, L_1] \geqslant 1$ and $[u_8, L_1] \geqslant 1$, then L_2 and L_3 are not connected. Thus, without loss of generality, $[u_8, L_1] = [u_9, L_1] = 0$, $[u_4, L_1] \geqslant [u_7, L_1]$, and $[u_5, L_1] \geqslant [u_6, L_1] = 1$.

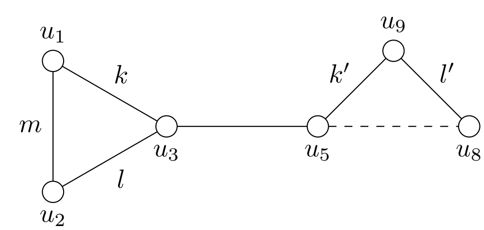
Lemma 3.2. *If* $[u_5, L_1] \geqslant 2$, *then* $[u_7, L_1] = 0$.

Proof. Assume that $[u_7, L_1] \geqslant 1$. The only possible subdiagram $\langle L_1, L_2, u_5, u_6 \rangle$ is shown below.

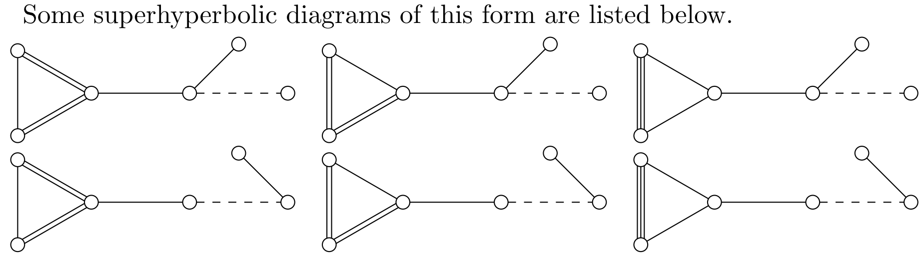


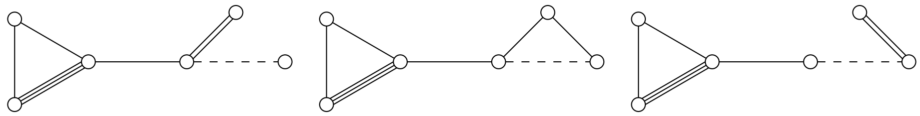
Then $[u_5, u_4] = [u_5, u_7] = [u_8, u_4] = [u_8, u_7] = 0$ and the subdiagrams L_2 and L_3 are not connected. □

We may suppose that $[u_5, L_1] = 1$ since otherwise we can swap L_2 and L_3. The vertex u_8 is joined to L_4 or the vertex u_9 is joined to L_3. Without loss of generality, u_9 is joined to L_3. The only possible diagram $\langle L_1, L_3, u_9 \rangle$ is shown below, $k' \geqslant 3$ or $l' \geqslant 3$.

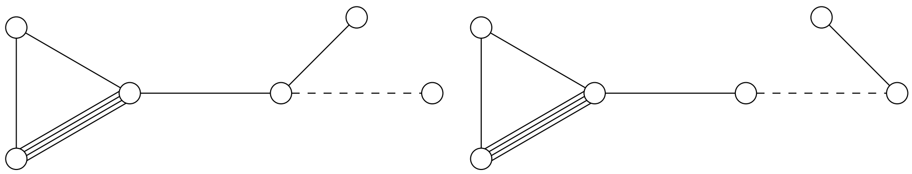


Some superhyperbolic diagrams of this form are listed below.



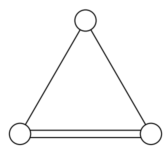


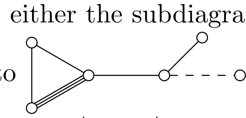
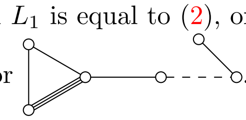
We also would like to note that the following diagrams contain a parabolic subdiagram.

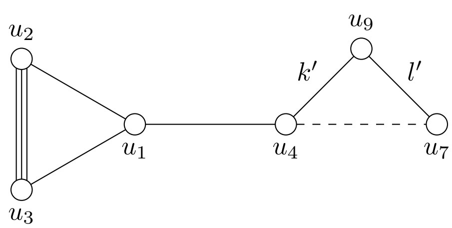


Lemma 3.3. *The diagram L_1 is equal to the following diagram.*

(2)



Proof. Combining Lemma 2.13, monotonicity of the function D, and a simple computation, we get that either the subdiagram L_1 is equal to (2), or the subdiagram $\langle L_1, L_3, u_9\rangle$ is equal to  or . Without loss of generality, the subdiagram $\langle L_2, u_9\rangle$ is connected, i.e. the subdiagram $\langle L_1, L_2, u_9\rangle$ is equal to the following diagram, $k' \geqslant 3$ or $l' \geqslant 3$.

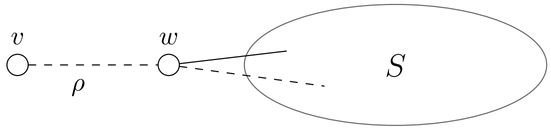


The diagram is superhyperbolic. □

Lemma 3.4. *Let S be a diagram that contains a hyperbolic subdiagram and let $v \notin S$ be a vertex that is joined with the only vertex $w \notin S$ by a dotted edge. If the inequality*

$$\det\big(\langle w, S\rangle\big) - \det(S) > 0$$

holds, then the diagram $\langle v, w, S\rangle$ is superhyperbolic.



Proof. Let us choose arbitrary labels on the dotted edges. Denote by ρ the label on the dotted edge between v and w. Direct calculation provides

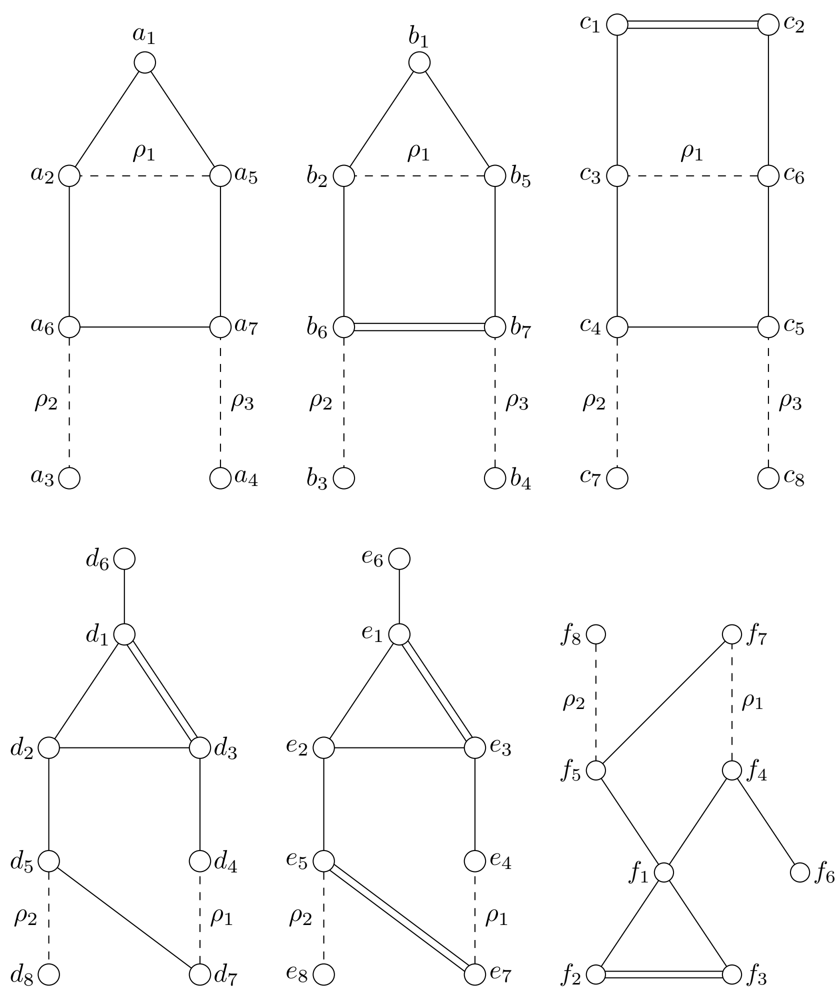
$$\det(\langle v, w, S\rangle) = \det(\langle w, S\rangle) - \rho^2 \det(S).$$

Suppose that the diagram $\langle v, w, S\rangle$ is hyperbolic. If $\det(S) < 0$, then

$$\rho \leqslant \sqrt{\frac{\det(\langle w, S\rangle)}{\det(S)}} = \sqrt{1 + \frac{\det(\langle w, S\rangle) - \det(S)}{\det(S)}} \leqslant 1.$$

We get $\det(S) = 0$ and $\det\big(\langle w, S\rangle\big) > 0$. Therefore, the diagram $\langle w, S\rangle$ is superhyperbolic. □

Corollary 3.5. *The diagrams below are superhyperbolic for any* $\rho_1, \rho_2, \rho_3 > 1$.



Proof. For $\rho_1, \rho_2, \rho_3 > 1$ we have

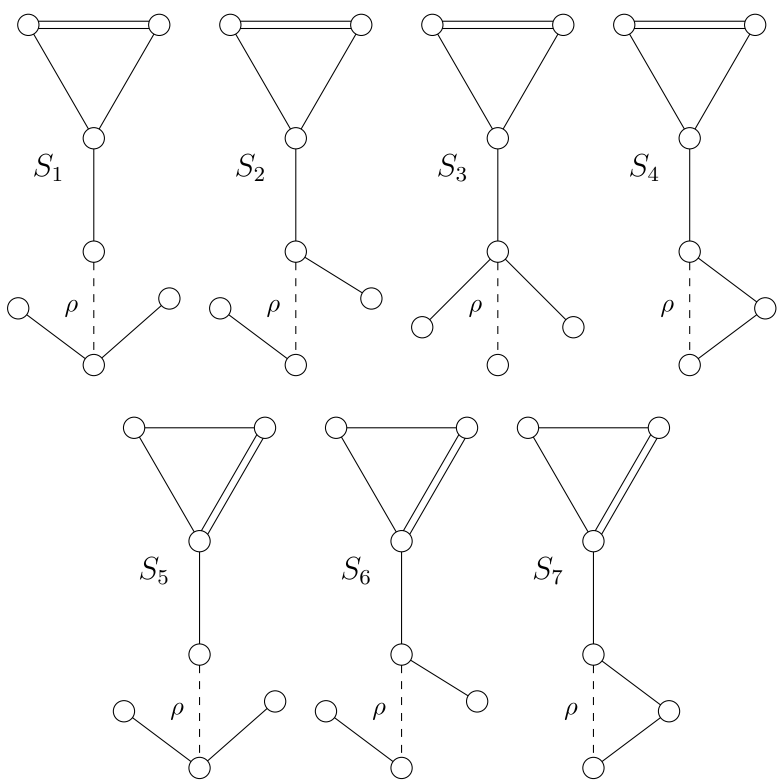
$$\det(\langle a_7, A\rangle) - \det(A) = \frac{1}{16}\big(3\rho_2^2 + 4\rho_1^2 - 2\rho_1 - 5\big) > 0,$$
$$\det(\langle b_7, B\rangle) - \det(B) = \frac{1}{16}\big(3\rho_2^2 + 8\rho_1^2 - 4(\sqrt{2}-1)\rho_1 - 6 - \sqrt{2}\big) > 0,$$
$$\det(\langle c_5, C\rangle) - \det(C) = \frac{1}{64}\big(4\rho_2^2 + 8\rho_1^2 - 4(2-\sqrt{2})\rho_1 - 2\sqrt{2} - 3\big) > 0,$$
$$\det(\langle d_5, D\rangle) - \det(D) = \frac{1}{32}\big(2\rho_1^2 - (3+2\sqrt{2})\rho_1 + 2\sqrt{2} + 2\big) > 0,$$
$$\det(\langle e_5, E\rangle) - \det(E) = \frac{1}{64}\big(4\rho_1^2 - 2(4+3\sqrt{2})\rho_1 + 8\sqrt{2} + 9\big) > 0,$$
$$\det(\langle f_5, F\rangle) - \det(F) = \frac{1}{64}\big(8\rho_1^2 - 8\rho_1 + 3\sqrt{2} - 4\big) > 0,$$

where

$$A = \langle a_1, a_2, a_3, a_5, a_6\rangle, \qquad D = \langle d_1, d_2, d_3, d_4, d_6, d_7\rangle,$$
$$B = \langle b_1, b_2, b_3, b_5, b_6\rangle, \qquad E = \langle e_1, e_2, e_3, e_4, e_6, e_7\rangle,$$
$$C = \langle c_1, c_2, c_3, c_4, c_6, c_7\rangle, \qquad F = \langle f_1, f_2, f_3, f_4, f_6, f_7\rangle.$$

□

Lemma 3.6. *The diagrams below are superhyperbolic for any $\rho > 1$.*

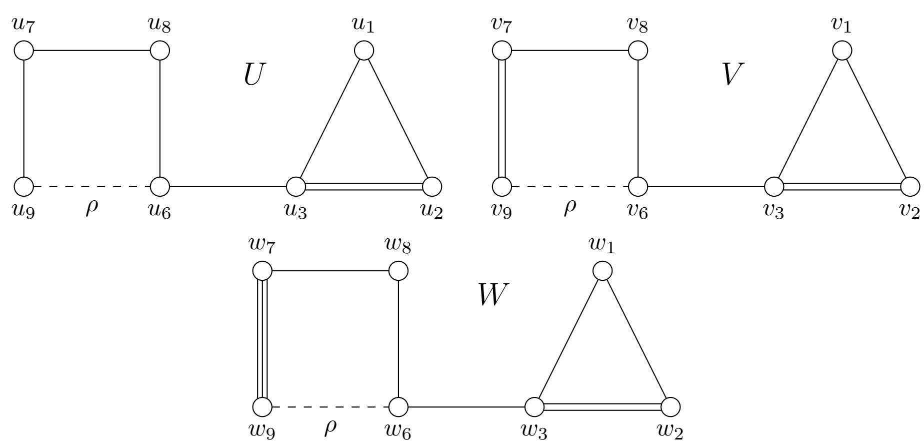


Proof. For $\rho > 1$ we have

$$\begin{aligned}
\det(S_1) &= \frac{1}{16}\Big(4\sqrt{2}\rho^2 - 2\sqrt{2} - 1\Big) > 0,\\
\det(S_2) &= \frac{1}{64}\Big(16\sqrt{2}\rho^2 - 9\sqrt{2} - 6\Big) > 0,\\
\det(S_3) &= \frac{1}{8}\Big(2\sqrt{2}\rho^2 - \sqrt{2} - 1\Big) > 0,\\
\det(S_4) &= \frac{1}{32}\Big(8\sqrt{2}\rho^2 + 4\sqrt{2}\rho - 4\sqrt{2} - 3\Big) > 0,\\
\det(S_5) &= \frac{1}{32}\Big(8\sqrt{2}\rho^2 - 4\sqrt{2} - 3\Big) > 0,\\
\det(S_6) &= \frac{1}{64}\Big(16\sqrt{2}\rho^2 - 9\sqrt{2} - 9\Big) > 0,\\
\det(S_7) &= \frac{1}{64}\Big(16\sqrt{2}\rho^2 + 8\sqrt{2}\rho - 8\sqrt{2} - 9\Big) > 0.
\end{aligned}$$

□

Lemma 3.7. *The diagrams below are superhyperbolic for any $\rho > 1$.*

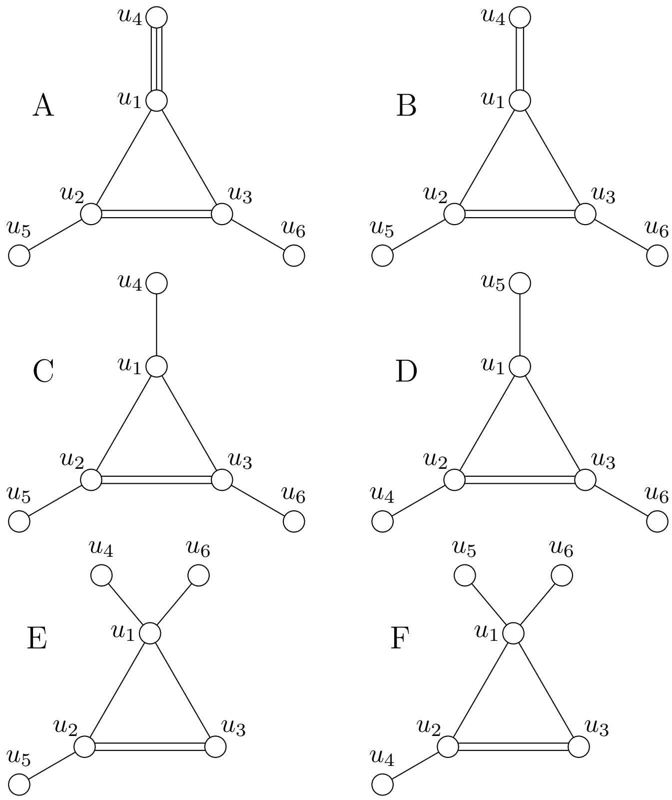


Proof. For $\rho > 1$ we have

$$\begin{aligned}
\det(U) &= \frac{1}{64}\Big(12\sqrt{2}\rho^2 + 4\sqrt{2}\rho - 5\sqrt{2} - 6\Big) > 0,\\
\det(V) &= \frac{1}{64}\Big(12\sqrt{2}\rho^2 + 8\rho - 2\sqrt{2} - 3\Big) > 0,\\
\det(W) &= \frac{1}{128}\Big(24\sqrt{2}\rho^2 + 4\sqrt{2}(1+\sqrt{5})\rho + 3\sqrt{10} + 3\sqrt{5} - 7\sqrt{2} - 9\Big) > 0.
\end{aligned}$$

□

Let us remind that we suppose that $[u_8, L_1] = [u_9, L_1] = 0$, $[u_4, L_1] \geqslant [u_7, L_1]$, and $[u_5, L_1] = [u_6, L_1] = 1$. Thus, the subdiagram $\langle L_1, u_4, u_5, u_6\rangle$ is one of the following.



3.3.1. *Case A.*

$$[u_5, u_4] = [u_5, u_7] = [u_4, u_8] = 0,$$
$$[u_4, u_6] = [u_4, u_9] = [u_6, u_7] = 0,$$
$$[u_5, u_6] = [u_5, u_9] = [u_6, u_8] = 0.$$

Otherwise, there is either a parabolic or hyperbolic subdiagram that must be elliptic. This implies that $[u_7, u_8] \neq 0$, $[u_8, u_9] \neq 0$, and $[u_9, u_7] \neq 0$. Then the subdiagram $\langle u_7, u_8, u_9 \rangle$ is not elliptic.

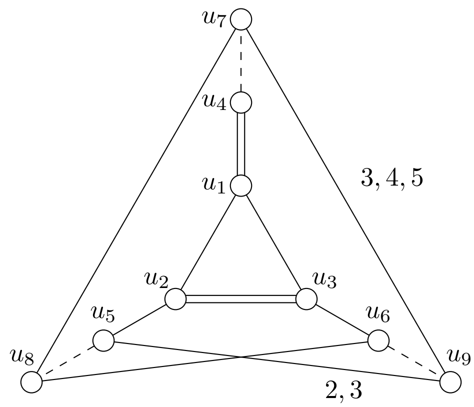
3.3.2. *Case B.* By the same argument we get

$$[u_5, u_4] = [u_5, u_7] = [u_4, u_8] = 0,$$
$$[u_4, u_6] = [u_4, u_9] = [u_6, u_7] = 0,$$
$$[u_5, u_6] = 0.$$

This yields that, without loss of generality,

$$1 \leqslant [u_7, u_8], \quad 1 \leqslant [u_7, u_9] \leqslant 3, \quad [u_8, u_9] = 0,$$
$$[u_6, u_8] = 1, \quad [u_7, u_8] = 1, \quad [u_5, u_9] \in \{0, 1\},$$
$$[u_7, u_1] = [u_7, u_2] = [u_7, u_3] = 0.$$

Therefore, the diagram is equal to the shown below.



From Lemma 3.7 it follows that the subdiagram $\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle$ is superhyperbolic.

3.3.3. *Case C.*

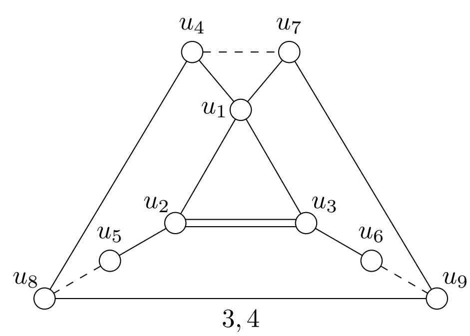
$$[u_5, u_4] = [u_5, u_7] = [u_6, u_4] = [u_6, u_7] = 0.$$

Let $[u_7, u_1] = 0$. Suppose that $[u_8, u_7] = 0$. Then

$$1 \leqslant [u_4, u_8] \leqslant 2, \quad [u_6, u_5] = [u_6, u_8] = 0.$$

Corollary 3.5 (D and E) implies that the diagram $\langle u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\rangle$ is superhyperbolic. Therefore, $[u_8, u_7] \geqslant 1$. For similar reasons, $[u_9, u_7] \geqslant 1$. Without loss of generality, $[u_8, u_6] = 1$, $[u_8, u_7] = 1$, and $1 \leqslant [u_9, u_7] \leqslant 3$. The subdiagram $\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle$ is superhyperbolic due to Lemma 3.7.

Let $[u_7, u_1] = 1$, then the only possible diagram is shown below.



Corollary 3.5 (A and B) implies that the subdiagram $\langle u_1, u_4, u_5, u_6, u_7, u_8, u_9\rangle$ is superhyperbolic.

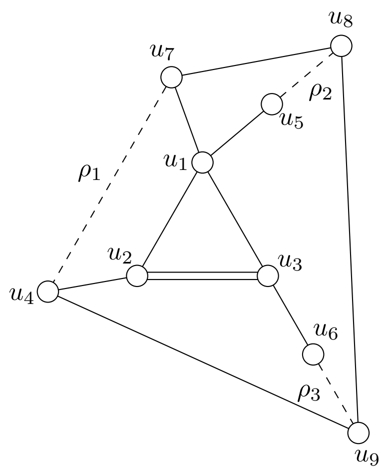
3.3.4. *Case D.* The case $[u_7, L_1] = 0$ is considered in the previous paragraph, so $[u_7, L_1] \neq 0$. Moreover, $[u_7, u_3] = 0$. Suppose that $[u_7, u_2] \geqslant 1$. Then the diagram $\langle L_2, L_3\rangle$ is not connected. Therefore, $[u_7, u_2] = 0$ and $[u_7, u_1] = 1$. The equality

$$[u_4, u_5] = [u_4, u_8] = [u_7, u_5] = 0$$

implies that $[u_7, u_8] \neq 0$. It is easy to check that

$$[u_4, u_6] = [u_7, u_6] = [u_5, u_6] = [u_8, u_6] = 0.$$

Suppose that $[u_5, u_9] \geqslant 1$. Then $[L_2, u_9] = 0$ and the subdiagram $\langle L_2, L_4\rangle$ is not connected. Therefore, $[u_5, u_9] = 0$, $[u_8, u_9] \geqslant 1$, $[u_7, u_9] = 0$, and $[u_4, u_9] \geqslant 1$. The only possible diagram is shown below.

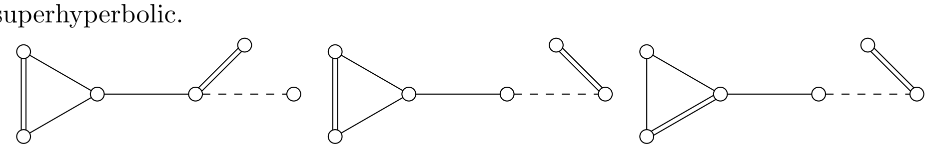


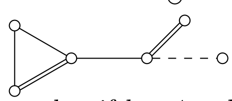
But this diagram is superhyperbolic since

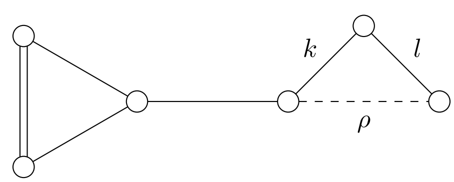
$$\det(\langle u_1, u_2, u_3, u_5, u_7, u_8, u_9\rangle) = \frac{1}{32}\Big(4\big(2\sqrt{2}+1\big)\rho_2^2 - 4\rho_2 - \big(4\sqrt{2}+5\big)\Big) > 0$$

for all $\rho_2 > 1$.

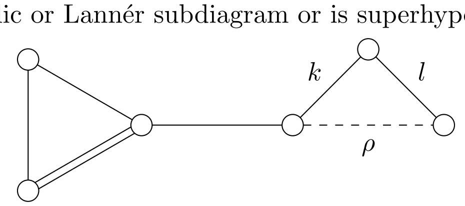
3.3.5. *Case E.* Let $[u_7, L_1] = 0$. Lemma 2.13 implies that the diagrams below are superhyperbolic.



The diagram  contains a parabolic subdiagram. Using Remark 2.14, we get that if $k \geqslant 4$ or $l \geqslant 4$, then the diagram below is superhyperbolic for any $\rho > 1$.



By the same argument, if $k \geqslant 4$ or $l \geqslant 4$, then the diagram below either contains an unwanted parabolic or Lannér subdiagram or is superhyperbolic for any $\rho > 1$.



Therefore, the multiplicity of every edge between the subdiagrams L_2, L_3, and L_4 does no exceed 1.

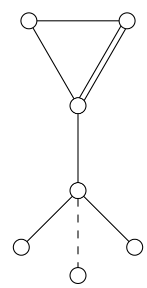
Applying Lemma 3.6 (S_1–S_4) to the subdiagram $\langle u_1, u_2, u_3, u_4, u_7, u_8, u_9\rangle$, we obtain that

$$[u_7, u_8] = [u_4, u_8] = 0 \quad \text{or} \quad [u_7, u_9] = [u_4, u_9] = 0.$$

By the same argument,

$$[u_9, u_8] = [u_6, u_8] = 0 \quad \text{or} \quad [u_9, u_7] = [u_6, u_7] = 0.$$

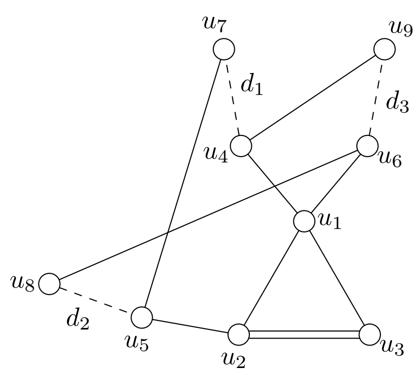
Note that the diagram below contains a parabolic subdiagram.



Thus, applying Lemma 3.6 (S_5–S_7) to the subdiagram $\langle u_1, u_2, u_3, u_5, u_7, u_8, u_9\rangle$, we obtain that

$$[u_8, u_7] = [u_5, u_7] = 0 \quad \text{or} \quad [u_8, u_9] = [u_5, u_9] = 0.$$

It is easy to check that, without loss of generality, the only diagram with such properties is shown below.

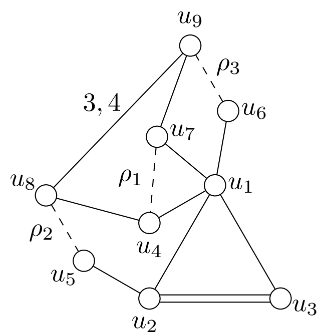


But Corollary 3.5 (F) implies that the subdiagram $\langle u_1, u_2, u_3, u_4, u_6, u_7, u_8, u_9\rangle$ is superhyperbolic.

Let $[u_7, L_1] \neq 0$. Then $[u_7, u_2] = 0$. We also may suppose that $[u_7, u_1] \neq 0$ since $[u_7, u_3] \neq 0$ is already considered in Case D.

$$[u_5, u_4] = [u_5, u_7] = [u_6, u_4] = [u_6, u_7] = 0.$$

Without loss of generality, the only such diagram is shown below.



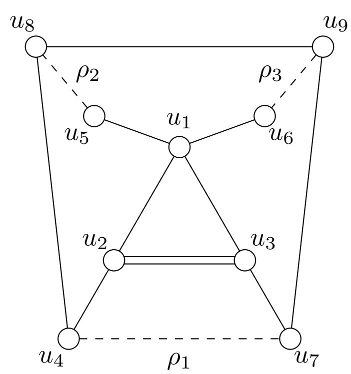
It is easy to calculate that for $\rho > 1$

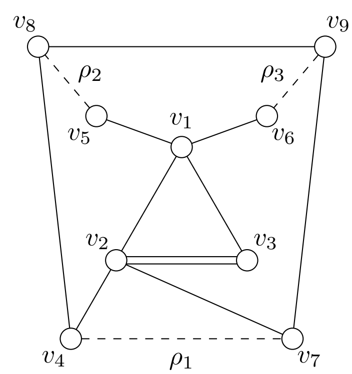
$$\det(\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle) = \frac{1}{32}\Big(4(1+2\sqrt{2})\rho_3^2 - 4\rho_3 - 4\sqrt{2} - 5\Big) > 0$$

and

$$\det(\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle) = \frac{1}{32}\Big(4(1+2\sqrt{2})\rho_3^2 - 4\rho_3 - 2\sqrt{2} - 3\Big) > 0.$$

3.3.6. *Case F.* Let $[u_7, L_1] \neq 0$. The opposite is considered in Case E. The only such diagrams are shown below.





Corollary 3.5 (C and A) implies that the subdiagrams $\langle u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\rangle$ and $\langle v_2, v_4, v_5, v_6, v_7, v_8, v_9\rangle$ are superhyperbolic.

4. Proof of Theorem C

We say that a polytope is 3-*free* if every set of facets with an empty intersection contains a pair of disjoint facets. Proposition 2.7 implies that the Coxeter diagram of a compact 3-free Coxeter polytope contains no Lannér subdiagrams of order $\geqslant 3$. Our aim is to prove Theorem C.

The proof is similar to the proof of [Bur22, Theorem 9.4], which is based on the proof of [Vin85, Theorem 6.1]. Thus, we need the Nikulin inequality.

Theorem 4.1 ([Nik81, Theorem 3.2.1])**.** *Let $\theta_0, \ldots, \theta_{k-1}$ be non-negative reals, $k \leqslant \left\lfloor \frac{d}{2} \right\rfloor$, and P a d-dimensional convex polytope. The following inequality holds*

$$\frac{1}{\alpha_k^P} \sum_{\substack{Q<P \\ \dim Q = k}} \sum_{i=0}^{k-1} \theta_i \alpha_i^Q < \sum_{i=0}^{k-1} \theta_i A_d^{(i,k)},$$

where α_k^R is a number of k-dimensional faces of a polytope R, the notation $Q < P$ means that Q is a face of P, and

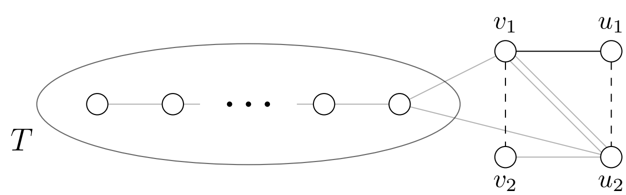
$$A_d^{(i,k)} = \binom{d-i}{k-i} \cdot \frac{\binom{\lceil d/2 \rceil}{i} + \binom{\lfloor d/2 \rfloor}{i}}{\binom{\lceil d/2 \rceil}{k} + \binom{\lfloor d/2 \rfloor}{k}}.$$

Corollary 4.2. *Consider a simple convex d-dimensional polytope, $d \geqslant 3$. The mean edge number of its 2-dimensional faces is less than*

$$A_d^{(1,2)} = \begin{cases} \frac{4(d-1)}{d-2}, & d \text{ is even;} \\ \frac{4d}{d-1}, & d \text{ is odd.} \end{cases}$$

Let $P \subset \mathbb{H}^d$ be a compact Coxeter polytope whose Coxeter diagram S contains no Lannér subdiagrams of order $\geqslant 3$. Denote by a_l the number of its l-dimensional faces and by $a_{2,k}$ the number of its 2-dimensional k-gonal faces. Note that the absence of high-order Lannér subdiagrams implies that $a_{2,3} = 0$.

Lemma 4.3. *Under these assumptions, $a_{2,4} \leqslant a_0 \cdot (d-1)$.*



Proof. Let T be the subdiagram of the diagram S that corresponds to a 4-gonal face. There are vertices v_1, v_2, u_1, u_2 of the diagram S such that the diagrams $\langle T, v_i, u_j \rangle$ are elliptic for $i, j \in \{1, 2\}$ and diagrams $\langle v_1, v_2 \rangle$ and $\langle u_1, u_2 \rangle$ are Lannér diagrams. Since the diagram $\langle v_1, v_2, u_1, u_2 \rangle$ is not superhyperbolic, then, without loss of generality, we may assume that $[v_1, u_1] \geqslant 1$.

Thus, the elliptic diagram $\langle T, v_1, u_1 \rangle$ with the edge $v_1 u_1$ provides an angle of the 4-gonal face. The number of such diagrams is equal to a_0. Every such diagram contains at most $d-1$ edges. □

Proof of Theorem C. Let $d \geqslant 13$. Assume that there exists a compact hyperbolic Coxeter polytope $P \subset \mathbb{H}^d$ whose Coxeter diagram S contains no Lannér subdiagrams of order $\geqslant 3$. From Corollary 4.2 it follows that the mean number of vertices in 2-dimensional faces $\varkappa = \binom{d}{2} \cdot \frac{a_0}{a_2}$ is less than $\frac{4 \cdot 13}{12} = 4\frac{1}{3}$. Since P contains no 2-dimensional triangular faces,

$$a_{2,4} > \frac{2}{3} \cdot a_2 = \frac{2}{3} \cdot \binom{d}{2} \cdot \frac{a_0}{\varkappa} > \frac{2}{3} \cdot \frac{13 \cdot 12}{2} \cdot \frac{a_0}{13/3} = 12 a_0.$$

On the other hand, Lemma 4.3 implies that $a_{2,4} \leqslant a_0 \cdot (d-1) \leqslant 12 a_0$. □

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