Locally conformally Hessian and statistical manifolds.

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Abstract

A statistical manifold (M, D, g) is a manifold M endowed with a torsion-free connection D and a Riemannian metric g such that the tensor Dg is totally symmetric. If D is flat then (M, g, D) is a Hessian manifold. A locally conformally Hessian (l.c.H) manifold is a quotient of a Hessian manifold (C, ∇, g) such that the monodromy group acts on C by Hessian homotheties, i.e. this action preserves ∇ and multiplies g by a group character. The l.c.H. rank is a rank of the image of this character considered as a function from the monodromy group to real numbers. A l.c.H. manifold is called radiant if the Lee vector field ξ is Killing and satisfies $\nabla\xi = \lambda\,\mathrm{Id}$. We prove that the subset set of radiant l.c.H. metrics of l.c.H. rank 1 is dense in the set of all radiant l.c.H. metrics. We prove the structure theorem for compact radiant l.c.H. manifold of l.c.H. rank 1. Every such manifold C is fibered over a circle, the fibers are statistical manifolds of constant curvature, the fibration is locally trivial, and C is reconstructed from the statistical structure on the fibers and the monodromy automorphism induced by this fibration.

Contents

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[2]Pavel Osipov is partially supported by the HSE University Basic Research Program and by the contest “Young Russian Mathematics”.

1 Introduction

A **flat affine manifold** is a differentiable manifold equipped with a flat torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all transition functions between charts are affine transformations (see [FGH] or [Sh]). A **Hessian manifold** is a flat affine manifold (C, ∇) with a metric g which is locally equivalent to a Hessian of a function. Equivalently, the metric g is Hessian if and only if the tensor ∇g is totally symmetric.

The metric g on a flat affine manifold called **locally conformally Hessian (l.c.H)** if for any open neighborhood $U \subset C$ there exists a function f on U such that the locally defined metric $e^{-f}g$ is Hessian. The main purpose of this paper is to describe compact flat affine manifolds with an l.c.H. metric.

A **statistical manifold** (C, D, g) is a manifold M endowed with a torsion-free connection D and a Riemannian metric g such that the tensor Dg is totally symmetric. The term "statistical manifolds" arose in information geometry (see [AN]). In this sense, statistical manifolds is a space of probability distributions endowed with the Fisher information metric. For example, the statistical manifold corresponding to the family of normal distributions is isometric to hyperbolic plane.

A statistical manifold (C, D, g) is said to be **of constant curvature** c if the curvature tensor Θ_D satisfies

$$\Theta_D(X, Y)Z = c\,(g(Y, Z)X - g(X, Z)Y)\,,$$

for any $X, Y, Z \in TM$. For example, a Riemannian manifold of constant section curvature is statistical manifold of constant curvature. The definition of statistical manifolds of constant curvature arose in the context of geometry of affine hypersurfaces ([Ku]). We describe this origin in Section 3.2 (Theorems 3.10 and 3.11). Note that Hessian manifolds are statistical manifolds of curvature 0. We assume that the curvature of a statistical manifold is not equal to 0.

Convex projective geometry provides a wide class of statistical manifolds. A domain $U \subset \mathbb{RP}^n$ is called properly convex if the closure of U is a compact convex set in some affine chart. If Γ is a discrete subgroup of the group of projective automorphisms of a properly convex domain $U \subset \mathbb{RP}^n$ such that $M = U/\Gamma$ is a manifold then M is called a **properly convex** $\mathbb{RP}^n$-manifold. For examples of compact properly convex $\mathbb{RP}^n$-manifolds see [B].

Theorem 1.1 ([KO])**.** Any properly convex $\mathbb{RP}^n$-manifold admits a statistical structure of negative constant curvature. Any compact statistical manifold of negative constant curvature admits a properly convex $\mathbb{RP}^n$ structure.

The proof of Theorem 1.1 is based on the characteristic of properly convex $\mathbb{RP}^n$-structures on compact manifolds given in [L]. This characteristic is equivalent to the following theorem.

Theorem 1.2 ([L] and [KO])**.** A compact statistical manifold (M, D, g) of negative constant curvature admits a properly convex $\mathbb{RP}^n$ structure if and only if M admits a D-flat volume form.

We will give an alternative proof of Theorem 1.1 using the results of the present paper.

A **self-similar manifold** (M, g, ξ) is a Riemannian manifold endowed with vector field ξ satisfying $\mathcal{L}_\xi g = 2g$ ([A]). A **self-similar Hessian manifold** (C, ∇, g, ξ) is a Hessian manifold (C, ∇, g) endowed with an affine vector field ξ such that (C, g, ξ) is a self-similar manifold ([Os], Definition 3.3). If the field ξ satisfies $\nabla\xi = \lambda \mathrm{Id}$ then we say that (M, ∇, g, ξ) is a **radiant Hessian manifold** (([Os], Definition 3.8)). The condition $\xi = \lambda \mathrm{Id}$ means that there exists a flat affine atlas on C such that in local coordinates we have $\xi = \lambda \sum \frac{\partial}{\partial x^i}$, i.e. ξ is proportional to the radiant vector field. The field ξ on a self-similar Hessian manifold (C, ∇, g, ξ) is a potential vector field(i.e. ξ equals to a gradient of a function) if and only if (C, ∇, g, ξ) is a direct product of radiant Hessian manifolds (([Os], Theorem 1.4)).

We construct the correspondence between radiant Hessian manifolds and statistical manifolds of constant curvature. Precisely, we show that a Riemannian cone $(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2)$ over a statistical manifold (M, g, D) of constant curvature admits a structure of a radiant Hessian manifold. Conversely, level sets of a Hessian potential on a radiant Hessian manifold are statistical manifolds of constant curvature.

By dd^c Lemma, Any Kähler form can be locally represented as a complex Hessian $dd^c\varphi$. Hence, we can consider Hessian manifold are a real analogue of Kähler manifolds. A **Sasakian manifold** is a Riemmanian manifold (M, g)

such that the cone metric $g = s^2 g_M + ds^2$ on $M \times \mathbb{R}^{>0}$ is Kähler with respect to a dilatation-invariant complex structure I (see [OV1]). Thus, we can consider statistical manifolds of constant curvature as an analogue of Sasakian manifolds.

We extend this analogy and define **locally conformally Hessian (l.c.H.) manifold** similarly to locally conformally Kähler manifolds (see e.g. [OV1]). An l.c.H. manifold (C, ∇, g, θ) is a flat affine manifold (C, ∇) endowed with a Riemannian metric g and closed 1-form θ such that $\nabla g - \theta \otimes g$ is a totally symmetric tensor. The form θ and the vector field $\xi = \theta^\sharp$ are called a **Lee form** and a **Lee vector field**. If locally $\theta = df$ then $e^{-f} g$ is a locally defined Hessian metric.

We study l.c.H. manifolds with an affine Killing Lee vector field. This class of manifolds is an analogue of Vaisman manifolds: a l.c.K. manifold is called Vaisman if the Lee vector field is Killing. The Lee vector field of any Vaisman manifold is holomorphic. The affine structure on an l.c.H. manifold take the same place as the complex structure on a l.c.K manifold. In contrast to the l.c.K. case, the Lee vector field on an l.c.H. manifold can be Killing but not affine (Example 4.5).

Let (C, ∇, g, θ) be a l.c.H. manifold with and ξ the Lee vector field. Then we say that (C, ∇, g, θ) is a **radiant l.c.H. manifold** if ξ is Killing and there exists a constant $\mu \in \mathbb{R}$ such that $\nabla \xi = \mu\,\mathrm{Id}$. Equivalently, a l.c.H. manifold if it is quotient if radiant Hessian manifold $\left(\widetilde{C}, \widetilde{\nabla}, \widetilde{g}\right)$ such that the monodromy group acts on $\widetilde{C}$ by Hessian homotheties (see Definition 4.14 and Proposition 4.15). The following theorem motivates us to consider radiant l.c.H. manifolds.

Theorem 1.3. Let (C, ∇, g, θ) be a compact l.c.H. manifold with an affine Killing Lee vector field ξ. Suppose (C, g) is not a locally conformally flat Riemannian manifold. Then (C, ∇, θ, ξ) is a radiant Hessian manifold.

An l.c.H. manifold (C, ∇, g, θ) admits a covering $\widetilde{C}$ endowed with a Hessian metric $\widetilde{g}$ such that the deck group $\mathrm{Aut}_C(\widetilde{C})$ acts on $\widetilde{C}$ by Hessian homotheties. It defines a character

$$\chi : \mathrm{Aut}_C(\widetilde{C}) \to \mathbb{R}^{>0}.$$

We call the rank of the group $\mathrm{Im}\chi \subset \mathbb{R}^{>0}$ by l.c.H. rank of (C, ∇, g, θ). The l.c.H. rank of an l.c.H. manifold (M, ∇, g, θ) is equal to 1 if and only if $[\theta] \in H^1(M, \mathbb{Q})$.

Now we describe the main results of the present paper.

Theorem 1.4. Let (C, ∇) be a compact flat affine manifolds and $L \subset H^1(C, \mathbb{R})$ a set of cohomology classes of Lee forms of radiant l.c.H. structures. Then L is open.

It follows from Theorem 1.4 that any radiant l.c.H. structure can be approximated by a radiant l.c.H. structure of rank 1.

Corollary 1.5. Let (C, ∇, g, θ) be a radiant l.c.H. manifold. Then there exist a metric g' and a 1-form θ' on C such that (C, ∇, g', θ') is a radiant l.c.H. manifold of l.c.H. rank 1.

Theorem 1.6. Let φ be an automorphism of a statistical manifold (M, g_M, D) of constant curvature. Consider the automorphism

$$\varphi_q : M \times \mathbb{R}^{>0} \to M \times \mathbb{R}^{>0}, \quad \varphi_q(m, t) = (\varphi(m), qt).$$

Then $M \times \mathbb{R}^{>0}/_{\varphi_q}$ admits a radiant l.c.H. structure of l.c.H. rank 1.

If the Lee vector field ξ on a l.c.H. manifold (C, ∇, g, ξ) is Killing then $a := g(\xi, \xi)$ is a constant (Corollary 4.11).

Theorem 1.7. Let (C, ∇, g, θ) be a compact radiant l.c.H. manifold of l.c.H. rank 1, ξ the Lee vector field, and $\nabla \xi = \mu \,\mathrm{Id}$, where μ is a constant. Then (C, ∇, g, θ) can be constructed from a statistical manifold (M, D, g_M) of constant curvature as in Theorem 1.6. Moreover, (M, D, g) is a statistical manifold of negative constant curvature if and only if $\mu \in (-\infty, -a) \cup (0, \infty)$, where $a = g(\xi, \xi)$.

Theorems 1.4, 1.6, and 1.7 are analogues to the structure theorems for compact Vaisman manifolds from [OV1] (except the condition on sign of curvature).

Combining Theorem 1.1, Corollary 1.5, and Theorem 1.7 we get the following corollary.

Theorem 1.8. Let (C, ∇, g, θ) be a radiant l.c.H. manifold, ξ the Lee vector field, and $\nabla \xi = \mu \,\mathrm{Id}$, where μ is a constant. Suppose $\mu \in (-\infty - a) \cup (0, \infty)$, where $a = g(\xi, \xi)$. Then the universal covering of C is isomorphic (as a flat affine manifold) to a convex cone without full straight lines.

2 Self-similar manifolds

2.1 Self-similar manifolds: the definition and basis properties.

Definition 2.1. A **self-similar manifold** (C, g, ξ) is a Riemannian manifold (C, g) endowed with a field ξ satisfying

$$\mathcal{L}_\xi g = 2g.$$

If ξ is complete then the manifold is called a **global self-similar manifold**.

It follows from the definition that a global self-similar manifold is a Riemannian manifold endowed with a 1-parameter group of homothetic automorphims $\{\varphi_t\}$ such that $\varphi_t^* g = e^{2t} g$. The term "self-similar" is motivated by the fact that for any $\lambda \in \mathbb{R}^{>0}$, a global self-similar manifold (C, g) is isometric to $(C, \lambda g)$.

Example 2.2. Let $(C = M \times \mathbb{R}^{>0}, g = s^2 g_M + ds^2)$ be a **Riemannian cone** and $\xi = s\frac{\partial}{\partial s}$. Then (C, g, ξ) is a globally self-similar manifold.

Example 2.3 ([Os])**.** Let φ and s are coordinates on S^1 and $\mathbb{R}^{>0}$ then the collection $\left(C = S^1 \times \mathbb{R}^{>0}, g = s^2 d\varphi^2 + sds \cdot d\varphi + ds^2, s\frac{\partial}{\partial s}\right)$ is a global self-similar manifold but (C, g) is not isometric to a Riemannian cone.

Definition 2.4. We say that (C, g, ξ) is a **self-similar manifold with a potential homothetic vector field** if (M, g, ξ) is a selfimilar manifold and ξ is locally defined as a gradient of a function. If $\xi = \operatorname{grad} f$ on a domain U then $\iota_\xi g|_U = df$. Moreover, a form is closed if and only if it is locally exact. Therefore, the vector field ξ is potential if and only if $d\iota_\xi g = 0$.

Theorem 2.5 ([Os])**.** Let (C, g, ξ) be a global self-similar manifold with a potential homothetic vector field.

(i) If ξ vanishes at a point then (C, g, ξ) is Euclidean space with a radiant vector field $\left(\mathbb{R}^n, \sum_{i=1}^n \left(dx^i\right)^2, \sum x^i \frac{\partial}{\partial x^i}\right)$.

(ii) If ξ does not vanish at any point then (C, g, ξ) is a Riemannian cone $\left(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2, s\frac{\partial}{\partial s}\right)$.

2.2 Self-similar Hessian manifolds

Definition 2.6. A **flat affine manifold** (C, ∇) is a differentiable manifold C equipped with a flat torsion-free connection ∇. Equivalently, it is a manifold equipped with an atlas such that all transition maps between charts are affine transformations (see e.g. [FGH]). A **radiant manifold** (C, ∇, ρ) is a flat affine manifold (C, ∇) endowed with a **radiant vector field** ρ i.e. a vector field satisfying

$$\nabla \rho = \mathrm{Id} \tag{2.1}$$

Equivalently, it is a manifold equipped with an atlas such that all transition maps between charts are linear transformations. In the corresponding local coordinates we have

$$\rho = \sum x^i \frac{\partial}{\partial x^i}$$

(see e.g. [Go]).

Definition 2.7. A **Hessian manifold** (C, ∇, g) is a flat affine manifold endowed with a Riemannian metric such that ∇g is a totally symmetric tensor.

Definition 2.8. Let (C, ∇) be a flat affine manifold. A vector field ξ on C is called **affine** if the flow along ξ preserves the connection ∇.

Proposition 2.9 ([Go])**.** Let X be a vector field on a flat affine manifold (C, ∇) then the following conditions are equivalent:

(i) X is affine.

(ii) ∇X is a ∇-flat 1-1 tensor.

(iii) For any local coordinates $(x^1, \dots, x^n)$, we have $X = \sum a_i(x) \frac{\partial}{\partial x^i}$, where $a_1(x), \dots, a_n(x)$ are linear functions.

Definition 2.10. A **self-similar Hessian manifold** (C, ∇, g, ξ) is a Hessian manifold (C, ∇, g) endowed with an affine vector field ξ such that $\mathcal{L}_\xi g = 2g$ and the flow along ξ preserves ∇.

Definition 2.11 ([Os])**.** A self-similar Hessian manifold (C, ∇, g, ξ) is a **radiant Hessian manifold** if it admits a radiant vector field ρ and a constant $\lambda \in \mathbb{R}$ such that $\xi = \lambda \rho$. Equivalently, there is a flat affine atlas such that in the corresponding local coordinates we have

$$\xi = \lambda \sum x^i \frac{\partial}{\partial x^i}$$

(see Definition 2.6).

Remark 2.1. The case $\lambda = 2$ is studied in [G-A] in the context of Equilibrium Thermodynamics. In this case, we have $\mathcal{L}_\rho g = g$ where $\rho = \sum x^i \frac{\partial}{\partial x^i}$ and $g = \text{Hess } \varphi$ locally. Then φ is linear along ρ and $\iota_\rho g = \iota_\rho(\text{Hess } \varphi) = 0$. Hence, if $\lambda = 2$ then g cannot be positive definite. If $\lambda 0$ then $\mathcal{L}_\xi g = 0 \neq 2g$. Thus, $\lambda \in \mathbb{R} \setminus \{0, 2\}$.

We say that a self-similar Hessian manifold (U, ∇, ξ, g) is a direct product of self-similar Hessian manifolds $(U_i, \nabla_i, \xi_i, g_i)$ if

$$U = \prod U_i, \quad \nabla = \sum \nabla_i, \quad g = \sum g_i, \quad \xi = \sum \xi_i.$$

Since the condition $\forall i \in \{1 \dots n\} : \mathcal{L}_{\xi_i} g_i = 2g_i$ implies $\mathcal{L}_\xi g = 2g$, the product of self-similar Hessian manifold is actually self-similar Hessian manifold.

Theorem 2.12. Let (C, ∇, g, ξ) be a self-similar Hessian manifold. Then the following conditions are equivalent.

(i) The vector field ξ is potential.

(ii) There are $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and a ∇-flat decomposition of the tangent bundle $TC = \bigoplus_{i=1^k} V_i$ such that for any $i \in \{1, \dots, k\}$ we have $\nabla\xi|_{V_i} = \lambda_i \text{Id}$.

(iii) For any point $p \in C$ there is a neighborhood $U \ni p$ and a collection of radiant Hessian manifolds $(U_i, \nabla_i, \xi_i, g_i)$ such that (U, ∇, ξ, g) is a direct product of self-similar Hessian manifolds $(U_i, \nabla_i, \xi_i, g_i)$ and the decomposition to the direct product $U = \prod U_i$ corresponds to $TU = \oplus V_i|_U$.

Theorem 2.12 can be reformulated in a short form.

Theorem 2.13. Let (C, ∇, g, ξ) be a self-similar Hessian manifold. Then ξ is potential if and only if (C, ∇, g, ξ) is locally isomorphic to a direct product of radiant Hessian manifolds.

Corollary 2.14. Let (C, ∇, g, ξ) be a self-similar Hessian manifold with a potential homothetic vector field ξ. Suppose the holonomy of (C, g) is irreducible. Then (C, ∇, g, ξ) is a radiant Hessian manifold.

3 Statistical manifolds

3.1 Statistical manifolds: the definition and examples

Definition 3.1. A **statistical manifold** (M, D, g) is a manifold M endowed with a torsion-free connection D and a Riemannian metric g such that the tensor Dg is totally symmetric. A statistical manifold (M, D, g) is said to be **of constant curvature** $c \in \mathbb{R}$ if the curvature tensor Θ_D satisfies

$$\Theta_D(X,Y)Z = c\,(g(Y,Z)X - g(X,Z)Y)\,, \tag{3.1}$$

for any $X, Y, Z \in TM$ ([Ku]).

The following example motivates the definition above.

Example 3.2. Let D be the Levi-Civita connection on a Riemannian manifold (M, g). Then (M, g, D) is a statistical manifold. The sectional curvature of M is constant and equals $c \in \mathbb{R}$ if and only Equation (3.1) is satisfies. Thus, (M, g, D) is a statistical manifold of constant curvature if and only if the sectional curvature of (M, g) is constant.

Example 3.3. Let $V \subset \mathbb{R}^{n+1}$ be a convex cone without full straight lines and

$$V^* = \{y \in \left(\mathbb{R}^{n+1}\right)^* \mid \forall x \in V : (x, y) > 0\}$$

be the dual cone. Consider the **characteristic function**

$$\psi(x) = \int_{V^*} e^{-(x,y)} dy$$

and the **characteristic hypersurface** $S = \{x \in V | \psi(x) = 1\}$. Then the bilinear the function ψ is smooth and form $g = \mathrm{Hess}\,(\ln \psi)$ is positive definite ([Vi]). Let ∇ be a standard connection on $\mathbb{R}^{n+1}$ and $\xi = \sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}$. Set a connection D on S and a bilinear form h on S as the projection of ∇ on components of the decomposition $TV|_S = TS \oplus \mathbb{R}\xi$, i.e. D and h are the connection D on S and the bilinear form h on S satisfying

$$\forall X, Y \in TS \quad : \quad \nabla(X, Y) = D_X Y + h(X, Y)\xi.$$

Then $(S, g|_S, D)$ is a statistical manifold of constant negative curvature ([Sh], Example 5.1 and Corollary 5.3). Let $\mathrm{Aut}_{\mathrm{SL}}(V)$ be the subgroup of automorphisms of V in $\mathrm{SL}\,(\mathbb{R}^{n+1})$. Then $\mathrm{Aut}_{\mathrm{SL}}(V)$ preserves $(S, g|_S, D)$ ([Vi]).

Remark 3.1. A convex cone V is called homogeneous if there is a transitive action on V by a group of Linear linear automorphisms. The characteristic function of a homogeneous cone is a solution of a Monge–Ampère equation and characteristic hypersufaces of cones are affine spheres of a negative constant curvature ([S]).

The following example is a reformulating of the previous one.

Example 3.4. We say that a domain $\Omega \subset \mathbb{RP}^n$ is a convex if the intersection of ω with any full projective line is connected. Let $\Omega \subset \mathbb{RP}^n$ be a convex domain without any full projective line, $\mathrm{Aut}(\Omega)$ the group of projective automorphisms of ω, and Γ a discrete subgroup of $\mathrm{Aut}(\Omega)$ such that $M = \Omega/\Gamma$ is a compact manifold. Then M is called a **properly convex $\mathbb{RP}^n$ manifold** (see [B]). Consider the projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$. A connected component $V \subset \mathbb{R}^{n+1}$ of the preimage $\pi^{-1}\Omega$ of a properly convex domain Ω is a convex cone without straight full lines. We can identify Ω with a characteristic hypersurface S of V and Γ with a subgroup of $\mathrm{Aut}_{\mathrm{SL}}(V)$. Hence, the $\mathrm{Aut}_{\mathrm{SL}}(V)$-invariant statistical structure on S from Example 3.3 can be identified with a $\mathrm{Aut}(\Omega)$-invariant statistical structure on U. Therefore, this statistical structure of constant curvature on $S \simeq U$ can be push forward to $M = U/\Gamma$. That is, any properly convex $\mathbb{RP}^n$ manifold admit a statistical structure of constant negative curvature.

It is proved in [KO] that any compact statistical manifold of negative constant curvature is a properly convex $\mathbb{RP}^n$ manifold.

Example 3.5 ([FHOSS])**.** A **Sasakian manifold** is a Riemannian manifold (M, g_M) such that the cone metric $g = s^2 g_M + ds^2$ on $M \times \mathbb{R}^{>0}$ is Kähler with respect to a dilatation-invariant complex structure I (see [OV1]). The field $\xi = I\frac{\partial}{\partial s} \in T(M \times 1) \simeq TM$ is called the **Reeb vector field**. For any $X, Y \in TM$ set $K(X,Y) = g(X,\xi)g(Y,\xi)\xi \in TM$. Then for any $f \in C^\infty M$ we have a statistical structure $(g, D^f := D + fK)$ on M (see [FHOSS]). In particular, if $M \simeq S^{2k+1}$ is a Sasakian sphere then for any $f \in C^\infty S^{2k+1}$ the collection $(S^{2k+1}, g, D^f = D + fK)$ is a statistical manifold of constant curvature 1.

3.2 Statistical manifolds of constant curvature, dual connections, and affine immersions

Definition 3.6. Let $M \subset \mathbb{R}^{n+1}$ be an n-dimensional hypersurface and ∇ be a standard connetion on $\mathbb{R}^{n+1}$. The section $\xi \in T\mathbb{R}^{n+1}|_M$ is a **transversal vector field** along M if

$$T\mathbb{R}^{n+1}|_M = TM \oplus \mathbb{R}\xi.$$

When ξ is given, we can define the **induced affine connection** D on M and the **second fundamental form** h on M as follows: for any $X, Y \in TM$

$$\nabla_X Y = D_X Y + h(X,Y)\xi.$$

If there exists a constant λ such that $\xi = \lambda \left(\sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i} \right)$ then the pair (M, ξ) is called a **centro-affine submanifold**.

Definition 3.7. Let D be an affine connection on M, $\iota : M \to \mathbb{R}^n$ an immersion and $\xi \in \iota^* T\mathbb{R}^{n+1}$ such that D is equal to the pullback of the induced affine connection M. Suppose for any neighborhood $U \subset M$ such that $\iota|_U$ is inclusion, the pair $(\iota U, \iota_* \xi)$ is a centro-affine submanifold. Then the pair (ι, ξ) is called a **centro-affine immersion** of (M, D).

Definition 3.8. Let (M, g) be a Riemannian manifold. Two affine connection D and $\overline{D}$ are called dual to each other (with respect to g) if

$$\mathcal{L}_X \left(g(Y, Z) \right) = g(D_X Y) + g(Y, \overline{D}_X Z).$$

Proposition 3.9. Let D and $\overline{D}$ be dual affine connections on a Riemannian manifold (M, g). Then (M, g, D) is statistical if and only if both ∇ and $\overline{\nabla}$ are torsion-free.

As a consequence, (M, g, D) is a statistical manifold if and only if $(M, g, \overline{D})$ is a statistical manifold.

Theorem 3.10 ([Ku])**.** Let (M, g, D) be a statistical manifold of dimension $d \geq 3$. If there exist centro-affine immersions (ι, ξ) of (M, D) and $(\bar{\iota}, \bar{\xi})$ of $(M, \overline{D})$ such that the pullbacks of the second fundamental forms are equal to g then (M, g, D) is a statistical manifold of a constant curvature c. Moreover, $\nabla \xi = -c\mathrm{Id}$ and $\nabla \bar{\xi} = -c\mathrm{Id}$, where ∇ is the standard flat connection o $\mathbb{R}^n$.

Theorem 3.11 ([Ku])**.** Let (M, g, D) be a statistical manifold. If (M, g, D) is a statistical manifold of a constant curvature c then there exist centro-affine immersions (ι, ξ) of (M, D) and $(\bar{\iota}, \bar{\xi})$ of $(M, \overline{D})$ such that the second fundamental form is equal to g and $\nabla \xi = -c\mathrm{Id}$ and $\nabla \bar{\xi} = -c\mathrm{Id}$, where ∇ is the standard flat connection o $\mathbb{R}^n$.

3.3 Statistical manifolds of constant curvature and radiant Hessian manifolds

Consider a Hessian metric $g = \mathrm{Hess}\, \varphi$ on $\mathbb{R}^n$ and the level set

$$M = \{ x \in \mathbb{R}^{n+1} \mid \varphi(x) = 1 \}.$$

Suppose that for any $x \in M^{n+1}$ we have $d\varphi_x \neq 0$. Then M is a submanifold in $\mathbb{R}^{n+1}$. Let E be the gradient of φ with respect to the metric g. Then the field E is transversal to M.

Proposition 3.12 ([Sh], Lemma 5.1 and Example 5.1)**.** Let

$$\varphi, \quad M = \{\varphi = 1\} \subset \mathbb{R}^{n+1}, \quad g = \text{Hess}\ \varphi, \quad E = \text{grad}\ \varphi$$

be as above. Consider the induced affine connection D on M and the second fundamental form h on M (see Definition 3.6). Then

$$h = -\frac{1}{E(\varphi)} g|_M.$$

Moreover, suppose (M, E) is a centro-affine immersion and $\nabla E = \mu \text{Id}$. Then $(D, g|_M)$ is a statistical structure of a constant curvature $c = \frac{\mu}{E(\varphi)}$.

Proposition 3.13 ([Os], Proposition 3.11)**.** Let (C, g, ∇, ξ) be a radiant Hessian manifold and $\nabla\xi = \lambda \text{Id}$, where $\lambda \in \mathbb{R}$. Then

$$g = \text{Hess}\left(\frac{g(\xi,\xi)}{4-2\lambda}\right).$$

Theorem 3.14. Let $\left(M \times \mathbb{R}^{>0}, g = s^2 g_M + ds^2, \nabla, s\frac{\partial}{\partial s}\right)$ be a radiant Hessian manifold and $\lambda \in \mathbb{R}$ a number which satisfies $\nabla\left(s\frac{\partial}{\partial s}\right) = \lambda \text{Id}$. Then there exists a connection D on M such that (M, g, D) is a statistical manifold of constant curvature $c = \lambda(2-\lambda)$.

Proof. According to the Proposition 3.13, the function $\varphi = \frac{s^2}{4-2\lambda}$ is a potential of g and $M \times \{1\}$ is a level set of the potential. Then the gradient of φ is

$$E = \frac{1}{2-\lambda} s \frac{\partial}{\partial s}.$$

Since $\nabla\left(s\frac{\partial}{\partial s}\right) = \lambda \text{Id}$, we have

$$\nabla E = \frac{\lambda}{2-\lambda} \text{Id}.$$

Let D be the affine connection induced by E. According to Proposition 3.12, (M, g, D) is a statistical structure of curvature

$$c = \frac{\lambda}{(2-\lambda)E(\varphi)} = \lambda(2-\lambda).$$

■

Remark 3.2. Consider assumption or Theorem 3.14. Let h be the second fundamental form of the pair $(M, s\frac{\partial}{\partial s})$ from the proof of Theorem 3.14. By Proposition 3.12, we have

$$h = -\frac{1}{E(\varphi)}g = -(2-\lambda)^2 g.$$

Moreover, the vector field

$$\rho = \frac{1}{\lambda}s\frac{\partial}{\partial s}$$

on $M \times \ R^{>0}$ is radiant and

$$E = \frac{\lambda}{2-\lambda}\rho.$$

Lemma 3.15. Let $(M \times \mathbb{R}^{>0}, \nabla)$ be a flat affine manifold s a coordinate on $\mathbb{R}^{>0}$, $\nabla\left(s\frac{\partial}{\partial s}\right) = \lambda \mathrm{Id}$, where $\lambda \in \mathbb{R}\setminus\{0,2\}$, and $\varphi = \left(\frac{s^2}{4-2\lambda}\right)$. Then $g = \operatorname{Hess}\varphi$ can be written as

$$g = s^2 g_M + ds^2,$$

where g_M is a symmetric bilinear form on M.

Proof. We have

$$\operatorname{Hess}\varphi\left(s\frac{\partial}{\partial s}, s\frac{\partial}{\partial s}\right) = s\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\left(\frac{s^2}{4-2\lambda}\right)\right) - \left(\nabla_{s\frac{\partial}{\partial s}}s\frac{\partial}{\partial s}\right)\left(\frac{s^2}{4-2\lambda}\right) =$$
$$= \frac{4s^2}{4-2\lambda} - \lambda s\frac{\partial}{\partial s}\left(\frac{s^2}{4-2\lambda}\right) = \frac{4s^2}{4-2\lambda} - \frac{2\lambda s^2}{4-2\lambda} = s^2$$

and for any $X \in TM$

$$\operatorname{Hess}\varphi\left(s\frac{\partial}{\partial s}, X\right) = s\frac{\partial}{\partial s}\left(X(\varphi)\right) - \nabla_{s\frac{\partial}{\partial s}}X(\varphi) = 0$$

because $X(\varphi) = 0$ and $\nabla_{s\frac{\partial}{\partial s}}X = \lambda X$. Moreover,

$$\mathcal{L}_{s\frac{\partial}{\partial s}}\operatorname{Hess}\left(\frac{s^2}{4-2\lambda}\right) = 2\operatorname{Hess}\left(\frac{s^2}{4-2\lambda}\right)$$

Thus, $g = s^2 g_M + ds^2$, where g_M is a symmetric bilinear form on M. ∎

Theorem 3.16. Let (M, g_M, D) be a statistical manifold of constant curvature $c \leq 1$ and λ a solution of the equation $\lambda(2-\lambda) = c$ (as in Theorem 3.14). Then there exists a connection ∇ on $M \times \mathbb{R}^{>0}$ such that

$$\left(M \times \mathbb{R}^{>0}, g = s^2 g_M + ds^2, \nabla, s\frac{\partial}{\partial s}\right)$$

is a radiant Hessian manifold and $\nabla\left(s\frac{\partial}{\partial s}\right) = \lambda\,\mathrm{Id}$.

Proof. This theorem is converse to Theorem 3.14. Let λ a solution of the equation $\lambda(2-\lambda) = c$ (as in Theorem 3.14). Then $(-(2-\lambda)^2 g, D)$ is a statistical structure of curvature $-\frac{\lambda}{2-\lambda}$. According to Theorem 3.11, there exists a centro-affine immersion $\iota : M \to \mathbb{R}^{n+1}$ with a transversal field $\xi = \frac{\lambda}{2-\lambda}\sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}$ and the second fundamental form $-(2-\lambda)^2 g$ (as in Remark 3.2). Set an immersion

$$\hat{\iota} : M \times \mathbb{R}^{>0} \to \mathbb{R}^{n+1}, \qquad \hat{\iota}(m \times s) = s^{\lambda}\iota(m).$$

Then we have $s\frac{\partial}{\partial s} = \hat{\iota}^*\left(\lambda \sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}\right)$. Set a connection ∇ on $M \times \mathbb{R}^{>0}$ as the pullback of the standard connection on $\mathbb{R}^{n+1}$ and

$$\widetilde{g} = \mathrm{Hess}\frac{s^2}{4-2\lambda}.$$

According to Lemma 3.15, $\widetilde{g}$ can be written as

$$\widetilde{g} = s^2 \widetilde{g}_M + ds^2.$$

The vector $E = \frac{1}{2-\lambda}s\frac{\partial}{\partial s}$ is the gradient of $\frac{s^2}{4-2\lambda}$ (as in Remark 3.2). According to Proposition 3.12,

$$\widetilde{g}_M = -E(\varphi)h = g_M.$$

Thus, $\left(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2, \nabla, s\frac{\partial}{\partial s}\right)$ is a radiant Hessian manifold. ∎

We constructed a correspondence between radiant Hessian manifolds and statistical manifolds of constant curvature.

4 Locally conformally Hessian manifold

4.1 Locally conformally Hessian manifold: the definition and examples

Definition 4.1. A **locally conformally Hessian (l.c.H.) manifold** (C, ∇, g, θ) is a flat affine manifold (C, ∇) endowed with a Riemannian metric g and a closed 1-form θ such that $\nabla g - \theta \otimes g$ is a totally symmetric tensor. The form θ and the vector field $\xi = \theta^\sharp$ are called a **Lee form** and a **Lee vector field**.

Suppose that (C, ∇, g, θ) is an l.c.H. manifold. The closed form θ is locally defined as a differential of a function $\theta = df$. Since we have

$$\nabla\left(e^{-f}g\right) = e^{-f}\nabla g - e^{-f}df \otimes g,$$

the tensor $\nabla\left(e^{-f}g\right)$ is totally symmetric. Hence, the locally defined metric $e^{-f}g$ is Hessian (see Definition 2.7).

We a interested in l.c.H. manifold with an affine Killing Lee vector field.

Example 4.2. A **Hopf manifold** is $H^n = \mathbb{R}^n \setminus \{0\}/_{x\sim aAx}$, where $a \in \mathbb{R}^{>0} \setminus \{1\}$ and $A \in \mathrm{O}(n)$. The universal covering $\mathbb{R}^n \setminus \{0\}$ posses the l.c.H. metric $\frac{\sum\left(dx^i\right)^2}{\sum\left(x^i\right)^2}$ which is invariant with respect to homotheties. Hence, this metric induces an l.c.H. structure on the Hopf manifold. The Lee vector equals to $\sum x^i\frac{\partial}{\partial x^i}$.

Example 4.3. Poincaré half-space $\left(\mathbb{R}^{>0} \times \mathbb{R}^n, \frac{\sum_{i=0}^n\left(dx^i\right)^2}{\left(x^0\right)^2}\right)$ is an l.c.H. manifold. The Lee vector field equal $x^0\frac{\partial}{\partial x^0}$. This field is affine but not Killing. Consider the action of $\mathbb{Z}^{n+1}$ on $\mathbb{R}^{>0} \times \mathbb{R}^n$ defined by

$$\left(z^0, z^1, \ldots, z^n\right)\left(x^0, x^1, \ldots, x^n\right) = \left(\lambda^{z^0}x^0, x^1 + z^1, \ldots, x^n + z^n\right),$$

where $\lambda \in \mathbb{R}^{>0} \setminus \{1\}$. This action preserves the affine structure and the metric $\frac{\sum_{i=1}^n\left(dx^i\right)^2}{\left(x^0\right)^2}$. Hence, this metric induces an l.c.H. structure on the torus $\mathbb{R}^{>0} \times \mathbb{R}^n/\mathbb{Z}^n = T^{n+1}$.

Example 4.4. Let V be a convex cone without any straight full line, $\mathrm{Aut}(V)$ the group of linear automorphisms of V, and ψ the characteristic function (see Example 3.3). Then

$$g_{H.} = \mathrm{Hess}\ln\psi \quad \text{and} \quad g_{l.c.H.} = \frac{\mathrm{Hess}\ \psi}{\psi}$$

are $\mathrm{Aut}(V)$-invariant Hessian and l.c.H. forms ([Vi]). The Lee form is equal to $-d\ln\varphi = -\frac{d\varphi}{\varphi}$. Then $\frac{\mathrm{Hess}\ \psi}{\psi}$ induces an l.c.H. structure on V/Γ. Suppose Γ is a subgroup of Linear automorphisms $\mathrm{Aut}(V)$ such that V/Γ is a compact manifold. Then $g_{H.}$ and $g_{l.c.H.}$ induces Hessian and l.c.H. metrics on V/Γ. Note that this situation differs from the Kähler case: a compact manifold cannot admits Kähler and l.c.K. structure at the same time ([Va]).

Example 4.5. Consider an affine plane $\mathbb{R}^2$ with affine coordinates (x, y) and conformally Hessian metric

$$g = \frac{\mathrm{Hess}\,(e^x + e^y)}{(1+e^{\frac{x}{2}})^2 + (1+e^{\frac{y}{2}})^2} = \frac{e^x dx^2 + e^y dy^2}{(1+e^{\frac{x}{2}})^2 + (1+e^{\frac{y}{2}})^2}.$$

Let us check that the Lee vector field is Killing but not affine. Set $\tilde{x} = 1+e^{\frac{x}{2}}$ and $\tilde{y} = 1+e^{\frac{y}{2}}$. Then

$$g = \frac{4d\tilde{x}^2 + 4d\tilde{y}^2}{\tilde{x}^2 + \tilde{y}^2}.$$

The Lee form equivalent to

$$d\ln\left(\frac{1}{\tilde{x}^2+\tilde{y}^2}\right) = \frac{-2\tilde{x}d\tilde{x} - 2\tilde{y}d\tilde{y}}{\tilde{x}^2+\tilde{y}^2}$$

The Lee vector field equals

$$\xi = -\frac{\tilde{x}}{2}\frac{\partial}{\partial\tilde{x}} - \frac{\tilde{y}}{2}\frac{\partial}{\partial\tilde{y}}$$

This vector field is Killing. In flat affine coordinates we have

$$\xi = -\frac{1+e^{\frac{x}{2}}}{2}\frac{\partial}{\partial\left(1+e^{\frac{x}{2}}\right)} - \frac{1+e^{\frac{y}{2}}}{2}\frac{\partial}{\partial\left(1+e^{\frac{y}{2}}\right)} = -\frac{\left(e^{\frac{x}{2}}+e^x\right)\frac{\partial}{\partial x} + \left(e^{\frac{y}{2}}+e^y\right)\frac{\partial}{\partial y}}{(1+e^{\frac{x}{2}})^2 + (1+e^{\frac{y}{2}})^2}.$$

Therefore the Lee vector field ξ is Killing but not affine.

Proposition 4.6. Let (C, ∇, g, θ) be a compact l.c.H manifold and $\theta \neq 0$ at any point. Then (g, ∇) is not globally conformally Hessian.

Proof. Suppose that g is globally conformally Hessian. That is, there exists a function p on C such that the tensor

$$\nabla(pg) = p\nabla g + dp \otimes \theta$$

is totally symmetric. Since C is compact, there exists a point $x \in C$ such that $dp|_{T_xC} = 0$. Hence,

$$\nabla g|_{T_xC} = 0$$

is totally symmetric. Since (C, ∇, θ) is an l.c.H. manifold, the tensor $\nabla g - \theta \otimes g.$ is totally symmetric. Thus,

$$\theta \otimes g|_{T_xC} = 0$$

is totally symmetric. Choose $X, Y \in T_xC$, such that $\theta(X) = 0$ and $\theta(Y) \neq 0$. Then
$$0 \neq \theta(Y) \otimes g(X, X) = \theta(X) \otimes g(X, Y) = 0.$$
We get the contradiction. Thus, g is not globally conformally Hessian. ∎

The examples of statistical manifolds of constant curvature from Section 3.1 combining with the following theorem provide examples of l.c.H. manifolds.

Theorem 4.7. Let φ be an automorphism of a statistical manifold (M, g, D) of constant curvature $c \leq 1$, λ a solution of the equation $c = \lambda(2 - \lambda)$, and $q \in \mathbb{R}^{>0}$. Consider the automorphism
$$\varphi_q : M \times \mathbb{R}^{>0} \to M \times \mathbb{R}^{>0}, \quad \varphi_q(m, t) = (\varphi(m), qt).$$
Then there is a connection ∇ on $M \times \mathbb{R}^{>0}/_{\varphi_q}$ such that
$$\left(M \times \mathbb{R}^{>0}/_{\varphi_q},\ g_M + \frac{ds^2}{s^2},\ \frac{-2ds}{s}\right)$$
is an l.c.H. manifold with an affine Killing Lee vector field $\xi = -2s\frac{\partial}{\partial s}$ satisfying $\nabla\xi = -2\lambda \operatorname{Id}$.

Remark 4.1. The vector field $s\frac{\partial}{\partial s}$ on $M \times \mathbb{R}^{>0}$ and the differential form $\frac{ds}{s}$ on $M \times \mathbb{R}^{>0}$ are invariant with are invariant with respect to φ_q. Hence, we can consider $s\frac{\partial}{\partial s}$, $\frac{ds}{s}$, and $\frac{ds^2}{s^2}$ as tensors on $M \times \mathbb{R}^{>0}/_{\varphi_q}$.

Proof of Theorem 4.7. According to Theorem 3.16, there exists a connection ∇ on $M \times \mathbb{R}^{>0}$ such that the collection $\left(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2, \nabla, s\frac{\partial}{\partial s}\right)$ is a radiant Hessian manifold. The automorphism φ_q preserves the metric $g_M + \frac{ds^2}{s^2}$. Moreover, since the field $s\frac{\partial}{\partial s}$ is affine, φ_q preserves the affine structure ∇. Therefore, the pair $\left(\nabla, g_M + \frac{ds^2}{s^2}\right)$ induces an l.c.H. structure on $M \times \mathbb{R}^{>0}/_{\varphi_q}$. The Lee form equals
$$d \ln \frac{1}{s^2} = \frac{-2ds}{s}$$
and the Lee vector equals $-2s\frac{\partial}{\partial s}$. This field is affine and Killing.

Therefore, $\left(M \times \mathbb{R}^{>0}/_{\varphi_q},\ g_M + \frac{ds^2}{s^2},\ \frac{-2ds}{s}\right)$ is a l.k.H. manifold with an affine Killing Lee vector field. ∎

Proof of Theorem 1.6. If (M, g_M, D) is a statistical manifold of curvature $c > 1$, we replace g_M by cg_M and get a statistical manifold of curvature 1. Using Theorem 4.7, we can construct a l.c.H. structure with an affine Killing Lee vector on $M \times \mathbb{R}^{>0}/_{\varphi_q}$. ∎

4.2 Minimal Hessian covering

Definition 4.8. Let (C, g, ∇, θ) be an l.c.H. manifold. Then the **weight bundle** is the trivial $\mathbb{R}$-bundle L endowed with a flat connection defined by θ.

Definition 4.9. Let (C, ∇, g, θ) be an l.c.H. manifold. Fix a point $x_0 \in C$. Consider the functional $L : \pi_1(C, x_0) \to \mathbb{R}$ defined by

$$L(\gamma) = \int_\gamma \theta.$$

Then the image of L is the monodromy group Γ of the weight bundle (L, θ). Let $\widetilde{C}$ be the covering of C with the fiber Γ. For any $\gamma \in \pi_1(\widetilde{C})$ we have

$$\int_\gamma \pi^* \theta = 0.$$

Therefore, there exists a function f on $\widetilde{C}$ such that $df = \pi^*\theta$. The metric $\widetilde{g} = e^{-f}\pi^* g$ is Hessian. We say that that $\left(\widetilde{C}, \pi^*\nabla, \widetilde{g}\right)$ is the **minimal Hessian covering** of the l.c.H. manifold (C, ∇, g, θ). The deck transform group of the covering $\widetilde{C} \to C$ act on $\widetilde{C}$ by Hessian homotheties.

Proposition 4.10 ([OV2]). Let θ be a 1-form on a Riemannian manifold (C, g), $\xi = \theta^\sharp$, and ∇_{LC} the Levi-Civita connection. Then the following conditions are equivalent:

(i) $\nabla_{\mathrm{LC}}\xi = 0$.

(ii) $\nabla_{\mathrm{LC}}\theta = 0$.

(iii) $d\theta = 0$ and $\mathcal{L}_\xi g = 0$.

Corollary 4.11. Let (C, ∇, g, θ) be an l.c.H. manifold with a Killing Lee vector ξ. Then $a := \theta(\xi) = g(\xi, \xi)$ is constant.

Proof. According to Proposition 4.10, $\nabla_{\text{L-C}}\xi = 0$. Therefore, $g(\xi, \xi)$ is constant. ∎

Lemma 4.12. Let (C, ∇, g, θ) is an l.c.H. manifold, ξ a vector field on C, $\left(\widetilde{C}, \pi^*\nabla, \widetilde{g}\right)$ is the minimal Hessian covering (where $\pi : \widetilde{C} \to C$ is the corresponding covering map), and $a = g(\xi, \xi) = \theta(\xi)$. Then $\mathcal{L}_\xi g = 0$ if and only if and $\left(\widetilde{C}, \widetilde{g}, \widetilde{\xi} := -\frac{2}{a}\pi^*\xi\right)$ is a self-similar manifold.

Proof. There exists a function f on $\widetilde{C}$ such that $\pi^*\theta = df$ and $\widetilde{g} = e^{-f}\pi^* g$ (see Definition 4.9). Therefore,

$$\widetilde{\xi}(f) = \left\langle \widetilde{\xi}, \pi^*\theta \right\rangle = \left\langle -\frac{2}{\theta(\xi)}\pi^*\xi, \pi^*\theta \right\rangle = -2$$

Hence,

$$\mathcal{L}_{\widetilde{\xi}}\widetilde{g} = \mathcal{L}_{\widetilde{\xi}}\left(e^{-f}\pi^* g\right) = -e^{-f}\widetilde{\xi}(f)\pi^* g + e^{-f}\mathcal{L}_{\widetilde{\xi}}\pi^* g = 2\widetilde{g} + e^{-f}\pi^*\left(\mathcal{L}_{\xi} g\right).$$

Thus, $\mathcal{L}_{\widetilde{\xi}}\widetilde{g} = 2\widetilde{g}$ if and only if $\mathcal{L}_{\xi} g = 0$. ■

Proposition 4.13. Let (C, ∇, g, θ) is an l.c.H. manifold with a Killing Lee vector ξ, $a = g(\xi, \xi)$, and $\left(\widetilde{C}, \pi^*\nabla, \widetilde{g}\right)$ its minimal Hessian covering. Then $\left(\widetilde{C}, \widetilde{g}, \widetilde{\xi} := -\frac{2}{a}\pi^*\xi\right)$ is a self-similar manifold with a potential homothetic vector field.

Proof. According to Lemma 4.12, $\left(\widetilde{C}, \widetilde{g}, -\frac{2}{a}\pi^*\xi\right)$ is a self-similar manifold.

There exists a function f on $\widetilde{C}$ such that $\pi^*\theta = df$ and $\widetilde{g} = e^{-f}\pi^* g$ (see Definition 4.9). Since $\iota_{\xi} g = \theta$, we have

$$\iota_{\pi^*\xi}\pi^* g = \pi^*\theta = df$$

and

$$\iota_{\pi^*\xi}\widetilde{g} = e^{-f}df = d\left(-e^{-f}\right).$$

Thus the field $\pi^*\xi$ is potential. According to Corollary 4.11, $g(\xi, \xi)$ is constant. Therefore, the homothetic field $-\frac{2}{a}\pi^*\xi$ is potential. ■

4.3 Radiant l.c.H. manifolds

Definition 4.14. Let (C, ∇, g, θ) be an l.c.H. manifold with a Killing Lee vector field ξ, $a = g(\xi, \xi)$, and $\left(\widetilde{C}, \pi^*\nabla, \widetilde{g}\right)$ its minimal Hessian covering. We say that (C, ∇, g, θ) is a **radiant l.c.H** manifold if $\left(\widetilde{C}, \widetilde{g}, \widetilde{\xi} = -\frac{2}{a}\pi^*\xi\right)$ is a radiant Hessian manifold.

Proposition 4.15. Let (C, ∇, g, θ) be a l.c.H. manifold with a Killing Lee vector field and $a = g(\xi, \xi)$. Then (C, ∇, g, θ) is a radiant l.c.H. manifold if and only if there exists a constant $\mu \in \mathbb{R} \setminus \{-a, 0\}$ such that $\nabla\xi = \mu\,\mathrm{Id}$.

Proof. Suppose there exist a constant μ such that $\nabla\xi = \mu\,\mathrm{Id}$. According to Proposition 4.13, $\left(\widetilde{C}, \widetilde{g}, \widetilde{\xi} = -\frac{2}{a}\pi^*\xi\right)$ is a self-similar manifold. Then

$$\pi^*\nabla\widetilde{\xi} = \lambda\,\mathrm{Id},$$

where

$$\lambda = -\frac{2}{a}\mu$$

is a constant. The proof in the opposite direction is analogical. We have $\lambda \in \mathbb{R} \setminus \{0, 2\}$ (see Remark 2.1). Therefore, $\mu \in \mathbb{R} \setminus \{-a, 0\}$. ∎

Theorem 4.16. [Ga] Let a group Γ acts on a Riemannian cone

$$\left(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2\right)$$

by homotheties and the quotient $M \times \mathbb{R}^{>0}/\Gamma$ is a compact manifold. Then the holonomy of $(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2)$ is irreducible or $(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2)$ is flat.

Proof of Theorem 1.3. According to Proposition 4.13, $\left(\widetilde{C}, \widetilde{g}, \widetilde{\xi}\right)$ is a self-similar manifold with the potential homothetic vector field $\widetilde{\xi}$. Since the manifold C is compact the vector field ξ is complete. Hence, the vector field $\widetilde{\xi}$ is complete. Combining it with Theorem 2.5, we get that $\left(\widetilde{C}, \widetilde{g}\right)$ is isometric to the Euclidean space with a radiant vector field $\left(\mathbb{R}^n, \sum_{i=1}^n \left(dx^i\right)^2, \sum x^i \frac{\partial}{\partial x^i}\right)$ or a Riemannian cone.

Suppose $\left(\widetilde{C}, \widetilde{g}, \widetilde{\xi}\right) = \left(\mathbb{R}^n, \sum_{i=1}^n \left(dx^i\right)^2, \sum x^i \frac{\partial}{\partial x^i}\right)$. Let Γ be the monodromy group of the covering $\pi : \widetilde{C} \to C$. Then Γ acts on $\left(\mathbb{R}^n, \sum_{i=1}^n \left(dx^i\right)^2\right)$ by homotheties preserving $\sum x^i \frac{\partial}{\partial x^i}$. Choose an element $\gamma \in \Gamma$ such that the action is not an isometry. Then the action of γ or the action of γ^{-1} is a contraction map preserving the point $0 \in \mathbb{R}^n$. Hence, the quotient $\widetilde{C}/\Gamma$ cannot be Hausdorff. However, $\widetilde{C}/\Gamma = C$ is a manifold.

Thus, $\left(\widetilde{C}, \widetilde{g}\right)$ is isometric to a Riemannian cone. According to Theorem 4.16, the holonomy of $\left(\widetilde{C}, \widetilde{g}\right)$ is irreducible or $\left(\widetilde{C}, \widetilde{g}\right)$ is flat. Since (C, g) is not locally conformally flat Riemannian manifold $\left(\widetilde{C}, \widetilde{g}\right)$ cannot be flat. Hence, the holonomy of $\left(\widetilde{C}, \widetilde{g}\right)$ is irreducible. According to Corollary 2.14, $\left(\widetilde{C}, \pi^*\nabla, \widetilde{g}, \widetilde{\xi}\right)$ is a radiant Hessian manifold. ∎

4.4 An l.c.H. metric expressed in terms of the Lee form

Proposition 4.17. Let (C, ∇, g, θ) be a compact radiant l.c.H. manifold ξ the Lee vector field, $\nabla\xi = \mu\,\mathrm{Id}$, and $a = g(\xi, \xi)$ the constant from Corollary 4.11. Set $u = -\mu - a$. Then we have

$$ug = \nabla\theta - \theta \otimes \theta,$$

where $u \neq 0$.

Proof. We have

$$(\nabla g)(\xi, X, Y) = \mathcal{L}_\xi(g(X, Y)) - g(\nabla_\xi X, Y) - g(X, \nabla_\xi Y).$$

Since ξ is Killing,

$$\mathcal{L}_\xi(g(X, Y)) = g([\xi, X], Y) + g(X, [\xi, Y]).$$

Moreover, the connection ∇ is torsion free. Thus,

$$(\nabla g)(\xi, X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi).$$

According to Theorem 1.3, there exist $\mu \in \mathbb{R}$ such that $\mu \notin \{0, -a\}$ and $\nabla\xi = \mu\,\mathrm{Id}$. Therefore,

$$(\nabla g)(\xi, X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) = -2\mu g(X, Y).$$

and

$$(\nabla g - \theta \otimes g)(\xi, X, Y) = (-2\mu - a)g(X, Y). \tag{4.1}$$

Using the identities $\iota_\xi g = \theta$ and $\nabla_X \xi = \mu \mathrm{Id}$ we obtain

$$(\nabla g - \theta \otimes g)(X, \xi, Y) = \mathcal{L}_X(\theta(Y)) - \theta(\nabla_X Y) - g(\nabla_X \xi, Y) - \theta(X)\theta(Y) = -\mu g(X, Y) + (\nabla\theta - \theta \otimes \theta)(X, Y). \tag{4.2}$$

By the definition of l.c.H. manifolds, the tenor $\nabla g - \theta \otimes g$ is totally symmetric. Combining this with (4.1) and (4.2), we get that

$$(-\mu - a)\, g = \nabla\theta - \theta \otimes \theta.$$

According to Corollary 4.11, we have $\mu \neq -a$. Hence, $u = -\mu - a \neq 0$. ∎

Proposition 4.18. Let (C, ∇) be a flat affine manifold and θ a closed 1-form such that the bilinear form $\nabla\theta - \theta \otimes \theta$ is positive definite. Then $(C, \nabla, g_\theta = \nabla\theta - \theta \otimes \theta, \theta)$ is an l.c.H. manifold.

Proof. It is enough to check that $(C, \nabla, g_\theta, \theta)$ is an l.c.H. manifold locally. The form θ is locally expressed as $\theta = df$. Then

$$\nabla\left(d\left(-e^{-f}\right)\right) = \nabla\left(e^{-f}df\right) = e^{-f}\nabla df - e^{-f}df \otimes df = e^{-f}g_\theta.$$

Thus the metric $e^{-f}g_\theta$ is Hessian and g is l.c.H. ∎

Proposition 4.19. Let u be a nonzero constant,

$$(C, \nabla, g = u^{-1}(\nabla\theta - \theta \otimes \theta), \theta)$$

a l.c.H. manifold and ξ be a Killing vector field on (C, g) such that $\mathcal{L}_\xi\theta = 0$ and $\nabla\xi = \mu\,\mathrm{Id}$, for a constant $\mu \in \mathbb{R}$. Then ξ coincides with the Lee vector field up to a constant multiplier.

Proof. Since $\mathcal{L}_\xi\theta = 0$, and θ is closed, the value $\theta(\xi)$ is constant. Hence, for any $X \in TC$

$$(\nabla\theta)(\xi, X) = \mathcal{L}_X(\theta(\xi)) - \theta(\nabla_X\xi) = -\mu\theta(X).$$

Therefore,

$$\iota_\xi g = \frac{\iota_\xi(\nabla\theta - \theta \otimes \theta)}{u} = \frac{(-\mu - \theta(\xi))\theta}{u},$$

i.e. $\iota_\xi g$ is proportional to θ. Thus, ξ coincides with the Lee vector field up to multiplication on a constant. ∎

Combining Propositions 4.17 and 4.19 we get the following.

Theorem 4.20. Let (C, ∇, g, θ) a compact radiant l.c.H. manifold and ξ be a Killing vector field on (C, g) such that $\mathcal{L}_\xi\theta = 0$ and $\nabla\xi = \mu\,\mathrm{Id}$, for a constant $\mu \in \mathbb{R}$. Then ξ coincides with the Lee vector field up to a constant multiplier.

Definition 4.21 ([Sh])**.** A Hessian manifold (C, ∇, g) is said to be of **Koszul type** if there exists a (globally defined) closed 1-form θ such that $g = \nabla\theta$.

Theorem 4.22 ([Ko])**.** Let (C, ∇, g) be a compact Hessian manifold of Koszul type. Then the universal covering $\widetilde{C}$ is a convex cone without full straight lines. Moreover, the lifting of the Hessian metric equals to $\mathrm{Hess}(\ln\psi)$ up to a constant multiplier, where ψ is the characteristic function of the cone (see example 3.3).

Theorem 4.23. Let (C, ∇, g, θ) be a compact radiant l.c.H. manifold ξ the Lee vector field, $\nabla\xi = \mu\,\mathrm{Id}$, and $a = g(\xi, \xi)$. Suppose $-\mu - a > 0$. Then $(C, \nabla, \nabla\theta)$ is a Hessian manifold of Koszul type.

Proof. According to Proposition 4.17, we have

$$ug = \nabla\theta - \theta \otimes \theta,$$

where $u = -\mu - a > 0$. Therefore, the from $\nabla\theta$ is positive definite and $(C, \nabla, \nabla\theta)$ is compact Hessian manifold of Koszul type. ∎

Combining Theorems 4.23 and 4.22 we get the following.

Corollary 4.24. Let (C, ∇, g, θ) be a compact radiant l.c.H. manifold ξ the Lee vector field, $\nabla\xi = \mu\,\mathrm{Id}$, and $a = g(\xi, \xi)$. Suppose $-\mu - a > 0$. Then the universal covering of C is a convex cone without full straight lines.

Theorem 4.25. Let φ be an automorphism of a statistical manifold (M, g, D) of constant curvature $c < 0$. Consider the automorphism

$$\varphi_q : M \times \mathbb{R}^{>0} \to M \times \mathbb{R}^{>0}, \quad \varphi_q(m, t) = (\varphi(m), qt).$$

Then $M \times \mathbb{R}^{>0}/_{\varphi_q}$ admits a Hessian structure of Koszul type.

Proof. Let λ be the positive solution of the equation $c = \lambda(2 - \lambda)$.

$$\varphi_q : M \times \mathbb{R}^{>0} \to M \times \mathbb{R}^{>0}, \quad \varphi_q(m, t) = (\varphi(m), qt).$$

According to Theorem 4.7, there is a connection ∇ on $M \times \mathbb{R}^{>0}/_{\varphi_q}$ such that

$$\left(M \times \mathbb{R}^{>0}/_{\varphi_q}, \nabla,\ g = g_M + \frac{ds^2}{s^2},\ \theta = \frac{-2ds}{s} \right)$$

is a radiant l.c.H. manifold with Lee vector field $\xi = -2s\frac{\partial}{\partial s}$ satisfying $\nabla\xi = -2\lambda$.

Since $\lambda > 2$, the constant $\mu = -2\lambda$ and $a = g(\xi, \xi) = 4$ satisfies the condition of Theorem 4.23. Therefore, $\left(M \times \mathbb{R}^{>0}/_{\varphi_q}, \nabla, \nabla\theta\right)$ is Hessian manifold of Koszul type. ∎

Corollary 4.26. Let (M, g_M, D) be a compact statistical manifold of negative constant curvature. Then the universal covering $\left(\widetilde{M}, \widetilde{g_M}\right)$ is isometric to a characteristic hypersurface of a cone.

Proof. According to Theorems 4.25, there is a Hessian structure of Koszul type $(\nabla, g = \nabla\theta)$ on $M \times \mathbb{R}^{>0}/_{\varphi_q}$, where $\theta = \frac{-2ds}{s}$. According to Theorem [Ku], the universal covering $\left(\widetilde{M} \times \mathbb{R}^{>0}, \widetilde{g}\right)$ is isometric to convex cone without straight lines with the metric $\mathrm{Hess} \ln \psi$, where ψ is the characteristic function (see Example 3.3). Hence, $\theta = d\ln\psi$. The set $\widetilde{M}\times 1$ is an integral hypersurface of the 1-form θ. That is, $\widetilde{M}\times 1$ is a level set of the function ψ. Therefore, $\left(\widetilde{M}, \widetilde{g_M}\right) = \left(\widetilde{M}, \widetilde{g}|_{\widetilde{M}}\right)$ is the characteristic hypersurface of the cone $\widetilde{M}\times\mathbb{R}^{>0}$. ■

We can identify a characteristic hypersurface of a cone with a properly convex daomain in $\mathbb{RP}^n$ (see Example 3.4). Therefore, Corollary 4.26 is a equivalent to the second part of Theorem 1.1.

4.5 L.c.H manifolds of rank 1

Definition 4.27. We say that an l.c.H. manifold (C, ∇, g, θ) is of of rank 1 if the monodromy group of the weight bundle (L, θ) is isomorphic to $\mathbb{Z}$.

Proposition 4.28. Let θ be a closed 1-form on a manifold C. Then monodromy group of the weight bundle (L, θ) is isomorphic to $\mathbb{Z}$ if and only if $[\theta] \in H^1(C, \mathbb{Q})$.

Proof. The proof is coinsides with the proof of the analogical proposition for l.c.K. rank in [OV1]. ■

Proof of Theorem 1.4. Consider a cohomology class $[\alpha] \in H^1(M;\mathbb{R})$ and let α be its harmonic representative. According to Proposition 4.17, g is proportional to $\nabla\theta - \theta\otimes\theta$. If α is chosen sufficiently small than $g_{\theta'} = \nabla\theta' - \theta'\otimes\theta'$ is positive definite. According to Proposition 4.18, $(C, \nabla, g_{\theta'}, \theta')$ is an l.c.H. manifold.

Let us show that the Lee vector field ξ of (C, ∇, g, θ) coincides with the Lee vector of $(C, \nabla, g_{\theta'}, \theta')$ up to a constant multiplier. Since the flow along the Lee vector field ξ acts on (C, g) by isometries and the form α is harmonic, this flow preserves α. Hence it preserves the form θ'. Moreover the flow along ξ preserves ∇, the flow along ξ preserves $g_{\theta'} = \nabla\theta' - \theta'\otimes\theta'$. According to Proposition 4.19, there exists a constant $a \in \mathbb{R}$ such that $a\xi$ is the Lee vector field on (C, ∇, g', θ'). Since the field ξ is affine and Killing with respect to $g_{\theta'}$, the field $a\xi$ is affine and Killing too. Thus, (C, ∇, g', θ') is a radiant l.c.H. manifold of l.c.H. rank 1. ■

Combining Theorem 1.4 and Proposition 4.28, we get the following.

Theorem 4.29. Let (C, ∇, g, θ) be a compact radiant l.c.H. manifold. Then g can be approximate by a sequence of Riemannian metrics which are conformally equivalent to an l.c.H. metric of rank 1 with an affine Killing Lee vector.

Proof of Theorem 1.7. The minimal Hessian covering $(\widetilde{C}, \widetilde{\nabla}, \widetilde{g})$ endowed with the vector field $\widetilde{\xi} = -\frac{2}{a}\pi^*\xi$ is a radiant Hessian manifold. We have

$$\widetilde{\nabla}\widetilde{\xi} = \lambda, \quad \text{where} \quad \lambda = -\frac{2}{a}.$$

By Corollary 4.11, $a = g(\xi, \xi)$ is constant. Hence, ξ is nonzero at any point. According to Theorem 3.14, $(\widetilde{C}, \widetilde{g})$ is isometric to a cone $(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2)$ over a statistical manifold (M, g_M, D) of constant curvature $c = \lambda(2 - \lambda)$. Since $\lambda = -\frac{2}{a}$, the constant c is negative if and only if $\mu \in (-\infty, -a) \cup (0, \infty)$.

Since M is an l.c.H. manifold of the rank 1, the deck group Γ of $\widetilde{C}$ is isometric to $\mathbb{Z}$. The manifold C is obtained from $\widetilde{C}$ as a factor $C = \widetilde{C}/\Gamma$, where Γ acts on $\widetilde{C} = M \times \mathbb{R}^{>0}$ be homotheties. Let the generator γ of Γ act on the first component of $M \times \mathbb{R}^{>0}$ by an isometry φ and the second component by multiplication on $q \in \mathbb{R}^{>0}$ and on the. Set a submersion

$$\sigma_0 \ : \ C \simeq (M \times \mathbb{R}^{>0})/\Gamma \ \to \ \mathbb{R}^{>0}/\{a\} \simeq S^1$$

using the diagram

$$\begin{array}{ccc} M \times \mathbb{R}^{>0} & \xrightarrow{\sigma} & \mathbb{R}^{>0} \\ \pi\big\downarrow & & \pi_0\big\downarrow \\ (M \times \mathbb{R}^{>0})/\Gamma & \xrightarrow{\sigma_0} & \mathbb{R}^{>0}/\{q\} \end{array}.$$

For any $p \in \mathbb{R}^{>0}/\{a\}$ we have

$$\sigma^{-1}\pi_0^{-1} = \{M \times q^k \mid k \in \mathbb{Z}\}.$$

and

$$\sigma_0^{-1}p = \pi\{M \times q^k \mid k \in \mathbb{Z}\} \simeq M.$$

Thus, the fibers of σ_0 are isometric to a statistical manifold (M, g_M) of constant curvature and the manifold C is isomorphic $M \times \mathbb{R}^{>0}/\varphi_q$. ■

Acknowledgements. Many thanks to Misha Verbitsky for fruitful discussions and help with the preparation of the paper.

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