Complex curves in hypercomplex nilmanifolds with $\mathbb{H}$-solvable Lie algebras

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Abstract

An operator I on a real Lie algebra $\mathfrak{g}$ is called a complex structure operator if $I^2 = -\operatorname{Id}$ and the $\sqrt{-1}$-eigenspace $\mathfrak{g}^{1,0}$ is a Lie subalgebra in the complexification of $\mathfrak{g}$. A hypercomplex structure on a Lie algebra $\mathfrak{g}$ is a triple of complex structures I, J and K on $\mathfrak{g}$ satisfying the quaternionic relations. We call a hypercomplex nilpotent Lie algebra **$\mathbb{H}$-solvable** if there exists a sequence of $\mathbb{H}$-invariant subalgebras

$$\mathfrak{g}_1^{\mathbb{H}} \supset \mathfrak{g}_2^{\mathbb{H}} \supset \cdots \supset \mathfrak{g}_{k-1}^{\mathbb{H}} \supset \mathfrak{g}_k^{\mathbb{H}} = 0,$$

such that $[\mathfrak{g}_i^{\mathbb{H}}, \mathfrak{g}_i^{\mathbb{H}}] \subset \mathfrak{g}_{i+1}^{\mathbb{H}}$. We give examples of $\mathbb{H}$-solvable hypercomplex structures on a nilpotent Lie algebra and conjecture that all hypercomplex structures on nilpotent Lie algebras are $\mathbb{H}$-solvable. Let (N, I, J, K) be a compact hypercomplex nilmanifold associated to an $\mathbb{H}$-solvable hypercomplex Lie algebra. We prove that, for a general complex structure L induced by quaternions, there are no complex curves in a complex manifold (N, L).

Contents

1 Introduction

1.1 Complex nilmanifolds

Recall that **a nilmanifold** N is a compact manifold that admits a transitive action of a nilpotent Lie group G. Any nilmanifold is diffeomorphic to a quotient of a connected, simply connected nilpotent Lie group G by a cocompact lattice Γ [Mal].

A complex nilmanifold could be defined in two different ways. **A complex parallelizable nilmanifold** [W] is a compact quotient of a complex nilpotent Lie group by a discrete, cocompact subgroup. This is not the definition we use. We define **a complex nilmanifold** as a quotient of a nilpotent Lie group with a left-invariant complex structure by a left action of a cocompact lattice.

Nilmanifolds provide examples of non-Kähler complex manifolds. One of the first examples of a non-Kähler complex manifold was given by Kodaira [Has1], see also [Th]. It is a complex surface called **the Kodaira surface** with the first Betti number $b_1 = 3$. It was proven in [BG] that a complex nilmanifold does not admit a Kähler structure unless it is a torus. Moreover, in [Has] Hasegawa proved that a nilmanifold that is not a torus is never homotopically equivalent to any Kähler manifold.

Example 1.1: The Kodaira surface could be obtained as a quotient of the group of matrices of the form

$$G := \left\{ \begin{pmatrix} 1 & x & z & t \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ : \ x, y, z, t \in \mathbb{R} \right\}$$

by the subgroup Γ of the similar matrices with integer entries.

Example 1.2: The example of a complex nilmanifold that is obtained from a complex Lie group is **an Iwasawa manifold**. It is a compact quotient of the 3-dimensional complex Heisenberg group G by a cocompact, discrete subgroup Γ of the corresponding matrices with the Gaussian integer entries. Unlike the Kodaira surface, the Iwasawa manifold is parallelizable, that is, its tangent bundle is trivial as a holomorphic bundle.

1.2 $\mathbb{H}$-solvable Lie algebras

Let $\mathfrak{g} = \operatorname{Lie} G$ denote the Lie algebra of a nilpotent Lie group G.

Definition 1.3: A subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ which satisfies $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$ defines **a complex structure operator** $I \in \operatorname{End}(\mathfrak{g})$. Subspaces $\mathfrak{g}^{1,0}$ and $\overline{\mathfrak{g}^{1,0}} = \mathfrak{g}^{0,1}$ are $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of the operator I respectively.

Let $\mathbb{H}$ be the quaternion algebra. Recall that $\mathbb{H}$ is generated by I, J and K satisfying the quaternionic relations:

$$I^2 = J^2 = K^2 = -\mathrm{Id}, \quad IJ = -JI = K. \tag{1.1}$$

Definition 1.4: Let $\mathfrak{g}$ be a nilpotent Lie algebra. **A hypercomplex structure** on $\mathfrak{g}$ is a triple of endomorphisms $I, J, K \in \operatorname{End}(\mathfrak{g})$ which satisfies the conditions (1.1) and defines the complex structures in the sense of Definition 1.3.

Denote by $\mathfrak{g}_i^I := \mathfrak{g}_i + I\mathfrak{g}_i$ the smallest I-invariant Lie subalgebra which contains the commutator subalgebra $\mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}]$, $i \in \mathbb{Z}_{>0}$. Reformulating the result of S. Salamon [S, Theorem 1.3], D. Millionschikov in [Mil, Proposition 2.5] has shown that

$$[\mathfrak{g}_k^I, \mathfrak{g}_k^I] \subset \mathfrak{g}_{k+1}^I$$

and

$$\mathfrak{g}_1^I := [\mathfrak{g}, \mathfrak{g}] + I[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}.$$

Hence, a sequence of complex-invariant Lie subalgebras

$$\mathfrak{g} \supset \mathfrak{g}_1^I \supset \cdots \supset \mathfrak{g}_n^I = 0$$

terminates for some $n \in \mathbb{Z}_{>0}$. It is natural to ask a similar question about the hypercomplex nilpotent Lie algebras: is the algebra $\mathbb{H}\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}] + I[\mathfrak{g}, \mathfrak{g}] + J[\mathfrak{g}, \mathfrak{g}] + K[\mathfrak{g}, \mathfrak{g}]$ equal to $\mathfrak{g}$ or not?

We introduce a notion of **an** $\mathbb{H}$**-solvable** nilpotent Lie algebra. Define inductively $\mathbb{H}$-invariant Lie subalgebras: $\mathfrak{g}_i^{\mathbb{H}} := \mathbb{H}[\mathfrak{g}_{i-1}^{\mathbb{H}}, \mathfrak{g}_{i-1}^{\mathbb{H}}]$, where $\mathfrak{g}_1^{\mathbb{H}} = \mathbb{H}[\mathfrak{g}, \mathfrak{g}]$.

Definition 1.5: A hypercomplex nilpotent Lie algebra $\mathfrak{g}$ is called $\mathbb{H}$**-solvable** if there exists $k \in \mathbb{Z}_{>0}$ such that

$$\mathfrak{g}_1^{\mathbb{H}} \supset \mathfrak{g}_2^{\mathbb{H}} \supset \cdots \supset \mathfrak{g}_{k-1}^{\mathbb{H}} \supset \mathfrak{g}_k^{\mathbb{H}} = 0.$$

Such a filtration corresponds to an iterated hypercomplex toric bundle, see [AV]. Clearly, this holds if and only if $\mathfrak{g}_{i-1}^{\mathbb{H}} \subsetneq \mathfrak{g}_i^{\mathbb{H}}$ for any $i \in \mathbb{Z}_{>0}$.

There are no known examples of hypercomplex Lie algebras which are not $\mathbb{H}$-solvable.

Conjecture 1.6: All hypercomplex structures on a nilpotent Lie algebra $\mathfrak{g}$ are $\mathbb{H}$-solvable.

1.3 Examples of $\mathbb{H}$-solvable algebras

An example of an $\mathbb{H}$-solvable Lie algebra is given by **an abelian complex structure**. Let us recall the definition.

Definition 1.7: Let $\mathfrak{g}$ be a nilpotent Lie algebra with a complex structure. Suppose that $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] = 0$. This complex structure is called **an abelian complex structure**.

It was already known that a nilpotent Lie algebra that admits an abelian hypercomplex structure is $\mathbb{H}$-solvable [AV, Proposition 4.5], see also [Rol, Corollary 3.11].

The main purpose of this article is to prove the following theorem:

Theorem 1.8: Let (N, I, J, K) be a hypercomplex nilmanifold, and assume that the corresponding Lie algebra $\mathfrak{g}$ is $\mathbb{H}$-solvable. Then there are no complex curves in the complex nilmanifold (N, L), where $L = aI + bJ + cK$, $L^2 = -\operatorname{Id}$ for all $(a, b, c) \in S^2$ except of a countable set $R \subset S^2$.

To give an example of an $\mathbb{H}$-solvable Lie algebra with a non-abelian hypercomplex structure, we need a construction described below. **The quaternionic double** was introduced in the work [SV]. Let (X, I_X) be a complex manifold which admits a torsion-free flat connection $\nabla : TX \longrightarrow TX \otimes \Lambda^1 X$ which also satisfies $\nabla I = 0$. For a fixed point $x \in X$ consider the monodromy group $\operatorname{Mon}(\nabla) \subset \operatorname{GL}(T_x X)$. Suppose that there exists a lattice $\Lambda_x \subset T_x X$ in a fiber which is preserved by the action of the monodromy group: $\operatorname{Mon}(\nabla)\Lambda_x = \Lambda_x$. Then we can construct the set $\Lambda \subset TX$ via all parallel transportation of Λ_x. Define a manifold $X^+ := TX/\Lambda$. It is fibered over X with fibers $T_x X/\Lambda_x$ are compact tori. It makes sense since for each $x \in X$ the intersection $\Lambda \cap T_x X$ is a lattice continuously depending on $x \in X$. The manifold $X^+ = TX/\Lambda$ is called **the quaternionic double**. It was shown in [SV] that there is a pair of almost complex structures

$$I := \begin{pmatrix} I_X & 0 \\ 0 & -I_X \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -\operatorname{Id} \\ \operatorname{Id} & 0 \end{pmatrix}$$

on X^+ which are integrable and satisfy the quaternionic relations.

Theorem 1.9: (Soldatenkov, Verbitsky) Let X^+ be the quaternionic double of an affine complex manifold X. If X is non-Kähler, then X^+ does not admit an HKT-metric [SV].

It was shown in [DF], see also [BDV], that any abelian hypercomplex nilmanifold is HKT.

Example 1.10: To provide an example of an $\mathbb{H}$-solvable Lie algebra with a non-abelian hypercomplex structure, we first define the Kodaira surface, following [Has], see also [AV, Example 1.7]. Consider the Lie algebra $\mathfrak{g} =$

$\langle x, y, z, t\rangle$, such that the only non-zero commutator is $[x, y] = z$. The complex structure is given by $Ix = y$ and $Iz = t$. There exists an operator $\nabla^+ : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by the formula

$$\nabla^+_a b := \frac{1}{2}([a, b] + I[Ia, b]), a, b \in \mathfrak{g}$$

We extended ∇^+ to a left-invariant connection on the Lie group G, also denoted by ∇^+. It is easy to see that the connection ∇^+ is complex-linear, torsion-free and flat. Therefore, we could define the quaternionic double of the Kodaira surface. It is a nilmanifold associated with the Lie algebra $\mathfrak{g}^+ = \mathfrak{g} \oplus \mathfrak{g}$, with the commutator defined as follows:

$$[(a, b), (c, d)] := ([a, b], \nabla^+_a d - \nabla^+_c b).$$

The hypercomplex structure on $\mathfrak{g}^+$ is defined as follows:

$$I(a, b) = (Ia, -Ib), \quad J(a, b) = (-b, a), \quad K(a, b) = (-Ib, -Ia).$$

Then

$$\mathfrak{g}^+_1 := [\mathfrak{g}^+, \mathfrak{g}^+] = [\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g} \oplus \mathfrak{g}] = \langle \lambda z, \mu z\rangle, \lambda, \mu \in \mathbb{R}.$$

Hence $\mathfrak{g}^+_2 = 0$ (because there are no non-trivial commutators on the second step), which implies the $\mathbb{H}$-solvability of $\mathfrak{g}^+$. From Theorem 1.9 it is clear that the hypercomplex structure on $\mathfrak{g}^+$ is non-abelian.

This gives an example of an $\mathbb{H}$-solvable Lie algebra with a non-abelian complex structure.

Example 1.11: Let (N, I, J, K) be a hypercomplex manifold. Fix a point $p \in N$ and consider the set $\mathfrak{g}^{(d)} \subset \mathfrak{g}$ of smooth vector fields such that $X \in \mathfrak{g}^{(d)}$ has zero of order d at p. Notice that the filtration $\mathfrak{g}^{(i)}$ is $\mathbb{H}$-invariant with the commutator $[\mathfrak{g}^{(d)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(d+k-1)}$. Therefore, the quotient $\mathfrak{g}^{(2)}/\mathfrak{g}^{(n)}$ is an $\mathbb{H}$-solvable algebra.

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2 Preliminaries

2.1 Nilpotent Lie algebras

Let G be a real nilpotent Lie group, and $\mathfrak{g}$ its Lie algebra. **The descending central series** of a Lie algebra $\mathfrak{g}$ is the chain of ideals defined inductively:

$$\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k \supset \ldots,$$

where $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}]$. It is also called **the lower central series** of $\mathfrak{g}$.

Definition 2.1: A Lie algebra $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}_k = 0$ for some $k \in \mathbb{Z}_{>0}$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. Recall that for any $\alpha \in \mathfrak{g}^*$ the Chevalley–Eilenberg differential $d : \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^*$ is defined as follows

$$d\alpha(\xi, \theta) = -\alpha([\xi, \theta]),$$

where $\xi, \theta \in \mathfrak{g}$. It extends to a finite-dimensional complex

$$0 \longrightarrow \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^* \longrightarrow \cdots \longrightarrow \Lambda^n\mathfrak{g}^* \longrightarrow 0 \tag{2.1}$$

by the Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\widetilde{\alpha}}\alpha \wedge d\beta$, where $\alpha, \beta \in \mathfrak{g}^*$ and $n = \dim_{\mathbb{R}} \mathfrak{g}$. The condition $d^2 = 0$ is equivalent to the Jacobi identity.

2.2 A short review of the Maltsev theory

Following Maltsev's papers [Mal] and [Mal2] we are going to consider only nilpotent groups without torsion.

Definition 2.2: **A nilmanifold** is a compact manifold which admits a transitive action of a nilpotent Lie group.

Recall that **a lattice** Γ is a discrete subgroup of a topological group G such that there exists a regular finite G-invariant measure on the quotient $\Gamma\backslash G$.

The famous **Maltsev's theorem** states that any nilmanifold N is diffeomorphic to a quotient of a connected, simply connected nilpotent Lie group

G by a cocompact lattice Γ. The group Γ is isomorphic to the fundamental group $\pi_1(N)$ of the nilmanifold N and G is the **Maltsev completion** of the group $\Gamma \approx \pi_1(N)$ [Mal].

Definition 2.3: A group G is called **complete** if for each $g \in G$ there exists $n \in \mathbb{Z}_{>0}$ such that the equation $x^n = g$ has solutions in G.

Definition 2.4: Let G be a subgroup of a complete nilpotent group $\hat{G}$. Suppose that for any $g \in \hat{G}$ there exists $n \in \mathbb{Z}_{>0}$ such that $g^n \in G$. Then $\hat{G}$ is called **the Maltsev completion** of a group G.

Let $\mathfrak{g}_{\mathbb{Q}}$ be a nilpotent Lie algebra over the field of rational numbers $\mathbb{Q}$. We identify $\mathfrak{g}_{\mathbb{Q}}$ with a subspace of a real nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$, and call $\mathfrak{g}_{\mathbb{Q}}$ **the rational lattice** of $\mathfrak{g}$.

Definition 2.5: **A rational structure** in a real nilpotent Lie algebra $\mathfrak{g}$ is a rational lattice $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$.

Definition 2.6: Let Γ be a lattice in a connected, simply connected nilpotent Lie group G. Then its **associated rational structure** is the $\mathbb{Q}$-span of $\log \Gamma \subset \mathfrak{g}$, where $\mathfrak{g} = \operatorname{Lie} G$ is the Lie algebra. If $\mathfrak{g}$ has a rational structure related to a $\mathbb{Q}$-algebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ then there exists a discrete subgroup Γ such that $\log \Gamma \subset \mathfrak{g}_{\mathbb{Q}}$ and the quotient $\Gamma \backslash G$ is compact [CG, Theorem 5.1.7].

Let Γ be a discrete subgroup of a connected, simply connected Lie group G. By Theorem 2.7 the Maltsev completion $\hat{\Gamma}$ is the unique closed connected subgroup of G such that a left quotient $\Gamma \backslash \hat{\Gamma}$ is compact.

Theorem 2.7: (Maltsev) Let G be a finitely generated nilpotent group without torsion. Then there exists a nilpotent, complete and torsion-free group $\hat{G}$ such that $\hat{G}$ is a completion of G. Moreover, all the completions of G are isomorphic [Mal2]. Finally, $\hat{G} = \exp \mathfrak{g}_{\mathbb{Q}}$ is the set of rational points in a real nilpotent Lie group G with the Lie algebra $\mathfrak{g}$ admitting a rational lattice $\mathfrak{g}_{\mathbb{Q}}$.
■

2.3 Hypercomplex nilmanifolds

Let X be a smooth manifold. Recall that **an almost complex structure** on X is an endomorphism $I \in \operatorname{End}(TX)$ satisfying $I^2 = -\operatorname{Id}$. The Nijenhuis tensor N_I associated to the almost complex structure I is given by the formula

$$N_I(X,Y) = [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY].$$

An almost complex structure is called **integrable** if its Nijenhuis tensor vanishes.

Remark 2.8: $N_I = 0$ if and only if $[TX^{1,0}, TX^{1,0}] \subset TX^{1,0}$.

Theorem 2.9: (Newlander–Nirenberg) If I is an integrable almost complex structure on X, then X admits the structure of a complex manifold compatible with I.

Definition 2.10: A complex nilmanifold is a pair (N, I), where $N = \Gamma\backslash G$ is a nilmanifold obtained from a nilpotent Lie group G and I an integrable left-invariant almost complex structure on G.

By definition, I is left-invariant if the left translations $L_g : (G, I) \longrightarrow (G, I)$ are holomorphic. Notice that the Lie group G does not need to be a complex Lie group, but in the case when it does both left and right translations on G are holomorphic.

Let X be a smooth manifold equipped with three integrable almost complex structures $I, J, K \in \operatorname{End}(TX)$, satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\operatorname{Id}$ and $IJ = K = -JI$. Such a quadruple (X, I, J, K) is called **a hypercomplex manifold**. Obata [Ob] proved that there exists a unique torsion-free connection ∇^{Ob} preserving the complex structures:

$$\nabla^{Ob} I = \nabla^{Ob} J = \nabla^{Ob} K = 0.$$

The connection ∇^{Ob} is called **the Obata connection**.

A hypercomplex structure induces a complex structure $L = aI + bJ + cK$ for each $(a, b, c) \in \mathbb{R}^3$ such that $a^2 + b^2 + c^2 = 1$ and the set of such structures is identified in a natural way with $S^2 \approx \mathbb{C}\mathrm{P}^1$.

Consider the product $X \times \mathbb{C}\mathrm{P}^1$, where X is a hypercomplex manifold. **The twistor space** $\operatorname{Tw}(X)$ of the hypercomplex manifold X is a complex manifold where the complex structure is defined as follows. For any point $(x, L) \in X \times \mathbb{C}\mathrm{P}^1$ the complex structure on $T_{(x,L)} \operatorname{Tw}(X)$ is L on $T_x X$ and the standard complex structure $I_{\mathbb{C}P^1}$ on $T_L \mathbb{C}\mathrm{P}^1$. This almost complex structure on the twistor space of a hypercomplex manifold is always integrable [K], [Besse, Theorem 14.68]. The space $\operatorname{Tw}(X)$ is equipped with the canonical holomorphic projection $\pi : \operatorname{Tw}(X) \longrightarrow \mathbb{C}\mathrm{P}^1$. The fiber $\pi^{-1}(L)$ at a point $L \in \mathbb{C}\mathrm{P}^1$ is biholomorphic to the complex manifold (X, L).

Definition 2.11: Let Γ be a cocompact lattice in a nilpotent Lie group G with a left-invariant hypercomplex structure. Then the manifold $N = \Gamma\backslash G$ is called **a hypercomplex nilmanifold**.

2.4 Positive bivectors on a Lie algebra

Consider a nilpotent Lie group G with a left-invariant complex structure $I \in \operatorname{End}(TG)$. Recall that **a complex structure operator** on a Lie algebra $\mathfrak{g}$ can be given by a decomposition of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes\mathbb{C}$ satisfying $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}}$, where $\mathfrak{g}^{1,0} = \{X \mid X \in \mathfrak{g}_{\mathbb{C}},\ I(X) = \sqrt{-1}X\}$ and $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$ by Definition 1.3.

Denote the k-th exterior power of $\mathfrak{g}^{1,0}$ (resp. $\mathfrak{g}^{0,1}$) by $\Lambda^{k,0}\mathfrak{g}$ (resp. $\Lambda^{0,k}\mathfrak{g}$). Consider the graded algebra of (p,q)-multivectors $\Lambda^*\mathfrak{g} \otimes \mathbb{C} = \bigoplus_{p,q} \Lambda^{p,q}\mathfrak{g}$.

Definition 2.12: The elements of the space $\Lambda^{1,1}\mathfrak{g} \subset \Lambda^2\mathfrak{g}$ are called **(1,1)-bivectors** or just **bivectors**.

A non-zero real bivector $\xi \in \Lambda^{1,1}\mathfrak{g}$ is called **positive** if for any non-zero $\alpha \in \Lambda^{1,0}\mathfrak{g}^*$ one has $\xi(\alpha, I\alpha) \geqslant 0$.

The Lie bracket gives a linear mapping $\delta_1 : \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}$, defined as $\delta_1 : x \wedge y \mapsto [x, y]$. Such a mapping extends by the formula (2.2) below to the finite-dimensional complex of k**-multivectors**, i.e. $\delta_m : \Lambda^{m+1}\mathfrak{g} \longrightarrow \Lambda^m\mathfrak{g}$

$$0 \longrightarrow \Lambda^{2n}\mathfrak{g} \xrightarrow{\delta_{2n-1}} \cdots \longrightarrow \Lambda^2\mathfrak{g} \xrightarrow{\delta_1} \mathfrak{g} \xrightarrow{\delta_0} 0$$

and it is dual to the Chevalley-Eilenberg complex (2.1). The boundary operator δ_{k-1} can be written as follows

$$\delta_{k-1}(x_1\wedge\cdots\wedge x_k) = \sum_{r<s}(-1)^{r+s+1}[x_r, x_s]\wedge x_1\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge x_k. \quad (2.2)$$

Definition 2.13: **A complex curve** in a complex manifold (X, I) is a 1-dimensional compact complex subvariety $C_I \subset X$.

Let $C_I \subset N$ be a complex curve in a complex nilmanifold (N, I) and $\omega \in \Lambda^2\mathfrak{g}^*$ a two-form. We identify $\Lambda^2\mathfrak{g}^*$ with the space of left-invariant 2-forms on the Lie group G, which descends to the space of 2-forms $\Lambda^2(N)$ on the nilmanifold $N = \Gamma\backslash G$.

Consider a functional ξ on the space of 2-forms $\Lambda^2\mathfrak{g}^*$:

$$\xi_{C_I}(\omega) := \int_{C_I} \omega. \tag{2.3}$$

Such a functional defines a bivector $\xi \in \Lambda^2\mathfrak{g}$[1].

3 Positive bivectors on a quaternionic vector space

We start with a sequence of linear-algebraic lemmas. Let V be a finite-dimensional vector space over $\mathbb{C}$ and V^* its dual. Denote by (V, I) the pair of a vector space V with a complex structure $I \in \operatorname{End}(V)$ on it.

Recall that **the kernel** of a bivector $\xi \in \Lambda^{1,1}V$ is the following set:

$$\ker \xi = \{x \in V^* \,|\, \xi(x, \cdot) = 0\} \subset V^*. \tag{3.1}$$

We denote the space of positive bivectors with respect to the complex structure I on a vector space V by $\Lambda^{1,1}_{I,pos}V$.

Lemma 3.1: Let (V, I) be a vector space with a complex structure I and $\xi \in \Lambda^{1,1}_{I,\text{pos}}V$ a non-zero positive bivector. Let $V_1^* := \{x \in V^* \,|\, \xi(x, Ix) = 0\}$. Then $V_1^* = \ker \xi$.

Proof: From the definition (3.1) it is obvious that $\ker \xi \subset V_1^*$. Suppose that $x \in V_1^*$ and $x \notin \ker \xi$. Then $0 \neq [x] \in V/\ker \xi$. On the space $V/\ker \xi$ the bivector ξ is positive definite because it has no kernel and it is diagonalizable. Hence, $\xi(x, Ix) > 0$, which is a contradiction. ■

Let W_1 be a subspace of a vector space (W, I) and consider two maps: $p : W \longrightarrow W/W_1$ and $\widetilde{p} : \Lambda^2 W \longrightarrow \Lambda^2(W/W_1)$. There are two subspaces of

[1]Since the homology $H_*(N) = H_*(\mathfrak{g})$ by Theorem 4.1 a complex curve C_L corresponds to the bivector ξ_{C_L}

$\Lambda^2 W$: $\Lambda^2 \ker p = W_1 \wedge W_1$ and $\ker \widetilde{p} = W \wedge W_1$. It is obvious that $\Lambda^2 \ker p \subset \ker \widetilde{p}$. We are going to show that $\Lambda^{1,1}_{I,pos} W_1 \cap \ker \widetilde{p} = \Lambda^{1,1}_{I,pos} W_1 \cap \Lambda^2 \ker p$.

Lemma 3.2: Let W_1 be a subspace of a vector space (W, I) and $\xi \in \Lambda^{1,1}_{I,pos} W$ a positive (1,1)-bivector. Assume that $\xi \in \ker \widetilde{p}$. Then $\xi \in \Lambda^2 W_1$.

Proof: Denote by $W_1^{\perp} \subset W^*$ the annihilator of the subspace W_1; it is isomorphic to the dual of the quotient $(W/W_1)^*$. Since $\xi \in W \wedge W_1$, we have $\xi(W_1^{\perp}, W_1^{\perp}) = 0$. Therefore, $W_1^{\perp} \subset \ker \xi$ by Lemma 3.1. So, $\xi_{|_{W^* \wedge W_1^{\perp}}} = 0$, which implies that $\xi \in \Lambda^2 W_1$. ■

Recall that for any pair of orthogonal complex structures $I, J \in \mathbb{H}$, one has

$$J(\Lambda^{p,q}_I V) = \Lambda^{q,p}_I V, \tag{3.2}$$

where the action of the complex structures I and J extended from $\Lambda^{1,0} V$ and $\Lambda^{0,1} V$ to (p, q)-bivectors by a multiplicativity. Indeed, I and J anticommute on $\Lambda^1(V)$ which implies $J(\Lambda^{1,0}_I) = \Lambda^{0,1}_I$ and $J(\Lambda^{0,1}_I) = \Lambda^{1,0}_I$

Consider the operator $W_I : \Lambda^* V_I \longrightarrow \Lambda^* V_I$ defined by the formula $W_I(\xi) = \sqrt{-1}(p-q)\xi$, where $\xi \in \Lambda^{p,q}_I V$. Notice that the elements W_I, W_J and W_K generate the Lie algebra $\mathfrak{su}(2)$ and the complex structures I, J and K are the the elements of the Lie group $SU(2)$ related to them, $I = \exp \frac{\pi W_I}{2}$, $J = \exp \frac{\pi W_J}{2}$ and $K = \exp \frac{\pi W_K}{2}$ [V2].

Lemma 3.3: Let V be a quaternionic vector space and $\Lambda^{1,1}_{I,\text{pos}} V \subset \Lambda^2 V$ the space of positive $(1,1)$-bivectors on (V, I). Then $\Lambda^{1,1}_{I,\text{pos}} V \cap \Lambda^{1,1}_{I',\text{pos}} V = 0$ for distinct complex structures I and I'.

Proof: Denote the intersection $R_{I,I'} := \Lambda^{1,1}_{I,\text{pos}} V \cap \Lambda^{1,1}_{I',\text{pos}} V$. First, suppose that $I' = -I$, the non-zero bivector $\xi \in R_{I,-I}$, and let $\alpha \in V^*$. Then $\xi(\alpha, I\alpha) > 0$ and $\xi(\alpha, -I\alpha) < 0$, so there is no such a bivector ξ.

Assume that $I' \neq -I$. Then suppose that $J \in \mathbb{H}$ is orthogonal to I and $IJ = -JI$, $J^2 = -\operatorname{Id}$. It is clear that as $I \neq \pm I'$, I is not proportional to I', then I' can be written in a form $I' = aI + bJ$ for some $a, b \in \mathbb{R}$, $b \neq 0$.

Let $\xi \in R_{I,I'}$. Since ξ is a $(1,1)$-bivector, $W_I(\xi) = W_{I'}(\xi) = 0$. Hence, $W_K(\xi) = 0$ because W_I and W_{aI+bJ} generate the Lie algebra $\mathfrak{su}(2)$ [V2]. We obtain that ξ is an $\mathfrak{su}(2)$-invariant bivector. It is therefore invariant under the multiplicative action $J(\xi)(\alpha, \beta) := \xi(J\alpha, J\beta)$ of $J \in SU(2)$. Consider

$$0 \leqslant \xi(\alpha, I\alpha) = \xi(J\alpha, JI\alpha) = -\xi(J\alpha, IJ\alpha) = -\xi(\beta, I\beta), \tag{3.3}$$

where $\beta = J\alpha$. However, $-\xi(\beta, I\beta) \leqslant 0$, hence $\xi = 0$. ∎

Corollary 3.4: The intersection of the set of positive bivectors and $SU(2)$-invariant bivectors contains only zero bivector.

Proof: Follows from the formula (3.3). An invariant bivector has to be positive for different complex structures, which is impossible by Lemma 3.3. ∎

4 Homology of a leaf of a foliation

Recall that a CW-space X with the fundamental group $\pi_1(X) = \pi$ and the higher homotopy groups $\pi_i(X) = 0$ for $i > 1$ is called **a $K(\pi, 1)$-space of Eilenberg–MacLane** or just **$K(\pi, 1)$-space**. Since the universal covering of a nilpotent Lie group is contractible, the nilmanifold $N = \Gamma\backslash G$ with the fundamental group $\pi_1(N) \approx \Gamma$ is a $K(\Gamma, 1)$-space. **The cohomology** of the group Γ is defined as $H^*(K(\Gamma, 1), \mathbb{Q})$.

In [N] Nomizu showed that the de Rham cohomology of a nilmanifold $N = \Gamma\backslash G$ can be computed using the left-invariant differential forms on the Lie group G.

Theorem 4.1: (**Nomizu**, [N, Theorem 1]) Let $N = \Gamma\backslash G$ be a nilmanifold and $(\Lambda^*\mathfrak{g}^*, d)$ is the Chevalley–Eilenberg complex. The natural inclusion of the complex of the left-invariant differential forms $\Omega^{inv}(G)$ on the nilpotent Lie group G into the de Rham algebra on the nilmanifold $\Omega^{inv}(N)$ induces the isomorphism of the corresponding cohomology $H^*(\mathfrak{g}, \mathbb{R}) \approx H^*(N, \mathbb{R})$. ∎

Since the nilmanifold N is $K(\Gamma, 1)$, the homology $H_*(N, \mathbb{R}) \approx H_*(\Gamma, \mathbb{R})$, hence $H_*(\Gamma, \mathbb{R}) \approx H_*(\mathfrak{g}, \mathbb{R})$. Pickel showed that instead of real coefficients we can take the rational ones, i.e. $H^*(\Gamma, \mathbb{Q}) \approx H^*(\mathfrak{g}, \mathbb{Q})$ as well [P].

Consider a nilpotent Lie group G and let Γ be its discrete subgroup, $\hat{\Gamma} \subset G$ the Maltsev completion of Γ, and define the Lie group $\hat{\Gamma}_{\mathbb{R}} := \exp(\log(\hat{\Gamma}) \otimes \mathbb{R})$. By Theorem 2.7, Γ is a lattice in $\hat{\Gamma}_{\mathbb{R}}$. Since the quotients $\Gamma\backslash G$ and $\Gamma\backslash\hat{\Gamma}_{\mathbb{R}}$ are both $K(\Gamma, 1)$, we have $H_*(\Gamma\backslash G) = H_*(\Gamma\backslash\hat{\Gamma}_{\mathbb{R}})$. From Theorem 4.1 follows that $H_*(\Gamma\backslash\hat{\Gamma}_{\mathbb{R}}) = H_*(\operatorname{Lie}\hat{\Gamma}_{\mathbb{R}})$, where $\operatorname{Lie}\hat{\Gamma}_{\mathbb{R}} \subset \mathfrak{g}$ is the Lie algebra of $\hat{\Gamma}_{\mathbb{R}}$ and $H_*(\operatorname{Lie}\hat{\Gamma}_{\mathbb{R}})$ denotes the cohomology of the Chevalley–Eilenberg homology complex (2.1).

Let $N = \Gamma\backslash G$ be a nilmanifold, $\mathfrak{g}$ the Lie algebra of the Lie group G and $\mathfrak{f} \subset \mathfrak{g}$ a Lie subalgebra. Let $F := \exp \mathfrak{f} \subset G$ be the corresponding Lie group. For each $x \in G$ define a subgroup of the lattice Γ as follows:

$$\Gamma_x = \{\gamma \in \Gamma \,|\, x\gamma x^{-1} \in F\}. \tag{4.1}$$

In other words, $\Gamma_x = \Gamma \cap x^{-1}Fx$.

Recall that **a distribution** on a smooth manifold X is a sub-bundle $\Sigma \subset TX$. The distribution called **involutive** if it is closed under the Lie bracket. **A leaf** of the involutive distribution Σ is the maximal connected, immersed submanifold $L \subset X$ such that $TL = \Sigma$ at each point of L. The set of all leaves is called **a foliation**.

The algebra $\mathfrak{f}$ defines a left-invariant foliation Σ on G. The leaves $\mathfrak{L}_x$ of the corresponding foliation on $\Gamma\backslash G$ are diffeomorphic to $\Gamma_x\backslash xF$ for each $x \in G$.

Definition 4.2: A subalgebra $\mathfrak{f} \subset \mathfrak{g}$ is said to be **rational** with respect to a given rational structure $\mathfrak{g}_{\mathbb{Q}}$ on $\mathfrak{g}$ if $\mathfrak{f}_{\mathbb{Q}} := \mathfrak{g}_{\mathbb{Q}} \cap \mathfrak{f}$ is a rational structure for $\mathfrak{f}$, i.e. $\mathfrak{f} = \mathfrak{f}_{\mathbb{Q}} \otimes \mathbb{R}$.

Remark 4.3: The rational homology of the leaf $\mathfrak{L}_x$ of the foliation Σ is equal to

$$H_*(\mathfrak{L}_x, \mathbb{Q}) := H_*(\Gamma_x\backslash xF, \mathbb{Q}) = H_*(\mathfrak{f}_{\mathbb{Q}}), \tag{4.2}$$

where $H_*(\mathfrak{f}_{\mathbb{Q}})$ is the homology of the complex dual to the rational Chevalley–Eilenberg complex (2.1). The last equality makes sense because of Theorem 4.1 and [P].

Consider the natural map of homology

$$j : H_*(\mathfrak{L}_x, \mathbb{Q}) \longrightarrow H_*(N, \mathbb{Q}), \tag{4.3}$$

associated with the immersion $\mathfrak{L}_x \longrightarrow N$. Notice that j does not have to be injective.

Claim 4.4: Let N be a nilmanifold and $X \subset N$ a subvariety tangent to the foliation Σ generated by the left translates of a Lie subalgebra $\mathfrak{f} \subset \mathfrak{g}$. Then the fundamental class $[X] \in H_*(N, \mathbb{Q})$ belongs to the image of $H_*(\mathfrak{f}_{\mathbb{Q}})$

in $H_*(N,\mathbb{Q})$, where the map $\tau : H_*(\mathfrak{f}_{\mathbb{Q}}) \longrightarrow H_*(N,\mathbb{Q})$ is obtained from (4.2) and (4.3).

Proof: Let X be a subvariety in a leaf of the foliation $\Sigma \subset TN$ and let $F := \exp \mathfrak{f}$. A leaf of Σ is diffeomorphic to $\Gamma_x \backslash xF$, hence $[X] \in \tau(H_*(\Gamma_x\backslash xF,\mathbb{Q}))$, where we identified $H_*(\Gamma_x\backslash xF,\mathbb{Q}) = H_*(\mathfrak{f}_{\mathbb{Q}})$. ∎

5 Finale

Let $(\mathfrak{g}, I, J, K)$ be a hypercomplex nilpotent Lie algebra. Define inductively

$$\mathfrak{g}_i^{\mathbb{H}} := \mathbb{H}[\mathfrak{g}_{i-1}^{\mathbb{H}}, \mathfrak{g}_{i-1}^{\mathbb{H}}],$$

where $\mathfrak{g}_1^{\mathbb{H}} = \mathbb{H}[\mathfrak{g},\mathfrak{g}]$ and let $\mathfrak{a}_i := \mathfrak{g}_{i-1}^{\mathbb{H}}/\mathbb{H}[\mathfrak{g}_{i-1}^{\mathbb{H}}, \mathfrak{g}_{i-1}^{\mathbb{H}}]$ be the corresponding commutative quotient algebra, $i \in \mathbb{Z}_{>0}$.

Observe that for any commutative Lie algebra $\mathfrak{a}$ its second homology group coincides with the space of all bivectors, $H_2(\mathfrak{a},\mathbb{R}) = \Lambda^2\mathfrak{a}$. Denote by $\Lambda^{1,1}_{L,pos}\mathfrak{a}$ the set of positive $(1,1)$-bivectors with respect to the complex structure L.

Proposition 5.1: Let $\mathfrak{a}$ be a commutative hypercomplex Lie algebra, and $\mathfrak{s} \subset \Lambda^2\mathfrak{a}$ a countable set of non-zero bivectors. Then for all $L \in \mathbb{C}\mathrm{P}^1$ except at most a countable number, the intersection $\Lambda^{1,1}_{L,pos}\mathfrak{a} \cap \mathfrak{s} = \emptyset$.

Proof: By Lemma 3.3 for any non-zero $\xi \in \mathfrak{s}$ there exists at most one complex structure $L_\xi \in \mathbb{C}\mathrm{P}^1$ such that $\xi \in \Lambda^{1,1}_{L_\xi,pos}\mathfrak{a}$. The union $\bigcup_{\xi\in\mathfrak{s}} L_\xi$ is at most countable, hence for any $L \in \mathbb{C}\mathrm{P}^1\backslash \bigcup_{\xi\in\mathfrak{s}} L_\xi$ the intersection $\Lambda^{1,1}_{L,pos}\mathfrak{a} \cap \mathfrak{s}$ is empty. ∎

Let Σ_i be the foliation on a nilmanifold N generated by the left-translates of the Lie subalgebra $\mathfrak{g}_i^{\mathbb{H}} \subset \mathfrak{g} = T_eG$ and $\mathfrak{L}_{x,i} \subset N$ a leaf of the foliation Σ_i. The leaf $\mathfrak{L}_{x,i}$ is diffeomorphic to the left quotient $\Gamma_x\backslash xF_i$, where $F_i = \exp \mathfrak{g}_i^{\mathbb{H}} \subset G$.

Consider the natural projection

$$p_i : \mathfrak{g}_{i-1}^{\mathbb{H}} \longrightarrow \mathfrak{a}_i.$$

Let r_i be the corresponding map of the second homology:

$$r_i : H_2(\mathfrak{g}_{i-1}^{\mathbb{H}}) \longrightarrow H_2(\mathfrak{a}_i) = \Lambda^2\mathfrak{a}_i.$$

Then the image of a homology class in $H_2(\mathfrak{g}_{i-1}^{\mathbb{H}})$ defines a bivector on the commutative Lie algebra $\mathfrak{a}_i$.

Denote by $\mathfrak{g}_{i,\mathbb{Q}}^{\mathbb{H}} = \mathfrak{g}_i^{\mathbb{H}} \cap \mathfrak{g}_{\mathbb{Q}}$. Let $\mathfrak{s}_i := r_i(H_2(\mathfrak{g}_{i-1,\mathbb{Q}}^{\mathbb{H}})) \subset \Lambda^2\mathfrak{a}_i$ and $R \subset \mathbb{C}\mathrm{P}^1$ be a union of

$$R_i := R[\mathfrak{s}_i] \subset \mathbb{C}\mathrm{P}^1 \tag{5.1}$$

the set of complex structures L such that there exists a positive bivector $\xi \in \mathfrak{s}_i \cap \Lambda^{1,1}_{L,pos}\mathfrak{a}$. By Proposition 5.1, the set R_i is countable.

Definition 5.2: Let Σ_k be a holomorphic foliation obtained from the Lie subalgebra $\mathfrak{g}_k^I = \mathfrak{g}_k + I\mathfrak{g}_k$. **A transversal Kähler form** ω_k with respect to the holomorphic foliation Σ_k is a closed positive (1,1)-form, such that $\ker\omega_k$ is precisely the tangent space of the foliation, i.e. $\omega_k(\Sigma_k) = 0$.

Proposition 5.3: Let C_L be a complex curve in a complex nilmanifold (N, L), where $L \in \mathbb{C}\mathrm{P}^1 \backslash R_i$ and the set $R_i \subset \mathbb{C}\mathrm{P}^1$ is defined in (5.1). Suppose that C_L is tangent to the foliation Σ_{i-1} defined by $\mathfrak{g}_{i-1}^{\mathbb{H}}$ as above. Then it is also tangent to Σ_i.

Proof: From Claim 4.4 it follows that the fundamental class $[C_L] \in H_2(N, \mathbb{Q})$ of the curve C_L belongs to $j(H_2(\mathfrak{L}_{x,i-1}, \mathbb{Q})) \subset H_2(N, \mathbb{Q})$, where j is the standard map on the rational second homology (4.3). Theorem 4.1 allows us to identify the fundamental class $[C_L] \in H_2(N, \mathbb{Q})$ with the bivector (2.3) $\xi_{C_L} =: \xi$. Under the projection r_i the fundamental class $[C_L]$ is mapped to the bivector $r_i(\xi) \in \Lambda^2\mathfrak{a}_i$. From the definition of the set R_i we know that $\xi \in \ker r_i$ and from Lemma 3.2 follows that $\xi \in \Lambda^{1,1}_{L,pos}\ker p_i = \Lambda^{1,1}_{L,pos}\mathfrak{g}_i^{\mathbb{H}}$.

Suppose that $\omega_{i-1} \in r_i^*(\Lambda^{1,1}_L\mathfrak{a}_i^*)$ is a transversal Kähler form of the foliation Σ_{i-1}. Then $\int_{C_L}\omega_{i-1} > 0$ unless C_L lies in the leaf of the foliation Σ_{i-1}. However, $\int_{C_L}\omega = 0$ because ω_{i-1} is closed (otherwise, referring to the analogue of Stokes' theorem, we obtain that the volume of a compact manifold is equal zero). Since $\mathfrak{g}_{i-1}^{\mathbb{H}} \supset \mathfrak{g}_i^{\mathbb{H}}$ we have $\omega_{i-1} \in \Lambda^{1,1}_{L,pos}(\mathfrak{g}_{i-1}^{\mathbb{H}})^* \subset \Lambda^{1,1}_{L,pos}(\mathfrak{g}_i^{\mathbb{H}})^*$. Hence, $\mathfrak{g}_i^{\mathbb{H}} \subset \ker\omega_{i-1}$. Hence, ω_{i-1} is a transversal Kähler form with respect to the foliation Σ_i and C_L lies in a leaf of Σ_i. ∎[1]

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Assume that for some $k \in \mathbb{Z}_{>0}$ the following sequence terminates:

$$\mathfrak{g}_1^{\mathbb{H}} \supset \mathfrak{g}_2^{\mathbb{H}} \supset \cdots \supset \mathfrak{g}_{k-1}^{\mathbb{H}} \supset \mathfrak{g}_k^{\mathbb{H}} = 0, \tag{5.2}$$

i.e. the Lie algebra $\mathfrak{g}$ is $\mathbb{H}$-solvable, see also Definition 1.5.

Corollary 5.4: Let $L \in \mathbb{C}\mathrm{P}^1 \backslash R$, where $R = \bigcup R_i$ is the countable subset defined in (5.1), and assume that the sequence (5.2) terminates to zero. Then the complex nilmanifold (N, L) contains no complex curves.

Proof: Suppose that the sequence (5.2) vanishes on the k-th step, i.e. $\Sigma_k = \{0\}$. Then Corollary 5.4 follows from the Proposition 5.3 and the induction on i. ■

Theorem 5.5: Let (N, I, J, K) be a hypercomplex nilmanifold and assume that the corresponding Lie algebra is $\mathbb{H}$-solvable. Then there are no complex curves in the general fiber of the holomorphic twistor projection $\mathrm{Tw}(N) \longrightarrow \mathbb{C}\mathrm{P}^1$.

Proof: Follows from Corollary 5.4. ■

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