Newton Polytopes and Irreducibility

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Abstract

In this paper we develop a new technique for dealing with the geometric irreducibility of a variety in the algebraic torus $(k\backslash 0)^n$ given by a general system of equations with fixed monomials and linear relations on coefficients.

We illustrate the technique via two examples. First we generalize the Khovanskii Irreducibility Theorem (criterion for a system $f_1 = \cdots = f_m = 0$ to give an irreducible variety if the monomials of $f_1, \ldots, f_m$ are fixed and their coefficients are generic enough) from $\mathbb{C}$ to an arbitrary field. Then we establish a sufficient combinatorial condition for a set of monomials A, so that the system $f = f'_x = 0$ yields a geometricallly irreducible variety in $(k\backslash 0)^n$ for a general f with monomials from A.

Contents

1. Introduction

1.1. Setup

Background Let k be an arbitrary field, $T^n := \operatorname{Spec} k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]$ be the algebraic n-dimensional torus over k, $f \in k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]$ a Laurent polynomial in n variables. Then we call the finite set

$$\operatorname{Supp} f := \left\{ x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \,\middle|\, c_{\mathbf{d}} \neq 0 \right\}, \text{ where } f = \sum_{\mathbf{d} \in \mathbb{Z}^n} c_{\mathbf{d}} x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$$

the support of the Laurent polynomial f. If we embed $\mathbb{Z}^n$ into $\mathbb{R}^n$, we could view $\operatorname{Supp} f$ as a subset of the n-dimensional euclidean space, so we could consider its convex hull $\Delta(f)$, which is called ***the Newton Polytope of*** f. The aim of the Newton Polytope Theory is to determine the geometric properties of the variety cut out by the system $f_1 = \cdots = f_m = 0$ in T^n in terms of the geometric properties of the support sets of $f_1, \dots, f_m$ and in terms of their Newton polytopes (hence the name). In this paper we address the property of (geometric) irreducibility.

Evidently, if we allow the coefficients of the system $f_1 = \cdots = f_m = 0$ to be arbitrary, then we could not hope to predict any properties just knowing the support sets. Though if we view the system as a point in the vector space $k^{A_\bullet} := k^{A_1} \times \cdots \times k^{A_m}$, where

$$k^{A_i} := \left\{ f \in k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right] \,\middle|\, \operatorname{Supp} f \subset A_i \right\},$$

then one could study the properties of the *general* system $\mathbf{f} = (f_1, \dots, f_m) \in k^{A_\bullet}$. The case when $k = \mathbb{C}$ is well-studied, for example the irreducibility question is thoroughly researched in [KH16] — we partially generalize this result to an arbitrary field, see Theorem 4.1.

However, if one studies the varieties given by the systems $f_1 = \cdots = f_m = 0$ with some additional relations on coefficients, then all such systems may be contained in a closed subset of $k^{A_\bullet}$, so the results about general systems from $k^{A_\bullet}$ do not apply. Such special systems could be of much interest, for example we may study the systems of the form $f(x, y, z) = f(y, x, z) = 0$ — this case is researched in details in [EL22], or we could study the critical loci (introduced in [E17]), i.e. systems of the form $f = f'_x = 0$ or $f'_x = f'_y = g_1 = g_2 = 0$, etc.

The Goal We aim to develop an approach to solve the following kind of problem. Let $V \subset k^{A_\bullet}$ be a vector subspace that correspond to a "reasonable" form of a system supported at $A_1, \dots, A_m$. We want to find a condition on $A_1, \dots, A_m$ so that for the general $\mathbf{f} \in V$ the system $f_1 = \cdots = f_m = 0$ cuts out a geometrically irreducible variety in $(k \backslash 0)^n$.

In this paper we take a step towards this goal by working out the necessary technical equipment. We illustrate the applicability of the approach by generalizing the Khovanskii Irreducibility Theorem to an arbitrary field and by establishing a combinatorial sufficient condition on the support set A so that for the general $f \in k^A$ the variety $f = f'_x = 0$ is geometrically irreducible.

1.2. The structure of the paper

In the following section 2 we fix the notation, prove a few technical propositions, and introduce Definition 2.1 that will be important in section 5.

In section 3 we prove all the necessary statements for our technique: Theorem 3.1 and Corollary 3.1. We give a detailed explanation of our approach in paragraphs "Idea" and "Filling in the gaps" of subsection 3.2. Shortly, the idea is that in order to prove that the general fibre of a morphism is geometrically irreducible it is sufficient to show that the fibred square of the morphism is irreducible. It turns out, that after throwing out a small[1] subvariety (that does not affect the

[1] i.e. at each closed point its dimension is strictly smaller than the dimension of the ambient variety.

irreducibility) we could easily show that the fibred square is irreducible as it turns out to be the total space of a vector bundle over an open subset of $T^n \times T^n$.

Sections 4 and 5 could be read independently. Following very similar paths in section 4 we generalize the Khovanskii Irreducibility Theorem — Theorem 4.1 and in section 5 we find a combinatorial condition on A so that for the general $f \in k^A$ the variety defined in the $(n+2)$-dimensional algebraic torus by $f = f'_x = 0$ is geometrically irreducible — Theorem 5.1. These two statements, of course, may represent interest on their own, but also they show the two key ingredients of the method. In section 4 it is shown to what limit and how one could carefully refine the sufficient condition produced by our technique so that it basically becomes a criterion. In section 5 we show how studying the subspace $V \subset k^{A_\bullet}$ gives us the condition so that for the general $\mathbf{f} \in V$ the variety $\mathbf{f} = 0$ is geometrically irreducible.

Finally, in section 6 we formulate three directions in which the further research could be conducted.

2. Preparatory Work

Notation Let $M \simeq \mathbb{Z}^n$ be the character lattice of the algebraic torus $T^n = \operatorname{Spec} k\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]$, $A_1, \dots, A_m$ be the maximal support sets, and $V \subset k^{A_\bullet}$ be a vector subspace. Consider the following family of varieties parameterized by V:

$$\mathscr{X} := \{(\mathbf{f}, x) \in V \times T^n |\ f_1(x) = \dots = f_m(x) = 0\}$$

If we are to study the properties of the general system $\mathbf{f} \in V$, then we need to study the general fibre of the projection $\mathscr{X} \to V$.

Obvious though useful observations Here we gather some simple remarks that apply equally to the case $V = k^{A_\bullet}$ and to the case $V = \left\{(f,g) \in k^A \times k^A \mid g = x_1 f'_{x_1}\right\}$.

Definition 2.1. A subset $P \subset M$ is called x_1**-dense** if there are $x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}, x_1^{b_1} \cdot \ldots \cdot x_n^{b_n} \in P$ such that $a_1 - b_1$ is not divisible by $\operatorname{char} k$.

Remark **2.1.** We could also give an invariant definition: let $h \in M^* := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be an element of the dual lattice. Then $P \subset M$ is called h-dense iff there are $p_1, p_2 \in P$ such that

$$h(p_1) \not\equiv h(p_2) \mod \operatorname{char} k.$$

In the above definition we implicitly identify the basis of the lattice M with the dual basis of the dual lattice M^*. In particular, it makes sense to call a subset t-dense for some $t \in M$ iff there is a distinguished basis of M (in our case such basis is $x_1, \dots, x_n$).

Claim **2.1.** $\mathscr{X} \to V$ is the kernel of the vector bundle morphism

$$V \times T^n \to \mathbb{A}_k^m \times T^n, \quad (\mathbf{f}, x) \mapsto (\mathbf{f}(x), x).$$

If $V = k^{A_\bullet}$ or $V = \left\{(f,g) \in k^A \times k^A \mid g = x f'_x\right\}$ and A is x_1-dense, then $\mathscr{X} \to T^n$ is a vector subbundle of the trivial bundle $V \times T^n \to T^n$. In the first case the corank of the vector subbundle is m, in the second case the corank is 2.

Proof. To prove that $\mathscr{X}$ is a vector bundle it is sufficient to show that the rank of the morhpism $V \times T^n \to \mathbb{A}_k^m \times T^n$ is constant.

First we deal with the case when $V = k^{A_\bullet}$. Then the matrix of $V \times T^n \to \mathbb{A}_k^m \times T^n$ over the point $x \in T^n$ is

$$\begin{pmatrix} t_{11}(x) & t_{12}(x) & \cdots & t_{1s_1}(x) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & t_{21}(x) & \cdots & t_{2s_2}(x) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & t_{m1}(x) & \cdots & t_{ms_m}(x) \end{pmatrix},$$

where $A_i = \{t_{i1}, \dots t_{is_i}\}$ and $t_{ij}(x) \in k(x)$. For all i, j we have $t_{ij}(x) \neq 0$ because $x \in T^n$, so the rank of the matrix is m for all $x \in T^n$. Hence, the rank of $V \times T^n \to \mathbb{A}_k^m \times T^n$ is constant and equals m, so its kernel $\mathscr{X}$ is a vector subbundle and its corank is equal to $n - m$.

Now we deal with the case $V = \left\{(f,g) \in k^A \times k^A \mid g = x_1 f'_{x_1}\right\}$. Let $A = \{t_1, \dots, t_s\}$. We introduce the following coordinates on V: to the row $(c_1, \dots, c_s) \in k^s$ we associate the point $\left(\sum_{i=1}^s c_i t_i, \sum_{i=1}^s d_i c_i t_i\right) \in V$, where $d_i := \deg_{x_1} t_i$. Now the matrix of the vector bundle morphism $V \times T^n \to \mathbb{A}_k^2 \times T^n$ over the point $x \in T^n$ with respect to the chosen coordinates is

$$\begin{pmatrix} t_1(x) & t_2(x) & \cdots & t_s(x) \\ d_1 t_1(x) & d_2 t_2(x) & \cdots & d_s t_s(x) \end{pmatrix}.$$

Since A is x_1-dense, we have that for some $1 \le i, j \le s$ the difference $d_i - d_j$ is not divisible by $\operatorname{char} k$. Without loss of generality $d_1 - d_2$ is not divisible by $\operatorname{char} k$. Now, for any $x \in T^n$ we have $\operatorname{char} k(x) = \operatorname{char} k$, so $d_1 - d_2$ is not divisible by $\operatorname{char} k(x)$ for all $x \in T^n$. Then the matrix above contains the submatrix

$$\begin{pmatrix} t_1(x) & t_2(x) \\ d_1 t_1(x) & d_2 t_2(x) \end{pmatrix}, \quad \det \begin{pmatrix} t_1(x) & t_2(x) \\ d_1 t_1(x) & d_2 t_2(x) \end{pmatrix} = (d_1 - d_2) t_1(x) t_2(x) \neq 0.$$

Therefore, the rank of $V \times T^n \to \mathbb{A}^2_k \times T^n$ is constant and equals 2, so its kernel $\mathscr{X}$ is a vector subbundle and its corank is equal to 2. ■

Corollary **2.1.** In the above two cases we have that $\mathscr{X}$ is an irreducible variety. In the case $V = k^{A_\bullet}$ we have $\dim \mathscr{X} = \dim V + n - m$, in the other case we have $\dim \mathscr{X} = \dim V + n - 2$.

Claim **2.2.** Either the general fibre of $\mathscr{X} \to V$ is of dimension $\dim \mathscr{X} - \dim V$ or the general fibre is empty.

Proof. If $\mathscr{X} \to V$ is dominant, then the dimension of the general fibre is equal to the relative dimension of the morphism: $\mathscr{X} \to V$ is clearly closed, thus restricting to an appropriate irreducible component and applying [Stacks, Tag 05F7], [Stacks, Tag 02JX] we get what we want. If $\mathscr{X} \to V$ is not dominant, then the general fiber is empty. ■

Remark **2.2.** It follows, that we always have to study just the following two cases: the degenerate case when the general system $\mathbf{f} \in V$ is not consistent, i.e. $\mathbf{f} = 0$ has no solutions and the case when the general system $\mathbf{f} = 0$ cuts out the variety of expected dimension. Note, that in the same fashion as in the above proof one could show that if there is just one system $\mathbf{f} \in V$ such that $\dim\{x \in T^n \mid f_1(x) = \cdots = f_m(x)\} \le D$, then the same inequality holds for the general system.

Remark **2.3.** Consider any field extensions K/k. From [EGA, IV.3, 9.7.8] it follows that the general fibre of $\mathscr{X} \to V$ is geometrically irreducible iff the general fibre of $\mathscr{X}_K \to V_K$ is[2], so we could from now on that k is algebraically closed (though it will not be necessary).

[2] by $\mathscr{X}_K \to V_K$ we denote the base change of $\mathscr{X} \to V$ w.r.t. $\operatorname{Spec} K \to \operatorname{Spec} k$

3. General Technique

In this section we setup the technical tools to prove the irreducibility of varieties cut out in the algebraic k-torus by general systems of equations of specific form and of prescribed support sets.

3.1. Main theorem

Theorem 3.1. If $X \to Y$ is a finitely presented morphism of integral schemes and the fibre square $X \times_Y X$ is irreducible, then the general fibre of $X \to Y$ is geometrically irreducible, i.e. there is a non-empty open subset $U \subset Y$ such that for any $y \in U$ the fibre X_y is geometrically irreducible.

Proof. We will assume without loss of generality that Y is affine. Let $\eta \in Y$ be the generic point of Y. By [EGA, IV.3, 9.7.8], we only need to show that the generic fibre X_η is geometrically irreducible.

In fact, we could also assume that X is affine. Indeed, let $X = U_1 \cup \cdots \cup U_n$ be an open covering such that U_i are affine and non-empty. Since the generic point of X lies in each U_i, we get that the fibres $U_{i\eta}$ give an affine open covering of X_η such that for all indicies i, j the intersection $U_{i\eta} \cap U_{j\eta}$ is non-empty. Then if we prove that all $U_{i\eta}$ are geometrically irreducible, we will also get that X_η is geometrically irreducible. So, we assume without loss of generality that X is affine.

Since $X \times_Y X$ is irreducible, we get that $(X \times_Y X)_\eta = X_\eta \times_\eta X_\eta$ is also irreducible. Now we are left with an algebraic statement to prove: if A is a K-algebra ($K := k(\eta)$) with no zero divisors and $A \otimes_K A$ has no zero divisors except nilpotents, then $\operatorname{Spec} A$ is geometrically irreducible. By [Stacks, Tag 037N] and [Stacks, Tag 0G33] we could just prove that K is separably closed in A. Assume the contrary: There is $\alpha \in A$ that is separably algebraic over K and $\alpha \notin K$. Then $K(\alpha) \otimes_K K(\alpha)$ contains non-nilpotent (because α is separable) zero divisors. Since $K(\alpha) \otimes_K K(\alpha)$ is a subalgebra of $A \otimes_K A$, we get that $A \otimes_K A$ also has non-nilpotent zero divisors, which is a contradiction. Hence, K is algebraically closed in A and $\operatorname{Spec} A$ is geometrically irreducible. ■

3.2. Excision Toolkit

Idea Now it is time to elaborate on what our technique is all about. Whenever we are to study the properties of the general system of specific kind with given support, we get a closed subscheme $\mathscr{X}$ in the space $V \times T^n$, where $V \subset k^{A_\bullet}$ is a closed subvariety. In this setting the properties of the general system $\mathbf{f} \in V$ are exactly the properties of the general fiber of the projection $\mathscr{X} \to V$. In the examples that are of primary interest V is a vector space. Usually, the projection $\mathscr{X} \to T^n$ is a vector bundle which, for example, gives us that $\mathscr{X}$ is irreducible. The theorem above shows that studying the projection $\mathscr{X} \times_V \mathscr{X} \to V$ can give us the irreducibility of the general fiber. Also just like before we get the projection $\mathscr{X} \times_V \mathscr{X} \to T^n \times T^n$ that should help us in studying the former projection. Of course, one could not show in the same fashion as before that the latter projection $\mathscr{X} \times_V \mathscr{X} \to T^n \times T^n$ is a vector bundle, because it is not even flat. However, the following discussion provide a way around.

Lemma **3.1.** Let X be a Jacobson scheme, $Z \subset X$ be a subset with the induced subspace topology and assume that for any point $p \in Z$ that is closed in X we have $\dim_p Z < \dim_p X$. Then $X \backslash Z$ is dense in X.

Proof. Assume the contrary: then there is an open subset $U \subset X$ such that $U \cap (X \backslash Z) = \varnothing$, i.e. $U \subset Z$. Since X is Jacobson, there is a point $p \in U$ that is closed in X. We have $\dim_p X = \dim_p U = \dim_p Z$, which is a contradiction. ■

Corollary **3.1** (Irrelevant fibres)**.** Let $X \to Y$ be a morphism of k-schemes locally of finite type. Let $Z \subset Y$ be an immersed subscheme such that for all closed $p \in Z$ and all closed $x \in X_p$ we have

$$\dim_x X_p < \dim_x X - \dim_p Z \quad \forall x \in X_p(k)$$

Then[3] $X \backslash X_Z$ is dense in X.

Proof. The question is local on Y, so we may assume that Z is closed in Y. Now, X_Z is a closed subscheme of X, hence points of X_Z that are closed in X are just closed points of X_Z. Restricting to appropriate irreducible comoponents of X_Z, Z and applying [Stacks, Tag 02JT], [Stacks, Tag 0D4H] we get

$$\dim_x X_p = \dim_x (X_Z)_p \geq \dim_x X_Z - \dim_p Z$$

Regrouping the terms and applying the inequality from the assumption we get

$$\dim_x X_Z \leq \dim_x X_p - \dim_p Z < \dim_x X.$$

As schemes locally of finite type over a field are Jacobson, we are done. ∎

Filling in the gaps Finally, we could formulate the whole idea of the technique: to establish the geometric irreducibility of the variety cut out by the general $\mathbf{f} \in V$, we are going to prove that $\mathscr{X} \times_V \mathscr{X}$ is irreducible. We will do it by studying the projection $\mathscr{X} \times_V \mathscr{X} \to T^n \times T^n$. Specifically, we are going to show that outside of a closed subscheme $S \subset T^n \times T^n$ the projection $\mathscr{X} \times_V \mathscr{X} \to T^n \times T^n$ is a vector bundle and that S satisfy corollary Corollary 3.1, i.e. throwing out its preimage does not affect the irreducibility of $\mathscr{X} \times_V \mathscr{X}$.

[3] by X_Z we denote the pre-image of Z under the morphism $X \to Y$

4. Khovanskii Irreducibility Theorem over an Arbitrary Field

We use the same notation as in section 2. In particular, we denote by $A_1, \dots, A_m \subset M$ some finite subsets that should be thought of as maximal support sets, we also assume that $m < n$. By Δ_i we denote the corresponding polytopes, i.e. the convex hulls of A_i w.r.t. the embedding $M \hookrightarrow M \otimes \mathbb{R}$. For a subset $J \subset \{1, \dots, m\}$ we define the number $N_J := \dim \sum_{j \in J} \Delta_j - |J|$, where by the sum we mean the Minkowski sum. i.e. $B + C := \{b + c \mid b \in B, c \in C\}$.

In this section we prove the following theorem.

Theorem 4.1. If for all non-empty subsets $J \subset \{1, \dots, m\}$ we have $N_J > 0$, then for the general $\mathbf{f} \in k^{A_\bullet}$ the variety defined in T^n by the system $f_1 = \dots = f_m = 0$ is geometrically irreducible.

The formulation of the theorem and the proof for the case $k = \mathbb{C}$ is due to Khovanskii, cf. [KH16].

Remark **4.1.** The condition that $\forall J \subset \{1, \dots, m\}, J \neq \varnothing$ we have $N_J > 0$ is absolutely natural. In fact, N_J determines the actual number of independent variables of the subsystem of $\mathbf{f}_J$ — the subsystem of $\mathbf{f}$ that corresponds to J. Indeed, if $N_J \leq 0$, then after a proper monomial change of coordinates and regrouping the equations we could assume that $J = \{1, \dots, r\}$ and $f_i = f_i(x_1, \dots, x_r)$ for $i \leq r$. By the Kouchnirenko-Bernstein-Khovanskii Theorem the system $\mathbf{f}_J$ then has $MVol_{\mathbb{Z}}(\Delta_1, \dots, \Delta_r)$ solutions in T^r. If $MVol_{\mathbb{Z}}(\Delta_1, \dots, \Delta_r) > 1$, then clearly the expected number of irreducible components is greater than one.

Definition 4.1. We define the subvariety of singularities $\mathcal{S} \subset T^n \times T^n$ of all the points such that the projection $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X} \to T^n \times T^n$ is degenerate over them, i.e.

$$\mathcal{S} := \{p \in T^n \times T^n \mid \operatorname{rk}_p \Phi < 2m\},$$

where Φ is the morphism $k^{A_\bullet} \times T^n \times T^n \to \mathbb{A}^{2m} \times T^n \times T^n$, $(\mathbf{f}, x, y) \mapsto (\mathbf{f}(x), \mathbf{f}(y), x, y)$.

4.1. Preparations

Claim **4.1.** $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X} \to T^n \times T^n$ is the kernel of Φ. If $\mathcal{S}$ satisfies the condition of Corollary 3.1, then $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X}$ is irreducible.

Proof. It is obvious that $\mathscr{X} \times_k^{A_\bullet} \mathscr{X} \to T^n \times T^n = \operatorname{Ker} \Phi$. We denote by $(\mathscr{X} \times_k^{A_\bullet} \mathscr{X})_{\mathcal{S}}$ the preimage of $\mathcal{S}$. Clearly the morphism $(\mathscr{X} \times_k^{A_\bullet} \mathscr{X}) \backslash (\mathscr{X} \times_k^{A_\bullet} \mathscr{X})_{\mathcal{S}} \to (T^n \times T^n) \backslash \mathcal{S}$ is a vector bundle, in particular, $\mathscr{X} \times_k^{A_\bullet} \mathscr{X} (\mathscr{X} \times_k^{A_\bullet} \mathscr{X})_{\mathcal{S}}$ is irreducible as it is the total space of a vector bundle over an irreducible base. Therefore, if $\mathcal{S}$ satisfies the condition of Corollary 3.1, then $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X}$ must also be irreducible. ∎

Remark **4.2.** Until the end of this section we will denote $A_i = \{t_{i1}, \dots, t_{is_i}\}$. Then we have that the matrix of Φ over the point $p \in T^n \times T^n$ is

$$\begin{pmatrix}
t_{11}(u_p) & t_{12}(u_p) & \cdots & t_{1s_1}(u_p) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & t_{21}(u_p) & \cdots & t_{2s_2}(u_p) & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & t_{m1}(w_p) & \cdots & t_{ms_m}(w_p) \\
t_{11}(w_p) & t_{12}(w_p) & \cdots & t_{1s_1}(w_p) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & t_{21}(w_p) & \cdots & t_{2s_2}(w_p) & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & t_{m1}(w_p) & \cdots & t_{ms_m}(w_p)
\end{pmatrix},$$

where $u_p, w_p \in k(p)^n$ are the values of the coordinate functions $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ that come from one same system of coordinated on T^n. Then $p \in \mathcal{S}$ iff i-th and $(i+m)$-th row of the above matrix are proprotional for some i.

Note that we could shift any subset A_i by any monomial as it corresponds to multiplying the i-th equation of the system by the same monomial. Therefore we will assume without loss of generality that $t_{i1} = 1\ \forall i$. Now we have that the i-th and the $(i+m)$-th rows of the above matrix are proportional iff they coincide. Hence we see that $p \in \mathcal{S}$ iff there is i such that $t_{ij}(x_p) = t_{ij}(y_p)\ \forall j$. We keep all the said assumption until the end of the section.

4.2. Stratification of singularities

Definition 4.2. Let $J \subset \{1, \ldots, m\}$ be a non-empty subset. We define the constructible subset

$$\mathcal{S}_J := \big\{p \in T^n \times T^n \mid \forall i \in J, \forall t \in A_i\ t(x_p) = t(y_p) \text{ and } \forall i \notin J \exists t \in A_i : t(x_p) \neq t(y_p)\big\}.$$

In other words, $\mathcal{S}_J$ consists of all the points $p \in T^n \times T^n$ such that $i \in J$ iff the i-th and the $(i+m)$-th rows of the matrix of Φ over p coincide.

Remark **4.3.** $\mathcal{S}_J$ form a stratification of $\mathcal{S}$, i.e. $\mathcal{S} = \bigsqcup_J \mathcal{S}_J$, where J runs over all non-empty subsets of $\{1, \ldots, m\}$. In particular, $\mathcal{S}$ satisfies the condition of Corollary 3.1 iff $\mathcal{S}_J$ does for all non-empty $J \subset \{1, \ldots, m\}$.

Lemma **4.1.** The following inequality holds: $\operatorname{codim}_{T^n \times T^n} \mathcal{S}_J \geq \dim \sum_{j \in J} \Delta_j$.

Proof. $\mathcal{S}_J$ is an open subset in the closed subset $\overline{\mathcal{S}_J}$ defined by the equations

$$t(x) = (y)\ \forall i \in J\ \forall t \in A_i$$

Recall that by M we denote the character lattice. Let $L \subset M \times M$ be the sublattice generated by $t_{ij}(x)t_{ij}^{-1}(y), i \in J$. Clearly $\overline{\mathcal{S}_J}$ is defined by the equations $l(x,y) = 1\ \forall l \in L$. We have that L is the image of $L' \subset M$ under the antidiagonal embedding $M \to M \times M, t \mapsto (t, t^{-1}) = t(x)t^{-1}(y)$, where L' is generated by all $t \in A_i$ for $i \in J$. Let $d = \dim \sum_{j \in J} \Delta_j$. We have $d = \operatorname{rk} L' = \operatorname{rk} L$. Hence, by the Smith Normal Form Theorem after a proper monomial change of coordinates we could assume that L' is generated by the monomials $x_1^{r_1}, \ldots, x_d^{r_d}, r_i > 0$. Then $\overline{\mathcal{S}_J}$ is defined by equations $x_i^{r_i} = y_i^{r_i}$. Therefore:

$$\operatorname{codim} \mathcal{S}_J \geq \operatorname{codim} \overline{\mathcal{S}_j} = d = \dim \sum_{i \in J} \Delta_I.$$

■

Lemma **4.2.** For all $p \in \mathcal{S}_J$ we have $\dim(\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X})_p = \dim k^{A_\bullet} - 2m + |J|$.

Proof. Obviously the rank of Φ over $p \in \mathcal{S}_J$ is exactly $2m - |J|$. Hence, the fibre $(\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X})_p = \operatorname{Ker} \Phi_p$ is nothing but a vector subspace in $k(p)^{A_\bullet}$ of codimension $2m - |J|$. ■

Lemma **4.3.** For all closed $x \in \mathscr{X} \times_{k^{A_\bullet}} \mathscr{X}$ we have $\dim_x \mathscr{X} \times_{k^{A_\bullet}} \mathscr{X} \geq \dim k^{A_\bullet} + 2n - 2m$.

Proof. All closed points of $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X}$ are closed points of $k^{A_\bullet} \times T^n \times T^n$ as well. At all of its closed points the dimension of $k^{A_\bullet} \times T^n \times T^n$ is $\dim k^{A_\bullet} + 2n$. The subvariety $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X}$ is cut out by $2m$ equations, so the dimension at each point may drop at most by $2m$. ■

Proof of Theorem 4.1 Now we see that for any closed $p \in \mathcal{S}_J$ and any closed $x \in (\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X})_p$ we have

$$\dim_x \mathscr{X} \times_{k^{A_\bullet}} \mathscr{X} - \dim_x (\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X})_p - \dim_p \mathcal{S}_J \geq \\ \geq \dim k^{A_\bullet} + 2n - 2m - \dim k^{A_\bullet} + 2m - |J| - (2n - \operatorname{codim}_{T^n \times T^n} \mathcal{S}_J) \geq N_J,$$

which shows that $\mathcal{S}_J$ satisfies the condition of Corollary 3.1 for all non-empty $J \subset \{1, \dots, m\}$, i.e. $\mathcal{S}$ satisfies the condition of Corollary 3.1 (see Remark 4.3), so $\mathscr{X} \times_{k^{A_\bullet}} \mathscr{X}$ is irreducible. Hence, by Theorem 3.1 we proved Theorem 4.1.

5. Critical Loci

In the previous section we showed that our approach could be helpful in generalizing to an arbitrary characteristic the already existing results about the irreducibility. Now it is time to show that our technique could also help in finding some new answers.

In this section we derive a sufficient combinatorial condition on $A \subset M$ (we still use the notation of section 2) so that for the general $f \in k^A$ the variety cut out by the equations $f = f'_{x_1} = 0$ in T^n is geometrically irreducible, see Theorem 5.1.

5.1. Preparations

Notation We slightly change the notation, so it is more convenient in the context of our specific problem of critical loci. First we change our coordinates from $x_1, \ldots, x_n, n > 2$ to $x, y_1, \ldots, y_{n-1}$ and from now on we will take the partial derivative with respect to x instead of x_1. Also to make our notation less complicated we change $n-1$ to n, i.e. now we have the coordinates $x, y_1, \ldots, y_n, n > 1$. We identify $V \cong k^A$ via $k^A \to V,\ f \mapsto (f, xf'_x)$.

We follow the path similar to what we did in section 4.

Definition 5.1. Let us define the subvariety of singularities $\mathcal{S} \subset T^{n+1} \times T^{n+1}$ of all the points such that the projection $\mathscr{X} \times_V \mathscr{X} \to T^{n+1} \times T^{n+1}$ is degenerate over them, i.e.

$$\mathcal{S} := \{p \in T^{n+1} \times T^{n+1} \mid \operatorname{rk}_p \Phi < 4\},$$

where Φ is the morphism $V \times T^{n+1} \times T^{n+1} \to \mathbb{A}^4 \times T^{n+1} \times T^{n+1}$, $(f, p, q) \mapsto (f(p), f(q), f'_x(p), f'_x(q), p, q)$.

Claim **5.1.** $\mathscr{X} \times_V \mathscr{X} \to T^{n+1} \times T^{n+1}$ is the kernel of Φ. If $\mathcal{S}$ satisfies the condition of Corollary 3.1, then $\mathscr{X} \times_V \mathscr{X}$ is irreducible.

Proof. The same as in Claim 4.1. ∎

Remark **5.1.** If A is x-dense (recall Definition 2.1), then for all $p \in T^n \times T^n$ we have $\operatorname{rk} \Phi_p \geq 2$. Indeed, the matrix of Φ w.r.t. the standard coordinates on[4] k^A is

$$\begin{pmatrix} t_1(u_p) & \ldots & t_s(u_p) \\ t_1(w_p) & \ldots & t_s(w_p) \\ d_1 t_1(u_p) & \ldots & d_2 t_s(u_p) \\ d_1 t_1(w_p) & \ldots & d_s t_s(w_p) \end{pmatrix},$$

where $A := \{t_1, \ldots, t_s\}$, $d_i := \deg_x t_i$ and $u_1, \ldots, u_{n+1}, w_1, \ldots, w_{n+1}$ are coordinates on $T^{n+1} \times T^{n+1}$ that come from one same system of coordinates on T^{n+1}. We have that the first and the third rows are not proportional because A is x-dense[5], i.e. $\operatorname{rk} \Phi_p \geq 2$. We keep the notation for the coordinates until the end of the section.

Claim **5.2.** If A is x-dense and $\dim \mathcal{S} \leq n+1$, then $\mathcal{S}$ satisfies the condition of Corollary 3.1.

Proof. The above remark tells us that $\forall p \in T^n \times T^n$ we have $\dim(\mathscr{X} \times_V \mathscr{X})_p \leq \dim V - 2$, because the fibre over p is nothing but the kernel of the morphism $\Phi_p : V \to \mathbb{A}^4$, and the rank of Φ_p is at least 2. For any closed $x \in V \times T^{n+1} \times T^{n+1}$ we have $\dim_x V \times T^{n+1} \times T^{n+1} = \dim V + 2(n+1)$. Since $\mathscr{X} \times_V \mathscr{X}$ is cut out in $V \times T^{n+1} \times T^{n+1}$ by 4 equations, we have that $\dim_x \mathscr{X} \times_V \mathscr{X} \geq \dim V + 2(n+1) - 4$ for any closed $x \in \mathscr{X} \times_V \mathscr{X}$. Putting it all together we get that for any closed $p \in \mathcal{S}$ and any closed $x \in (\mathscr{X} \times_V \mathscr{X})_p$ the following inequality holds:

$$\begin{aligned} &\dim_x \mathscr{X} \times_V \mathscr{X} - \dim_x (\mathscr{X} \times_V \mathscr{X})_p - \dim_p \mathcal{S} \geq \\ &\qquad \geq \dim V + 2(n+1) - 4 - (\dim V - 2) - \dim \mathcal{S} \geq 2n - (n+1) = n - 1 > 0, \end{aligned}$$

i.e. the condition of Corollary 3.1 is satisfied. ∎

[4] recall that $V \cong k^A$

[5] as in the proof of Claim 2.1

5.2. The sufficient condition

Theorem 5.1. If $A \subset M$ contains[6] two non-degenerate $(n+1)$-simplicies such that their equivalence classes modulo the sublattice $\langle x_1^{\operatorname{char} k}, y_1, \ldots, y_n \rangle$ are disjoint, then for the general $f \in k^A$ the variety cut out in T^{n+1} by $f = f'_x = 0$ is geometrically irreducible.

Proof. Let $l_0, l_1, \ldots, l_{n+1}, q_1, \ldots, q_{n+1} \in A$ be such that the sublatticies $L := \langle l_i l_j^{-1} \rangle_{1 \le i,j \le n+1}$ and $Q := \langle q_i q_j^{-1} \rangle_{1 \le i,j \le n+1}$ are of the rank $n+1$ and $\forall i,j$ we have $l_i q_j^{-1} \notin \langle x_1^{\operatorname{char} k}, y_1, \ldots, y_n \rangle$. Let $\mathcal{L}$ and $\mathcal{Q}$ be the subvarieties of $T^{n+1} \times T^{n+1}$ defined by the equations[7] $t(u) = t(w) \forall t \in L$ and $t(u) = \ t(w)\ \forall t \in Q$ respectively. $\dim \mathcal{L} = \dim \mathcal{Q} = n+1$ because the codimensions of these subvarieties are equal to the ranks[8] of L, Q. Obviously A is x-dense, so if we show that $\mathcal{S} \subset \mathcal{L} \cup \mathcal{Q}$, then by the above claims $\mathscr{X} \times_V \mathscr{X}$ is irreducible and by Theorem 3.1 we will be done.

We need to show that for all $p \in T^{n+1} \times T^{n+1} \backslash (\mathcal{L} \cup \mathcal{Q})$ we have $\operatorname{rk}_p \Phi = 4$. The key observation for us will be that $\forall p \notin \mathcal{L}$ the rows[9] $(l_1(u_p), \ldots, l_{n+1}(u_p))$ and $(l_1(w_p), \ldots, l_{n+1}(w_p))$ are not proportional. Indeed, otherwise we have that for all $1 \le i,j \le n+1$

$$\frac{l_i(u_p)}{l_i(w_p)} = \frac{l_j(u_p)}{l_j(w_p)} \iff l_i(u_p) l_j^{-1}(u_p) = l_i(w_p) l_j^{-1}(w_p).$$

Since $l_i l_j^{-1}$ generate the lattice L, it would imply that $p \in \mathcal{L}$. The same way, if $p \notin \mathcal{Q}$, then the rows $(q_1(u_p), \ldots, q_{n+1}(u_p))$ and $(q_1(w_p), \ldots, q_{n+1}(w_p))$ are not proportional as well. Now let us fix $p \notin \mathcal{L} \cup \mathcal{Q}$. Without loss of generality the rows $(l_1(u_p), l_2(u_p))$ and $(l_1(w_p), l_2(w_p))$ are not proportional, and the rows $(q_1(u_p), q_2(u_p))$ and $(q_1(w_p), q_2(w_p))$ are not proportional. We introduce the following notation:

$$\begin{aligned} a_1 &= l_1(u_p), & a_2 &= l_2(u_p), & a_3 &= q_1(u_p), & a_4 &= q_2(u_p), \\ b_1 &= l_1(w_p), & b_2 &= l_2(w_p), & b_3 &= q_1(w_p), & b_4 &= q_2(w_p), \\ d_1 &= \deg_x l_1, & d_2 &= \deg_x l_2, & d_3 &= \deg_x q_1, & d_4 &= \deg_x q_2. \end{aligned}$$

The following matrix is a submatrix of Φ_p:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ d_1 a_1 & d_2 a_2 & d_3 a_3 & d_4 a_4 \\ d_1 b_1 & d_2 b_2 & d_3 b_3 & d_4 b_4 \end{pmatrix}$$

we will show that this submatrix is of rank 4, thus Φ_p will also be of rank 4 and the theorem will be proved. We show that the rank of the submatrix is 4 via straightforward Gaussian Elimination:

$$\operatorname{rk} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ d_1 a_1 & d_2 a_2 & d_3 a_3 & d_4 a_4 \\ d_1 b_1 & d_2 b_2 & d_3 b_3 & d_4 b_4 \end{pmatrix} = \operatorname{rk} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & (d_2 - d_1) a_2 & (d_3 - d_1) a_3 & (d_4 - d_1) a_4 \\ 0 & (d_2 - d_1) b_2 & (d_3 - d_1) b_3 & (d_4 - d_1) b_4 \end{pmatrix} =$$

$$[b'_i := b_i - (b_1/a_1) a_i] \quad = \operatorname{rk} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & b'_2 & b'_3 & b'_4 \\ 0 & (d_2 - d_1) a_2 & (d_3 - d_1) a_3 & (d_4 - d_1) a_4 \\ 0 & (d_2 - d_1) b'_2 & (d_3 - d_1) b'_3 & (d_4 - d_1) b'_4 \end{pmatrix} =$$

[6] i.e. there are $l_0, l_1, \ldots, l_{n+1}, q_1, \ldots, q_{n+1} \in A$ such that the sublatticies $\langle l_i l_j^{-1} \rangle, \langle q_i q_j^{-1} \rangle$ are of the rank $n+1$ and $\forall i,j$ we have $l_i q_j^{-1} \notin \langle x_1^{\operatorname{char} k}, y_1, \ldots, y_n \rangle$.

[7] recall that $u_1, \ldots, u_{n+1}, w_1, \ldots, w_{n+1}$ are the coordinates on $T^{n+1} \times T^{n+1}$.

[8] in fact, they are equal to the ranks of the images of these sublattice under the anti-diagonal embedding $M \hookrightarrow M \times M$, $t \mapsto t(u) t^{-1}(w)$, but these ranks are the same.

[9] recall that u_p is the vector $(u_1, \ldots, u_{n+1})$ that is evaluated at p and the same for w_p.

we need $b'_2 \neq 0$ for the following identity to be true. It is proved after the Gaussian Elimination.

$$[a'_i := b'_2 a_i / a_2] \quad = \operatorname{rk}\begin{pmatrix} a'_1 & b'_2 & a'_3 & a'_4 \\ 0 & b'_2 & b'_3 & b'_4 \\ 0 & (d_2-d_1)b'_2 & (d_3-d_1)a'_3 & (d_4-d_1)a'_4 \\ 0 & (d_2-d_1)b'_2 & (d_3-d_1)b'_3 & (d_4-d_1)b'_4 \end{pmatrix} =$$

$$= \operatorname{rk}\begin{pmatrix} a'_1 & b'_2 & a'_3 & a'_4 \\ 0 & b'_2 & b'_3 & b'_4 \\ 0 & 0 & (d_3-d_2)a'_3 & (d_4-d_2)a'_4 \\ 0 & 0 & (d_3-d_2)b'_3 & (d_4-d_2)b'_4 \end{pmatrix} = \operatorname{rk}\begin{pmatrix} a'_1 & b'_2 \\ 0 & b'_2 \end{pmatrix} + \operatorname{rk}\begin{pmatrix} (d_3-d_2)a'_3 & (d_4-d_2)a'_4 \\ (d_3-d_2)b'_3 & (d_4-d_2)b'_4 \end{pmatrix} =$$

$$= \operatorname{rk}\begin{pmatrix} a'_1 & a'_2 \\ b'_1 & b'_2 \end{pmatrix} + \operatorname{rk}\left(\begin{pmatrix} a'_3 & a'_4 \\ b'_3 & b'_4 \end{pmatrix}\begin{pmatrix} d_3-d_2 & 0 \\ 0 & d_4-d_2 \end{pmatrix}\right) = (*)$$

Now, the condition that $l_i q_j^{-1} \notin \langle x^{\operatorname{char} k}, y_1, \ldots, y_n\rangle$ means exactly that $\deg_x l_i - \deg_x q_j$ is not divisible by $\operatorname{char} k$, i.e. $\deg_x l_i - \deg_x q_j \neq 0$ in $k(p) \forall p \in T^{n+1} \times T^{n+1}$. Therefore, we have that $d_2 - d_3 \neq 0$ and $d_2 - d_4 \neq 0$ in $k(p)$. Thus

$$\operatorname{rk}\left(\begin{pmatrix} a'_3 & a'_4 \\ b'_3 & b'_4 \end{pmatrix}\begin{pmatrix} d_3-d_2 & 0 \\ 0 & d_4-d_2 \end{pmatrix}\right) = \operatorname{rk}\begin{pmatrix} a'_3 & a'_4 \\ b'_3 & b'_4 \end{pmatrix}$$

Furthermore, $a_i, b_i \mapsto a'_i, b'_i$ is a linear automorphism: to prove it we need to show that

$$b'_2 = b_2 - (b_1/a_1)a_2 \neq 0.$$

It is the case because the rows $(a_1\ a_2)$ and $(b_1\ b_2)$ are non-proportional. Hence, $a_i, b_i \mapsto a'_i, b'_i$ is indeed a linear isomorphism, so

$$(*) = \operatorname{rk}\begin{pmatrix} a'_1 & a'_2 \\ b'_1 & b'_2 \end{pmatrix} + \operatorname{rk}\begin{pmatrix} a'_3 & a'_4 \\ b'_3 & b'_4 \end{pmatrix} = \begin{pmatrix} a_1 & b_2 \\ b_1 & b_2 \end{pmatrix} + \operatorname{rk}\begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \end{pmatrix} = 2 + 2 = 4.$$

■

REMARK **5.2.** In case $\operatorname{char} k = 0$, there is a more geometric formulation of our condition: if there are non-degenerate simplicies $\Delta, \Delta' \subset A$ such that their projections onto $\langle x\rangle$ along the sublattice $\langle y_1, \ldots, y_n\rangle$ are disjoint, then for the general $f \in k^A$ the variety $f = f'_x = 0$ is geometrically irreducible. In case $\operatorname{char} k > 0$ we need to replace $\langle x\rangle$ with the quotient lattice $\langle x\rangle / \langle x^{\operatorname{char} k}\rangle$.

5.3. Examples

In this subsection we gather a few important examples. In particular, we show some limitations of the found condition.

EXAMPLE **5.1** (Sufficient though not necessary)**.** The following example shows that the found condition is indeed not a criterion. Consider a finite set of monomials:

$$\Delta_d := \left\{ x^a y_1^{b_1} \cdot \ldots \cdot y_n^{b_n} \middle| a + b_1 + \ldots + b_n \leq d; a, b_i \geq 0 \right\}.$$

For a polynomial $f \in k[x, y_1, \ldots, y_n]$ we have $\operatorname{Supp} f \subset \Delta_d \iff \deg f \leq d$. For $d = 1$ the variety $f = f'_x = 0$ is empty for the general $f \in \Delta_1$, because f'_x is a non-zero constant. For $d \geq 3$ the set Δ_d satisfies the condition of Theorem 5.1. The case $d = 2$ drops out: it does not satisfy the condition of our theorem and one can manually check[10], that for the general $f \in k^{\Delta_2}$ the solution set of $f = f'_x = 0$ is a non-empty geometrically irreducible variety.

[10] Indeed: $f'_x = 0$ is a hyperplane and the restriction of $f = 0$ to the hyperplane gives a quadric. The general quadric is geometrically irreducible.

Example **5.2** (Actual counter-example)**.** The following example shows a situation when for the general $f \in k^A$ the system $f = f'_x = 0$ defines a variety that is not irreducible. Let $B \subset \langle y_1, \ldots, y_n\rangle$ be an arbitrary finite subset, $L \subset \langle x, \mathbf{y}_1, \ldots, y_n\rangle$ be a line segment disjoint from B. Then we could consider $A := B \subset L$. For any polynomial f supported at A there is the decomposition of the following form:

$$f(x, y) = g(y) + h(x, y), \quad \operatorname{Supp} g \subset B, \operatorname{Supp} h \subset L$$

Then $f'_x = h'_x$. Since $\operatorname{Supp} h$ is contained in a line segment, we see that there are monomials $v, z \in \langle x, y_1, \ldots, y_n\rangle$ such that $h(x, y) = v \cdot p(z)$, where $p \in k[T]$ is a polynomial in one variable. Therefore $f'_x = 0$ defines a finite number of shifted subtori and $f = f'_x = 0$ is the intersection of the hypersurface $f = 0$ with these subtori. In particular, if at least two intersection are non-empty, then the variety $f = f'_x = 0$ is not irreducible.

To give a concrete example let us consider the case when $n = 2$, $B = \{1, y_1, y_2\}$, $L = \{x, \ldots, x^{m+1}\}$, and k is algebraically closed. Then for the general $f, \operatorname{Supp} f \subset (B \cup L)$ the equation $f'_x = 0$ defines m parallel hyperplanes $x = c_i, 1 \leq i \leq m$ and the system $f = f'_x = 0$ defines m disjoint planes of codimension 2.

Example **5.3** (Characteristic matters)**.** Here we illustrate that the characteristic of the base field is significant. Consider $A = A_0 \cup \{x\} \cup x^3 A_3 \cup \{x^5\}$, where $A_0, A_3 \subset \langle y_1, \ldots, y_n\rangle$ are arbitrary finite subsets that contain non-degenerate n-simplicies. Then any $f \in k^A$ is of the form

$$f = ax^5 + x^3 h(y) + bx + g(y), \quad a, b \in k.$$

If $\operatorname{char} k > 3$, then $A_0 \cup \{x^5\}$ and $\{x\} \cup x^3 A_3$ each contain a non-degenerate $(n+1)$-simplex and the projections of these sets onto $\langle x\rangle$ are disjont modulo $x^{\operatorname{char} k}$, i.e. this case satisfies the condition of Theorem 5.1. Though it is different for $\operatorname{char} k = 3$. If $\operatorname{char} k = 3$, then $f'_x = 5ax^4 + b$ and the variety $f = f'_x = 0$ has 4 geometric irreducibility components for the general $f \in k^A$.

6. Further Work

A Unified Approach It is hard not to notice the great resemblance of the proofs of Theorem 4.1 and Theorem 5.1 — they are very similar in the geometric part, though they differ when it comes to analyzing the fibres of the projection $\mathscr{X} \times_V \mathscr{X} \to T^n \times T^n$ which is completely determined by the vector subspace V. In the same way as in these theorems, we could find a condition on the support sets so that the general systems of the kind $f'_x = f'_y = 0$, $f = f'_x = g_1 = \cdots = g_m = 0$, etc. give a geometrically irreducible variety. Clearly, we are not bound just to critical loci and we could study other systems with fixed support sets and linear relations on coefficients. There must be a way to do this in a unified manner, i.e. there must be a correspondence between some special properties of $V \subset k^{A_\bullet}$ and the conditions on $A_\bullet$ so that for the general $\mathbf{f} \in V$ the variety $\mathbf{f} = 0$ is geometrically irreducible.

Discriminant Conditions Suppose we already know that for the general $\mathbf{f} \in V$ the variety $\mathbf{f} = 0$ is geometrically irreducible. Then it is natural to seek an explicit condition (like the non-degeneracy conditions in [Ber75] and[KH78]) on $f \in V$ so that $\mathbf{f}$ is general, i.e. $\mathbf{f} = 0$ is geometrically irreducible. The exact criterion may be hard to find, but adapting a common toric geometry technique[11] one should be able to find a sufficient condition.

Sufficient though not necessary The approach produces only sufficient conditions. In the case of the Khovanskii Irreducibility Theorem the condition is virtually a criterion, but it is because the general systems in $k^{A_\bullet}$ is a relatively nice setting. As we see things get more complicated if a slightly more difficult problem is considered. Therefore, it would be useful to find a way to estimate the roughness of the produced conditions.

[11] for example, see [EL22][Sec. 7, p.30]

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