BIRATIONAL AUTOMORPHISM GROUPS OF SEVERI–BRAUER SURFACES OVER THE FIELD OF RATIONAL NUMBERS

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Abstract. We prove that the only non-trivial finite subgroups of birational automorphism group of non-trivial Severi–Brauer surfaces over the field of rational numbers are $\mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/3\mathbb{Z})^2$. Moreover, we show that $(\mathbb{Z}/3\mathbb{Z})^2$ is contained in $\mathrm{Bir}(V)$ for any Severi–Brauer surface V over a field of characteristic zero, and $(\mathbb{Z}/3\mathbb{Z})^3$ is contained in $\mathrm{Bir}(V)$ for any Severi–Brauer surface V over a field of characteristic zero which contains a non-trivial cube root of unity.

1. Introduction

The Cremona group $\mathrm{Cr}_n(\mathbf{F})$ is a group of birational automorphisms of $\mathbb{P}^n$ over a field $\mathbf{F}$. It is difficult to describe this group, except the case $n = 1$, when we have $\mathrm{Cr}_1(\mathbf{F}) \simeq \mathrm{PGL}_2(\mathbf{F})$. Even the classification of finite subgroups seems extremely hard. Nowadays, we know the description of conjugacy classes of finite subgroups only for $\mathrm{Cr}_2(\mathbb{C})$ (see [6]).

It is natural to ask how birational automorphisms of forms of projective spaces behave.

Definition 1.1. An n-dimensional variety V over a field $\mathbf{F}$ is called a *Severi–Brauer variety* if

$$V \times_{\mathrm{Spec}(\mathbf{F})} \mathrm{Spec}(\overline{\mathbf{F}}) \simeq \mathbb{P}^n_{\overline{\mathbf{F}}},$$

where $\overline{\mathbf{F}}$ is an algebraic closure of $\mathbf{F}$. Such a variety V is called non-trivial if it is not isomorphic to $\mathbb{P}^n_{\mathbf{F}}$.

Like $\mathrm{Cr}_n(\mathbf{F})$, the group $\mathrm{Bir}(V)$ of birational automorphisms of a Severi–Brauer variety V also has complicated structure (cf. [8] and [19]). A classification of finite groups that appear as subgroups of $\mathrm{Bir}(V)$ for non-trivial Severi–Brauer surfaces over various fields of characteristic zero was given in [15, Theorem 1.3]. On the other hand, there is no simple way to decide which of them are realized for a given field, or for a given Severi–Brauer surface. Meanwhile, we have a very simple description of finite subgroups of $\mathrm{Bir}(V)$ over the field $\mathbb{Q}$ of rational numbers.

Theorem 1.2 ([15, Corollary 1.4])**.** *Let V be a non-trivial Severi–Brauer surface over $\mathbb{Q}$ and let $G \subset \mathrm{Bir}(V)$ be a finite subgroup. Then we have $G \subset (\mathbb{Z}/3\mathbb{Z})^3$.*

The goal of this paper is to prove the following result which is a strengthening of Theorem 1.2.

Theorem 1.3. *Let V be a non-trivial Severi–Brauer surface over the field $\mathbb{Q}$ and let G be a finite group. Then G is isomorphic to a subgroup of $\mathrm{Bir}(V)$ if and only if $G \subset (\mathbb{Z}/3\mathbb{Z})^2$, and G is isomorphic to a subgroup of $\mathrm{Aut}(V)$ if and only if $G \subset \mathbb{Z}/3\mathbb{Z}$.*

Recall the following result of A.Beauville which is a particular case of [1, Theorem] and [1, Lemma 3.1].

Theorem 1.4. *Let* $\mathbf{F}$ *be an algebraically closed field of characteristic zero. Then we have*

(i) $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{F}}) \supset (\mathbb{Z}/3\mathbb{Z})^3$;
(ii) $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{F}}) \not\supset (\mathbb{Z}/3\mathbb{Z})^4$;
(iii) $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{F}}) \not\supset (\mathbb{Z}/3\mathbb{Z})^3$.

In the process of proving Theorem 1.3 we obtain the following result which can be considered as an analogue of Theorem 1.4 for arbitrary Severi–Brauer surfaces.

Proposition 1.5. *Let* V *be a Severi–Brauer surface over a field* $\mathbf{F}$ *of characteristic zero. Then*

(i) $\mathrm{Bir}(V) \supset (\mathbb{Z}/3\mathbb{Z})^2$;
(ii) $\mathrm{Bir}(V) \supset (\mathbb{Z}/3\mathbb{Z})^3$ *if and only if* $\mathbf{F}$ *contains a non-trivial cube root of unity;*
(iii) $\mathrm{Bir}(V) \not\supset (\mathbb{Z}/3\mathbb{Z})^4$;
(iv) $\mathrm{Aut}(V) \supset \mathbb{Z}/3\mathbb{Z}$;
(v) $\mathrm{Aut}(V) \supset (\mathbb{Z}/3\mathbb{Z})^2$ *if and only if* $\mathbf{F}$ *contains a non-trivial cube root of unity;*
(vi) $\mathrm{Aut}(V) \not\supset (\mathbb{Z}/3\mathbb{Z})^3$.

Remark 1.6. The existence of birational actions of the group $(\mathbb{Z}/3\mathbb{Z})^3$ on certain non-trivial Severi–Brauer surfaces was proved in [16, Theorem 1.2] (see also [16, Section 3] for construction of an example of such an action). Proposition 1.5(ii) strengthens this result by showing that such an action exists on *every* Severi–Brauer surface over a field of characteristic zero containing a non-trivial cube root of unity.

Let us briefly explain the idea of the proof of Proposition 1.5(i) and (ii) as long as Theorem 1.3 immediately follows from it. We show that $\mathbb{Z}/3\mathbb{Z}$ acts biregularly on V. But $(\mathbb{Z}/3\mathbb{Z})^2$ does not if $\mathbf{F}$ does not contain a non-trivial cube root of unity. However, the group $(\mathbb{Z}/3\mathbb{Z})^2$ acts birationally on every Severi–Brauer surface. To this end we blow up the Severi–Brauer surface $\mathbb{Z}/3\mathbb{Z}$-equivariantly, obtain a smooth cubic surface and observe that it is isomorphic to the Fermat cubic surface over an algebraic closure of $\mathbf{F}$. Studying 3-subgroups in the automorphism group of the Fermat cubic surface, which commute with the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ we get that $(\mathbb{Z}/3\mathbb{Z})^2$ acts biregularly on this cubic surface. The remaining assertions in Proposition 1.5 are easy.

The plan of the paper is as follows. In Section 2 we prove some supplementary lemmas. In Section 3 we collect some basic facts about Severi–Brauer surfaces and study finite subgroups of their automorphism groups. In Section 4 we collect some auxiliary facts about cubic surfaces and construct a birational action of $(\mathbb{Z}/3\mathbb{Z})^2$ on Severi–Brauer surfaces. In Section 5 we study 3-groups in the birational automorphism groups of Severi–Brauer surfaces an we prove Proposition 1.5. In Section 6 we prove Theorem 1.3.

Notation. Let X be a variety defined over $\mathbf{F}$. If $\mathbf{F} \subset \mathbf{L}$ is an extension of $\mathbf{F}$, then we will denote by $X_{\mathbf{L}}$ the variety

$$X_{\mathbf{L}} = X \times_{\mathrm{Spec}(\mathbf{F})} \mathrm{Spec}(\mathbf{L}).$$

By $\overline{\mathbf{F}}$ we denote an algebraic closure of $\mathbf{F}$. A geometric point of X is a point of $X_{\overline{\mathbf{F}}}$. A geometric line on X is a line on $X_{\overline{\mathbf{F}}}$.

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2. Preliminaries

In this section we collect some auxiliary facts.

Lemma 2.1. *Let* $\mathbf{F}$ *be a field of zero characteristic which does not contain non-trivial cube roots of unity. Then* $(\mathbb{Z}/3\mathbb{Z})^3 \not\subset \mathrm{GL}_4(\mathbf{F})$.

Proof. Assume that there is a linear action of $(\mathbb{Z}/3\mathbb{Z})^3$ on the vector space $\mathbf{F}^4$. As this group is abelian we get that the matrices, which represent the elements of the group, are diagonalizable simultaneously over $\overline{\mathbf{F}}$. Thus, the elements of the group are conjugate to

$$\begin{pmatrix} \omega^a & 0 & 0 & 0 \\ 0 & \omega^b & 0 & 0 \\ 0 & 0 & \omega^c & 0 \\ 0 & 0 & 0 & \omega^d \end{pmatrix},$$

where ω is a non-trivial cube root of unity and $a, b, c, d \in \{0, 1, 2\}$. Note that the determinant and the trace of these matrices belong to $\mathbf{F}$. Therefore, such matrices have to satisfy the following:

$$a + b + c + d \equiv 0 \mod 3; \tag{2.1}$$

$$\omega^a + \omega^b + \omega^c + \omega^d \in \mathbf{F}. \tag{2.2}$$

The condition (2.1) gives us only 27 matrices. The condition (2.2) decreases this number because, for example, the matrix with the eigenvalues $\omega, \omega, \omega, 1$ does no satisfy the condition (2.2), but satisfies the condition (2.1). □

Let $X \subset \mathbb{P}^n$ be a projective variety over a field $\mathbf{F}$. Denote by $\mathrm{Ir}(X_{\overline{\mathbf{F}}})$ the set of irreducible components of $X_{\overline{\mathbf{F}}}$. For an element $\phi \in \mathrm{PGL}_{n+1}(\overline{\mathbf{F}})$ let

$$\phi_{\mathrm{Ir}} : \mathrm{Ir}(X_{\overline{\mathbf{F}}}) \to \mathrm{Ir}(\phi(X_{\overline{\mathbf{F}}}))$$

be the induced map between the sets of irreducible components.

Lemma 2.2. *Let* $\mathbf{F}$ *be an arbitrary field of characteristic zero. Let* $X \subset \mathbb{P}^n$ *be a projective variety defined over* $\mathbf{F}$*. Let* $\phi \in \mathrm{PGL}_{n+1}(\overline{\mathbf{F}})$ *be an element such that* $X' = \phi(X)$ *is defined over* $\mathbf{F}$*. Assume also that* ϕ_{Ir} *commutes with the action of the Galois group* $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ *on* $\mathrm{Ir}(X_{\overline{\mathbf{F}}})$ *and* $\mathrm{Ir}(X'_{\overline{\mathbf{F}}})$*. Assume that any irreducible component of* $X_{\overline{\mathbf{F}}}$ *is a linear subspace in* $\mathbb{P}^n_{\overline{\mathbf{F}}}$*. Then there is* $\psi \in \mathrm{PGL}_{n+1}(\mathbf{F})$ *such that* $\psi(X) = X'$ *and* $\psi_{\mathrm{Ir}} = \phi_{\mathrm{Ir}}$.

Proof. Both $X_{\overline{\mathbf{F}}}$ and $X'_{\overline{\mathbf{F}}}$ decompose into unions of linear subspaces over $\overline{\mathbf{F}}$, i.e.

$$X_{\overline{\mathbf{F}}} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_m, \quad X'_{\overline{\mathbf{F}}} = \mathcal{L}'_1 \cup \ldots \cup \mathcal{L}'_m.$$

Since ϕ_{Ir} commutes with the Galois group, we can renumber $\mathcal{L}_i$ and $\mathcal{L}'_i$ so that

$$\phi(\mathcal{L}_i) = \mathcal{L}'_i$$

and the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ acts on $\mathrm{Ir}(X_{\overline{\mathbf{F}}})$ and $\mathrm{Ir}(X'_{\overline{\mathbf{F}}})$ as follows:

$$g(\mathcal{L}_i) = \mathcal{L}_{g(i)} \quad \text{and} \quad g(\mathcal{L}'_i) = \mathcal{L}'_{g(i)}$$

for all $g \in \mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$. For all $1 \leqslant i \leqslant m$ denote by $\mathcal{F}_i$ the set of linear homogeneous polynomials

$$f \in \overline{\mathbf{F}}[x_0, \ldots, x_n]$$

such that $f|_{\mathcal{L}'_i} = 0$. Denote by

$$\mathcal{M} \subset \mathbb{P}\big(\mathrm{Mat}_{n+1}(\overline{\mathbf{F}})\big) \simeq \mathbb{P}^{(n+1)^2-1}_{\overline{\mathbf{F}}}$$

the set of all non-degenerate matrices ψ such that $\psi(\mathcal{L}_i) = \mathcal{L}'_i$ for all $1 \leqslant i \leqslant m$.

For every i, $P \in \mathcal{L}_i$ and $f \in \mathcal{F}_i$ denote by R^i_{Pf} the linear relation

$$f(\psi(P)) = 0$$

on the entries of the matrix ψ. Let $\mathcal{R}$ be the set of all such $\mathcal{R}^i_{Pf}$ for all

$$i \in \{1, \ldots, m\}, \quad P \in \mathcal{L}_i, \quad f \in \mathcal{F}_i.$$

So $\mathcal{M}$ is an intersection of $\mathrm{PGL}_{n+1}(\overline{\mathbf{F}}) \subset \mathbb{P}^{(n+1)^2-1}_{\overline{\mathbf{F}}}$ with a linear subspace $\widetilde{\mathcal{M}}$ which is defined by the equations R^i_{Pf}. Take an element g of the Galois group. We obtain

$$g(R^i_{Pf}) = R^{g(i)}_{g(P)g(f)}.$$

Let us prove that $R^{g(i)}_{g(P)g(f)} \in \mathcal{R}$. Indeed, we have

$$g(\mathcal{L}_i) = \mathcal{L}_{g(i)}, \quad g(P) \in \mathcal{L}_{g(i)} \quad \text{and} \quad g(f)|_{\mathcal{L}'_{g(i)}} = 0,$$

where the last equality holds because ψ_{Ir} commutes with the Galois group. Therefore, $\widetilde{\mathcal{M}}$ is Galois-invariant. Thus, $\widetilde{\mathcal{M}}$ is defined over $\mathbf{F}$ and there is a dense set of $\mathbf{F}$-points on $\widetilde{\mathcal{M}}$. The intersection $\mathcal{M} = \widetilde{\mathcal{M}} \cap \mathrm{PGL}_{n+1}(\mathbf{F})$ is non-empty because it contains ϕ. Therefore, it contains an $\mathbf{F}$-point.

□

3. Severi–Brauer surfaces

In this section we study finite subgroups of the automorphism groups of Severi–Brauer surfaces. Let us mention some properties of Severi–Brauer varieties (for more details see, for instance, [7] and [10]). If V is a Severi–Brauer variety over a field $\mathbf{F}$, then V is non-trivial if and only if $V(\mathbf{F}) = \varnothing$ (see [7, Theorem 5.1.3]). There is a bijection between Severi–Brauer varieties of dimension n over a field $\mathbf{F}$ and central simple algebras of dimension $(n+1)^2$ over $\mathbf{F}$ (see e.g. [7, §5.2]).

Theorem 3.1 ([7, Theorems 2.1.3 and 5.2.1])**.** *Let V be a non-trivial Severi–Brauer variety of dimension n such that $n+1$ is a prime number. Then the central simple algebra A which is associated to V is a division algebra.*

The following theorem describes automorphism groups of Severi–Brauer varieties.

Theorem 3.2 (see, for example, [4, p. 266] or [17, Lemma 4.1])**.** *If a central simple algebra A corresponds to a Severi–Brauer variety V over a field $\mathbf{F}$ then we have* $\mathrm{Aut}(V) \simeq A^*/\mathbf{F}^*$.

Theorem 3.2 allows to obtain restrictions on the orders of automorphisms of Severi–Brauer varieties.

Example 3.3 (cf. [15, Lemma 5.2])**.** Let $\mathbf{F}$ be a field of characteristic zero. Let V be a non-trivial Severi–Brauer surface and let $x \in \mathrm{Aut}(V)$ be an element of prime order $p \neq 3$. We claim that $p \equiv 1 \pmod 3$. Indeed, let A be a central simple algebra which corresponds to V. Then A is a division algebra by Theorem 3.1. By Theorem 3.2 we have $\mathrm{Aut}(V) \simeq A^*/\mathbf{F}^*$.

Let $\bar{x}$ be any element in the preimage of x under the homomorphism $A^* \to A^*/\mathbf{F}^*$. Then we have $\bar{x}^p = a \in \mathbf{F}^*$. Let $B \subset A$ be a field which is generated by $\bar{x}$. Since

$$\dim_{\mathbf{F}} A = \dim_B A \cdot \dim_{\mathbf{F}} B,$$

we obtain $\dim_{\mathbf{F}} B = 3$, because $\dim_{\mathbf{F}} B = 9$ is impossible as B is a commutative algebra while A is not.

Let $f(t)$ be a minimal polynomial of $\bar{x}$ over $\mathbf{F}$. Its degree is equal to 3. Moreover, the polynomial $f(t)$ divides $t^p - a$. In particular, the polynomial $t^p - a$ is reducible over $\mathbf{F}$. So by [11, Theorem VI.9.1] we get $a = c^p$ for some $c \in \mathbf{F}^*$. This means that the roots of $f(t)$ in $\overline{\mathbf{F}}$ are $c\xi_1$, $c\xi_2$ and $c\xi_3$, where ξ_i are pairwise different p-th roots of unity. Hence these ξ_i for $1 \leqslant i \leqslant 3$ form a $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$-orbit. Let Γ be the image of $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ in the automorphism group

$$\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/(p-1)\mathbb{Z},$$

where $\mathbb{Z}/p\mathbb{Z}$ is considered as the multiplicative group of p-th roots of unity. Then the group Γ has an orbit of order 3. Therefore, the order of Γ is divisible by 3. Thus, 3 divides $p-1$ and we are done.

The following lemma is a well-known fact about subgroups of the automorphism group of Severi–Brauer surface. We reproduce its proof for the convenience of the reader.

Lemma 3.4 (see e.g. [15, Example 4.1 and Remark 4.2])**.** *Let V be a Severi–Brauer surface over a field $\mathbf{F}$ of characteristic zero. Then* $\mathrm{Aut}(V)$ *contains a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$. If $\mathbf{F}$ contains a non-trivial cube root of unity then* $\mathrm{Aut}(V)$ *contains a subgroup isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^2$.

Proof. If $V \simeq \mathbb{P}^2$, then $\mathrm{Aut}(V) \simeq \mathrm{PGL}_3(\mathbf{F})$, and there is a group of order 3 in $\mathrm{PGL}_3(\mathbf{F})$ which is generated by the element cyclically permuting the coordinates. If $\mathbf{F}$ contains a non-trivial cube root ω of unity then the elements

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \in \mathrm{PGL}_3(\mathbf{F}) \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \in \mathrm{PGL}_3(\mathbf{F})$$

generate a group $(\mathbb{Z}/3\mathbb{Z})^2 \subset \mathrm{PGL}_3(\mathbf{F})$.

Now assume that V is a non-trivial Severi–Brauer surface. Then the Severi–Brauer surface V corresponds to a division algebra A by Theorem 3.1, and this algebra is a cyclic algebra by [7, Chapter 7, Exercise 9]. By [7, Proposition 2.5.2] the algebra A is generated by a Galois extension $\mathbf{F} \subset \mathbf{L}$ of degree 3 and an element $\alpha \in A$ such that $\alpha \notin \mathbf{F}^*$ and $\alpha^3 \in \mathbf{F}^*$. Furthermore, one has $\alpha\lambda = \sigma(\lambda)\alpha$

for all $\lambda \in \mathbf{L}$, where σ is a generator of the Galois group of the extension. The element α gives us an automorphism of order 3. Therefore, one has $\mathbb{Z}/3\mathbb{Z} \subset \mathrm{Aut}(V)$.

Finally, if $\mathbf{F}$ contains a non-trivial cube root ω of unity then by Kummer theory (see, for example, [2, Chapter III, §2, Lemma 2]) we obtain $\mathbf{L} = \mathbf{F}(\beta)$ for some $\beta \notin \mathbf{F}$ such that $\beta^3 \in \mathbf{F}^*$ and for a generator $\sigma \in \mathrm{Gal}(\mathbf{L}/\mathbf{F})$ one has $\sigma(\beta) = \omega\beta$. We have the following relations

$$\alpha^3 \in \mathbf{F}^*,\ \beta^3 \in \mathbf{F}^*,\ \alpha\beta = \omega\beta\alpha.$$

Therefore, the image of α and β under the homomorphism $A^* \to A^*/\mathbf{F}^*$ generate the group $(\mathbb{Z}/3\mathbb{Z})^2 \subset A^*/\mathbf{F}^* \simeq \mathrm{Aut}(V)$.

□

Now we are going to study 3-subgroups in the automorphism groups of Severi–Brauer surfaces. First of all, let us prove the following lemma about central simple algebras of dimension 9.

Lemma 3.5. *Let A be a central simple algebra of dimension 9 over a field $\mathbf{F}$ of characteristic zero. Assume that $\widehat{T} = (\mathbb{Z}/3\mathbb{Z})^2 \subset A^*/\mathbf{F}^*$, and let x and y be generators of $\widehat{T}$. If $\bar{x}$ and $\bar{y}$ are any elements in the preimages of x and y under the natural homomorphism $A^* \to A^*/\mathbf{F}^*$, respectively, then*

$$\bar{x}\bar{y}\bar{x}^{-1} = \omega\bar{y},$$

where ω is a non-trivial cube root of unity.

Proof. We have $xyx^{-1}y^{-1} = 1$ in $A^*/\mathbf{F}^*$, because x and y commute with each other. Therefore, we obtain

$$\bar{x}\bar{y}\bar{x}^{-1} = a\bar{y}$$

for some $a \in \mathbf{F}^*$. As $\bar{x}^3 \in \mathbf{F}^*$, we get

$$\bar{y} = \bar{x}^3\bar{y}\bar{x}^{-3} = a^3\bar{y}.$$

Thus, we have $a^3 = 1$.

Assume that $a = 1$. So the elements $\bar{x}$ and $\bar{y}$ commute with each other. Consider the subalgebra B of A generated by the elements

$$1, \bar{x}, \bar{y}, \bar{x}^2, \bar{y}^2, \bar{x}\bar{y}, \bar{x}^2\bar{y}, \bar{x}\bar{y}^2, \bar{x}^2\bar{y}^2. \tag{3.1}$$

Let us prove that these elements are linearly independent over $\mathbf{F}$. Let $z \neq 1$ be an element from the set (3.1). Thus, by definition the trace of the matrix corresponding to the multiplication by z in B is zero. Indeed, it is not hard to see if we take as basis of B the linear independent elements from (3.1) which generate the basis.

If the elements in (3.1) are linearly dependent we have a relation

$$\alpha_0 + \alpha_1\bar{x} + \alpha_2\bar{y} + \alpha_3\bar{x}^2 + \alpha_4\bar{y}^2 + \alpha_5\bar{x}\bar{y} + \alpha_6\bar{x}^2\bar{y} + \alpha_7\bar{x}\bar{y}^2 + \alpha_8\bar{x}^2\bar{y}^2 = 0 \tag{3.2}$$

over $\mathbf{F}$. We can assume that $\alpha_0 \neq 0$. Indeed, at least one of the coefficients in (3.2), say α_j, is non-zero. Let $\alpha_j\bar{t}$ be the corresponding summand in (3.2). Multiplying (3.2) by $\bar{t}^2$, we obtain a non-zero free term $\alpha_j\bar{t}^3$. On the one hand, the trace of the multiplication by the left hand side of (3.2) is equal to the trace of α_0 which is non-zero. On the other hand, it is equal to zero. Therefore, the elements (3.1) are linearly independent. This give us $\dim_{\mathbf{F}} B = 9 = \dim_{\mathbf{F}} A$. But A is a central simple algebra and B is a commutative algebra. This contradiction gives us that a is a non-trivial cube root of unity.

□

Lemma 3.6. *Let* $\mathbf{F}$ *be a field of zero characteristic which does not contain non-trivial cube roots of unity. Let* V *be a Severi–Brauer surface over* $\mathbf{F}$. *Then* $(\mathbb{Z}/3\mathbb{Z})^2$ *is not a subgroup of* $\mathrm{Aut}(V)$.

Proof. Let A be a central simple algebra corresponding to the Severi–Brauer surface V. Then by Theorem 3.2 we obtain $\mathrm{Aut}(V) \simeq A^*/\mathbf{F}^*$. Assume that

$$(\mathbb{Z}/3\mathbb{Z})^2 \subset A^*/\mathbf{F}^*.$$

Let x and y be generators of $(\mathbb{Z}/3\mathbb{Z})^2$. Let $\bar{x}$ and $\bar{y}$ be any elements in the preimages of x and y under the natural homomorphism $A^* \to A^*/\mathbf{F}^*$, respectively. By Lemma 3.5 we obtain

$$\bar{x}\bar{y}\bar{x}^{-1} = \omega\bar{y}$$

where ω is a non-trivial cube root of unity. However, the field $\mathbf{F}$ does not contain non-trivial cube root of unity. This contradiction gives us that

$$(\mathbb{Z}/3\mathbb{Z})^2 \not\subset \mathrm{Aut}(V).$$

□

Corollary 3.7 (cf. [15, Corollary 6.3]). *Let* V *be a non-trivial Severi–Brauer surface over a field* $\mathbf{F}$ *of characteristic zero. Let* $\widehat{T} = (\mathbb{Z}/3\mathbb{Z})^2$ *be a subgroup in the automorphism group of* V *generated by the elements* x *and* y. *Then the sets of fixed points of* x *and* y *are disjoint.*

Proof. According to Lemma 3.6 the field $\mathbf{F}$ contains a non-trivial cube root of unity. Let A be a central simple algebra corresponding to V. Then by Theorem 3.2 we have $\widehat{T} \subset A^*/\mathbf{F}^*$. Therefore, by Lemma 3.5 for any elements $\bar{x}$ and $\bar{y}$ in the preimages of x and y under the natural homomorphism $A^* \to A^*/\mathbf{F}^*$, respectively we get

$$\bar{x}\bar{y} = \omega\bar{y}\bar{x}, \tag{3.3}$$

where ω is a non-trivial cube root of unity. Let us consider the elements $\bar{x}$ and $\bar{y}$ as automorphisms of $V_{\overline{\mathbf{F}}} \simeq \mathbb{P}^2_{\overline{\mathbf{F}}}$. Then these automorphisms correspond to 3×3 matrices over $\overline{\mathbf{F}}$ and satisfy relation (3.3). Thus, $\bar{x}$ and $\bar{y}$ cannot have a common eigenvector and hence, the elements x and y have no common fixed points.

□

It appears that the action of the group $\mathbb{Z}/3\mathbb{Z}$ on a non-trivial Severi–Brauer surface V, which exists by Lemma 3.4, can be lifted to a smooth cubic surface obtained as a blowup of V.

Lemma 3.8. *Let* $\mathbf{F}$ *be a field of characteristic zero, and let* V *be a non-trivial Severi–Brauer surface over* $\mathbf{F}$. *Let* $T \simeq \mathbb{Z}/3\mathbb{Z}$ *be a subgroup of* $\mathrm{Aut}(V)$. *Then*

(i) *there is a unique triple of geometric points* p_1, p_2, p_3 *on* V *which are fixed by the group* T *(in particular, the triple* p_1, p_2 *and* p_3 *is defined over* $\mathbf{F}$*);*
(ii) *there is an orbit of* T *consisting of a triple of geometric points* p_4, p_5, p_6, *which is defined over the field* $\mathbf{F}$;
(iii) *for any choice of* p_4, p_5, p_6 *as in* (ii) *the blowup of* V *at* $p_1, \ldots, p_6$ *is a smooth cubic surface.*

Proof. First of all, let us prove (i). By [18, Lemma 4], we have that the set of T-fixed points consists of 3 geometric points of V. Denote them by p_1, p_2, p_3. Observe that such three geometric points do not lie on one line in $V_{\overline{\mathbf{F}}} \simeq \mathbb{P}^2_{\overline{\mathbf{F}}}$. Indeed, otherwise we get that the action of T on the line has exactly 3 fixed points, but that is impossible. The triple p_1, p_2 and p_3 is defined over $\mathbf{F}$ because the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ commutes with T.

Now let us prove (ii). By [18, Theorem 2], the quotient V/T is $\mathbf{F}$-rational. As $\mathbf{F}$-points in V/T is a dense subset, so the preimage of a general point $p \in V/T$ consists of 3 geometric points. Denote them by p_4, p_5 and p_6. Note that they do not lie on one line l in $V_{\overline{\mathbf{F}}}$, because otherwise the group T fixes this line and thus, by [18, Proposition 3] the quotient l/T does not contain $\mathbf{F}$-points, which contradicts the fact that $p \in l/T$ is an $\mathbf{F}$-rational point.

Let us prove (iii). We have to show that the blow up of the points $p_1, \ldots, p_6$ is a smooth cubic surface. For this it is enough to prove that any 3 points among these points do not lie on a line and all 6 points do not lie on a conic. First of all, let us prove that any triple of the set $p_1, \ldots, p_6$ does not lie on one line. Indeed, first of all, assume that p_1, p_2 and p_4 lie on the line l over $\overline{\mathbf{F}}$. Then as T permutes p_4, p_5 and p_6, these points also lie on l. But this is impossible by the above argument.

Now assume that p_4, p_5 and p_1 lie on one line over $\overline{\mathbf{F}}$. Then under the action of a non-trivial element α of the group T this line maps to the line passing through p_1, p_6 and one of the points p_4 and p_5, which means that p_4, p_5 and p_6 lie on one line, which contradicts the above arguments.

Finally, note that no non-trivial automorphism fixes 3 points on a conic. Therefore, the points p_1, p_2, p_3, p_4, p_5, p_6 do not lie on one conic. So the blowup of these 6 points gives us a smooth cubic surface.

□

Remark 3.9. Let $V \simeq \mathbb{P}^2$. There is a biregular action of $T \simeq \mathbb{Z}/3\mathbb{Z}$ on V which is generated by the element cyclically permuting the coordinates. The geometric points

$$p_1 = [1 : 1 : 1], \quad p_2 = [\omega : 1 : \omega^2], \quad p_3 = [\omega^2 : 1 : \omega], \tag{3.4}$$

where ω is a non-trivial cube of unity, are fixed by T. Note that the union $p_1 \cup p_2 \cup p_3$ is Galois-invariant. The geometric points

$$p_4 = [1 : 0 : 0], \quad p_5 = [0 : 1 : 0], \quad p_6 = [0 : 0 : 1] \tag{3.5}$$

form an orbit of T. It is not hard to see that any 3 points among these 6 ones do not lie on a line and all 6 points do not lie on a conic. Therefore, the blowup of p_1, p_2, p_3, p_4, p_5, p_6 is a smooth cubic surface.

If the field $\mathbf{F}$ contains a non-trivial cube root of unity then by Lemma 3.4 on any Severi–Brauer surface over $\mathbf{F}$ there is a biregular action of $(\mathbb{Z}/3\mathbb{Z})^2$. It turns out that this action can be lifted to a smooth cubic surface which is a blowup of V provided that V is a non-trivial Severi–Brauer surface.

Lemma 3.10. *Let* $\mathbf{F}$ *be a field of characteristic zero and let* V *be a non-trivial Severi–Brauer surface over* $\mathbf{F}$*. Assume that the group* $\widehat{T} \simeq (\mathbb{Z}/3\mathbb{Z})^2$ *is contained in* $\mathrm{Aut}(V)$*. Let* b *and* c *be generators of* $\widehat{T}$*. Then*

(i) *there is a unique triple of geometric points p_1, p_2, p_3 on V which are fixed by the subgroup generated by b (in particular, the triple p_1, p_2 and p_3 is defined over $\mathbf{F}$);*
(ii) *there is a unique triple of geometric points p_4, p_5, p_6 on V which are fixed by the subgroup generated by c (in particular, the triple p_4, p_5 and p_6 is defined over $\mathbf{F}$);*
(iii) *the two triples p_1, p_2, p_3 and p_4, p_5, p_6 have no common points;*
(iv) *the element b cyclically permutes p_4, p_5, p_6;*
(v) *the element c cyclically permutes p_1, p_2, p_3;*
(vi) *the blowup of V at $p_1, \ldots, p_6$ is a smooth cubic surface.*

Proof. Assertions (i) and (ii) follow directly from Lemma 3.8(i). Assertion (iii) follows from Corollary 3.7. Assertions (iv) and (v) follow from the fact that b and c commute with each other which means that the element b fixes the set of fixed points of c and vice versa. Assertion (vi) follows from Lemma 3.8(iii).

□

Remark 3.11. Assume that $V \simeq \mathbb{P}^2$ over a field $\mathbf{F}$ which contains a non-trivial cube root ω of unity. Then there is a biregular action of $\mathbb{Z}/3\mathbb{Z}$ on V which was constructed in Remark 3.9. Let b be a generator of this group. Consider the element

$$c = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \in \mathrm{PGL}_3(\mathbf{F}).$$

Together with b it generates the subgroup $\widehat{T} = (\mathbb{Z}/3\mathbb{Z})^2$ in $\mathrm{Aut}(V)$. While the element b fixes p_1, p_2 and p_3 from (3.4) and cyclically permutes p_4, p_5 and p_6 from (3.5), the element c on the contrary fixes p_4, p_5 and p_6 and cyclically permutes p_1, p_2 and p_3. In particular, the set $\{p_1, \ldots, p_6\}$ is $\widehat{T}$-invariant. It is straightforward to check that the blowup of V at these 6 points is a smooth cubic surface.

4. Cubic surfaces

In this section we study cubic surfaces over a field of characteristic zero. First of all, we make the following observation.

Lemma 4.1. *Let $\mathbf{F}$ be a field of zero characteristic which does not contain the non-trivial cube root of unity. Let S be a smooth cubic surface over $\mathbf{F}$. Then the group $(\mathbb{Z}/3\mathbb{Z})^3$ does not act biregularly on S.*

Proof. Assume that there is an action of $(\mathbb{Z}/3\mathbb{Z})^3$ on S. Then we get an induced action of this group on $\mathbb{P}^3$. Also the action of $(\mathbb{Z}/3\mathbb{Z})^3$ induces a linear action on

$$H^0(S, -K_S) \simeq \mathbf{F}^4.$$

However, by Lemma 2.1 the group $(\mathbb{Z}/3\mathbb{Z})^3$ does not act linearly on $\mathbf{F}^4$.

□

The following lemma states that the map between smooth cubic surfaces is defined uniquely by the image of pairwise skew lines $E_1, \ldots, E_6$.

Lemma 4.2. *Let $\mathbf{F}$ be an algebraically closed field. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface with pairwise skew lines $E_1, \ldots, E_6$. Assume that there is an element $\theta \in \mathrm{PGL}_4(\mathbf{F})$ such that $\theta(E_i) = E_i$ for all i. Then $\theta = id$.*

Proof. First of all, assume that θ preserves S. Then we can blow down $E_1, \ldots, E_6$ and get the induced automorphism θ on $\mathbb{P}^2$ with 6 fixed points in general position. Therefore, θ acts trivially on $\mathbb{P}^2$ and so on S and $\mathbb{P}^3$.

Now assume that $\theta(S) \neq S$. Denote by S' the image of cubic surface $\theta(S)$. The intersection $S \cdot S'$ of these two cubic surfaces in $\mathbb{P}^3$ is a possibly non-reduced curve of degree 9. Our 6 lines $E_1, \ldots, E_6$ are contained in this curve. It is well-known that for any 5 lines among $E_1, \ldots, E_6$ there is a unique line on S which intersects all these 5 lines. This line is a strict transform of the conic through 5 points in general position which is unique. Any of these lines lies in S' because it has at least 5 common points with S'. And so we get that the curve of degree 9 contains 12 lines, which is a contradiction.

□

Let S be a smooth cubic surface over a field $\mathbf{F}$. Then there is a natural action of the Weyl group $W(\mathrm{E}_6)$ on the Picard group $\mathrm{Pic}(S_{\overline{\mathbf{F}}})$ of $S_{\overline{\mathbf{F}}}$ (see, for example, [12, Corollary 25.1.1]). Namely, the group $W(\mathrm{E}_6)$ consists of all automorphisms of the lattice $\mathrm{Pic}(S_{\overline{\mathbf{F}}}) \simeq \mathbb{Z}^7$ preserving the intersection form and fixing the canonical class K_S. In particular, for every choice of 6 pairwise skew lines $E_1, \ldots, E_6$ on $S_{\overline{\mathbf{F}}}$ there is an action of the symmetric group S_6 on the Picard group $\mathrm{Pic}(S_{\overline{\mathbf{F}}})$ which permutes the classes of these 6 lines. It is well-known that the automorphism group of S is embedded in the Weyl group $W(\mathrm{E}_6)$ (see [5, Corollary 8.2.40]). Note also that the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ maps to $W(\mathrm{E}_6)$.

Lemma 4.3. *Let S be a smooth cubic surface over a field $\mathbf{F}$ of characteristic zero. Let $\phi \in \mathrm{Aut}(S_{\overline{\mathbf{F}}})$ be an automorphism of $S_{\overline{\mathbf{F}}}$ which commutes with the image of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ in $W(\mathrm{E}_6)$. Then the automorphism ϕ is defined over $\mathbf{F}$, i.e. $\phi \in \mathrm{Aut}(S)$.*

Proof. Denote by E_i for $1 \leqslant i \leqslant 27$ all lines lying on $S_{\overline{\mathbf{F}}}$. Then the curve

$$\mathcal{E} = E_1 + \ldots + E_{27}$$

is Galois-invariant, thus, is defined over $\mathbf{F}$. Applying Lemma 2.2 to the curve $\mathcal{E}$ and the element $\phi \in \mathrm{PGL}_4(\overline{\mathbf{F}})$ we obtain $\psi \in \mathrm{PGL}_4(\mathbf{F})$ such that $\phi(E_i) = \psi(E_i)$ for all $1 \leqslant i \leqslant 27$. Hence, applying Lemma 4.2 to the element $\phi \circ \psi^{-1}$ we get $\phi = \psi$. Thus, we have $\phi \in \mathrm{Aut}(S)$.

□

We are mostly interested in two conjugacy classes of elements in $W(\mathrm{E}_6)$, which are conjugacy classes of type A_2 and A_2^2 in the notation of [3]. They consist of the elements whose eigenvalues on the vector space corresponding to the root system E_6 are

$$\omega, \omega^2, 1, 1, 1, 1$$

and

$$\omega, \omega, \omega^2, \omega^2, 1, 1,$$

respectively; here ω is a non-trivial cube root of unity. We will say that an automorphism of a smooth cubic surface is of type A_2 (of type A_2^2), if its image in the Picard group is an element in the conjugacy class of type A_2 (of type A_2^2).

Example 4.4. Let S be a smooth cubic surface over a field of characteristic zero. Let $E_1, \ldots, E_6$ be pairwise skew geometric lines on S. Let us consider the element (456) in $W(\mathrm{E}_6)$ which cyclically permutes E_4, E_5 and E_6 and fixes E_1, E_2 and E_3. Then by [13, Table 1] this element is of type A_2.

Now let us discuss the automorphism group of the Fermat cubic surface, i.e. the cubic surface which is defined by the equation $x^3 + y^3 + z^3 + t^3 = 0$.

Example 4.5. Let S be the Fermat cubic surface over algebraically closed field $\mathbf{F}$ of characteristic zero. Then by [13, Lemma 2.4 and Table 1] an element in $\mathrm{Aut}(S)$ is of type A_2 if and only if its quadruple of eigenvalues in $\mathrm{PGL}_4(\mathbf{F})$ up to multiplication by a non-zero element in the field $\mathbf{F}$ is of the form $1, 1, \omega, \omega$, where ω is a non-trivial cube root of unity. An element in $\mathrm{Aut}(S)$ is of type A_2^2 if and only if its quadruple of eigenvalues in $\mathrm{PGL}_4(\mathbf{F})$ up to multiplication by a non-zero element in the field $\mathbf{F}$ is of the form $1, 1, \omega, \omega^2$.

Theorem 4.6 ([6, Theorem 6.10])**.** *Let S be a smooth cubic surface over an algebraically closed field of characteristic zero admitting an automorphism of type* A_2. *Then S is isomorphic to the Fermat cubic.*

Corollary 4.7. *Let S be a smooth cubic surface over an algebraically closed field of characteristic zero. Let $E_1, \ldots, E_6$ be pairwise skew lines on S. Assume that there is a biregular action of the group $T \simeq \mathbb{Z}/3\mathbb{Z}$ on S which fixes E_1, E_2 and E_3 and cyclically permutes E_4, E_5 and E_6. Then S is isomorphic to the Fermat cubic.*

Proof. Consider the subgroup $\mathrm{S}_6 \subset W(\mathrm{E}_6)$ acting on $E_1, \ldots, E_6$ by permutations. Then the image of the group T in $W(\mathrm{E}_6)$ is generated by the element $(456) \in \mathrm{S}_6$. By Example 4.4 the conjugacy class of the element (456) has type A_2. Therefore, by Theorem 4.6 the cubic surface S is isomorphic to the Fermat cubic. □

Lemma 4.8. *Let S be the Fermat cubic surface over an algebraically closed field of characteristic zero. Then there are exactly* 6 *elements in* $\mathrm{Aut}(S)$ *of type* A_2 *and they commute with each other. Moreover, the centralizer of any element of type* A_2 *in* $\mathrm{Aut}(S)$ *is isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

Proof. The Fermat cubic surface S is defined by the equation $x^3 + y^3 + z^3 + t^3 = 0$ in $\mathbb{P}^3$. The automorphism group of S is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathrm{S}_4$ (see, for instance, [5, Theorem 9.5.6]). The group S_4 acts by the permutations of the coordinates x, y, z, and t. The group $(\mathbb{Z}/3\mathbb{Z})^3$ acts by the multiplication of the coordinates by cube roots of unity. Any element in $(\mathbb{Z}/3\mathbb{Z})^3$ can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^a & 0 & 0 \\ 0 & 0 & \omega^b & 0 \\ 0 & 0 & 0 & \omega^c \end{pmatrix} \in \mathrm{PGL}_4(\overline{\mathbf{F}}), \tag{4.1}$$

where ω is a non-trivial cube root of unity and $a, b, c \in \{0, 1, 2\}$. From Example 4.5 we get that the element of type A_2 has the form (4.1) if and only if

$$a = b,\ c = 0; \quad \text{or} \quad a = c,\ b = 0; \quad \text{or} \quad b = c,\ a = 0.$$

Therefore, these are 6 elements of type A_2.

By Example 4.5 all elements of order 3 in the group S_4 are of type A_2^2, because the eigenvalues of the corresponding matrices are $\omega, \omega^2, 1, 1$. Let us consider all elements in $\mathrm{Aut}(S)$ of order 3 of the form gh, where $g \in (\mathbb{Z}/3\mathbb{Z})^3$ and h is a non-trivial element in S_4. By direct computation one can see that the eigenvalues of such elements are $1, \omega, \omega^2, \omega^t$, where $t \in \{0, 1, 2\}$. Therefore, by Example 4.5 such elements are of type A_2^2. So there are exactly 6 elements of type A_2 in $\mathrm{Aut}(S)$, and they commute with each other.

Let us prove that the centralizer of any element of type A_2 in $\mathrm{Aut}(S)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. Indeed, as we showed above all such elements lie in the normal subgroup $(\mathbb{Z}/3\mathbb{Z})^3 \subset \mathrm{Aut}(S)$. Let us fix the element $t \in \mathrm{Aut}(S)$ of type A_2 and consider the elements in S_4 which commute with t. These are the elements which permute the eigenvectors of t with the same eigenvalues. Recall from Example 4.5 that eigenvalues in $\mathrm{PGL}_4(\mathbf{F})$ up to multiplication by non-zero element of t are $1, 1, \omega, \omega$. Thus, the centraliser is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

□

Lemma 4.9. *Let S be a smooth cubic surface over a field $\mathbf{F}$ of characteristic zero. Let $E_1, \ldots, E_6$ be pairwise skew lines on $S_{\overline{\mathbf{F}}}$ such that their union is Galois-invariant. Suppose that there is an element $b \in \mathrm{Aut}(S)$ of order 3 such that it fixes E_1, E_2 and E_3 and cyclically permutes E_4, E_5 and E_6. Then 3-Sylow subgroups in the centralizers of the element b in both $\mathrm{Aut}(S_{\overline{\mathbf{F}}})$ and $W(\mathrm{E}_6)$ are unique and isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. By Corollary 4.7 we get that $S_{\overline{\mathbf{F}}}$ is a Fermat cubic surface. Therefore, we have $\mathrm{Aut}(S_{\overline{\mathbf{F}}}) \simeq (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathrm{S}_4$. By Lemma 4.8 the centralizer C of the element b in $\mathrm{Aut}(S_{\overline{\mathbf{F}}})$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. Obviously, the Sylow 3-subgroup Z in the group C is unique. In particular, it is normal in C and is preserved by the automorphism group of C.

Let us find the order of the centralizer $\mathcal{G}$ for the element b in $W(\mathrm{E}_6)$. By Example 4.4 the element b is of type A_2. By the classification of conjugacy classes of elements in $W(\mathrm{E}_6)$ (see [3, Table 9]) the number of elements in the conjugacy class of the element b is equal to 240. Therefore, the order of centralizer is equal to

$$\frac{51840}{240} = 216 = 2^3 \cdot 3^3.$$

Let us prove that $\mathcal{G}$ contains a unique Sylow 3-subgroup. Indeed, we have $\mathcal{G} \supset C$, because $\mathrm{Aut}(S_{\overline{\mathbf{F}}}) \subset W(\mathrm{E}_6)$. The subgroup C is normal in $\mathcal{G}$ because its index is 2. In particular, it is preserved by conjugations. Thus, the group Z is preserved by conjugation in $\mathcal{G}$. So Z is normal in $\mathcal{G}$ as well. As it is a Sylow subgroup in $\mathcal{G}$ we get that it is a unique Sylow 3-subgroup. Hence, a Sylow 3-subgroup in $\mathcal{G}$ and C is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$.

□

Let us prove the auxiliary proposition about endomorphisms of the Picard group of a smooth cubic surface which is needed for the lemmas below.

Proposition 4.10. *Let S be a smooth cubic surface over an algebraically closed field $\mathbf{F}$ of characteristic zero and set $\mathrm{Pic}(S)_{\mathbb{Q}} = \mathrm{Pic}(S) \otimes \mathbb{Q}$. Let α be an endomorphism of a $\mathbb{Q}$-vector space $\mathrm{Pic}(S)_{\mathbb{Q}}$ such that α fixes the canonical class K_S and maps pairwise skew lines $E_1, \ldots, E_6$ to other pairwise skew lines $\alpha(E_1), \ldots, \alpha(E_6)$. Then α is an automorphism of $\mathrm{Pic}(S)_{\mathbb{Q}}$ which restricts to an automorphism of $\mathrm{Pic}(S)$ and preserves the intersection form on $\mathrm{Pic}(S)$. In particular, it lies in $W(\mathrm{E}_6)$.*

Proof. The divisors $K_S, E_1, \ldots, E_6$ and $K_S, \alpha(E_1), \ldots, \alpha(E_6)$ are two bases of the vector space $\mathrm{Pic}(S)_{\mathbb{Q}}$, so α is an automorphism of $\mathrm{Pic}(S)_{\mathbb{Q}}$. Also by definition of the automorphism α it preserves the intersection form on $\mathrm{Pic}(S)_{\mathbb{Q}}$. The Picard group $\mathrm{Pic}(S)$ is a lattice generated by $E_1, \ldots, E_6$ and the divisor

$$L = \frac{1}{3}\left(-K_S + E_1 + \ldots + E_6\right),$$

which is a pull-back of a line via the morphism $\pi : S \to \mathbb{P}^2$ contracting $E_1, \ldots, E_6$. The automorphism α maps L to

$$\alpha(L) = \frac{1}{3}\left(-K_S + \alpha(E_1) + \ldots + \alpha(E_6)\right).$$

We can blow down the skew lines $\alpha(E_1), \ldots, \alpha(E_6)$ and obtain a map $\widetilde{\pi} : S \to \mathbb{P}^2$ such that $\alpha(L) = \widetilde{\pi}^* l$, where l is a line on $\mathbb{P}^2$. Therefore, the Picard group of S is generated by $\alpha(L), \alpha(E_1), \ldots, \alpha(E_6)$. Hence, the element $\alpha \in \mathrm{Aut}\left(\mathrm{Pic}(S)_{\mathbb{Q}}\right)$ restricts to an automorphism of $\mathrm{Pic}(S)$ and, thus, lies in $W(\mathrm{E}_6)$.

□

Now we are going to prove two lemmas which are the main ingredients of the proof of Proposition 1.5.

Lemma 4.11. *Let S be a smooth cubic surface over a field $\mathbf{F}$ of characteristic zero. Let $E_1, \ldots, E_6$ be pairwise skew lines on $S_{\overline{\mathbf{F}}}$ such that their union is Galois-invariant. Suppose that there is an element $b \in W(\mathrm{E}_6)$ such that it fixes E_1, E_2 and E_3, cyclically permutes E_4, E_5 and E_6, and commutes with the image of the Galois group* $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ *in $W(\mathrm{E}_6)$. Then there is an element $r \in W(\mathrm{E}_6)$ such that it commutes with the image of the Galois group in $W(\mathrm{E}_6)$, and the elements r and b generate the group $(\mathbb{Z}/3\mathbb{Z})^2$. Moreover, suppose that c is an element in $W(\mathrm{E}_6)$ such that it fixes E_4, E_5 and E_6 and cyclically permutes E_1, E_2 and E_3. Then r can be chosen in such a way that r, b and c generate the group $(\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. On $S_{\overline{\mathbf{F}}}$ we denote by Q_i for all $i = 1, \ldots, 6$ the pairwise skew lines such that

$$E_i \cdot Q_i = 0 \quad \text{and} \quad E_i \cdot Q_j = 1 \quad \text{for} \quad i \neq j.$$

For $i, j \in \{1, \ldots, 6\}$ and $i < j$ we denote by L_{ij} the remaining 15 lines on $S_{\overline{\mathbf{F}}}$ such that

$$L_{ij} \cdot E_k = 0 \quad \text{if} \quad k \notin \{i,j\} \quad \text{and} \quad L_{ij} \cdot E_k = 1 \quad \text{if} \quad k \in \{i,j\};$$
$$L_{ij} \cdot Q_k = 0 \quad \text{if} \quad k \notin \{i,j\} \quad \text{and} \quad L_{ij} \cdot Q_k = 1 \quad \text{if} \quad k \in \{i,j\};$$
$$L_{ij} \cdot L_{kl} = 0 \quad \text{if} \quad \{i,j\} \cap \{k,l\} \neq \varnothing \quad \text{and} \quad L_{ij} \cdot L_{kl} = 1 \quad \text{if} \quad \{i,j\} \cap \{k,l\} = \varnothing.$$

By Proposition 4.10 there is an element $r \in W(\mathrm{E}_6)$ which fixes the canonical class K_S and maps the skew lines E_1, E_2, E_3, E_4, E_5, E_6 to the other skew lines Q_1, Q_2, Q_3, L_{56}, L_{46}, L_{45}, respectively. Furthermore, one can see that r has order 3. Indeed, the element r^2 maps the skew lines E_1, E_2, E_3, E_4, E_5, E_6 to other skew lines L_{23}, L_{13}, L_{12}, Q_4, Q_5, Q_6, respectively, and $r^3(E_i) = E_i$ for all $i \in \{1, \ldots, 6\}$.

Let us prove that the element r commutes with the image Γ of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ in $W(\mathrm{E}_6)$. It is enough to check this on the curves $E_1, \ldots, E_6$. As Γ commutes with b, it permutes E_1, E_2 and E_3 with each other and permutes E_4, E_5 and E_6 with each other. Meanwhile, the (-1)-curve $r(E_i)$ for $1 \leqslant i \leqslant 3$ is defined uniquely by the property that it intersects exactly two lines from the set E_1, E_2

and E_3 and intersects the lines E_4, E_5 and E_6. Therefore, as Γ is an isometry on $W(\mathrm{E}_6)$, the element $g \in \Gamma$ maps $r(E_i)$ to $r(E_j)$ such that $g(E_i) = E_j$. The same holds for $r(E_i)$ with $4 \leqslant i \leqslant 6$. In this case $r(E_i)$ is defined uniquely by the property that it intersects exactly two lines from the set E_4, E_5 and E_6 and does not intersect the lines E_1, E_2 and E_3. Thus, every element $g \in \Gamma$ maps $r(E_i)$ to $r(E_j)$ such that $g(E_i) = E_j$. So the element r commutes with the Galois group.

For the elements b and r one has $br(K_S) = K_S = rb(K_S)$ and

$$\begin{aligned} br(E_1) = Q_1 = rb(E_1); \quad br(E_4) = L_{46} = rb(E_4);\\ br(E_2) = Q_2 = rb(E_2); \quad br(E_5) = L_{45} = rb(E_5);\\ br(E_3) = Q_3 = rb(E_3); \quad br(E_6) = L_{56} = rb(E_6). \end{aligned} \tag{4.2}$$

As $K_S, E_1, \ldots, E_6$ is a basis of $\mathrm{Pic}(S_{\overline{\mathbf{F}}}) \otimes \mathbb{Q}$, the elements b and r commute. Moreover, we obtain $r \neq b, b^2$, because b and b^2 fix the set $E_1, \ldots, E_6$, while r does not. Thus, the elements b and r generate the group $(\mathbb{Z}/3\mathbb{Z})^2$.

Let $c \in W(\mathrm{E}_6)$ be an element such that it fixes E_4, E_5 and E_6 and cyclically permutes E_1, E_2 and E_3. The elements b and c commute because they correspond to the elements (456) and (123) in $\mathrm{S}_6 \subset W(\mathrm{E}_6)$, respectively. The elements c and r also commute, which can be seen from a computation similar to (4.2). Hence, the elements r, b and c generate a group $(\mathbb{Z}/3\mathbb{Z})^n$, where $n \leqslant 3$. The elements b and c generate a subgroup $H \simeq (\mathbb{Z}/3\mathbb{Z})^2$ which preserves the set $E_1, \ldots, E_6$, while the element r does not preserves this set. Thus, they generate a group $(\mathbb{Z}/3\mathbb{Z})^3 \subset W(\mathrm{E}_6)$. □

Lemma 4.12. *Let S be a smooth cubic surface over a field $\mathbf{F}$ of characteristic zero. Let $E_1, \ldots, E_6$ be pairwise skew lines on $S_{\overline{\mathbf{F}}}$ such that their union be Galois-invariant. Suppose that there is an element $b \in \mathrm{Aut}(S)$ of order 3 such that it fixes E_1, E_2 and E_3 and cyclically permutes E_4, E_5 and E_6. Then there is a biregular action of $(\mathbb{Z}/3\mathbb{Z})^2$ on S. Moreover, assume that c is an element in $\mathrm{Aut}(S)$ such that it fixes E_4, E_5 and E_6 and cyclically permutes E_1, E_2 and E_3. Then r can be chosen in such a way that b, c and r generate a group $(\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. Let us consider the element $r \in W(\mathrm{E}_6)$ from Lemma 4.11, so that b and r generate the group $(\mathbb{Z}/3\mathbb{Z})^2$ in $W(\mathrm{E}_6)$, and r commutes with the image of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ in $W(\mathrm{E}_6)$. Moreover, if there is an element $c \in \mathrm{Aut}(S)$ such that it fixes E_4, E_5 and E_6 and cyclically permutes E_1, E_2 and E_3, then the element r can be chosen so that b, c and r are generate the group $(\mathbb{Z}/3\mathbb{Z})^3$ in $W(\mathrm{E}_6)$.

It remains to check that r lies in $\mathrm{Aut}(S)$. The element r commutes with the element b, and therefore, by Lemma 4.9 it lies in the centralizer of b in $\mathrm{Aut}(S_{\overline{\mathbf{F}}}) \subset W(\mathrm{E}_6)$. Moreover, since the element r commutes with the image of the Galois group in $W(\mathrm{E}_6)$, by Lemma 4.3 it is contained in $\mathrm{Aut}(S)$. □

5. 3-groups in the birational automorphism groups

In this section we prove Proposition 1.5. First of all, let us study the 3-groups in the automorphism groups of del Pezzo surfaces and conic bundles. Recall that the degree of a del Pezzo surface S is defined as $d = (K_S)^2$. The following fact is well-known.

Lemma 5.1 (see, for instance, [9, Chapter V, §2])**.** *Let $\mathbf{F}$ be an algebraically closed field of characteristic zero. Then $\mathrm{PGL}_2(\mathbf{F})$ does not contain $(\mathbb{Z}/3\mathbb{Z})^2$.*

Let us study 3-subgroups in the automorphisms groups of del Pezzo surfaces of degree d. Recall that $1 \leqslant d \leqslant 9$.

Lemma 5.2. *Let $\mathbf{F}$ be an algebraically closed field of characteristic zero. Let S be a del Pezzo surface of degree $d \neq 3$ over $\mathbf{F}$. Then its automorphism group does not contain $(\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. Suppose that $d = 9$, i.e. $S \simeq \mathbb{P}^2$. Then $\mathrm{Aut}(S) \simeq \mathrm{PGL}_3(\mathbf{F})$, and by Theorem 1.4(iii) we obtain $(\mathbb{Z}/3\mathbb{Z})^3 \not\subset \mathrm{Aut}(S)$.

Suppose that $d = 8$ and $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We get that

$$\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq ((\mathrm{PGL}_2(\mathbf{F}) \times \mathrm{PGL}_2(\mathbf{F})) \rtimes \mathbb{Z}/2\mathbb{Z}$$

which does not contain $(\mathbb{Z}/3\mathbb{Z})^3$ because $\mathrm{PGL}_2(\mathbf{F})$ does not contain $(\mathbb{Z}/3\mathbb{Z})^2$ by Lemma 5.1.

Suppose that either $d = 8$ and $S \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$, or $d = 7$. Then we get an $\mathrm{Aut}(S)$-equivariant map $S \to \mathbb{P}^2$. So we have $\mathrm{Aut}(S) \subset \mathrm{PGL}_3(\mathbf{F})$. Thus, the group $(\mathbb{Z}/3\mathbb{Z})^3$ is not contained in $\mathrm{Aut}(S)$.

Suppose that $d = 6$. Then by [5, Theorem 8.4.2] we get

$$\mathrm{Aut}(S) \simeq (\mathbf{F}^*)^2 \rtimes \mathrm{D}_6.$$

The subgroup $\mathrm{D}_6 \simeq \mathrm{S}_3 \times \mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Aut}(S)$ is the dihedral group of order 12 acting on the graph of (-1)-curves, which is a hexagon, and $(\mathbf{F}^*)^2 \subset \mathrm{Aut}(S)$ acts trivially on this graph. If there is a subgroup $(\mathbb{Z}/3\mathbb{Z})^3 \subset \mathrm{Aut}(S)$ then the projection of this subgroup to D_6 gives us either $\mathbb{Z}/3\mathbb{Z}$, or the trivial group. Hence there is a $(\mathbb{Z}/3\mathbb{Z})^3$-invariant triple of pairwise skew (-1)-curves. Thus, we can blow them down $(\mathbb{Z}/3\mathbb{Z})^3$-equivariantly and get $\mathbb{P}^2$ with the action of the group $(\mathbb{Z}/3\mathbb{Z})^3$ which is impossible.

Suppose that $d = 5$ or $d = 4$. By [5, Corollary 8.2.40] the automorphism group $\mathrm{Aut}(S)$ is contained in the Weyl group $W(\mathrm{A}_4)$ or $W(\mathrm{D}_5)$, respectively. The order of $W(\mathrm{A}_4)$ is equal to $120 = 2^3 \cdot 3 \cdot 5$ and the order of $W(\mathrm{D}_5)$ is equal to $1920 = 2^7 \cdot 3 \cdot 5$. Therefore, the automorphism group of S does not contain $(\mathbb{Z}/3\mathbb{Z})^3$.

Suppose that $d = 2$. Then the anticanonical linear system gives us a double cover

$$\phi_{|-K_S|} : S \to \mathbb{P}^2.$$

Therefore, $\phi_{|-K_S|}$ induces the following exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P}^2).$$

Hence, $\mathrm{Aut}(S)$ does not contain $(\mathbb{Z}/3\mathbb{Z})^3$.

Suppose that $d = 1$. Then the base locus of the linear system $|-K_S|$ is a point p. Hence, the point p is fixed by $\mathrm{Aut}(S)$. Assume that $(\mathbb{Z}/3\mathbb{Z})^3 \subset \mathrm{Aut}(S)$. Therefore, the group $(\mathbb{Z}/3\mathbb{Z})^3$ acts faithfully on the Zariski tangent space $T_p(S)$ to S at p. Thus, we obtain

$$(\mathbb{Z}/3\mathbb{Z})^3 \subset \mathrm{GL}\left(T_p(S)\right) \simeq \mathrm{GL}_2(\mathbf{F}),$$

which is impossible.

□

Lemma 5.3. *Let $\mathbf{F}$ be a field of characteristic zero. Let $\phi : S \to \mathbb{P}^1$ be a conic bundle over $\mathbf{F}$. Let Γ be a subgroup in $\mathrm{Aut}(S)$ which consists of the elements mapping every fiber of ϕ to a fiber of ϕ. Then Γ does not contain $(\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. We have the following exact sequence

$$1 \to \mathrm{Aut}_\phi(S) \to \Gamma \to \mathrm{Aut}(\mathbb{P}^1),$$

where $\mathrm{Aut}_\phi(S)$ is the group of all automorphisms of S which map every fiber of ϕ to itself. The group $\mathrm{Aut}_\phi(S)$ is contained in the automorphism group of the scheme-theoretic generic fiber C of ϕ, which is isomorphic to $\mathbb{P}^1_{\mathbf{F}(t)}$, where t is a transcendental variable. One has

$$\mathrm{Aut}(\mathbb{P}^1_{\mathbf{F}(t)}) \simeq \mathrm{PGL}_2(\mathbf{F}(t)).$$

So by Theorem 5.1 we get that neither $\mathrm{Aut}_\phi(S)$, nor $\mathrm{Aut}(\mathbb{P}^1)$ contains $(\mathbb{Z}/3\mathbb{Z})^2$. Therefore, $\mathrm{Aut}(S)$ does not contain $(\mathbb{Z}/3\mathbb{Z})^3$.

□

Corollary 5.4. *Let* $\mathbf{F}$ *be a field of characteristic zero which does not contain non-trivial cube roots of unity. Let* V *be a Severi–Brauer surface over* $\mathbf{F}$*. Then*

$$\mathrm{Bir}(V) \not\supset (\mathbb{Z}/3\mathbb{Z})^3.$$

Proof. Assume that $G \simeq (\mathbb{Z}/3\mathbb{Z})^3$ is contained in $\mathrm{Bir}(V)$. Then G acts biregularly either on a del Pezzo surface, or on a conic bundle $\phi : S \to B$ over a geometrically rational curve B such that ϕ is equivariant with respect to G. By Lemma 5.3 the latter case is impossible. By Lemma 5.2 the group G does not act biregularly on a del Pezzo surface of degree $d \neq 3$. Finally, by Lemma 4.1 the group G does not act biregularly on a del Pezzo surface of degree $d = 3$.

□

Now we are ready to prove Proposition 1.5.

Proof of Proposition 1.5. Let us prove (i). We can blow up V and get a smooth cubic surface S such that the group $\mathbb{Z}/3\mathbb{Z}$ acts on S fixing 3 exceptional curves E_1, E_2 and E_3 and cyclically permuting the other 3 exceptional curve E_4, E_5 and E_6. This follows from Lemmas 3.4 and 3.8 if V is a non-trivial Severi–Brauer surface, and from Remark 3.9 for $V \simeq \mathbb{P}^2$. By Lemma 4.12 there is a biregular action of the group $(\mathbb{Z}/3\mathbb{Z})^2$ on S. This gives a birational action of $(\mathbb{Z}/3\mathbb{Z})^2$ on V.

Now let us prove (ii). Assume that $\mathbf{F}$ contains a non-trivial cube root of unity. Then we can blow up V and get a smooth cubic surface S such that the group $(\mathbb{Z}/3\mathbb{Z})^2$ generated by the elements b and c acts on S as follows: the element b fixes 3 exceptional curves E_1, E_2 and E_3 and cyclically permutes E_4, E_5 and E_6, while the element c fixes exceptional curves E_4, E_5 and E_6 and cyclically permutes E_1, E_2 and E_3. This follows from Lemmas 3.4 and 3.10 if V is a non-trivial Severi–Brauer surface, and from Remark 3.11 for $V \simeq \mathbb{P}^2$. By Lemma 4.12 there is a biregular action of the group $(\mathbb{Z}/3\mathbb{Z})^3$ on S. This gives a birational action of $(\mathbb{Z}/3\mathbb{Z})^3$ on V. If $\mathbf{F}$ does not contain non-trivial cube roots of unity then by Corollary 5.4 the group $(\mathbb{Z}/3\mathbb{Z})^3$ does not act birationally on V.

Assertion (iii) follows from Theorem 1.4(ii).

Recall from Lemma 3.4 that the group $\mathrm{Aut}(V)$ contains $\mathbb{Z}/3\mathbb{Z}$. This gives (iv). If $\mathbf{F}$ does not contain a non-trivial cube root of unity then by Lemma 3.6 the group $\mathrm{Aut}(V)$ does not contain $(\mathbb{Z}/3\mathbb{Z})^2$. Otherwise, if $\mathbf{F}$ contains a non-trivial cube root of unity then by Lemma 3.4 we get that $\mathrm{Aut}(V)$ contains $(\mathbb{Z}/3\mathbb{Z})^2$. This gives (v). Finally, from Theorem 1.4(iii) we obtain (vi).

□

6. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

Proof of Theorem 1.3. Let $G \subset \mathrm{Bir}(V)$ be a finite subgroup of birational automorphisms of V. By Theorem 1.2 one has $G \subset (\mathbb{Z}/3\mathbb{Z})^3$. By Proposition 1.5(ii) the group G is contained in $(\mathbb{Z}/3\mathbb{Z})^2$ and by Proposition 1.5(i) the group $(\mathbb{Z}/3\mathbb{Z})^2$ is contained in $\mathrm{Bir}(V)$.

Let $G \subset \mathrm{Aut}(V)$ be a finite subgroup of the automorphisms group of V. By Theorem 1.2 one has $G \subset (\mathbb{Z}/3\mathbb{Z})^3$. By Proposition 1.5(v) the group G is contained in $\mathbb{Z}/3\mathbb{Z}$ and by Proposition 1.5(iv) the group $\mathbb{Z}/3\mathbb{Z}$ is contained in $\mathrm{Bir}(V)$.

□

Remark 6.1. One of the facts used in the proof of [15, Theorem 1.3] is the following assertion: the automorphism group of a Severi–Brauer surface V over $\mathbb{Q}$ does not contain elements of prime order $p \geqslant 5$. This fact follows, for instance, from [14, Theorem 6]. Also, there are two alternative proofs of this fact provided in [15, Lemma 7.1]. Unfortunately, the first of these proofs contains a gap: it treats identification $V_{\overline{\mathbb{Q}}} \simeq \mathbb{P}^2_{\overline{\mathbb{Q}}}$ as a Galois-invariant isomorphism, while it is obviously not Galois-invariant if V is non-trivial. The same kind of gap is present in the proof of [15, Lemma 5.2] (cf. [15, Lemma 5.3], where a similar trouble is luckily avoided). One can find a corrected proof of [15, Lemma 5.2] in Example 3.3.

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