Mixed volume of infinite-dimensional convex compact sets

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Abstract

In 1985, B. S. Tsirelson discovered a deep connection between Gaussian processes and important geometric characteristics of a convex compact sets in a separable Hilbert space, called intrinsic volumes. In this work, we generalize Tsirelson's theorem to the mixed volumes of the infinite-dimensional convex compact sets, first introducing this notion and studying its properties. Using the obtained result we compute the mixed volume of the closed convex hulls of the two orthogonal Wiener spirals. Moreover, we prove an analogue of the Tsirelson's theorem for Grassmann angles of infinite-dimensional convex cones.

Contents

1 Introduction

1.1 Intrinsic volumes

Let $K \subset \mathbb{R}^d$ be a non-empty convex compact set and $\dim K$ be the dimension of K (that is, the dimension of the smallest affine subspace containing K). One of the most important geometric characteristics of K is its *intrinsic volumes* $V_0(K), \dots, V_d(K)$, which are defined as the coefficients in the Steiner formula (see, e.g., [19, relation 14.5])

$$\mathrm{Vol}_d(K + \lambda \mathbb{B}^d) = \sum_{k=0}^{d} \kappa_{d-k} V_k(K) \lambda^{d-k}, \quad \lambda \geqslant 0, \tag{1}$$

where $\mathrm{Vol}_d(\cdot)$ denotes the volume (d-dimensional Lebesgue measure), $\mathbb{B}^k$ is the k-dimensional unit ball and $\kappa_k := \mathrm{Vol}_k(\mathbb{B}^k) = \pi^{k/2}/\Gamma(\frac{k}{2}+1)$ is the volume of $\mathbb{B}^k$. In other words, the volume of the neighborhood is represented by a polynomial whose coefficients depend on the set K.

The intrinsic volumes play an important role in convex geometry (see, e.g., [18]). In particular, it can be shown [19, Section 6.2] that $V_d(\cdot)$ is the d-dimensional volume, $V_{d-1}(\cdot)$ is half the surface area for d-dimensional convex compact sets, $V_1(\cdot)$ is the mean width, up to a constant factor, and $V_0(\cdot) \equiv 1$.

Moreover, the normalization in (1) is chosen so that the intrinsic volumes of the set do not depend on the dimension of the ambient space. This means that if we embed K into $\mathbb{R}^N$ with $N \geqslant d$, the intrinsic volumes will be the same. This observation allowed Sudakov [20] and Chevet [3] to generalize the concept of intrinsic volume to the case of infinite-dimensional K as follows.

Let H be an infinite-dimensional separable Hilbert space. Then for an arbitrary non-empty convex set $K \subset H$ we define $V_k(K), k = 0, 1, \dots,$ by the formula

$$V_k(K) = \sup_{K' \subset K} V_k(K') \in [0, \infty], \tag{2}$$

where the supremum is taken over all finite-dimensional convex compact subsets K' of K.

In the next subsection, we formulate the well-known results demonstrating a deep connection between the intrinsic volumes of some convex compact sets and Gaussian processes.

1.2 Sudakov's and Tsirelson's theorems

A mean-zero Gaussian random process $(\xi(h))_{h\in H}$ over a separable Hilbert space H is called *isonormal* if its covariance function has the form

$$\operatorname{cov}(\xi(h), \xi(g)) = \langle h, g\rangle, \tag{3}$$

where $\langle\ ,\ \rangle$ denotes the inner product on H.

In his paper [20, Proposition 14] Sudakov discovered a connection between the first intrinsic volume and the expectation of the supremum of an isonormal process.

Theorem 1 (Sudakov)**.** *For a convex compact set $K \subset H$*

$$V_1(K) = \sqrt{2\pi}\,\mathbf{E} \sup_{h\in K} \xi(h). \tag{4}$$

Later Tsirelson [23, Theorem 6] generalized Theorem 1. Let $\{\xi_i(h)\colon h \in H\}$, $1 \leqslant i \leqslant k$, denote k independent copies of the isonormal process. Then the *k-dimensional spectrum* of a convex compact set $K \subset H$ is defined as the following random set:

$$\operatorname{Spec}_k K := \{(\xi_1(h), \dots, \xi_k(h))\colon h \in K\} \subset \mathbb{R}^k.$$

To formulate Tsirelson's result, we first need the notion of a GB-set. A subset K of a separable Hilbert space H is said to be a *GB-set* if there exists a modification of the isonormal process with index set K, which has almost surely bounded realizations (see Section 3 for detailed definitions and properties). It is known [20, Theorem 1] that the property of a convex K to be a GB-set is equivalent to $V_1(K) < \infty$. In the latter case $V_k(K) < \infty$ for all $k = 0, 1, \dots$ (see, e.g., [3]).

Theorem 2 (Tsirelson)**.** *For all convex compact GB-sets $K \subset H$ and all $k = 0, 1, \dots,$*

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbf{E} \operatorname{Vol}_k(\operatorname{Spec}_k K). \tag{5}$$

Remark 1. In the case when $K \subset \mathbb{R}^d$, $k \leqslant d$, the last formula can be rewritten as

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbf{E} \operatorname{Vol}_k(AK),$$

where A is a standard Gaussian matrix of size $k \times d$ (whose entries are independent standard normal random variables), $\operatorname{Spec}_k K = AK := \{Ax : x \in K\} \subset \mathbb{R}^k$.

Notice that in $\mathbb{R}^d$ for all non-empty convex compact sets $V_1(K) < \infty$; therefore, in the finite-dimensional case, it is not necessary to assume the GB-property, since it is certainly satisfied.

Remark 2. Strictly speaking, in Theorem 1 we need the existence of a *separable* modification of the process ξ, and in Theorem 2 we need the existence of a so-called *natural* modification of ξ_i (see Subsections 3.3, 3.4 for details). We will show that under the assumptions of Theorems 1, 2 the corresponding modifications do exist (see Statement 1).

The main goal of this work is to obtain a generalization of Theorem 2 to the *mixed volumes* defined in the next subsection.

1.3 Mixed volumes

In 1911, Minkowski proved [16] that for arbitrary non-empty convex compact sets $K_1, \dots, K_s \subset \mathbb{R}^d$ the functional $\mathrm{Vol}_d(\lambda_1 K_1 + \ldots + \lambda_s K_s)$ for $\lambda_1, \dots, \lambda_s \geqslant 0$ is a homogeneous polynomial of degree d with non-negative coefficients:

$$\mathrm{Vol}_d(\lambda_1 K_1 + \ldots + \lambda_s K_s) = \sum_{i_1=1}^{s} \cdots \sum_{i_d=1}^{s} \lambda_{i_1} \ldots \lambda_{i_d} \tilde{V}_d(K_{i_1}, \dots, K_{i_d}). \tag{6}$$

The coefficients $\tilde{V}_d(K_{i_1}, \dots, K_{i_d})$ are uniquely determined if we assume that they are symmetric with respect to the permutations of $K_{i_1}, \dots, K_{i_d}$. The coefficient $\tilde{V}_d(K_{i_1}, \dots, K_{i_d})$ is called the *mixed volume* of $K_{i_1}, \dots, K_{i_d}$.

It is easy to understand (see, e.g., [18, Section 5.1]) that intrinsic volumes are special cases of the mixed volumes, namely,

$$V_k(K) = \frac{\binom{d}{k}}{\kappa_{d-k}} \tilde{V}_d(\underbrace{K, \dots, K}_{k \text{ times}}, \mathbb{B}^d, \dots, \mathbb{B}^d). \tag{7}$$

The theory of mixed volumes finds wide application in convex and algebraic geometry [2, Chapter 4], inequalities [18] and the theory of Gaussian distributions [9]. Some of the properties of the mixed volumes are given in Subsection 3.5.

A well-known inequality related to the mixed volumes is the Aleksandrov–Fenchel inequality (see, e.g., [18, Section 7.3]), independently proven by Aleksandrov and Fenchel.

Theorem 3 (Aleksandrov, Fenchel)**.** *Let $K_1, K_2, \dots, K_d$ be non-empty convex compact sets in $\mathbb{R}^d$. Then*

$$\tilde{V}_d^2(K_1, K_2, \dots, K_d) \geqslant \tilde{V}_d(K_1, K_1, K_3, \dots, K_d) \tilde{V}_d(K_2, K_2, K_3, \dots, K_d). \tag{8}$$

Remark 3. If K_1 and K_2 are homothetic, i.e., $K_2 = \lambda K_1 + x$ for some $\lambda > 0$ and $x \in \mathbb{R}^d$ or one of K_1, K_2 is a one-pointed set, it is not hard to prove that equality in (8) holds (see Properties 3, 4 of the mixed volumes in Subsection 3.5). However, the complete classification of the equality cases is still unknown.

The Aleksandrov–Fenchel inequality generalizes many well-known inequalities, such as the Brunn-Minkowski inequality, the Minkowski's inequalities, the isoperimetric inequalities. We refer to the book of Schneider [18] for more detailed information.

In the next subsection, we consider conic counterparts of the intrinsic volumes, so-called *Grassmann angles*.

1.4 Convex cones and Grassmann angles

We begin this subsection with the well-known notion of a solid angle. Let $C \subseteq \mathbb{R}^d$ be a *convex cone* (equivalently, C is a closed convex set such that $\lambda x \in C$ for all $x \in C$ and $\lambda \geqslant 0$), U be a random vector uniformly distributed over the unit sphere in the linear span of C in $\mathbb{R}^d$ ($U \in \operatorname{lin} C \cap \mathbb{S}^{d-1}$). The *solid angle* of a convex cone $C \subseteq \mathbb{R}^d$ is defined as

$$\alpha(C) := \mathbf{P}[U \in C].$$

As above, we will denote by $\dim C$ the dimension of the cone C (i.e., the dimension of the smallest linear subspace containing C).

Note that if $C \subset \mathbb{R}^d$ has non-empty interior (i.e., $\dim C = d$) and

$$C \neq \mathbb{R}^d,$$

then its solid angle $\alpha(C)$ can be calculated as the one half of the probability to be non-trivially intersected with the random line W_1 passing through the origin randomly chosen with respect to the Haar measure:

$$\alpha(C) = \frac{1}{2}\mathbf{P}[C \cap W_1 \neq \{0\}].$$

This observation encouraged Grünbaum [8] to introduce the generalization of the solid angle in the following way. Let W_j be a j-dimensional linear subspace randomly chosen with respect to the Haar measure on the Grassmannian of all linear j-dimensional subspaces in $\mathbb{R}^d$. Define the j-th *Grassmann angle* of the convex cone $C \subseteq \mathbb{R}^d$ as the probability of the non-trivial intersection of C with the random $(d-j)$-plane W_{d-j}:

$$\gamma_j(C) := \mathbf{P}[C \cap W_{d-j} \neq \{0\}], \quad j = 0, 1, \ldots, d.$$

If C is a d-dimensional cone, its solid angle can be expressed in terms of the Grassmann angles:

$$\alpha(C) := \frac{1}{2}\gamma_{d-1}(C) + \frac{1}{2}\mathbb{1}[C = \mathbb{R}^d] = \frac{1}{2}\mathbf{P}[C \cap W_1 \neq \{0\}] + \frac{1}{2}\mathbb{1}[C = \mathbb{R}^d].$$

In [8], Grünbaum showed that, like the solid angle and the intrinsic volumes, the Grassmann angles do not depend on the dimension of the ambient space: if we embed C in $\mathbb{R}^N$ with $N \geqslant d$, the Grassmann angles remain the same. In particular, for a linear j-plane $L_j \subseteq \mathbb{R}^d$, $j = 1, \dots, d$, we have

$$\gamma_0(L_j) = \ldots = \gamma_{j-1}(L_j) = 1, \quad \gamma_j(L_j) = \ldots = \gamma_d(L_j) = 0.$$

For $C = \{0\}$, we have $\gamma_0(C) = \gamma_1(C) = \ldots = 0$.

It turns out that the Grassmann angles are conic counterparts of the intrinsic volumes. Generally speaking, there are at least two conic functionals that can be considered as conic analogs of intrinsic volumes with similar properties: Grassmann angles and the so-called *conic intrinsic volumes*. It is well known that these two concepts are related to each other by linear expression (see, e.g., [19, p. 261, Crofton formula]). At the same time, they have an important difference for us: Grassmann angles are monotone under set inclusion, while conic intrinsic volumes in general are not. In this paper, we are going to work with Grassmann angles and will not delve into the concept of the conic intrinsic volumes. We refer the reader to the works [15, 19] for more detailed information about conic intrinsic volumes.

Another goal of this paper is to find a conic analogue of Tsirelson's theorem for Grassmann angles.

Next, we formulate the main results of this work.

2 Main results

2.1 Generalization of Tsirelson's theorem

In order to generalize Theorem 2 to the case of mixed volumes, we first construct an isonormal Gaussian random process according to Tsirelson [22].

Consider a linear topological space with mean-zero Gaussian measure (E, γ) and its kernel $E_0 \subset E$ (see Section 3 for definitions and properties). Since the kernel is a Hilbert space, the inner product on E_0 is defined, which we will denote by $\langle\ ,\ \rangle_{E_0}$ (it is uniquely determined by the measure γ). For each $\theta \in E_0$ the linear functional $\langle \theta, \eta \rangle_{E_0}$ is continuous in $\eta \in E_0$ and has a unique (up to equality almost everywhere) extension to a linear functional, measurable in $x \in E$ (see [12, Section 9, Lemma 2] or [1, Corollary 2.10.8]), which we denote by $\langle \theta, x \rangle$. Moreover,

$$\int_E \langle \theta, x \rangle^2 \gamma(dx) = \|\theta\|^2 = \langle \theta, \theta \rangle. \tag{9}$$

Thus, for any set $K \subset E_0$, the isonormal Gaussian random process $\langle \theta, x \rangle$ is defined, where $\theta \in K$, x ranges over the space E equipped with the Gaussian measure γ.

To state and prove the main result, we will consider the kernel E_0 as H and the process $\langle \theta, \cdot \rangle$ as isonormal process.

Let us rewrite Theorem 2 according to the notation of this subsection for further convenience. Formula (5) for $k = 0, 1, \ldots$, becomes

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \int_E \int_E \ldots \int_E \mathrm{Vol}_k(\mathrm{Spec}(x_1, \ldots, x_k | K)) \gamma(dx_1) \ldots \gamma(dx_k).$$

Here $K \subset E_0$ is a convex compact GB-set and

$$\mathrm{Spec}(x_1, \ldots, x_k | K) := \{(\langle \theta, x_1 \rangle, \ldots, \langle \theta, x_k \rangle) : \theta \in K\} \subset \mathbb{R}^k$$

is the joint spectrum for $x_1, \ldots, x_k \in E$ on K.

Now we introduce the concept of mixed volume for infinite-dimensional convex sets similar to (2).

Let $K_1, \ldots, K_k \subset H$ be non-empty convex subsets of an infinite-dimensional separable Hilbert space H. Then the *mixed volume* $V(K_1, \ldots, K_k)$ of the sets $K_1, \ldots, K_k$ is defined as

$$V(K_1, \ldots, K_k) = \sup_{K_i' \subset K_i} \frac{\binom{d}{k}}{\kappa_{d-k}} \tilde{V}_d(K_1', \ldots, K_k', \underbrace{\mathbb{B}^d, \ldots, \mathbb{B}^d}_{d-k \text{ times}}), \tag{10}$$

where the supremum is taken over all $d \geqslant k$ and all finite-dimensional convex compact subsets $K_i' \subset K_i, \dim K_i' \leqslant d, i = 1, \ldots, k$.

Remark 4. The normalization in (10) is chosen so that for K_i with $\dim K_i \leqslant d$, the right-hand side of (10) does not depend on d, as well as in (7). Therefore, $V(K_1, \ldots, K_k)$ is well defined.

The proof of Remark 4 can be found in Subsection 4.

Remark 5. The author of this paper found only one source [2, Chapter 4, Section 25], which mentioned a possible generalization of the concept of mixed volume to the case of infinite-dimensional sets. However, the normalization given there is incorrect and for finite-dimensional sets the definition depends on the dimension d of the ambient space.

Now we are ready to formulate the main result of this subsection.

Theorem 4. *Fix $k \in \mathbb{N}$. For convex compact GB-sets $K_i \subset E_0$, $i = 1, \ldots, k$, we have*

$$V(K_1, \ldots, K_k) = \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbf{E}\, \tilde{V}_k(\mathrm{Spec}_k K_1, \ldots, \mathrm{Spec}_k K_k)$$

$$= \frac{(2\pi)^{k/2}}{k!\kappa_k} \int_E \ldots \int_E \tilde{V}_k(\mathrm{Spec}(x_1, \ldots, x_k|K_1), \ldots, \mathrm{Spec}(x_1, \ldots, x_k|K_k))\gamma(dx_1) \ldots \gamma(dx_k).$$

Remark 6. GB-property of the sets K_i ensures almost everywhere boundedness (and convexity) of the sets $\mathrm{Spec}(x_1, \ldots, x_k|K_i)$.

Remark 7. As mentioned in the introduction (see Remark 1), in the case when $K_i \subset \mathbb{R}^d$, $i = 1, \ldots, k$, we have $V_1(K_i) < \infty$, hence, the GB-property is automatically satisfied and need not be assumed.

2.2 Example: mixed volume of the closed convex hulls of two orthogonal Wiener spirals

Let us first recall the definition of the *Wiener spiral* introduced by Kolmogorov [11]. The set of functions

$$\{\mathbb{1}_{[0,t]}(\cdot) \colon t \in [0, 1]\} \subset L^2[0, 1]$$

is called the *Wiener spiral*. This set is an important object in functional analysis [11].

Recall that the *convex hull* of a set F is the smallest convex set containing F.

Gao and Vitale [6] calculated the intrinsic volumes of the closed convex hull K of the Wiener spiral (it is known that K is a compact subset of $L^2[0, 1]$):

$$V_k(K) = \frac{\kappa_k}{k!} = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2} + 1\right) k!}. \tag{11}$$

This was probably the first result that gave an explicit formula for the intrinsic volumes of a non-trivial infinite-dimensional convex compact set. In particular, (11) implies that $V_1(K) < \infty$, so K is a GB-set (see Theorem 10 in Subsection 3.4).

Later similar results were obtained for other infinite-dimensional convex compact sets [10].

Unsurprisingly, the Wiener spiral is closely related to the Wiener process. Let $\{W(t) \colon t \geqslant 0\}$ be the standard one-dimensional Brownian motion. Consider the standard two-dimensional Brownian motion

$$\{X^{(2)}(t) = (W_1(t), W_2(t)) \colon t \geqslant 0\},$$

where $W_1(t), W_2(t)$ are independent copies of $W(t)$. Looking at the definition of the Wiener spiral and Property (3) of the isonormal process, it is easy to see that $\mathrm{Spec}_2 K$ has the same distribution as the closed convex hull of the two-dimensional Brownian motion $\{X^{(2)}(t)\colon t \in [0,1]\}$.

Consider two Wiener spirals S_1 and S_2 in $L^2[0,2]$:

$$S_1 = \{\mathbb{1}_{[0,t]}(\cdot)\colon t \in [0,1]\} \subset L^2[0,2] \quad \text{and} \quad S_2 = \{\mathbb{1}_{[1+t,2]}(\cdot)\colon t \in [0,1]\} \subset L^2[0,2].$$

We denote the corresponding closed convex hulls by K_1 and K_2. In our next theorem, we compute $V(K_1, K_2)$.

Theorem 5. *For the closed convex hulls K_1 and K_2 of two orthogonal Wiener spirals we have*

$$V(K_1, K_2) = 2.$$

The proof of Theorem 5 uses Theorem 4 (see Section 6).

2.3 The Aleksandrov–Fenchel inequality

In this subsection, we generalize the Aleksandrov–Fenchel inequality to the case of the mixed volumes of *infinite-dimensional* convex compact sets.

Theorem 6. *Let $K_1, K_2, \dots K_k$ be non-empty convex compact sets in H. Then*

$$V^2(K_1, K_2, \dots, K_k) \geqslant V(K_1, K_1, K_3, \dots, K_k) V(K_2, K_2, K_3, \dots, K_k). \tag{12}$$

We provide the proof of Theorem 6 in Section 7.

Remark 8. As in the finite-dimensional case (see Remark 3), if K_1 and K_2 are homothetic, then equality in (12) holds.

Remark 9. Let $K_i,\ i = 1, \dots, k$, be convex compact GB-sets. Taking the expectation of both sides of (12) for random sets $\mathrm{Spec}_k K_i,\ i = 1, \dots, k$, we obtain

$$\begin{aligned} \mathbf{E}\ \tilde{V}_k^2(&\mathrm{Spec}_k K_1, \mathrm{Spec}_k K_2, \dots, \mathrm{Spec}_k K_k) \\ &\geqslant \mathbf{E}\ (\tilde{V}_k(\mathrm{Spec}_k K_1, \mathrm{Spec}_k K_1, \mathrm{Spec}_k K_3, \dots, \mathrm{Spec}_k K_k) \\ &\qquad \times\ \tilde{V}_k(\mathrm{Spec}_k K_2, \mathrm{Spec}_k K_2, \mathrm{Spec}_k K_3, \dots, \mathrm{Spec}_k K_k)). \end{aligned} \tag{13}$$

On the other hand, Theorem 4, together with the Aleksandrov–Fenchel inequality, gives us another estimate for the mean mixed volumes. More precisely, if we apply

Theorem 4 to the left- and right-hand sides of (12), we get

$$\begin{aligned}(\mathbf{E}\ \tilde{V}_k(&\mathrm{Spec}_k K_1, \mathrm{Spec}_k K_2, \dots, \mathrm{Spec}_k K_k))^2 \\ &\geqslant \mathbf{E}\ \tilde{V}_k(\mathrm{Spec}_k K_1, \mathrm{Spec}_k K_1, \mathrm{Spec}_k K_3, \dots, \mathrm{Spec}_k K_k) \\ &\quad \times \mathbf{E}\ \tilde{V}_k(\mathrm{Spec}_k K_2, \mathrm{Spec}_k K_2, \mathrm{Spec}_k K_3, \dots, \mathrm{Spec}_k K_k).\end{aligned} \tag{14}$$

The Cauchy–Schwarz inequality implies that the left-hand side of (14) is not greater than the left-hand side of (13):

$$\mathbf{E}\ \tilde{V}_k^2(\mathrm{Spec}_k K_1, \mathrm{Spec}_k K_2, \dots, \mathrm{Spec}_k K_k) \geqslant (\mathbf{E}\ \tilde{V}_k(\mathrm{Spec}_k K_1, \mathrm{Spec}_k K_2, \dots, \mathrm{Spec}_k K_k))^2.$$

Nevertheless, the right-hand sides of (13) and (14) are generally incomparable.

Thus, by Theorem 4 we obtain one more estimate for the mean mixed volumes different from (13).

2.4 Analogue of Tsirelson's theorem for Grassmann angles

As before, let H be a separable Hilbert space. A non-empty set $C \subset H$ is called a *convex cone* or simply a *cone,* if C is a closed convex set such that $\lambda C \subseteq C$ for all $\lambda \geqslant 0$.

We shall introduce the concept of Grassmann angles for infinite-dimensional convex cones similar to (2), (10). For $j = 0, 1, \dots,$ by definition, put

$$\gamma_j(C) := \sup_{C' \subset C} \gamma_j(C') = \sup_{C' \subset C} \mathbf{P}[C' \cap W_{d-j} \neq \{0\}], \tag{15}$$

where the supremum is taken over all $d \geqslant j$ and all finite-dimensional convex cones $C' \subset C,\ \dim C' \leqslant d$.

Remark 10. Since Grassmann angles for finite-dimensional cones are monotone under set inclusion and do not depend on the dimension of the ambient space, $\gamma_j(C)$ is well defined.

Remark 11. It suffices to consider the supremum in (15) only over polyhedral finite-dimensional cones (a cone is called *polyhedral* if it is the intersection of a finite number of closed half-spaces which have 0 on their boundary).

The proof of Remark 11 is postponed to Section 8.

By the *k-dimensional spectrum* of the cone C, similarly to the case of a compact set, we mean the following random set:

$$\mathrm{Spec}_k C := \{(\xi_1(h), \dots, \xi_k(h))\colon h \in C\} \subset \mathbb{R}^k,$$

where ξ_i, $1 \leqslant i \leqslant k$, are independent copies of the isonormal process.

We say that the cone C is a *GB_σ-cone* if C is represented as a countable union of GB-sets.

As above, to state and prove the following theorem, we use the kernel E_0 as a Hilbert space H and the process $\langle \theta, \cdot \rangle$ as an isonormal process.

Theorem 7. *Let $C \subset E_0$ be a GB_σ-cone, and let k be some fixed positive integer, $m := \min(\dim C, k) < \infty$. Then for all $j = 0, \ldots, m-1$ we have*

$$\mathbf{E}[\gamma_j(\mathrm{Spec}_k C)] = \gamma_j(C).$$

Remark 12. For finite-dimensional cones, Theorem 7 was proved by Götze, Kabluchko, and Zaporozhets in [7, Theorem 3.5].

The proof of Theorem 7 can be found in Section 8.

Let us conclude this introductory part by describing how the rest of the paper is organized. The next section contains the necessary concepts, definitions and facts from the theory of random processes and convex geometry, which supplement the information presented in the first two sections. In particular, in Subsection 3.4 we formulate and prove Statement 1 auxiliary to Theorem 4 about one of the interpretations of the GB-property of convex compact sets. Sections 4 and 5 contain proofs of Remark 4 and Theorem 4. The proofs of Theorems 5, 6 are presented in Sections 6, 7. Finally, in Section 8, one can find the proof of Theorem 7 about Grassmann angles.

3 Preliminaries

3.1 Gaussian vectors in linear spaces

Following [13, Chapters 1, 4] and [12, Sections 8, 9], we present the definition and basic properties of a Gaussian vector in a linear space.

Let E be a linear topological space, E^* be the space of continuous linear functionals on E. A *random vector* X taking values in E is defined as a measurable mapping from some probability space $(\Omega, \mathcal{B}, \mathbf{P})$ to E. At the same time, it is assumed that the corresponding σ-algebra of the space E is large enough: all continuous linear functionals on E are measurable with respect to it.

A random vector $X \in E$ is called *Gaussian* if $f(X)$ is a normal random variable for all $f \in E^*$.

An element $a \in E$ is said to be the *expectation* of X if $\mathbf{E} f(X) = f(a)$ for all $f \in E^*$. A linear operator $C : E^* \to E$ is called the *covariance operator* of X if for any $f_1, f_2 \in E^*$

$$\mathrm{cov}(f_1(X), f_2(X)) = f_1(Cf_2),$$

where $\mathrm{cov}(\cdot, \cdot)$ denotes the covariance between two random variables. The covariance operator C has the following properties:

1. $f(Cg) = g(Cf) \ \ \forall f, g \in E^*$ (symmetry);
2. $f(Cf) \geqslant 0 \ \ \forall f \in E^*$ (non-negative definiteness).

The definition of a Gaussian vector makes sense when the space of continuous linear functionals on E is rich enough. To this end, we will tacitly assume everywhere below that E is a locally convex linear topological space, and the distribution of X is a Radon measure. In this case, any Gaussian vector X has an expectation and a covariance operator [12, Section 8] that uniquely determine the distribution of X. Therefore, similarly to the finite-dimensional case, we denote by $N(a, C)$ the distribution of the Gaussian vector X with expectation a and covariance operator C. Under the above assumptions, the distributions of all Gaussian vectors have form $N(a, C)$.

In the following, we will be interested in mean-zero case when $a = 0$.

3.2 Measurable linear functionals and kernel

Consider a Gaussian vector X taking values in the linear space E. We will assume that $a = 0$. Denote by $\gamma = N(0, C)$ the distribution of X in E.

By definition of a Gaussian vector, the random variable $f(X)$ has normal distribution, so

$$\mathbf{E}f(X)^2 = \int_E |f(x)|^2 \gamma(dx) < \infty.$$

Thus, a canonical embedding I^* of the space E^* into the Hilbert space $L_2(E, \gamma)$ is well defined. The closure of the image $I^*(E^*)$ in $L_2(E, \gamma)$ is said to be the space of *measurable linear functionals* and denoted by E^*_γ.

The inner product in E^*_γ is inherited from $L_2(E, \gamma)$:

$$\langle g_1, g_2 \rangle_{E^*_\gamma} = \int_E g_1(x) g_2(x) \gamma(dx) = \mathbf{E} g_1(X) g_2(X);$$
$$\|g\|^2_{E^*_\gamma} = \mathbf{E} g(X)^2.$$

In what follows, we treat the operator I^* as the embedding $I^* : E^* \to E_\gamma^*$. We define the dual operator $I : E_\gamma^* \to E$ by the following relation:

$$f(Ig) = \langle I^* f, g\rangle_{E_\gamma^*} = \mathbf{E} f(X) g(X), \quad \forall f \in E^*, g \in E_\gamma^*.$$

It is known [13, Section 4.1] that under the assumptions stated in Subsection 3.1, the dual operator I exists, it is linear and injective, and, moreover, the covariance operator C can be factorized as

$$C = II^*.$$

Finally, the *kernel* is defined as the set $E_0 := I(E_\gamma^*) \subset E$ equipped with inner product

$$\langle \theta_1, \theta_2\rangle_{E_0} := \langle I^{-1}\theta_1, I^{-1}\theta_2\rangle_{E_\gamma^*}, \quad \theta_1, \theta_2 \in E_0,$$

and hence with norm

$$\|\theta\|^2 := \|\theta\|_{E_0}^2 = \langle \theta, \theta\rangle_{E_0}, \quad \theta \in E_0.$$

The norm is well defined since the operator I is injective.

Thus, the kernel is uniquely determined by the measure γ and provides the key information about it (see [13]).

We collect some properties of the kernel (see [13, Section 4.1]).

1. $C(E^*) \subset E_0 \subset E$. If the kernel is finite-dimensional, then in the nondegenerate case these three spaces coincide, otherwise they are all distinct.

2. If E_0 is infinite-dimensional, then $\gamma(E_0) = 0$.

3. The space E_0 is separable.

4. The balls $\{\theta \in E_0 : \|\theta\| \leqslant R\}$, $R > 0$, are compact sets in E.

3.3 Separable and natural modifications of process

Let $(\Omega, \mathcal{B}, \mathbf{P})$ be a probability space and T be a metric space. A random process $\xi(t, \omega)$, $t \in T$, $\omega \in \Omega$, is said to be *separable* if there exists at most countable set $S \subset T$ (a *separant* of the process) such that for any open set $U \subset T$ with probability 1 the following equalities hold:

$$\sup_{t \in U} \xi(t) = \sup_{t \in U \cap S} \xi(t), \quad \inf_{t \in U} \xi(t) = \inf_{t \in U \cap S} \xi(t).$$

Recall that a random process $(\eta(t))_{t\in T}$ is called a *modification* of the process $(\xi(t))_{t\in T}$ if these processes are defined on the same probability space and equality $\mathbf{P}(\xi(t) = \eta(t)) = 1$ holds for any $t \in T$. A *realization* of the process $(\xi(t))_{t\in T}$ is the function $t \mapsto \xi(t,\omega)$ for some fixed $\omega \in \Omega$. The following theorem (see, e.g., [1, Proposition 2.6.5]) provides a sufficient condition for the existence of a separable modification of a mean-zero Gaussian process.

Theorem 8. *Consider a mean-zero Gaussian random process $(\xi(t))_{t\in T}$ on a set T. Suppose that T with semimetric $d(t,s) = \sqrt{\mathbf{E}\,|\xi(t) - \xi(s)|^2}$ is separable. Then, there exists a separable modification $(\eta(t))_{t\in T}$ of the process $(\xi(t))_{t\in T}$.*

To prove Theorem 4, the existence of a separable modification of the process $\langle \theta, x\rangle$ is not sufficient. We need the so-called *natural* modification introduced by Tsirelson [21].

A modification $(\eta(t))_{t\in T}$ of the process $(\xi(t))_{t\in T}$ is called *natural* if there exists a metric ρ_1 on T such that (T,ρ_1) is a separable metric space and the process $(\eta(t))_{t\in T}$ has almost surely continuous realizations on (T,ρ_1).

Below we formulate a theorem (see [12, Section 7] or [1, Theorem 2.6.3, Proposition 2.6.4]) that allows us to check the existence of a natural modification in terms of the *oscillations* α.

Theorem 9. *Let (T,ρ) be a separable metric space and let $(\xi(t))_{t\in T}$ be a mean-zero separable Gaussian random process with the continuous covariance function*

$$(t,s) \mapsto \mathbf{E}\;\xi(t)\xi(s).$$

Then there exists a non-random function $\alpha : T \to [0,\infty]$ such that with probability 1 for all $t \in T$

$$\alpha(t) = \lim_{\varepsilon\to 0} \sup\{|\xi(u,\omega) - \xi(v,\omega)|, u, v \in B(t,\varepsilon)\},$$

where $B(t,\varepsilon)$ denotes the open ball of radius ε centered at t.

Moreover, if $\alpha(t) < \infty$ for all $t \in T$, then the process $(\xi(t))_{t\in T}$ has a natural modification.

3.4 GB-sets: equivalent definitions and properties

As mentioned in the introduction, a GB-set is a subset K of a separable Hilbert space H such that there exists a modification of the isonormal process with index set K, which has almost surely bounded realizations.

In this subsection, we formulate the results of Sudakov [20, Theorem 1] and Tsirelson [21, Theorem 3] on equivalent definitions of GB-set, and also prove an auxiliary Statement 1 about the connection between the GB-property of a set and the oscillation of a corresponding process.

Theorem 10 (Sudakov)**.** *Let $K \subset H$ be a convex subset of a Hilbert space H. The following statements are equivalent:*

1. *the set K is a GB-set;*
2. *$V_1(K) < \infty$.*

Theorem 11 (Tsirelson)**.** *Let $K \subset H$ be a subset of a Hilbert space H. The following statements are equivalent:*

1. *the isonormal process on the set K has a natural modification;*
2. *the set K is a GB_σ-set (that is a countable union of GB-sets).*

Statement 1. *Let $K \subset E_0$ be a convex compact GB-set. Then Theorem 9 holds for the process $\langle \theta, \cdot \rangle$ on $T = K$ with standard metric generated by the inner product. Thus, the process $\langle \theta, \cdot \rangle$ on K has a natural modification.*

Moreover, the converse also holds. Consider a convex compact set $K \subset E_0$ with standard metric satisfying all conditions of Theorem 9. Then K is a GB-set.

Remark 13. Note that Statement 1 can be deduced from Theorem 11. Nevertheless, we will provide an alternative proof of Statement 1 for the reader's convenience.

Remark 14. The existence of a separable modification of the process $\langle \theta, \cdot \rangle$ does not require the GB-property of the compact set K, as can be seen from the proof below.

Proof of Statement 1. First, let us check that the process $\langle \theta, x \rangle,\ \theta \in K$, has a separable modification.

We will use Theorem 8. Note that for $\theta_1, \theta_2 \in K$ we have

$$\begin{aligned} d(\theta_1, \theta_2) &:= \sqrt{\mathbf{E}|\langle \theta_1, x \rangle - \langle \theta_2, x \rangle|^2} = \sqrt{\mathbf{E}\left(\langle \theta_1, x \rangle^2 + \langle \theta_2, x \rangle^2 - 2\langle \theta_1, x \rangle \langle \theta_2, x \rangle\right)} \\ &= \sqrt{\langle \theta_1, \theta_1 \rangle + \langle \theta_2, \theta_2 \rangle - 2\langle \theta_1, \theta_2 \rangle} = \sqrt{\langle \theta_1 - \theta_2, \theta_1 - \theta_2 \rangle} = \|\theta_1 - \theta_2\|. \end{aligned}$$

Here in the third equality, we used the fact that the process $\langle \theta, x \rangle,\ \theta \in K$, is isonormal. Therefore, the semimetric d coincides with the standard metric on K, and hence K is separable with this semimetric. Then, by Theorem 8, we can assume without loss of generality that the process $\langle \theta, \cdot \rangle$ is separable.

Now we need to check the continuity of the covariance function.

Let $(\theta_1, \theta_2) \in K \times K$ and $\|\theta_1^n - \theta_1\| \to 0$, $\|\theta_2^n - \theta_2\| \to 0$ as $n \to \infty$. Let us show that

$$\mathbf{E}\langle\theta_1^n, x\rangle\langle\theta_2^n, x\rangle = \int_E \langle\theta_1^n, x\rangle\langle\theta_2^n, x\rangle\gamma(dx) \to \int_E \langle\theta_1, x\rangle\langle\theta_2, x\rangle\gamma(dx) = \mathbf{E}\langle\theta_1, x\rangle\langle\theta_2, x\rangle.$$

Indeed, by the isonormality of the process and the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\left|\int_E \langle\theta_1^n, x\rangle\langle\theta_2^n, x\rangle\gamma(dx) - \int_E \langle\theta_1, x\rangle\langle\theta_2, x\rangle\gamma(dx)\right| \\
&= |\langle\theta_1^n, \theta_2^n\rangle - \langle\theta_1, \theta_2\rangle| \\
&\leqslant |\langle\theta_1^n - \theta_1, \theta_2\rangle| + |\langle\theta_1^n - \theta_1, \theta_2^n - \theta_2\rangle| + |\langle\theta_2^n - \theta_2, \theta_1\rangle| \\
&\leqslant \|\theta_1^n - \theta_1\|\|\theta_2\| + \|\theta_1^n - \theta_1\|\|\theta_2^n - \theta_2\| + \|\theta_2^n - \theta_2\|\|\theta_1\|.
\end{aligned}$$

Letting $n \to \infty$ in the last inequality leads to the required relation.

Finally, let us verify that the oscillation α introduced in Theorem 9 is finite in the case when K is a convex compact GB-set.

We have $V_1(K) < \infty$ by Theorem 10. We will need formula (4) for $V_1(K)$:

$$V_1(K) = \sqrt{2\pi}\int_E (\sup_{\theta\in K}\langle\theta, x\rangle)\gamma(dx).$$

Assume that there exist $\theta \in K, E_1 \subset E, \gamma(E_1) > 0$ such that for $x \in E_1$

$$\alpha(\theta) = \lim_{\varepsilon\to 0}\sup\{|\langle\theta_1, x\rangle - \langle\theta_2, x\rangle|, \theta_1, \theta_2 \in B(\theta, \varepsilon)\} = \infty.$$

Since for $x \in E_1$ we have

$$\infty = \alpha(\theta) \leqslant 2\sup_{\theta\in K}|\langle\theta, x\rangle|, \quad \sup_{\theta\in K} -\langle\theta, x\rangle = \sup_{\theta\in K}\langle\theta, -x\rangle,$$

and the distribution γ is symmetric, we get a contradiction with finiteness of $V_1(K)$. This means that $\alpha(\theta) < \infty$.

Conversely, suppose that $\alpha(\theta) < \infty$ for all $\theta \in K$. Let us prove that in this case

$$\gamma(x \in E : \sup_{\theta\in K}|\langle\theta, x\rangle| < \infty) = 1.$$

We fix $x \in E_1$, where $E_1 \subset E$ is a set of full measure on which $\alpha(\theta) < \infty$ for all $\theta \in K$.

For each $\theta \in K$ we choose $\tilde{\varepsilon}(\theta), M(\theta) < \infty$ such that

$$\sup\{|\langle \theta_1, x\rangle - \langle \theta_2, x\rangle|, \theta_1, \theta_2 \in B(\theta, \tilde{\varepsilon}(\theta))\} < M(\theta). \tag{16}$$

Consider the covering of K by balls $\{B(\theta, \tilde{\varepsilon}(\theta))\}_{\theta \in K}$. Since K is compact, we can choose a finite subcovering of K of the form $\{B(\theta^i, \tilde{\varepsilon}(\theta^i))\}_{i=1}^N = \{B_i\}_{i=1}^N$.

Then by the linearity of the process $\langle \theta, \cdot\rangle$ and by relation (16), we have

$$\begin{aligned}\sup_{\theta \in K} |\langle \theta, x\rangle| &= \max_{1 \leqslant i \leqslant N} \sup_{\theta \in B_i} |\langle \theta - \theta^i, x\rangle + \langle \theta^i, x\rangle| \\ &\leqslant \max_{1 \leqslant i \leqslant N} |\langle \theta^i, x\rangle| + \max_{1 \leqslant i \leqslant N} \sup_{\theta \in B_i} |\langle \theta - \theta^i, x\rangle| \\ &\leqslant \max_{1 \leqslant i \leqslant N} |\langle \theta^i, x\rangle| + \max_{1 \leqslant i \leqslant N} M(\theta^i) < \infty.\end{aligned}$$

Since the last inequality holds for all $x \in E_1$ by our assumption, and $\gamma(E_1) = 1$, we have $\gamma(x \in E : \sup_{\theta \in K} |\langle \theta, x\rangle| < \infty) = 1$.

Thus, K is a GB-set. Then by Theorem 10, we obtain that $V_1(K) < \infty$.

The statement is proved. $\square$

3.5 Properties of mixed volumes

We collect the basic properties of the mixed volumes defined in the introduction, some of which we will need in the proofs of Theorems 4, 6. For a more detailed introduction to mixed volume theory, we refer to [2, Chapter 4] and [18, Chapter 5].

For any non-empty convex compact sets $K, K_1, \dots, K_d \subset \mathbb{R}^d$ we have:

1. $\tilde{V}_d(K, \dots, K) = \mathrm{Vol}_d(K)$.

2. Independence of order:

   $$\tilde{V}_d(K_1, \dots, K_d) = \tilde{V}_d(K_{\sigma_1}, \dots, K_{\sigma_d}),$$

   where σ is an arbitrary permutation of numbers $1, \dots, d$.

3. Non-negative multilinearity:

   $$\begin{aligned}&\tilde{V}_d(\lambda K_1 + \lambda' K_1', K_2 \dots, K_d) \\ &\qquad = \lambda \tilde{V}_d(K_1, K_2 \dots, K_d) + \lambda' \tilde{V}_d(K_1', K_2 \dots, K_d) \text{ for } \lambda, \lambda' \geqslant 0.\end{aligned}$$

4. Invariance with respect to a parallel translation:
$$\tilde{V}_d(K_1+a_1,\dots,K_d+a_d)=\tilde{V}_d(K_1,\dots,K_d)$$
for any $a_1,\dots,a_d\in\mathbb{R}^d$.

5. Invariance with respect to a unimodular affine transformation O:
$$\tilde{V}_d(OK_1,\dots,OK_d)=\tilde{V}_d(K_1,\dots,K_d).$$

6. Monotonicity with respect to each argument: let $L_i,\ i=1,\dots,d$, be convex compact sets such that $K_i\subset L_i$. Then
$$\tilde{V}_d(K_1,\dots,K_d)\leqslant\tilde{V}_d(L_1,\dots,L_d).$$
This property implies the non-negativity of the mixed volumes.

7. Additivity: if $A,B,A\cup B\subset\mathbb{R}^d$ are non-empty convex compact sets, then
$$\tilde{V}_d(\underbrace{A\cup B,\dots,A\cup B}_{i\text{ times}},K_{i+1},\dots,K_d)=\tilde{V}_d(A,\dots,A,K_{i+1},\dots,K_d)$$
$$+\ \tilde{V}_d(B,\dots,B,K_{i+1},\dots,K_d)-\tilde{V}_d(A\cap B,\dots,A\cap B,K_{i+1},\dots,K_d).$$

8. The mixed volume $\tilde{V}_d$ is continuous function on $(\mathcal{K}^d)^d$, where by $\mathcal{K}^d$ we denote the collection of all non-empty convex compact sets in $\mathbb{R}^d$ equipped with the Hausdorff metric d_H
($d_H(K_1,K_2):=\inf\{\varepsilon\geqslant 0\ :\ K_1\subset K_2+\varepsilon\mathbb{B}^d$ and $K_2\subset K_1+\varepsilon\mathbb{B}^d\}$).

4 Proof of Remark 4

Let $K_i\subset\mathbb{R}^d$. It is sufficient to prove that
$$\frac{\binom{d}{k}}{\kappa_{d-k}}\tilde{V}_d(K_1,\dots,K_k,\mathbb{B}^d,\dots,\mathbb{B}^d)=\frac{\binom{d+1}{k}}{\kappa_{d+1-k}}\tilde{V}_{d+1}(K_1,\dots,K_k,\mathbb{B}^{d+1},\dots,\mathbb{B}^{d+1}).$$

Indeed, according to Minkowski's formula (6),
$$\begin{aligned}\mathrm{Vol}_{d+1}&(\lambda_1K_1+\ldots+\lambda_kK_k+\lambda\mathbb{B}^{d+1})\\&=\int_{-\lambda}^{\lambda}\mathrm{Vol}_d(\lambda_1K_1+\ldots+\lambda_kK_k+\sqrt{\lambda^2-z^2}\mathbb{B}^d)dz\\&=\int_{-\lambda}^{\lambda}\sum_{i_1=1}^{k+1}\cdots\sum_{i_d=1}^{k+1}\lambda_{i_1}\ldots\lambda_{i_d}\tilde{V}_d(K_{i_1},\dots,K_{i_d})dz,\end{aligned}$$

where $\lambda_{k+1} = \sqrt{\lambda^2 - z^2}$, $K_{k+1} = \mathbb{B}^d$. To explain the first equality notice that the intersection of the set $\lambda_1 K_1 + \ldots + \lambda_k K_k + \lambda \mathbb{B}^{d+1}$ with the horizontal hyperplane $\mathbb{R}^d + (0, \ldots, 0, z)$ gives us the set $\lambda_1 K_1 + \ldots + \lambda_k K_k + \sqrt{\lambda^2 - z^2} \mathbb{B}^d$ and equality follows from Fubini's theorem.

Now let us look at the coefficient of the monomial $\lambda_1 \cdots \lambda_k$ on the left- and right-hand sides of the last equality.

By Minkowski's formula (6) applied to the left-hand side, we obtain the coefficient

$$k!\ \tilde{V}_{d+1}(K_1, \ldots, K_k, \mathbb{B}^{d+1}, \ldots, \mathbb{B}^{d+1}) \binom{d+1}{k} \lambda^{d+1-k}.$$

On the right-hand side, the coefficient of $\lambda_1 \cdots \lambda_k$ is

$$k!\ \tilde{V}_d(K_1, \ldots, K_k, \mathbb{B}^d, \ldots, \mathbb{B}^d) \binom{d}{k} \int_{-\lambda}^{\lambda} \sqrt{(\lambda^2 - z^2)}^{d-k} dz.$$

Notice that

$$\int_{-\lambda}^{\lambda} \sqrt{(\lambda^2 - z^2)}^{d-k} dz = \lambda^{d+1-k} \int_{-1}^{1} \sqrt{(1 - z^2)}^{d-k} dz = \lambda^{d+1-k} \frac{\sqrt{\pi}\Gamma(\frac{d-k}{2} + 1)}{\Gamma(\frac{d-k+3}{2})}.$$

Comparing the coefficients, we obtain

$$\begin{aligned} &k!\ \tilde{V}_{d+1}(K_1, \ldots, K_k, \mathbb{B}^{d+1}, \ldots, \mathbb{B}^{d+1}) \binom{d+1}{k} \lambda^{d+1-k} \\ &= k!\ \tilde{V}_d(K_1, \ldots, K_k, \mathbb{B}^d, \ldots, \mathbb{B}^d) \binom{d}{k} \lambda^{d+1-k} \int_{-1}^{1} \sqrt{(1 - z^2)}^{d-k} dz. \end{aligned}$$

Taking into account the value of $\kappa_k := \pi^{k/2}/\Gamma(\frac{k}{2} + 1)$ and the last equality, we have

$$\begin{aligned} &\binom{d+1}{k} \tilde{V}_{d+1}(K_1, \ldots, K_k, \mathbb{B}^{d+1}, \ldots, \mathbb{B}^{d+1}) \kappa_{d-k} \\ &= \binom{d}{k} \tilde{V}_d(K_1, \ldots, K_k, \mathbb{B}^d, \ldots, \mathbb{B}^d) \kappa_{d+1-k}, \end{aligned}$$

which completes the proof.

5 Proof of Theorem 4

We divide the proof of the theorem into two cases.

Case 1. $\dim K_i < \infty$ for all $i = 1, \ldots, k$. In this case, taking into account Remarks 1, 4, the statement of Theorem 4 can be rewritten in the following form.

Statement 2. *If $K_1, \ldots, K_k \subset \mathbb{R}^d$, $k = 1, \ldots, d$, then*

$$\tilde{V}_d(K_1, \ldots, K_k, \mathbb{B}^d, \ldots, \mathbb{B}^d) = c_{k,d} \mathbf{E}\, \tilde{V}_k(AK_1, \ldots, AK_k),$$

where $c_{k,d} = \frac{\kappa_{d-k}(2\pi)^{k/2}}{k!\binom{d}{k}\kappa_k}$, $AK_i := \{Ax : x \in K_i\} \subset \mathbb{R}^k$ and A is the standard Gaussian matrix of size $k \times d$.

Proof of Statement 2. Let us look at $\mathrm{Vol}_d\left(\sum_{i=1}^k \alpha_i K_i + (d-1)\lambda \mathbb{B}^d\right)$.

By Minkowski's theorem (6),

$$\mathrm{Vol}_d\left(\sum_{i=1}^{k} \alpha_i K_i + (d-1)\lambda \mathbb{B}^d\right) = \sum_{i_1=1}^{d+k-1} \cdots \sum_{i_d=1}^{d+k-1} \lambda_{i_1} \ldots \lambda_{i_d} \tilde{V}_d(K_{i_1}, \ldots, K_{i_d}), \tag{17}$$

where $\lambda_1 = \alpha_1, \ldots, \lambda_k = \alpha_k$, $\lambda_{k+1} = \ldots = \lambda_{d+k-1} = \lambda$, $K_{k+1} = \ldots = K_{d+k-1} = \mathbb{B}^d$. Therefore, in the sum on the right-hand side, the coefficient of λ^{d-k} is a polynomial in $\alpha_1, \ldots, \alpha_k$, and the coefficient of $\lambda^{d-k}\alpha_1 \cdots \alpha_k$ is equal to

$$k!\binom{d}{k} \tilde{V}_d(K_1, \ldots, K_k, \mathbb{B}^d, \ldots, \mathbb{B}^d).$$

On the other hand, considering $K := \sum_{i=1}^k \alpha_i K_i$, by Minkowski's theorem (6), we have

$$\mathrm{Vol}_d(K + (d-1)\lambda \mathbb{B}^d) = \sum_{i_1=1}^{d} \cdots \sum_{i_d=1}^{d} \lambda_{i_1} \ldots \lambda_{i_d} \tilde{V}_d(L_{i_1}, \ldots, L_{i_d}), \tag{18}$$

where $\lambda_1 = 1$, $\lambda_2 = \ldots = \lambda_d = \lambda$, $L_1 = K$, $L_2 = \ldots = L_d = \mathbb{B}^d$. In this case, since the mixed volumes are invariant with respect to permutations of the arguments, the coefficient of λ^{d-k} will be equal to

$$\binom{d}{k} \tilde{V}_d(\underbrace{K, \ldots, K}_{k \text{ times}}, \mathbb{B}^d, \ldots, \mathbb{B}^d).$$

Further,

$$\begin{aligned}
\tilde{V}_d(\underbrace{K, \ldots, K}_{k \text{ times}}, \mathbb{B}^d, \ldots, \mathbb{B}^d) &\overset{(7)}{=} \frac{\kappa_{d-k}}{\binom{d}{k}} V_k(K) = \frac{\kappa_{d-k}}{\binom{d}{k}} V_k\left(\sum_{i=1}^k \alpha_i K_i\right) \\
&\overset{(5)}{=} \frac{\kappa_{d-k}}{\binom{d}{k}} \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbf{E}\,\mathrm{Vol}_k\left(A\left(\sum_{i=1}^k \alpha_i K_i\right)\right) \\
&= \frac{\kappa_{d-k}}{\binom{d}{k}} \frac{(2\pi)^{k/2}}{k!\kappa_k} \mathbf{E}\,\mathrm{Vol}_k\left(\sum_{i=1}^k \alpha_i AK_i\right).
\end{aligned}$$

Applying again Minkowski's theorem (6), we conclude that $\mathbf{E}\,\mathrm{Vol}_k\left(\sum_{i=1}^k \alpha_i AK_i\right)$ is a homogeneous polynomial of degree k in $\alpha_1,\ldots,\alpha_k$ with coefficient of $\alpha_1\cdots\alpha_k$ equal to

$$k!\,\mathbf{E}\,\tilde{V}_k(AK_1,\ldots,AK_k).$$

Thus, on the right-hand side of (18) the coefficient of $\lambda^{d-k}\alpha_1\cdots\alpha_k$ equals

$$\kappa_{d-k}\frac{(2\pi)^{k/2}}{\kappa_k}\mathbf{E}\,\tilde{V}_k(AK_1,\ldots,AK_k).$$

So, the left-hand sides of relations (17) and (18) are the same. Hence, the coefficients of $\lambda^{d-k}\alpha_1\cdots\alpha_k$ are the same on the right-hand sides:

$$k!\binom{d}{k}\tilde{V}_d(K_1,\ldots,K_k,\mathbb{B}^d,\ldots,\mathbb{B}^d)=\kappa_{d-k}\frac{(2\pi)^{k/2}}{\kappa_k}\mathbf{E}\,\tilde{V}_k(AK_1,\ldots,AK_k).$$

This completes the proof of Statement 2.

□

Case 2. $\dim K_i=\infty$ for at least one index $i=1,\ldots,k$.

According to Statement 1, the GB-property of the compact sets K_i implies that the processes $\langle\theta,x\rangle,\theta\in K_i$, have a natural modification.

Next, we reduce Case 2 to the finite-dimensional one (Statement 2).

Let $K_{1,1}\subset K_{1,2}\subset\ldots\subset K_1,\ K_{2,1}\subset K_{2,2}\subset\ldots\subset K_2,\ldots,\ K_{k,1}\subset K_{k,2}\subset\ldots\subset K_k$. Here $K_{i,j}$ are finite-dimensional convex compact sets, and $\cup_{j=1}^\infty K_{i,j}$ is dense in K_i. Then by definition (10) and by Properties 6, 8 of mixed volumes, we get

$$V(K_1,\ldots,K_k)=\lim_{j\to\infty}V(K_{1,j},\ldots,K_{k,j}).$$

Now we formulate the lemma proved by Tsirelson in [23].

Lemma 1. *Let $\xi(t,\omega)$ be a natural modification of some random process, $\omega\in\Omega$, $t\in T$, and let $S\subset T$ be dense in T in the following sense: for any $t\in T$ there are $s_n\in S$, $n=1,2,\ldots$, such that $\xi(s_n,\omega)\to\xi(t,\omega)$ as $n\to\infty$ for almost all ω (the corresponding set of probability 1, generally speaking, depends on t). Then there exists a set $\Omega_0\subset\Omega$ of probability 1 with the following property: for any $t\in T$ there are $s'_n\in S$, $n=1,2,\ldots$, such that $\xi(s'_n,\omega)\to\xi(t,\omega)$ as $n\to\infty$ for all $\omega\in\Omega_0$.*

Corollary 1. Let $K\subset E_0$ be a convex compact GB-set. If $K_0\subset K$ is dense in K, then for almost all $(x_1,\ldots,x_k)$ the set $\mathrm{Spec}(x_1,\ldots,x_k|K_0)$ is dense in the set $\mathrm{Spec}(x_1,\ldots,x_k|K)$.

The above corollary is stated in [23] without proof. For the reader's convenience, we prove it here.

Proof of Corollary 1. Since K_0 is dense in K in the usual sense, for any $\theta \in K$ there exist $s_n \in K_0$, $n = 1, 2, \ldots$, such that $\|s_n - \theta\| \to 0$ as $n \to \infty$. Moreover, by Property (9),

$$\int_E \langle s_n - \theta, x\rangle^2 \gamma(dx) = \|s_n - \theta\|^2 \to 0$$

as $n \to \infty$. Therefore, the sequence $\langle s_n - \theta, x\rangle^2$ converges to 0 in measure. Then there is a subsequence (we will also denote it by s_n) such that $\langle s_n - \theta, x\rangle^2$ converges to 0 almost everywhere (the corresponding set of probability 1 depends on θ).

Since $K \subset E_0$ is a convex compact GB-set, the Gaussian process $\langle \theta, x\rangle$ has a natural modification by Statement 1. Hence, by Lemma 1, for any $\theta \in K$ there exist $s'_n \in K_0$ such that

$$\langle s'_n, x\rangle \to \langle \theta, x\rangle \tag{19}$$

as $n \to \infty$ for almost all x, and the corresponding set of probability 1 is common for all $\theta \in K$.

Then we can conclude that for almost all $(x_1, \ldots, x_k)$ the set

$$\mathrm{Spec}(x_1, \ldots, x_k | K_0) = \{(\langle \theta, x_1\rangle, \ldots, \langle \theta, x_k\rangle) : \theta \in K_0\} \subset \mathbb{R}^k$$

is dense in the set

$$\mathrm{Spec}(x_1, \ldots, x_k | K) = \{(\langle \theta, x_1\rangle, \ldots, \langle \theta, x_k\rangle) : \theta \in K\} \subset \mathbb{R}^k,$$

since the argument above implies a coordinate-wise density (19), and the corresponding set of probability 1 in $\mathbb{R}^k$ will also be common for all $\theta \in K$.

□

Using Corollary 1, we get that almost surely $\cup_{j=1}^{\infty} \mathrm{Spec}(x_1, \ldots, x_k | K_{i,j})$ is dense in $\mathrm{Spec}(x_1, \ldots, x_k | K_i)$.

Then we use Statement 2 for the finite-dimensional $K_{i,j}$:

$$\begin{aligned}
&V(K_1, \ldots, K_k) = \lim_{j\to\infty} V(K_{1,j}, \ldots, K_{k,j}) \\
&= \lim_{j\to\infty} \frac{(2\pi)^{k/2}}{k!\kappa_k} \int_E \ldots \int_E \tilde{V}_k(\mathrm{Spec}(x_1, \ldots, x_k | K_{1,j}), \ldots, \mathrm{Spec}(x_1, \ldots, x_k | K_{k,j}))\gamma(dx_1)\ldots\gamma(dx_k) \\
&= \frac{(2\pi)^{k/2}}{k!\kappa_k} \int_E \ldots \int_E \lim_{j\to\infty} \tilde{V}_k(\mathrm{Spec}(x_1, \ldots, x_k | K_{1,j}), \ldots, \mathrm{Spec}(x_1, \ldots, x_k | K_{k,j}))\gamma(dx_1)\ldots\gamma(dx_k) \\
&= \frac{(2\pi)^{k/2}}{k!\kappa_k} \int_E \ldots \int_E \tilde{V}_k(\mathrm{Spec}(x_1, \ldots, x_k | K_1), \ldots, \mathrm{Spec}(x_1, \ldots, x_k | K_k))\gamma(dx_1)\ldots\gamma(dx_k).
\end{aligned}$$

Here in the third equality we have used Lebesgue's dominated convergence theorem. The last equality also follows from Properties 6, 8 of mixed volumes and Corollary 1.

The proof of Theorem 4 is complete.

6 Proof of Theorem 5

Since K_1 and K_2 are compact GB-sets, we can apply Theorem 4 with $k = 2$:

$$V(K_1, K_2) = \frac{2\pi}{2\kappa_2}\ \mathbf{E}\ \tilde{V}_2(\mathrm{Spec}_2 K_1, \mathrm{Spec}_2 K_2).$$

Since $\kappa_2 = \pi$, we get

$$\begin{aligned} V(K_1,K_2) &= \mathbf{E}\ \tilde{V}_2(\mathrm{Spec}_2 K_1, \mathrm{Spec}_2 K_2) \\ &= \mathbf{E}\ \tilde{V}_2\left(\mathrm{conv}\left(\{X_1^{(2)}(t)\colon t\in[0,1]\}\right), \mathrm{conv}\left(\{X_2^{(2)}(t)\colon t\in[0,1]\}\right)\right), \end{aligned} \tag{20}$$

where $X_1^{(2)}(t)$, $X_2^{(2)}(t)$ are independent standard two-dimensional Brownian motions, and $\mathrm{conv}(F)$ denotes a convex hull of the set F (i.e., the smallest convex set containing F).

Therefore, our problem is reduced to finding the mean mixed area $\mathbf{E}\ \tilde{V}_2$ of the convex hulls of independent two-dimensional Brownian motions on $[0, 1]$.

Further, for calculation, we will use an analogue of the technique given in [14]. The main tools of this technique are the support function and the associated Cauchy's formulae.

Let C be an arbitrary closed smooth convex curve in a plane. Let us represent the curve C as

$$C = \{(x(s), y(s)),\ s \in C\}.$$

Now we recall the notion of the support function of the curve C.

For $\varphi \in [0, 2\pi)$ the value of the *support function* $M(\varphi)$ of the curve C is defined by

$$M(\varphi) = \max_{s\in C}\{x(s)\cos\varphi + y(s)\sin\varphi\}.$$

The Cauchy's formulae (see, e.g., [14, pp. 48-49]) allow us to express the length L of the curve C and the area A of the figure bounded by the curve C in terms of the support function:

$$L = \int_0^{2\pi} M(\varphi) d\varphi, \tag{21}$$

$$A = \frac{1}{2} \int_0^{2\pi} \left((M(\varphi))^2 - (M'(\varphi))^2 \right) d\varphi. \tag{22}$$

In the case when the curve C is random (for example, the boundary of a convex hull of a two-dimensional Brownian motion is almost surely smooth [4]), $M(\varphi)$ and $M'(\varphi)$ are random variables.

Taking the expectation of both sides of relations (21), (22), we get

$$\mathbf{E}L = \int_0^{2\pi} \mathbf{E}M(\varphi) d\varphi, \tag{23}$$

$$\mathbf{E}A = \frac{1}{2} \int_0^{2\pi} \left(\mathbf{E}(M(\varphi))^2 - \mathbf{E}(M'(\varphi))^2 \right) d\varphi. \tag{24}$$

Note that the distribution of the two-dimensional Brownian motion is invariant under rotations. Hence, the distribution of the support function $M(\varphi)$ does not depend on φ. Therefore, it is sufficient to consider $\varphi = 0, M(\varphi) \stackrel{d}{=} M(0)$, where $\stackrel{d}{=}$ is equality in distribution. Relations (23), (24) in this case can be written in the form

$$\begin{aligned} \mathbf{E}L &= 2\pi \ \mathbf{E}M(0), \\ \mathbf{E}A &= \pi \left(\mathbf{E}(M(0))^2 - \mathbf{E}(M'(0))^2 \right). \end{aligned} \tag{25}$$

The following expression (see, e.g., [17, pp. 4-5]) is an analogue of the Cauchy's formulae for computation of the mixed area of two convex compact sets F_1, F_2 in a plane with smooth boundary:

$$\tilde{V}_2(F_1, F_2) = \frac{1}{2} \int_0^{2\pi} \left(M_1(\varphi) M_2(\varphi) - M_1'(\varphi) M_2'(\varphi) \right) d\varphi,$$

where M_1 and M_2 are the support functions of the curves representing the boundaries of F_1 and F_2, respectively.

Similarly to (24), for random F_1, F_2 we get

$$\mathbf{E} \ \tilde{V}_2(F_1, F_2) = \frac{1}{2} \int_0^{2\pi} \left(\mathbf{E}(M_1(\varphi) M_2(\varphi)) - \mathbf{E}(M_1'(\varphi) M_2'(\varphi)) \right) d\varphi. \tag{26}$$

Now consider $\mathrm{conv}\left(\{X_1^{(2)}(t)\colon t\in[0,1]\}\right)$ and $\mathrm{conv}\left(\{X_2^{(2)}(t)\colon t\in[0,1]\}\right)$ as F_1 and F_2, respectively. By formula (26) and the independence of $X_1^{(2)}(t)$ and $X_2^{(2)}(t)$, we have

$$\begin{aligned}
&\mathbf{E}\,\tilde{V}_2\left(\mathrm{conv}\left(\{X_1^{(2)}(t)\colon t\in[0,1]\}\right), \mathrm{conv}\left(\{X_2^{(2)}(t)\colon t\in[0,1]\}\right)\right)\\
&\quad=\frac{1}{2}\int_0^{2\pi}\left(\mathbf{E}M_1(\varphi)\mathbf{E}M_2(\varphi)-\mathbf{E}M_1'(\varphi)\mathbf{E}M_2'(\varphi)\right)d\varphi\\
&\quad=\frac{1}{2}\,2\pi\left((\mathbf{E}M_1(0))^2-(\mathbf{E}M_1'(0))^2\right)=\pi\left((\mathbf{E}M_1(0))^2-(\mathbf{E}M_1'(0))^2\right). \qquad (27)
\end{aligned}$$

Here the second equality follows from relation (25) and the fact that M_1 and M_2 are identically distributed.

Thus, it remains to calculate $\mathbf{E}M_1(0)$ and $\mathbf{E}M_1'(0)$, where M_1 is the support function of the boundary of the convex hull of the two-dimensional Brownian motion on $[0,1]$.

Recall that

$$\{X^{(2)}(t)\colon t\in[0,1]\}=\{(W_1(t),W_2(t))\colon t\in[0,1]\},$$

where $W_1(t)$ and $W_2(t)$ are independent standard one-dimensional Brownian motions.

We fix a direction φ. For $t\in[0,1]$, consider projections on the direction φ and perpendicular to it:

$$\begin{aligned}
z_\varphi(t)&=W_1(t)\cos\varphi+W_2(t)\sin\varphi,\\
h_\varphi(t)&=-W_1(t)\sin\varphi+W_2(t)\cos\varphi.
\end{aligned}$$

Then z_φ and h_φ are independent standard one-dimensional Brownian motions on $[0,1]$.

Consequently, the support function

$$M_1(\varphi)=\max_{t\in[0,1]} z_\varphi(t)$$

is the maximum of the one-dimensional Brownian motion z_φ on $[0,1]$.

Let $t^*\in[0,1]$ be the time when this maximum is attained. Then

$$M_1(\varphi)=z_\varphi(t^*)=W_1(t^*)\cos\varphi+W_2(t^*)\sin\varphi. \qquad (28)$$

Differentiating (28) with respect to φ, we have

$$M_1'(\varphi)=-W_1(t^*)\sin\varphi+W_2(t^*)\cos\varphi=h_\varphi(t^*).$$

In other words, $M_1(\varphi)$ is the maximum of the first Brownian motion z_φ, and $M_1'(\varphi)$ corresponds to the value of the second, independent Brownian motion h_φ at time t^* when the first one attains its maximum.

In particular, for $\varphi = 0$ we obtain $z_0(t) = W_1(t)$, $h_0(t) = W_2(t)$, and

$$\begin{aligned} M_1(0) &= \max_{t\in[0,1]} W_1(t), \\ M_1'(0) &= W_2(t^*). \end{aligned}$$

The cumulative distribution function of the maximum of one-dimensional Brownian motion on $[0,1]$ is well known (see, e.g., [5]), namely

$$F(m) = \mathbf{P}\left(\max_{t\in[0,1]} W_1(t) \leqslant m\right) = \operatorname{erf}\left(\frac{m}{\sqrt{2}}\right),$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}\,du$. The first moment of this distribution is easily calculated:

$$\mathbf{E}M_1(0) = \mathbf{E}\max_{t\in[0,1]} W_1(t) = \sqrt{\frac{2}{\pi}}. \tag{29}$$

Let us show that

$$\mathbf{E}M_1'(0) = \mathbf{E}W_2(t^*) = 0. \tag{30}$$

Indeed, since t^* and $W_2(t)$ are independent, we see that

$$\mathbf{E}M_1'(0) = \mathbf{E}W_2(t^*) = \int_0^1 \int_{-\infty}^{\infty} x p_1(x,t)dx\; p_2(t)dt. \tag{31}$$

Here p_1 denotes the density of the normal distribution $N(0,t)$ under the condition that $t^* = t$, and p_2 denotes the density of the random variable t^* (an explicit formula for p_2 can be found in [14]). Since for a fixed $t \in [0,1]$ the inner integral in (31) equals 0, we have $\mathbf{E}M_1'(0) = 0$.

Combining (20), (27), (29) and (30), we get

$$V(K_1,K_2) = \pi\left((\mathbf{E}M_1(0))^2 - (\mathbf{E}M_1'(0))^2\right) = 2.$$

7 Proof of Theorem 6

Taking into account Remark 4, we can write

$$V(K_1, K_2, \dots, K_k) = \sup_{K_i' \subset K_i} V(K_1', K_2', \dots, K_k'),$$

where the supremum is taken over all finite-dimensional convex compact subsets $K_i' \subset K_i, i = 1, \dots, k$.

So, we want to prove that

$$\left(\sup_{K_i' \subset K_i} V(K_1', K_2', \dots, K_k') \right)^2 \geqslant \sup_{K_i'' \subset K_i} V(K_1'', K_1'', \dots, K_k'') \sup_{K_i''' \subset K_i} V(K_2''', K_2''', \dots, K_k'''). \tag{32}$$

Let us start with the right-hand side of (32):

$$\begin{aligned}
&V(K_1, K_1, K_3, \dots, K_k) V(K_2, K_2, K_3, \dots, K_k) \\
&= \sup_{K_i'' \subset K_i} V(K_1'', K_1'', K_3'', \dots, K_k'') \sup_{K_i''' \subset K_i} V(K_2''', K_2''', K_3''', \dots, K_k''') \\
&= \sup_{K_i'' \subset K_i, K_i''' \subset K_i} [V(K_1'', K_1'', K_3'', \dots, K_k'') V(K_2''', K_2''', K_3''', \dots, K_k''')] \\
&\leqslant \sup_{K_i'' \subset K_i, K_i''' \subset K_i} [V(K_1'', K_1'', \mathrm{conv}(K_3'' \cup K_3'''), \dots, \mathrm{conv}(K_k'' \cup K_k''')) \\
&\qquad \times V(K_2''', K_2''', \mathrm{conv}(K_3'' \cup K_3'''), \dots, \mathrm{conv}(K_k'' \cup K_k'''))] \\
&\leqslant \sup_{K_i'' \subset K_i, K_i''' \subset K_i} V^2(K_1'', K_2''', \mathrm{conv}(K_3'' \cup K_3'''), \dots, \mathrm{conv}(K_k'' \cup K_k''')) \\
&\qquad \leqslant \left(\sup_{K_i' \subset K_i} V(K_1', K_2', \dots, K_k') \right)^2 = V^2(K_1, K_2, \dots, K_k).
\end{aligned}$$

Here, the first inequality holds by monotonicity Property 6 of the mixed volumes and the second inequality holds by the Aleksandrov–Fenchel inequality (8) for finite-dimensional sets.

8 Proof of Theorem 7

Throughout this section we use the following notation:

- the closed conic (or positive) hull of the set F we denote by

$$\operatorname{pos} F := \{\lambda x : x \in \overline{\operatorname{conv}}(F), \lambda \geqslant 0\},$$

  where $\overline{\operatorname{conv}}(F)$ denotes a closed convex hull of the set F.

  We will write $\operatorname{pos}(v_1, \ldots, v_l)$ for conic hull of vectors $v_1, \ldots, v_l$.

- the linear span of F we denote by $\operatorname{lin} F$;
- d_H denotes the Hausdorff metric on the sphere;
- for two orthogonal vectors x and y we write $x \perp y$;
- $\langle x, y \rangle$ denotes the inner product of vectors x, y and $\|x\|$ denotes the norm of x, it will be clear from the context in which space the inner product and norm are taken.

To prove Remark 11 and Theorem 7, we need two statements.

Statement 3. *Suppose that the cones $C, \{C_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ satisfy $C_n \cap \mathbb{S}^{d-1} \to C \cap \mathbb{S}^{d-1}$ in the Hausdorff metric d_H on the sphere $\mathbb{S}^{d-1}$. Then for $j = 0, 1, \ldots, d$ we have*

$$\gamma_j(C_n) \to \gamma_j(C)$$

as $n \to \infty$.

Remark 15. This statement is not new, see [15, Proposition 8.2], [19, Theorem 6.5.2(b)]. In [15] it was proved that the conic intrinsic volumes mentioned in the introduction are continuous with respect to the so-called conic Hausdorff metric. This implies Statement 3. For the reader's convenience, here we present the proof of Statement 3 without using the concept of conic intrinsic volumes.

Proof of Statement 3. Denote $S = C \cap \mathbb{S}^{d-1}, S_n = C_n \cap \mathbb{S}^{d-1}$ (see Figure 1) and let $d_H(S_n, S) = \varepsilon_n$. We have to show that

$$|\mathbf{P}[C_n \cap W_{d-j} \neq \{0\}] - \mathbf{P}[C \cap W_{d-j} \neq \{0\}]| \to 0, \qquad n \to \infty,$$

which will follow from the two convergences:

$$\begin{aligned} &\mathbf{P}[C_n \cap W_{d-j} \neq \{0\}, C \cap W_{d-j} = \{0\}] \to 0, \qquad n \to \infty; \\ &\mathbf{P}[C_n \cap W_{d-j} = \{0\}, C \cap W_{d-j} \neq \{0\}] \to 0, \qquad n \to \infty. \end{aligned}$$

If $j = d$, then the claim of the statement is trivial. If $j = d-1$, then it is equivalent to convergence of the Lebesgue measure of S_n to the Lebesgue measure of S. Further, we proceed by induction on the difference $d - j$.

Let us denote by $G_{j'}(H)$ the Grassmannian of j'-dimensional linear subspaces of the affine space H. The Haar measure $\mathbf{P}$ of the required events (for $j < d-1$) can be represented as (see, e.g., [19, Theorem 7.1.1])

$$\mathbf{P}[C_n \cap W_{d-j} \neq \{0\}, C \cap W_{d-j} = \{0\}] \tag{33}$$
$$= \int_{\mathbb{S}^{d-1}} \int_{G_{d-j-1}(H_x)} \mathbb{1}[C_n \cap W_{d-j-1}(H_x) \neq \emptyset, C \cap W_{d-j-1}(H_x) = \emptyset]\, d\mu\, dx,$$

and

$$\mathbf{P}[C_n \cap W_{d-j} = \{0\}, C \cap W_{d-j} \neq \{0\}]$$
$$= \int_{\mathbb{S}^{d-1}} \int_{G_{d-j-1}(H_x)} \mathbb{1}[C_n \cap W_{d-j-1}(H_x) = \emptyset, C \cap W_{d-j-1}(H_x) \neq \emptyset]\, d\mu\, dx,$$

where H_x denotes the hyperplane $\{y \in \mathbb{R}^d \colon \langle x, y\rangle = 1\}$ with x as the origin, dx is the normalized Lebesgue measure on the sphere $\mathbb{S}^{d-1}$, $d\mu$ is the Haar measure on $G_{d-j-1}(H_x)$. By Lebesgue's dominated convergence theorem it is sufficient to prove that the inner integrals in (33) tend to 0 as n tends to ∞ for almost every $x \in \mathbb{S}^{d-1}$. If $x \in \mathrm{Int}S$ (where $\mathrm{Int}S$ denotes the interior of the set S taken on the sphere), then this claim is trivial, since for large n we have $x \in S_n$. The set of x such that $x \in S \setminus \mathrm{Int}S$ has Lebesgue measure 0. Hence further we can assume $x \notin S$ and $x \notin S_n$ for sufficiently large n.

Consider the sets $B_x = C \cap H_x$ and $B_{x,n} = C_n \cap H_x$ and let $\Pi_x \colon H_x \to \mathbb{S}^{d-2}(H_x)$ be the projection on the unit sphere in H_x, i.e., the mapping defined by $x + h \mapsto x + \frac{h}{\|h\|}$ for $h \perp x$. It is clear that the following events coincide:

$$\{C \cap W_{d-j-1}(H_x) = \emptyset\} = \{B_x \cap W_{d-j-1}(H_x) = \emptyset\}$$
$$= \{\Pi_x B_x \cap W_{d-j-1}(H_x) = \emptyset\} = \{\mathrm{pos}_{H_x}(\Pi_x B_x) \cap W_{d-j-1}(H_x) = \{x\}\},$$

where pos_{H_x} denotes the positive hull in H_x. Similarly, for C_n we get

$$\{C_n \cap W_{d-j-1}(H_x) \neq \emptyset\} = \{B_{x,n} \cap W_{d-j-1}(H_x) \neq \emptyset\}$$
$$= \{\Pi_x B_{x,n} \cap W_{d-j-1}(H_x) \neq \emptyset\} = \{\mathrm{pos}_{H_x}(\Pi_x B_{x,n}) \cap W_{d-j-1}(H_x) \neq \{x\}\}.$$

Hence, we have to show that for almost every $x \in \mathbb{S}^{d-1}$

$$\mu\left[\Pi_x B_{x,n} \cap W_{d-j-1}(H_x) \neq \emptyset, \Pi_x B_x \cap W_{d-j-1}(H_x) = \emptyset\right] \to 0, \quad \text{as } n \to \infty;$$
$$\mu\left[\Pi_x B_{x,n} \cap W_{d-j-1}(H_x) = \emptyset, \Pi_x B_x \cap W_{d-j-1}(H_x) \neq \emptyset\right] \to 0, \quad \text{as } n \to \infty.$$

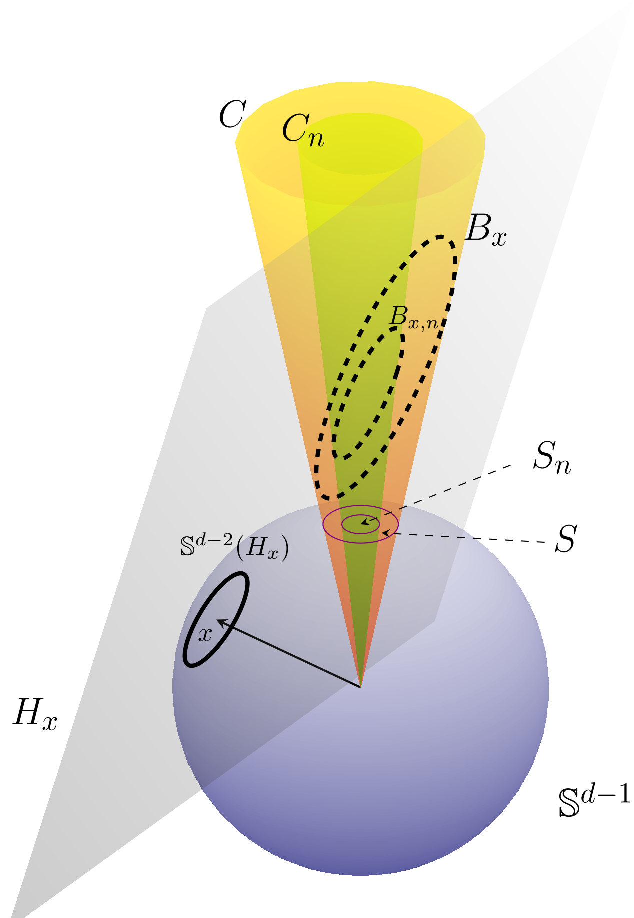

Figure 1: Illustration for the proof of Statement 3.

The latter will follow from the convergence

$$\Pi_x B_{x,n} \to \Pi_x B_x, \quad \text{as } n \to \infty,$$

in the Hausdorff metric on H_x by the induction hypothesis. First let us show that $\Pi_x B_{x,n}$ is a subset of a small neighbourhood of $\Pi_x B_x$. Consider $p \in \Pi_x B_{x,n}$. We have to show that there exist a point $p' \in \Pi_x B_x$ close to p. By the definition of Π_x and the Hausdorff metric there exist $q \in S$, $\lambda > 0$ and $v \in \mathbb{B}^d$ such that $\lambda(q + \varepsilon_n v) \in H_x$ and $\Pi_x(\lambda q + \lambda \varepsilon_n v) = p$. Expanding $q = \alpha x + h$ and $v = \alpha_v x + h_v$, where $\alpha, \alpha_v \in \mathbb{R}$, $h, h_v \perp x$, we get

$$p = \Pi_x(\lambda q + \lambda \varepsilon_n v) = \Pi_x(\lambda(\alpha x + h) + \lambda \varepsilon_n(\alpha_v x + h_v)) = x + \frac{h + \varepsilon_n h_v}{\|h + \varepsilon_n h_v\|}.$$

Notice that $\alpha + \varepsilon_n \alpha_v > 0$, since $\lambda > 0$ and the coefficient of x is $\lambda(\alpha + \varepsilon_n \alpha_v) = 1$. Hence, $-\alpha < \varepsilon_n \alpha_v \leqslant \varepsilon_n$.

If $\alpha > 0$, then $\frac{1}{\alpha} q = x + \frac{h}{\alpha} \in B_x$ and we put $p' = x + \frac{h}{\|h\|} \in \Pi_x B_x$. Since $x \notin S$, $\inf_{q \in S} \|h\| > 0$. Therefore, $\|p - p'\| = O(\varepsilon_n)$, because the projection Π_x is $\frac{1}{r}$-Lipschitz on the sets $\{x + h \colon \|h\| > r\}$ for all $r > 0$.

The case $\alpha < 0$ is a bit more complicated.

Assume that there exists $q_x \in S$ such that $\Delta = \langle q_x, x \rangle > 0$. Consider $\varepsilon_n' = \frac{\varepsilon_n}{\Delta}$. For some $h_x \perp x$ we have $q_x = \Delta x + h_x$. Let

$$q' = (1 - \varepsilon_n') q + \varepsilon_n' q_x = ((1 - \varepsilon_n')\alpha + \Delta \varepsilon_n') x + (1 - \varepsilon_n') h + \varepsilon_n' h_x.$$

It is clear that $\alpha' := (1 - \varepsilon_n')\alpha + \Delta \varepsilon_n' > \alpha + \Delta \varepsilon_n' \geqslant 0$ for sufficiently small ε_n. Hence, $\frac{1}{\alpha'} q' \in B_x$ and we put

$$p' = x + \frac{h + \varepsilon_n'(h_x - h)}{\|h + \varepsilon_n'(h_x - h)\|} \in \Pi_x B_x.$$

Similarly to the previous situation we get the uniform bound $\|p - p'\| = O(\varepsilon_n)$.

When $\langle q, x \rangle < 0$ for all $q \in S$ we have $\Pi_x B_{x,n} = \Pi_x B_x = \emptyset$ for all sufficiently large n. Consequently, we are left with the case when $\sup_{q \in S} \langle q, x \rangle = 0$. This set of possible x has measure 0 and hence can be dismissed.

To show that $\Pi_x B_x$ is a subset of a small neighbourhood of $\Pi_x B_{x,n}$ we proceed similarly. For $p \in \Pi_x B_x$ we take $q_n \in S_n$, $\lambda_n \geqslant 0$ and $v_n \in \mathbb{B}^d$ such that $\lambda(q_n + \varepsilon_n v_n) \in H_x$ and $\Pi_x(\lambda_n q_n + \lambda_n \varepsilon_n v_n) = p$. All of the constants in $O(\varepsilon_n)$ depended only on $\inf_{q \in S_n} \|h\|$, which is uniformly bounded for all S_n for large n. Hence, the same bounds will hold. □

Proof of Remark 11. Let $\{C_n'\}_{n \in \mathbb{N}}$ be a sequence of polyhedral finite-dimensional cones approximating an arbitrary finite-dimensional cone C': $C_1' \subset C_2' \subset \ldots \subset C'$, $\cup_{n=1}^{\infty} C_n' \subset C'$. If $\dim C' = d$, then $C_n' \cap \mathbb{S}^{d-1} \to C' \cap \mathbb{S}^{d-1}$ in the Hausdorff metric d_H on the sphere $\mathbb{S}^{d-1}$. Therefore, by Statement 3, $\gamma_j(C_n') = \mathbf{P}[C_n' \cap W_{d-j} \neq \{0\}] \to \gamma_j(C') = \mathbf{P}[C' \cap W_{d-j} \neq \{0\}]$. Thus, it suffices to take the supremum in (15) over polyhedral finite-dimensional cones. □

Statement 4. *Consider the finite-dimensional cones $C_n \subset E_0$, $n \in \mathbb{N}$: $C_1 \subset C_2 \subset \ldots \subset C$ that approximate the cone $C \subset E_0$ from the inside: $\cup_{n=1}^{\infty} C_n$ is dense in C. Then we have*

$$\gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty} C_n)) = \gamma_j(C).$$

Proof of Statement 4. Clearly, $\gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty} C_n)) \leqslant \gamma_j(C)$ because of $\cup_{n=1}^{\infty} C_n \subset C$ and monotonicity of Grassmann angles. Let us show that $\gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty} C_n)) \geqslant \gamma_j(C)$. By definition,

$$\gamma_j(C) = \sup_{C' \subset C} \mathbf{P}[C' \cap W_{d-j} \neq \{0\}],$$

where the supremum can be taken only over polyhedral finite-dimensional cones by Remark 11. Consequently, it is sufficient to prove that for every finite-dimensional polyhedral cone $C' \subset C$, $d := \dim C'$, we have

$$\gamma_j(C') = \mathbf{P}[W_{d-j} \cap C' \neq \{0\}] \leqslant \gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty} C_n)).$$

The cone C' is polyhedral, hence there are unit vectors $v_1, \dots, v_l \in \mathrm{lin}\, C'$ such that $C' = \mathrm{pos}(v_1, \dots, v_l)$. Consider $\varepsilon > 0$. The set $\cup_{n=1}^{\infty} C_n$ is dense in C, hence for every v_i there exists a unit vector $u_i \in \cup_{n=1}^{\infty} C_n$ such that $d_H(u_i, v_i) < \varepsilon$. Therefore, the cone $C'_\varepsilon = \mathrm{pos}(u_1, \dots, u_l)$ is close to C', i.e., $d_H(C'_\varepsilon \cap \mathbb{S}(E_0), C' \cap \mathbb{S}(E_0)) < \varepsilon$, where $\mathbb{S}(E_0)$ denotes the unit sphere in E_0. Let ε_k be a sequence of positive numbers such that $\lim \varepsilon_k = 0$ as $k \to \infty$. We want to use Statement 3 to show that $\lim_{k\to\infty} \gamma_j(C'_{\varepsilon_k}) = \gamma_j(C')$. To do this formally we need to place all C'_{ε_k} into one common finite-dimensional space with C'.

Notice that $\dim C'_\varepsilon \leqslant l$ for every ε, hence $\dim \mathrm{lin}(C'_\varepsilon \cup C') \leqslant d + l$. This means that for every ε there exist a linear subspace $U_\varepsilon \subset E_0$ with $\dim U_\varepsilon = d + l$ such that $C'_\varepsilon \cup C' \subset U_\varepsilon$. Decompose U_ε into the orthogonal sum $U_\varepsilon = \mathrm{lin}\, C' \oplus V_\varepsilon$. Now fix $(d+l)$-dimensional space $\mathbb{R}^{d+l} = \mathbb{R}^d \oplus \mathbb{R}^l$ and the isometry $\mathcal{I}$ between $\mathrm{lin}\, C'$ and $\mathbb{R}^d$. Let $\mathcal{J}_\varepsilon \colon U_\varepsilon \to \mathbb{R}^{d+l}$ be an isometric operator such that $\mathcal{J}_\varepsilon$ coincides with $\mathcal{I}$ on $\mathrm{lin}\, C'$. Then $\gamma_j(C') = \gamma_j(\mathcal{I}C')$, $\gamma_j(C'_\varepsilon) = \gamma_j(\mathcal{J}_\varepsilon C'_\varepsilon)$ and $d_H(C'_\varepsilon \cap \mathbb{S}(E_0), C' \cap \mathbb{S}(E_0)) = d_H(\mathcal{J}_\varepsilon C'_\varepsilon \cap \mathbb{S}^{d+l-1}, \mathcal{I}C' \cap \mathbb{S}^{d+l-1})$. Therefore, $\mathcal{J}_{\varepsilon_k} C'_{\varepsilon_k} \to \mathcal{I}C'$, as $k \to \infty$, Statement 3 applies and

$$\gamma_j(C') = \gamma_j(\mathcal{I}C') = \lim_{k\to\infty} \gamma_j(\mathcal{J}_{\varepsilon_k} C'_{\varepsilon_k}) = \lim_{k\to\infty} \gamma_j(C'_{\varepsilon_k}).$$

To conclude the proof notice that for every $\varepsilon > 0$, we have $C'_\varepsilon \subset \mathrm{pos}(\cup_{n=1}^{\infty} C_n)$, hence $\gamma_j(C'_\varepsilon) \leqslant \gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty} C_n))$.

$\square$

We return to the proof of Theorem 7.

For cones C such that $\dim C < \infty$, this theorem is proved in [7, Theorem 3.5]. It remains to check the case $\dim C = \infty$. We approximate the cone C from the inside by finite-dimensional cones C_n, $n \in \mathbb{N}$: $C_1 \subset C_2 \subset \ldots \subset C$; $\cup_{n=1}^{\infty} C_n$ is dense in C.

We will use the same argument as applied above to the mixed volumes (see Subsection 2.1 and Section 5).

The proof of Corollary 1 does not use the compactness of the set K, the key property in the proof is the existence of a natural modification of the process $\langle \theta, x\rangle$ on K. Therefore, by Theorem 11 we can repeat the proof of Corollary 1 for a convex GB_σ-cones C.

Thus, using Corollary 1, we get that $\cup_{n=1}^{\infty}\mathrm{Spec}(x_1,\dots,x_k|C_n)$ is almost surely dense in $\mathrm{Spec}(x_1,\dots,x_k|C) = \{(\langle\theta,x_1\rangle,\dots,\langle\theta,x_k\rangle) : \theta\in C\}$. Hence,

$$\begin{aligned}
\mathbf{E}[\gamma_j(\mathrm{Spec}_k C)] &= \mathbf{E}[\gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty}\mathrm{Spec}_k C_n))] = \mathbf{E}[\lim_{n\to\infty}\gamma_j(\mathrm{Spec}_k C_n)] \\
&= \lim_{n\to\infty}\mathbf{E}[\gamma_j(\mathrm{Spec}_k C_n)] = \lim_{n\to\infty}\gamma_j(C_n) = \gamma_j(\mathrm{pos}(\cup_{n=1}^{\infty}C_n)) = \gamma_j(C).
\end{aligned}$$

Here, the first and last equalities hold by Statement 4, the second and fifth by definition of Grassmann angle (15). In the third equality we have used Lebesgue's dominated convergence theorem. Finally, the fourth equality is the assertion of the theorem for finite-dimensional cones C_n.

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