DEVIATION OF THE RANK AND CRANK MODULO 11

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ABSTRACT. In this paper, we build on recent results of Frank Garvan and Rishabh Sarma as well as classical results of Bruce Berndt in order to establish the 11-dissection of the deviations of the rank and crank modulo 11. Using our new dissections we re-derive results of Garvan, Atkin, Swinnerton-Dyer, Hussain, Ekin and Chern. By developing and exploiting positivity conditions for quotients of theta functions, we will also prove new rank-crank inequalities and make several conjectures. We would like to point out that Kathrin Bringmann and Badri Vishal Pandey have recently solved one of our conjectures. For other applications of our methods, in this paper we will also prove new congruences for rank moments as well as the Andrews' smallest parts function and Eisenstein series.

1. INTRODUCTION

A partition of a positive integer n is a weakly-decreasing sequence of positive integers whose sum is n. We denote the number of partitions of n by $p(n)$. Among the most famous results in the theory of partitions are Ramanujan's congruences:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In 1944, Dyson [15] conjectured combinatorial interpretations of the first two congruences. He defined the rank of a partition as the largest part minus the number of parts and conjectured that the rank modulo 5 divided the partitions of $5n+4$ into 5 equal classes and that the rank modulo 7 divided the partitions of $7n+5$ into 7 equal classes. For example the partitions of the number 4 are $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ and their ranks are $3, 1, 0, -1, -3$ respectively, giving an equinumerous distribution of the partitions of 4 into the five residue classes modulo 5. His modulo 5 and modulo 7 rank conjectures were proved by Atkin and Swinnerton-Dyer [7].

Although the rank does not explain Ramanujan's third congruence, Dyson conjectured another function, which he called the crank, that would divide the partitions of $11n+6$ into 11 equal classes. Andrews and Garvan later discovered the crank [3]. For a partition π, let $\lambda(\pi)$ denote the largest part, $\vartheta(\pi)$ the number of ones, and $\mu(\pi)$ the number of parts larger than $\vartheta(\pi)$. The crank of π, denoted $c(\pi)$, is defined as follow

$$c(\pi) := \begin{cases} \lambda(\pi), & \text{when } \ \vartheta(\pi) = 0, \\ \mu(\pi) - \vartheta(\pi), & \text{otherwise.} \end{cases}$$

The cranks of the five partitions of 4 are $4, 0, 2, -2, -4$ respectively, giving an equinumerous distribution of the partitions of 4 into the five residue classes modulo 5.

Let $N(m,n)$ denote the number of partitions of n with rank m and $N(a,r,n)$ denote the number of partitions of n with rank $\equiv a \pmod r$. Note that there is a symmetric property $N(a,r,n) = N(r-a,r,n)$. Let $M(m,n)$ denote the number of partitions of n with crank m and $M(a,r,n)$

Date: 15 September 2023.

2020 *Mathematics Subject Classification.* 11P83, 11F27, 11F30, 11B65, 11F20, 11F33.

Key words and phrases. Partitions, Dyson's rank, crank, dissections, inequalities, congruences.

denote the number of partitions of n with crank $\equiv a \pmod{r}$. As with the rank, the crank has the symmetry property $M(a,r,n) = M(r-a,r,n)$. Andrews and Garvan [3] showed

$$M(a,5,5n+4) = \frac{p(5n+4)}{5}, \quad \text{for } 0 \le a \le 4,$$
$$M(a,7,7n+5) = \frac{p(7n+5)}{7}, \quad \text{for } 0 \le a \le 6,$$
$$M(a,11,11n+6) = \frac{p(11n+6)}{11}, \quad \text{for } 0 \le a \le 10.$$

Let $q := e^{2\pi i z}$ be a nonzero complex number with $\mathrm{Im}(z) > 0$. Using the terminology above, we define the deviation of the rank from the expected value to be

$$D(a,r) = D(a,r;q) := \sum_{n=0}^{\infty} \left(N(a,r,n) - \frac{p(n)}{r} \right) q^n$$

and the deviation of the crank from the expected value to be

$$D_C(a,r) = D_C(a,r;q) := \sum_{n=0}^{\infty} \left(M(a,r,n) - \frac{p(n)}{r} \right) q^n.$$

Recall the q-Pochhammer notation, defined by

$$(x)_n = (x;q)_n := \prod_{i=0}^{n-1} (1 - xq^i).$$

In his last letter to Hardy, Ramanujan gave a list of seventeen functions which he called “mock theta functions.“ He stated that they have certain asymptotic properties similar to the properties of ordinary theta functions, but that they are not theta functions [32]. Recall the definition of the universal mock theta function, which gives many mock theta functions as a special cases,

$$g(x;q) := x^{-1} \left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n} \right).$$

Also recall the definition of the theta function

$$j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k,$$

where the equivalence of product and sum follows from Jacobi's triple product identity. Let a and m be integers with m positive. We introduce

$$J_{a,m} := j(q^a;q^m), \quad J_m := J_{m,3m} = (q^m;q^m)_\infty \ \text{ and } \ P_i := J_{i,11}, \quad X_i := J_{11i,121}. \tag{1.1}$$

To formulate our main results we need the following notation.

Definition 1.1. We define

$$\begin{aligned} v_{11}&(a_0,a_1,a_2,a_3,a_4,a_5,a_7,a_8,a_9,a_{10}) \\ &:= \frac{J_{121}^2}{11} \Big(a_0 \frac{1}{X_1} + a_1 q \frac{X_5}{X_2X_3} + a_2 q^2 \frac{X_3}{X_1X_4} + a_3 q^3 \frac{X_2}{X_1X_3} + a_4 q^4 \frac{1}{X_2} \\ &\quad + a_5 q^5 \frac{X_4}{X_2X_5} + a_7 q^7 \frac{1}{X_3} + a_8 q^{19} \frac{X_1}{X_4X_5} + a_9 q^9 \frac{1}{X_4} + a_{10} q^{10} \frac{1}{X_5} \Big), \end{aligned} \tag{1.2}$$

$$\begin{aligned} G_{11}(b_0,b_4,b_7,b_9,b_{10}) &:= b_0 q^{22} g(q^{22};q^{121}) + b_4 q^{37} g(q^{44};q^{121}) \\ &\quad + b_7 q^{40} g(q^{55};q^{121}) + b_9 q^{31} g(q^{33};q^{121}) + b_{10}[q^{-1} + q^{10} g(q^{11};q^{121})], \end{aligned}$$

and

$$\vartheta(a_1,a_2,a_3,a_4,a_5) := \frac{J_{11}^6}{J_1^2}\Big[a_1\frac{q^2}{P_4P_5^2}+a_2\frac{1}{P_1^2P_3}+a_3\frac{q}{P_1P_4^2}+a_4\frac{q}{P_2^2P_5}+a_5\frac{q}{P_2P_3^2}\Big]. \tag{1.3}$$

The next two theorems give the 11-dissections for the deviations of the crank and rank modulo 11.

Theorem 1.2. *We have*

$$\begin{aligned}
D_C(0,11) &= v_{11}(10,-12,-2,8,6,4,-4,-6,-8,2),\\
D_C(1,11) &= v_{11}(-1,10,-2,-3,-5,4,7,5,3,2),\\
D_C(2,11) &= v_{11}(-1,-1,9,-3,6,-7,-4,-6,3,2),\\
D_C(3,11) &= v_{11}(-1,-1,-2,8,-5,4,-4,5,3,-9),\\
D_C(4,11) &= v_{11}(-1,-1,-2,-3,6,-7,7,5,-8,2),\\
D_C(5,11) &= v_{11}(-1,-1,-2,-3,-5,4,-4,-6,3,2).
\end{aligned}$$

Theorem 1.3. *We have*

$$\begin{aligned}
D(0,11) &= G_{11}(-2,0,0,0,0)+v_{11}(10,-12,-2,8,6,4,18,-6,-8,2)+\sum_{m=0}^{10}\vartheta_{0,m}(q^{11})q^m,\\
D(1,11) &= G_{11}(1,0,-1,0,0)+v_{11}(-1,10,-2,-3,-5,4,-4,-6,3,2)+\sum_{m=0}^{10}\vartheta_{1,m}(q^{11})q^m,\\
D(2,11) &= G_{11}(0,0,1,0,-1)+v_{11}(-1,-1,9,-3,6,-7,-4,5,3,2)+\sum_{m=0}^{10}\vartheta_{2,m}(q^{11})q^m,\\
D(3,11) &= G_{11}(0,0,0,1,1)+v_{11}(-1,-1,-2,-3,17,-7,-4,5,3,-9)+\sum_{m=0}^{10}\vartheta_{3,m}(q^{11})q^m,\\
D(4,11) &= G_{11}(0,1,0,-1,0)+v_{11}(-1,-1,-2,-3,6,4,-4,5,-8,13)+\sum_{m=0}^{10}\vartheta_{4,m}(q^{11})q^m,\\
D(5,11) &= G_{11}(0,-1,0,0,0)+v_{11}(-1,-1,-2,-3,-5,4,-4,-6,3,-9)+\sum_{m=0}^{10}\vartheta_{5,m}(q^{11})q^m,
\end{aligned}$$

where

$$\begin{aligned}
\vartheta_{0,6}(q) &= \vartheta(0,0,2,2,-2),\\
\vartheta_{1,6}(q) &= \vartheta(-1,1,-1,-2,1),\\
\vartheta_{2,6}(q) &= \vartheta(1,0,-1,2,0),\\
\vartheta_{3,6}(q) &= \vartheta(1,0,1,-1,-1),\\
\vartheta_{4,6}(q) &= \vartheta(0,-1,1,0,2),\\
\vartheta_{5,6}(q) &= \vartheta(-1,0,-1,0,-1),
\end{aligned}$$

and the other $\vartheta_{a,m}(q)$ are given explicitly in Section 4.

Remark 1.4. The dissections for the deviations of the rank and crank modulo 5, 7 and 4, 8 were found by Hickerson and Mortenson [23, 29] and we use the setting from there.

1.1. **A prelude to new proofs of classical results.** Using Theorems 1.2 and 1.3, we give new proofs of classical results. In Section 5.1 we re-derive crank equalities found by Garvan [20] such as, for $n \geq 0$,

$$M(0,11,11n+1)+M(1,11,11n+1)=2M(2,11,11n+1),$$
$$M(2,11,11n+1)=M(3,11,11n+1)=M(4,11,11n+1)=M(5,11,11n+1).$$

In Section 6.1 we re-derive crank-crank inequalities, which were first proved by Ekin [16] and Berkovich and Garvan [8], such as, for $n \geq 0$,

$$M(1,11,11n+1) \geq \frac{p(11n+1)}{11} \geq M(2,11,11n+1) \geq M(0,11,11n+1).$$

In Section 5.2 we re-derive congruences for the partition function, which were establish by Atkin and Swinnerton-Dyer [7], such as

$$\sum_{n=0}^{\infty} p(11n)q^n \equiv \frac{J_{11}^2}{P_1} \pmod{11}.$$

In Section 5.3 we re-derive linear rank congruences due to Atkin and Hussain [6], such as, for $n \geq 0$,

$$N(2,11;11n)-5N(3,11;11n)-2N(4,11;11n)+6N(5,11;11n) \equiv 0 \pmod{11}.$$

1.2. **A prelude to new results and new conjectures.** In Section 2 we present new rank and rank-crank inequalities, such as, for $n \geq 0$,

$$2N(2,11,11n)+N(3,11,11n)+N(5,11,11n) \geq 4N(4,11,11n),$$

which follow directly from the positivity of Fourier coefficients of theta quotients as described in Section 6.2. Then as a corollary we derive two-term, four-term and six-term inequalities, such as, for $n \geq 0$,

$$M(1,11,11n) \geq N(4,11,11n),$$
$$N(2,11,11n)+N(3,11,11n) \geq N(4,11,11n)+M(1,11,11n),$$
$$N(2,11,11n)+2N(3,11,11n) \geq N(5,11,11n)+2M(1,11,11n).$$

Using a numerical computing environment, it is possible to generate higher order inequalities for eight-terms, ten-terms, etc.

As another application of Theorem 1.3 we present in Section 2 new congruences for rank and crank moments, such as

$$\sum_{n=0}^{\infty}\left(\sum_{m=-\infty}^{\infty} m^2 N(m,11n+6)\right) q^n \equiv \vartheta(-4,3,1,5,-2) \pmod{11}.$$

and congruences for the Andrews' smallest parts function $\operatorname{spt}(n)$, such as

$$\sum_{n=0}^{\infty} \operatorname{spt}(11n+6)q^n \equiv \vartheta(2,4,5,3,1) \pmod{11}.$$

In Section 6.4 we state new conjectural two-term rank and rank-crank inequalities, such as

$$N(0,11,11n) \geq_3 N(1,11,11n) \geq N(2,11,11n) \geq_1 M(0,11,11n) \geq \frac{p(11n)}{11} \geq$$
$$\geq M(1,11,11n) \geq N(3,11,11n) \geq_2 N(4,11,11n) \geq N(5,11,11n),$$

where $A_n \geq B_n$ means that $A_n \geq B_n$ for all $n \geq 0$ and $A_n \geq_m B_n$ means that $A_n \geq B_n$ for all $n \geq m$. Recently, our inequalities among rank and crank from Conjecture 6.15 were fully solved by Bringmann and Pandey using techniques connected with the Circle method [13]. Also in Section 6.4 we state Conjecture 6.19, which is the generalization of our observations on the positivity of

Fourier coefficients of theta quotients provided in Section 6.2 and we state Conjecture 6.20, which is the generalization of our new rank and rank-crank inequalities from Section 2.

1.3. **A guide to the paper.** In Section 2 we state our new rank-crank inequalities and new congruences as described in Section 1.2. In Section 3 we prove Theorem 1.2, which gives us the 11-dissection of the deviation of the crank modulo 11. In Section 4 we present all the dissection elements of the deviation of the rank and prove Theorem 1.3.

In Section 5 we provide new proofs for classical results as described in Section 1.1. In Section 6.1 we observe known results on crank inequalities as described in Section 1.2. In Section 6.2 we develop techniques to exploit the positivity of Fourier coefficients. In Section 6.3 we use results from Section 6.2 to prove some examples of new rank-crank inequalities. Proofs of other new rank-crank inequalities are straightforward and similar but for the sake of prosperity and to underscore the role played by the dissections of Theorems 1.2 and 1.3, we place in Section 8 all the proofs for the new results found in Section 2. In Section 6.4 we state Conjecture 6.15, which presents rank-crank inequalities, and state Conjecture 6.19, which describes in general the positivity of Fourier coefficients of sums of theta quotients. The rank-crank inequalities in Conjecture 6.15 have recently been solved by Bringmann and Pandey [13]. Also in Section 6.4 we state Conjecture 6.20, which describes the general case of rank-crank inequalities. In Section 7 we prove new congruences for rank and crank moments, Andrews' smallest parts function $\mathrm{spt}(n)$ and Eisenstein series.

2. The Main Results in Full

Using Theorem 1.2 and Theorem 1.3 we are able to find new rank and rank-crank inequalities.

Theorem 2.1. *Consider $n \geq 0$. For $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ we have*

$$\begin{aligned} N_0 + 2N_1 + M_1 &\geq 2N_2 + N_4 + M_0, \\ N_0 + 2N_1 + 3N_2 + M_1 &\geq 3N_3 + 3N_5 + M_0, \\ 2N_2 + N_3 + N_5 &\geq 4N_4, \\ N_2 + 5N_3 + 3N_4 + M_0 &\geq N_0 + 2N_1 + 6N_5 + M_1. \end{aligned}$$

For $N_i = N(i, 11, 11n+1)$ and $M_i = M(i, 11, 11n+1)$ we have

$$\begin{aligned} N_0 + 4N_2 + N_4 + M_0 &\geq 2N_1 + 3N_3 + N_5 + M_2, \\ N_0 + 3N_1 + 6N_3 + M_0 &\geq 4N_2 + 6N_4 + M_2, \\ 2N_1 + 6N_4 &\geq N_2 + 3N_3 + 4N_5, \\ N_1 + 2N_2 + 3N_5 + 3M_2 &\geq N_0 + N_3 + 4N_4 + 2M_0 + M_1, \\ 3N_2 + 2N_3 + N_4 + 3M_2 &\geq N_0 + N_1 + 4N_5 + 2M_0 + M_1. \end{aligned}$$

For $N_i = N(i, 11, 11n+2)$ and $M_i = M(i, 11, 11n+2)$ we have

$$\begin{aligned} 3N_2 + N_4 &\geq 2N_0 + 2N_5 \\ 2N_1 + 2N_3 + N_5 + 2M_0 &\geq 2N_2 + 3N_4 + 2M_2 \\ 3N_0 + 2N_2 + M_2 &\geq N_1 + N_3 + 3N_5 + M_0 \\ 2N_0 + N_1 + N_3 + 3N_4 + M_0 &\geq 4N_2 + 3N_5 + M_2 \\ N_0 + 3N_1 + N_2 + 8N_5 + 3M_0 &\geq 8N_3 + 5N_4 + 3M_2. \end{aligned}$$

For $N_i = N(i, 11, 11n+3)$ *and* $M_i = M(i, 11, 11n+3)$ *we have*

$$N_0 + 2N_3 + M_1 \geq N_1 + 2N_5 + M_0,$$
$$5N_1 + 2N_2 + 2N_4 \geq 2N_0 + 4N_3 + 3N_5,$$
$$2N_0 + 4N_3 + N_5 + M_0 \geq N_1 + 3N_2 + 3N_4 + M_1,$$
$$6N_2 + 3N_5 + 5M_1 \geq N_0 + N_1 + 2N_3 + 5N_4 + 5M_0,$$
$$4N_0 + 2N_1 + 4N_4 + 3M_0 \geq 7N_2 + 3N_3 + 3M_1.$$

For $N_i = N(i, 11, 11n+4)$ *and* $M_i = M(i, 11, 11n+4)$ *we have*

$$4N_1 + 3N_3 + 5M_1 \geq 2N_0 + 3N_2 + N_4 + N_5 + 5M_0,$$
$$4N_0 + 5N_2 + 3M_0 \geq 5N_1 + 2N_3 + N_4 + N_5 + 3M_1,$$
$$3N_1 + N_2 + M_0 \geq 2N_0 + 2N_3 + M_1,$$
$$3N_0 + N_4 + N_5 + M_1 \geq 3N_2 + 2N_3 + M_0.$$

For $N_i = N(i, 11, 11n+5)$ *and* $M_i = M(i, 11, 11n+5)$ *we have*

$$3N_0 + N_2 + N_5 + 2M_2 \geq 2N_1 + 3N_3 + 2M_0,$$
$$7N_1 + N_4 + 3M_2 \geq 3N_0 + 2N_2 + N_3 + 2N_5 + 3M_0,$$
$$2N_0 + N_2 + 3N_3 + N_5 + 4M_0 \geq 3N_1 + 4N_4 + 4M_2,$$
$$4N_2 + 7N_3 + 2N_4 \geq 4N_0 + 2N_1 + 7N_5,$$
$$4N_1 + N_2 + 2N_4 + N_5 + 5M_2 \geq 2N_0 + 6N_3 + 5M_0.$$

For $N_i = N(i, 11, 11n+7)$ *and* $M_i = M(i, 11, 11n+7)$ *we have*

$$N_0 + 2N_4 + 2M_0 \geq 2N_3 + N_5 + 2M_1,$$
$$3N_0 + N_1 + N_2 + 7N_3 + 4N_5 \geq 5N_4 + 4M_0 + 7M_1,$$
$$4N_1 + 4N_2 + 3N_4 + 4M_1 \geq 4N_0 + 6N_3 + N_5 + 4M_0,$$
$$N_0 + 5N_3 + 2N_4 \geq 2N_1 + 2N_2 + 4N_5.$$

For $N_i = N(i, 11, 11n+8)$ *and* $M_i = M(i, 11, 11n+8)$ *we have*

$$N_0 + 2N_2 + 3N_4 + 5M_0 \geq 3N_1 + 3N_5 + 5M_1,$$
$$2N_1 + 4N_3 + 2N_5 + 3M_1 \geq N_0 + 2N_2 + 5N_4 + 3M_0,$$
$$2N_1 + 5N_2 + 2N_5 + M_1 \geq 3N_0 + 5N_3 + N_4 + M_0,$$
$$7N_0 + 4N_1 + 5N_4 + 4N_5 \geq 8N_2 + N_3 + 4M_0 + 7M_1,$$
$$5N_0 + 6N_1 + 4N_3 + 2N_4 \geq N_2 + 5N_5 + 6M_0 + 5M_1.$$

For $N_i = N(i, 11, 11n+9)$ *and* $M_i = M(i, 11, 11n+9)$ *we have*

$$4N_2 + M_0 \geq 2N_1 + N_3 + N_4 + M_1,$$
$$4N_1 + N_3 + N_4 \geq N_0 + 3N_2 + 2N_5,$$
$$4N_0 + 3N_3 + 3N_4 + 4M_0 \geq 2N_1 + 5N_2 + 3N_5 + 4M_1,$$
$$3N_1 + 4N_2 + 7N_5 + 3M_1 \geq 2N_0 + 6N_3 + 6N_4 + 3M_0.$$

For $N_i = N(i, 11, 11n+10)$ *and* $M_i = M(i, 11, 11n+10)$ *we have*

$$3N_1 + 2N_5 + 2M_3 \geq 2N_2 + 2N_3 + N_4 + 2M_0,$$
$$3N_0 + N_2 + N_3 + M_0 \geq 3N_1 + 2N_4 + M_3,$$
$$2N_1 + 4N_4 + M_3 \geq N_0 + N_2 + N_3 + 3N_5 + M_0,$$
$$6N_2 + 6N_3 + 6M_0 + 5M_3 \geq 6N_0 + 6N_1 + 3N_4 + 8N_5.$$

As a corollary of Theorem 2.1 we can derive two-term and four-term rank-crank inequalities modulo 11. Similar inequalities among rank and crank of different modulus were studied by many authors. For example, two-term rank-rank and crank-crank inequalities modulo $2, 3, 4$ were studied by Andrews and Lewis [4, 27], two-term rank-rank and crank-crank inequalities modulo $5, 7, 11$ were studied by Garvan and Ekin [16, 20], two-term rank-crank inequalities modulo 8 were studied by Lewis and Mortenson [26, 29], two-term and four-term rank-rank inequalities modulo 10 were studied by Mao, Alwaise, Iannuzzi, Swisher [1, 28] and two-term rank-rank inequalities modulo 12 were studied by Fan, Xia, Zhao [18].

Corollary 2.2. *Consider $n \geq 0$. For $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ we have*

$$\begin{aligned} M_1 &\geq N_4, \\ N_2 + N_3 &\geq N_4 + M_1. \end{aligned}$$

For $N_i = N(i, 11, 11n + 1)$ and $M_i = M(i, 11, 11n + 1)$ we have

$$\begin{aligned} N_2 &\geq M_2, \\ M_2 &\geq N_4, \\ N_2 + N_4 &\geq N_3 + N_5. \end{aligned}$$

For $N_i = N(i, 11, 11n + 2)$ and $M_i = M(i, 11, 11n + 2)$ we have

$$\begin{aligned} N_1 &\geq M_2, \\ M_0 &\geq N_5, \\ 2M_0 &\geq N_2 + N_5, \\ N_2 + M_0 &\geq N_0 + N_5, \\ N_1 + N_3 &\geq M_0 + M_2, \\ N_1 + N_5 &\geq N_4 + M_2. \end{aligned}$$

For $N_i = N(i, 11, 11n + 3)$ and $M_i = M(i, 11, 11n + 3)$ we have

$$\begin{aligned} N_0 &\geq M_0, \\ M_1 &\geq N_4, \\ N_0 + M_0 &\geq 2N_2, \\ N_2 + M_1 &\geq N_4 + M_0, \\ N_0 + N_3 &\geq N_2 + M_1. \end{aligned}$$

For $N_i = N(i, 11, 11n + 4)$ and $M_i = M(i, 11, 11n + 4)$ we have

$$\begin{aligned} N_1 &\geq M_0, \\ 2N_1 &\geq N_0 + M_0, \\ N_1 + M_1 &\geq N_2 + M_0, \\ N_1 + N_3 &\geq 2M_0. \end{aligned}$$

For $N_i = N(i, 11, 11n+5)$ *and* $M_i = M(i, 11, 11n+5)$ *we have*

$$\begin{aligned} N_2 &\geq M_0, \\ M_2 &\geq N_5, \\ N_2 + N_3 &\geq M_0 + M_2, \\ N_2 + M_2 &\geq N_0 + N_5, \\ N_0 + M_0 &\geq N_1 + N_4, \\ N_2 + N_3 &\geq N_1 + N_5, \\ N_0 + N_2 &\geq N_1 + M_0. \end{aligned}$$

For $N_i = N(i, 11, 11n+7)$ *and* $M_i = M(i, 11, 11n+7)$ *we have*

$$\begin{aligned} N_0 &\geq M_1, \\ M_0 &\geq N_4, \\ N_3 + N_4 &\geq N_5 + M_0. \end{aligned}$$

For $N_i = N(i, 11, 11n+8)$ *and* $M_i = M(i, 11, 11n+8)$ *we have*

$$\begin{aligned} N_2 &\geq M_1, \\ M_0 &\geq N_3, \\ M_0 + M_1 &\geq N_2 + N_4, \\ N_1 + N_5 &\geq 2N_3, \\ N_0 + M_0 &\geq 2N_2, \\ N_3 + M_0 &\geq N_4 + M_1, \\ N_2 + N_3 &\geq 2M_1, \\ 2M_0 &\geq N_1 + N_5. \end{aligned}$$

For $N_i = N(i, 11, 11n+9)$ *and* $M_i = M(i, 11, 11n+9)$ *we have*

$$\begin{aligned} N_2 &\geq M_1, \\ M_0 &\geq N_5, \\ M_1 + M_0 &\geq N_2 + N_5, \\ N_0 + M_0 &\geq N_2 + M_1, \\ N_2 + N_5 &\geq N_3 + N_4. \end{aligned}$$

For $N_i = N(i, 11, 11n+10)$ *and* $M_i = M(i, 11, 11n+10)$ *we have*

$$\begin{aligned} N_1 &\geq M_0, \\ M_3 &\geq N_4, \\ N_0 + M_0 &\geq N_1 + N_4, \\ N_2 + N_3 &\geq 2M_0, \\ N_1 + N_4 &\geq 2M_0, \\ N_1 + M_3 &\geq N_2 + N_3. \end{aligned}$$

We can also consider six-term rank-crank inequalities.

Corollary 2.3. *Consider $n \geq 0$. For $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ we have*

$$\begin{aligned} N_2 + 2N_3 &\geq N_5 + 2M_1, \\ 2N_3 + N_4 &\geq 2N_5 + M_1, \\ 3M_1 &\geq N_2 + N_4 + N_5, \\ N_3 + 2M_1 &\geq N_2 + 2N_5. \end{aligned}$$

For $N_i = N(i, 11, 11n+1)$ and $M_i = M(i, 11, 11n+1)$ we have

$$\begin{aligned} 2N_1 + N_3 &\geq N_2 + 2N_4, \\ 2N_1 + N_4 &\geq N_2 + 2N_5, \\ N_1 + 2N_4 &\geq 2N_5 + M_0, \\ N_1 + N_3 + N_5 &\geq 3N_4, \\ 2N_2 + N_4 &\geq 2N_5 + M_0. \end{aligned}$$

For $N_i = N(i, 11, 11n+2)$ and $M_i = M(i, 11, 11n+2)$ we have

$$\begin{aligned} N_0 + N_1 + N_4 &\geq N_2 + M_0 + M_2, \\ 3M_0 &\geq N_2 + N_3 + N_4, \\ N_1 + N_2 + N_5 &\geq 2M_0 + M_2, \\ 3M_0 &\geq N_0 + N_4 + N_5, \\ N_1 + N_3 + M_0 &\geq N_0 + 2N_5, \\ N_1 + N_3 + M_0 &\geq N_2 + N_5 + M_2, \\ N_1 + 2M_0 &\geq N_2 + N_4 + M_2, \\ N_1 + N_2 + N_3 &\geq N_0 + N_5 + M_2, \\ N_1 + N_2 + M_0 &\geq N_0 + N_4 + M_2. \end{aligned}$$

For $N_i = N(i, 11, 11n+3)$ and $M_i = M(i, 11, 11n+3)$ we have

$$\begin{aligned} 2N_1 + N_4 &\geq N_3 + N_5 + M_1, \\ N_0 + N_1 + N_5 &\geq M_0 + 2M_1, \\ N_0 + N_1 + N_4 &\geq N_2 + 2M_1, \\ M_0 + 2M_1 &\geq N_2 + N_3 + N_5, \\ 2N_0 + N_5 &\geq 2N_2 + N_4, \\ N_0 + N_3 + N_5 &\geq N_4 + M_0 + M_1, \\ 3M_1 &\geq N_1 + 2N_4, \\ 2N_2 + N_5 &\geq N_4 + 2M_0, \\ N_0 + N_3 + M_1 &\geq N_1 + 2N_4, \\ N_0 + 2N_3 &\geq M_0 + 2M_1. \end{aligned}$$

For $N_i = N(i, 11, 11n+4)$ and $M_i = M(i, 11, 11n+4)$ we have

$$\begin{aligned} 2N_0 + M_0 &\geq N_1 + 2N_3, \\ 2N_0 + N_2 &\geq N_1 + N_3 + M_1, \\ N_0 + N_2 + M_1 &\geq N_1 + N_4 + N_5, \\ N_0 + N_2 + N_3 &\geq N_4 + N_5 + M_0, \\ N_0 + 2M_1 &\geq 2N_2 + N_3, \\ 3M_1 &\geq N_2 + N_4 + N_5. \end{aligned}$$

For $N_i = N(i, 11, 11n+5)$ *and* $M_i = M(i, 11, 11n+5)$ *we have*

$$\begin{aligned}
2N_3 + M_0 &\geq 2N_5 + M_2, \\
N_2 + 2N_3 &\geq N_0 + 2N_5, \\
N_3 + M_0 + M_2 &\geq N_0 + 2N_5, \\
N_3 + 2M_0 &\geq N_1 + N_4 + N_5, \\
2N_2 + N_3 &\geq N_0 + N_5 + M_0, \\
N_1 + N_2 + N_5 &\geq N_3 + 2M_0, \\
M_0 + 2M_2 &\geq N_0 + N_3 + N_5, \\
N_1 + N_2 + N_4 &\geq 3M_0, \\
N_0 + N_2 + N_5 &\geq 2M_0 + M_2.
\end{aligned}$$

For $N_i = N(i, 11, 11n+7)$ *and* $M_i = M(i, 11, 11n+7)$ *we have*

$$\begin{aligned}
N_1 + N_2 + M_1 &\geq N_0 + N_5 + M_0, \\
N_1 + N_2 + M_1 &\geq N_0 + N_3 + N_4, \\
M_0 + 2M_1 &\geq N_0 + 2N_5, \\
N_0 + 2N_3 &\geq M_0 + 2M_1, \\
N_1 + N_2 + N_4 &\geq 2M_0 + M_1.
\end{aligned}$$

For $N_i = N(i, 11, 11n+8)$ *and* $M_i = M(i, 11, 11n+8)$ *we have*

$$\begin{aligned}
N_1 + N_5 + M_1 &\geq N_2 + N_3 + N_4, \\
M_0 + 2M_1 &\geq N_0 + 2N_4, \\
N_1 + N_2 + N_5 &\geq N_0 + 2N_4, \\
N_1 + N_3 + N_5 &\geq N_4 + M_0 + M_1, \\
N_1 + N_5 + M_0 &\geq N_2 + 2N_4, \\
N_0 + N_1 + N_5 &\geq 2N_2 + N_3, \\
N_2 + N_3 + M_0 &\geq N_0 + 2N_4, \\
N_2 + M_0 + M_1 &\geq N_0 + N_3 + N_4, \\
N_1 + N_2 + N_5 &\geq M_0 + 2M_1, \\
N_0 + N_1 + N_5 &\geq N_2 + 2M_1, \\
2N_2 + M_0 &\geq N_0 + N_4 + M_1, \\
N_0 + N_3 + M_0 &\geq N_2 + 2M_1, \\
2N_2 + M_0 &\geq N_0 + 2N_3, \\
N_0 + N_4 + M_0 &\geq N_2 + N_3 + M_1, \\
N_0 + N_3 + N_4 &\geq 3M_1, \\
2N_2 + N_4 &\geq 3M_1.
\end{aligned}$$

For $N_i = N(i, 11, 11n + 9)$ *and* $M_i = M(i, 11, 11n + 9)$ *we have*

$$\begin{aligned}
2N_1 + M_1 &\geq N_0 + N_3 + N_4,\\
N_0 + N_3 + N_4 &\geq N_2 + N_5 + M_1,\\
M_0 + 2M_1 &\geq N_2 + N_3 + N_4,\\
N_0 + N_5 + M_0 &\geq N_2 + N_3 + N_4,\\
N_1 + N_2 + N_5 &\geq M_0 + 2M_1,\\
N_0 + N_3 + N_4 &\geq 3M_1,\\
N_0 + 2M_0 &\geq N_1 + N_2 + N_5,\\
N_0 + N_2 + N_5 &\geq 3M_1,\\
N_0 + 2M_0 &\geq N_1 + N_3 + N_4,\\
N_2 + M_0 + M_1 &\geq N_1 + N_3 + N_4,\\
N_0 + N_2 + M_0 &\geq N_1 + 2M_1,\\
N_0 + N_2 + M_0 &\geq 2N_1 + N_5.
\end{aligned}$$

For $N_i = N(i, 11, 11n + 10)$ *and* $M_i = M(i, 11, 11n + 10)$ *we have*

$$\begin{aligned}
N_0 + N_2 + N_3 &\geq N_1 + N_4 + M_0,\\
2M_0 + M_3 &\geq N_1 + N_4 + N_5,\\
N_2 + N_3 + M_3 &\geq N_1 + N_4 + N_5,\\
N_4 + M_0 + M_3 &\geq N_0 + 2N_5,\\
N_2 + N_3 + M_3 &\geq N_0 + N_5 + M_0,\\
2M_0 + M_3 &\geq N_2 + N_3 + N_4,\\
N_0 + N_1 + N_5 &\geq N_2 + N_3 + N_4,\\
M_0 + 2M_3 &\geq N_0 + N_4 + N_5,\\
2N_1 + N_5 &\geq N_2 + N_3 + M_0,\\
N_1 + 2M_3 &\geq N_0 + N_5 + M_0,\\
2N_1 + N_5 &\geq 3M_0.
\end{aligned}$$

Remark 2.4. Note that the four-term and six-term inequalities in Corollary 2.2 and Corollary 2.3 are selected so that they are corollaries of inequalities in Theorem 2.1 but they are not corollaries of Conjecture 6.15 and they cannot be obtained from each other using Conjecture 6.15.

For residue 6 we have the following results.

Theorem 2.5. *Consider* $n \geq 0$. *For* $N_i = N(i, 11, 11n + 6)$ *we have*

$$\begin{aligned}
2N_1 + N_2 + 2N_4 &\geq 2N_3 + 3N_5,\\
2N_0 + N_2 + 2N_5 &\geq 2N_1 + N_3 + 2N_4,\\
N_0 + N_1 + 3N_3 + N_4 &\geq 4N_2 + 2N_5,\\
N_0 + 6N_1 + 4N_2 &\geq 2N_3 + 5N_4 + 4N_5.
\end{aligned}$$

Corollary 2.6. *Consider* $n \geq 0$. *For* $N_i = N(i, 11, 11n + 6)$ *we have*

$$N_0 \geq \frac{p(11n + 6)}{11} \quad \textit{and} \quad \frac{p(11n + 6)}{11} \geq N_5.$$

Corollary 2.7. *Consider* $n \geq 0$. *For* $N_i = N(i, 11, 11n+6)$ *we have*

$$\begin{aligned}
N_0 + N_3 &\geq N_1 + N_4,\\
N_1 + N_2 + N_4 &\geq N_3 + 2N_5,\\
N_2 + 2N_3 &\geq N_1 + N_4 + N_5,\\
N_0 + N_1 + N_4 &\geq N_2 + 2N_5,\\
3N_3 &\geq N_2 + 2N_5,\\
N_0 + 2N_3 &\geq 2N_2 + N_5.
\end{aligned}$$

We prove Theorem 2.1, Corollary 2.2, Corollary 2.3, Theorem 2.5, Corollary 2.6 and Corollary 2.7 in Section 6.3. In the spirit of Corollary 2.2 and Corollary 2.6 we introduce new conjectural rank-crank inequalities in Section 6.4.

Recall

$$P_i := J_{i,11}.$$

Define the following sums of theta quotients.

Definition 2.8. We define

$$[c_1, c_2, c_3, c_4, c_5] := \frac{J_{11}^2}{J_1^3}\Big(c_1 P_5^2 P_4 + c_2 q^2 P_1^2 P_3 + c_3 q P_4^2 P_1 + c_4 q P_2^2 P_5 + c_5 q P_3^2 P_2\Big).$$

Definition 2.9. We define

$$\begin{aligned}
[c_1, c_2, c_3, c_4, c_5; c_6]_0 &:= \frac{1}{P_1}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q P_1 P_3 P_4 P_5}{P_2^2},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_1 &:= \frac{P_5}{P_2 P_3}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q P_2^2 P_4}{P_1},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_2 &:= \frac{P_3}{P_1 P_4}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q^3 P_1^2 P_2}{P_5},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_3 &:= \frac{P_2}{P_1 P_3}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q P_3^2 P_5}{P_4},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_4 &:= \frac{1}{P_2}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q^2 P_1 P_2 P_3 P_5}{P_4^2},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_5 &:= \frac{P_4}{P_2 P_5}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q P_1 P_5^2}{P_3},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_7 &:= \frac{1}{P_3}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q^2 P_1 P_2 P_3 P_4}{P_5^2},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_8 &:= \frac{q P_1}{P_4 P_5}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{P_3 P_4^2}{P_2},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_9 &:= \frac{1}{P_4}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q P_1 P_2 P_4 P_5}{P_3^2},\\
[c_1, c_2, c_3, c_4, c_5; c_6]_{10} &:= \frac{1}{P_5}[c_1, c_2, c_3, c_4, c_5] + c_6 \frac{J_{11}^2}{J_1^3} \frac{q^{-1} P_2 P_3 P_4 P_5}{P_1^2}.
\end{aligned}$$

Recall the following notation:

$$\vartheta(a_1, a_2, a_3, a_4, a_5) := \frac{J_{11}^6}{J_1^2}\Big[a_1 \frac{q^2}{P_4 P_5^2} + a_2 \frac{1}{P_1^2 P_3} + a_3 \frac{q}{P_1 P_4^2} + a_4 \frac{q}{P_2^2 P_5} + a_5 \frac{q}{P_2 P_3^2}\Big].$$

Rank and crank moments are defined as [5]

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n),$$
$$M_k(n) := \sum_{m=-\infty}^{\infty} m^k M(m,n)$$

for even $k \in \mathbb{N}$. Define $T_{a,m}(q)$ to be the elements of the 11-dissection of the generating functions for rank and crank moments:

$$\sum_{n=0}^{\infty} N_k(n)q^n =: \sum_{m=0}^{10} T_{k,m}(q^{11})q^m,$$
$$\sum_{n=0}^{\infty} M_k(n)q^n =: \sum_{m=0}^{10} T^C_{k,m}(q^{11})q^m.$$

The reformulation of this definition then reads

$$T_{k,m}(q) := \sum_{n=0}^{\infty} N_k(11n+m)q^n, \tag{2.1}$$

$$T^C_{k,m}(q) := \sum_{n=0}^{\infty} M_k(11n+m)q^n. \tag{2.2}$$

As an another application of Theorem 1.2 and Theorem 1.3 we derive new congruences for the rank and crank moments and for the Andrews' smallest parts function $\mathrm{spt}(n)$, where $\mathrm{spt}(n)$ denotes the number of smallest parts in the partitions of n. For example, the partitions of the number 4 are $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ with $1,1,2,2,4$ being the number of smallest parts respectively, so we see $\mathrm{spt}(4) = 10$. Garvan [19, Theorem 5.1] considered congruences for $T_{k,6}(q)$ in terms of Eisenstein series. We will provide congruences for $T^C_{k,m}(q)$ for residues $m \neq 6$ and congruences for $T_{k,m}(q)$ for residues $m \in \{1,2,3,5,6,8\}$ in terms of theta quotients $\vartheta(a_1,a_2,a_3,a_4,a_5)$ from Definition 1.1 and $[c_1,c_2,c_3,c_4,c_5;c_6]_m$ from Definition 2.9.

Theorem 2.10. *Using the notation of Definition 1.1 and Definition 2.9, we have*

$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+1)q^n \equiv [1,5,-4,1,-5;1]_1 \pmod{11},$$
$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+2)q^n \equiv [3,-5,3,-2,5;-5]_2 \pmod{11},$$
$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+3)q^n \equiv [5,2,-2,-4,4;4]_3 \pmod{11},$$
$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+5)q^n \equiv [3,1,3,4,-4;-3]_5 \pmod{11},$$
$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+6)q^n \equiv \vartheta(2,4,5,3,1) \pmod{11},$$
$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+8)q^n \equiv [-1,-2,2,-1,3;2]_8 \pmod{11}.$$

Theorem 2.11. *Using the notation of Definition 1.1 and Definition 2.9, for residue* 1 *modulo* 11 *we have*

$$\begin{aligned}
T_{2,1}(q) &\equiv [0,-1,-5,-4,-3;-2]_1 \pmod{11},\\
T_{4,1}(q) &\equiv 0 \pmod{11},\\
T_{6,1}(q) &\equiv [0,-1,-4,2,5;3]_1 \pmod{11},\\
T_{8,1}(q) &\equiv [0,5,1,4,-4;-3]_1 \pmod{11}.
\end{aligned}$$

For residue 2 *modulo* 11 *we have*

$$\begin{aligned}
T_{2,2}(q) &\equiv [2,2,-3,-4,4;-1]_2 \pmod{11},\\
T_{4,2}(q) &\equiv [2,5,-1,0,3;1]_2 \pmod{11},\\
T_{6,2}(q) &\equiv [2,4,-2,2,5;3]_2 \pmod{11},\\
T_{8,2}(q) &\equiv [2,-1,2,4,1;5]_2 \pmod{11}.
\end{aligned}$$

For residue 3 *modulo* 11 *we have*

$$\begin{aligned}
T_{2,3}(q) &\equiv [-3,0,-3,1,-4;3]_3 \pmod{11},\\
T_{4,3}(q) &\equiv [-1,-4,2,5,4;5]_3 \pmod{11},\\
T_{6,3}(q) &\equiv [-4,0,3,3,-2;5]_3 \pmod{11},\\
T_{8,3}(q) &\equiv [-5,2,0,4,-5;5]_3 \pmod{11}.
\end{aligned}$$

For residue 5 *modulo* 11 *we have*

$$\begin{aligned}
T_{2,5}(q) &\equiv [-2,5,1,-1,4;-5]_5 \pmod{11},\\
T_{4,5}(q) &\equiv [-4,3,-3,5,-4;-2]_5 \pmod{11},\\
T_{6,5}(q) &\equiv [-5,2,-1,0,2;1]_5 \pmod{11},\\
T_{8,5}(q) &\equiv [4,0,-2,2,5;-3]_5 \pmod{11}.
\end{aligned}$$

For residue 6 *modulo* 11 *we have*

$$\begin{aligned}
T_{2,6}(q) &\equiv \vartheta(-4,3,1,5,-2) \pmod{11},\\
T_{4,6}(q) &\equiv \vartheta(-2,-4,-5,-3,-1) \pmod{11},\\
T_{6,6}(q) &\equiv \vartheta(1,5,-5,4,2) \pmod{11},\\
T_{8,6}(q) &\equiv \vartheta(-5,-5,1,-2,-2) \pmod{11}.
\end{aligned}$$

For residue 8 *modulo* 11 *we have*

$$\begin{aligned}
T_{2,8}(q) &\equiv [-1,-4,-1,5,-3;-4]_8 \pmod{11},\\
T_{4,8}(q) &\equiv [-3,5,-5,-3,-2;-5]_8 \pmod{11},\\
T_{6,8}(q) &\equiv [3,4,3,-3,1;3]_8 \pmod{11},\\
T_{8,8}(q) &\equiv [0,2,-1,5,-1;5]_8 \pmod{11}.
\end{aligned}$$

Theorem 2.12. *We have*

$$
\begin{aligned}
T^C_{2,0}(q) \equiv T^C_{4,0}(q) \equiv T^C_{6,0}(q) \equiv T^C_{8,0}(q) &\equiv 0 \pmod{11},\\
T^C_{2,1}(q) \equiv T^C_{4,1}(q) \equiv T^C_{6,1}(q) \equiv T^C_{8,1}(q) &\equiv 2\frac{J_{11}^2 P_5}{P_2P_3} \pmod{11},\\
7T^C_{2,2}(q) \equiv 10T^C_{4,2}(q) \equiv 8T^C_{6,2}(q) \equiv 2T^C_{8,2}(q) &\equiv \frac{J_{11}^2 P_3}{P_1P_4} \pmod{11},\\
8T^C_{2,3}(q) \equiv 7T^C_{4,3}(q) \equiv 2T^C_{6,3}(q) \equiv 10T^C_{8,3}(q) &\equiv \frac{J_{11}^2 P_2}{P_1P_3} \pmod{11},\\
8T^C_{2,4}(q) \equiv 9T^C_{4,4}(q) \equiv 3T^C_{6,4}(q) \equiv 6T^C_{8,4}(q) &\equiv \frac{J_{11}^2}{P_2} \pmod{11},\\
3T^C_{2,5}(q) \equiv 2T^C_{4,5}(q) \equiv 8T^C_{6,5}(q) \equiv 5T^C_{8,5}(q) &\equiv \frac{J_{11}^2 P_4}{P_2P_5} \pmod{11},\\
T^C_{2,7}(q) \equiv 7T^C_{4,7}(q) \equiv 10T^C_{6,7}(q) \equiv 5T^C_{8,7}(q) &\equiv \frac{J_{11}^2}{P_3} \pmod{11},\\
7T^C_{2,8}(q) \equiv 9T^C_{4,8}(q) \equiv 9T^C_{6,8}(q) \equiv 7T^C_{8,8}(q) &\equiv \frac{J_{11}^2 qP_1}{P_4P_5} \pmod{11},\\
T^C_{2,9}(q) \equiv 9T^C_{4,9}(q) \equiv 4T^C_{6,9}(q) \equiv 3T^C_{8,9}(q) &\equiv \frac{J_{11}^2}{P_4} \pmod{11},\\
3T^C_{2,10}(q) \equiv 4T^C_{4,10}(q) \equiv 9T^C_{6,10}(q) \equiv T^C_{8,10}(q) &\equiv \frac{J_{11}^2}{P_5} \pmod{11}.
\end{aligned}
$$

Remark 2.13. The congruences among $M_k(11n+m)$ given by Theorem 2.12 were initially found by Chern [14, (5.6)-(5.15)] using general formulas found by Atkin and Garvan [5, (6.6)-(6.8)]. In Theorem 2.12 we extend Chern's results to congruences among generating functions of crank moments and theta quotients.

Using [19, Theorem 5.1] we can also deduce congruences for Eisenstein series E_4 and E_6 defined as

$$E_j(q) := 1 - \frac{2n}{B_n}\sum_{n=1}^{\infty} \sigma_{j-1}(n)q^n,$$

where B_n is the n-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$.

Corollary 2.14. *Using the notation of Definition 1.1, we have*

$$
\begin{aligned}
E_4(q) &\equiv \frac{1}{J_1^2 J_{11}}\vartheta(-1,1,1,1,1) \pmod{11},\\
E_6(q) &\equiv \frac{1}{J_1^2 J_{11}}\vartheta(-3,1,5,4,-2) \pmod{11}.
\end{aligned}
$$

3. Proof of Theorem 1.2

In this section we demonstrate how to obtain the 11-dissection for the crank deviation as stated in Theorem 1.2.

Proof of Theorem 1.2. From [3] we know that the two-variable generating function for the crank has the form

$$F(z;q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}.$$

Here for $n \leq 1$ we set

$$M(m,n) := \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

From this we can find the formula for the deviation of crank, which was also mentioned in [29, (2.12)]:

$$D_C(a,r) = \frac{1}{r} \sum_{j=1}^{r-1} \zeta_r^{-aj} \frac{(q)_\infty}{(\zeta_r^j q)_\infty (\zeta_r^{-j} q)_\infty} = \frac{1}{r} \sum_{j=1}^{r-1} \zeta_r^{-aj} F(\zeta_r^j; q) \tag{3.1}$$

Let us take the 11-dissection for $F(\zeta_{11}; q)$, which is given in [9, Theorem 7.1] and apply an identity, which can be verified by rearranging terms:

$$X_1 X_2 X_3 X_4 X_5 = J_{11} J_{121}^4,$$

where X_i is defined in (1.1). Then we will obtain

$$\begin{aligned} F(\zeta_{11}^j; q) = J_{121}^2 \Big(& \frac{1}{X_1} + (A_1 - 1) q \frac{X_5}{X_2 X_3} + A_2 q^2 \frac{X_3}{X_1 X_4} + (A_3 + 1) q^3 \frac{X_2}{X_1 X_3} \\ & + (A_2 + A_4 + 1) q^4 \frac{1}{X_2} - (A_2 + A_4) q^5 \frac{X_4}{X_2 X_5} + (A_1 + A_4) q^7 \frac{1}{X_3} \\ & - (A_2 + A_5 + 1) q^{19} \frac{X_1}{X_4 X_5} - (A_4 + 1) q^9 \frac{1}{X_4} - A_3 q^{10} \frac{1}{X_5} \Big), \end{aligned}$$

where $A_n = \zeta_{11}^{jn} + \zeta_{11}^{-jn}$ and $\zeta_r := e^{2\pi i/r}$ is a primitive root of unity. We put the previous expressions into (3.1) with $r = 11$ and directly obtain Theorem 1.2. □

4. 11-DISSECTION FOR THE DEVIATION OF THE RANK

Define $Q_{a,m}(q)$ to be the elements of the 11-dissection of the deviation of the rank:

$$D(a, 11) =: \sum_{m=0}^{10} Q_{a,m}(q^{11}) q^m.$$

The reformulation of this definition then reads

$$Q_{a,m}(q) := \sum_{n=0}^{\infty} \left(N(a, 11, 11n + m) - \frac{p(11n+m)}{11} \right) q^n. \tag{4.1}$$

For convenience we will decompose

$$Q_{a,m}(q) = Q_{a,m}^{\text{th}}(q) + Q_{a,m}^{\text{mck}}(q),$$

where $Q_{a,m}^{\text{mck}}$ is a mock part of $Q_{a,m}$, that is, it corresponds to the terms in G_{11}. Note that $Q_{a,m}^{\text{mck}}$ can be non-zero only for $(a,m) \in \{(0,0), (1,0), (4,4), (5,4), (1,7), (2,7), (3,9), (4,9), (2,10), (3,10)\}$ as stated in Theorem 1.3. For residue 0 we have

$$Q_{0,0}^{\text{mck}}(q) = -2q^2 g(q^2; q^{11}), \tag{4.2}$$

$$Q_{1,0}^{\text{mck}}(q) = q^2 g(q^2; q^{11}). \tag{4.3}$$

For residue 4 we have

$$Q_{4,4}^{\text{mck}}(q) = q^3 g(q^4; q^{11}), \tag{4.4}$$

$$Q_{5,4}^{\text{mck}}(q) = -q^3 g(q^4; q^{11}). \tag{4.5}$$

For residue 7 we have

$$Q^{\text{mck}}_{1,7}(q) = -q^3 g(q^5; q^{11}), \tag{4.6}$$
$$Q^{\text{mck}}_{2,7}(q) = q^3 g(q^5; q^{11}). \tag{4.7}$$

For residue 9 we have

$$Q^{\text{mck}}_{3,9}(q) = q^2 g(q^3; q^{11}), \tag{4.8}$$
$$Q^{\text{mck}}_{4,9}(q) = -q^2 g(q^3; q^{11}). \tag{4.9}$$

For residue 10 we have

$$Q^{\text{mck}}_{2,10}(q) = -[q^{-1} + g(q; q^{11})], \tag{4.10}$$
$$Q^{\text{mck}}_{3,10}(q) = q^{-1} + g(q; q^{11}). \tag{4.11}$$

Remark 4.1. We can establish mock parts of $Q_{a,m}$ by using [23, Theorem 4.1].

The remainder $Q_{a,m} - Q^{\text{mck}}_{a,m}$ we will call the theta part and denote $Q^{\text{th}}_{a,m}$. Also recall a useful identity found by O'Brien [30, p. 6]:

$$J_1^3 = P_5^2 P_4 - q^2 P_1^2 P_3 - q P_4^2 P_1 - q P_2^2 P_5 - q P_3^2 P_2.$$

By Definition 2.8 this identity is transformed to

$$J_{11}^2 = [1, -1, -1, -1, -1]. \tag{4.12}$$

Note that by using identity (4.12) we can convert v_{11}-term of $Q_{a,m}$ to the form $[c_1, c_2, c_3, c_4, c_5; c_6]_m$. For example we know

$$\frac{J_{11}^2 P_5}{P_2 P_3} = [1, -1, -1, -1, -1; 0]_1.$$

Then we can formulate Theorem 1.3 in its full form, that is, write $\vartheta_{a,m}(q)$ explicitly. The mock parts of the dissection elements for residue 0 are stated in (4.2) and (4.3). The theta parts of the dissection elements for residue 0 are

$$Q^{\text{th}}_{0,0}(q) = \frac{1}{11}[10, 56, -32, -10, -10; 22]_0 = \frac{10}{11}\frac{J_{11}^2}{P_1} + [0, 6, -2, 0, 0; 2]_0 =: \frac{10}{11}\frac{J_{11}^2}{P_1} + \vartheta_{0,0}(q),$$
$$Q^{\text{th}}_{1,0}(q) = \frac{1}{11}[-1, -10, 23, 1, 1; -11]_0 = -\frac{1}{11}\frac{J_{11}^2}{P_1} + [0, -1, 2, 0, 0; -1]_0 =: -\frac{1}{11}\frac{J_{11}^2}{P_1} + \vartheta_{1,0}(q),$$
$$Q^{\text{th}}_{2,0}(q) = \frac{1}{11}[-1, -32, 1, 12, 1; 0]_0 = -\frac{1}{11}\frac{J_{11}^2}{P_1} + [0, -3, 0, 1, 0; 0]_0 =: -\frac{1}{11}\frac{J_{11}^2}{P_1} + \vartheta_{2,0}(q),$$
$$Q^{\text{th}}_{3,0}(q) = \frac{1}{11}[-1, 23, -21, 1, 12; 0]_0 = -\frac{1}{11}\frac{J_{11}^2}{P_1} + [0, 2, -2, 0, 1; 0]_0 =: -\frac{1}{11}\frac{J_{11}^2}{P_1} + \vartheta_{3,0}(q),$$
$$Q^{\text{th}}_{4,0}(q) = \frac{1}{11}[-1, -10, 23, -21, 1; 0]_0 = -\frac{1}{11}\frac{J_{11}^2}{P_1} + [0, -1, 2, -2, 0; 0]_0 =: -\frac{1}{11}\frac{J_{11}^2}{P_1} + \vartheta_{4,0}(q),$$
$$Q^{\text{th}}_{5,0}(q) = \frac{1}{11}[-1, 1, -10, 12, -10; 0]_0 = -\frac{1}{11}\frac{J_{11}^2}{P_1} + [0, 0, -1, 1, -1; 0]_0 =: -\frac{1}{11}\frac{J_{11}^2}{P_1} + \vartheta_{5,0}(q).$$

Dissection elements for residue 1 are

$$
\begin{aligned}
Q_{0,1}(q) &= \frac{1}{11}[10,12,12,12,-10;-22]_1 = -\frac{12}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + [2,0,0,0,-2;-2]_1 =: -\frac{12}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + \vartheta_{0,1}(q),\\
Q_{1,1}(q) &= \frac{1}{11}[-1,-10,-10,1,23;0]_1 = \frac{10}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + [-1,0,0,1,3;0]_1 =: \frac{10}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + \vartheta_{1,1}(q),\\
Q_{2,1}(q) &= \frac{1}{11}[-1,12,1,-10,1;11]_1 = -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + [0,1,0,-1,0;1]_1 =: -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + \vartheta_{2,1}(q),\\
Q_{3,1}(q) &= \frac{1}{11}[-1,-10,12,1,-21;11]_1 = -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + [0,-1,1,0,-2;1]_1 =: -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + \vartheta_{3,1}(q),\\
Q_{4,1}(q) &= \frac{1}{11}[-1,1,-10,12,-10;0]_1 = -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + [0,0,-1,1,-1;0]_1 =: -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + \vartheta_{4,1}(q),\\
Q_{5,1}(q) &= \frac{1}{11}[-1,1,1,-10,12;-11]_1 = -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + [0,0,0,-1,1;-1]_1 =: -\frac{1}{11}\frac{J_{11}^2 P_5}{P_2 P_3} + \vartheta_{5,1}(q).
\end{aligned}
$$

Dissection elements for residue 2 are

$$
\begin{aligned}
Q_{0,2}(q) &= \frac{1}{11}[-2,-20,24,2,2;-22]_2 = -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + [0,-2,2,0,0;-2]_2 =: -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + \vartheta_{0,2}(q),\\
Q_{1,2}(q) &= \frac{1}{11}[9,13,-31,2,2;11]_2 = -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + [1,1,-3,0,0;1]_2 =: -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + \vartheta_{1,2}(q),\\
Q_{2,2}(q) &= \frac{1}{11}[-2,-9,24,-9,2;11]_2 = \frac{9}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + [-1,0,3,0,1;1]_2 =: \frac{9}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + \vartheta_{2,2}(q),\\
Q_{3,2}(q) &= \frac{1}{11}[-2,13,13,2,-9;-22]_2 = -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + [0,1,1,0,-1;-2]_2 =: -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + \vartheta_{3,2}(q),\\
Q_{4,2}(q) &= \frac{1}{11}[-2,-9,-20,13,2;22]_2 = -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + [0,-1,-2,1,0;2]_2 =: -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + \vartheta_{4,2}(q),\\
Q_{5,2}(q) &= \frac{1}{11}[-2,2,2,-9,2;-11]_2 = -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + [0,0,0,-1,0;-1]_2 =: -\frac{2}{11}\frac{J_{11}^2 P_3}{P_1 P_4} + \vartheta_{5,2}(q).
\end{aligned}
$$

Dissection elements for residue 3 are

$$
\begin{aligned}
Q_{0,3}(q) &= \frac{1}{11}[8,36,-8,-8,14;-22]_3 = \frac{8}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + [0,4,0,0,2;-2]_3 =: \frac{8}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + \vartheta_{0,3}(q),\\
Q_{1,3}(q) &= \frac{1}{11}[-3,-30,3,3,3;22]_3 = -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + [0,-3,0,0,0;2]_3 =: -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + \vartheta_{1,3}(q),\\
Q_{2,3}(q) &= \frac{1}{11}[8,14,3,3,-8;-22]_3 = -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + [1,1,0,0,-1;-2]_3 =: -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + \vartheta_{2,3}(q),\\
Q_{3,3}(q) &= \frac{1}{11}[-3,3,-8,3,-8;22]_3 = -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + [0,0,-1,0,-1;2]_3 =: -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + \vartheta_{3,3}(q),\\
Q_{4,3}(q) &= \frac{1}{11}[-3,14,14,-8,3;-11]_3 = -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + [0,1,1,-1,0;-1]_3 =: -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + \vartheta_{4,3}(q),\\
Q_{5,3}(q) &= \frac{1}{11}[-3,-19,-8,3,3;0]_3 = -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + [0,-2,-1,0,0;0]_3 =: -\frac{3}{11}\frac{J_{11}^2 P_2}{P_1 P_3} + \vartheta_{5,3}(q).
\end{aligned}
$$

The mock parts of the dissection elements for residue 4 are stated in (4.4) and (4.5). The theta parts of the dissection elements for residue 4 are

$$Q^{\text{th}}_{0,4}(q) = \frac{1}{11}[6,-6,16,-28,16;0]_4 = \frac{6}{11}\frac{J_{11}^2}{P_2} + [0,0,2,-2,2;0]_4 =: \frac{6}{11}\frac{J_{11}^2}{P_2} + \vartheta_{0,4}(q),$$
$$Q^{\text{th}}_{1,4}(q) = \frac{1}{11}[6,5,-17,5,5;0]_4 = -\frac{5}{11}\frac{J_{11}^2}{P_2} + [1,0,-2,0,0;0]_4 =: -\frac{5}{11}\frac{J_{11}^2}{P_2} + \vartheta_{1,4}(q),$$
$$Q^{\text{th}}_{2,4}(q) = \frac{1}{11}[-5,-6,5,27,-6;0]_4 = \frac{6}{11}\frac{J_{11}^2}{P_2} + [-1,0,1,3,0;0]_4 =: \frac{6}{11}\frac{J_{11}^2}{P_2} + \vartheta_{2,4}(q),$$
$$Q^{\text{th}}_{3,4}(q) = \frac{1}{11}[6,5,16,-17,-17;0]_4 = \frac{17}{11}\frac{J_{11}^2}{P_2} + [-1,2,3,0,0;0]_4 =: \frac{17}{11}\frac{J_{11}^2}{P_2} + \vartheta_{3,4}(q),$$
$$Q^{\text{th}}_{4,4}(q) = \frac{1}{11}[-5,-6,-17,27,-6;-11]_4 = \frac{6}{11}\frac{J_{11}^2}{P_2} + [-1,0,-1,3,0;-1]_4 =: \frac{6}{11}\frac{J_{11}^2}{P_2} + \vartheta_{4,4}(q),$$
$$Q^{\text{th}}_{5,4}(q) = \frac{1}{11}[-5,5,5,-28,16;11]_4 = -\frac{5}{11}\frac{J_{11}^2}{P_2} + [0,0,0,-3,1;1]_4 =: -\frac{5}{11}\frac{J_{11}^2}{P_2} + \vartheta_{5,4}(q).$$

Dissection elements for residue 5 are

$$Q_{0,5}(q) = \frac{1}{11}[4,18,-4,-4,-4;22]_5 = \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + [0,2,0,0,0;2]_5 =: \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + \vartheta_{0,5}(q),$$
$$Q_{1,5}(q) = \frac{1}{11}[4,-15,7,7,-4;0]_5 = \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + [0,-1,1,1,0;0]_5 =: \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + \vartheta_{1,5}(q),$$
$$Q_{2,5}(q) = \frac{1}{11}[4,7,-4,7,7;-22]_5 = -\frac{7}{11}\frac{J_{11}^2P_4}{P_2P_5} + [1,0,-1,0,0;-2]_5 =: -\frac{7}{11}\frac{J_{11}^2P_4}{P_2P_5} + \vartheta_{2,5}(q),$$
$$Q_{3,5}(q) = \frac{1}{11}[-7,-4,-4,7,7;11]_5 = -\frac{7}{11}\frac{J_{11}^2P_4}{P_2P_5} + [0,-1,-1,0,0;1]_5 =: -\frac{7}{11}\frac{J_{11}^2P_4}{P_2P_5} + \vartheta_{3,5}(q),$$
$$Q_{4,5}(q) = \frac{1}{11}[4,7,7,-37,-4;11]_5 = \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + [0,1,1,-3,0;1]_5 =: \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + \vartheta_{4,5}(q),$$
$$Q_{5,5}(q) = \frac{1}{11}[-7,-4,-4,18,-4;-11]_5 = \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + [-1,0,0,2,0;-1]_5 =: \frac{4}{11}\frac{J_{11}^2P_4}{P_2P_5} + \vartheta_{5,5}(q).$$

The mock parts of the dissection elements for residue 7 are stated in (4.6) and (4.7). The theta parts of the dissection elements for residue 7 are

$$Q^{\text{th}}_{0,7}(q) = \frac{1}{11}[18,26,4,-18,-18;0]_7 = \frac{18}{11}\frac{J_{11}^2}{P_3} + [0,4,2,0,0;0]_7 =: \frac{18}{11}\frac{J_{11}^2}{P_3} + \vartheta_{0,7}(q),$$
$$Q^{\text{th}}_{1,7}(q) = \frac{1}{11}[-4,-29,-7,4,37;11]_7 = -\frac{4}{11}\frac{J_{11}^2}{P_3} + [0,-3,-1,0,3;1]_7 =: -\frac{4}{11}\frac{J_{11}^2}{P_3} + \vartheta_{1,7}(q),$$
$$Q^{\text{th}}_{2,7}(q) = \frac{1}{11}[7,26,4,15,-29;-11]_7 = -\frac{4}{11}\frac{J_{11}^2}{P_3} + [1,2,0,1,-3;-1]_7 =: -\frac{4}{11}\frac{J_{11}^2}{P_3} + \vartheta_{2,7}(q),$$
$$Q^{\text{th}}_{3,7}(q) = \frac{1}{11}[-4,-18,4,-7,15;0]_7 = -\frac{4}{11}\frac{J_{11}^2}{P_3} + [0,-2,0,-1,1;0]_7 =: -\frac{4}{11}\frac{J_{11}^2}{P_3} + \vartheta_{3,7}(q),$$
$$Q^{\text{th}}_{4,7}(q) = \frac{1}{11}[-4,15,-7,-7,4;0]_7 = -\frac{4}{11}\frac{J_{11}^2}{P_3} + [0,1,-1,-1,0;0]_7 =: -\frac{4}{11}\frac{J_{11}^2}{P_3} + \vartheta_{4,7}(q),$$
$$Q^{\text{th}}_{5,7}(q) = \frac{1}{11}[-4,-7,4,4,-18;0]_7 = -\frac{4}{11}\frac{J_{11}^2}{P_3} + [0,-1,0,0,-2;0]_7 =: -\frac{4}{11}\frac{J_{11}^2}{P_3} + \vartheta_{5,7}(q).$$

Dissection elements for residue 8 are

$$Q_{0,8}(q) = \frac{1}{11}[38,6,6,-16,6;0]_8 = -\frac{6}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + [4,0,0,-2,0;0]_8 =: -\frac{6}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + \vartheta_{0,8}(q),$$
$$Q_{1,8}(q) = \frac{1}{11}[-17,-5,-5,6,6;11]_8 = -\frac{6}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + [-1,-1,-1,0,0;1]_8 =: -\frac{6}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + \vartheta_{1,8}(q),$$
$$Q_{2,8}(q) = \frac{1}{11}[16,6,-5,6,-5;0]_8 = \frac{5}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + [1,1,0,1,0;0]_8 =: \frac{5}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + \vartheta_{2,8}(q),$$
$$Q_{3,8}(q) = \frac{1}{11}[-6,-5,17,-5,-5;0]_8 = \frac{5}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + [-1,0,2,0,0;0]_8 =: \frac{5}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + \vartheta_{3,8}(q),$$
$$Q_{4,8}(q) = \frac{1}{11}[-17,6,-16,-5,-5;0]_8 = \frac{5}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + [-2,1,-1,0,0;0]_8 =: \frac{5}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + \vartheta_{4,8}(q),$$
$$Q_{5,8}(q) = \frac{1}{11}[5,-5,6,6,6;-11]_8 = -\frac{6}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + [1,-1,0,0,0;-1]_8 =: -\frac{6}{11}\frac{J_{11}^2 qP_1}{P_4P_5} + \vartheta_{5,8}(q).$$

The mock parts of the dissection elements for residue 9 are stated in (4.8) and (4.9). The theta parts of the dissection elements for residue 9 are

$$Q_{0,9}^{\text{th}}(q) = \frac{1}{11}[14,8,-36,8,8;0]_9 = -\frac{8}{11}\frac{J_{11}^2}{P_4} + [2,0,-4,0,0;0]_9 =: -\frac{8}{11}\frac{J_{11}^2}{P_4} + \vartheta_{0,9}(q),$$
$$Q_{1,9}^{\text{th}}(q) = \frac{1}{11}[3,-14,19,-3,8;0]_9 = \frac{3}{11}\frac{J_{11}^2}{P_4} + [0,-1,2,0,1;0]_9 =: \frac{3}{11}\frac{J_{11}^2}{P_4} + \vartheta_{1,9}(q),$$
$$Q_{2,9}^{\text{th}}(q) = \frac{1}{11}[3,19,8,-3,-3;0]_9 = \frac{3}{11}\frac{J_{11}^2}{P_4} + [0,2,1,0,0;0]_9 =: \frac{3}{11}\frac{J_{11}^2}{P_4} + \vartheta_{2,9}(q),$$
$$Q_{3,9}^{\text{th}}(q) = \frac{1}{11}[3,-14,19,-3,-14;-11]_9 = \frac{3}{11}\frac{J_{11}^2}{P_4} + [0,-1,2,0,-1;-1]_9 =: \frac{3}{11}\frac{J_{11}^2}{P_4} + \vartheta_{3,9}(q),$$
$$Q_{4,9}^{\text{th}}(q) = \frac{1}{11}[-8,8,-14,8,-3;11]_9 = -\frac{8}{11}\frac{J_{11}^2}{P_4} + [0,0,-2,0,-1;1]_9 =: -\frac{8}{11}\frac{J_{11}^2}{P_4} + \vartheta_{4,9}(q),$$
$$Q_{5,9}^{\text{th}}(q) = \frac{1}{11}[-8,-3,-14,-3,8;0]_9 = \frac{3}{11}\frac{J_{11}^2}{P_4} + [-1,0,-1,0,1;0]_9 =: \frac{3}{11}\frac{J_{11}^2}{P_4} + \vartheta_{5,9}(q).$$

The mock parts of the dissection elements for residue 10 are stated in (4.10) and (4.11). The theta parts of the dissection elements for residue 10 are

$$Q_{0,10}^{\text{th}}(q) = \frac{1}{11}[2,-24,20,20,-2;0]_{10} = \frac{2}{11}\frac{J_{11}^2}{P_5} + [0,-2,2,2,0;0]_{10} =: \frac{2}{11}\frac{J_{11}^2}{P_5} + \vartheta_{0,10}(q),$$
$$Q_{1,10}^{\text{th}}(q) = \frac{1}{11}[13,20,-13,-2,-2;0]_{10} = \frac{2}{11}\frac{J_{11}^2}{P_5} + [1,2,-1,0,0;0]_{10} =: \frac{2}{11}\frac{J_{11}^2}{P_5} + \vartheta_{1,10}(q),$$
$$Q_{2,10}^{\text{th}}(q) = \frac{1}{11}[-42,-13,-2,-24,-2;11]_{10} = \frac{2}{11}\frac{J_{11}^2}{P_5} + [-4,-1,0,-2,0;1]_{10} =: \frac{2}{11}\frac{J_{11}^2}{P_5} + \vartheta_{2,10}(q),$$
$$Q_{3,10}^{\text{th}}(q) = \frac{1}{11}[46,9,9,9,9;-11]_{10} = -\frac{9}{11}\frac{J_{11}^2}{P_5} + [5,0,0,0,0;-1]_{10} =: -\frac{9}{11}\frac{J_{11}^2}{P_5} + \vartheta_{3,10}(q),$$
$$Q_{4,10}^{\text{th}}(q) = \frac{1}{11}[-9,-13,-13,20,-2;0]_{10} = \frac{13}{11}\frac{J_{11}^2}{P_5} + [-2,0,0,3,1;0]_{10} =: \frac{13}{11}\frac{J_{11}^2}{P_5} + \vartheta_{4,10}(q),$$
$$Q_{5,10}^{\text{th}}(q) = \frac{1}{11}[-9,9,9,-13,-2;0]_{10} = -\frac{9}{11}\frac{J_{11}^2}{P_5} + [0,0,0,-2,-1;0]_{10} =: -\frac{9}{11}\frac{J_{11}^2}{P_5} + \vartheta_{5,10}(q).$$

In the next proof we will show how to obtain these formulas.

Proof of Theorem 1.3. Recall that the rank generating function has the form

$$R(z;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}.$$

Let the $P_{j,m}(q)$ to be elements of the 11-dissection of $R(\zeta_{11}^j; q)$:

$$R(\zeta_{11}^j; q) =: \sum_{m=0}^{10} P_{j,m}(q^{11}) q^m.$$

So reformulation of this definition is

$$P_{j,m}(q) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{10} N(k, 11, 11n+m) \zeta_{11}^{kj} \right) q^n.$$

Note that $P_{j,m}(q)$ can be found from $P_{1,m}(q)$ by taking ζ_{11}^j instead of ζ_{11}. Recall the definition of the Dedekind eta-function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$$

and the notation of Biagioli [10]

$$f_{p,k}(z) := q^{\frac{(p-2k)^2}{8p}} (q^k; q^p)_\infty (q^{p-k}; q^p)_\infty (q^p; q^p)_\infty,$$
$$j(p, \overrightarrow{n}, z) := \eta(pz)^{n_0} \prod_{k=1}^{\frac{1}{2}(p-1)} f_{p,k}(z)^{n_k}.$$

where $p \geq 1$, $p \nmid k$, $\overrightarrow{n} \in \mathbb{Z}^{\frac{1}{2}(p+1)}$. In [21, (6.13)] Garvan found $P_{1,6}(q)$

$$(q^{11}; q^{11})_\infty \sum_{n=0}^{\infty} \left(\sum_{k=0}^{10} N(k, 11, 11n+6) \zeta_{11}^k \right) q^{n+1} = \sum_{r=1}^{5} c_{6,r} j(11, \pi_r(\overrightarrow{n}), z) \tag{4.13}$$

where $c_{6,r} \in \mathbb{Z}[\zeta_{11}]$ are given explicitly in [21, Section 6.4], $\overrightarrow{n} = (15, -4, -2, -3, -2, -2)$ and π_r is the permutation on $\{1,2,3,4,5\}$, defined as $\pi_r(i) = i'$, where $ri' \equiv \pm i \pmod{11}$. As described in [22, (1.9)] for $p > 3$ prime, $1 \leq a \leq \frac{1}{2}(p-1)$ we denote

$$\Phi_{p,a}(q) := \begin{cases} \sum\limits_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1} (q^{p-a};q^p)_n}, & \text{if } 0 < 6a < p, \\ -1 + \sum\limits_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1} (q^{p-a};q^p)_n}, & \text{if } p < 6a < 3p. \end{cases}$$

Recently Garvan and Sarma found $P_{1,7}(q)$ [22, (7.5)]

$$q^{\frac{1}{11}} (q^{11}; q^{11})_\infty \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{10} N(k, 11, 11n+7) \zeta_{11}^k \right) q^{n+1} + a_7 q^{-1} \Phi_{11,5}(q) \right)$$
$$= \frac{f_{11,5}(z)}{f_{11,1}(z)} \sum_{r=1}^{5} c_{7,r} j(11, \pi_r(\overrightarrow{n}), z) + \frac{f_{11,4}(z)}{f_{11,5}(z)} \sum_{r=1}^{5} d_{7,r} j(11, \pi_r(\overrightarrow{n}), z) \tag{4.14}$$

where $a_7, c_{7,r}, d_{7,r} \in \mathbb{Z}[\zeta_{11}]$ are given explicitly in [22, Section 7.1.2]. Also they found $P_{1,8}(q)$ [22, (7.6)]

$$q^{\frac{2}{11}}(q^{11};q^{11})_\infty \sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n+8)\zeta_{11}^k\right) q^{n+1}$$
$$= \frac{f_{11,4}(z)}{f_{11,1}(z)}\sum_{r=1}^{5} c_{8,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,3}(z)}{f_{11,4}(z)}\sum_{r=1}^{5} d_{8,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.15)$$

where $c_{8,r}, d_{8,r} \in \mathbb{Z}[\zeta_{11}]$ are given explicitly in [22, Section 7.1.3]. Then using [22, Theorem 4.11] we can find other $P_{1,m}(q)$ using formulas for $P_{1,7}(q)$ and $P_{1,8}(q)$. We derive

$$q^{\frac{5}{11}}(q^{11};q^{11})_\infty \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n)\zeta_{11}^k\right) q^{n} + a_0\Phi_{11,2}(q)\right)$$
$$= \frac{f_{11,2}(z)}{f_{11,4}(z)}\sum_{r=1}^{5} c_{0,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,5}(z)}{f_{11,2}(z)}\sum_{r=1}^{5} d_{0,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.16)$$

$$q^{\frac{6}{11}}(q^{11};q^{11})_\infty \sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n+1)\zeta_{11}^k\right) q^{n}$$
$$= \frac{f_{11,2}(z)}{f_{11,5}(z)}\sum_{r=1}^{5} c_{1,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,4}(z)}{f_{11,2}(z)}\sum_{r=1}^{5} d_{1,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.17)$$

$$q^{\frac{7}{11}}(q^{11};q^{11})_\infty \sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n+2)\zeta_{11}^k\right) q^{n}$$
$$= \frac{f_{11,1}(z)}{f_{11,3}(z)}\sum_{r=1}^{5} c_{2,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,2}(z)}{f_{11,1}(z)}\sum_{r=1}^{5} d_{2,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.18)$$

$$q^{\frac{8}{11}}(q^{11};q^{11})_\infty \sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n+3)\zeta_{11}^k\right) q^{n}$$
$$= \frac{f_{11,3}(z)}{f_{11,2}(z)}\sum_{r=1}^{5} c_{3,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,5}(z)}{f_{11,3}(z)}\sum_{r=1}^{5} d_{3,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.19)$$

$$q^{\frac{9}{11}}(q^{11};q^{11})_\infty \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n+4)\zeta_{11}^k\right) q^{n} + a_4 q^{-1}\Phi_{11,4}(q)\right)$$
$$= \frac{f_{11,4}(z)}{f_{11,3}(z)}\sum_{r=1}^{5} c_{4,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,1}(z)}{f_{11,4}(z)}\sum_{r=1}^{5} d_{4,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.20)$$

$$q^{\frac{10}{11}}(q^{11};q^{11})_\infty \sum_{n=0}^{\infty}\left(\sum_{k=0}^{10} N(k,11,11n+5)\zeta_{11}^k\right) q^{n}$$
$$= \frac{f_{11,5}(z)}{f_{11,4}(z)}\sum_{r=1}^{5} c_{5,r} j(11,\pi_r(\overrightarrow{n}),z) + \frac{f_{11,1}(z)}{f_{11,5}(z)}\sum_{r=1}^{5} d_{5,r} j(11,\pi_r(\overrightarrow{n}),z), \quad (4.21)$$

$$q^{\frac{3}{11}}(q^{11};q^{11})_\infty\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{10}N(k,11,11n+9)\zeta_{11}^k\right)q^{n+1}+a_9\Phi_{11,3}(q)\right)$$
$$=\frac{f_{11,3}(z)}{f_{11,5}(z)}\sum_{r=1}^{5}c_{9,r}j(11,\pi_r(\overrightarrow{n}),z)+\frac{f_{11,2}(z)}{f_{11,3}(z)}\sum_{r=1}^{5}d_{9,r}j(11,\pi_r(\overrightarrow{n}),z),\quad (4.22)$$

$$q^{\frac{4}{11}}(q^{11};q^{11})_\infty\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{10}N(k,11,11n+10)\zeta_{11}^k\right)q^{n+1}+a_{10}\Phi_{11,1}(q)\right)$$
$$=\frac{f_{11,1}(z)}{f_{11,2}(z)}\sum_{r=1}^{5}c_{10,r}j(11,\pi_r(\overrightarrow{n}),z)+\frac{f_{11,3}(z)}{f_{11,1}(z)}\sum_{r=1}^{5}d_{10,r}j(11,\pi_r(\overrightarrow{n}),z),\quad (4.23)$$

where $a_i, c_{i,r}, d_{i,r}\in\mathbb{Z}[\zeta_{11}]$ can be found explicitly from $a_7, c_{7,r}, d_{7,r}$ and $c_{8,r}, d_{8,r}$ by [22, Theorem 4.11].

Then we transfer formulas (4.13)-(4.23) to a different notation using

$$f_{p,k}(z)=q^{\frac{(p-2k)^2}{8p}}J_{k,p}$$

and

$$\Phi_{p,a}(q)=\begin{cases}1+q^a g(q^a;q^p), & \text{if } 0<6a<p,\\ q^a g(q^a;q^p), & \text{if } p<6a<3p.\end{cases}$$

Recall the formula for the deviation of the rank [29, (2.10)]

$$D(a,r)=\frac{1}{r}\sum_{j=1}^{r-1}\zeta_r^{-aj}\left(1-\zeta_r^j\right)\left(1+\zeta_r^j g(\zeta_r^j;q)\right)=\frac{1}{r}\sum_{j=1}^{r-1}\zeta_r^{-aj}R(\zeta_r^j;q).$$

As with the crank, if we know the 11-dissection elements $P_{j,m}(q)$ for $R(\zeta_r^j;q)$, we can deduce the 11-dissection elements $Q_{a,m}(q)$ for $D(a,r)$ using the previous formula. We defined earlier

$$\vartheta(a_1,a_2,a_3,a_4,a_5):=\frac{J_{11}^6}{J_1^2}\Big[a_1\frac{q^2}{P_4P_5^2}+a_2\frac{1}{P_1^2P_3}+a_3\frac{q}{P_1P_4^2}+a_4\frac{q}{P_2^2P_5}+a_5\frac{q}{P_2P_3^2}\Big].\quad (4.24)$$

Also recall an important identity, which can be obtained by rearranging terms:

$$P_1P_2P_3P_4P_5=J_1J_{11}^4.\quad (4.25)$$

Note that (4.24) corresponds to the right-hand side of (4.13) after changing the notation and applying (4.25):

$$j(11,\pi_1(\overrightarrow{n}),z)=\frac{qJ_{11}^{15}}{P_1^4P_2^2P_3^3P_4^2P_5^2}=\frac{J_{11}^7}{J_1^2}\cdot\frac{q}{P_1^2P_3},$$
$$j(11,\pi_2(\overrightarrow{n}),z)=\frac{q^2J_{11}^{15}}{P_1^2P_2^4P_3^2P_4^2P_5^3}=\frac{J_{11}^7}{J_1^2}\cdot\frac{q^2}{P_2^2P_5},$$
$$j(11,\pi_3(\overrightarrow{n}),z)=\frac{q^2J_{11}^{15}}{P_1^2P_2^3P_3^4P_4^2P_5^2})=\frac{J_{11}^7}{J_1^2}\cdot\frac{q^2}{P_2P_3^2},$$
$$j(11,\pi_4(\overrightarrow{n}),z)=\frac{q^2J_{11}^{15}}{P_1^3P_2^2P_3^2P_4^4P_5^2}=\frac{J_{11}^7}{J_1^2}\cdot\frac{q^2}{P_1P_4^2},$$
$$j(11,\pi_5(\overrightarrow{n}),z)=\frac{q^3J_{11}^{15}}{P_1^2P_2^2P_3^2P_4^3P_5^4}=\frac{J_{11}^7}{J_1^2}\cdot\frac{q^3}{P_4P_5^2}.$$

By calculations we see from (4.13) that

$$Q_{0,6}(q) = \frac{1}{11}\sum_{j=1}^{10} P_{j,6}(q) = \vartheta(0,0,2,2,-2),$$

$$Q_{1,6}(q) = \frac{1}{11}\sum_{j=1}^{10} \zeta_{11}^{-j} P_{j,6}(q) = \vartheta(-1,1,-1,-2,1),$$

$$Q_{2,6}(q) = \frac{1}{11}\sum_{j=1}^{10} \zeta_{11}^{-2j} P_{j,6}(q) = \vartheta(1,0,-1,2,0),$$

$$Q_{3,6}(q) = \frac{1}{11}\sum_{j=1}^{10} \zeta_{11}^{-3j} P_{j,6}(q) = \vartheta(1,0,1,-1,-1),$$

$$Q_{4,6}(q) = \frac{1}{11}\sum_{j=1}^{10} \zeta_{11}^{-4j} P_{j,6}(q) = \vartheta(0,-1,1,0,2),$$

$$Q_{5,6}(q) = \frac{1}{11}\sum_{j=1}^{10} \zeta_{11}^{-5j} P_{j,6}(q) = \vartheta(-1,0,-1,0,-1).$$

In the same way from (4.14) we see that

$$\begin{aligned}
Q_{0,7}(q) &= q^{-1}\frac{P_5}{11P_1}\vartheta(-18,0,-18,26,10) + \frac{P_4}{11P_5}\vartheta(0,0,0,0,4),\\
Q_{1,7}(q) &= -q^3 g(q^5;q^{11}) + q^{-1}\frac{P_5}{11P_1}\vartheta(37,0,4,-29,21) + \frac{P_4}{11P_5}\vartheta(11,0,0,0,-7),\\
Q_{2,7}(q) &= q^3 g(q^5;q^{11}) + q^{-1}\frac{P_5}{11P_1}\vartheta(-29,0,15,26,-34) + \frac{P_4}{11P_5}\vartheta(-11,0,0,0,4),\\
Q_{3,7}(q) &= q^{-1}\frac{P_5}{11P_1}\vartheta(15,0,-7,-18,21) + \frac{P_4}{11P_5}\vartheta(0,0,0,0,4),\\
Q_{4,7}(q) &= q^{-1}\frac{P_5}{11P_1}\vartheta(4,0,-7,15,-12) + \frac{P_4}{11P_5}\vartheta(0,0,0,0,-7),\\
Q_{5,7}(q) &= q^{-1}\frac{P_5}{11P_1}\vartheta(-18,0,4,-7,-1) + \frac{P_4}{11P_5}\vartheta(0,0,0,0,4).
\end{aligned} \tag{4.26}$$

From (4.15) we see that

$$\begin{aligned}
Q_{0,8}(q) &= q^{-1}\frac{P_4}{11P_1}\vartheta(6,0,42,-36,-6) + \frac{P_3}{11P_4}\vartheta(0,0,-16,6,0),\\
Q_{1,8}(q) &= q^{-1}\frac{P_4}{11P_1}\vartheta(6,0,-24,30,5) + \frac{P_3}{11P_4}\vartheta(0,0,6,-5,0),\\
Q_{2,8}(q) &= q^{-1}\frac{P_4}{11P_1}\vartheta(-5,0,9,-14,5) + \frac{P_3}{11P_4}\vartheta(0,0,6,6,0),\\
Q_{3,8}(q) &= q^{-1}\frac{P_4}{11P_1}\vartheta(-5,0,9,8,-17) + \frac{P_3}{11P_4}\vartheta(0,0,-5,-5,0),\\
Q_{4,8}(q) &= q^{-1}\frac{P_4}{11P_1}\vartheta(-5,0,-13,-3,16) + \frac{P_3}{11P_4}\vartheta(0,0,-5,6,0),\\
Q_{5,8}(q) &= q^{-1}\frac{P_4}{11P_1}\vartheta(6,0,-2,-3,-6) + \frac{P_3}{11P_4}\vartheta(0,0,6,-5,0).
\end{aligned} \tag{4.27}$$

Formulas for $Q_{a,m}(q)$ in terms of $\vartheta(a_1,a_2,a_3,a_4,a_5)$ for other residues can be also of interest, so we write them explicitly. From (4.16) we see that

$$\begin{aligned}
Q_{0,0}(q) &= -2q^2g(q^2,q^{11}) + q\frac{P_2}{11P_4}\vartheta(-10,-52,0,56,32) + \frac{P_5}{11P_2}\vartheta(0,10,0,22,0),\\
Q_{1,0}(q) &= q^2g(q^2,q^{11}) + q\frac{P_2}{11P_4}\vartheta(1,25,0,-10,-23) + \frac{P_5}{11P_2}\vartheta(0,-1,0,-11,0),\\
Q_{2,0}(q) &= q\frac{P_2}{11P_4}\vartheta(1,14,0,-32,-1) + \frac{P_5}{11P_2}\vartheta(0,-1,0,0,0),\\
Q_{3,0}(q) &= q\frac{P_2}{11P_4}\vartheta(12,-8,0,23,21) + \frac{P_5}{11P_2}\vartheta(0,-1,0,0,0),\\
Q_{4,0}(q) &= q\frac{P_2}{11P_4}\vartheta(1,3,0,-10,-23) + \frac{P_5}{11P_2}\vartheta(0,-1,0,0,0),\\
Q_{5,0}(q) &= q\frac{P_2}{11P_4}\vartheta(-10,-8,0,1,10) + \frac{P_5}{11P_2}\vartheta(0,-1,0,0,0).
\end{aligned}$$

From (4.17) we see that

$$\begin{aligned}
Q_{0,1}(q) &= q\frac{P_2}{11P_5}\vartheta(0,-6,-12,-4,-12) + \frac{P_4}{11P_2}\vartheta(0,10,0,-12,0),\\
Q_{1,1}(q) &= q\frac{P_2}{11P_5}\vartheta(0,5,-1,-4,10) + \frac{P_4}{11P_2}\vartheta(0,-1,0,10,0),\\
Q_{2,1}(q) &= q\frac{P_2}{11P_5}\vartheta(0,16,10,-15,-1) + \frac{P_4}{11P_2}\vartheta(0,-1,0,-12,0),\\
Q_{3,1}(q) &= q\frac{P_2}{11P_5}\vartheta(0,-6,-1,18,-12) + \frac{P_4}{11P_2}\vartheta(0,-1,0,10,0),\\
Q_{4,1}(q) &= q\frac{P_2}{11P_5}\vartheta(0,-6,-12,18,10) + \frac{P_4}{11P_2}\vartheta(0,-1,0,-1,0),\\
Q_{5,1}(q) &= q\frac{P_2}{11P_5}\vartheta(0,-6,10,-15,-1) + \frac{P_4}{11P_2}\vartheta(0,-1,0,-1,0).
\end{aligned}$$

From (4.18) we see that

$$\begin{aligned}
Q_{0,2}(q) &= q\frac{P_1}{11P_3}\vartheta(-12,30,2,-20,0) + \frac{P_2}{11P_1}\vartheta(2,-2,0,0,0),\\
Q_{1,2}(q) &= q\frac{P_1}{11P_3}\vartheta(-12,-36,2,13,0) + \frac{P_2}{11P_1}\vartheta(2,9,0,0,0),\\
Q_{2,2}(q) &= q\frac{P_1}{11P_3}\vartheta(21,19,-9,-9,0) + \frac{P_2}{11P_1}\vartheta(2,-2,0,0,0),\\
Q_{3,2}(q) &= q\frac{P_1}{11P_3}\vartheta(-1,8,2,13,0) + \frac{P_2}{11P_1}\vartheta(-9,-2,0,0,0),\\
Q_{4,2}(q) &= q\frac{P_1}{11P_3}\vartheta(10,-3,13,-9,0) + \frac{P_2}{11P_1}\vartheta(2,-2,0,0,0),\\
Q_{5,2}(q) &= q\frac{P_1}{11P_3}\vartheta(-12,-3,-9,2,0) + \frac{P_2}{11P_1}\vartheta(2,-2,0,0,0).
\end{aligned}$$

From (4.19) we see that

$$\begin{aligned}
Q_{0,3}(q) &= \frac{P_3}{11P_2}\vartheta(14,8,4,0,-12) + \frac{P_5}{11P_3}\vartheta(0,0,-8,0,-8),\\
Q_{1,3}(q) &= \frac{P_3}{11P_2}\vartheta(3,-3,4,0,21) + \frac{P_5}{11P_3}\vartheta(0,0,3,0,3),\\
Q_{2,3}(q) &= \frac{P_3}{11P_2}\vartheta(-8,8,-18,0,-12) + \frac{P_5}{11P_3}\vartheta(0,0,3,0,3),\\
Q_{2,3}(q) &= \frac{P_3}{11P_2}\vartheta(-8,-3,15,0,-1) + \frac{P_5}{11P_3}\vartheta(0,0,3,0,-8),\\
Q_{2,4}(q) &= \frac{P_3}{11P_2}\vartheta(3,-3,15,0,-23) + \frac{P_5}{11P_3}\vartheta(0,0,-8,0,14),\\
Q_{2,5}(q) &= \frac{P_3}{11P_2}\vartheta(3,-3,-18,0,21) + \frac{P_5}{11P_3}\vartheta(0,0,3,0,-8).
\end{aligned}$$

From (4.20) we see that

$$\begin{aligned}
Q_{0,4}(q) &= \frac{P_4}{11P_3}\vartheta(16,6,-28,26,0) + q\frac{P_1}{11P_4}\vartheta(0,0,0,-6,0),\\
Q_{1,4}(q) &= \frac{P_4}{11P_3}\vartheta(5,6,5,-18,0) + q\frac{P_1}{11P_4}\vartheta(0,0,0,5,0),\\
Q_{2,4}(q) &= \frac{P_4}{11P_3}\vartheta(-6,-5,27,4,0) + q\frac{P_1}{11P_4}\vartheta(0,0,0,-6,0),\\
Q_{3,4}(q) &= \frac{P_4}{11P_3}\vartheta(-17,6,-17,-7,0) + q\frac{P_1}{11P_4}\vartheta(0,0,0,5,0),\\
Q_{4,4}(q) &= q^3g(q^4;q^{11}) + \frac{P_4}{11P_3}\vartheta(-6,-5,27,-18,0) + q\frac{P_1}{11P_4}\vartheta(0,0,-11,-6,0),\\
Q_{5,4}(q) &= -q^3g(q^4;q^{11}) + \frac{P_4}{11P_3}\vartheta(16,-5,-28,26,0) + q\frac{P_1}{11P_4}\vartheta(0,0,11,5,0).
\end{aligned}$$

From (4.21) we see that

$$\begin{aligned}
Q_{0,5}(q) &= \frac{P_5}{11P_4}\vartheta(6,4,0,-18,24) + q\frac{P_1}{11P_5}\vartheta(-4,0,0,0,4),\\
Q_{1,5}(q) &= \frac{P_5}{11P_4}\vartheta(-5,4,0,15,-9) + q\frac{P_1}{11P_5}\vartheta(-4,0,0,0,-7),\\
Q_{2,5}(q) &= \frac{P_5}{11P_4}\vartheta(17,4,0,-7,-9) + q\frac{P_1}{11P_5}\vartheta(7,0,0,0,4),\\
Q_{3,5}(q) &= \frac{P_5}{11P_4}\vartheta(6,-7,0,4,24) + q\frac{P_1}{11P_5}\vartheta(7,0,0,0,4),\\
Q_{4,5}(q) &= \frac{P_5}{11P_4}\vartheta(-27,4,0,-7,-20) + q\frac{P_1}{11P_5}\vartheta(-4,0,0,0,-7),\\
Q_{5,5}(q) &= \frac{P_5}{11P_4}\vartheta(6,-7,0,4,2) + q\frac{P_1}{11P_5}\vartheta(-4,0,0,0,4).
\end{aligned}$$

From (4.22) we see that

$$
\begin{aligned}
Q_{0,9}(q) &= \frac{P_3}{11P_5}\vartheta(0,14,2,-8,-36) + \frac{P_2}{11P_3}\vartheta(0,0,8,0,0),\\
Q_{1,9}(q) &= \frac{P_3}{11P_5}\vartheta(0,3,-9,14,19) + \frac{P_2}{11P_3}\vartheta(0,0,-3,0,0),\\
Q_{2,9}(q) &= \frac{P_3}{11P_5}\vartheta(0,3,13,-19,8) + \frac{P_2}{11P_3}\vartheta(0,0,-3,0,0),\\
Q_{3,9}(q) &= q^2 g(q^3;q^{11}) + \frac{P_3}{11P_5}\vartheta(0,3,-31,14,19) + \frac{P_2}{11P_3}\vartheta(0,0,-3,0,-11),\\
Q_{4,9}(q) &= -q^2 g(q^3;q^{11}) + \frac{P_3}{11P_5}\vartheta(0,-8,13,-8,-14) + \frac{P_2}{11P_3}\vartheta(0,0,8,0,11),\\
Q_{5,9}(q) &= \frac{P_3}{11P_5}\vartheta(0,-8,13,3,-14) + \frac{P_2}{11P_3}\vartheta(0,0,-3,0,0).
\end{aligned}
$$

From (4.23) we see that

$$
\begin{aligned}
Q_{0,10}(q) &= \frac{P_1}{11P_2}\vartheta(16,2,20,0,20) + q^{-1}\frac{P_3}{11P_1}\vartheta(-2,0,0,0,0),\\
Q_{1,10}(q) &= \frac{P_1}{11P_2}\vartheta(5,13,-2,0,-13) + q^{-1}\frac{P_3}{11P_1}\vartheta(-2,0,0,0,0),\\
Q_{2,10}(q) &= -[q^{-1} + g(q;q^{11})] + \frac{P_1}{11P_2}\vartheta(-39,-42,-24,0,-2) + q^{-1}\frac{P_3}{11P_1}\vartheta(-2,11,0,0,0),\\
Q_{3,10}(q) &= q^{-1} + g(q;q^{11}) + \frac{P_1}{11P_2}\vartheta(27,46,9,0,9) + q^{-1}\frac{P_3}{11P_1}\vartheta(9,-11,0,0,0),\\
Q_{4,10}(q) &= \frac{P_1}{11P_2}\vartheta(-6,-9,20,0,-13) + q^{-1}\frac{P_3}{11P_1}\vartheta(-2,0,0,0,0),\\
Q_{5,10}(q) &= \frac{P_1}{11P_2}\vartheta(5,-9,-13,0,9) + q^{-1}\frac{P_3}{11P_1}\vartheta(-2,0,0,0,0).
\end{aligned}
$$

To obtain $Q_{a,m}$ in the form $[c_1,c_2,c_3,c_4,c_5;c_6]_m$, we need to use the three-term Weierstrass relation for theta functions [24, Proposition 2.1]

$$P_{a+c}P_{a-c}P_{b+d}P_{b-d} = P_{a+d}P_{a-d}P_{b+c}P_{b-c} + q^{b-c}P_{a+b}P_{a-b}P_{c+d}P_{c-d}.$$

Let us consider ten cases of the three-term Weierstrass relation, mentioned in [7, (4.6)-(4.10)] and [17, (b1)-(b5)]:

$$P_2P_4P_5^2 - P_3^2P_4P_5 + q^2P_1^2P_2P_3 = 0, \tag{4.28}$$

$$P_1P_4P_5^2 - P_2P_3P_4^2 + qP_1P_2P_3^2 = 0, \tag{4.29}$$

$$P_1P_3P_5^2 - P_2^2P_4P_5 + qP_1^2P_3P_4 = 0, \tag{4.30}$$

$$P_1P_4^2P_5 - P_2P_3^2P_5 + qP_1P_2^2P_4 = 0, \tag{4.31}$$

$$P_1P_3P_4^2 - P_2^2P_3P_5 + qP_1^2P_2P_5 = 0, \tag{4.32}$$

and

$$P_3P_5^3 - P_5P_4^3 + q^3P_2P_1^3 = 0, \tag{4.33}$$

$$P_2P_5^3 - P_3P_4^3 + q^2P_1P_2^3 = 0, \tag{4.34}$$

$$P_2P_4^3 - P_5P_3^3 + q^2P_4P_1^3 = 0, \tag{4.35}$$

$$P_1P_5^3 - P_4P_3^3 + qP_3P_2^3 = 0, \tag{4.36}$$

$$P_1P_3^3 - P_4P_2^3 + qP_5P_1^3 = 0. \tag{4.37}$$

Also note that another form of ϑ can be obtained by applying (4.25):

$$\vartheta(a_1,a_2,a_3,a_4,a_5)=\frac{J_{11}^2}{J_1^3}\Big[a_1\frac{q^2P_1P_2P_3}{P_5}+a_2\frac{P_2P_4P_5}{P_1}+a_3\frac{qP_2P_3P_5}{P_4}+a_4\frac{qP_1P_3P_4}{P_2}+a_5\frac{qP_1P_4P_5}{P_3}\Big].$$

Let us consider the case residue 7. From (4.26) we have

$$\begin{aligned}Q^{\mathrm{th}}_{a,7}(q)=q^{-1}\frac{P_5}{11P_1}\vartheta(a_1,0,a_3,a_4,a_5)+\frac{P_4}{11P_5}\vartheta(b_1,0,0,0,b_5)=\\ \frac{J_{11}^2}{J_1^3}\left(a_1qP_2P_3+a_3\frac{P_2P_3P_5^2}{P_1P_4}+a_4\frac{P_3P_4P_5}{P_2}+a_5\frac{P_4P_5^2}{P_3}+b_1\frac{q^2P_1P_2P_3P_4}{P_5^2}+b_5\frac{qP_1P_4^2}{P_3}\right).\end{aligned}$$

Then we consider

$$\begin{aligned}(4.31)\times\frac{P_5}{P_1P_3P_4}\quad&:\quad\frac{P_2P_3P_5^2}{P_1P_4}=\frac{P_4P_5^2}{P_3}+\frac{qP_2^2P_5}{P_3},\\(4.28)\times\frac{1}{P_2P_3}\quad&:\quad\frac{P_3P_4P_5}{P_2}=q^2P_1^2+\frac{P_4P_5^2}{P_3}.\end{aligned}$$

After changing the terms to the new ones using the above expressions, we obtain

$$Q^{\mathrm{th}}_{a,7}(q)=\frac{J_{11}^2}{J_1^3}\left(c_1\frac{P_4P_5^2}{P_3}+c_2q^2P_1^2+c_3\frac{qP_1P_4^2}{P_3}+c_4\frac{qP_2^2P_5}{P_3}+c_5qP_2P_3+c_6\frac{q^2P_1P_2P_3P_4}{P_5^2}\right).$$

As a result, we obtain $[c_1,c_2,c_3,c_4,c_5;c_6]_7$. Similar calculations can be done for residues $0,4,9,10$.

Another case we are going to consider as an example is residue 8. From (4.27) we have

$$\begin{aligned}Q_{a,8}(q)=q^{-1}\frac{P_4}{11P_1}\vartheta(a_1,0,a_2,a_3,a_4)+\frac{P_3}{11P_4}\vartheta(0,0,b_3,b_4,0)=\\ \frac{J_{11}^2}{J_1^3}\left(a_1\frac{qP_2P_3P_4}{P_5}+a_2\frac{P_2P_3P_5}{P_1}+a_3\frac{P_3P_4^2}{P_2}+a_4\frac{P_4^2P_5}{P_3}+b_3\frac{qP_2P_3^2P_5}{P_4^2}+b_4\frac{qP_1P_3^2}{P_2}\right).\end{aligned}$$

Then we consider

$$\begin{aligned}(4.29)\times\frac{q}{P_4P_5}\quad&:\quad\frac{qP_2P_3P_4}{P_5}=qP_1P_5+\frac{q^2P_1P_2P_3^2}{P_4P_5},\\(4.32)\times\frac{1}{P_1P_2}\quad&:\quad\frac{P_2P_3P_5}{P_1}=\frac{P_3P_4^2}{P_2}+qP_1P_5,\\(4.28)\times\frac{P_4}{P_2P_3P_5}\quad&:\quad\frac{P_4^2P_5}{P_3}=\frac{P_3P_4^2}{P_2}-\frac{q^2P_1^2P_4}{P_5},\\(4.31)\times\frac{q}{P_4^2}\quad&:\quad\frac{qP_2P_3^2P_5}{P_4^2}=qP_1P_5+\frac{q^2P_1P_2^2}{P_4},\\(4.28)\times\frac{qP_1}{P_2P_4P_5}\quad&:\quad\frac{qP_1P_3^2}{P_2}=\frac{q^3P_1^3P_3}{P_4P_5}+qP_1P_5.\end{aligned}$$

After changing the terms to the new ones using the above expressions, we obtain

$$Q_{a,8}(q)=c_1qP_1P_5+c_2\frac{q^3P_1^3P_3}{P_4P_5}+c_3\frac{q^2P_1^2P_4}{P_5}+c_4\frac{q^2P_1P_2^2}{P_4}+c_5\frac{q^2P_1P_2P_3^2}{P_4P_5}+c_6\frac{P_3P_4^2}{P_2}.$$

As a result, we obtain $[c_1,c_2,c_3,c_4,c_5;c_6]_8$. Similar calculations can be done for residues $1,2,3,5$. □

5. Properties of partition function, crank and rank modulo 11

5.1. **Equalities between cranks modulo 11.** Garvan [20, (1.51)-(1.67)] found equalities between cranks modulo 11. We present his results in the following theorem.

Theorem 5.1 ([20, (1.51)-(1.67)]). *Consider $n \geq 0$. For $M_i = M(i, 11, 11n)$ we have*

$$M_1 = M_2 = M_3 = M_4 = M_5.$$

For $M_i = M(i, 11, 11n+1)$ we have

$$M_0 + M_1 = 2M_2 \quad \text{and} \quad M_2 = M_3 = M_4 = M_5.$$

For $M_i = M(i, 11, 11n+2)$ we have

$$M_0 = M_1 = M_3 = M_4 = M_5.$$

For $M_i = M(i, 11, 11n+3)$ we have

$$M_0 = M_3 \quad \text{and} \quad M_1 = M_2 = M_4 = M_5.$$

For $M_i = M(i, 11, 11n+4)$ we have

$$M_0 = M_2 = M_4 \quad \text{and} \quad M_1 = M_3 = M_5.$$

For $M_i = M(i, 11, 11n+5)$ we have

$$M_0 = M_1 = M_3 = M_5 \quad \text{and} \quad M_2 = M_4.$$

For $M_i = M(i, 11, 11n+6)$ we have

$$M_0 = M_1 = M_2 = M_3 = M_4 = M_5 = \frac{p(11n+6)}{11}.$$

For $M_i = M(i, 11, 11n+7)$ we have

$$M_0 = M_2 = M_3 = M_5 \quad \text{and} \quad M_1 = M_4.$$

For $M_i = M(i, 11, 11n+8)$ we have

$$M_0 = M_2 = M_5 \quad \text{and} \quad M_1 = M_3 = M_4.$$

For $N_i = N(i, 11, 11n+9)$ and $M_i = M(i, 11, 11n+9)$ we have

$$M_0 = M_4 \quad \text{and} \quad M_1 = M_2 = M_3 = M_5.$$

For $N_i = N(i, 11, 11n+10)$ and $M_i = M(i, 11, 11n+10)$ we have

$$M_0 = M_1 = M_2 = M_4 = M_5.$$

Proof of Theorem 5.1. By comparing respective coefficients in the 11-dissections in Theorem 1.2 we can directly deduce the following equalities. Also note that there is no element corresponding to q^6 in 11-dissection of the deviation of the crank, so we have the corresponding equality for residue 6. □

5.2. **Partition function congruences modulo** 11**.** Atkin and Swinnerton-Dyer in [7, Theorem 3] realised the following congruences.

Theorem 5.2 ([7, Theorem 3])**.** *We have*

$$\sum_{n=0}^{\infty} p(11n)q^n \equiv \frac{J_{11}^2}{P_1} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+1)q^n \equiv \frac{J_{11}^2 P_5}{P_2 P_3} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+2)q^n \equiv 2\frac{J_{11}^2 P_3}{P_1 P_4} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+3)q^n \equiv 3\frac{J_{11}^2 P_2}{P_1 P_3} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+4)q^n \equiv 5\frac{J_{11}^2}{P_2} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+5)q^n \equiv 7\frac{J_{11}^2 P_4}{P_2 P_5} \pmod{11}$$
$$\sum_{n=0}^{\infty} p(11n+7)q^n \equiv 4\frac{J_{11}^2}{P_3} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+8)q^n \equiv 6\frac{J_{11}^2 q P_1}{P_4 P_5} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+9)q^n \equiv 8\frac{J_{11}^2}{P_4} \pmod{11},$$
$$\sum_{n=0}^{\infty} p(11n+10)q^n \equiv 9\frac{J_{11}^2}{P_5} \pmod{11}.$$

Proof of Theorem 5.2. Using Theorem 1.2 it is obvious how to obtain such congruences. For example, we can take

$$\sum_{n=0}^{\infty} \left(M(0,11,11n) - \frac{p(11n)}{11} \right) q^n = \frac{10}{11} \frac{J_{11}^2}{P_1},$$

multiply it by -11 and take modulo 11. □

Remark 5.3. Elements of 11-dissection for $p(n)$ can be obtained using [25, Lemma 4] and they are found explicitly in terms of theta quotients by Bilgici and Ekin [11]. Another forms for the 11-dissection element $\sum_{n\geq 0} p(11n+6)q^n$, which represent explicitly its modular properties, can be found in [31].

5.3. **Linear rank congruences modulo** 11**.** Atkin and Hussain [6] studied the rank modulo 11 and for each residue they found linear congruences between ranks [6, (9.16)]. We present their results in the following theorem.

Theorem 5.4 ([6, (9.16)])**.** *Consider* $n \geq 0$*. For* $N_i = N(i, 11, 11n)$ *we have*

$$N_2 - 5N_3 - 2N_4 + 6N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n+1)$ *we have*

$$N_1 - 6N_2 + 4N_3 + 3N_4 - 2N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 2)$ *we have*

$$N_0 + 4N_2 - 6N_4 + N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 3)$ *we have*

$$N_0 + 3N_1 - N_2 + 2N_3 - N_4 - 4N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 4)$ *we have*

$$N_0 + 3N_1 - 2N_2 - 4N_3 + N_4 + N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 5)$ *we have*

$$N_0 - 5N_1 - N_2 + N_3 + 5N_4 - N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 6)$ *we have*

$$N_1 - 5N_2 - N_3 + N_4 + 4N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 7)$ *we have*

$$N_0 - 2N_1 - 2N_2 + 5N_3 + 2N_4 - 4N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 8)$ *we have*

$$N_0 + 5N_1 + 2N_2 + N_3 - 3N_4 - 6N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 9)$ *we have*

$$N_0 - 4N_1 + 3N_2 - N_3 - N_4 + 2N_5 \equiv 0 \pmod{11}.$$

For $N_i = N(i, 11, 11n + 10)$ *we have*

$$N_0 - 6N_1 + N_4 + 4N_5 \equiv 0 \pmod{11}.$$

Proof of Theorem 5.4. It is straightforward now to obtain such congruences by using the calculations of the 11-dissection of the deviation of the rank found in Section 4. For example, let us consider the case of residue 0. We find that

$$Q_{2,0}(q) - 5Q_{3,0}(q) - 2Q_{4,0}(q) + 6Q_{5,0}(q) = 11[0, -1, 0, 1, -1; 0]_0.$$

As the sum of coefficients in the above sum is zero, we see that the coefficient of q^n on the left-hand side of the previous sum is

$$N(2, 11, 11n) - 5N(3, 11, 11n) - 2N(4, 11, 11n) + 6N(5, 11, 11n).$$

Taking expression modulo 11 we obtain the desired result. □

6. Rank-crank inequalities, positivity techniques, and positivity conjectures

Firstly we give new proofs of work of Ekin [16] in order to motivate our use of positivity techniques and to give context to our positivity conjectures.

6.1. **Inequalities between cranks modulo** 11. Ekin [16, (21)-(28)] considered inequalities between cranks modulo 11. We state the results in the following theorem and conjecture.

Theorem 6.1 ([16, (21)-(28)]). *For $n \geq 0$ we have*

$$M(0,11,11n) \geq \frac{p(11n)}{11} \geq M(1,11,11n),$$
$$M(1,11,11n+1) \geq \frac{p(11n+1)}{11} \geq M(2,11,11n+1) \geq M(0,11,11n+1),$$
$$M(2,11,11n+2) \geq \frac{p(11n+2)}{11} \geq M(0,11,11n+2),$$
$$M(0,11,11n+3) \geq \frac{p(11n+3)}{11} \geq M(1,11,11n+3),$$
$$M(0,11,11n+4) \geq \frac{p(11n+4)}{11} \geq M(1,11,11n+4),$$
$$M(0,11,11n+5) \geq \frac{p(11n+5)}{11} \geq M(2,11,11n+5),$$
$$M(1,11,11n+7) \geq \frac{p(11n+7)}{11} \geq M(0,11,11n+7),$$
$$M(1,11,11n+9) \geq \frac{p(11n+9)}{11} \geq M(0,11,11n+9),$$
$$M(0,11,11n+10) \geq \frac{p(11n+10)}{11} \geq M(3,11,11n+10).$$

Conjecture 6.2. *For $n \geq 0$, $n \neq 2$ we have*

$$M(1,11,11n+8) \geq \frac{p(11n+8)}{11} \geq M(0,11,11n+8).$$

Lemma 6.3. *We have*

$$\frac{J_{11}^2}{P_a} \geq 0$$

for $a \in \{1,2,3,4,5\}$ and

$$\frac{J_{11}^2 P_{2a}}{P_a P_{3a}} \geq 0$$

for $a \in \{1,2,3,4\}$.

Proof of Lemma 6.3. By the Jacobi's triple product identity we see that for $a \in \{1,2,3,4,5\}$

$$\frac{J_{11}^2}{P_a} = \frac{(q^{11};q^{11})_\infty}{(q^a;q^{11})_\infty (q^{11-a};q^{11})_\infty} = \frac{(-q^a;q^{11})_\infty (-q^{11-a};q^{11})_\infty (q^{11};q^{11})_\infty}{(q^{2a};q^{22})_\infty (q^{22-2a};q^{22})_\infty}$$
$$= \frac{1}{(q^{2a};q^{22})_\infty (q^{22-2a};q^{22})_\infty} \sum_{n=-\infty}^{\infty} q^{11\binom{n}{2}+an}.$$

By the quintuple product identity [16, (42)] we see that for $a \in \{1,2,3\}$

$$\frac{J_{11}^2 P_{2a}}{P_a P_{3a}} = \frac{J_{33}^3}{J_{3a,33} J_{11-3a,33}} + \frac{q^a J_{33}^3}{J_{3a,33} J_{22-3a,33}}. \tag{6.1}$$

In the same way as in the previous case but with change $q \to q^3$ we see that both of the terms in the right-hand side of the expression above have non-negative Fourier coefficients. The case of $a = 4$ is proved in [8, Corollary 4.8] using a different representation of $\frac{J_{11}^2 P_3}{P_1 P_4}$. □

Proof of Theorem 6.1. To obtain Theorem 6.1 we use Lemma 6.3 and refer to the positivity or negativity of the coefficients in Theorem 1.2. □

Remark 6.4. Conjecture 6.2 is partially solved. Using analytic methods it is known from [33] that there is explicit N, such that for $n > N$ we have

$$M(1, 11, 11n + 8) \geq M(0, 11, 11n + 8).$$

6.2. **Positivity of theta quotients.** Denote

$$\tilde{P}_a := (q^a; q^{11})_\infty (q^{11-a}; q^{11})_\infty = \frac{P_a}{J_{11}}.$$

Definition 6.5. We define the notation $F(q) \geq G(q)$ if the Fourier coefficients of $F(q) - G(q)$ are non-negative.

Lemma 6.6. *We have*

$$J_{11}^a q^b \tilde{P}_1^{\alpha_1} \tilde{P}_2^{\alpha_2} \tilde{P}_3^{\alpha_3} \tilde{P}_4^{\alpha_4} \tilde{P}_5^{\alpha_5} \geq 0 \tag{6.2}$$

where $\alpha_i \leq 0$ for $1 \leq i \leq M$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = r$ and $0 \leq a \leq r$, $b \in \mathbb{Z}$.

Proof of Lemma 6.6. By Lemma 6.3 we see that for $a \in \{1, 2, 3, 4, 5\}$

$$\frac{J_{11}}{\tilde{P}_a} \geq 0$$

and obviously

$$\frac{1}{\tilde{P}_a} \geq 0.$$

Lemma 6.6 follows directly from inequalities above. □

Remark 6.7. By Lemma 6.6 all terms in sums of theta quotients in (1.3), Definition 2.8 and Definition 2.9 have non-negative Fourier coefficients. For example, by applying (4.25) we have

$$\vartheta(1,0,0,0,0) = \frac{J_{11}^6}{J_1^2} \cdot \frac{q^2}{P_4 P_5^2} = J_{11}^{14} \cdot \frac{q^2}{P_1^2 P_2^2 P_3^2 P_4^3 P_5^4} = J_{11} \cdot \frac{q^2}{\tilde{P}_1^2 \tilde{P}_2^2 \tilde{P}_3^2 \tilde{P}_4^3 \tilde{P}_5^4} \geq 0.$$

Remark 6.8. Let $\alpha_1 < 0$ in (6.2). Then we can establish not just non-negativity of Fourier coefficients of (6.2) but also that the coefficients of (6.2) are monotonically non-decreasing by multiplying the expression by $(1 - q)$. In this sense we have that if

$$P(q) := \sum_{i \in \mathbb{Z}} c_i q^i$$

and

$$(1 - q)P(q) = \sum_{i \in \mathbb{Z}} (c_i - c_{i-1}) q^i \geq 0,$$

then we have $c_i \geq c_{i-1}$ for any $i \in \mathbb{Z}$.

Let us compare Fourier coefficients of theta quotients in $[c_1, c_2, c_3, c_4, c_5]$. We have the following inequalities

Proposition 6.9. *We have*

$$[0, 1, 0, 0, 0] \leq [0, 0, 1, 0, 0] \leq [0, 0, 0, 1, 0] \leq [0, 0, 0, 0, 1]. \tag{6.3}$$

Proof of Proposition 6.9. Inequalities can be proved using the three-term Weierstrass relations (4.28)-(4.32) in the following form

$$[0, -1, 1, 0, 0] = \frac{J_{11}^2}{J_1^3} q P_3^2 P_2 - \frac{J_{11}^2}{J_1^3} q P_2^2 P_5 = \frac{J_{11}^2}{J_1^3} \frac{q^3 P_1^2 P_2^2 P_3}{P_4 P_5} \geq 0,$$

$$[0, 0, -1, 1, 0] = \frac{J_{11}^2}{J_1^3} q P_2^2 P_5 - \frac{J_{11}^2}{J_1^3} q P_4^2 P_1 = \frac{J_{11}^2}{J_1^3} \frac{q^2 P_1^2 P_2 P_5}{P_3} \geq 0,$$

$$[0, 0, 0, -1, 1] = \frac{J_{11}^2}{J_1^3} q P_4^2 P_1 - \frac{J_{11}^2}{J_1^3} q^2 P_1^2 P_3 = \frac{J_{11}^2}{J_1^3} \frac{q P_1^2 P_4 P_5^2}{P_2 P_3} \geq 0.$$

Here we can see the positivity of the Fourier coefficients of the right-hand side of equations above by Lemma 6.6 as the theta quotients in the right-hand side can be transformed into (6.2) by applying (4.25). □

Remark 6.10. We can strengthen Proposition 6.9 to

$$J_1[0,1,0,0,0] \le J_1[0,0,1,0,0] \le J_1[0,0,0,1,0] \le J_1[0,0,0,0,1]. \tag{6.4}$$

We are able to compare Fourier coefficients of theta quotients in $[c_1, c_2, c_3, c_4, c_5; c_6]_i$ which was defined in Definition 2.9.

Proposition 6.11. *For every residue $i \neq 6$ we have*

$$[0,1,0,0,0;0]_i \le [0,0,1,0,0;0]_i \le [0,0,0,1,0;0]_i \le [0,0,0,0,1;0]_i.$$

Proof of Proposition 6.11. For $i \neq 8$ this result can directly obtained from Proposition 6.9. For example for residue 0 we need to multiply (6.3) by $\frac{1}{P_1} \ge 0$. For $i = 8$ we use Remark 6.10 and multiply (6.4) by $\frac{qP_1}{J_1P_4P_5} \ge 0$. □

For residue 6 we have the following comparison formulas.

Proposition 6.12. *We have*

$$\vartheta(0,0,0,0,1) \le \vartheta(0,0,0,1,0) \le \vartheta(0,0,1,0,0) \le \vartheta(0,1,0,0,0).$$

Proof of Proposition 6.12. Inequalities can be proved using the three-term Weierstrass relations (4.28)-(4.37) in the following form.

$$\begin{aligned}
\vartheta(0,0,0,1,-1) &= \frac{J_{11}^2}{J_1^3}\frac{qP_1P_3P_4}{P_2} - \frac{J_{11}^2}{J_1^3}\frac{qP_1P_4P_5}{P_3} = \frac{J_{11}^2}{J_1^3}\frac{q^3P_1^3}{P_5} \ge 0,\\
\vartheta(0,0,1,-1,0) &= \frac{J_{11}^2}{J_1^3}\frac{qP_2P_3P_5}{P_4} - \frac{J_{11}^2}{J_1^3}\frac{qP_1P_3P_4}{P_2} = \frac{J_{11}^2}{J_1^3}\frac{q^2P_1^2P_5}{P_4} \ge 0,\\
\vartheta(0,1,-1,0,0) &= \frac{J_{11}^2}{J_1^3}\frac{P_2P_4P_5}{P_1} - \frac{J_{11}^2}{J_1^3}\frac{qP_2P_3P_5}{P_4} = \frac{J_{11}^2}{J_1^3}\frac{P_5^3}{P_3} \ge 0.
\end{aligned}$$
□

We also provide some additional comparison formulas among theta quotients.

Lemma 6.13. *We have*

$$\begin{aligned}
&[0,0,0,0,-1;1]_1 \ge 0,\\
&[0,1,0,0,0;-1]_2 \ge 0,\\
&[0,0,0,1,0;-1]_3 \ge 0,\\
&[0,0,-1,0,0;1]_3 \ge 0,\\
&[0,0,1,0,0;-1]_5 \ge 0,\\
&[0,0,0,-1,0;1]_8 \ge 0.
\end{aligned}$$

Proof of Lemma 6.13. We need to use the three-term Weierstrass relations (4.28)-(4.37)

$$[0,0,0,0,-1;1]_1 = \frac{J_{11}^2}{J_1^3}\left(\frac{qP_2^2P_4}{P_1} - \frac{P_5}{P_2P_3}\cdot qP_3^2P_2\right) = \frac{J_{11}^2}{J_1^3}\frac{q^2P_1P_3P_4}{P_5} \geq 0,$$
$$[0,1,0,0,0;-1]_2 = \frac{J_{11}^2}{J_1^3}\left(\frac{P_3}{P_1P_4}\cdot q^2P_1^2P_3 - \frac{q^3P_1^2P_2}{P_5}\right) = \frac{J_{11}^2}{J_1^3}\frac{q^2P_1^2P_4}{P_2} \geq 0,$$
$$[0,0,0,1,0;-1]_3 = \frac{J_{11}^2}{J_1^3}\left(\frac{P_2}{P_1P_3}\cdot qP_2^2P_5 - \frac{qP_3^2P_5}{P_4}\right) = \frac{J_{11}^2}{J_1^3}\frac{q^2P_1^2P_5^2}{P_3P_4} \geq 0,$$
$$[0,0,-1,0,0;1]_3 = \frac{J_{11}^2}{J_1^3}\left(\frac{qP_3^2P_5}{P_4} - \frac{P_2}{P_1P_3}\cdot qP_4^2P_1\right) = \frac{J_{11}^2}{J_1^3}\frac{q^3P_1^3}{P_3} \geq 0,$$
$$[0,0,1,0,0;-1]_5 = \frac{J_{11}^2}{J_1^3}\left(\frac{P_4}{P_2P_5}\cdot qP_4^2P_1 - \frac{qP_1P_5^2}{P_3}\right) = \frac{J_{11}^2}{J_1^3}\frac{q^3P_1^2P_2^2}{P_3P_5} \geq 0,$$
$$[0,0,0,-1,0;1]_8 = \frac{J_{11}^2}{J_1^3}\left(\frac{P_3P_4^2}{P_2} - \frac{qP_1}{P_4P_5}\cdot qP_2^2P_5\right) = \frac{J_{11}^2}{J_1^3}\frac{P_5^3}{P_4} \geq 0.$$ □

6.3. **Proofs of rank-crank inequalities.** Define $Q^C_{a,m}(q)$ to be the elements of the 11-dissection of the deviation of the crank:

$$D_C(a,11) =: \sum_{m=0}^{10} Q^C_{a,m}(q^{11})q^m.$$

So the reformulation of this definition is

$$Q^C_{a,m}(q) := \sum_{n=0}^{\infty}\left(M(a,11,11n+m) - \frac{p(11n+m)}{11}\right)q^n. \tag{6.5}$$

Proof of Theorem 2.1. As an example we consider inequalities corresponding to residue 0. For $n \geq 0$, $N_i = N(0,11,11n)$ and $M_i = M(i,11,11n)$ we want to establish

$$N_0 + 2N_1 + M_1 \geq 2N_2 + N_4 + M_0,$$
$$N_0 + 2N_1 + 3N_2 + M_1 \geq 3N_3 + 3N_5 + M_0,$$
$$2N_2 + N_3 + N_5 \geq 4N_4,$$
$$N_2 + 5N_3 + 3N_4 + M_0 \geq N_0 + 2N_1 + 6N_5 + M_1.$$

We sum up

$$Q_{0,0}(q) + 2Q_{1,0}(q) + Q^C_{1,0}(q) - 2Q_{2,0}(q) - Q_{4,0}(q) - Q^C_{0,0}(q) = [0,11,0,0,0;0]_0 \geq 0,$$
$$Q_{0,0}(q) + 2Q_{1,0}(q) + 3Q_{2,0}(q) + Q^C_{1,0}(q) - 3Q_{3,0}(q) - 3Q_{5,0}(q) - Q^C_{0,0}(q) = [0,-11,11,0,0;0]_0 \geq 0,$$
$$2Q_{2,0}(q) + Q_{3,0}(q) + Q_{5,0}(q) - 4Q_{4,0}(q) = [0,0,-11,11,0;0]_0 \geq 0$$

and

$$Q_{2,0}(q) + 5Q_{3,0}(q) + 3Q_{4,0}(q) + Q^C_{0,0}(q) - Q_{0,0}(q) - 2Q_{1,0}(q) - 6Q_{5,0}(q) - Q^C_{1,0}(q)$$
$$= [0,0,0,-11,11;0]_0 \geq 0.$$

and apply Proposition 6.11. For other residues Lemma 6.13 is also used. □

Proof of Corollary 2.2. As an example let us consider the inequality

$$N(2,11,11n+1) \geq M(2,11,11n+1),$$

where $n \geq 0$. To prove it we sum up

$$Q_{2,1}(q) - Q^C_{2,1}(q) = [0,1,0,-1,0;1]_1 \geq 0$$

and apply Proposition 6.11 and Lemma 6.13. □

Proof of Corollary 2.3. As an example let us consider the inequality

$$N(1, 11, 11n + 1) + 2N(4, 11, 11n + 1) \geq 2N(5, 11, 11n + 1) + M(0, 11, 11n + 1),$$

where $n \geq 0$. To prove it we sum up

$$Q_{1,1}(q) + 2Q_{4,1}(q) - Q_{5,1}(q) - Q^C_{0,1}(q) = [1, -2, -4, 3, -3; 2]_1 =$$
$$= [1, -1, -1, -1, -1, 0]_1 + [0, -1, -3, 4, -2; 2]_1 \geq 0$$

and apply Proposition 6.11, Lemma 6.13 and note that

$$[1, -1, -1, -1, -1; 0]_1 = \frac{J^2_{11} P_5}{P_2 P_3} \geq 0$$

by Lemma 6.3. □

Proof of Corollary 2.6. We consider

$$Q_{0,6}(q) = \vartheta(0, 0, 1, 1, -1) \geq 0,$$
$$-Q_{5,6}(q) = \vartheta(1, 0, 1, 0, 1) \geq 0,$$

where $n \geq 0$ and apply Proposition 6.12. □

Proof of Corollary 2.7. As an example let us consider the inequality

$$N(0, 11, 11n + 6) + N(3, 11, 11n + 6) \geq N(1, 11, 11n + 6) + N(4, 11, 11n + 6),$$

where $n \geq 0$. To prove it we sum up

$$Q_{0,6}(q) + Q_{3,6}(q) - Q_{1,6}(q) - Q_{4,6}(q) = \vartheta(2, 0, 3, 3, -6) \geq 0$$

and apply Proposition 6.12. □

Remark 6.14. If it is known that

$$[c_1, c_2, c_3, c_4, c_5; c_6]_m \geq 0$$

with some $c_k \in \mathbb{Q}$, $1 \leq k \leq 6$ and residue $m \neq 6$, then you can construct some rank-crank inequality with $N_i = N(i, 11, 11n + m)$ and $M_i = M(i, 11, 11n + m)$. We need to look at $\{Q_{a,m} \mid 0 \leq a \leq 4\}$ and $Q^C_{0,m}$ as linear independent vectors in $\mathbb{R}^6$ as described in Section 4, that is, we need to consider equation

$$\sum_{j=0}^{4} a_j Q_{j,m} + b_0 Q^C_{0,m} = [c_1, c_2, c_3, c_4, c_5; c_6]_m$$

as a linear system over unknowns $\{a_0, a_1, a_2, a_3, a_4, b_0\}$ with a unique rational solution. The same is true in case of residue 6. If it is known that

$$\vartheta(c_1, c_2, c_3, c_4, c_5) \geq 0$$

with some $c_k \in \mathbb{Q}$, $1 \leq k \leq 5$, then you can construct some rank inequality with $N_i = N(i, 11, 11n+6)$ and $M_i = M(i, 11, 11n+6)$. We need to look at $\{\vartheta_{a,6} \mid 0 \leq a \leq 4\}$ as linear independent vectors in $\mathbb{R}^5$ as described in Theorem 1.3, that is, we need to consider equation

$$\sum_{j=0}^{4} a_j \vartheta_{j,6} = \vartheta(c_1, c_2, c_3, c_4, c_5)$$

as a linear system over unknowns $\{a_0, a_1, a_2, a_3, a_4\}$ with a unique rational solution. Then we might need to remove a partition function term by applying

$$p(n) = N(0, 11, n) + 2N(1, 11, n) + \cdots + 2N(5, 11, n).$$

6.4. **Conjectural rank-crank inequalities.** In the sense of rank-crank inequalities in Corollary 2.2 and Corollary 2.6 we can conjecture the following stronger two-term rank-crank inequalities.

Conjecture 6.15. *For $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ we have*

$$N_0 \geq_3 N_1 \geq N_2 \geq_1 M_0 \geq \frac{p(11n)}{11} \geq M_1 \geq N_3 \geq_2 N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+1)$ and $M_i = M(i, 11, 11n+1)$ we have

$$N_0 \geq N_1 \geq N_2 \geq_1 M_1 \geq \frac{p(11n+1)}{11} \geq M_2 \geq M_0 \geq_1 N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+2)$ and $M_i = M(i, 11, 11n+2)$ we have

$$N_0 \geq_3 N_1 \geq N_2 \geq_1 M_2 \geq \frac{p(11n+2)}{11} \geq M_0 \geq N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+3)$ and $M_i = M(i, 11, 11n+3)$ we have

$$N_0 \geq_2 N_1 \geq_1 N_2 \geq M_0 \geq \frac{p(11n+3)}{11} \geq M_1 \geq N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+4)$ and $M_i = M(i, 11, 11n+4)$ we have

$$N_0 \geq_3 N_1 \geq N_2 \geq_1 M_0 \geq \frac{p(11n+4)}{11} \geq M_1 \geq_1 N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+5)$ and $M_i = M(i, 11, 11n+5)$ we have

$$N_0 \geq N_1 \geq N_2 \geq M_0 \geq \frac{p(11n+5)}{11} \geq M_2 \geq N_3 \geq_1 N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+6)$ and $M_i = M(i, 11, 11n+6)$ we have

$$N_0 \geq_1 N_1 \geq N_2 \geq \frac{p(11n+6)}{11} \geq N_3 \geq N_4 \geq_1 N_5.$$

For $N_i = N(i, 11, 11n+7)$ and $M_i = M(i, 11, 11n+7)$ we have

$$N_0 \geq N_1 \geq_1 N_2 \geq M_1 \geq \frac{p(11n+7)}{11} \geq M_0 \geq N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+8)$ and $M_i = M(i, 11, 11n+8)$ we have

$$N_0 \geq_3 N_1 \geq N_2 \geq M_1 \geq_3 \frac{p(11n+8)}{11} \geq_3 M_0 \geq N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+9)$ and $M_i = M(i, 11, 11n+9)$ we have

$$N_0 \geq_2 N_1 \geq N_2 \geq M_1 \geq \frac{p(11n+9)}{11} \geq M_0 \geq_1 N_3 \geq N_4 \geq N_5.$$

For $N_i = N(i, 11, 11n+10)$ and $M_i = M(i, 11, 11n+10)$ we have

$$N_0 \geq_3 N_1 \geq N_2 \geq M_0 \geq \frac{p(11n+10)}{11} \geq M_3 \geq_1 N_3 \geq N_4 \geq N_5.$$

Notation $A_n \geq B_n$ means that $A_n \geq B_n$ for all $n \geq 0$ and $A_n \geq_m B_n$ means that $A_n \geq B_n$ for all $n \geq m$.

Remark 6.16. Some of the inequalities of Conjecture 6.15 are proved in Corollary 2.2.

Remark 6.17. Inequalities between ranks where considered by Bringmann and Kane [12] by using analytic methods. For $0 \leq a < b \leq 5$ and for $n > N_{a,b}$, where $N_{a,b}$ is an explicit constant, we have the inequality

$$N(a, 11, n) > N(b, 11, n).$$

Recently, Conjecture 6.15 were fully proved using analytic methods by Bringmann and Pandey [13].

As a generalization of rank-crank inequalities we can state the following conjectures in terms of Definition 2.9 and Definition 1.1.

Definition 6.18. We define notation $F(q) \geq_m G(q)$ if for $F(q) = \sum_{n\geq 0} a(n)q^n$ and for $G(q) = \sum_{n\geq 0} b(n)q^n$ we have $a(n) \geq b(n)$ for $n \geq m$.

Conjecture 6.19. *For any $c_k \in \mathbb{Q}$, $1 \leq k \leq 6$ and residue $i \neq 6$ modulo 11 there is $N \in \mathbb{N}_0$ such that*

$$[c_1, c_2, c_3, c_4, c_5; c_6]_i \geq_N 0 \quad or \quad [c_1, c_2, c_3, c_4, c_5; c_6]_i \leq_N 0,$$

For any $a_k \in \mathbb{Q}$, $1 \leq k \leq 5$ there is $N \in \mathbb{N}_0$ such that

$$\vartheta(a_1, a_2, a_3, a_4, a_5) \geq_N 0 \quad or \quad \vartheta(a_1, a_2, a_3, a_4, a_5) \leq_N 0.$$

We have the following corollary of Conjecture 6.19.

Conjecture 6.20. *For any $a_k, b_k \in \mathbb{Z}$, $0 \leq k \leq 5$, such that*

$$\sum_{k=0}^{5}(a_k + b_k) = 0,$$

and residue m modulo 11 there is $N \in \mathbb{N}_0$ such that

$$\sum_{k=0}^{5}\Big[a_k N(k, 11, 11n + m) + b_k M(k, 11, 11n + m)\Big] \geq_N 0 \quad or$$
$$\sum_{k=0}^{5}\Big[a_k N(k, 11, 11n + m) + b_k M(k, 11, 11n + m)\Big] \leq_N 0.$$

Notation $A_n \geq_m B_n$ means that $A_n \geq B_n$ for all $n \geq m$.

7. Proofs of new congruences

Proof of Theorem 2.12. By properties of the crank we are able to deduce

$$M_k(n) \equiv \sum_{m=1}^{10} m^k M(m, 11, n) \pmod{11}.$$

Then consider

$$M_2(n) \equiv 2M(1, 11, n) - 3M(2, 11, n) - 4M(3, 11, n) - M(4, 11, n) + 6M(5, 11, n) \pmod{11},$$
$$M_4(n) \equiv 2M(1, 11, n) - M(2, 11, n) - 3M(3, 11, n) + 6M(4, 11, n) - 4M(5, 11, n) \pmod{11},$$
$$M_6(n) \equiv 2M(1, 11, n) - 4M(2, 11, n) + 6M(3, 11, n) - 3M(4, 11, n) - M(5, 11, n) \pmod{11},$$
$$M_8(n) \equiv 2M(1, 11, n) + 6M(2, 11, n) - M(3, 11, n) - 4M(4, 11, n) - 3M(5, 11, n) \pmod{11}.$$

Taking $n = 11l + m$ in congruences above and using Theorem 1.2 we obtain Theorem 2.12. For example using notation (2.2) with $m = 1$ we have

$$T^C_{2,1}(q) = \sum_{n=0}^{\infty} M_2(11n + 1)q^n \equiv 2Q^C_{1,1}(q) - 3Q^C_{2,1}(q) - 4Q^C_{3,1}(q) - Q^C_{4,1}(q) + 6Q^C_{5,1}(q)$$
$$\equiv 2\frac{J_{11}^2 P_5}{P_2 P_3} \pmod{11}. \quad \square$$

Proof of Theorem 2.11. By properties of the rank we are able to deduce

$$N_k(n) \equiv \sum_{m=1}^{10} m^k N(m, 11, n) \pmod{11}.$$

Then consider

$$N_2(n) \equiv 2N(1,11,n) - 3N(2,11,n) - 4N(3,11,n) - N(4,11,n) + 6N(5,11,n) \pmod{11},$$
$$N_4(n) \equiv 2N(1,11,n) - N(2,11,n) - 3N(3,11,n) + 6N(4,11,n) - 4N(5,11,n) \pmod{11},$$
$$N_6(n) \equiv 2N(1,11,n) - 4N(2,11,n) + 6N(3,11,n) - 3N(4,11,n) - N(5,11,n) \pmod{11},$$
$$N_8(n) \equiv 2N(1,11,n) + 6N(2,11,n) - N(3,11,n) - 4N(4,11,n) - 3N(5,11,n) \pmod{11}.$$

Taking $n = 11l + m$ in congruences above and using Theorem 1.3 and calculations in Section 4 we obtain Theorem 2.11. For example using notation (2.1) with $m = 1$ we have

$$T_{2,1}(q) = \sum_{n=0}^{\infty} N_2(11n+1)q^n \equiv 2Q_{1,1}(q) - 3Q_{2,1}(q) - 4Q_{3,1}(q) - Q_{4,1}(q) + 6Q_{5,1}(q)$$
$$\equiv [0,-1,-5,-4,-3;-2]_1 \pmod{11}. \quad \square$$

Proof of Theorem 2.10. Andrews [2] showed that $\mathrm{spt}(n)$ is related to the second rank moment

$$\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n). \tag{7.1}$$

As an example let us consider congruence

$$\sum_{n=0}^{\infty} \mathrm{spt}(11n+1)q^n \equiv [1,5,-4,1,-5;1]_1 \pmod{11}.$$

We know that

$$\sum_{n=0}^{\infty} (11n+1)p(11n+1)q^n \equiv [1,-1,-1,-1,-1;0]_1 \pmod{11}$$

by Theorem 5.2 and

$$\sum_{n=0}^{\infty} N_2(11n+1)q^n \equiv [0,-1,-5,-4,-3;-2]_1 \pmod{11}$$

by Theorem 2.11. Using (7.1) we obtain the desired congruence. □

Proof of Corollary 2.14. In terms of notation (2.1) from [19, Theorem 5.1] we know

$$T_{2,6}(q) \equiv 3J_1^{13} \pmod{11},$$
$$T_{6,6}(q) \equiv J_1^{13}(4 + E_4(q)) \pmod{11},$$
$$T_{8,6}(q) \equiv J_1^{13}(5 + 6E_4(q) + 6E_6(q)) \pmod{11}.$$

Using Theorem 2.11 and applying

$$J_1^{13} \equiv J_1^2 J_{11} \pmod{11},$$

we obtain Corollary 2.14. □

Remark 7.1. We also can deduce congruences modulo 11 for rank moments and $\mathrm{spt}(n)$ corresponding to residues $i \in \{0,4,7,9,10\}$, but they consist of the universal mock theta functions $g(x;q)$. For example for residue 0 in terms of notation (2.1) we have

$$T_{2,0}(q) \equiv T_{4,0}(q) \equiv 2q^2 g(q^2;q^{11}) + [0,0,4,5,1;-2]_0 \pmod{11},$$
$$T_{6,0}(q) \equiv 2q^2 g(q^2;q^{11}) + [0,3,-2,1,-4;-2]_0 \pmod{11},$$
$$T_{8,0}(q) \equiv 2q^2 g(q^2;q^{11}) + [0,4,1,0,2;-2]_0 \pmod{11},$$

and

$$\sum_{n=0}^{\infty} \mathrm{spt}(11n)q^n \equiv -q^2 g(q^2;q^{11}) + [0,0,-2,3,5;1]_0 \pmod{11}.$$

8. Full proofs for new crank-rank inequalities

In this section we give the full account of calculations for the proofs of Theorem 2.1, Corollary 2.2, Corollary 2.3, Theorem 2.5 and Corollary 2.7.

Definition 8.1. Using the notation of (4.1) and (6.5), we define

$$\begin{aligned}
(a_0,a_2,a_3,a_4,a_5;b_0,b_1)_0 &:= \sum_{i=0}^{5} a_i Q_{i,0}(q) + b_0 Q^C_{0,0}(q) + b_1 Q^C_{1,0}(q),\\
(a_0,a_2,a_3,a_4,a_5;b_0,b_1,b_2)_1 &:= \sum_{i=0}^{5} a_i Q_{i,1}(q) + \sum_{i=0}^{2} b_i Q^C_{i,1}(q),\\
(a_0,a_2,a_3,a_4,a_5;b_0,b_2)_2 &:= \sum_{i=0}^{5} a_i Q_{i,2}(q) + b_0 Q^C_{0,2}(q) + b_2 Q^C_{2,2}(q),\\
(a_0,a_2,a_3,a_4,a_5;b_0,b_1)_3 &:= \sum_{i=0}^{5} a_i Q_{i,3}(q) + b_0 Q^C_{0,3}(q) + b_1 Q^C_{1,3}(q),\\
(a_0,a_2,a_3,a_4,a_5;b_0,b_1)_4 &:= \sum_{i=0}^{5} a_i Q_{i,4}(q) + b_0 Q^C_{0,4}(q) + b_1 Q^C_{1,4}(q),\\
(a_0,a_2,a_3,a_4,a_5;b_0,b_2)_5 &:= \sum_{i=0}^{5} a_i Q_{i,5}(q) + b_0 Q^C_{0,5}(q) + b_2 Q^C_{2,5}(q),\\
(a_0,a_1,a_2,a_3,a_4,a_5)_6 &:= \sum_{i=0}^{5} a_i Q_{i,6}(q),\\
(a_0,a_1,a_2,a_3,a_4,a_5;b_0,b_1)_7 &:= \sum_{i=0}^{5} a_i Q_{i,7}(q) + b_0 Q^C_{0,7}(q) + b_1 Q^C_{1,7}(q),\\
(a_0,a_1,a_2,a_3,a_4,a_5;b_0,b_1)_8 &:= \sum_{i=0}^{5} a_i Q_{i,8}(q) + b_0 Q^C_{0,8}(q) + b_1 Q^C_{1,8}(q),\\
(a_0,a_1,a_2,a_3,a_4,a_5;b_0,b_1)_9 &:= \sum_{i=0}^{5} a_i Q_{i,9}(q) + b_0 Q^C_{0,9}(q) + b_1 Q^C_{1,9}(q),\\
(a_0,a_1,a_2,a_3,a_4,a_5;b_0,b_3)_{10} &:= \sum_{i=0}^{5} a_i Q_{i,10}(q) + b_0 Q^C_{0,10}(q) + b_3 Q^C_{3,10}(q).
\end{aligned}$$

Full proof of Theorem 2.1. The calculations below can be derived directly from calculations of the dissection elements $Q_{a,m}(q)$ from Section 4 and calculations of the dissection elements $Q^C_{a,m}(q)$ from Theorem 1.2. The positivity of sums of theta quotients can be derived from Proposition 6.11 and Lemma 6.13. The inequalities with $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ are equivalent to

$$\begin{aligned}
(1,2,-2,0,-1,0;-1,1)_0 &= [0,11,0,0,0;0]_0 \geq 0,\\
(1,2,3,-3,0,-3;-1,1)_0 &= [0,-11,11,0,0;0]_0 \geq 0,\\
(0,0,2,1,-4,1;0,0)_0 &= [0,0,-11,11,0;0]_0 \geq 0,\\
(-1,-2,1,5,3,-6;1,-1)_0 &= [0,0,0,-11,11;0]_0 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+1)$ and $M_i = M(i, 11, 11n+1)$ are equivalent to

$$\begin{aligned}
(1,-2,4,-3,1,-1;1,0,-1)_1 &= [0,11,0,0,0;0]_1 \geq 0,\\
(1,3,-4,6,-6,0;1,0,-1)_1 &= [0,-11,11,0,0;0]_1 \geq 0,\\
(0,2,-1,-3,6,-4;0,0,0)_1 &= [0,0,-11,11,0;0]_1 \geq 0,\\
(-1,1,2,-1,-4,3;-2,-1,3)_1 &= [0,0,0,-11,11;0]_1 \geq 0,\\
(-1,-1,3,2,1,-4;-2,-1,3)_1 &= [0,0,0,0,-11;11]_1 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+2)$ and $M_i = M(i, 11, 11n+2)$ are equivalent to

$$\begin{aligned}
(-2,0,3,0,1,-2;0,0)_2 &= [0,0,0,0,0;11]_2 \geq 0\\
(0,2,-2,2,-3,1;2,-2)_2 &= [0,11,0,0,0;-11]_2 \geq 0\\
(3,-1,2,-1,0,-3;-1,1)_2 &= [0,-11,11,0,0;0]_2 \geq 0\\
(2,1,-4,1,3,-3;1,-1)_2 &= [0,0,-11,11,0;0]_2 \geq 0\\
(1,3,1,-8,-5,8;3,-3)_2 &= [0,0,0,-11,11;0]_2 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+3)$ and $M_i = M(i, 11, 11n+3)$ are equivalent to

$$\begin{aligned}
(1,-1,0,2,0,-2;-1,1)_3 &= [0,11,0,0,0;0]_3 \geq 0,\\
(-2,5,2,-4,2,-3;0,0)_3 &= [0,-11,11,0,0;0]_3 \geq 0,\\
(2,-1,-3,4,-3,1;1,-1)_3 &= [0,0,-11,0,0;11]_3 \geq 0,\\
(-1,-1,6,-2,-5,3;-5,5)_3 &= [0,0,0,11,0;-11]_3 \geq 0,\\
(4,2,-7,-3,4,0;3,-3)_3 &= [0,0,0,-11,11;0]_3 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+4)$ and $M_i = M(i, 11, 11n+4)$ are equivalent to

$$\begin{aligned}
(-2,4,-3,3,-1,-1;-5,5)_4 &= [0,11,0,0,0;0]_4 \geq 0,\\
(4,-5,5,-2,-1,-1;3,-3)_4 &= [0,-11,11,0,0;0]_4 \geq 0,\\
(-2,3,1,-2,0,0;1,-1)_4 &= [0,0,-11,11,0;0]_4 \geq 0,\\
(3,0,-3,-2,1,1;-1,1)_4 &= [0,0,0,-11,11;0]_4 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+5)$ and $M_i = M(i, 11, 11n+5)$ are equivalent to

$$\begin{aligned}
(3,-2,1,-3,0,1;-2,2)_5 &= [0,11,0,0,0;0]_5 \geq 0,\\
(-3,7,-2,-1,1,-2;-3,3)_5 &= [0,-11,11,0,0;0]_5 \geq 0,\\
(2,-3,1,3,-4,1;4,-4)_5 &= [0,0,-11,11,0;0]_5 \geq 0,\\
(-4,-2,4,7,2,-7;0,0)_5 &= [0,0,0,-11,11;0]_5 \geq 0,\\
(-2,4,1,-6,2,1;-5,5)_5 &= [0,0,11,0,0;-11]_5 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+7)$ and $M_i = M(i, 11, 11n+7)$ are equivalent to

$$\begin{aligned}
(1,0,0,-2,2,-1;2,-2)_7 &= [0,11,0,0,0;0]_7 \geq 0,\\
(3,1,1,7,-5,4;-4,-7)_7 &= [0,-11,11,0,0;0]_7 \geq 0,\\
(-4,4,4,-6,3,-1;-4,4)_7 &= [0,0,-11,11,0;0]_7 \geq 0,\\
(1,-2,-2,5,2,-4;0,0)_7 &= [0,0,0,-11,11;0]_7 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 8)$ and $M_i = M(i, 11, 11n + 8)$ are equivalent to

$$\begin{aligned}
(1, -3, 2, 0, 3, -3; 5, -5)_8 &= [0, 11, 0, 0, 0; 0]_8 \geq 0,\\
(-1, 2, -2, 4, -5, 2; -3, 3)_8 &= [0, -11, 11, 0, 0; 0]_8 \geq 0,\\
(-3, 2, 5, -5, -1, 2; -1, 1)_8 &= [0, 0, -11, 11, 0; 0]_8 \geq 0,\\
(7, 4, -8, -1, 5, 4; -4, -7)_8 &= [0, 0, 0, -11, 11; 0]_8 \geq 0,\\
(5, 6, -1, 4, 2, -5; -6, -5)_8 &= [0, 0, 0, -11, 0; 11]_8 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 9)$ and $M_i = M(i, 11, 11n + 9)$ are equivalent to

$$\begin{aligned}
(0, -2, 4, -1, -1, 0; 1, -1)_9 &= [0, 11, 0, 0, 0; 0]_9 \geq 0,\\
(-1, 4, -3, 1, 1, -2; 0, 0)_9 &= [0, -11, 11, 0, 0; 0]_9 \geq 0,\\
(4, -2, -5, 3, 3, -3; 4, -4)_9 &= [0, 0, -11, 11, 0; 0]_9 \geq 0,\\
(-2, 3, 4, -6, -6, 7; -3, 3)_9 &= [0, 0, 0, -11, 11; 0]_9 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 10)$ and $M_i = M(i, 11, 11n + 10)$ are equivalent to

$$\begin{aligned}
(0, 3, -2, -2, -1, 2; -2, 2)_{10} &= [0, 11, 0, 0, 0; 0]_{10} \geq 0,\\
(3, -3, 1, 1, -2, 0; 1, -1)_{10} &= [0, -11, 11, 0, 0; 0]_{10} \geq 0,\\
(-1, 2, -1, -1, 4, -3; -1, 1)_{10} &= [0, 0, -11, 11, 0; 0]_{10} \geq 0,\\
(-6, -6, 6, 6, -3, -8; 6, 5)_{10} &= [0, 0, 0, -11, 11; 0]_{10} \geq 0.
\end{aligned}$$ □

Full proof of Corollary 2.2. The calculations below can be derived directly from calculations of the dissection elements $Q_{a,m}(q)$ from Section 4 and calculations of the dissection elements $Q^C_{a,m}(q)$ from Theorem 1.2. The positivity of sums of theta quotients can be derived from Proposition 6.11 and Lemma 6.13. The inequalities with $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ are equivalent to

$$\begin{aligned}
(0, 0, 0, 0, -1, 0; 0, 1)_0 &= [0, 1, -2, 2, 0; 0]_0 \geq 0,\\
(0, 0, 1, 1, -1, 0; 0, -1)_0 &= [0, 0, -4, 3, 1; 0]_0 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 1)$ and $M_i = M(i, 11, 11n + 1)$ are equivalent to

$$\begin{aligned}
(0, 0, 1, 0, 0, 0; 0, 0, -1)_1 &= [0, 1, 0, -1, 0; 1]_1 \geq 0,\\
(0, 0, 0, 0, -1, 0; 0, 0, 1)_1 &= [0, 0, 1, -1, 1; 0]_1 \geq 0,\\
(0, 0, 1, -1, 1, -1; 0, 0, 0)_1 &= [0, 2, -2, 1, 0; 1]_1 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 2)$ and $M_i = M(i, 11, 11n + 2)$ are equivalent to

$$\begin{aligned}
(0, 1, 0, 0, 0, 0; 0, -1)_2 &= [0, 2, -2, 1, 1; 1]_2 \geq 0\\
(0, 0, 0, 0, 0, -1; 1, 0)_2 &= [0, 0, 0, 1, 0; 1]_2 \geq 0,\\
(0, 0, -1, 0, 0, -1; 2, 0)_2 &= [0, 1, -2, 2, 0; 0]_2 \geq 0,\\
(-1, 0, 1, 0, 0, -1; 1, 0)_2 &= [0, 1, 0, 0, 0; 4]_2 \geq 0,\\
(0, 1, 0, 1, 0, 0; -1, -1)_2 &= [0, 3, -1, 1, 0; -1]_2 \geq 0,\\
(0, 1, 0, 0, -1, 1; 0, -1)_2 &= [0, 3, 0, -1, 1; -2]_2 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 3)$ and $M_i = M(i, 11, 11n + 3)$ are equivalent to

$$(1, 0, 0, 0, 0, 0; -1, 0)_3 = [0, 4, 0, 0, 2; -2]_3 \geq 0,$$
$$(0, 0, 0, 0, -1, 0; 0, 1)_3 = [0, -1, -1, 1, 0; 1]_3 \geq 0,$$
$$(1, 0, -2, 0, 0, 0; 1, 0)_3 = [0, 0, -2, -2, 2; 2]_3 \geq 0,$$
$$(0, 0, 1, 0, -1, 0; -1, 1)_3 = [0, 1, 0, 2, 0; -1]_3 \geq 0,$$
$$(1, 0, -1, 1, 0, 0; 0, -1)_3 = [0, 2, -2, -1, 1; 2]_3 \geq 0.$$

The inequalities with $N_i = N(i, 11, 11n + 4)$ and $M_i = M(i, 11, 11n + 4)$ are equivalent to

$$(0, 1, 0, 0, 0, 0; -1, 0)_4 = [0, 1, -1, 1, 1; 0]_4 \geq 0,$$
$$(-1, 2, 0, 0, 0, 0; -1, 0)_4 = [0, 2, -4, 4, 0; 0]_4 \geq 0,$$
$$(0, 1, -1, 0, 0, 0; -1, 1)_4 = [0, 2, -1, -1, 2; 0]_4 \geq 0,$$
$$(0, 1, 0, 1, 0, 0; -2, 0)_4 = [0, 2, 1, 0, 0; 0]_4 \geq 0.$$

The inequalities with $N_i = N(i, 11, 11n + 5)$ and $M_i = M(i, 11, 11n + 5)$ are equivalent to

$$(0, 0, 1, 0, 0, 0; -1, 0)_5 = [0, 1, 0, 1, 1; -2]_5 \geq 0,$$
$$(0, 0, 0, 0, 0, -1; 0, 1)_5 = [0, 1, 1, -1, 1; 1]_5 \geq 0,$$
$$(0, 0, 1, 1, 0, 0; -1, -1)_5 = [0, 0, -1, 1, 1; -1]_5 \geq 0,$$
$$(-1, 0, 1, 0, 0, -1; 0, 1)_5 = [0, 0, 1, 0, 2; -3]_5 \geq 0,$$
$$(1, -1, 0, 0, -1, 0; 1, 0)_5 = [0, 2, -2, 2, 0; 1]_5 \geq 0,$$
$$(0, -1, 1, 1, 0, -1; 0, 0)_5 = [0, 2, -1, -1, 2; 0]_5 \geq 0,$$
$$(1, -1, 1, 0, 0, 0; -1, 0)_5 = [0, 4, -1, 0, 1; 0]_5 \geq 0.$$

The inequalities with $N_i = N(i, 11, 11n + 7)$ and $M_i = M(i, 11, 11n + 7)$ are equivalent to

$$(1, 0, 0, 0, 0, 0; 0, -1)_7 = [1, 3, 1, -1, -1; 0]_7 \geq 0,$$
$$(0, 0, 0, 0, -1, 0; 1, 0)_7 = [0, -1, 1, 1, 0; 0]_7 \geq 0,$$
$$(0, 0, 0, 1, 1, -1; -1, 0)_7 = [0, 0, -1, -2, 3; 0]_7 \geq 0.$$

The inequalities with $N_i = N(i, 11, 11n + 8)$ and $M_i = M(i, 11, 11n + 8)$ are equivalent to

$$(0, 0, 1, 0, 0, 0; 0, -1)_8 = [1, 1, 0, 1, 0; 0]_8 \geq 0,$$
$$(0, 0, 0, -1, 0, 0; 1, 0)_8 = [0, 1, -1, 1, 1; 0]_8 \geq 0,$$
$$(0, 0, -1, 0, -1, 0; 1, 1)_8 = [0, -1, 2, 0, 1; 0]_8 \geq 0,$$
$$(0, 1, 0, -2, 0, 1; 0, 0)_8 = [0, 0, -3, 2, 2; 0]_8 \geq 0,$$
$$(1, 0, -2, 0, 0, 0; 1, 0)_8 = [0, 0, 2, -2, 2; 0]_8 \geq 0,$$
$$(0, 0, 0, 1, -1, 0; 1, -1)_8 = [0, 0, 4, 1, 1; 0]_8 \geq 0,$$
$$(0, 0, 1, 1, 0, 0; 0, -2)_8 = [0, 1, 2, 1, 0; 0]_8 \geq 0,$$
$$(0, -1, 0, 0, 0, -1; 2, 0)_8 = [0, 2, 1, 0, 0; 0]_8 \geq 0.$$

The inequalities with $N_i = N(i, 11, 11n + 9)$ and $M_i = M(i, 11, 11n + 9)$ are equivalent to

$$(0, 0, 1, 0, 0, 0; 0, -1)_9 = [0, 2, 1, 0, 0; 0]_9 \geq 0,$$
$$(0, 0, 0, 0, 0, -1; 1, 0)_9 = [0, 1, 2, 1, 0; 0]_9 \geq 0,$$
$$(0, 0, -1, 0, 0, -1; 1, 1)_9 = [0, -1, 1, 1, 0; 0]_9 \geq 0,$$
$$(1, 0, -1, 0, 0, 0; 1, -1)_9 = [0, 0, -3, 2, 2; 0]_9 \geq 0,$$
$$(0, 0, 1, -1, -1, 1; 0, 0)_9 = [0, 2, -1, -1, 2; 0]_9 \geq 0.$$

The inequalities with $N_i = N(i, 11, 11n + 10)$ and $M_i = M(i, 11, 11n + 10)$ are equivalent to

$$\begin{aligned}
(0,1,0,0,0,0;-1,0)_{10} &= [1,2,-1,0,0;0]_{10} \geq 0,\\
(0,0,0,0,-1,0;0,1)_{10} &= [0,2,2,-1,1;0]_{10} \geq 0,\\
(1,-1,0,0,-1,0;1,0)_{10} &= [0,-3,4,0,0;0]_{10} \geq 0,\\
(0,0,1,1,0,0;-2,0)_{10} &= [0,0,1,-1,1;0]_{10} \geq 0,\\
(0,1,0,0,1,0;-2,0)_{10} &= [0,1,-2,2,0;0]_{10} \geq 0,\\
(0,1,-1,-1,0,0;0,1)_{10} &= [0,3,-1,2,0;0]_{10} \geq 0.
\end{aligned}$$

□

Full proof of Corollary 2.3. The calculations below can be derived directly from calculations of the dissection elements $Q_{a,m}(q)$ found in Section 4 and calculations of the dissection elements $Q^C_{a,m}(q)$ found in Theorem 1.2. The positivity of sums of theta quotients can be derived from Proposition 6.11 and Lemma 6.13. The inequalities with $N_i = N(i, 11, 11n)$ and $M_i = M(i, 11, 11n)$ are equivalent to

$$\begin{aligned}
(0,0,1,2,0,-1;0,-2)_0 &= [0,1,-3,0,3;0]_0 \geq 0,\\
(0,0,0,2,1,-2;0,-1)_0 &= [0,3,0,-4,4;0]_0 \geq 0,\\
(0,0,-1,0,-1,-1;0,3)_0 &= [0,4,-1,0,1;0]_0 \geq 0,\\
(0,0,-1,1,0,-2;0,2)_0 &= [0,5,0,-3,3;0]_0 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 1)$ and $M_i = M(i, 11, 11n + 1)$ are equivalent to

$$\begin{aligned}
(0,2,-1,1,-2,0;0,0,0)_1 &= [0,-4,1,-1,4;0]_1 \geq 0,\\
(0,2,-1,0,1,-2;0,0,0)_1 &= [0,-3,-3,4,1;1]_1 \geq 0,\\
(0,1,0,0,2,-2;-1,0,0)_1 &= [1,-2,-4,3,-3;2]_1 \geq 0,\\
(0,1,0,1,-3,1;0,0,0)_1 &= [0,-2,3,-4,4;0]_1 \geq 0,\\
(0,0,2,0,1,-2;-1,0,0)_1 &= [1,1,-2,0,-4;4]_1 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 2)$ and $M_i = M(i, 11, 11n + 2)$ are equivalent to

$$\begin{aligned}
(1,1,-1,0,1,0;-1,-1)_2 &= [0,0,-4,3,1;0]_2 \geq 0,\\
(0,0,-1,-1,-1,0;3,0)_2 &= [0,1,-1,0,1;-1]_2 \geq 0,\\
(0,1,1,0,0,1;-2,-1)_2 &= [0,1,0,-1,1;1]_2 \geq 0,\\
(-1,0,0,0,-1,-1;3,0)_2 &= [0,3,0,0,0;1]_2 \geq 0,\\
(-1,1,0,1,0,-2;1,0)_2 &= [1,4,-4,2,-1;3]_2 \geq 0,\\
(0,1,-1,1,0,-1;1,-1)_2 &= [0,4,-3,3,0;-1]_2 \geq 0,\\
(0,1,-1,0,-1,0;2,-1)_2 &= [0,4,-2,1,1;-2]_2 \geq 0,\\
(-1,1,1,1,0,-1;0,-1)_2 &= [0,4,-1,1,0;3]_2 \geq 0,\\
(-1,1,1,0,-1,0;1,-1)_2 &= [0,4,0,-1,1;2]_2 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+3)$ and $M_i = M(i, 11, 11n+3)$ are equivalent to

$$\begin{aligned}
(0,2,0,-1,1,-1;0,-1)_3 &= [0,-3,3,-1,1;1]_3 \geq 0,\\
(1,1,0,0,0,1;-1,-2)_3 &= [0,-1,-1,0,2;0]_3 \geq 0,\\
(1,1,-1,0,1,0;0,-2)_3 &= [0,0,0,-2,2;1]_3 \geq 0,\\
(0,0,-1,-1,0,-1;1,2)_3 &= [0,0,1,-1,1;0]_3 \geq 0,\\
(2,0,-2,0,-1,1;0,0)_3 &= [0,1,-4,-1,4;1]_3 \geq 0,\\
(1,0,0,1,-1,1;-1,-1)_3 &= [0,1,-3,1,1;1]_3 \geq 0,\\
(0,-1,0,0,-2,0;0,3)_3 &= [0,1,-2,2,0;0]_3 \geq 0,\\
(0,0,2,0,-1,1;-2,0)_3 &= [0,1,0,3,0;-3]_3 \geq 0,\\
(1,-1,0,1,-2,0;0,1)_3 &= [1,4,-4,1,0;0]_3 \geq 0,\\
(1,0,0,2,0,0;-1,-2)_3 &= [0,4,-2,0,0;2]_3 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+4)$ and $M_i = M(i, 11, 11n+4)$ are equivalent to

$$\begin{aligned}
(2,-1,0,-2,0,0;1,0)_4 &= [0,-3,1,-3,5;0]_4 \geq 0,\\
(2,-1,1,-1,0,0;0,-1)_4 &= [0,-3,3,-2,3;0]_4 \geq 0,\\
(1,-1,1,0,-1,-1;0,1)_4 &= [0,-1,5,0,0;0]_4 \geq 0,\\
(1,0,1,1,-1,-1;-1,0)_4 &= [1,0,5,-1,-1;0]_4 \geq 0,\\
(1,0,-2,-1,0,0;0,2)_4 &= [0,1,0,-5,5;0]_4 \geq 0,\\
(0,0,-1,0,-1,-1;0,3)_4 &= [0,2,2,-1,1;0]_4 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+5)$ and $M_i = M(i, 11, 11n+5)$ are equivalent to

$$\begin{aligned}
(0,0,0,2,0,-2;1,-1)_5 &= [1,-1,-1,-3,1;4]_5 \geq 0,\\
(-1,0,1,2,0,-2;0,0)_5 &= [0,-1,0,-1,3;0]_5 \geq 0,\\
(-1,0,0,1,0,-2;1,1)_5 &= [0,-1,1,-2,2;1]_5 \geq 0,\\
(0,-1,0,1,-1,-1;2,0)_5 &= [0,0,-2,1,1;1]_5 \geq 0,\\
(-1,0,2,1,0,-1;-1,0)_5 &= [0,0,0,1,3;-4]_5 \geq 0,\\
(0,1,1,-1,0,1;-2,0)_5 &= [0,0,1,3,0;-4]_5 \geq 0,\\
(-1,0,0,-1,0,-1;1,2)_5 &= [0,0,2,-1,1;-2]_5 \geq 0,\\
(0,1,1,0,1,0;-3,0)_5 &= [0,1,2,-1,1;-1]_5 \geq 0,\\
(1,0,1,0,0,1;-2,-1)_5 &= [0,2,-1,2,0;-1]_5 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n+7)$ and $M_i = M(i, 11, 11n+7)$ are equivalent to

$$\begin{aligned}
(-1,1,1,0,0,-1;-1,1)_7 &= [0,-3,-2,2,3;0]_7 \geq 0,\\
(-1,1,1,-1,-1,0;0,1)_7 &= [0,-3,-1,4,0;0]_7 \geq 0,\\
(-1,0,0,0,0,-2;1,2)_7 &= [0,-2,-2,0,4;0]_7 \geq 0,\\
(1,0,0,2,0,0;-1,-2)_7 &= [0,0,2,-2,2;0]_7 \geq 0,\\
(0,1,1,0,1,0;-2,-1)_7 &= [0,1,-1,1,1;0]_7 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 8)$ and $M_i = M(i, 11, 11n + 8)$ are equivalent to

$$\begin{aligned}
(0, 1, -1, -1, -1, 1; 0, 1)_8 &= [0, -2, 0, 1, 2; 0]_8 \geq 0,\\
(-1, 0, 0, 0, -2, 0; 1, 2)_8 &= [0, -2, 2, 2, 0; 0]_8 \geq 0,\\
(-1, 1, 1, 0, -2, 1; 0, 0)_8 &= [0, -2, 2, 4, 1; 0]_8 \geq 0,\\
(0, 1, 0, 1, -1, 1; -1, -1)_8 &= [0, -2, 3, 1, 1; 0]_8 \geq 0,\\
(0, 1, -1, 0, -2, 1; 1, 0)_8 &= [0, -2, 4, 2, 3; 0]_8 \geq 0,\\
(1, 1, -2, -1, 0, 1; 0, 0)_8 &= [0, -1, 0, -1, 3; 0]_8 \geq 0,\\
(-1, 0, 1, 1, -2, 0; 1, 0)_8 &= [0, -1, 4, 3, 0; 0]_8 \geq 0,\\
(-1, 0, 1, -1, -1, 0; 1, 1)_8 &= [0, 0, -1, 3, 0; 0]_8 \geq 0,\\
(0, 1, 1, 0, 0, 1; -1, -2)_8 &= [0, 0, 0, 2, 1; 0]_8 \geq 0,\\
(1, 1, -1, 0, 0, 1; 0, -2)_8 &= [0, 0, 2, 0, 3; 0]_8 \geq 0,\\
(-1, 0, 2, 0, -1, 0; 1, -1)_8 &= [0, 1, 1, 4, 0; 0]_8 \geq 0,\\
(1, 0, -1, 1, 0, 0; 1, -2)_8 &= [0, 1, 4, -1, 2; 0]_8 \geq 0,\\
(-1, 0, 2, -2, 0, 0; 1, 0)_8 &= [0, 2, -4, 4, 0; 0]_8 \geq 0,\\
(1, 0, -1, -1, 1, 0; 1, -1)_8 &= [0, 2, -1, -1, 2; 0]_8 \geq 0,\\
(1, 0, 0, 1, 1, 0; 0, -3)_8 &= [0, 2, 2, -1, 1; 0]_8 \geq 0,\\
(0, 0, 2, 0, 1, 0; 0, -3)_8 &= [0, 3, -1, 2, 0; 0]_8 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 9)$ and $M_i = M(i, 11, 11n + 9)$ are equivalent to

$$\begin{aligned}
(-1, 2, 0, -1, -1, 0; 0, 1)_9 &= [0, -3, 6, -2, 2; 0]_9 \geq 0,\\
(1, 0, -1, 1, 1, -1; 0, -1)_9 &= [1, -1, -2, 2, -1; 0]_9 \geq 0,\\
(0, 0, -1, -1, -1, 0; 1, 2)_9 &= [0, -1, -1, 0, 2; 0]_9 \geq 0,\\
(1, 0, -1, -1, -1, 1; 1, 0)_9 &= [0, 0, -5, 1, 4; 0]_9 \geq 0,\\
(0, 1, 1, 0, 0, 1; -1, -2)_9 &= [0, 0, 1, -1, 1; 0]_9 \geq 0,\\
(1, 0, 0, 1, 1, 0; 0, -3)_9 &= [0, 1, -2, 2, 0; 0]_9 \geq 0,\\
(1, -1, -1, 0, 0, -1; 2, 0)_9 &= [0, 2, -3, 3, 1; 0]_9 \geq 0,\\
(1, 0, 1, 0, 0, 1; 0, -3)_9 &= [0, 3, -3, 1, 2; 0]_9 \geq 0,\\
(1, -1, 0, -1, -1, 0; 2, 0)_9 &= [0, 4, -4, 2, 3; 0]_9 \geq 0,\\
(0, -1, 1, -1, -1, 0; 1, 1)_9 &= [0, 4, -1, 0, 1; 0]_9 \geq 0,\\
(1, -1, 1, 0, 0, 0; 1, -2)_9 &= [0, 5, -3, 2, 1; 0]_9 \geq 0,\\
(1, -2, 1, 0, 0, -1; 1, 0)_9 &= [1, 6, -4, 2, -1; 0]_9 \geq 0.
\end{aligned}$$

The inequalities with $N_i = N(i, 11, 11n + 10)$ and $M_i = M(i, 11, 11n + 10)$ are equivalent to

$$\begin{aligned}
(1, -1, 1, 1, -1, 0; -1, 0)_{10} &= [0, -3, 5, -1, 1; 0]_{10} \geq 0,\\
(0, -1, 0, 0, -1, -1; 2, 1)_{10} &= [0, -1, 2, 0, 1; 0]_{10} \geq 0,\\
(0, -1, 1, 1, -1, -1; 0, 1)_{10} &= [0, -1, 3, -1, 2; 0]_{10} \geq 0,\\
(-1, 0, 0, 0, 1, -2; 1, 1)_{10} &= [0, 0, -4, 3, 1; 0]_{10} \geq 0,\\
(-1, 0, 1, 1, 0, -1; -1, 1)_{10} &= [0, 2, -1, -1, 2; 0]_{10} \geq 0,
\end{aligned}$$

$$\begin{aligned}
(0,0,-1,-1,-1,0;2,1)_{10} &= [0,2,1,0,0;0]_{10} \geq 0,\\
(1,1,-1,-1,-1,1;0,0)_{10} &= [1,2,2,0,-1;0]_{10} \geq 0,\\
(-1,0,0,0,-1,-1;1,2)_{10} &= [0,4,0,-1,2;0]_{10} \geq 0,\\
(0,2,-1,-1,0,1;-1,0)_{10} &= [1,5,-2,0,-1;0]_{10} \geq 0,\\
(-1,1,0,0,0,-1;-1,2)_{10} &= [0,5,-2,1,2;0]_{10} \geq 0,\\
(0,2,0,0,0,1;-3,0)_{10} &= [1,5,-1,-1,0;0]_{10} \geq 0.
\end{aligned}$$ □

Full proof of Theorem 2.5. The calculations below can be derived directly from calculations of the dissection elements $Q_{a,m}(q)$ from Theorem 1.3. The positivity of sums of theta quotients can be derived from Proposition 6.12. The inequalities are equivalent to

$$\begin{aligned}
(0,2,1,-2,2,-3)_6 &= \vartheta(0,0,0,0,11) \geq 0,\\
(2,-2,1,-1,-2,2)_6 &= \vartheta(0,0,0,11,-11) \geq 0,\\
(1,1,-4,3,1,-2)_6 &= \vartheta(0,0,11,-11,0) \geq 0,\\
(1,6,4,-2,-5,-4)_6 &= \vartheta(0,11,-11,0,0) \geq 0.
\end{aligned}$$ □

Full proof of Corollary 2.7. The calculations below can be derived directly from calculations of the dissection elements $Q_{a,m}(q)$ from Theorem 1.3. The positivity of sums of theta quotients can be derived from Proposition 6.12. The inequalities are equivalent to

$$\begin{aligned}
(1,-1,0,1,-1,0)_6 &= \vartheta(2,0,3,3,-6) \geq 0,\\
(0,1,1,-1,1,-2)_6 &= \vartheta(1,0,0,1,6) \geq 0,\\
(0,-1,1,2,-1,-1)_6 &= \vartheta(5,0,2,2,-4) \geq 0,\\
(1,1,-1,0,1,-2)_6 &= \vartheta(0,0,5,-2,3) \geq 0,\\
(0,0,-1,3,0,-2)_6 &= \vartheta(4,0,6,-5,-1) \geq 0,\\
(1,0,-2,2,0,-1)_6 &= \vartheta(1,0,7,-4,-3) \geq 0.
\end{aligned}$$ □

Acknowledgements

I would like to offer my gratitude to my scientific advisor Eric T. Mortenson for the idea to work through deviations of the crank and rank modulo 11 and for helpful comments and crucial suggestions on references. I also want to thank Wadim Zudilin for pointing out [31] and Jeremy Lovejoy for suggesting corrections to the article.

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