Adams–Hilton models and higher Whitehead brackets for polyhedral products

Elizaveta Zhuravleva

Abstract. In this paper, we construct Adams–Hilton models for the polyhedral products of spheres $(\underline{S})^{\mathcal{K}}$ and Davis–Januszkiewicz spaces $(\mathbb{C}P^\infty)^{\mathcal{K}}$. We show that in these cases the Adams–Hilton model can be chosen so that it coincides with the cobar construction of the homology coalgebra. We apply the resulting models to the study of iterated higher Whitehead products in $(\mathbb{C}P^\infty)^{\mathcal{K}}$. Namely, we explicitly construct a chain in the cobar construction representing the homology class of the Hurewicz image of a Whitehead product.

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Introduction

For each simplicial complex $\mathcal{K}$ the following two topological spaces are defined, which are the main objects in toric topology: the moment–angle complex $\mathcal{Z}_{\mathcal{K}}$ and the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^{\mathcal{K}}$. These spaces are particular examples of a more general construction, the polyhedral product $(\underline{X}, \underline{A})^{\mathcal{K}}$; see [**9**] for the details of this construction.

The homotopy theory of polyhedral products is an active research area, see the survey [**8**]. In this paper, we look into the Pontryagin algebra $H_*(\Omega\,(\mathbb{C}P^\infty)^{\mathcal{K}})$ of the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^{\mathcal{K}}$. The Pontryagin algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ of the moment–angle complex can be identified with the commutator subalgebra of $H_*(\Omega\,(\mathbb{C}P^\infty)^{\mathcal{K}})$.

In [**16**], the Pontryagin algebra $H_*(\Omega\,(\mathbb{C}P^\infty)^{\mathcal{K}})$ was studied from the categorical point of view. In particular, the Pontryagin algebra was described completely there by generators and relations in the case when $\mathcal{K}$ is a flag complex. A set of multiplicative generators for $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ was presented in [**12**]. When $\mathcal{K}$ is not flag, nontrivial higher Whitehead products appear in the Pontryagin algebra $H_*(\Omega\,(\mathbb{C}P^\infty)^{\mathcal{K}})$ and the problem of describing its generators and relations becomes difficult.

Higher Whitehead products are invariants of unstable homotopy type, they have been studied since the 1960s, see [**14**], [**17**], [**20**]. Nowadays, they play an important role in the study of polyhedral products. A large class of simplicial complexes is formed by those $\mathcal{K}$ for which $\mathcal{Z}_{\mathcal{K}}$ is a wedge of spheres, and each sphere is mapped as a higher iterated Whitehead product to the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^{\mathcal{K}}$ [**16, 2**]. However, in general it is not true that if the moment–angle complex $\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent a wedge of spheres, then each sphere is represented by some Whitehead product [**1**].

Adams' cobar construction [**3**] on the singular chains of X gives a model for the Pontryagin algebra $H_*(\Omega X)$; however, it is not easy to work with. We show that for some polyhedral products one may work directly with the cobar construction of the homology coalgebra. The resulting algebraic models can be calculated explicitly. This constitutes our first result.

Theorem (Prop. 3.8). *Let $(\underline{S})^{\mathcal{K}}$ be a polyhedral product of simply connected spheres. We regard its homology $H_*((\underline{S})^{\mathcal{K}})$ with integer coefficients as a dg coalgebra with zero differential. Then its cobar construction is given by*

$$\operatorname{Cobar} H_*((\underline{S})^{\mathcal{K}}) \cong (T(U), \partial),\ U = \langle b_J,\ J \in \mathcal{K},\ J \neq \varnothing\rangle,$$

$$\partial(b_{j_1\cdots j_s}) = \sum_{p=1}^{s-1} \sum_{\theta\in\widetilde{S}(p,s-p)} \varepsilon(\theta)(-1)^{|b_{j_{\theta(1)}\cdots j_{\theta(p)}}|+1}\big[b_{j_{\theta(1)}\cdots j_{\theta(p)}}, b_{j_{\theta(p+1)}\cdots j_{\theta(s)}}\big].$$

The homology of $\operatorname{Cobar} H_*((\underline{S})^{\mathcal{K}})$ *is isomorphic to the Pontryagin algebra $H_*(\Omega((\underline{S})^{\mathcal{K}}))$.*

A similar statement holds for the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^{\mathcal{K}}$ [**9**].

For a simply connected CW-complex X, Adams and Hilton [**4**] constructed a dga $\mathbf{AH}(X)$ over $\mathbb{Z}$. Its underlying algebra $T(V)$ is free on a set of generators V corresponding to the cells of X. The homology of the Adams–Hilton model is the Pontryagin algebra $H_*(\Omega X)$. Adams–Hilton models have already arisen in the study of polyhedral products [**13, 11**]. There is an indeterminacy in the construction of $\mathbf{AH}(X)$, so there is a family of Adams–Hilton models for a given X. However, in the case of our polyhedral products, we prove that there exists a canonical Adams–Hilton model, which coincides with the cobar construction of the homology coalgebra. In general, it is not true that an Adams–Hilton model is isomorphic to the cobar construction of some coalgebra.

Theorem (Th. 4.16). *Let $\mathcal{K}$ be an arbitrary simplicial complex. Then there exist Adams–Hilton models $\mathbf{AH}((\underline{S})^{\mathcal{K}})$ and $\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}})$, such that*

$$\mathbf{AH}((\underline{S})^{\mathcal{K}}) \cong \operatorname{Cobar} H_*((\underline{S})^{\mathcal{K}}), \qquad \mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}) \cong \operatorname{Cobar} H_*((\mathbb{C}P^\infty)^{\mathcal{K}})$$

in the category DGA.

Adams–Hilton models behave nicely with respect to CW-maps. If we fix models $\mathbf{AH}(X)$ and $\mathbf{AH}(Y)$ and a CW-map $f\colon X \to Y$, there is a map $AH(f)\colon \mathbf{AH}(X) \to \mathbf{AH}(Y)$, which in some cases can be calculated explicitly. This turns out to be useful when we work with higher Whitehead products.

Theorem (Th. 5.6). *Let $[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5] \in \pi_7(\Omega\,(\mathbb{C}P^\infty)^{\mathcal{K}})$ be a canonical iterated Whitehead product in $(\mathbb{C}P^\infty)^{\mathcal{K}}$, given by the composite*

$$S^8 \longrightarrow T(S^5, S^2, S^2) \longrightarrow (S^2)^{\mathcal{K}} \longrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Then Adams–Hilton models can be constructed explicitly for each map in the sequence above:

$$\mathbf{AH}(S^8) \longrightarrow \mathbf{AH}(T(S^5, S^2, S^2)) \longrightarrow \mathbf{AH}((S^2)^{\mathcal{K}}) \longrightarrow \mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}).$$

Furthermore, the Hurewicz image of the higher Whitehead product $[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5]$ is represented by the following cycle in the cobar construction $\operatorname{Cobar} H_*((\mathbb{C}P^\infty)^{\mathcal{K}}) \cong \mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}})$*:*

$$-\partial\big([\chi_{123}, \chi_{45}] + [\chi_{1234}, \chi_5] + [\chi_{1235}, \chi_4]\big).$$

1. Preliminaries

A *simplicial complex* $\mathcal{K}$ on the set $[m] = \{1, 2, \ldots, m\}$ is a collection of subsets $I \subset [m]$ closed under taking any subsets. We assume that $\mathcal{K}$ contains $\varnothing$ and all one-element subsets $\{i\},\ i = 1, \ldots, m$. We refer to $I \in \mathcal{K}$ as a *simplex* of $\mathcal{K}$. Denote by Δ^{m-1} or by $\Delta(1, \ldots, m)$ the *full simplex* on the set $[m]$. It is a simplicial complex consisting of all subsets of $[m]$. Similarly, denote by $\Delta(I)$ the full simplex on the set $I = \{i_1, \ldots, i_s\}$. The condition $I \in \mathcal{K}$ implies $\Delta(I) \subset \mathcal{K}$. The simplicial complex $\partial\Delta(I)$ consists of all simplices J, $J \in \Delta(I)$, except the simplex I.

Construction 1.1. Let $\mathcal{K}$ be a simplicial complex on the set $[m]$. Suppose we are given an ordered set of m pairs of based CW-complexes

$$(\underline{X}, \underline{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}.$$

For each $I \in \mathcal{K}$ we consider the set

$$(\underline{X}, \underline{A})^I = \{(x_1, \ldots, x_m) \in \prod_{k=1}^{m} X_k \mid x_k \in A_k \text{ if } k \notin I\}.$$

We define the *polyhedral product* of pairs $(\underline{X}, \underline{A})$ corresponding to $\mathcal{K}$ as $(\underline{X}, \underline{A})^{\mathcal{K}}$, where

$$(\underline{X}, \underline{A})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\underline{X}, \underline{A})^I = \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in \mathcal{K}} X_i \times \prod_{i \notin \mathcal{K}} A_i \Big) \subset \prod_{k=1}^{m} X_k.$$

If $X = X_i$ and $A = A_i$ for all i we use the notation $(X, A)^{\mathcal{K}}$ instead of $(\underline{X}, \underline{A})^{\mathcal{K}}$. If $A_i = pt$ for all i we write $(\underline{X})^{\mathcal{K}}$.

Example 1.2. The polyhedral product $(D^2, S^1)^{\mathcal{K}}$ is called the *moment–angle complex*; it is denoted by $\mathcal{Z}_{\mathcal{K}}$.

Example 1.3. The polyhedral product $(\mathbb{C}P^\infty)^{\mathcal{K}} = (\mathbb{C}P^\infty, pt)^{\mathcal{K}}$ is homotopy equivalent to the *Davis–Januszkiewicz space* $DJ(\mathcal{K})$.

Example 1.4. Denote by $(\underline{S})^{\mathcal{K}} = (\underline{S}, pt)^{\mathcal{K}}$ the polyhedral product of spheres:

$$(\underline{S}, pt)^{\mathcal{K}} = \big((S^{n_1}, pt), \ldots, (S^{n_m}, pt)\big)^{\mathcal{K}}, \quad n_i \geqslant 2 \text{ for each } i.$$

Note that we assume all spheres to be simply connected. A special case is the space $(S^2)^{\mathcal{K}}$.

Theorem 1.5 ([**9**, Th. 4.3.2]). *The moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is the homotopy fibre of the canonical inclusion $(\mathbb{C}P^\infty)^{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m$.*

The fibre inclusion $\mathcal{Z}_{\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}$ can be described explicitly. Consider the map of pairs $(D^2, S^1) \to (\mathbb{C}P^\infty, pt)$ which sends the interior of D^2 homeomorphically onto the 2-dimensional cell in $\mathbb{C}P^1$, and sends the boundary of the disk to the basepoint. Then we have the induced map of the polyhedral products $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}} \to (\mathbb{C}P^\infty, pt)^{\mathcal{K}}$.

Corollary 1.6. *There is an exact sequence of rational homotopy Lie algebras*

$$0 \to \pi_*(\Omega\mathcal{Z}_{\mathcal{K}}) \otimes \mathbb{Q} \to \pi_*(\Omega\, (\mathbb{C}P^\infty)^{\mathcal{K}}) \otimes \mathbb{Q} \to \langle u_1, \ldots, u_m \rangle_{\mathbb{Q}} \to 0$$

where $\langle u_1, \ldots, u_m \rangle_{\mathbb{Q}}$ denotes the m-dimensional vector space with trivial Lie bracket, $|u_i| = 1$, and an exact sequence of Pontryagin algebras

$$1 \to H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k}) \to H_*(\Omega\, (\mathbb{C}P^\infty)^{\mathcal{K}}; \boldsymbol{k}) \to \Lambda[u_1, \ldots, u_m] \to 0$$

for any commutative ring $\boldsymbol{k}$ with unit.

Corollary 1.7. *Any map $S^n \to (\mathbb{C}P^\infty)^{\mathcal{K}}$ with $n \geqslant 3$ lifts to a map $S^n \to \mathcal{Z}_{\mathcal{K}}$.*

The following combinatorial construction is crucial for the definition of canonical higher Whitehead brackets for polyhedral products; it can be found in [**2**].

Definition 1.8. Let $\mathcal{K}$ be a simplicial complex on the set of vertices $[m]$, and let $\mathcal{K}_1,\dots,\mathcal{K}_m$ be an ordered collection of simplicial complexes such that sets of vertices of each two $\mathcal{K}_i$ and $\mathcal{K}_j$ do not intersect. The *substitution complex* $\mathcal{K}(\mathcal{K}_1,\dots,\mathcal{K}_m)$ is defined as follows:

$$\mathcal{K}(\mathcal{K}_1,\dots,\mathcal{K}_m)=\{I_{j_1}\sqcup\dots\sqcup I_{j_k}\mid I_{j_l}\in\mathcal{K}_{j_l},\ l=1,\dots,k,\ \{j_1,\dots,j_k\}\in\mathcal{K}\}.$$

It is easy to see that $\mathcal{K}(\mathcal{K}_1,\dots,\mathcal{K}_m)$ is a simplicial complex.

Example 1.9. Suppose each $\mathcal{K}_i$ is a single vertex $\{i\}$; then $\mathcal{K}(\mathcal{K}_1,\dots,\mathcal{K}_m)=\mathcal{K}$. In particular,

$$\partial\Delta(1,\dots,m)=\partial\Delta^{m-1},\quad \Delta(1,\dots,m)=\Delta^{m-1},$$

in accordance with the previous notation.

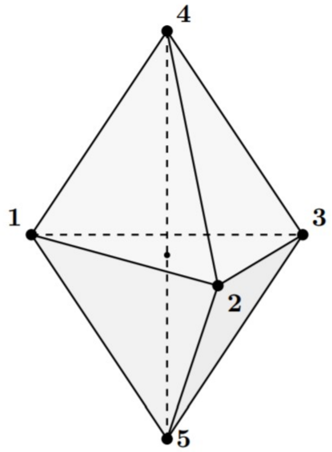

Figure 1. Substitution complex $\partial\Delta(\partial\Delta(1,2,3),4,5)$.

Example 1.10. Let $\mathcal{K}=\partial\Delta^2$, let $\mathcal{K}_1=\partial\Delta(1,2,3)$, let $\mathcal{K}_2=\{\varnothing,\{4\}\}$, and let $\mathcal{K}_2=\{\varnothing,\{5\}\}$. Then the substitution complex has the following form:

$$\mathcal{K}(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3)=\partial\Delta\big(\partial\Delta(1,2,3),4,5\big).$$

It is shown in Fig. 1.

2. Canonical higher Whitehead products for the Davis–Januszkiewicz space

First, we give the definition of a (general) higher Whitehead product, following [**5**], [**14**]. Given the spheres $S^{n_1},\dots,S^{n_k}, n_j\geqslant 2$, we consider the wedge $S^{n_1}\vee\dots\vee S^{n_k}$ and the *fat wedge* $T\,(S^{n_1},\dots,S^{n_k})$, which is defined by the equation

$$S^{n_1}\times\dots\times S^{n_k}=T\,(S^{n_1},\dots,S^{n_k})\cup_\omega e^N,$$

where e^N, $N=n_1+\dots+n_k$, is the top cell in the product $S^{n_1}\times\dots\times S^{n_k}$. Let $\omega\colon S^{N-1}\to T\,(S^{n_1},\dots,S^{n_k})$ be the attaching map that corresponds to this cell.

Definition 2.1. Let X be a topological space. Suppose given homotopy classes $x_j\in\pi_{n_j}(X)$, $j=1,\dots,k$. Denote by g the induced map $g=x_1\vee\ldots\vee x_k\colon S^{n_1}\vee\dots\vee S^{n_k}\to X$. The *(general) higher Whitehead product* $[x_1,\dots,x_k]_W\subset\pi_{N-1}(X)$ is the set of homotopy classes

$$\{f\circ\omega\mid f\colon T\,(S^{n_1},\dots,S^{n_k})\to X \text{ is an extension of } g\}.$$

This is described by the diagram

$$\begin{array}{ccc} & S^{n_1}\vee\cdots\vee S^{n_k} & \xrightarrow{\ g\ } X \\ & \big\downarrow & \nearrow_{f} \\ S^{N-1} \xrightarrow{\ \omega\ } & T\left(S^{n_1},\ldots,S^{n_k}\right) & \end{array}$$

We say that the higher Whitehead product is *trivial* if $0\in[x_1,\ldots,x_k]_W$.

Remark 2.2. In the case $k=2$, the fat wedge $T(S^{n_1},S^{n_2})$ coincides with the wedge $S^{n_1}\vee S^{n_2}$ and we obtain the classical definition of the Whitehead product:

$$[x_1,x_2]_W\colon S^{N-1}\xrightarrow{\omega}S^{n_1}\vee S^{n_2}\xrightarrow{g}X.$$

From now on we work with the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^{\mathcal{K}}$. As defined above, a (general) higher Whitehead product is a set of homotopy classes. However, in the case of $(\mathbb{C}P^\infty)^{\mathcal{K}}$ a canonical representative can be chosen in this set.

We define 2-dimensional classes μ_i:

$$\mu_i\colon S^2\cong\mathbb{C}P^1\hookrightarrow\mathbb{C}P^\infty\hookrightarrow(\mathbb{C}P^\infty)^{\vee m}\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}}.$$

The map $\mathbb{C}P^\infty\hookrightarrow(\mathbb{C}P^\infty)^{\vee m}$ is the inclusion of the ith wedge summand. The inclusion $(\mathbb{C}P^\infty)^{\vee m}\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}}$ is well defined since $\mathcal{K}$ contains all $\{i\},\ i=1,\ldots,m$.

Consider the Whitehead product of μ_i and μ_j, $i\neq j$:

$$\begin{aligned}[]
[\mu_i,\mu_j]_W\colon S^3\cong\partial(D^2\times D^2)\cong D^2\times S^1\cup S^1\times D^2\to S^2_i\times pt\cup pt\times S^2_j\\
\cong(S^2)^{\partial\Delta(i,j)}\hookrightarrow(\mathbb{C}P^\infty)^{\partial\Delta(i,j)}\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}}.
\end{aligned}$$

Here the composite $S^3\to S^2\vee S^2$ is the attaching of the top cell in $S^2\times S^2$. The map $S^2\vee S^2\to(\mathbb{C}P^\infty)^{\mathcal{K}}$ coincides with $\mu_i\vee\mu_j$.

Definition 2.3. Let the indices $i_1,\ldots,i_n$ be pairwise different. Suppose that $\partial\Delta(i_1,\ldots,i_n)\subset\mathcal{K}$. The *canonical higher Whitehead product* $[\mu_{i_1},\ldots,\mu_{i_n}]$ is the element in $\pi_{2n-1}((\mathbb{C}P^\infty)^{\mathcal{K}})$ defined by

$$[\mu_{i_1},\ldots,\mu_{i_n}]\colon S^{2n-1}\cong\partial(D^2_{i_1}\times\cdots\times D^2_{i_n})\cong\bigcup_{k=1}^{n}(D^2_{i_1}\times\cdots\times S^1_{i_k}\times\cdots\times D^2_{i_n})\to$$

$$\to\bigcup_{k=1}^{n}(S^2_{i_1}\times\cdots\times pt\times\cdots\times S^2_{i_n})\cong(S^2)^{\partial\Delta(i_1,\ldots,i_n)}\hookrightarrow(\mathbb{C}P^\infty)^{\partial\Delta(i_1,\ldots,i_n)}\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Here the map to the fat wedge $S^{2n-1}\to(S^2)^{\partial\Delta(i_1,\ldots,i_n)}\cong T(S^2,\ldots,S^2)$ coincides with the attaching map of the top cell in the product $(S^2)^n$. In the case of the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^{\mathcal{K}}$, there is a canonical map from the fat wedge $T(S^2,\ldots,S^2)\cong(S^2)^{\partial\Delta(i_1,\ldots,i_n)}\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}}$ (the indices $i_1,\ldots,i_n$ are pairwise different). Since the map $(S^2)^{\partial\Delta(i_1,\ldots,i_n)}\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}}$ extends the map

$$\mu_{i_1}\vee\cdots\vee\mu_{i_n}\colon S^2\vee\cdots\vee S^2\hookrightarrow(\mathbb{C}P^\infty)^{\mathcal{K}},$$

the canonical higher Whitehead product $[\mu_{i_1},\ldots,\mu_{i_n}]$ is contained in the (general) Whitehead product $[\mu_{i_1},\ldots,\mu_{i_n}]_W$, defined as a set in Definition 2.1.

The canonical higher Whitehead product $[\mu_{i_1},\ldots,\mu_{i_n}]$ is well defined if and only if $\partial\Delta(i_1,\ldots,i_n)\subset\mathcal{K}$. It is easy to show that this homotopy class equals zero when $\Delta(i_1,\ldots,i_n)\subset\mathcal{K}$.

For the general higher Whitehead product $[\mu_{i_1},\ldots,\mu_{i_n}]_W$ to be well defined, it is necessary that $[\mu_{i_1},\ldots,\widehat{\mu_{i_k}},\ldots,\mu_{i_n}]_W=0,\ k=1,\ldots,n$.

Proposition 2.4. *The (general) higher Whitehead product* $[\mu_{i_1},\ldots,\mu_{i_n}]_W$ *is*

(1) *defined in* $\pi_{2n-1}((\mathbb{C}P^\infty)^{\mathcal{K}})$ *if and only if* $\partial\Delta(i_1,\dots,i_n) \subset \mathcal{K}$;
(2) *trivial if and only if* $\Delta(i_1,\dots,i_n) \subset \mathcal{K}$.

Further we consider a bracket sequence with 2-dimensional classes μ_i such that this sequence contains only pairwise different indices, for example:

$$\Big[\mu_1,\mu_2,[\mu_3,\mu_4,\mu_5],\big[\mu_6,[\mu_7,\mu_8,\mu_9],\mu_{10}\big],[\mu_{11},\mu_{12}]\Big].$$

Construction 2.5. [**2**, Constr. 4.4] Here we define the substitution simplicial complex $\partial\Delta_w$ corresponding to a bracket sequence w with pairwise different indices. The bracket $[\mu_{i_1},\dots,\mu_{i_n}]$ corresponds to the complex $\partial\Delta(i_1,\dots,i_n)$. Suppose we have a bracket sequence w of the form

$$[w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}],$$

where $w_1,\dots,w_q$ are bracket sequences with the corresponding simplicial complexes $\partial\Delta_{w_1},\dots,\partial\Delta_{w_q}$. Then we assign to the the bracket sequence w the following substitution simplicial complex (see Definition 1.8):

$$\partial\Delta_w := \partial\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q},i_1,\dots,i_p).$$

We also define

$$\Delta_w := \Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q},i_1,\dots,i_p).$$

Definition 2.6. Assume that $\partial\Delta_w \subset \mathcal{K}$. The *canonical higher Whitehead product* w that corresponds to the bracket sequence $[w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]$ is defined inductively as follows. The bracket $[\mu_{i_1},\dots,\mu_{i_n}] \in \pi_{2n-1}((\mathbb{C}P^\infty)^{\mathcal{K}})$ is described in Definition 2.3. By induction, suppose that canonical higher Whitehead products $w_1,\dots,w_q$, $\deg w_i = k_i$, are already defined. Following [**2**, Lemma 3.2], define $[w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]$ as the following element of $\pi_*((\mathbb{C}P^\infty)^{\mathcal{K}})$:

$$\begin{aligned}
&[w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]: S^{k_1+\dots+k_q+2p-1} \cong \partial(D^{k_1}\times\dots\times D^{k_q}\times D^2_{i_1}\times\dots\times D^2_{i_p})\\
&\cong\left(\Big(\bigcup_{i=1}^q(D^{k_1}\times\dots\times S^{k_i-1}\times\dots\times D^{k_q})\Big)\times D^2_{i_1}\times\dots\times D^2_{i_p}\right)\cup\\
&\qquad\cup\left(D^{k_1}\times\dots\times D^{k_q}\times\Big(\bigcup_{j=1}^p(D^2_{i_1}\times\dots\times S^1_{i_j}\times\dots\times D^2_{i_p})\Big)\right)\\
&\xrightarrow{\alpha}\left(\Big(\bigcup_{i=1}^q(S^{k_1}\times\dots\times pt\times\dots\times S^{k_q})\Big)\times(D^2,S^1)^{\Delta(i_1,\dots,i_p)}\right)\cup\\
&\qquad\cup\left(S^{k_1}\times\dots\times S^{k_q}\times(D^2,S^1)^{\partial\Delta(i_1,\dots,i_p)}\right)\\
&\xrightarrow{\beta}\left(\Big(\bigcup_{i=1}^q(S^{k_1}\times\dots\times pt\times\dots\times S^{k_q})\Big)\times(S^2)^{\Delta(i_1,\dots,i_p)}\right)\cup\\
&\qquad\cup\left(S^{k_1}\times\dots\times S^{k_q}\times(S^2)^{\partial\Delta(i_1,\dots,i_p)}\right)\\
&\xrightarrow{\gamma}\left(\Big(\bigcup_{i=1}^q((S^2)^{\partial\Delta_{w_1}}\times\dots\times pt\times\dots\times(S^2)^{\partial\Delta_{w_q}})\Big)\times(S^2)^{\Delta(i_1,\dots,i_p)}\right)\cup\\
&\qquad\cup\left((S^2)^{\partial\Delta_{w_1}}\times\dots\times(S^2)^{\partial\Delta_{w_q}}\times(S^2)^{\partial\Delta(i_1,\dots,i_p)}\right)\\
&\cong\left((S^2)^{\partial\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q})}\times(S^2)^{\Delta(i_1,\dots,i_p)}\right)\cup\left((S^2)^{\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q})}\times(S^2)^{\partial\Delta(i_1,\dots,i_p)}\right)
\end{aligned}$$

$$\cong (S^2)^{\partial\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q},i_1,\dots,i_p)} \cong (S^2)^{\partial\Delta_w} \cong (\mathbb{C}P^1)^{\partial\Delta_w} \overset{\delta}{\hookrightarrow} (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Here the map α is induced by collapsing the boundaries of the disks D^{k_i} to the point. The analogous map of the pairs $(D^2, S^1) \to (S^2, pt)$ induces the map of polyhedral products $(D^2, S^1)^{\mathcal{K}} \to (S^2, pt)^{\mathcal{K}}$, which we use to define the map β. The map γ is defined using the maps $S^{k_i} \xrightarrow{w_i} (S^2)^{\partial\Delta_{w_i}} \hookrightarrow (\mathbb{C}P^\infty)^{\partial\Delta_{w_i}}$ from the previous step. Since $\mathcal{K} \supset \partial\Delta_w = \partial\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q}, i_1,\dots,i_p)$, the last map δ is well defined.

Proposition 2.7. *Suppose that $\partial\Delta_w \subset \mathcal{K}$. Then the canonical higher Whitehead product w is contained in the corresponding (general) Whitehead product.*

Proof. The domain of the map γ from Definition 2.6,

$$\Bigg(\Big(\bigcup_{i=1}^{q}(S^{k_1}\times\dots\times pt\times\dots\times S^{k_q})\Big)\times(S^2)^{\Delta(i_1,\dots,i_p)}\Bigg)\cup\Bigg(S^{k_1}\times\dots\times S^{k_q}\times(S^2)^{\partial\Delta(i_1,\dots,i_p)}\Bigg)$$

is the fat wedge $T(S^{k_1},\dots,S^{k_q},S^2_{i_1},\dots,S^2_{i_p})$. The composition $\beta\circ\alpha$

$$S^{k_1+\dots+k_q+2p-1} \xrightarrow{\beta\circ\alpha} T(S^{k_1},\dots,S^{k_q},S^2_{i_1},\dots,S^2_{i_p})$$

coincides with the attaching map of the top cell in the product $S^{k_1}\times\dots\times S^{k_q}\times S^2_{i_1}\times\dots\times S^2_{i_p}$. The restriction of the map $\delta\circ\gamma$

$$T(S^{k_1},\dots,S^{k_q},S^2_{i_1},\dots,S^2_{i_p}) \xrightarrow{\gamma} (S^2)^{\partial\Delta_w} \overset{\delta}{\hookrightarrow} (\mathbb{C}P^\infty)^{\mathcal{K}}$$

to the wedge $S^{k_1}\vee\dots\vee S^{k_q}\vee S^2_{i_1}\vee\dots\vee S^2_{i_p}$ is the wedge of maps $w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}$. Therefore, the composition $w = \delta\circ\gamma\circ\beta\circ\alpha$

$$w = [w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}] : S^{k_1+\dots+k_q+2p-1}$$
$$\xrightarrow{\beta\circ\alpha} T(S^{k_1},\dots,S^{k_q},S^2_{i_1},\dots,S^2_{i_p}) \xrightarrow{\delta\circ\gamma} (\mathbb{C}P^\infty)^{\mathcal{K}},$$

is indeed an element of the (general) higher Whitehead product from Definition 2.1. □

Corollary 2.8. *Assume that $\partial\Delta_w \subset \mathcal{K}$. Then the (general) Whitehead product w is nonempty.*

Question 2.9. *Let $w = [w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]_W$ be a (general) Whitehead product. Suppose that the product w is defined in $\pi_*((\mathbb{C}P^\infty)^{\mathcal{K}})$, in other words, there exists at least one extension from the wedge to the fat wedge in Definition 2.1. Does it imply that $\partial\Delta_w \subset \mathcal{K}$?*

Theorem 2.10 ([**2**, Th. 5.1]). *Let $\mathcal{K} = \partial\Delta_w$. Then the canonical higher Whitehead product $w = [w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]$ is nontrivial in $\pi_{k_1+\dots+k_q+2p-1}((\mathbb{C}P^\infty)^{\mathcal{K}})$.*

Construction 2.11. Let w be a canonical higher Whitehead product. Suppose that $\mathcal{K} \supset \Delta_w$. Then the *canonical extension* of w to the disk is defined:

$$D^{k_1+\dots+k_q+2p} \cong D^{k_1}\times\dots\times D^{k_q}\times D^2_{i_1}\times\dots\times D^2_{i_p} \to S^{k_1}\times\dots\times S^{k_q}\times S^2_{i_1}\times\dots\times S^2_{i_p}$$
$$\to (S^2)^{\partial\Delta_{w_1}}\times\dots\times(S^2)^{\partial\Delta_{w_q}}\times S^2_{i_1}\times\dots\times S^2_{i_p} \cong (S^2)^{\partial\Delta_{w_1}}\times\dots\times(S^2)^{\partial\Delta_{w_q}}\times(S^2)^{\Delta(i_1,\dots,i_p)}$$
$$\cong (S^2)^{\partial\Delta_{w_1}*\dots*\partial\Delta_{w_q}*\Delta(i_1,\dots,i_p)} \cong (S^2)^{\Delta(\partial\Delta_{w_1},\dots,\partial\Delta_{w_q},i_1,\dots,i_p)} \hookrightarrow (\mathbb{C}P^\infty)^{\mathcal{K}}.$$

This implies that $w = [w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}] = 0$.

It follows that a canonical higher Whitehead product $w = [w_1,\dots,w_q,\mu_{i_1},\dots,\mu_{i_p}]$ is well defined if and only if $\partial\Delta_w \subset \mathcal{K}$ and equals zero in $\pi_*((\mathbb{C}P^\infty)^{\mathcal{K}})$ if $\Delta_w \subset \mathcal{K}$, as observed in [**2**].

QUESTION 2.12. *Let $w = [w_1, \dots, w_q, \mu_{i_1}, \dots, \mu_{i_p}]_W$ be a canonical higher Whitehead product. Suppose that $w_j \neq 0$ for $j = 1, \dots, q$ and $w = 0$ in $\pi_*((\mathbb{C}P^\infty)^\mathcal{K})$. Does it imply that $\Delta_w \subset \mathcal{K}$?*

The answer is positive for products of depth at most two:

THEOREM 2.13 ([**2**, Th. 5.2]). *Let $w_j = [\mu_{j_1}, \dots, \mu_{j_{p_j}}]$, $j = 1, \dots, q$, be nontrivial canonical higher Whitehead products. Consider the following higher Whitehead product:*

$$w = [w_1, \dots, w_q, \mu_{i_1}, \dots, \mu_{i_p}].$$

Then $w = 0$ in $\pi_*((\mathbb{C}P^\infty)^\mathcal{K})$ if and only if $\mathcal{K}$ contains Δ_w as a subcomplex.

QUESTION 2.14. *Suppose that the higher Whitehead product $w = [w_1, \dots, w_q, \mu_{i_1}, \dots, \mu_{i_p}]_W$ is trivial and w_j are nontrivial for $j = 1, \dots, q$. Does it imply that $\Delta_w \subset \mathcal{K}$?*

3. Cobar construction for polyhedral products

We consider the cobar construction (the algebraic loop functor) Cobar from the category of 1-connected differential graded coalgebras $DGC_{1,\mathbb{Z}}$ to the category of differential graded connected algebras $DGA_{0,\mathbb{Z}}$. Its right adjoint is the bar construction (the algebraic classifying functor) Bar:

$$\mathrm{Cobar}\colon DGC_{1,\mathbb{Z}} \leftrightarrows DGA_{0,\mathbb{Z}} : \mathrm{Bar}\,.$$

The *cobar functor* Cobar assigns to a dg coalgebra (C, ∂_C) with $C_0 = \mathbb{Z}$ and $C_1 = 0$ the dg algebra

$$\mathrm{Cobar}\, C = (T(s^{-1}\overline{C}), \partial)$$

that is the free associative algebra on the desuspended module $\overline{C} = \mathrm{Coker}(C_0 \cong \mathbb{Z} \hookrightarrow C)$. The differential ∂ is given by

$$\partial(s^{-1}c) = -s^{-1}\partial_C(c) + \sum_i (-1)^{|x_i|} s^{-1}x_i \otimes s^{-1}y_i \tag{3.1}$$

where $c \in C$ with comultiplication $\Delta c = c \otimes 1 + 1 \otimes c + \sum_i x_i \otimes y_i$.

THEOREM 3.1 ([**3**]). *For a simply connected pointed space X and a commutative ring $\mathbf{k}$ with unit, there is a natural isomorphism of graded algebras*

$$H(\mathrm{Cobar}\, C_*(X; \mathbf{k})) \cong H_*(\Omega X; \mathbf{k}),$$

where $C_*(X; \mathbf{k})$ denotes the suitably reduced singular chain complex of X.

Now we turn to polyhedral products. Recall that $\underline{X}$ stands for the sequence of topological spaces $(X_1, \dots, X_m)$. Also, by $\underline{X}^\mathcal{K}$ we denote the polyhedral product $\big((X_1, pt), \dots, (X_m, pt)\big)^\mathcal{K}$.

THEOREM 3.2 ([**9**, Th. 8.1.2]). *If each space X_i in $\underline{X} = (X_1, \dots, X_m)$ is formal, then the polyhedral product $\underline{X}^\mathcal{K}$ is also formal.*

It follows that the Davis–Januszkiewicz space $(\mathbb{C}P^\infty)^\mathcal{K}$ and the polyhedral product of spheres $(\underline{S})^\mathcal{K} = \big((S^{n_1}, pt), \dots, (S^{n_k}, pt)\big)^\mathcal{K}$ are formal. As shown in [**15**], the space $(\mathbb{C}P^\infty)^\mathcal{K}$ is also $\mathbb{Z}$-formal. The same holds for $(\underline{S})^\mathcal{K}$.

The *face algebra* $\mathbb{Z}[\mathcal{K}]$ is the quotient dg algebra with zero differential

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}_\mathcal{K}, \ |v_i| = 2,$$

where $\mathcal{I}_\mathcal{K} = (v_I \mid I \notin \mathcal{K})$ is the ideal generated by those monomials $v_I = v_{i_1} \cdots v_{i_s}$ for which $I = \{i_1, \dots, i_s\}$ is not a simplex in $\mathcal{K}$.

The *face coalgebra* $\mathbb{Z}\langle\mathcal{K}\rangle$ is the graded dual of the face algebra $\mathbb{Z}[\mathcal{K}]$. We consider multisets σ of m elements,

$$\sigma = \{\underbrace{1,\dots,1}_{k_1},\underbrace{2,\dots,2}_{k_2},\dots,\underbrace{m,\dots,m}_{k_m}\}$$

such that the *support* of σ (i.e. the set $I_\sigma = \{i \in [m] \mid k_i \neq 0\}$) is a simplex in $\mathcal{K}$. Then $\mathbb{Z}\langle\mathcal{K}\rangle$ has an additive basis consisting of the elements c_σ, each c_σ is dual to the monomial $v_1^{k_1}v_2^{k_2}\cdots v_m^{k_m} \in \mathbb{Z}[\mathcal{K}]$. The coproduct is given by

$$\Delta c_\sigma = \sum_{(\tau,\tau'),\ \sigma=\tau\sqcup\tau'} c_\tau \otimes c_{\tau'},$$

where the sum ranges over all ordered partitions of σ into submultisets τ and τ'.

PROPOSITION 3.3. *We have*

(a) *the cohomology algebra* $H^*((\mathbb{C}P^\infty)^\mathcal{K};\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}[\mathcal{K}]$,
(b) *the homology coalgebra* $H_*((\mathbb{C}P^\infty)^\mathcal{K};\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}\langle\mathcal{K}\rangle$.

Consider the cobar construction $\operatorname{Cobar}\mathbb{Z}\langle\mathcal{K}\rangle$. By the definition above, we obtain

$$\operatorname{Cobar}\mathbb{Z}\langle\mathcal{K}\rangle = T(U),\ U = \langle\chi_\sigma \mid I_\sigma \in \mathcal{K},\ \sigma\neq\varnothing\rangle,\ |\chi_\sigma| = 2|\sigma|-1. \tag{3.2}$$

Since the differential on $\mathbb{Z}\langle\mathcal{K}\rangle$ is zero, we get

$$\partial\chi_\sigma = \sum_{(\tau,\tau'),\ \sigma=\tau\sqcup\tau'} \chi_\tau\otimes\chi_{\tau'}. \tag{3.3}$$

For example,

$$\partial\chi_{ii} = \chi_i\otimes\chi_i, \quad \partial\chi_{ij} = \chi_i\otimes\chi_j + \chi_j\otimes\chi_i.$$

PROPOSITION 3.4 ([**9**, Prop. 8.4.10]). *There is an isomorphism of graded algebras*

$$H_*(\Omega\,(\mathbb{C}P^\infty)^\mathcal{K};\mathbb{Z}) \cong H(\operatorname{Cobar}\mathbb{Z}\langle\mathcal{K}\rangle).$$

We want to obtain a similar description for the cobar construction and the Pontryagin algebra for the polyhedral product of spheres $(\underline{S})^\mathcal{K}$. We start by describing explicitly the cobar construction $\operatorname{Cobar} H_*((\underline{S})^\mathcal{K};\mathbb{Z})$.

THEOREM 3.5 ([**7**]). *Let $\underline{X} = (X_1,\dots,X_m)$ be a sequence of pointed cell complexes, and let* $\mathbf{k}$ *be a ring such that the natural map*

$$H^*(X_{j_1};\mathbf{k})\otimes\cdots\otimes H^*(X_{j_k};\mathbf{k}) \to H^*(X_{j_1}\times\cdots\times X_{j_k};\mathbf{k})$$

is an isomorphism for any $\{j_1,\dots,j_k\}\subset[m]$. Then there is an isomorphism of algebras

$$H^*(\underline{X}^\mathcal{K};\mathbf{k}) \simeq (H^*(X_1;\mathbf{k})\otimes\cdots\otimes H^*(X_m;\mathbf{k}))/\mathcal{I},$$

where $\mathcal{I}$ is the generalised Stanley–Reisner ideal, generated by elements $x_{j_1}\otimes\cdots\otimes x_{j_k}$, $x_{j_i}\in\widetilde{H}^*(X_{j_i};\mathbf{k})$ and $\{j_1,\dots,j_k\}\notin\mathcal{K}$. Furthermore, the inclusion $\underline{X}^\mathcal{K}\hookrightarrow X_1\times\cdots\times X_m$ induces the quotient projection in cohomology.

COROLLARY 3.6. *Let $(\underline{S})^\mathcal{K} = \big((S^{n_1},pt),\dots,(S^{n_k},pt)\big)^\mathcal{K}$ be a polyhedral product of spheres. Then the cohomology algebra is given by*

$$H^*((\underline{S})^\mathcal{K};\mathbb{Z}) = \big(\mathbb{Z}[a_1]/(a_1^2)\otimes\ldots\otimes\mathbb{Z}[a_m]/(a_m^2)\big)/\big(a_{j_1}\otimes\cdots\otimes a_{j_s}\ \textit{if}\ \{j_1,\dots,j_s\}\notin\mathcal{K}\big),$$

where $|a_i| = n_i$.

Given $J = \{j_1,\dots,j_s\}\subset[m]$, $j_1<\cdots<j_s$, we denote by α_J the element of $H_*((\underline{S})^\mathcal{K};\mathbb{Z})$ dual to $a_{j_1}\otimes\cdots\otimes a_{j_s}\in H^*((\underline{S})^\mathcal{K};\mathbb{Z})$.

Proposition 3.7. *Let* $(\underline{S})^{\mathcal{K}} = \big((S^{n_1}, pt), \ldots, (S^{n_k}, pt)\big)^{\mathcal{K}}$ *be a polyhedral product of spheres. Then the homology coalgebra is*

$$H_*((\underline{S})^{\mathcal{K}}; \mathbb{Z}) = \langle \alpha_J,\ J \in \mathcal{K}\rangle, \quad J = \{j_1, \ldots, j_s\}, \quad |\alpha_J| = n_{j_1} + \cdots + n_{j_s}.$$

The coproduct is given by

$$\Delta\alpha_{j_1\cdots j_s} = \sum_{p=1}^{s-1} \sum_{\theta\in\widetilde{S}(p,s-p)} \varepsilon(\theta)\big(\alpha_{j_{\theta(1)}\cdots j_{\theta(p)}} \otimes \alpha_{j_{\theta(p+1)}\cdots j_{\theta(s)}} + {} + (-1)^{|\alpha_{\ldots}||\alpha_{\ldots}|}\alpha_{j_{\theta(p+1)}\cdots j_{\theta(s)}} \otimes \alpha_{j_{\theta(1)}\cdots j_{\theta(p)}}\big), \tag{3.4}$$

where $\widetilde{S}(p, s-p)$ *denotes the set of shuffle permutations such that* $\theta(1) = 1$*, and* $\varepsilon(\theta)$ *is the Koszul sign of the elements* $a_{j_1}, \ldots, a_{j_s}$*. Equivalently,*

$$\Delta\alpha_J = \sum_{(I,L),\, I\sqcup L=J,\, i_1=j_1} \varepsilon(I,L)\big(\alpha_I \otimes \alpha_L + (-1)^{|a_L||a_I|}\alpha_L \otimes \alpha_I\big) \tag{3.5}$$

$$= \sum_{(I,L),\, I\sqcup L=J} \varepsilon(I,L)\alpha_I \otimes \alpha_L. \tag{3.6}$$

Here $I = \{i_1, \ldots, i_p\}$ *and* $L = \{l_1, \ldots, l_{s-p}\}$ *are subsets of* $[m]$*,* $i_1 < \cdots < i_p$*,* $l_1 < \cdots < l_{s-p}$*, and* $\varepsilon(I,L)$ *is the Koszul sign:*

$$a_{j_1} \otimes \cdots \otimes a_{j_s} = \varepsilon(I,L)a_{i_1} \otimes \cdots a_{i_p} \otimes a_{l_1} \otimes \cdots a_{l_{s-p}}.$$

Proof. Since $H^*((\underline{S})^{\mathcal{K}}; \mathbb{Z})$ is free, we obtain an isomorphism of modules $H_*((\underline{S})^{\mathcal{K}}; \mathbb{Z}) \cong \mathrm{Hom}(H^*((\underline{S})^{\mathcal{K}}); \mathbb{Z})$, which is also an isomorphism of coalgebras. It remains to prove the coproduct formulae.

We denote the algebra $H^*((\underline{S})^{\mathcal{K}}; \mathbb{Z})$ by A and the coalgebra $H_*((\underline{S})^{\mathcal{K}}; \mathbb{Z})$ by A^*. Let $\mu\colon A \otimes A \to A$ be the product in A. The algebra A is finite-dimensional, thus $(A \otimes A)^* \cong A^* \otimes A^*$. The coproduct $\Delta\colon A^* \to A^* \otimes A^*$ is defined by $\Delta\alpha_J = \alpha_J \circ \mu$. We have

$$\mu(a_I \otimes a_L) = \varepsilon(I,L)a_J.$$

It follows that $\langle a_I \otimes a_L, \Delta\alpha_J\rangle = \varepsilon(I,L)$, where we use the sign convention

$$\langle a_I \otimes a_L, \alpha_I \otimes \alpha_L\rangle = 1.$$

Decompose $\Delta\alpha_J$ into a sum of basis elements in $A^* \otimes A^*$:

$$\Delta\alpha_J = \sum_{(I,L),\, i_1=j_1} \varepsilon(I,L)\alpha_I \otimes \alpha_L + \varepsilon(L,I)\alpha_L \otimes \alpha_I$$

$$= \sum_{(I,L),\, i_1=j_1} \varepsilon(I,L)\Big(\alpha_I \otimes \alpha_L + (-1)^{|a_L||a_I|}\alpha_L \otimes \alpha_I\Big)$$

$$= \sum_{(I,L),\, I\sqcup L=J} \varepsilon(I,L)\alpha_I \otimes \alpha_L.$$

This gives (3.5) and (3.6), and (3.4) follows easily. □

Proposition 3.8. *Let* $(\underline{S})^{\mathcal{K}} = \big((S^{n_1}, pt), \ldots, (S^{n_k}, pt)\big)^{\mathcal{K}}$ *be a polyhedral product of spheres. Then the cobar construction* $\mathrm{Cobar}\, H_*((\underline{S})^{\mathcal{K}}; \mathbb{Z})$ *is isomorphic to the dg algebra* $T(U)$*, where*

$$U = \langle b_J,\ J \in \mathcal{K},\ J \neq \varnothing\rangle,\ J = \{j_1, \ldots, j_s\},\ |b_J| = n_{j_1} + \cdots + n_{j_s} - 1,$$

the differential is given by

$$\partial b_J = \sum_{(I,L), I\sqcup L=J} \varepsilon(I,L)(-1)^{|b_I|+1} b_I \otimes b_L, \tag{3.7}$$

where $\varepsilon(I,L)$ is the Koszul sign of the suspended elements $s(b_i), |s(b_i)| = n_i$:

$$sb_{j_1}\otimes\cdots\otimes sb_{j_s} = \varepsilon(I,L)sb_{i_1}\otimes\cdots sb_{i_p}\otimes sb_{l_1}\otimes\cdots sb_{l_{s-p}}.$$

Equivalently,

$$\partial(b_{j_1\cdots j_s}) = \sum_{p=1}^{s-1}\sum_{\theta\in\widetilde{S}(p,s-p)}\varepsilon(\theta)(-1)^{|b_{j_{\theta(1)}\cdots j_{\theta(p)}}|+1}\big[b_{j_{\theta(1)}\cdots j_{\theta(p)}}, b_{j_{\theta(p+1)}\cdots j_{\theta(s)}}\big], \tag{3.8}$$

where $\varepsilon(\theta)$ is the Koszul sigh of the elements $sb_{j_1},\dots,sb_{j_s}$, and $\widetilde{S}(p,s-p)$ denotes the set of shuffle permutations such that $\theta(1)=1$.

Proof. We denote $b_J := s^{-1}\alpha_J$. Using formula (3.1) for the differential in the cobar construction together with (3.6) and noting that $\partial_C = 0$ in our case, we obtain

$$\partial(s^{-1}\alpha_J) = \sum_{(I,L),J=I\sqcup L}\varepsilon(I,L)(-1)^{|a_I|}s^{-1}\alpha_I\otimes s^{-1}\alpha_L = \sum_{(I,L),J=I\sqcup L}\varepsilon(I,L)(-1)^{|b_I|+1}b_I\otimes b_L.$$

It remains to prove (3.8):

$$\begin{aligned}\partial b_J &= \sum_{(I,L),J=I\sqcup L,i_1=j_1}\varepsilon(I,L)(-1)^{|a_I|}b_I\otimes b_L + \varepsilon(L,I)(-1)^{|a_L|}b_L\otimes b_I\\ &= \sum_{(I,L),J=I\sqcup L,i_1=j_1}\varepsilon(I,L)(-1)^{|a_I|}\Big(b_I\otimes b_L + (-1)^{|a_L|+|a_I|+|a_L||a_I|}b_L\otimes b_I\Big)\\ &= \sum_{(I,L),J=I\sqcup L,i_1=j_1}\varepsilon(I,L)(-1)^{|b_I|+1}\Big(b_I\otimes b_L - (-1)^{|b_L||b_I|}b_L\otimes b_I\Big).\end{aligned}$$

□

Remark 3.9. When all the spheres in $(\underline{S})^{\mathcal{K}}$ are even-dimensional we obtain the following formula differential in Cobar $H_*((\underline{S})^{\mathcal{K}};\mathbb{Z})$:

$$\partial b_J = \sum_{(I,L),I\sqcup L=J} b_I\otimes b_L.$$

This formula coincides with (3.3) when $\sigma = I$ (no multiple elements in σ). It follows that the cobar construction of $(S^2)^{\mathcal{K}}$ embeds canonically into the cobar construction of $(\mathbb{C}P^\infty)^{\mathcal{K}}$. This will be important when studying higher Whitehead products.

4. Adams–Hilton models

The main purpose of this section is to verify that Adams–Hilton models for $(\underline{S})^{\mathcal{K}}$ and $(\mathbb{C}P^\infty)^{\mathcal{K}}$ coincide with the cobar construction of homology coalgebra.

Given a simply connected CW-complex X with a single 0-dimensional cell and no 1-dimensional cells, one can construct an *Adams–Hilton model* $\mathbf{AH}(X) = (AH(X), d_X)$, a dg algebra over $\mathbb{Z}$ together with a quasi-isomorphism

$$\theta_X\colon \mathbf{AH}(X)\xrightarrow{\simeq} CU_*(\Omega X)$$

to normalised cubical chains of the loop space [**4**]. Suppose X is a CW-complex of the form

$$X = pt\cup(\bigcup_{\alpha\in S}e_\alpha),\quad |e_\alpha|\geqslant 2.$$

Then the underlying graded algebra $AH(X)$ is a free associative algebra,

$$AH(X) = T(V),\quad V = \langle v_\alpha\mid\alpha\in S\rangle,\quad |v_\alpha| = |e_\alpha|-1,$$

where each generator $v_\alpha\in V$ corresponds to a cell e_α in X.

The maps θ_X and ∂_X are defined inductively. Suppose we have already constructed an Adams–Hilton model for the nth skeleton:

$$\mathbf{AH}(X^n) = (T(V_{\leqslant n-1}), \partial_{X^n}), \quad \theta_{X^n}\colon T(V_{\leqslant n-1}), \partial_{X^n}) \xrightarrow{\simeq} CU_*(\Omega X^n).$$

Define $\partial_{X^{n+1}}(v_\alpha)$ for a generator v_α, $|v_\alpha| = n$, corresponding to an $(n+1)$-cell cell e_α. Consider the attaching map $f_\alpha\colon S^n \to X^n$ and the generator $\beta \in H_{n-1}(\Omega S^n) \cong \mathbb{Z}$. Then $H(\Omega f_\alpha)\beta \in H_{n-1}(\Omega X^n)$ and, by induction, there exists an element $z \in T(V_{\leqslant n-1})$, $|z| = n-1$, such that

$$(\theta_{X^n})_*[z] = H(\Omega f_\alpha)\beta.$$

Then we define

$$\partial_{X^{n+1}}(v_\alpha) = z.$$

For the definition of $\theta_{X^{n+1}}(v_\alpha)$, see [**4**].

Proposition 4.1. *Suppose there is an Adams–Hilton model $(AH(X), \partial_X)$. Fix a generator $v_\alpha \in AH(X) \cong T(V)$, $|v_\alpha| = n$, corresponding to $(n+1)$-cell e_α. Suppose $\partial_{X^{n+1}}(v_\alpha) = z$. Then for every $a \in T(V_{\leqslant n-1})$, $|a| = n$, there exists Adams–Hilton model $(AH(X), \widetilde{\partial}_X)$, for which*

$$\widetilde{\partial}_{X^n} = \partial_{X^n}, \quad \widetilde{\partial}_{X^{n+1}}(v_\alpha) = z + \partial_{X^n}(a), \quad \widetilde{\partial}_{X^{n+1}}(v_\beta) = \partial_{X^{n+1}}(v_\beta),$$

where v_β is an arbitrary generator of degree n in $AH(X)$, $v_\beta \neq v_\alpha$.

Proof. It follows from the inductive construction of ∂_X. □

Fix Adams–Hilton models $(\mathbf{AH}(X), \theta_X)$ and $(\mathbf{AH}(Y), \theta_Y)$ for X and Y, where $AH(X) \cong T(V)$, $AH(Y) \cong T(W)$. Then for any map $f\colon X \to Y$ there is an inductive construction [**4**] of a dga-morphism $\mathbf{AH}(f)\colon \mathbf{AH}(X) \to \mathbf{AH}(Y)$ and a chain homotopy $\psi_f\colon \mathbf{AH}(X) \to CU_*(\Omega Y)$ between $CU_*(\Omega f)\circ\theta_X$ and $\theta_Y \circ \mathbf{AH}(f)$ in the following diagram:

$$\begin{array}{ccc} \mathbf{AH}(X) & \xrightarrow{\theta_X} & CU_*(\Omega X) \\ \downarrow{\scriptstyle \mathbf{AH}(f)} & & \downarrow{\scriptstyle CU_*(\Omega f)} \\ \mathbf{AH}(Y) & \xrightarrow{\theta_Y} & CU_*(\Omega Y) \end{array}$$

We refer to $(\mathbf{AH}(f), \psi_f)$ as an *Adams–Hilton model for f*.

Similarly, there is an indeterminacy in the construction of $\mathbf{AH}(f)$ and ψ_f. We have

Proposition 4.2. *Suppose there is an Adams–Hilton model $\mathbf{AH}(f)$. Fix a generator $v_\alpha \in AH(X) \cong T(V)$, $|v_\alpha| = n$, corresponding to $(n+1)$-cell e_α. Suppose $\mathbf{AH}(f)(v_\alpha) = x$. Then for every $b \in T(W_{\leqslant n-1})$, $|b| = n+1$, there exists Adams–Hilton model $\widetilde{\mathbf{AH}(f)}$, for which*

$$\widetilde{\mathbf{AH}(f)}|_{T(V_{\leqslant n-1})} = \mathbf{AH}(f)|_{T(V_{\leqslant n-1})}, \quad \widetilde{\mathbf{AH}(f)}(v_\alpha) = x + \partial_{Y^n}(b), \quad \widetilde{\mathbf{AH}(f)}(v_\beta) = \mathbf{AH}(f)(v_\beta),$$

where v_β is an arbitrary generator of degree n in $AH(X)$, $v_\beta \neq v_\alpha$.

Proof. It follows from the inductive construction of an Adams–Hilton map. We use the notation of [**4**, Th. 3.1], where φ is an inductive map we construct, an element a is a generator for which we want to define $\varphi(a)$. The element g_2 is defined precisely by the condition $dg_2 = \varphi da$. It is easy to see that $\widetilde{g_2} = g_2 + db$ is an appropriate choice for such element. We have $\varphi(a) = g_2 + z_2$ for some particular element z_2 in [**4**, Th. 3.1]. Thus the element $\widetilde{\varphi(a)} = g_2 + db + z_2$ fits the inductive procedure. □

Theorem 4.3 ([**4**], [**6**]). *Adams–Hilton models have the following properties:*

(1) *If $f \simeq g$, then $\mathbf{AH}(f) \simeq \mathbf{AH}(g)$ in DGA. In particular, two models for the same map must be homotopic in DGA.*
(2) *A model for the identity map 1_X is $(1_{\mathbf{AH}(X)}, 0)$.*

(3) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *then* $\mathbf{AH}(g \circ f)$ *may be taken as* $\mathbf{AH}(g) \circ \mathbf{AH}(f)$.

(4) *Let* X_0 *be a subcomplex of* X, *say* $X = pt \cup \left(\bigcup_{\alpha \in S} e_\alpha\right)$, $X_0 = pt \cup \left(\bigcup_{\alpha \in S_0} e_\alpha\right)$, $S_0 \subseteq S$. *Given any model* $(AH(X_0), \partial_{X_0}, \theta_{X_0})$, *there is a model* $(AH(X), \partial_X, \theta_X)$ *for which* ∂_X *and* θ_X *are extensions over* $AH(X)$ *of* ∂_{X_0} *and* θ_{X_0}.

(5) *Under the hypotheses of* (4), *a model for the injection* $X_0 \hookrightarrow X$ *is the injection* $\mathbf{AH}(X_0) \hookrightarrow \mathbf{AH}(X)$.

(6) *Under the hypotheses of* (4), *let* $f\colon X \to Y$ *be a map and put* $f_0 = f|_{X_0}$. *Given any models* $(\mathbf{AH}(Y), \theta_Y)$ *and* $(\mathbf{AH}(f_0), \psi_{f_0})$, *there is a model* $(\mathbf{AH}(f), \psi_f)$ *for which* $\mathbf{AH}(f)$ *and* ψ_f *are extensions over* $AH(X)$ *of* $\mathbf{AH}(f_0)$ *and* ψ_{f_0}.

(7) *Let* $\{X_\beta\}$ *be a family of subcomplexes of a CW-complex* X, *and suppose* $X = \bigcup_\beta X_\beta$. *Suppose we have models* $\left(\mathbf{AH}(X_\beta), \theta_{X_\beta}\right)$ *satisfying the coherency conditions*

$$\partial_{X_\beta}\big|_{AH(X_\beta \cap X_\gamma)} = \partial_{X_\gamma}\big|_{AH(X_\beta \cap X_\gamma)},$$
$$\theta_{X_\beta}\big|_{AH(X_\beta \cap X_\gamma)} = \theta_{X_\gamma}\big|_{AH(X_\beta \cap X_\gamma)}$$

for each pair of indices (β, γ). *Then* $\operatorname{colim}\left\{\mathbf{AH}(X_\beta), \theta_{X_\beta}\right\}$ *is an Adams-Hilton model for* X.

(8) *Under the hypotheses of* (7), *let* $f\colon X \to Y$ *be a map and put* $f_\beta = f|_{X_\beta}$. *Fixing a model* $(\mathbf{AH}(Y), \theta_Y)$, *suppose we have models* $\left(\mathbf{AH}(f_\beta), \psi_{f_\beta}\right)$ *satisfying the coherency conditions*

$$\mathbf{AH}(f_\beta)|_{AH(X_\beta \cap X_\gamma)} = \mathbf{AH}(f_\gamma)|_{AH(X_\beta \cap X_\gamma)},$$
$$\psi_{f_\beta}\big|_{AH(X_\beta \cap X_\gamma)} = \psi_{f_\gamma}\big|_{AH(X_\beta \cap X_\gamma)}.$$

Then $\operatorname{colim}\left\{\mathbf{AH}(f_\beta), \psi_{f_\beta}\right\}$ *is an Adams–Hilton model for* f.

(9) *Let* $f_0\colon S^n \to X_0$, $n \geqslant 2$, *and extend* f_0 *to* $f\colon D^{n+1} \to X = X_0 \cup_{f_0} e^{n+1}$. *Choosing the standard three-cell decomposition of* D^{n+1} *we have* $AH\left(D^{n+1}\right) = T(z, z_0)$ *with* $|z_0| = n-1$, $|z| = n$, $\partial(z) = -z_0$. *Let* $(\mathbf{AH}(X_0), \theta_{X_0})$ *and* $(\mathbf{AH}(f_0), \psi_{f_0})$ *be models for* X_0 *and* f_0. *Then one for* X *is given by* $AH(X) = AH(X_0) \otimes T(v_f)$, $|v_f| = n$, $\partial_X(x) = \partial_{X_0}(x)$ *for* $x \in AH(X_0)$, $\partial_X(v_f) = -\mathbf{AH}(f_0)(z_0)$, $\theta_X(x) = \theta_{X_0}(x)$ *for* $x \in AH(X_0), \theta_X(v_f) = CU_*(\Omega f)(\theta_{D^{n+1}}(z)) + \psi_{f_0}(z_0)$.

Further properties concern the model for a product space $X \times Y$. *Suppose* $X = pt \cup \left(\bigcup_{\alpha \in S} e_\alpha\right)$ *and* $Y = pt \cup \left(\bigcup_{\alpha \in S'} e_\alpha\right)$, *then* $AH(X \times Y)$ *is the tensor algebra with the set of generators* $\{v_\alpha \mid \alpha \in S''\}$, *where*

$$S'' = S \cup S' \cup \left(S \times S'\right)$$

(Adams and Hilton point out that $X \times Y$ *need not be a CW complex for their construction to exist.) We define the ring homomorphism*

$$\nu_{XY}\colon AH(X \times Y) \to AH(X) \otimes AH(Y)$$

by $\nu_{XY}(v_\alpha) = v_\alpha \otimes 1$ *for* $\alpha \in S$, $\nu_{XY}(v_\alpha) = 1 \otimes v_\alpha$ *for* $\alpha \in S'$, $\nu_{XY}(v_\alpha) = 0$ *for* $\alpha \in (S \times S')$.

(10) *Given* $(\mathbf{AH}(X), \theta_X)$ *and* $(\mathbf{AH}(Y), \theta_Y)$, *it is possible to choose inductively* $\partial_{X \times Y}$ *and* $\theta_{X \times Y}$ *for* $AH(X \times Y)$ *such that* ν_{XY} *is a dga-morphism and the diagram*

$$\begin{array}{ccccc}
\mathbf{AH}(X \times Y) & & \xrightarrow{\quad \nu_{XY} \quad} & & \mathbf{AH}(X) \otimes \mathbf{AH}(Y) \\
\Big\downarrow{\scriptstyle \theta_{X \times Y}} & & & & \Big\downarrow{\scriptstyle \theta_X \otimes \theta_Y} \\
CU_*(\Omega(X \times Y)) & \longrightarrow & CU_*(\Omega X \times \Omega X) & \longrightarrow & CU_*(\Omega X) \otimes CU_*(\Omega Y)
\end{array}$$

commutes up to chain homotopy and leads to a commutative diagram of isomorphisms of homology rings.

Furthermore, letting $X \overset{p_X}{\leftarrow} X \times Y \overset{p_Y}{\rightarrow} Y$ *denote the projections, we may take* $\mathbf{AH}\,(p_X) = \pi_1 \nu_{XY}$ *and* $\mathbf{AH}\,(p_Y) = \pi_2 \nu_{XY}$*.*

(11) *Let* $AH(f)\colon AH(X_1) \to AH(Y_1)$ *and* $AH(g)\colon AH(X_2) \to AH(Y_2)$ *be associated with maps* $f\colon X_1 \to Y_1$ *and* $g\colon X_2 \to Y_2$*, and let* ∂*,* θ *be chosen on* $X_1 \times X_2$*,* $Y_1 \times Y_2$ *so that* $\nu_{X_1X_2}$*,* $\nu_{Y_1Y_2}$ *are chain mappings. Then we may choose a map*

$$\varphi\colon \mathbf{AH}(X_1 \times X_2) \to \mathbf{AH}(Y_1 \times Y_2)$$

so that the following diagram is commutative

$$\begin{array}{ccc} \mathbf{AH}(X_1 \times X_2) & \xrightarrow{\nu_{X_1X_2}} & \mathbf{AH}(X_1) \otimes \mathbf{AH}(X_2) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle \mathbf{AH}(f)\otimes\mathbf{AH}(g)} \\ \mathbf{AH}(Y_1 \times Y_2) & \xrightarrow{\nu_{Y_1Y_2}} & \mathbf{AH}(Y_1) \otimes \mathbf{AH}(Y_2) \end{array}$$

Moreover, any such map φ *that makes the diagram commutative is an Adams–Hilton model for* $f \times g$*.*

Remark 4.4. The properties (1)–(4), (6)–(10) of Theorem 4.3 are from [**6**, Th. 8.1]; the property (5) follows easily from the construction of Adams–Hilton models; the last property is [**4**, Cor. 4.1].

Theorem 4.5 ([**4**, Th. 3.4]). *An Adams–Hilton model for* $\mathbb{C}P^2$ *is given by*

$$\mathbf{AH}(\mathbb{C}P^2) = (T(a_1, a_2), \partial), \quad |a_1| = 1, \quad |a_2| = 3, \quad \partial a_1 = 0, \quad \partial a_2 = a_1^2.$$

Theorem 4.6. *An Adams–Hilton model for* $\mathbb{C}P^n$ *is given by*

$$\mathbf{AH}(\mathbb{C}P^n) = (T(a_1, \ldots, a_n), \partial), \quad |a_i| = 2i - 1, \tag{4.1}$$

$$\partial a_1 = 0, \quad \partial a_i = a_1 \otimes a_{i-1} + a_2 \otimes a_{i-2} + \cdots + a_{i-1} \otimes a_1, \quad i = 2, \ldots, n.$$

Proof. Suppose we have already constructed Adams–Hilton models $\mathbf{AH}(\mathbb{C}P^i)$ of the required form and the model $\mathbf{AH}(\mathbb{C}P^j)$ extends the model $\mathbf{AH}(\mathbb{C}P^i)$, for $i < j \leqslant n-1$. We extend the model $\mathbf{AH}(\mathbb{C}P^{n-1}) = (T(a_1, \ldots, a_{n-1}, \partial)$ of the form (4.1) to some Adams–Hilton model $\mathbf{AH}(\mathbb{C}P^n) = (T(a_1, \ldots, a_{n-1}, a_n), \widetilde{\partial})$. We have

$$\widetilde{\partial} a_i = \partial a_i = a_1 \otimes a_{i-1} + a_2 \otimes a_{i-2} + \cdots + a_{i-1} \otimes a_1, \ i = 2, \ldots, n-1, \qquad \widetilde{\partial} a_n = w,$$

where w is some element of degree $2n-2$. Let

$$v := a_1 \otimes a_{n-1} + a_2 \otimes a_{n-2} + \cdots + a_{n-1} \otimes a_1, \quad |v| = 2n-2.$$

We need to prove that there is an Adams–Hilton model $\mathbf{AH}(\mathbb{C}P^n)$ with $\partial a_n = v$. By direct computation we obtain that $\partial v = 0$ (or we can use the fact that v is a cycle in the cobar construction $\operatorname{Cobar} H_*(\mathbb{C}P^\infty)$). Using the Hopf fibration, we get $\Omega \mathbb{C}P^n \simeq \Omega S^{2n+1} \times S^1$, hence, $H_{2n-2}(\Omega \mathbb{C}P^n) = 0$. Therefore, v is a boundary. It follows that

$$v = k \widetilde{\partial} a_n + \partial x \quad \text{for some} \quad k \in \mathbb{Z}, \ x \in T(a_1, \ldots, a_{n-1}).$$

The differential on $T(a_1, \ldots, a_{n-1})$ increases the tensor length by one, hence, any summand of the form $a_i a_{n-i}$ can not appear in ∂x. It follows that each summand of v appears in $k\widetilde{\partial} a_n$. Therefore, $k = \pm 1$. Then

$$\widetilde{\partial} a_n = \pm v + \partial x, \ x \in T(a_1, \ldots, a_{n-1}).$$

Reorienting the cell corresponding to a_n if necessary, we obtain $\widetilde{\partial} a_n = v + \partial x$, $x \in T(a_1, \ldots, a_{n-1})$. Using Proposition 4.1, we obtain that there is another model $(\mathbf{AH}(\mathbb{C}P^n), \partial)$ with $\partial a_n = v$ and an appropriate choice of $\theta(a_n)$, extending $(\mathbf{AH}(\mathbb{C}P^{n-1}), \partial)$. ☐

Since Adams–Hilton models respect colimits, we obtain

Corollary 4.7. *An Adams–Hilton model for* $\mathbb{C}P^\infty$ *is given by*

$$\mathbf{AH}(\mathbb{C}P^\infty) = (T(a_1, a_2, \ldots), \partial), \quad |a_i| = 2i - 1,$$

$$\partial a_1 = 0, \quad \partial a_i = a_1 \otimes a_{i-1} + a_2 \otimes a_{i-2} + \cdots + a_{i-1} \otimes a_1, \quad i \geqslant 2,$$

where the generator a_i *corresponds to the* $2i$*-cell in* $\mathbb{C}P^\infty$*.*

The sphere S^n has the standard two-cell decomposition. The decomposition of the product $S^{n_1} \times \cdots \times S^{n_k}$ has cells of the form $e_J = \prod\limits_{j_t \in J} e_{j_t}$, $J \subset \{1, \ldots, k\}$. We denote by b_J the generator corresponding to the cell e_J.

Theorem 4.8 ([**4**, Th. 4.3]). *An Adams–Hilton model for* $X = S^{n_1} \times \cdots \times S^{n_k}$ *is given by*

$$\mathbf{AH}(S^{n_1} \times \cdots \times S^{n_k}) = (T(U), \partial), \quad U = \langle b_J,\ J \subseteq \{1, \ldots, k\},\ J \neq \varnothing \rangle,$$

$$|b_J| = |e_J| - 1 = n_{j_1} + \cdots + n_{j_s} - 1,$$

$$\partial b_J = \sum_{(I,L), I \sqcup L = J} (-1)^{\widetilde{\varepsilon}(I,L)} b_I b_L, \tag{4.2}$$

$$\widetilde{\varepsilon}(I, L) = \sum_{i \in I} n_i + \sum_{i \in I,\ l \in L,\ i > l} n_i\, n_l.$$

Moreover, this model extends the Adams–Hilton models over each product of $\leqslant k - 1$ *spheres. Namely, if* $K = S^{n_{j_1}} \times \cdots \times S^{n_{j_s}}$, $K \subseteq S^{n_1} \times \cdots \times S^{n_k}$*, then*

$$\partial_X|_K = \partial_K,\ \theta_X|_K = \theta_K.$$

Proposition 4.9. *The differential* ∂ *from Theorem 4.8 can be expressed in the following way:*

$$\partial b_{j_1 \cdots j_s} = \sum_{p=1}^{s-1} \sum_{\theta \in S(p, s-p)} \varepsilon(\theta)(-1)^{|b_{j_{\theta(1)} \cdots j_{\theta(p)}}| + 1} b_{j_{\theta(1)} \cdots j_{\theta(p)}} \otimes b_{j_{\theta(p+1)} \cdots j_{\theta(s)}} \tag{4.3}$$

$$= \sum_{p=1}^{s-1} \sum_{\theta \in \widetilde{S}(p, s-p)} \varepsilon(\theta)(-1)^{|b_{j_{\theta(1)} \cdots j_{\theta(p)}}| + 1} \left[b_{j_{\theta(1)} \cdots j_{\theta(p)}}, b_{j_{\theta(p+1)} \cdots j_{\theta(s)}} \right], \tag{4.4}$$

where $1 \leqslant j_1 < \cdots < j_s \leqslant k$*,* $S(p, s-p)$ *denotes the set of shuffle permutations,* $\widetilde{S}(p, s-p)$ *is the set of shuffle permutations such that* $\theta(1) = 1$*, and* $\varepsilon(\theta)$ *is the Koszul sign of the elements* $sb_{i_{\theta(1)}}, \ldots, sb_{i_{\theta(s)}}$*.*

Proof. We prove that (4.2) and (4.3) are equivalent. Suppose $J = \{j_1, \ldots, j_s\} \subset \{1, \ldots, k\}$, $I = \{j_{\theta(1)} \cdots j_{\theta(p)}\}$, $L = \{j_{\theta(p+1)} \cdots j_{\theta(s)}\}$. It suffices to prove that the signs coincide. First,

$$\sum_{i \in I} n_i = |e_I| = |b_{j_{\theta(1)} \cdots j_{\theta(p)}}| + 1.$$

Second, we have the Koszul sign $\varepsilon(\theta)$ of the elements sb_i, $|sb_i| = n_i$, $|b_i| = n_i - 1$,

$$sb_{j_1} \wedge \cdots \wedge sb_{j_s} = \varepsilon(\theta) sb_{j_{\theta(1)}} \wedge \cdots \wedge sb_{j_{\theta(s)}}.$$

Since θ is a shuffle permutation, it follows that

$$\varepsilon(\theta) = \mathrm{sgn}\big(\sum_{i \in I,\ l \in L,\ i > l} n_i\, n_l \big). \qquad \square$$

Theorem 4.10. *An Adams–Hilton model for $(\underline{S})^{\mathcal{K}}$ is given by*

$$\mathbf{AH}((\underline{S})^{\mathcal{K}}) = (T(U), \partial), \quad U = \langle b_J,\ J \in \mathcal{K},\ J \neq \varnothing \rangle, \quad |b_J| = n_{j_1} + \cdots + n_{j_s} - 1,$$

$$\partial b_{j_1 \cdots j_s} = \sum_{p=1}^{s-1} \sum_{\theta \in S(p,s-p)} \varepsilon(\theta)(-1)^{|b_{j_{\theta(1)} \cdots j_{\theta(p)}}|+1} b_{j_{\theta(1)} \cdots j_{\theta(p)}} \otimes b_{j_{\theta(p+1)} \cdots j_{\theta(s)}}. \tag{4.5}$$

Proof. The polyhedral product $(\underline{S})^{\mathcal{K}}$ is the colimit of products of spheres

$$(\underline{S})^{\mathcal{K}} = \operatorname*{colim}_{J \in \mathcal{K}} (\underline{S})^J.$$

Since Adams–Hilton models from Theorem 4.8 extend each other over subproducts, they satisfy the coherency conditions from Theorem 4.3 (7). For example, the Adams–Hilton model $\mathbf{AH}(S^{n_2} \times S^{n_3})$ extends to $\mathbf{AH}(S^{n_1} \times S^{n_2} \times S^{n_3})$ and $\mathbf{AH}(S^{n_2} \times S^{n_3} \times S^{n_4})$. Therefore, the Adams–Hilton model $\mathbf{AH}(S^{n_1} \times S^{n_2} \times S^{n_3} \cup_{S^{n_2} \times S^{n_3}} S^{n_2} \times S^{n_3} \times S^{n_4})$ is well defined. The same holds for the arbitrary products of spheres. Thus we obtain

$$\mathbf{AH}((\underline{S})^{\mathcal{K}}) = \operatorname*{colim}_{J \in \mathcal{K}} \mathbf{AH}((\underline{S})^J).$$

□

Corollary 4.11. *An Adams–Hilton model for $(S^2)^{\mathcal{K}}$ is given by*

$$\mathbf{AH}((S^2)^{\mathcal{K}}) = (T(U), \partial), \quad U = \langle b_J,\ J \in \mathcal{K},\ J \neq \varnothing \rangle,$$
$$\partial b_J = \sum_{(I,L), I \sqcup L = J} b_I \otimes b_L, \quad |b_J| = 2|J| - 1.$$

Theorem 4.12. *An Adams–Hilton model for $\mathbb{C}P^n \times \mathbb{C}P^k$ is given by*

$$\mathbf{AH}(\mathbb{C}P^n \times \mathbb{C}P^k) = \big(T(\chi_\sigma \mid \sigma = \{\underbrace{1, \ldots, 1}_{\leqslant n}, \underbrace{2, \ldots, 2}_{\leqslant k}\},\ \sigma \neq \varnothing), \partial\big),$$

$$\partial \chi_\sigma = \sum_{\sigma = \tau \sqcup \tau'} \chi_\tau \chi_{\tau'}, \quad |\chi_\sigma| = 2|\sigma| - 1. \tag{4.6}$$

Moreover, this model extends the models $\mathbf{AH}(\mathbb{C}P^s \times \mathbb{C}P^t)$, $0 \leqslant s \leqslant n$, $0 \leqslant t \leqslant k$. It also extends the model $\mathbf{AH}(S^2 \times S^2)$ from Theorem 4.8.

Proof. The proof is by induction on $N = n + k$. The theorem holds for $N = 2$ since the models $\mathbf{AH}(pt \times \mathbb{C}P^2)$, $\mathbf{AH}(\mathbb{C}P^2 \times pt)$ and $\mathbf{AH}(S^2 \times S^2)$ are already constructed and satisfy the coherency conditions on trivial intersections.

Assume the theorem holds for $N - 1$. By induction, we have already constructed $\mathbf{AH}(\mathbb{C}P^{n-1} \times \mathbb{C}P^k)$, $\mathbf{AH}(\mathbb{C}P^n \times \mathbb{C}P^{k-1})$, which extend the model $\mathbf{AH}(\mathbb{C}P^{n-1} \times \mathbb{C}P^{k-1})$, the maps ∂ and θ agree on the intersection. Thus we obtain the model on $(2N-2)$-skeleton

$$sk^{2N-2}(\mathbb{C}P^n \times \mathbb{C}P^k) = \mathbb{C}P^{n-1} \times \mathbb{C}P^k \bigcup_{\mathbb{C}P^{n-1} \times \mathbb{C}P^{k-1}} \mathbb{C}P^n \times \mathbb{C}P^{k-1},$$

the formula (4.6) holds.

By Theorem 4.3 (10), we may choose the maps ∂ and θ on $AH(\mathbb{C}P^n \times \mathbb{C}P^k)$ inductively in such a way that the ring homomorphism

$$\nu = \nu_{\mathbb{C}P^n \mathbb{C}P^k} \colon \mathbf{AH}(\mathbb{C}P^n \times \mathbb{C}P^k) \to \mathbf{AH}(\mathbb{C}P^n) \otimes \mathbf{AH}(\mathbb{C}P^k)$$

is a chain equivalence. We need to prove that model $\mathbf{AH}(sk^{2N-2}(\mathbb{C}P^n \times \mathbb{C}P^k))$ constructed above with differential (4.6) satisfies $\partial \nu = \nu \partial$. We have

$$\nu(\chi_{1 \cdots 1}) = \chi_{1 \cdots 1} \otimes 1, \quad \nu(\chi_{2 \cdots 2}) = 1 \otimes \chi_{2 \cdots 2}, \quad \nu(\chi_{1 \cdots 12 \cdots 2}) = 0.$$

Hence,

$$\partial \nu(\chi_{1 \cdots 1}) = \partial(\chi_{1 \cdots 1} \otimes 1) = \partial(\chi_{1 \cdots 1}) \otimes 1 = \nu \partial(\chi_{1 \cdots 1})$$

and similarly for $\chi_{2\cdots 2}$. If σ contains both 1 and 2, then

$$\nu\partial(\chi_\sigma) = \nu(\sum \chi_\tau\chi_{\tau'}) = \nu(\chi_{1\cdots 1}\chi_{2\cdots 2} + \chi_{2\cdots 2}\chi_{1\cdots 1}) = 0 = \partial\nu(\chi_\sigma).$$

By Theorem 4.3 (10), there is an Adams–Hilton model $\mathbb{C}P^n \times \mathbb{C}P^k$ that extends the model $\mathbf{AH}(sk^{2N-2}(\mathbb{C}P^n \times \mathbb{C}P^k))$, for which the equality $\nu\partial = \partial\nu$ holds.

Consider the generator $\chi_{\widetilde{\sigma}}$, $\widetilde{\sigma} = \{\underbrace{1,\dots,1}_{n},\underbrace{2,\dots,2}_{k}\}$, corresponding to maximal cell of $\mathbb{C}P^n \times \mathbb{C}P^k$. We need to identify $\widetilde{\partial}(\chi_{\widetilde{\sigma}})$, where $\widetilde{\partial}$ denotes the differential in the extended model $\mathbf{AH}(\mathbb{C}P^n \times \mathbb{C}P^k)$. Let

$$x := \sum_{\tau,\,\tau',\,\tau\sqcup\tau'=\widetilde{\sigma}} \chi_\tau\chi_{\tau'}.$$

Then

$$\nu x = \nu(\sum \chi_\tau\chi_{\tau'}) = \nu(\chi_{1\cdots 1}\chi_{2\cdots 2} + \chi_{2\cdots 2}\chi_{1\cdots 1}) = 0.$$

By direct computation we obtain $\partial x = 0$. The map ν is a chain equivalence, hence x is a boundary. Then

$$x = k\widetilde{\partial}(\chi_{\widetilde{\sigma}}) + \partial y, \quad y \in \mathbf{AH}(sk^{2N-2}(\mathbb{C}P^n \times \mathbb{C}P^k)).$$

The differential ∂ in $\mathbf{AH}(sk^{2N-2}(\mathbb{C}P^n \times \mathbb{C}P^k))$ preserves multisets and increases the tensor length by one. As in the proof of Theorem 4.6, we have $k = 1$. Using Proposition 4.1, we obtain that $\partial\chi_{\widetilde{\sigma}} = x$ is a valid choice of $\partial\chi_{\widetilde{\sigma}}$. □

THEOREM 4.13. *An Adams–Hilton model for $\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n}$ is given by*

$$\mathbf{AH}(\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n}) = \big(T(\chi_\sigma \mid \sigma = \{\underbrace{1,\dots,1}_{\leqslant k_1},\underbrace{2,\dots,2}_{\leqslant k_2},\dots,\underbrace{n,\dots,n}_{\leqslant k_n}\},\ \sigma \neq \varnothing), \partial\big),$$

$$\partial\chi_\sigma = \sum_{\sigma=\tau\sqcup\tau'} \chi_\tau\chi_{\tau'}, \quad |\chi_\sigma| = 2|\sigma| - 1. \tag{4.7}$$

Moreover, this model extends models $\mathbf{AH}(\mathbb{C}P^{s_1} \times \cdots \times \mathbb{C}P^{s_n})$, $0 \leqslant s_i \leqslant n_i$. It also extends the model $\mathbf{AH}(S^2 \times \cdots \times S^2)$ from Theorem 4.8.

PROOF. Analogously, the proof is by induction on $N = k_1 + \cdots + k_n$. The case $N = 2$ is similar to the one from the previous theorem.

Assume the theorem holds for $N-1$. By induction, we have already constructed models

$$\mathbf{AH}(\mathbb{C}P^{k_1-1} \times \mathbb{C}P^{k_2} \times \cdots \times \mathbb{C}P^{k_n}), \dots, \mathbf{AH}(\mathbb{C}P^{k_1} \times \mathbb{C}P^{k_2} \times \cdots \times \mathbb{C}P^{k_n-1}),$$

the maps ∂ and θ agree on intersections.

Thus we obtain the model on $(2N-2)$-skeleton

$$sk^{2N-2}(\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n}) = \mathbb{C}P^{k_1-1} \times \cdots \times \mathbb{C}P^{k_n} \cup \cdots \cup \mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n-1},$$

the formula (4.7) holds.

By Theorem 4.3 (10), we may choose the maps ∂ and θ on $AH(\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n})$ inductively in such a way that the ring homomorphism

$$\nu\colon \mathbf{AH}(\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n}) \to \mathbf{AH}(\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_{n-1}}) \otimes \mathbf{AH}(\mathbb{C}P^{k_n})$$

is a chain mapping, and hence a chain equivalence. We prove that the constructed model $\mathbf{AH}(sk^{2N-2}(\mathbb{C}P^{k_1} \times \cdots \times \mathbb{C}P^{k_n}))$ satisfies $\partial\nu = \nu\partial$. We have

$\nu(\chi_\sigma) = \chi_\sigma \otimes 1$, $n \notin \sigma$;

$\nu(\chi_\sigma) = 1 \otimes \chi_\sigma$, $\sigma = \{n,\dots,n\}$;

$\nu(\chi_\sigma) = 0$, otherwise.

Suppose $n \notin \sigma$, then

$$\partial\nu(\chi_\sigma) = \partial(\chi_\sigma \otimes 1) = \partial(\chi_\sigma) \otimes 1 = \nu\partial(\chi_\sigma).$$

Suppose $n \in \sigma$, $i \in \sigma$, $i = 1, \dots, n-1$, then $\sigma = \alpha \sqcup \beta$, $n \notin \alpha$, $\beta = \{n, \dots, n\}$,

$$\nu\partial(\chi_\sigma) = \nu(\sum \chi_\tau\chi_{\tau'}) = // \text{ the rest summands vanish } //$$
$$= \nu(\chi_\alpha\chi_\beta + \chi_\beta\chi_\alpha) = 0 = \partial\nu(\chi_\sigma).$$

By Theorem 4.3 (10), there is an Adams–Hilton model $\mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n}$ that extends the model $\mathbf{AH}(sk^{2N-2}(\mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n}))$, for which the equality $\nu\partial = \partial\nu$ holds.

Consider the generator $\chi_{\widetilde{\sigma}}$, $\widetilde{\sigma} = \{\underbrace{1, \dots, 1}_{k_1}, \underbrace{2, \dots, 2}_{k_2}, \dots, \underbrace{n, \dots, n}_{k_n}\}$. Denote by $\widetilde{\partial}(\chi_{\widetilde{\sigma}})$ the differential in the extended model $\mathbf{AH}(\mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n})$.

Denote by x the element

$$x := \sum_{\tau, \tau', \tau \sqcup \tau' = \widetilde{\sigma}} \chi_\tau \chi_{\tau'}.$$

Then

$$\nu x = \nu(\sum \chi_\tau\chi_{\tau'}) = \nu(\chi_\alpha\chi_\beta + \chi_\beta\chi_\alpha) = 0,$$

where $\sigma = \alpha \sqcup \beta$, $n \notin \alpha$, $\beta = \{n, \dots, n\}$. By direct computation we obtain $\partial x = 0$. The map ν is a chain equivalence, hence an element x is a boundary. Then

$$x = k\widetilde{\partial}(\chi_{\widetilde{\sigma}}) + \partial y, \quad y \in \mathbf{AH}(sk^{2N-2}(\mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n}).$$

To conclude the proof it remains to use the argument from the previous theorem. □

Corollary 4.14. *An Adams–Hilton model for the product of n copies of $\mathbb{C}P^\infty$ is given by*

$$\mathbf{AH}(\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty) = \big(T(\chi_\sigma \mid \sigma = \{1, \dots, 1, 2, \dots, 2, \dots, n, \dots, n\},\, \sigma \neq \varnothing), \partial\big),$$
$$\partial\chi_\sigma = \sum_{\sigma = \tau \sqcup \tau'} \chi_\tau\chi_{\tau'}, \quad |\chi_\sigma| = 2|\sigma| - 1.$$

Moreover, this model extends the model $\mathbf{AH}(S^2 \times \dots \times S^2)$ from Theorem 4.8.

Theorem 4.15. *An Adams–Hilton model for $(\mathbb{C}P^\infty)^{\mathcal{K}}$ is given by*

$$\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}) = \big(T(\chi_\sigma \mid I_\sigma \in \mathcal{K},\, \sigma \neq \varnothing), \partial\big),$$

$$\partial\chi_\sigma = \sum_{\sigma = \tau \sqcup \tau'} \chi_\tau\chi_{\tau'}, \quad |\chi_\sigma| = 2|\sigma| - 1, \tag{4.8}$$

where I_σ denotes the support of a multiset σ. Moreover, this model extends the model $\mathbf{AH}((S^2)^{\mathcal{K}})$ from Corollary 4.11.

Proof. The polyhedral product $(\mathbb{C}P^\infty)^{\mathcal{K}}$ is the colimit of products

$$(\mathbb{C}P^\infty)^{\mathcal{K}} = \operatorname*{colim}_{J \in \mathcal{K}} (\mathbb{C}P^\infty)^J.$$

Adams–Hilton models from Corollary 4.14 extend each other over subproducts, thus they satisfy the coherency conditions, see Theorem 4.3 (7). Thus we obtain

$$\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}) = \operatorname*{colim}_{J \in \mathcal{K}} \mathbf{AH}((\mathbb{C}P^\infty)^J).$$ □

Theorem 4.16. *Let $\mathcal{K}$ be an arbitrary simplicial complex. Consider Adams–Hilton models $\mathbf{AH}((\underline{S})^{\mathcal{K}})$ and $\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}})$ from Theorem 4.10 and Theorem 4.15. Then*

$$\mathbf{AH}((\underline{S})^{\mathcal{K}}) \cong \operatorname{Cobar} H_*((\underline{S})^{\mathcal{K}}), \qquad \mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}) \cong \operatorname{Cobar} H_*((\mathbb{C}P^\infty)^{\mathcal{K}})$$

in the category DGA.

Proof. In case $(\underline{S})^{\mathcal{K}}$ compare Theorem 4.10 and Proposition 3.8. In case $(\mathbb{C}P^\infty)^{\mathcal{K}}$ compare Theorem 4.15 and formulae (3.2), (3.3). □

5. Adams–Hilton models for maps and chains in cobar construction corresponding to Whitehead products

The main purpose of this section is to describe the chain in cobar construction Cobar $H_*((\mathbb{C}P^\infty)^\mathcal{K}) \cong \mathbf{AH}((\mathbb{C}P^\infty)^\mathcal{K})$ corresponding to the Hurewicz image of the higher Whitehead product $[[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5]$. It is possible since Adams–Hilton models behave nicely with respect to CW-maps.

Further we only consider Adams–Hilton models for $(\underline{S})^\mathcal{K}$, $(S^2)^\mathcal{K}$, $(\mathbb{C}P^\infty)^\mathcal{K}$ from Theorem 4.10, Corollary 4.11, Theorem 4.15. These models coincide with the corresponding cobar constructions.

Proposition 5.1. *Let $\mathcal{K}$ be an arbitrary simplicial complex. Then the inclusion $\mathbf{AH}((S^2)^\mathcal{K}) \hookrightarrow \mathbf{AH}((\mathbb{C}P^\infty)^\mathcal{K})$ is an Adams–Hilton model $\mathbf{AH}(i)$ for the inclusion $(S^2)^\mathcal{K} \overset{i}{\hookrightarrow} (\mathbb{C}P^\infty)^\mathcal{K}$.*

Proof. By Theorem 4.15, the model $\mathbf{AH}((\mathbb{C}P^\infty)^\mathcal{K})$ extends the model $\mathbf{AH}((S^2)^\mathcal{K})$. The property (5) of Theorem 4.3 gives us the required. □

Corollary 5.2. *Let $\mathcal{K}$ be an arbitrary simplicial complex. Then the inclusion $(S^2)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^\mathcal{K}$ and the inclusion of dgas*

$$\mathrm{Cobar}(H_*((S^2)^\mathcal{K})) \hookrightarrow \mathrm{Cobar}(H_*((\mathbb{C}P^\infty)^\mathcal{K}))$$

induce the same map

$$H_*(\Omega(S^2)^\mathcal{K}) \to H_*(\Omega\,(\mathbb{C}P^\infty)^\mathcal{K})$$

in homology groups.

Proposition 5.3. *Let*

$$\omega\colon S^{N-1} \to (\underline{S})^{\partial\Delta(1,\dots,k)} \cong T(S^{n_1}, \dots, S^{n_k}),\ N = n_1 + \dots + n_k,$$

be the attaching map corresponding to the top cell in $S^{n_1} \times \dots \times S^{n_k}$. Suppose $\mathbf{AH}((\underline{S})^{\partial\Delta})$ is the Adams–Hilton model from Theorem 4.10. Then the Adams–Hilton model $\mathbf{AH}(\omega)$ is given by

$$\mathbf{AH}(\omega)\colon \mathbf{AH}(S^{N-1}) \cong (T(a), 0) \longrightarrow \mathbf{AH}(T(S^{n_1}, \dots, S^{n_k})),$$
$$\mathbf{AH}(T(S^{n_1}, \dots, S^{n_k})) \cong (T(b_J, J \in \partial\Delta, J \neq \varnothing), \partial),$$
$$a \mapsto \sum_{p=1}^{k-1} \sum_{\theta \in S(p,k-p)} \varepsilon(\theta)(-1)^{|b_{\theta(1)\cdots\theta(p)}|+1} b_{\theta(1)\cdots\theta(p)} \otimes b_{\theta(p+1)\cdots\theta(k)}\ (= \partial b_{1\cdots k}).$$

Proof. Suppose X is a CW complex, and $(n+1)$-skeleton X^{n+1} is obtained from X^n by attaching the $(n+1)$-dimensional cell e, suppose v is the corresponding generator in $\mathbf{AH}(X^{n+1})$. Let $f\colon S^n \to X^n$ be the attaching map, an Adams–Hilton model for f makes the following diagram commutative up to chain homotopy:

$$\begin{array}{ccc} \mathbf{AH}(S^n) & \longrightarrow & CU_*(\Omega S^n) \\ \big\downarrow {\scriptstyle \mathbf{AH}(f)} & & \big\downarrow {\scriptstyle CU_*(\Omega f)} \\ \mathbf{AH}(X^n) & \xrightarrow{\theta_{X^n}} & CU_*(\Omega X^n) \end{array}$$

Since $\mathbf{AH}(S^n) \cong (T(a), 0)$, it is suffice to define $\mathbf{AH}(f)$ on a. Assume $\mathbf{AH}(f)(a) = z$. Using the diagram, we get

$$(\theta_{X^n})_*[z] = H(\Omega f)[a].$$

We observe that this formula is exactly the definition of the differential

$$\partial_{X^{n+1}}(v) = z\,(+\partial_{X^n} b).$$

Using Proposition 4.2, we obtain that $\partial_{X^{n+1}}(v)$ is a suitable choice for $\mathbf{AH}(f)(a)$.

Applying this to the map $\omega\colon S^{N-1} \to (\underline{S})^{\partial\Delta(1,\dots,k)} \cong T(S^{n_1},\dots,S^{n_k})$, we conclude that we can take $\mathbf{AH}(\omega)(a) = \partial b_{1\dots k}$, where $b_{1\dots k}$ is the generator corresponding to the top cell in $S^{n_1}\times\dots\times S^{n_k}$. □

THEOREM 5.4. *Consider the map*

$$f\colon S^5\times S^2 \to T(S^2,S^2,S^2)\times S^2 \cong (S^2)^{\partial\Delta(1,2,3)*\{4\}}$$

that is the product of the map $S^5 \to T(S^2,S^2,S^2)$ attaching the top cell in $S^2\times S^2\times S^2$ and the identity map $S^2\to S^2$. Then the Adams–Hilton model $\mathbf{AH}(f)$ is given by

$$\mathbf{AH}(S^5\times S^2)\cong T(b_1,b_2,b_{12}),\quad |b_1|=4,\quad |b_2|=1,\quad |b_{12}|=6,$$
$$\partial b_i = 0,\quad \partial b_{12} = -[b_1,b_2];$$
$$\mathbf{AH}(T(S^2,S^2,S^2)\times S^2)\cong T(c_J,\ J\in\partial\Delta(1,2,3)*\{4\},\ J\neq\varnothing),$$
$$\partial c_J = \sum_{(I,L),I\sqcup L=J} c_I\otimes c_L;$$
$$b_1 \xmapsto{\mathbf{AH}(f)} \text{“}\partial c_{123}\text{''} = [c_1,c_{23}]+[c_{12},c_3]+[c_{13},c_2],\quad b_2\xmapsto{\mathbf{AH}(f)} c_4,$$
$$b_{12}\xmapsto{\mathbf{AH}(f)} [c_{124},c_3]+[c_{134},c_2]+[c_1,c_{234}]+[c_{12},c_{34}]+[c_{13},c_{24}]+[c_{14},c_{23}].$$

PROOF. Adams–Hilton models for $S^5\times S^2$ and $T(S^2,S^2,S^2)\times S^2$ are described in Theorem 4.10 and Corollary 4.11. It is easy to see that the maps of algebras defined in Theorem 4.3 (10)

$$\mathbf{AH}(S^5\times S^2)\xrightarrow{\nu_{S^5S^2}}\mathbf{AH}(S^5)\otimes\mathbf{AH}(S^2),$$
$$\mathbf{AH}(T(S^2,S^2,S^2)\times S^2)\xrightarrow{\nu_{TS^2}}\mathbf{AH}(T(S^2,S^2,S^2))\otimes\mathbf{AH}(S^2),$$

commute with differentials, therefore these maps are dga-morphisms. The model for the attaching map $\omega\colon S^5\to T(S^2,S^2,S^2)$ is given in Proposition 5.3:

$$\mathbf{AH}(S^5)\cong(T(b_1),0),\quad |b_1|=4,$$
$$\mathbf{AH}(T(S^2,S^2,S^2))\cong T(c_1,c_2,c_3,c_{12},c_{13},c_{23}),\quad |c_i|=1,\quad |c_{ij}|=3,$$
$$\partial c_i=0,\quad \partial c_{ij}=[c_i,c_j]=c_i\otimes c_j+c_j\otimes c_i,$$
$$b_1\xmapsto{\mathbf{AH}(\omega)}\text{“}\partial c_{123}\text{”} = [c_1,c_{23}]+[c_{12},c_3]+[c_{13},c_2].$$

The identity map $S^2\to S^2$ is modeled by the identity map of Adams–Hilton models.

$$\mathbf{AH}(S^2)\cong(T(b_2),0)\to\mathbf{AH}(S^2)\cong(T(c_4),0),\quad b_2\mapsto c_4$$

By property (11) from Theorem 4.3 for a map

$$\varphi\colon \mathbf{AH}(S^5\times S^2)\to\mathbf{AH}(T(S^2,S^2,S^2)\times S^2)$$

to be an Adams–Hilton model for a product of maps it suffice to make the following diagram commutative

$$\begin{array}{ccc}
\mathbf{AH}(S^5\times S^2) & \xrightarrow{\nu_{S^5S^2}} & \mathbf{AH}(T(S^2,S^2,S^2))\otimes\mathbf{AH}(S^2)\\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mathbf{AH}(\omega)\otimes 1_{\mathbf{AH(S^2)}}}\\
\mathbf{AH}(T(S^2,S^2,S^2)\times S^2) & \xrightarrow{\nu_{TS^2}} & \mathbf{AH}(T(S^2,S^2,S^2))\otimes\mathbf{AH}(S^2)
\end{array}$$

Since $\mathbf{AH}(S^5\times S^2)$ extends the model $\mathbf{AH}(S^5\vee S^2)$ and $\mathbf{AH}(S^5\times S^2)\cong(T(b_1,b_2,b_{12}),\partial)$ we only have to define

$$b_{12}\overset{\varphi}{\mapsto}[c_{124},c_3]+[c_{134},c_2]+[c_1,c_{234}]+[c_{12},c_{34}]+[c_{13},c_{24}]+[c_{14},c_{23}].$$

We check that φ commutes with the differential. We have

$$b_1\overset{\varphi}{\mapsto}\text{“}\partial c_{123}\text{”},\quad b_2\overset{\varphi}{\mapsto}c_4,\quad \partial b_{12}=-[b_1,b_2]\overset{\varphi}{\mapsto}-[\text{“}\partial c_{123}\text{”},c_4].$$

We use the inclusion of models $\mathbf{AH}(T(S^2,S^2,S^2)\times S^2)\hookrightarrow \mathbf{AH}((S^2)^{\Delta(1,2,3,4)})$ to conclude

$$\partial c_{1234}=[c_{123},c_4]+[c_{124},c_3]+[c_{134},c_2]+[c_1,c_{234}]+[c_{12},c_{34}]+[c_{13},c_{24}]+[c_{14},c_{23}].$$

By taking ∂ we obtain

$$\partial[c_{123},c_4]=[\partial c_{123},c_4]=-\partial([c_{124},c_3]+[c_{134},c_2]+[c_1,c_{234}]+[c_{12},c_{34}]+[c_{13},c_{24}]+[c_{14},c_{23}]),$$

and this expression is well defined in $\mathbf{AH}(T(S^2,S^2,S^2)\times S^2)$. Note that

$$b_{12}\overset{\varphi}{\mapsto} \text{“}\partial c_{1234}-[c_{123},c_4]\text{”} = [c_{124},c_3]+[c_{134},c_2]+[c_1,c_{234}]+[c_{12},c_{34}]+[c_{13},c_{24}]+[c_{14},c_{23}]. \tag{5.1}$$

It only remains to check that φ makes the diagram commutative. We have $\nu_{S^5S^2}(b_{12})=0$ by definition of $\nu_{S^5S^2}$. Since ν_{TS^2} is a ring homomorphism, it commutes with the Lie bracket, and we obtain $\nu_{TS^2}\varphi(b_{12})=0$. □

THEOREM 5.5. *Let* $\mathcal{K}=\partial\Delta(\partial\Delta(1,2,3),4,5)$. *Consider the map*

$$g\colon T(S^5,S^2,S^2)\to (S^2)^{\mathcal{K}},$$

induced by the maps $S^5\xrightarrow{\omega}T(S^2,S^2,S^2)$, $S^2\xrightarrow{1_{S^2}}S^2$. *Then the Adams–Hilton model* $\mathbf{AH}(g)$ *is given by*

$$\mathbf{AH}(T(S^5,S^2,S^2)\cong T(b_1,b_2,b_3,b_{12},b_{13},b_{23}),$$

$$|b_1|=4,\quad |b_2|=|b_3|=1,\quad |b_{12}|=|b_{13}|=6,\quad |b_{23}|=3,$$

$$\partial b_i=0,\quad \partial b_{12}=-[b_1,b_2],\quad \partial b_{13}=-[b_1,b_3],\quad \partial b_{23}=[b_2,b_3];$$

$$\mathbf{AH}((S^2)^{\mathcal{K}})\cong T(c_J,\ J\in\mathcal{K},\ J\neq\varnothing),$$

$$\partial c_J=\sum_{(I,L),I\sqcup L=J} c_I\otimes c_L;$$

$$b_1\xmapsto{\mathbf{AH}(g)}\text{“}\partial c_{123}\text{”}=[c_1,c_{23}]+[c_{12},c_3]+[c_{13},c_2],\quad b_2\xmapsto{\mathbf{AH}(g)}c_4,\quad b_3\xmapsto{\mathbf{AH}(g)}c_5, \tag{5.2}$$

$$b_{12}\xmapsto{\mathbf{AH}(g)}[c_{124},c_3]+[c_{134},c_2]+[c_1,c_{234}]+[c_{12},c_{34}]+[c_{13},c_{24}]+[c_{14},c_{23}],$$

$$b_{13}\xmapsto{\mathbf{AH}(g)}[c_{125},c_3]+[c_{135},c_2]+[c_1,c_{235}]+[c_{12},c_{35}]+[c_{13},c_{25}]+[c_{15},c_{23}],$$

$$b_{23}\xmapsto{\mathbf{AH}(g)}c_{45}.$$

PROOF. It follows from the property (8) of Theorem 4.3 since the map g is the colimit of maps f from Theorem 5.4 and the identity map. □

THEOREM 5.6. *Let* $\mathcal{K}=\partial\Delta(\partial\Delta(1,2,3),4,5)$. *Let* $[[\mu_1,\mu_2,\mu_3],\mu_4,\mu_5]\in\pi_7(\Omega\,(\mathbb{C}P^\infty)^{\mathcal{K}})$ *be a canonical iterated Whitehead product in* $(\mathbb{C}P^\infty)^{\mathcal{K}}$, *given by the composite*

$$S^8\xrightarrow{\omega}T(S^5,S^2,S^2)\xrightarrow{g}(S^2)^{\mathcal{K}}\xrightarrow{i}(\mathbb{C}P^\infty)^{\mathcal{K}}.$$

Then Adams–Hilton models can be constructed explicitly for each map in the sequence above:

$$\mathbf{AH}(S^8)\longrightarrow\mathbf{AH}(T(S^5,S^2,S^2))\longrightarrow\mathbf{AH}((S^2)^{\mathcal{K}})\longrightarrow\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}).$$

Furthermore, the Hurewicz image of the higher Whitehead product $[[\mu_1,\mu_2,\mu_3],\mu_4,\mu_5]$ *is represented by the following cycle in the cobar construction* $\operatorname{Cobar}H_*((\mathbb{C}P^\infty)^{\mathcal{K}})\cong\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}})$:

$$-\partial\big([\chi_{123},\chi_{45}]+[\chi_{1234},\chi_5]+[\chi_{1235},\chi_4]\big).$$

Proof. Adams–Hilton models for S^8, $T(S^5, S^2, S^2)$, $(S^2)^{\mathcal{K}}$ and $(\mathbb{C}P^\infty)^{\mathcal{K}}$ are constructed in Theorem 4.10, Corollary 4.11 and Theorem 4.15. Adams-Hilton models for maps ω, g and i are constructed in Proposition 5.3, Theorem 5.5 and Proposition 5.1. By the property (3) of Theorem 4.3 we may take $\mathbf{AH}(i \circ g \circ \omega)$ as Adams–Hilton model for iterated higher Whitehead product $i \circ g \circ \omega = [[\mu_1, \mu_2, \mu_3], \mu_4, \mu_5]$.

It only remains to calculate the image of $\mathbf{AH}(i \circ g \circ \omega)$. We have

$$\mathbf{AH}(S^8) \cong (T(a_1), 0), \quad a_1 \xmapsto{\mathbf{AH}(\omega)} \text{``}\partial b_{123}\text{''} = -[b_1, b_{23}] - [b_{12}, b_3] - [b_{13}, b_2].$$

We use formulae (5.1), (5.2) and inclusions of models $\mathbf{AH}((S^2)^{\mathcal{K}}) \hookrightarrow \mathbf{AH}((S^2)^{\Delta^4})$ and $\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}) \hookrightarrow \mathbf{AH}((\mathbb{C}P^\infty)^{\Delta^4})$ to conclude

$$\begin{aligned}
\text{``}\partial b_{123}\text{''} \xmapsto{\mathbf{AH}(g)} & -[\text{``}\partial c_{123}\text{''}, c_{45}] - [\text{``}\partial c_{1234}\text{''}, c_5] + [[\text{``}c_{123}\text{''}, c_4], c_5] - [\text{``}\partial c_{1235}\text{''}, c_4] + [[\text{``}c_{123}\text{''}, c_5], c_4] \\
&= -[\text{``}\partial c_{123}\text{''}, c_{45}] - [\text{``}\partial c_{1234}\text{''}, c_5] - [[c_4, c_5], \text{``}c_{123}\text{''}] - [\text{``}\partial c_{1235}\text{''}, c_4] \\
&= -\partial\left([\text{``}c_{123}\text{''}, c_{45}] + [\text{``}c_{1234}\text{''}, c_5] + [\text{``}c_{1235}\text{''}, c_4]\right) \\
&\xmapsto{\mathbf{AH}(i)} -\partial\left([\text{``}\chi_{123}\text{''}, \chi_{45}] + [\text{``}\chi_{1234}\text{''}, \chi_5] + [\text{``}\chi_{1235}\text{''}, \chi_4]\right).
\end{aligned}$$

We also may apply the differential ∂ to obtain well defined chain in $\mathbf{AH}((\mathbb{C}P^\infty)^{\mathcal{K}}) \cong \operatorname{Cobar} H_*((\mathbb{C}P^\infty)^{\mathcal{K}})$. □

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