Milnor K-groups and differential forms

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1 Introduction

Let R be a commutative associative ring with a unit. Denote by R^* its multiplicative group. Then its Milnor K-group $K_n^M(R)$ of degree n can be defined as the n-th graded component of the quotient of the tensor ring $(R^*)^{\otimes\bullet}$ by a two-sided ideal, generated by elements of the type $r \otimes (1-r)$, where both r and $1-r$ are invertible. Such elements are called Steinberg relations.

Milnor K-groups are important algebraic invariants that play a fundamental role in various domains of algebra and arithmetics, such as class field theory. Unfortunately, they are usually quite hard to commute, since their definition involves a delicate interplay between the additive and multiplicative structures in the ring R.

At the same time, the R-module of (absolute) differential forms Ω^n_R of degree n is relatively easy to calculate explicitly. There is a functorial group homomorphism $d\log\colon K_n^M(R) \to \Omega^n_R$, however in the general case it is far from being an isomorphism.

Let $I \subset R$ be a nilpotent ideal of degree N such that there exists a section of the quotient map $R \to R/I$ that is also a ring homomorphism (in this case we call the pair (R, I) a split nilpotent extension of the ring R/I). By definition, the corresponding Milnor K-group $K_n^M(R, I)$ is the kernel of the natural homomorphism $K_n^M(R) \to K_n^M(R/I)$. In 1975 Bloch [4, § 1] constructed canonical integral of the relative map $d\log$, that is, the functorial group homomorphism

$$\mathrm{B} \;:\; K_n^M(R, I) \longrightarrow \Omega^{n-1}_{R,I}/d\,\Omega^{n-2}_{R,I}\,, \qquad n \geqslant 1\,,$$

such that there is an equality

$$d \circ \mathrm{B} = d\log \;:\; K_n^M(R, I) \longrightarrow \Omega^n_{R,I}\,.$$

It was done under the assumption that all the natural numbers from 1 to N are invertible in R.

Later the combination of results, obtained by Bloch [4, theorem 0.1], Maazen and Stienstra [22, § 3.12], van der Kallen [18, corollary 8.5] and Dribus [14] showed that under the additional assumption of R being 5-fold stable the map B is an isomorphism. This result might be interpreted as a variant of famous Goodwillie Theorem [8] with Milnor K-groups replacing algebraic K-groups.

Some time later Gorchinskiy and Osipov [9, Theor. 2.9] proved that the map B is an isomorphism in the case $R = S[\varepsilon]$, $I = (\varepsilon)$, where is a formal variable such that $\varepsilon^2 = 0$ and S is a weakly 5-fold stable ring such that 2 is invertible in it. They applied this result to the study of the higher-dimensional Contou-Carrère symbol. The approach in [9] was based on the explicit analysis of elements in Milnor K-groups.

First major result of this paper (which we also call the isomorphism theorem for Bloch map) is Theorem 5.1 which states that in order for the map B to be an isomorphism it is enough for R to be weakly 5-fold stable. This result was published in paper [10], written together with S.O. Gorchinsky (see [10, Theorem 2.12]). Note that the condition of R being weakly 5-fold stable is substantially more general that the condition of R being just 5-fold stable (a good example is a ring of Laurent series with a suitable ring of coefficients). In addition, the proof of Theorem 5.1 was carried in a much more simpler way, than the proof described in the articles mentioned above. In particular, the proof is reduced to the case of [9, Theor. 2.9] by using the fact that relative Milnor K-groups and modules of differential forms commute with a certain class of non-filtered colimits and also applying several new tricks to deal with elements in Milnor K-groups.

Now let us fix some prime p bigger than two. Note that in case of p-adically complete ring R with all natural numbers exept the ones divisible by p being invertible in it, the integration of $d\log$ is not possible in general. However, it turns out that one can define a p-adic equivalent of the Bloch map B. Moreover, one might actually not regress to the relative case for some nilpotent ideal. However, in order to do that one must consider the (derived) p-adic completions of the corresponding modules of differential forms.

Originally, Katou [19, § I.3] defined such a p-adic equivalent of the Bloch map for the case of smooth schemes over the ring of Witt vectors of some perfect field with characteristic $p > 2$, equipped with a lifting of the Frobenius homomorphism. (In fact, Katou defined this map for a more general case of syntomic schemes over the ring of Witt vectors, without any chosen lifting of the Frobenius homomorphism; in this case the image of this map lies in syntomic cohomologies). The main non-trivial fact here is that the constructed map satisfies the Steinberg property, that is, it sends all the Steinberg relations to zero (see [19, proposition I.3.2]). The proof of the Steinberg property provided by Katou is based on two statements. Firstly, one shows that p-adic Bloch map in a right way does not depend on the choice of a lifting of the Frobenius homomorphism (see [19, p. 212]). For this purpose one has to reduce the syntomic cohomologies to the crystallic ones. Secondly, one considers the separate case of the ring $\mathbb{Z}_p[x, x^{-1}, (1-x)^{-1}]$, equipped with a lifting of the Frobenius homomorphism that maps x to x^p [19, p. 217] (compare this to steps 5 and 6 in the proof of Theorem 6.5). For this purpose the proof is reduced to the case of the ring $\mathbb{Z}_p((x))$ of Laurent series. However, we think that the last reduction in [19] is not entirely clear.

Note that (affine) smooth schemes over the ring of Witt vectors, considered by Katou, can be viewed as a special case of a δ-ring. The notion of a δ-ring was firstly introduced by Joyal [17] and was later studied by Buium [7], who called them rings, equipped with p-derivations. The article of Bhatt–Scholze [2, § 2] can also serve as a great source. Briefly, by a δ-structure on the ring R one means a map $\delta\colon R \to R$ that satisfies the set of certain properties, from which, in particular, it follows that the map $\varphi\colon r \mapsto r^p + p\delta(r)$ is a correctly defined endomorphism of the ring R and thus is also a lifting to the Frobenius homomorphism (see § 2.5). Notably, if the ring R has trivial p-torsion, then the notions of a δ-structure and a lifting of the Frobenius homomorphism are equivalent.

Second major result of this paper is a generalization of Katou's result for the case of p-adically complete δ-rings. It is easy to show that a δ-structure on a R allows to define a group homomorphism $\frac{\varphi}{p^n}$ on the module Ω^n_R, that coincides with the natural action of φ on Ω^n_R after being multiplied by p^n and commutes with the differential map (see Proposition 2.19).

Then there exists a canonical integral of the map $\left(1 - \frac{\varphi}{p}\right) d\log$. In other words for any p-adically complete δ-ring (R, δ) there exists a functorial group homomorphism (see Proposition 6.4)

$$\mathrm{B}_\delta \ : \ (R^*)^{\otimes n} \longrightarrow {}^D\widehat{\Omega}^{n-1}_R / d\, {}^D\widehat{\Omega}^{n-2}_R \,, \qquad n \geqslant 1$$

that satisfies the equality

$$d \circ \mathrm{B}_\delta = \left(1 - \frac{\varphi}{p^n}\right) d\log \ : \ (R^*)^{\otimes n} \longrightarrow {}^D\widehat{\Omega}^n_R \,.$$

Here, by ${}^D\widehat{\Omega}^n_R$ we denote the derived p-adic completion of the group ${}^D\widehat{\Omega}^n_R$.

Second major result of this paper is Theorem 6.5, that states that the map B_δ quotients through Steinberg relations. Thus, there is a group homomorphism

$$\mathrm{B}_\delta \ : \ K^M_n(R) \longrightarrow {}^D\widehat{\Omega}^{n-1}_R / d\, {}^D\widehat{\Omega}^{n-2}_R \,.$$

The proof of this theorem is explicit and does not use divided powers theory or chrystallic cohomologies. We call the homomorphism B_δ the Bloch–Artin–Hasse map, because in case $n = 1$ the corresponding group homomorphism from R^* to R might be considered as a generalization of the classic Artin–Hasse logarithm, which is an isomorphism the groups $1 + t\mathbb{Z}_p[[t]] \xrightarrow{\sim} t\mathbb{Z}_p[[t]]$, mapping element $1+t$ to $\sum_{p \nmid i} (-1)^{i-1} \frac{t^i}{i}$ (see [34, § 1]).

Now let $R = S \oplus I$ be a split nilpotent extension of S such that both rings R and S are p-adically complete and have trivial p-torsion and $I^N = 0$ for some $N \in \mathbb{N}$. Suppose that there is a δ-structure on R such that $\delta(S) \subset S$ and $\delta(I) \subset I$. It is easy to see that the restriction Bloch–Artin–Hasse map B_δ defines the homomorphism

$$\mathrm{B}_\delta : {}^d\widehat{K}^M_{n+1}(R, I) \to {}^D\widehat{\Omega}^n_{R,I} / d\, {}^D\widehat{\Omega}^{n-1}_{R,I} .$$

Analogously to Theorem 5.1, there is a reason to believe that under some additional assumptions this map is an isomorphism. For instance, it is easy to show that if a δ-ring R has trivial p-torsion and there is also an inclusion $\delta(I) \subset I^2$ then the corresponding Bloch–Artin–Hasse map $\mathrm{B}_\delta \colon 1 + I \xrightarrow{\sim} I$ is an isomorphism (compare this to the results of [13], and also compare the particular case of the ring $\mathbb{Z}_p[[t]]$ with [34, Proposition 1]).

Our third major result is Theorem 7.4 that states that if S is a p-adically complete weakly 5-fold stable δ-ring with trivial p-torsion, then for any $N \in \mathbb{N}$ and for any extension of the δ-structure, such that $\delta(I_N) \subset I_N^2$ the homomorphism $B_\delta : {}^D\widehat{K}^M_2(R_N, I_N) \to {}^D\widehat{\Omega}^1_{R_N, I_N}/dI_N$ is an isomorphism. Here by R_N we denote the ring $S[t]/(t^N)$ and by I_N — its nilpotent ideal $(\bar{t})$. The proof is carried by induction on N and actively uses the machinery, developed in paper [10] (see Subsection 3). We would also like to note that, while Theorem 6.5 stays true for the case of classic p-adic completion, in order to achieve this particular result we had to turn to derived p-adic completion, since classic p-adic completion does not satisfy some necessary conditions that are required for the proof (for example, the cokernel of a map of p-adically complete modules can fail to be p-adically complete).

In summary, there is a list of our main results:

(i) (Theorem 5.1) Let $I \subset R$ be a nilpotent ideal and $N \geqslant 1$ be a natural number such that $I^N = 0$. Suppose that the quotient map $R \to R/I$ admits a splitting by a ring homomorphism $R/I \to R$, that $N!$ is invertible in R, and that R is weakly 5-fold stable. Then for any natural number $n \geqslant 0$, the Bloch map is an isomorphism

$$\mathrm{B} \ : \ K^M_{n+1}(R, I) \xrightarrow{\sim} \Omega^n_{R,I} / d\, \Omega^{n-1}_{R,I} .$$

(ii) (Proposition 6.4, Theorem 6.5) There exists a group homomorphism

$$\mathrm{B}_\delta \ : \ K_n^M(R) \longrightarrow {}^D\widehat{\Omega}_R^{n-1}/d\,{}^D\widehat{\Omega}_R^{n-2}$$

that is functorial on the category of p-adically complete δ-rings and satisfies the equality

$$d \circ \mathrm{B}_\delta = \Big(1 - \frac{\varphi}{p^n}\Big) d\log \ .$$

(iii) (Theorem 7.4) If S is a p-adically complete weakly 5-fold stable δ-ring with trivial p-torsion, then for any $N \in \mathbb{N}$ and for any extension of the δ-structure, such that $\delta(I_N) \subset I_N^2$ the homomorphism

$$B_\delta : {}^d\widehat{K}_2^M(R_N, I_N) \to {}^D\widehat{\Omega}^1_{R_N,I_N}/dI_N$$

is an isomorphism.

2 Preliminaries and supplementary results

Throughout the paper, by a ring we mean a commutative associative unitary ring.

Let R be a ring. If we need auxiliary assumptions on R, we say this explicitly in what follows. Let $n \geqslant 0$ be a natural number.

2.1 Milnor K-groups

By R^* we denote the multiplicative group of invertible elements in R. The *n-th Milnor K-group* of R is defined as the quotient

$$K_n^M(R) := (R^*)^{\otimes n}/\mathrm{St}_n(R)\,.$$

Here, $\mathrm{St}_n(R)$ is the subgroup of $(R^*)^{\otimes n}$ generated by so-called *Steinberg relations*, which are elements of type

$$r_1 \otimes \ldots \otimes r_i \otimes r \otimes (1-r) \otimes r_{i+1} \otimes \ldots \otimes r_{n-2}\,,$$

where $0 \leqslant i \leqslant n-2$ and $r_1, \ldots, r_{n-2}, r, 1-r \in R^*$.

For example, $K_0^M(R) = \mathbb{Z}$ and $K_1^M(R) = R^*$. For $n \geqslant 2$ the class of a tensor $r_1 \otimes \ldots \otimes r_n$ in $K_n^M(R)$ is denoted by $\{r_1, \ldots, r_n\}$. Elements of Milnor K-groups are often called *symbols*. The group law in $K_n^M(R)$ is written additively except for the case $n=1$.

The assignment of $K_n^M(R)$ to R is functorial with respect to the ring R. Given a homomorphism of rings, for simplicity, we denote the corresponding map between their Milnor K-groups similarly as the ring homomorphism.

Let $I\subset R$ be an ideal such that all elements in $1+I$ are invertible in R. Equivalently, an element in R is invertible if and only if its image in R/I is invertible. An example is when all elements in I are nilpotent or topologically nilpotent. For instance, this holds for R being the ring $S[[t]]$ of formal power series in a formal variable t with coefficients in a ring S and $I=(t)$.

The natural homomorphism $R^*\to(R/I)^*$ is surjective and it follows that the homomorphism $K_n^M(R)\to K_n^M(R/I)$ is surjective as well.

Definition 2.1. The *relative n-th Milnor K-group* is given by the formula

$$K_n^M(R,I):=\mathrm{Ker}\big(K_n^M(R)\to K_n^M(R/I)\big)\,.$$

In particular, there is an equality $K_1^M(R,I)=1+I$ between subgroups of $K_1^M(R)=R^*$.

The following simple lemma is needed for what follows.

Lemma 2.2. *The relative Milnor K-group $K_n^M(R,I)$ is generated by elements of type $\{r_1,\ldots,r_i,1+x,r_{i+1},\ldots,r_{n-1}\}$, where $0\leqslant i\leqslant n-1$, $r_1,\ldots,r_{n-1}\in R^*$, and $x\in I$.*

Proof. By definition of Milnor K-groups, we have the following commutative diagram with exact rows and with vertical maps α, β, and γ being induced by the quotient map $R\to R/I$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{St}_n(R) & \longrightarrow & (R^*)^{\otimes n} & \longrightarrow & K_n^M(R) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & \mathrm{St}_n(R/I) & \longrightarrow & \big((R/I)^*\big)^{\otimes n} & \longrightarrow & K_n^M(R/I) & \longrightarrow & 0
\end{array}$$

Take an element

$$a_1\otimes\ldots\otimes a_i\otimes a\otimes(1-a)\otimes a_{i+1}\otimes\ldots\otimes a_{n-2}\in\mathrm{St}_n(R/I)\,,$$

where $0\leqslant i\leqslant n-2$ and $a_1,\ldots,a_{n-2},a,1-a\in(R/I)^*$. Let $r_1,\ldots,r_{n-2},r\in R$ be any preimages of $a_1,\ldots,a_{n-2},a\in R/I$, respectively. Then the elements $r_1,\ldots,r_{n-2},r,1-r$ are invertible in R and we have the equality

$$\alpha(r_1\otimes\ldots\otimes r_i\otimes r\otimes(1-r)\otimes r_{i+1}\otimes\ldots\otimes r_{n-2})=$$

$$= a_1 \otimes \ldots \otimes a_i \otimes a \otimes (1-a) \otimes a_{i+1} \otimes \ldots \otimes a_{n-2}\,.$$

Thus the map α is surjective. Therefore, by the snake lemma, the natural map

$$\mathrm{Ker}(\beta) \longrightarrow \mathrm{Ker}(\gamma) = K_n^M(R, I)$$

is surjective as well.

On the other hand, we have an exact sequence

$$1 \longrightarrow (1+I) \longrightarrow R^* \longrightarrow (R/I)^* \longrightarrow 1\,.$$

Since tensor product is right exact, we obtain a right exact sequence

$$\bigoplus_{i=0}^{n-1} (R^*)^{\otimes i} \otimes (1+I) \otimes (R^*)^{(n-i-1)} \longrightarrow (R^*)^{\otimes n} \xrightarrow{\beta} \big((R/I)^*\big)^{\otimes n} \longrightarrow 1\,.$$

This gives an explicit description of $\mathrm{Ker}(\beta)$, which finishes the proof. □

It follows directly from Definition 2.1 that for any ideal $J \subset R$ contained in I, there is an exact sequence

$$0 \longrightarrow K_n^M(R, J) \longrightarrow K_n^M(R, I) \longrightarrow K_n^M(R', I') \longrightarrow 0\,, \qquad (2.1)$$

where we put $R' = R/J$, $I' = I/J$.

2.2 Weak k-stability

The following notion goes back to Morrow [27, Def. 3.1].

Definition 2.3. Given a natural number $k \geqslant 2$, a ring R is called *weakly k-fold stable* if for any collection of elements $r_1, \ldots, r_{k-1} \in R$, there exists $r \in R^*$ such that $r_1 + r, \ldots, r_{k-1} + r \in R^*$.

Example 2.4. A ring R is weakly 2-fold stable if and only if any element in R is a sum of two invertible elements.

Example 2.5. It is easy to see that, given an ideal $I \subset R$ such that all elements in $1+I$ are invertible in R, the quotient R/I is weakly k-fold stable if and only if the initial ring R is weakly k-fold stable.

In particular, S is a weakly k-fold stable ring if and only if the same holds for $S[[t]]$. See more details on weak stability in [9, § 2.2].

The following lemma, which was proved by Morrow [27, Lem. 3.6] by using a method which goes back to Nesterenko and Suslin [28, Lem. 3.2] (see also Lemma 2.2 from the paper of Kerz [21]). It will be playing a significant role in what follows.

Lemma 2.6. *Let R be a weakly 5-fold stable ring. Then for all elements $r, s \in R^*$, there is an equality $\{r, s\} = -\{s, r\}$ in $K_2^M(R)$.*

2.3 Differential forms

By Ω^1_R we denote the R-module of (absolute) differential forms of R. Recall that the R-module Ω^1_R is generated by elements dr, $r \in R$, subject to linearity $d(r+s) = dr + ds$ and the Leibniz rule $d(rs) = rds + sdr$, where $r, s \in R$.

The R-module Ω^n_R of *differential forms of degree n* is defined as the wedge power

$$\Omega^n_R := \bigwedge\nolimits^n_R \Omega^1_R \,.$$

By definition, $\Omega^0_R = R$. Explicitly, Ω^n_R is the quotient of the R-module $(\Omega^1_R)^{\otimes n}_R$ over the R-submodule generated by elements of type

$$dr_1 \otimes \ldots \otimes dr_i \otimes dr \otimes dr \otimes dr_{i+1} \otimes \ldots \otimes dr_{n-2} \,,$$

where $0 \leqslant i \leqslant n-2$ and $r_1, \ldots, r_{n-2}, r \in R$.

We have a group homomorphism

$$d \;:\; R \longrightarrow \Omega^1_R \,, \qquad r \longmapsto dr \,,$$

which defines also a group homomorphism

$$d \;:\; \Omega^n_R \longrightarrow \Omega^{n+1}_R \,, \qquad sdr_1 \wedge \ldots \wedge dr_n \longmapsto ds \wedge dr_1 \wedge \ldots \wedge dr_n \,,$$

called *de Rham differential*. Since $d^2 = 0$, we have a complex

$$R \xrightarrow{d} \Omega^1_R \xrightarrow{d} \ldots \xrightarrow{d} \Omega^i_R \xrightarrow{d} \ldots \,,$$

called *de Rham complex* of R. Its cohomology groups are called *de Rham cohomology* of R and are denoted by $H^i_{dR}(R)$, $i \geqslant 0$.

By $(\Omega^n_R)^{cl}$ denote the group of closed differential forms, that is, put

$$(\Omega^n_R)^{cl} := \mathrm{Ker}\big(d \colon \Omega^n_R \to \Omega^{n+1}_R\big) \,.$$

The assignment of Ω^n_R to R is functorial with respect to the ring R. As in the case of Milnor K-groups, given a homomorphism of rings, we denote the corresponding maps between their modules of differential forms and de Rham cohomology similarly as the ring homomorphism.

Let $I \subset R$ be an ideal. The natural morphism of R-modules $\Omega^n_R \to \Omega^n_{R/I}$ is surjective.

Definition 2.7. The *relative R-module of differential forms of degree n* is given by the formula

$$\Omega^n_{R,I} := \mathrm{Ker}\big(\Omega^n_R \to \Omega^n_{R/I}\big)\,.$$

Note that we use this term by analogy with relative Milnor K-groups (see Definition 2.1). The reader is warned that usually the term "relative differential forms" has another meaning in the context of a homomorphism between rings.

In particular, there is an equality $\Omega^0_{R,I} = I$ between R-submodules of $\Omega^0_R = R$.

Relative modules of differential forms give a subcomplex $\Omega^\bullet_{R,I}$ of $\Omega^\bullet_R$, called a *relative de Rham complex*. Its cohomology groups are denoted by $H^i_{dR}(R,I)$, $i \geqslant 0$, and are called *relative de Rham cohomology*.

The following simple lemma is needed for what follows.

Lemma 2.8. *The relative module of differential forms $\Omega^n_{R,I}$ is generated additively as an abelian group by differential forms of type $x\,dr_1 \wedge \ldots \wedge dr_n$ and by differential forms of type $r_1\,dx \wedge dr_2 \wedge \ldots \wedge dr_n$, where $r_1, \ldots, r_n \in R$ and $x \in I$.*

Proof. By [24, Theor. 25.2], there is an exact sequence

$$I \longrightarrow \Omega^1_R/(I \cdot \Omega^1_R) \longrightarrow \Omega^1_{R/I} \longrightarrow 0\,,$$

where the first map sends an element $x \in I$ to the class of dx in the quotient. In other words, there is an isomorphism of R-modules

$$\Omega^1_{R/I} \simeq \Omega^1_R/(I \cdot \Omega^1_R + dI)\,.$$

Taking the wedge power, we obtain isomorphisms

$$\Omega^n_{R/I} \simeq \Omega^n_R/\big((I \cdot \Omega^1_R + dI) \wedge \Omega^{n-1}_R\big) \simeq \Omega^n_R/(I \cdot \Omega^n_R + dI \wedge \Omega^{n-1}_R)\,,$$

which proves the lemma. □

Similarly to Milnor K-groups, given an ideal $J \subset R$ contained in I, there is an exact sequence of relative de Rham complexes

$$0 \longrightarrow \Omega^{\bullet}_{R,J} \longrightarrow \Omega^{\bullet}_{R,I} \longrightarrow \Omega^{\bullet}_{R',I'} \longrightarrow 0\,,$$

where, as above, we put $R' = R/J$, $I' = I/J$. This gives a long exact sequence of relative de Rham cohomology

$$\ldots \longrightarrow H^{n-1}_{dR}(R',I') \longrightarrow H^n_{dR}(R,J) \longrightarrow H^n_{dR}(R,I) \longrightarrow H^n_{dR}(R',I') \longrightarrow \ldots\,, \tag{2.2}$$

Besides, truncating the relative de Rham complexes in degrees greater than n and taking the corresponding long exact sequence of cohomology, we obtain an exact sequence

$$\ldots \longrightarrow H^{n-1}_{dR}(R',I') \longrightarrow \Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J} \longrightarrow \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} \longrightarrow \Omega^n_{R',I'}/d\,\Omega^{n-1}_{R',I'} \longrightarrow 0\,. \tag{2.3}$$

2.4 The map $d\log$

Let $n \geqslant 0$ be a natural number. It is easy to check that there is a homomorphism of groups

$$d\log\ :\ K^M_{n+1}(R) \longrightarrow \Omega^{n+1}_R\,, \qquad \{r_1,\ldots,r_{n+1}\} \longmapsto \frac{dr_1}{r_1}\wedge\ldots\wedge\frac{dr_{n+1}}{r_{n+1}}\,,$$

which is functorial with respect to R.

Example 2.9. It is easy to check that the composition $d\circ d\log$ is equal to zero. Thus, the image of $d\log$ is contained in the subgroup $(\Omega^{n+1}_R)^{cl} \subset \Omega^{n+1}_R$.

Example 2.10. Let $I \subset R$ be a nilpotent ideal. Since $d\log$ is functorial, for any natural n, the restriction of $d\log$ to the subgroup $K^M_n(R,I)$ gives a homomorphism between relative groups

$$d\log\ :\ K^M_n(R,I) \longrightarrow (\Omega^n_{R,I})^{cl}\,,$$

which we denote by the same symbol.

Example 2.11. Let S be a ring such that 2 is invertible in it. Let ε be a formal variable such that $\varepsilon^2 = 0$. Using the equalities $\varepsilon\,d\varepsilon = \frac{1}{2}d(\varepsilon^2) = 0$, one shows that there is a decomposition

$$\Omega^{n+1}_{S[\varepsilon],(\varepsilon)} \simeq (\varepsilon\,\Omega^{n+1}_S) \oplus (d\varepsilon \wedge \Omega^n_S)\,.$$

Denote by $\mathrm{pr}\colon \Omega^{n+1}_{S[\varepsilon],(\varepsilon)} \to \Omega^1_S$ the projection to the second summand $d\varepsilon \wedge \Omega^n_S \simeq \Omega^n_S$.

The following theorem was proved by S.O. Gorchinskiy and D.V. Osipov (see [9, Theor. 2.9]).

Theorem 2.12. *Let S be a weakly 5-stable ring. Then the homomorphism*

$$K^M_{n+1}(S[\varepsilon],(\varepsilon)) \xrightarrow{d\log} \Omega^{n+1}_{S[\varepsilon],(\varepsilon)} \xrightarrow{\mathrm{pr}} \Omega^1_S \tag{2.4}$$

is an isomorphism.

This statement will play a crucial role in our proof of one of the main results, namely, Theorem 5.1.

For all $a \in S$ and $b_1, \ldots, b_n \in S^*$, the isomorphism in Theorem 2.12 sends the element $\{1 + ab_1 \ldots b_n\varepsilon, b_1, \ldots, b_n\}$ to the form $adb_1 \wedge \ldots \wedge db_n$. Note that since S is weakly 2-fold stable, elements of this type generate the group $K^M_{n+1}(S[\varepsilon],(\varepsilon))$.

2.5 δ-rings

Let p be a prime number not equal to 2.

A *δ-structure* on a ring R is a map $\delta\colon R \to R$ such that for all $r, s \in R$, the following identities are satisfied:

$$\delta(r+s) = \delta(r) + \delta(s) + F_p(r,s)\,,$$

$$\delta(rs) = r^p\delta(s) + s^p\delta(r) + p\delta(r)\delta(s)\,, \qquad \delta(1) = \delta(0) = 1\,,$$

where $F_p(x,y) = \big(x^p + y^p - (x+y)^p\big)/p \in \mathbb{Z}[x,y]$ is a polynomial with integer coefficients. In this case, the pair (R,δ) is called a *δ-ring*. Morphisms between δ-rings are defined in a natural way: a ring homomorphism $f\colon R \to R'$ is a morphism of δ-rings $(R,\delta) \to (R',\delta')$ if it satisfies the equality $f \circ \delta = \delta' \circ f$. Note that a ring can be equipped with more than one δ-structure. Nevertheless, hereinafter we will omit sometimes the symbol δ in the notation of a δ-ring (R,δ).

For a δ-ring (R,δ), consider the map

$$\varphi \;:\; R \longrightarrow R\,, \qquad \varphi(r) = r^p + p\delta(r)\,.$$

It follows from the properties of the δ-structure that the map φ is an endomorphism of the ring R and is a lift of the Frobenius on R/p. If a ring R has trivial p-torsion, then there is a bijective correspondence between δ-structures on R and Frobenius lifts on R (however, this is not true in general).

The following lemma easily follows from the additivity formula of δ-structure:

Lemma 2.13. *Let R be a δ-ring. If $I \subset R$ is an ideal of R such that $\delta(I) \subset I$ then the quotient ring R/I admits a unique δ-structure compatible with the one on R.*

Example 2.14.

(i) It is easy to show that there is a unique δ-structure on the ring $\mathbb{Z}$, which maps an arbitrary $n \in \mathbb{Z}$ to $(n - n^p)/p$. In particular, for any $i \geqslant 1$, the number $\delta(p^i)$ is divisible by p^{i-1}.

(ii) The forgetful functor from the category of δ-rings to the category of sets has a left adjoint. Thus, one can define a free δ-ring $\mathbb{Z}[x_s,\, s \in S]_\delta$ for an arbitrary set S. Namely, $\mathbb{Z}[x_s,\, s \in S]_\delta$ is the ring $\mathbb{Z}[\delta^i x_s,\, s \in S,\, i \geqslant 0]$ of polynomials in formal variables $\delta^i x_s$, $s \in S$, $i \geqslant 0$, and the δ-structure on it is defined by equalities $\delta(\delta^i x_s) = \delta^{i+1} x_s$, $s \in S$, $i \geqslant 0$. In particular, for any δ-ring (R, δ) and an element $r \in R$, there is a uniquely defined morphism of δ-rings $\mathbb{Z}[x]_\delta \to R$ that sends x to r.

Remark 2.15. It is easy to see that for any δ-ring R there is an isomorphism

$$\operatorname*{colim}_{(P,f)} P \xrightarrow{\sim} R\,, \tag{2.5}$$

where a pair (P, f) consists of a free δ-ring P and a morphism of δ-rings $f \colon P \to R$. Note that this colimit is not directed.

For an arbitrary abelian group A, denote its p-adic completion $\varprojlim_{i \in \mathbb{N}} A/p^i$ by $\widehat{A}$ or $A^\wedge$. Note that the p-adic completion of a ring is also a ring.

Lemma 2.16.

(i) *For any δ-ring (R, δ), the map $\delta \colon R \to R$ is continuous in p-adic topology. More precisely, if $r, r' \in R$ satisfy $r \equiv r' \pmod{p^i}$, $i \geqslant 1$, then $\delta(r) \equiv \delta(r') \pmod{p^{i-1}}$.*

(ii) *There is a unique δ-structure on $\widehat{R}$ such that the natural ring homomorphism $R \to \widehat{R}$ is a morphism of δ-rings.*

Proof. (i) Let $r' = r + p^i s$ for some $s \in R$. It follows from the additive property of δ-structure that there is an equality

$$\delta(r') = \delta(r + p^i s) = \delta(r) + \delta(p^i s) + F_p(r, p^i s)\,.$$

It follows from the multiplicative property of δ-structure that there is an equality

$$\delta(p^i s) = p^{pi}\delta(s) + s^p\delta(p^i) + p\delta(p^i)\delta(s)\,.$$

Thus, since $\delta(p^i)$ is divisible by p^{i-1} (see Example 2.14(i)), the element $\delta(p^i s)$ of R is divisible by p^{i-1}. Moreover, it is easy to see that the coefficients of $F_p(x, p^i y)$ are divisible by p^i. Altogether this proves (i).

(ii) The statement follows directly from (i). Namely, there are maps $\delta_i \colon R/p^i \to R/p^{i-1}$, $i \geqslant 1$, such that $\delta_j \equiv \delta_i \pmod{p^{i-1}}$ for $j \geqslant i$. Taking the inverse limit, we obtain the map

$$\widehat{\delta} = \varprojlim_i \delta_i \;:\; \widehat{R} \longrightarrow \widehat{R}\,,$$

which commutes with δ with respect to the natural homomorphism $R \to \widehat{R}$ and is also p-adically continuous. Since the image of R is dense in $\widehat{R}$, it follows that the map $\widehat{\delta}$ satisfies all properties of a δ-structure. Uniqueness of the map $\widehat{\delta}$ follows also from the fact that the image of R is p-adically dense in $\widehat{R}$. □

In particular, it follows from Example 2.14(i) and Lemma 2.16(ii) that there is a unique δ-structure on $\mathbb{Z}_p$.

Note that if a ring R has trivial p-torsion then so is the ring $\widehat{R}$. Therefore, in this case, the statement of Lemma 2.16(ii) is equivalent to the existence of the unique extension of the map φ to $\widehat{R}$. The later follows from the map φ being additive.

Lemma 2.17. *Let (R, δ) be a δ-ring and M be a multiplicative subset R. Then there is a unique δ-structure on $(M^{-1}R)^\wedge$ such that the natural homomorphism $R \to (M^{-1}R)^\wedge$ is a morphism of δ-rings. This morphism is initial among all morphisms of δ-rings $f \colon R \to S$, where S is a p-adically complete δ-ring and $f(M) \subset S^*$.*

Proof. It follows from [2, Lemma 2.15] that there exists a unique δ-structure on $(M+pR)^{-1}R$ such that the natural homomorphism $R \to (M+pR)^{-1}R$ is a morphism of δ-rings. By Lemma 2.16(ii), there exists a unique δ-structure on $\big((M+pR)^{-1}R\big)^{\wedge}$ such that the natural homomorphism $(M+pR)^{-1}R \to \big((M+pR)^{-1}R\big)^{\wedge}$ is a morphism of δ-rings. Note that the natural ring homomorphism

$$\psi \;:\; (M^{-1}R)^{\wedge} \longrightarrow \big((M+pR)^{-1}R\big)^{\wedge}$$

is actually an isomorphism. Indeed, since elements of $M+pR$ are invertible in $(M^{-1}R)^{\wedge}$, there is a correctly defined homomorphism that is inverse to ψ. Using the fact that ψ is an isomorphism, one can prove all the required properties of the ring $(M^{-1}R)^{\wedge}$. □

2.6 Differential forms of δ-ring

Let (R,δ) be a δ-ring.

Lemma 2.18. *For any $n \geqslant 0$, the isomorphism* (2.5) *from Remark 2.15 induces an isomorphism*

$$\operatorname*{colim}_{(P,f)} \Omega^n_P \xrightarrow{\sim} \Omega^n_R\,.$$

Proof. Denote by $\mathcal{C}_R$ the category of pairs (F,g), where F is a free ring, that is, the ring of polynomials over $\mathbb{Z}$, and $g\colon F \to R$ is a ring homomorphism. Denote by $(\mathcal{C}_R)_\delta$ the category of pairs (P,f), where P is a free δ-ring and $f\colon P \to R$ is a morphism of δ-rings. In particular, the colimit in the statement of the lemma is taken over the category $(\mathcal{C}_R)_\delta$.

Note that the natural embedding of categories $(\mathcal{C}_R)_\delta \to \mathcal{C}_R$ has a left adjoint that sends a ring homomorphism $u\colon \mathbb{Z}[x_s,\, s\in S] \to R$ to the morphism of δ-rings $v\colon \mathbb{Z}[x_s,\, s\in S]_\delta \to R$ that sends x_s to $u(x_s)$. Thus, the subcategory $(\mathcal{C}_R)_\delta \to \mathcal{C}_R$ is cofinal.

It is a well-known fact that the natural map $\operatorname*{colim}_{(F,g)\in\mathcal{C}_R} \Omega^n_F \to \Omega^n_R$ is an isomorphism. Together with the fact that the category $(\mathcal{C}_R)_\delta$ is cofinal, this proves the lemma. □

For an arbitrary δ-ring (R,δ), we denote the group homomorphism $\Omega^n_R \to \Omega^n_R$ induced by $\varphi\colon R \to R$ by the same symbol φ.

Proposition 2.19. *For all natural numbers $m \leqslant n$ and a δ-ring (R, δ), there is a unique group homomorphism*

$$\frac{\varphi}{p^m} \;:\; \Omega^n_R \longrightarrow \Omega^n_R$$

that satisfies the equality $p^m \cdot \frac{\varphi}{p^m} = \varphi$ between maps from Ω^n_R to itself and is functorial with respect to δ-rings. Moreover, $\frac{\varphi}{p^m}$ is a φ-semilinear morphism of R-modules and commutes with de Rham differential d.

Proof. For all $r, s_1, \ldots, s_n \in R$, there are equalities

$$\varphi(r ds_1 \wedge \ldots \wedge ds_n) = \varphi(r) d\big(\varphi(s_1)\big) \wedge \ldots \wedge d\big(\varphi(s_n)\big) =$$
$$= \varphi(r) d\big(s_1^p + p\delta(s_1)\big) \wedge \ldots \wedge d\big(s_n^p + p\delta(s_n)\big) =$$
$$= p^n\, \varphi(r)\big(s_1^{p-1} ds_1 + d\delta(s_1)\big) \wedge \ldots \wedge \big(s_n^{p-1} ds_n + d\delta(s_n)\big)\,.$$

Thus, there is an embedding $\varphi(\Omega^n_R) \subset p^n \cdot \Omega^n_R$. Therefore, if for a δ-ring (R, δ) the modules of differential forms Ω^n_R, $n \geqslant 0$, have trivial p-torsion, then for any $m \leqslant n$ there is a well-defined φ-semilinear morphism of R-modules $\frac{\varphi}{p^m} \colon \Omega^n_R \to \Omega^n_R$ that satisfies all the requirements of the proposition.

Note that according to Example 2.14(ii), free δ-rings are polynomial rings with coefficients in $\mathbb{Z}$, so their modules of differential forms have trivial p-torsion. Thus, it follows from Lemma 2.18 that the map $\frac{\varphi}{p^m}$ can be constructed for the case of arbitrary δ-rings. $\square$

Note that for an arbitrary δ-ring (R, δ) and elements $r, s_1, \ldots, s_n \in R$, there is an equality

$$\frac{\varphi}{p^m}(r ds_1 \wedge \ldots \wedge ds_n) = p^{n-m}\, \varphi(r)\big(s_1^{p-1} ds_1 + d\delta(s_1)\big) \wedge \ldots \wedge \big(s_n^{p-1} ds_n + d\delta(s_n)\big)\,. \tag{2.6}$$

In fact, there is a more direct proof of Proposition 2.19 that does not involve Remark 2.15 or Lemma 2.18, but instead relies on a direct checking that formula (2.6) defines correctly a map from Ω^n_R to itself that satisfies all the needed requirements.

Remark 2.20. There is a unique extension of de Rham differential d to p-adic completions $\widehat{\Omega}^n_R$, which, in turn, defines the complex $\widehat{\Omega}^\bullet_R$. In a similar way, for a δ-ring (R, δ), the endomorphism $\varphi \colon \Omega^n_R \to \Omega^n_R$ extends uniquely to the group endomorphism $\varphi \colon \widehat{\Omega}^n_R \to \widehat{\Omega}^n_R$, and the group endomorphism $\frac{\varphi}{p^m} \colon \Omega^n_R \to \Omega^n_R$ from Proposition 2.19 extends uniquely to the group endomorphism $\frac{\varphi}{p^m} \colon \widehat{\Omega}^n_R \to \widehat{\Omega}^n_R$ preserving all its properties.

The following technical lemma will be used further.

Lemma 2.21. *For an arbitrary ring R, the group homomorphism $\widehat{\Omega}^n_R \to \widehat{\Omega}^n_{\widehat{R}}$ induced by the natural map $R \to \widehat{R}$ is actually an isomorphism.*

Proof. Note that for any ideal $I \subset R$ there is a canonical isomorphism $\Omega^n_R/\big(I\Omega^n_R + dI \wedge \Omega^{n-1}_R\big) \simeq \Omega^n_{R/I}$. In particular, if $I = (p^i)$, then there is a canonical isomorphism $\Omega^n_R/p^i \simeq \Omega^n_{R/p^i}$. Applying the analogous isomorphism with the ring $\widehat{R}$ instead of R and using the ring isomorphism $R/p^i \simeq \widehat{R}/p^i$, one obtains the isomorphisms

$$\Omega^n_R/p^i \simeq \Omega^n_{R/p^i} \simeq \Omega^n_{\widehat{R}/p^i} \simeq \Omega^n_{\widehat{R}}/p^i\,.$$

Thus, the proof follows from taking an inverse limit over i. □

2.7 Construction with filtered dg-rings

In this subsection, we explain some of the constructions in [19, § 2].

Consider two morphisms of dg-rings $f, g\colon B^\bullet \to A^\bullet$. Put

$$C^\bullet := cone\big(B^\bullet \stackrel{f-g}{\longrightarrow} A^\bullet\big)[-1]\,.$$

In particular, there is an equality $C^n = B^n \oplus A^{n-1}$, and the differential $d\colon C^n \to C^{n+1}$ is defined by the formula $d(b,a) = \big(db, f(b) - g(b) - da\big)$ for $b \in B^n$, $a \in A^{n-1}$. One can consider the complex $C^\bullet$ as a derived equaliser of the morphisms f and g. The following lemma can proved directly.

Lemma 2.22. *The element $(1,0) \in C^0$ and the morphism of complexes*

$$C^\bullet \otimes C^\bullet \longrightarrow C^\bullet\,, \qquad (b,a)\otimes(b',a') \longmapsto \big(bb', (-1)^n f(b)a' + ag(b')\big)\,,$$

where $(b,a) \in B^n \oplus A^{n-1}$ and $(b',a') \in B^{n'} \oplus A^{n'-1}$, define the structure of a dg-ring on $C^\bullet$.

Additionally, the natural morphism $C^\bullet \to B^\bullet$ is a morphism of dg-rings.

Let $F^nA^\bullet$, $n \geqslant 0$, be a multiplicative decreasing filtration by subcomplexes on the dg-ring $A^\bullet$ such that $1 \in F^0A^\bullet$. Equivalently, one can consider the graded dg-subring $B^\bullet = \bigoplus\limits_{n\geqslant 0} F^nA^\bullet \cdot t^n$ in the graded dg-ring

$A^\bullet[t] = \bigoplus_{n\geqslant 0} A^\bullet \cdot t^n$, where the grading is defined by degrees of the element t. Denote by τ the natural embeddings $F^nA^\bullet \hookrightarrow A^\bullet$, $n \geqslant 0$, and $B^\bullet \hookrightarrow A^\bullet[t]$. Suppose that we are given a collection of morphisms of complexes $\lambda_n \colon F^nA^\bullet \to A^\bullet$, $n \geqslant 0$, such that for all $a \in F^nA^i$ and $a' \in F^{n'}A^{i'}$, there is an equality $\lambda_n(a) \cdot \lambda_{n'}(a') = \lambda_{n+n'}(a \cdot a')$. Note that this is equivalent to the existence of a morphism of dg-rings $\lambda \colon B^\bullet \to A^\bullet[t]$. Then Lemma 2.22 defines the structure of a dg-ring on the complex

$$C^\bullet = cone\big(B^\bullet \stackrel{\tau-\lambda}{\longrightarrow} A^\bullet[t]\big)[-1] = \bigoplus_{n\geqslant 0} C^\bullet(n)t^n\,,$$

where $C^\bullet(n) = cone\big(F^nA^\bullet \stackrel{\tau-\lambda_n}{\longrightarrow} A^\bullet\big)[-1]$. In particular, we obtain a graded ring $\bigoplus_{n\geqslant 0} H^n\big(C^\bullet(n)\big)$ and a graded morphism $\bigoplus_{n\geqslant 0} H^n\big(C^\bullet(n)\big) \to \bigoplus_{n\geqslant 0} H^n(F^nA^\bullet)$.

Now let $F^nA^\bullet$ be the stupid filtration, that is, $(F^nA^\bullet)^i = A^i$ for $i \geqslant n$ and $(F^nA^\bullet)^i = 0$ for $i < n$. Then $H^n(F^nA^\bullet)$ coincides with the group of closed elements $(A^n)^{cl}$, and there is an isomorphism

$$H^n\big(C^\bullet(n)\big) \simeq \big\{(b,[a]) \in (A^n)^{cl} \oplus A^{n-1}/dA^{n-2} \;\mid\; (1-\lambda_n)b = da\big\}\,. \qquad (2.7)$$

Product of elements $(b,[a]) \in H^n\big(C^\bullet(n)\big)$ and $(b',[a']) \in H^{n'}\big(C^\bullet(n')\big)$ is defined by the formula

$$(b,[a]) \cdot (b',[a']) = \big(bb', [(-1)^n ba' + a\lambda_{n'}(b')]\big) \qquad (2.8)$$

and the graded morphism $\bigoplus_{n\geqslant 0} H^n\big(C^\bullet(n)\big) \to \bigoplus_{n\geqslant 0} (A^n)^{cl}$ sends $(b,[a])$ to b.

2.8 Derived p-adic completion

As we have mentioned above, the classic p-adic completion is not enough for our goals. One of the problems is the the p-adic completion functor is not exact from either side. Another related problem is that the category of p-adically complete abelian groups is not closed under cokernels and is not abelian. Because of this, we have to turn to a derived version of p-adic completion.

Let $\mathcal{A}b$ denote the category of abelian groups. The category of *derived p-adically complete abelian groups* is the minimal abelian subcategory in $\mathcal{A}b$

that contains all classically p-adically complete abelian groups. It turns out that all derived p-adically complete groups are obtained as cokernels of morphisms between classic p-adic completions of free abelian groups. The inclusion of the category of derived p-adically complete abelian groups to $\mathcal{A}b$ has a left-adjoint, which is called a *derived p-adic completion*. Given an abelian group A, denote its derived p-adic completion by ${}^D\widehat{A}$. By definition, an abelian group A is derived p-adically complete if the natural group homomorphism $A \to {}^D\widehat{A}$ is an isomorphism.

Here are some properties of the derived p-adic completion. The derived p-adic completion functor is right exact. For any abelian group A, there is a functorial homomorphism ${}^D\widehat{A} \to \widehat{A}$ and its composition with the homomorphism $A \to {}^D\widehat{A}$ coincides with the natural homomorphism $A \to \widehat{A}$. If an abelian group A is classically p-adically complete, then the homomorphism ${}^D\widehat{A} \to \widehat{A}$ is an isomorphism, that is, classically p-adically complete groups are also derived p-adically complete.

The following property will be extremely useful for us (see the proof in http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/).

Lemma 2.23. *If an abelian group A has trivial p-torsion, then the natural map ${}^D\widehat{A} \to \widehat{A}$ is an isomorphism.*

In particular, this allows to define explicitly the derived p-adic completion of an arbitrary abelian group. Namely, consider an exact sequence

$$F_1 \longrightarrow F_2 \longrightarrow A \longrightarrow 0\,,$$

where F_1, F_2 are free abelian groups. Since the derived p-adic completion functor is right exact, it follows from Lemma 2.23 that ${}^D\widehat{A}$ is the cokernel of the map $F_1 \to F_2$.

Corollary 2.24. *Let R be a ring such that Ω^n_R has trivial p-torsion for some $n \geqslant 0$. Then the map ${}^D\widehat{\Omega}^n_R \to \widehat{\Omega}^n_R$ and the map ${}^D\widehat{\Omega}^n_R \to {}^D\widehat{\Omega}^n_{{}^d\widehat{R}}$ induced by the natural homomorphism $R \to {}^D\widehat{R}$ are isomorphisms.*

Proof. This follows directly from Lemma 2.21 and Lemma 2.23. ☐

Remark 2.25. All statements of Remark 2.20 remain true for derived p-adic completions ${}^D\widehat{\Omega}^n{}_R$.

3 Milnor K-group of the ring of formal power series

Let S be a ring such that 2 is invertible in it and S is weakly 5-fold stable (see Definition 2.3). In this section, we consider the Milnor K-group $K_2^M\big(S[[t]]\big)$. The main results are Proposition 3.2 an Corollary 3.3.

3.1 Filtrations on the Milnor K-group

One has a decreasing filtration on $S[[t]]^*$ given by the formula

$$U_0 := S[[t]]^* = K_1^M\big(S[[t]]\big)\,, \qquad U_k := 1+t^k S[[t]] = K_1^M\big(S[[t]],(t^k)\big)\,, \qquad k \geqslant 1\,.$$

This induces naturally a filtration on Milnor K-groups of $S[[t]]$. Namely, let the subgroup

$$V_k \subset K_2^M\big(S[[t]]\big)\,, \qquad k \geqslant 0\,, \tag{3.1}$$

be generated by symbols $\{f, g\}$, where $f \in U_i$, $g \in U_j$ for $i, j \geqslant 0$ such that $i + j \geqslant k$. Also, put

$$W_0 := K_2^M\big(S[[t]]\big)\,, \qquad W_k := K_2^M\big(S[[t]],(t^k)\big)\,, \qquad p \geqslant 1\,. \tag{3.2}$$

It follows from Lemma 2.2 that there are embeddings

$$V_k \supset W_k \supset V_{2k-1}\,, \qquad k \geqslant 0\,.$$

Lemma 3.1. *If $k \geqslant 1$ is invertible in S then $k \cdot V_k = V_k$*

Proof. The lemma easily follows form the fact that any element $f \in U_1$ admits a root of degree k that also belong to U_1. □

Proposition 3.2. *For any $k \geqslant 0$, there are embeddings*

$$V_{k+1} \subset W_k\,, \qquad k \cdot V_k \subset W_k$$

of subgroups in $K_2^M\big(S[[t]]\big)$. In particular, if $k \geqslant 1$ is invertible in S, then there is an equality $V_k = W_k$.

Proposition 3.2 is proved in Subsection 3.4. Here is an important corollary of Proposition 3.2, which is, actually, its equivalent formulation.

Corollary 3.3. *For all elements $a, b \in S$ and natural numbers $i, j, k \geqslant 0$, the following statements hold.*

(i) If $i + j \geqslant k + 1$, then there is a vanishing

$$\{1 + a\bar{t}^i, 1 + b\bar{t}^j\} = 0$$

in $K_2^M\big(S[t]/(t^k)\big)$, where $\bar{t}$ denotes the image of t in $S[t]/(t^k)$.

(ii) If $i + j \geqslant k$, then there is a vanishing

$$k \cdot \{1 + a\bar{t}^i, 1 + b\bar{t}^j\} = 0$$

in $K_2^M\big(S[t]/(t^k)\big)$. In particular, if $k \geqslant 1$ is invertible in S, then for all elements $a, b \in S$ and natural numbers $i, j \geqslant 0$ such that $i + j \geqslant k$, there is a vanishing $\{1 + a\bar{t}^i, 1 + b\bar{t}^j\} = 0$ in $K_2^M\big(S[t]/(t^k)\big)$.

We will use systematically an auxiliary S-algebra

$$S' = S[[x]]$$

and also the algebra $S'[[t]] = S[[x, t]]$ over $S[[t]]$. Let the subgroups

$$U'_k \subset S'[[t]]^*\,, \qquad V'_k, W'_k \subset K_2^M\big(S'[[t]]\big)\,, \qquad k \geqslant 0\,,$$

be defined similarly as U_k, V_k, W_k with S being replaced by S'. Define the homomorphisms of algebras over $S[[t]]$

$$\theta_0 \;:\; S'[[t]] \longrightarrow S[[t]]\,, \qquad f(x,t) \longmapsto f(0,t)\,,$$

$$\theta_n \;:\; S'[[t]] \longrightarrow S[[t]]\,, \qquad f(x,t) \longmapsto f(t^n,t)\,,$$

where $n \geqslant 1$. In other words, θ_0 is the quotient map to $S'[[t]]/(x) \simeq S[[t]]$.

The following lemma gives a way to make induction with respect to the indices in the filtration V_k, $k \geqslant 0$. The possibility of making such induction is the reason to introduce this filtration.

Lemma 3.4. *For all $n, k \geqslant 0$, there is an embedding*

$$(\theta_n - \theta_0)(V'_k) \subset V_{k+n}$$

of subgroups in $K_2^M\big(S[[t]]\big)$, where we use the homomorphism of groups

$$\theta_n - \theta_0 \;:\; K_2^M\big(S'[[t]]\big) \longrightarrow K_2^M\big(S[[t]]\big)\,. \qquad (3.3)$$

Proof. Consider a symbol $\{f,g\}\in V'_k$, where $f\in U'_i$, $g\in U'_j$, and $i+j\geqslant k$. Write

$$f=\theta_0(f)\cdot\tilde f\,,\qquad g=\theta_0(g)\cdot\tilde g\,,$$

where $\theta_0(f),\theta_0(g)\in S[[t]]^*\subset S'[[t]]^*$ and $\tilde f,\tilde g\in 1+xS'[[t]]$. We have equalities in $S[[t]]^*$

$$\theta_n(f)=\theta_0(f)\cdot\theta_n(\tilde f)\,,\qquad \theta_n(g)=\theta_0(g)\cdot\theta_n(\tilde g)\,.$$

Thus there are equalities in $K_2^M\big(S[[t]]\big)$

$$(\theta_n-\theta_0)\,\{f,g\}=\{\theta_n(f),\theta_n(g)\}-\{\theta_0(f),\theta_0(g)\}=$$

$$=\{\theta_0(f),\theta_n(\tilde g)\}+\{\theta_n(\tilde f),\theta_0(g)\}+\{\theta_n(\tilde f),\theta_n(\tilde g)\}\,.$$

Clearly, $\theta_0(f)\in U_i\subset U'_i$ and $\theta_0(g)\in U_j\subset U'_j$. Therefore

$$\tilde f\in U'_i\cap\big(1+xS'[[t]]\big)=1+xt^iS'[[t]]$$

and, similarly, $\tilde g\in 1+xt^jS'[[t]]$. Hence we have $\theta_n(\tilde f)\in U_{i+n}$ and $\theta_n(\tilde g)\in U_{j+n}$, which finishes the proof. □

Lemma 3.4 implies immediately the following fact.

Corollary 3.5. *For all $n,k\geqslant 0$, the homomorphism in formula* (3.3) *induces a homomorphism*

$$K_2^M\big(S'[[t]]\big)/V'_k\longrightarrow K_2^M\big(S[[t]]\big)/V_{k+n}\,,$$

which we denote also by $\theta_n-\theta_0$ for simplicity.

3.2 Construction of symbols from differential forms

Let us give a way to construct elements in the quotients $K_2^M\big(S[[t]]\big)/V_{k+1}$, $k\geqslant 1$, from differential forms in Ω^1_S.

Since 2 is invertible in S and the ring S is weakly 5-fold stable, by Theorem 2.12, there is an isomorphism

$$K_2^M\big(S[t]/(t^2),(\bar t\,)\big)\xrightarrow{\sim}\Omega^1_S\,,\qquad (3.4)$$

which sends a symbol $\{1+ab\bar t,b\}$ to the differential form adb for any $a\in S$ and $b\in S^*$ (see also Example 4.5), where, as above, $\bar t$ denotes the image of t in the quotient $S[t]/(t^2)$.

For any $k \geqslant 1$, define the homomorphism of S-algebras

$$\lambda_k \;:\; S[[t]] \longrightarrow S[[t]]\,, \qquad f(t) \longmapsto f(t^k)\,.$$

There are embeddings $\lambda_k(W_2) \subset W_{2k} \subset V_{2k} \subset V_{k+1}$. Thus the homomorphism

$$\lambda_k \;:\; K_2^M\big(S[[t]]\big) \longrightarrow K_2^M\big(S[[t]]\big)$$

induces a homomorphism

$$K_2^M\big(S[[t]]\big)/W_2 \longrightarrow K_2^M\big(S[[t]]\big)/V_{k+1}\,,$$

which we denote also by λ_k for simplicity.

Consider the composition

$$\phi_k \;:\; \Omega^1_S \longrightarrow K_2^M\big(S[[t]]\big)/W_2 \xrightarrow{\lambda_k} K_2^M\big(S[[t]]\big)/V_{k+1}\,,$$

where the first map is the inverse of the isomorphism (3.4) followed by the embedding

$$K_2^M\big(S[t]/(t^2),(\bar{t}\,)\big) \subset K_2^M\big(S[t]/(t^2)\big) \simeq K_2^M\big(S[[t]]\big)/W_2\,.$$

In what follows, we denote similarly elements in $K_2^M\big(S[[t]]\big)$ and their images under the quotient map $K_2^M\big(S[[t]]\big) \to K_2^M\big(S[[t]]\big)/V_{k+1}$ to ease notation. For any $a \in S$ and $b \in S^*$, we have an equality in $K_2^M\big(S[[t]]\big)/V_{k+1}$

$$\phi_k(adb) = \{1 + abt^k, b\}\,. \tag{3.5}$$

To avoid confusion, we denote by ϕ'_k the map ϕ_k with S' in place of S, that is, we define the maps

$$\phi'_k \;:\; \Omega^1_{S'} \longrightarrow K_2^M\big(S'[[t]]\big)/V'_{k+1}\,, \qquad k \geqslant 1\,.$$

Lemma 3.6. *For all $k \geqslant 1$, $n \geqslant 0$, and $a, b \in S$, there is an equality*

$$\big((\theta_n - \theta_0) \circ \phi'_k\big)(axdb) = \phi_{k+n}(adb)$$

in $K_2^M\big(S[[t]]\big)/V_{k+n+1}$, where we use the homomorphism of groups

$$\theta_n - \theta_0 \;:\; K_2^M\big(S'[[t]]\big)/V'_{k+1} \longrightarrow K_2^M\big(S[[t]]\big)/V_{k+n+1}$$

from Corollary 3.5.

Proof. Since the ring S is weakly 5-fold stable and, in particular, is weakly 2-fold stable, S is generated additively by invertible elements. Hence it is enough to prove the lemma in the case when b is invertible. Then, by formula (3.5), there are equalities

$$\big((\theta_q-\theta_0)\circ\phi'_k\big)(axdb) = (\theta_q-\theta_0)\,\{1+axbt^k,b\} = \{1+abt^{p+q},b\} = \phi_{k+q}(adb)$$

in $K_2^M\big(S[[t]]\big)/V_{k+q+1}$. □

3.3 Key result

Here is a key result to prove further Proposition 3.2.

Proposition 3.7. *For all elements $a,b\in S$ and natural numbers $i,j\geqslant 1$, there is an equality*

$$(i+j)\{1+at^i,1+bt^j\} = \phi_{i+j}(iadb-jbda) \tag{3.6}$$

in $K_2^M\big(S[[t]]\big)/V_{i+j+1}$.

Proof. The proof is by induction on the sum $i+j$. First we consider the base of the induction, that is, the case $i+j=2$ or, equivalently, $i=j=1$. The proof repeats that of [9, Lemma 3.3] with a minor refinement. We need to prove that for any pair (a,b) of elements in S, there is an equality

$$2\{1+at,1+bt\} = \phi_2(adb-bda) \tag{3.7}$$

in $K_2^M\big(S[[t]]\big)/V_3$. Note that both sides of (3.7) are linear in a and b. Since S is generated additively by invertible elements, we may assume that $a,b\in S^*$.

Moreover, since S is weakly 4-fold stable, there is $c\in S^*$ such that the elements

$$\frac{a+b}{2}+c\,,\qquad a+c\,,\qquad b+c$$

are invertible in S. By bilinearity, it is enough to prove (3.7) for the pairs $(a+c,b+c)$, (a,c), (c,b), and (c,c). Note that each of these pairs satisfies the following condition: its both terms and their sum are invertible. Thus we may assume that $a,b,a+b\in S^*$.

We have the Steinberg relations in $K_2^M\big(S[[t]]\big)$

$$\left\{\frac{b}{a+b}(1+at),\frac{a}{a+b}(1-bt)\right\} = \left\{\frac{b}{a+b},\frac{a}{a+b}\right\} = 0\,.$$

Subtracting the second symbol from the first one, we obtain an equality in $K_2^M\big(S[[t]]\big)$

$$\left\{\frac{b}{a+b}, 1-bt\right\} + \left\{1+at, \frac{a}{a+b}\right\} + \{1+at, 1-bt\} = 0\,. \tag{3.8}$$

Applying the automorphism of the S-algebra $S[[t]]$ that sends a series $f(t)$ to $f(-t)$, we get

$$\left\{\frac{b}{a+b}, 1+bt\right\} + \left\{1-at, \frac{a}{a+b}\right\} + \{1-at, 1+bt\} = 0\,. \tag{3.9}$$

Besides, the equalities

$$\{1+at, 1-bt\} = \{1-at, 1+bt\} = -\{1+at, 1+bt\}$$

hold in the quotient $K_2^M\big(S[[t]]\big)/V_3$. Hence, taking the sum of (3.8) and (3.9), we obtain an equality in $K_2^M\big(S[[t]]\big)/V_3$

$$\left\{\frac{b}{a+b}, 1-b^2t^2\right\} + \left\{1-a^2t^2, \frac{a}{a+b}\right\} - 2\{1+at, 1+bt\} = 0\,.$$

Since the ring $S[[t]]$ is weakly 5-fold stable, applying Lemma 2.6, we get equalities in $K_2^M\big(S[[t]]\big)/V_3$

$$2\{1+at, 1+bt\} = -\left\{1-b^2t^2, \frac{b}{a+b}\right\} + \left\{1-a^2t^2, \frac{a}{a+b}\right\} =$$

$$= -\{1-b^2t^2, b\} + \{1-b^2t^2, a+b\} + \{1-a^2t^2, a\} - \{1-a^2t^2, a+b\} =$$
$$= -\{1-b^2t^2, b\} + \{1-a^2t^2, a\} + \{1+(a^2-b^2)t^2, a+b\}\,.$$

By formula (3.5) from Subsection 3.2, the latter expression is equal to

$$\phi_2\big(bdb - ada + (a-b)d(a+b)\big) = \phi_2(adb - bda)\,,$$

which proves the base of the induction.

Let us make the induction step to arbitrary $i+j$. First suppose that $i=j$. There are embeddings $\lambda_i(V_3) \subset V_{3i} \subset V_{2i+1}$. Hence the homomorphism λ_i induces a homomorphism

$$K_2^M\big(S[[t]]\big)/V_3 \longrightarrow K_2^M\big(S[[t]]\big)/V_{2i+1}\,,$$

which we denote also by λ_i for simplicity.

Let us apply this homomorphism to equality (3.7). Note that the composition

$$\Omega^1_S \xrightarrow{\phi_2} K_2^M\big(S[[t]]\big)/V_3 \xrightarrow{\lambda_i} K_2^M\big(S[[t]]\big)/V_{2i+1}$$

is equal to ϕ_{2i}, because $\lambda_i \circ \lambda_2 = \lambda_{2i}$. Hence we obtain the equality

$$2\{1+at^i, 1+bt^i\} = \phi_{2i}(adb - bda)$$

in $K_2^M\big(S[[t]]\big)/V_{2i+1}$. This gives the statement in the case $i=j$.

Now suppose that $i \neq j$. We can assume that $i<j$ (the other case is done similarly or one can use Lemma 2.6). Since S is weakly 5-fold stable, the same holds for the ring $S' = S[[x]]$. Apply the induction hypothesis to the ring S' in place of S, to the elements $a, bx \in S'$, and to the exponents i, $j-i$. We obtain the equality

$$j\{1+at^i, 1+bxt^{j-i}\} = \phi'_j\big(iad(bx) - (j-i)bxda\big)$$

in $K_2^M\big(S'[[t]]\big)/V'_{j+1}$.

Applying the map

$$\theta_i - \theta_0 \ :\ K_2^M\big(S'[[t]]\big)/V'_{j+1} \longrightarrow K_2^M\big(S[[t]]\big)/V_{i+j+1}$$

from Corollary 3.5, we get the equality

$$j\{1+at^i, 1+bt^j\} = \big((\theta_i-\theta_0)\circ\phi'_j\big)\big(iad(bx)-(j-i)bxda\big) \tag{3.10}$$

in $K_2^M\big(S[[t]]\big)/V_{i+j+1}$.

Let us compare formula (3.10) with the needed formula (3.6). By Lemma 3.6, the right hand side of formula (3.6) is equal to

$$\phi_{i+j}(iadb - jbda) = \big((\theta_i-\theta_0)\circ\phi'_j\big)(iaxdb - jbxda)\,.$$

Besides, there are equalities in Ω^1_S

$$\big(iad(bx)-(j-i)bxda\big)-(iaxdb-jbxda) = iabdx+ibxda = ibd(ax) = ibd(1+ax)\,.$$

Therefore the difference between the right hand side of formula (3.10) and the right hand side of formula (3.6) is equal to

$$\big((\theta_i-\theta_0)\circ\phi'_j\big)\big(ibd(1+ax)\big) = i(\theta_i-\theta_0)\,\{1+b(1+ax)t^j, 1+ax\} =$$

$$= i\{1 + bt^j + abt^{i+j}, 1 + at^i\} = i\{1 + bt^j, 1 + at^i\} = -i\{1 + at^i, 1 + bt^j\}\,,$$

where all equalities are in $K_2^M\big(S[[t]]\big)/V_{i+j+1}$, we use formula (3.5) from Subsection 3.2 for the first equality and we use Lemma 2.6 applied to the weakly 5-stable ring $S[[t]]$ for the last equality. The latter expression is equal to the difference between the left hand side of formula (3.10) and the left hand side of the needed formula (3.6). This proves the proposition. □

The following lemma is needed in the proof of an important corollary of Proposition 3.7 below. Recall that we are assuming that $2 \in S$ is invertible.

Lemma 3.8. *For all natural $i, j \geqslant 1$, the group $K_2^M(S[[t]], (t))/V_{i+j+1}$ is uniquely 2-divisible.*

Proof. Given that the group $K_2^M(S[[t]], (t))$ coincides with V_1 there is a finite decreasing filtration of $K_2^M(S[[t]], (t))/V_{i+j+1}$ by its subgroups V_k/V_{i+j+1} for $1 \leqslant k \leqslant i + j + 1$. Consider an automorphism of the ring $S[[t]]$ that sends t to $2t$. It is easy to see that for any $k \geqslant 1$, this automorphism preserves the subgroups V_k in $K_2^M(S[[t]])$. Therefore, it induces an automorphism on the adjoint quotients V_k/V_{k+1}. Moreover, on the quotient V_k/V_{k+1} this automorphism coincides with multiplication by a power of 2. Thus, multiplication by 2 is a bijection on the adjoints quotients of the filtration on $K_2^M(S[[t]], (t))/V_{i+j+1}$ and therefore it is a bijection on the group $K_2^M(S[[t]], (t))/V_{i+j+1}$ itself. □

Corollary 3.9. *For all elements $a, b \in S$ and natural numbers $i, j \geqslant 1$, there is an equality*

$$\{1 + at^i, 1 + bt^j\} - \{1 + t^i, 1 + abt^j\} = -\phi_{i+j}(bda) \tag{3.11}$$

in $K_2^M(S[[t]])/V_{i+j+1}$.

Proof. The proof is similar to that of Proposition 3.7. We use induction on $i + j$. The base of the induction is $i = j = 1$. By Proposition 3.7, for all elements $a, b \in S$, there are equalities

$$2\big(\{1 + a\bar{t}, 1 + b\bar{t}\} - \{1 + \bar{t}, 1 + ab\bar{t}\}\big) = \phi_2(adb - bda - d(ba)) = -2\phi_2(bda)$$

in $K_2^M(S[[t]])/V_3$. It follows from formula (3.5) that the element $\phi_2(bda)$ belongs to the subgroup $K_2^M\big(S[[t]], (t)\big)/V_3 \subset K_2^M(S[[t]])/V_3$. By Lemma 3.8, the group $K_2^M(S[[t]], (t))/V_3$ is uniquely 2-divisible. This proves the base of the induction.

Now let us make the induction step. If $i = j$, then the needed statement follows from the equality $\lambda_i \circ \phi_2 = \lambda_{2i}$ and the base of the induction. Now suppose that $i \neq j$. We can can assume that $i < j$. Applying the induction hypothesis to the elements $a, bx \in S'$ and numbers $i, j-i$, we obtain the equality

$$\{1+at^i, 1+bxt^{j-i}\} - \{1+t^i, 1+abxt^{j-i}\} = -\phi'_j(bxda) \in K_2^M(S'[[t]])/V'_{j+1}\,.$$

Applying homomorphism $\theta_i - \theta_0 : K_2^M(S'[[t]])/V'_{j+1} \to K_2^M(S[[t]])/V_{i+j+1}$, we obtain the equality

$$\{1+at^i, 1+bt^j\} - \{1+t^i, 1+abt^j\} = \big(\theta_i - \theta_0\big)(-\phi'_j(bxda)) \qquad (3.12)$$

in $K_2^M(S[[t]])/V_{i+j+1}$. Finally, by Lemma 3.6, the right hand side in (3.12) is equal to $-\phi_{i+j}(bda)$ in $K_2^M(S[[t]])/V_{i+j+1}$. □

3.4 Proof of Proposition 3.2

First we deduce a useful corollary from Propostion 3.7.

Corollary 3.10. *For any $k \geqslant 0$, there is an embedding*

$$k \cdot V_k \subset W_k + V_{k+1}$$

of subgroups in $K_2^M\big(S[[t]]\big)$.

Proof. The case $k = 0$ is trivial. For $k \geqslant 1$, note that the group V_k is generated by symbols of type $\{1+at^k, b\}$, $\{b, 1+at^k\}$, where $a \in S$, $b \in S^*$, by symbols of type $\{1+at^i, 1+bt^j\}$, where $i, j \geqslant 1$, $i+j = k$, $a, b \in S$, and by elements of the subgroup V_{k+1}.

Clearly, symbols of the first type belong to W_k. By Proposition 3.7, symbols of the second type satisfy the condition

$$k\{1+at^i, 1+bt^j\} \in \phi_k(iadb - jbda) + V_{k+1}\,.$$

Furthermore, we claim that $\phi_k(iadb - jbda) \in W_k$. Indeed, since S is generated additively by invertible elements, we may assume that $a, b \in S^*$ and use formula (3.5) from Subsection 3.2. Hence, symbols of the second type multiplied by k belong to $W_k + V_{k+1}$. □

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. Let us prove that for any n with $0 \leqslant n \leqslant k-2$, there is an embedding

$$k \cdot V_{k+n} \subset W_k + V_{k+n+1}\,. \tag{3.13}$$

of subgroups in $K_2^M\big(S[[t]]\big)$.

Since S' is weakly 5-fold stable, by Corollary 3.10 applied with S' in place of S, we have

$$k \cdot V'_k \subset W'_k + V'_{k+1}\,. \tag{3.14}$$

Since $\theta_n(t^k) = t^k$ and $\theta_0(t^k) = t^k$, we see that $\theta_n(W'_k) \subset W_k$ and $\theta_0(W'_k) \subset W_k$. In particular, we have

$$(\theta_n - \theta_0)(W'_k) \subset W_k\,.$$

By Lemma 3.4, there are embeddings

$$(\theta_n - \theta_0)(V'_k) \subset V_{k+n}\,, \qquad (\theta_q - \theta_0)(V'_{k+1}) \subset V_{k+n+1}\,.$$

Moreover, we claim that there is an equality

$$(\theta_n - \theta_0)(V'_k) = V_{k+n}\,. \tag{3.15}$$

Indeed, the group V_{k+n} is generated by symbols of type $\{1+at^i, 1+bt^j\}$, where $i,j \geqslant 0$, $i+j \geqslant k+n$, and $a,b \in S$. Since $n<k$, either i or j is greater than n. We may assume that $i>n$. Then we have

$$(\theta_n - \theta_0)\,\{1+axt^{i-n}, 1+bt^j\} = \{1+at^i, 1+bt^j\}\,,$$

where $\{1+axt^{i-n}, 1+bt^j\} \in V'_k$. This proves equality (3.15).

Now, applying the map $\theta_n - \theta_0$ to formula (3.14), we obtain formula (3.13).

Further, using formula (3.13) repeatedly, we get the embeddings

$$k^{k-1}\cdot V_k \subset k^{k-2}\cdot(W_k+V_{k+1}) \subset W_k+k^{k-2}\cdot V_{k+1} \subset \ldots \subset W_k+k\cdot V_{2k-2} \subset W_k+V_{2k-1}\,.$$

Since $V_{2k-1} \subset W_k$, this proves that there is an emebedding

$$k^{k-1} \cdot V_k \subset W_k\,. \tag{3.16}$$

In particular, we have

$$k^{k-1} \cdot V_{k+1} \subset W_k\,,$$

because, trivially, $V_{k+1} \subset V_k$.

On the other hand, replacing k by $k+1$ in formula (3.16), we obtain embeddings

$$(k+1)^k \cdot V_{k+1} \subset W_{k+1} \subset W_k \,.$$

Since the numbers k^{k-1} and $(k+1)^k$ are coprime, we get the first required embedding $V_{k+1} \subset W_k$. Combining this with Corollary 3.10, we obtain the second required embedding $k \cdot V_k \subset W_k$. □

Note that in the proof of Proposition 3.2 we use only Corollary 3.10 and not Proposition 3.7 itself. However, it is not clear to us whether it is possible to prove Corollary 3.10 directly by induction. This is why we have replaced it with a more precise and stronger statement, namely, with Proposition 3.7, which admits an inductive proof.

4 Bloch map

4.1 Bloch map for a split nilpotent extension

Let $I \subset R$ be a nilpotent ideal. According to Example 2.10 there is a homomorphism

$$d\log \;:\; K^M_{n+1}(R,I) \longrightarrow (\Omega^{n+1}_{R,I})^{cl} \,.$$

Let $N \geqslant 1$ be a natural number such that $I^N = 0$. Until the end of this subsection, we assume that $(N-1)!$ is invertible in R.

For an element $x \in I$, put

$$\log(1+x) := x - \frac{x^2}{2} + \ldots + (-1)^N \frac{x^{N-1}}{N-1} \in I \,.$$

Note that there are equalities

$$d\big(\log(1+x)\big) = \frac{d(1+x)}{1+x} = (d\log)(1+x) \,.$$

Lemma 4.1. *The image of the map* $d\log \colon K^M_{n+1}(R,I) \to (\Omega^{n+1}_{R,I})^{cl}$ *is contained in the relative subgroup of exact differential forms* $d\,\Omega^n_{R,I} \subset (\Omega^{n+1}_{R,I})^{cl}$, *that is, we have a map*

$$d\log \;:\; K^M_{n+1}(R,I) \longrightarrow d\,\Omega^n_{R,I} \,.$$

Proof. For any $0 \leqslant i \leqslant n$, $r_1, \ldots, r_n \in R^*$, and $x \in I$, there is an equality

$$d\log\{r_1, \ldots, r_i, 1+x, r_{i+1}, \ldots, r_n\} = d\left((-1)^i \log(1+x)\frac{dr_1}{r_1} \wedge \ldots \wedge \frac{dr_n}{r_n}\right). \tag{4.1}$$

Thus, applying Lemma 2.2 with n being replaced by $n+1$ finishes the proof. □

Our main object of study is the following map, which was introduced originally by Bloch [4, § 1] (previous versions of this map were constructed by van der Kallen [18] and Bloch [3]).

Definition 4.2. A homomorphism of groups

$$\mathrm{B} \ : \ K^M_{n+1}(R, I) \longrightarrow \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$$

is called a *Bloch map* if for all i, $0 \leqslant i \leqslant n$, $r_1, \ldots, r_n \in R^*$, and $x \in I$, we have (cf. formula (4.1))

$$\mathrm{B}\,\{r_1, \ldots, r_i, 1+x, r_{i+1}, \ldots, r_n\} = (-1)^i \log(1+x)\frac{dr_1}{r_1} \wedge \ldots \wedge \frac{dr_n}{r_n}, \tag{4.2}$$

where, for simplicity, we denote similarly elements in $\Omega^n_{R,I}$ and their images under the quotient map $\Omega^n_{R,I} \to \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$.

Sometimes we denote the Bloch map as in Definition 4.2 by $\mathrm{B}_{R,I}$ to specify the ring and the ideal. One can consider the Bloch map as an integral of the map $d\log$ from Lemma 4.1.

Let us discuss some general properties of the Bloch map. By Lemma 2.2, a Bloch map is unique whenever it exists. For $n = 0$, the Bloch map always exists and coincides with the isomorphism $\log\colon 1 + I \xrightarrow{\sim} I$.

Lemma 4.3. *Suppose that any element in R is a sum of invertible elements and that the Bloch map* $\mathrm{B}\colon K^M_{n+1}(R, I) \to \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ *exists. Then the Bloch map is surjective.*

Proof. For all $r_1, \ldots, r_n \in R$ and $x \in I$, there is an equality in $\Omega^n_{R,I}$

$$r_1\,dx \wedge dr_2 \wedge \ldots \wedge dr_n = d(r_1x\,dr_2 \wedge \ldots \wedge dr_n) - x\,dr_1 \wedge dr_2 \wedge \ldots \wedge dr_n\,.$$

Hence by Lemma 2.8, it is enough to show that classes in $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ of differential forms of type $x\,dr_1 \wedge \ldots \wedge dr_n$ are in the image of the Bloch map.

Since R is generated additively by invertible elements, it is enough to consider the case when $r_1, \ldots, r_n \in R^*$. Then there is an equality

$$\mathrm{B}\,\{\exp(xr_1 \ldots r_n), r_1, \ldots, r_n\} = x\,dr_1 \wedge \ldots \wedge dr_n\,,$$

where for an element $y \in I$, we put

$$\exp(y) := 1 + y + \frac{y}{2} + \ldots + \frac{y^{N-1}}{(N-1)!} \in R\,.$$

This proves the lemma. $\square$

The following simple observation claims the existence of the Bloch map in some special cases.

Remark 4.4. The Bloch map $\mathrm{B}\colon K^M_{n+1}(R,I) \to \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ exists when there is a vanishing $H^n_{dR}(R,I) = 0$. The reason is that the latter condition is equivalent to requiring that the de Rham differential gives an isomorphism

$$d\;:\;\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} \xrightarrow{\sim} d\,\Omega^n_{R,I}\,.$$

The Bloch map is equal to the composition of the map $d\log$ from Lemma 4.1 and the inverse of this isomorphism.

Consider an example, which was treated also in [9].

Example 4.5. Let S be a ring such that 2 is invertible in it. Let ε be a formal variable such that $\varepsilon^2 = 0$. Then the ring $R = S[\varepsilon]$ and the ideal $I = (\varepsilon)$ satisfy $H^n_{dR}(R,I) = 0$ for any $n \geqslant 0$ (see Lemma 4.10 and Proposition 4.11 below for generalizations of this fact). Indeed, using the equalities $\varepsilon\,d\varepsilon = \frac{1}{2}d(\varepsilon^2) = 0$, one shows that there is a decomposition

$$\Omega^{n+1}_{S[\varepsilon],(\varepsilon)} \simeq (\varepsilon\,\Omega^{n+1}_S) \oplus (d\varepsilon \wedge \Omega^n_S)\,.$$

This implies that there are isomorphisms

$$\Omega^n_S \xrightarrow{\sim} \Omega^n_{S[\varepsilon],(\varepsilon)}/d\,\Omega^{n-1}_{S[\varepsilon],(\varepsilon)}\,, \qquad \omega \longmapsto \varepsilon\,\omega\,, \tag{4.3}$$

$$\Omega^n_S \xrightarrow{\sim} d\,\Omega^n_{S[\varepsilon],(\varepsilon)}\,, \qquad \omega \longmapsto d(\varepsilon\,\omega) = \varepsilon\,d\omega + d\varepsilon \wedge \omega\,.$$

Therefore, there is an isomorphism

$$d\;:\;\Omega^n_{S[\varepsilon],(\varepsilon)}/d\,\Omega^{n-1}_{S[\varepsilon],(\varepsilon)} \xrightarrow{\sim} d\,\Omega^n_{S[\varepsilon],(\varepsilon)}\,.$$

Thus by Remark 4.4, the Bloch map exists in this case.

Moreover, one checks directly that the Bloch map coincides with composition (2.4) from Theorem 2.12. In particular, the Bloch map is an isomorphism and

for $a \in S$ and $b_1, \ldots, b_n \in S^*$, we have

$$\mathrm{B}\,\{1 + ab_1 \ldots b_n\varepsilon, b_1, \ldots, b_n\} = \varepsilon\, adb_1 \wedge \ldots \wedge db_n \in \Omega^n_{S[\varepsilon],(\varepsilon)}/d\,\Omega^{n-1}_{S[\varepsilon],(\varepsilon)}\,.$$

The following statement asserts the existence of the Bloch map in the case when the quotient R/I splits out of R.

Theorem 4.6. *Let $I \subset R$ be a nilpotent ideal and $N \geqslant 1$ be a natural number such that $I^N = 0$. Suppose that the quotient map $R \to R/I$ admits a splitting by a ring homomorphism $R/I \to R$ and that $N!$ is invertible in R. Then for any natural number $n \geqslant 0$, the Bloch map*

$$\mathrm{B}\ :\ K^M_{n+1}(R, I) \longrightarrow \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$$

exists.

Remark 4.7. We show in Proposition 5.7 below that the Bloch map need not exist if we drop the assumption that R/I splits out of R. So, Theorem 4.6 fails in a non-split case.

We will prove theorem 4.6 later in this section.

4.2 Category of split nilpotent extensions

Fix a ring S. Let $N \geqslant 1$ be a natural number, which will be the nilpotency degree.

We will work with the following objects.

Definition 4.8.

(i) A *split nilpotent extension of S of nilpotency degree N* is a pair (R, I), where R is an S-algebra and $I \subset R$ is a nilpotent ideal such that $I^N = 0$ and the summation map $S \oplus I \to R$ is an isomorphism of S-modules.

(ii) A *morphism of split nilpotent extensions* $(R, I) \to (R', I')$ is a morphism of S-algebras $f\colon R \to R'$ such that $f(I) \subset I'$.

(iii) By

$$\mathrm{SNil}_N(S)$$

denote the category of split nilpotent extensions of S of nilpotency degree N.

In particular, a S-algebra R as in Definition 4.8(i) is augmented, that is, it is fixed a (surjective) homomorphism of S-algebras $R \to S$ whose kernel is I. Note that giving a split nilpotent extension of S of nilpotency degree N is the same as giving a S-module I with a commutative associative product map $I \otimes_S I \to I$ of nilpotency degree N. Indeed, such S-module I corresponds to the split nilpotent extension $(S \oplus I, I)$ of S.

A morphism of split nilpotent extensions is the same as a homomorphism of augmented S-algebras.

We will use the following notation: if a ring R is a quotient of the ring $S[t_1, \ldots, t_m]$ of polynomials in formal variables $t_1, \ldots, t_m$, $m \geqslant 1$, we denote the image of t_i in R by $\bar{t}_i$ for each i, $1 \leqslant i \leqslant m$.

Definition 4.9. An object in $\mathrm{SNil}_N(S)$ is *finite free* if it is isomorphic to

$$(R_{N,m}, I_{N,m}) := \left(S[t_1, \ldots, t_m]/(t_1, \ldots, t_m)^N, (\bar{t}_1, \ldots, \bar{t}_m)\right)$$

for some natural number $m \geqslant 0$.

Split nilpotent extensions as in Definition 4.9 are indeed finite free objects in $\mathrm{SNil}_N(S)$ in the following sense: they are values on finite sets of the left adjoint functor to the functor from $\mathrm{SNil}_N(S)$ to the category of sets that sends (R, I) to the ideal I considered as a set.

4.3 Vanishing of relative de Rham cohomology

Proposition 4.11 below claims the vanishing of relative de Rham cohomology (see Subsection 2.3) for finite free split nilpotent extensions under an additional invertibility condition. The proof of this fact is based on the following lemma, whose main argument is the action of the Euler vector field on differential forms by a Lie derivative.

Lemma 4.10. *Let $J \subset S[t]$ be an ideal such that $t^N \in J$ and J is generated by monomials in t (which may be of degree zero, that is, be elements of S). Let $R = S[t]/J$ and $I = (\bar{t}\,) \subset R$. Suppose that $N!$ is invertible in S. Then for all $n \geqslant 0$, we have $H^n_{dR}(R, I) = 0$.*

Proof. Consider a standard graded ring structure on $S[t]$ such that elements of S have degree zero and t has degree one. The de Rham complex $\Omega^\bullet_{S[t]}$ becomes naturally a graded complex. We call the degree that corresponds to this grading an internal degree in order to distinguish it from the degree of differential forms. Explicitly, for all $n, i \geqslant 0$, the homogenous component $\left(\Omega^n_{S[t]}\right)_i$ in

$$\Omega^n_{S[t]} \simeq \Omega^n_S[t] \oplus \Omega^{n-1}_S[t]dt$$

of internal degree i consists of differential forms of type

$$\omega \cdot t^i + t^{i-1}\eta \wedge dt\,, \tag{4.4}$$

where $\omega \in \Omega^n_S$ and $\eta \in \Omega^{n-1}_S$. In particular, we have $\left(\Omega^n_{S[t]}\right)_0 = \Omega^n_S$ for $n \geqslant 0$.

For all $n, i \geqslant 0$, define a S-linear map

$$h \;:\; \left(\Omega^n_{S[t]}\right)_i \longrightarrow \left(\Omega^{n-1}_{S[t]}\right)_i\,, \qquad h(\omega \cdot t^i + t^{i-1}\eta \wedge dt) = (-1)^{n-1}\eta \cdot t^i\,.$$

By definition, h vanishes on $\Omega^0_{S[t]}$ and on $\left(\Omega^n_{S[t]}\right)_0$, where $n \geqslant 1$. One checks directly that the restriction of the map $d \circ h + h \circ d$ to the homogenous component $\left(\Omega^n_{S[t]}\right)_i$ coincides with multiplication by i.

Note that an ideal in $S[t]$ is generated by monomials if and only if it is homogenous. In particular, the ideal J is homogenous. Hence, we obtain a graded ring structure on R, which induces a graded complex structure on the de Rham complex $\Omega^\bullet_R$. The quotient map $\Omega^\bullet_{S[t]} \to \Omega^\bullet_R$ preserves the internal degree and $\left(\Omega^n_R\right)_i$ is the quotient of the S-module $\left(\Omega^n_{S[t]}\right)_i$ for all $n, i \geqslant 0$. We claim that there is an equality

$$\Omega^\bullet_{R,I} = \bigoplus_{i=1}^{N-1} \left(\Omega^\bullet_R\right)_i\,. \tag{4.5}$$

Indeed, formula (4.4) implies that $\left(\Omega^\bullet_R\right)_i \subset \Omega^\bullet_{R,I}$ for any $i \geqslant 1$. In addition, the quotient R/I splits out of R as the homogenous component of degree zero and we have an isomorphism $\left(\Omega^\bullet_R\right)_0 \simeq \Omega^\bullet_{R/I}$. Therefore, $\Omega^\bullet_{R,I} = \bigoplus_{i\geqslant 1} \left(\Omega^\bullet_R\right)_i$. On the other hand, there is an equality

$$0 = d(\bar{t}^N) = N\bar{t}^{N-1}d\bar{t} \tag{4.6}$$

in Ω^1_R. Since N is invertible in S and in R, this implies that $\left(\Omega^\bullet_R\right)_i = 0$ for all $i \geqslant N$.

Furthermore, for each $i \geqslant 0$, the map $h\colon \big(\Omega^n_{S[t]}\big)_i \to \big(\Omega^{n-1}_{S[t]}\big)_i$ descends to a map $\bar{h}\colon \big(\Omega^n_R\big)_i \to \big(\Omega^{n-1}_R\big)_i$. One way to show this is just to check it directly using the description of the kernel of the quotient map $\big(\Omega^n_{S[t]}\big)_i \to \big(\Omega^n_R\big)_i$ given by Lemma 2.8. A more conceptual way is to observe that $\bar{h}$ is the contraction with the derivation ∂ on R that acts as multiplication by i on the homogenous component of degree i in R (sometimes ∂ is called an Euler vector field).

Clearly, the equality $d\circ h + h\circ d = i$ implies the equality $d\circ \bar{h} + \bar{h}\circ d = i$. Alternatively, $d\circ \bar{h} + \bar{h}\circ d$ is the Lie derivative defined by the derivation ∂, which is known to act as multiplication by i on differential forms of internal degree i in Ω^n_R.

We see that for each $i \geqslant 0$, multiplication by i on the complex $\big(\Omega^\bullet_R\big)_i$ is homotopically trivial. Using that $(N-1)!$ is invertible in S and formula (4.5), we conclude that the complex $\Omega^\bullet_{R,I}$ is homotopically trivial, whence its cohomology groups vanish. □

Proposition 4.11. *Suppose that $N!$ is invertible in S. Then for all $m, n \geqslant 0$, we have $H^n_{dR}(R_{N,m}, I_{N,m}) = 0$.*

Proof. The proof is by induction on m. The case $m = 0$ is trivial as the corresponding relative module of differential forms just vanishes.

Let us make an induction step from $m-1$ to m. We have natural isomorphisms

$$R_{N,m-1} \simeq R_{N,m}/(\bar{t}_m)\,, \qquad I_{N,m-1} \simeq I_{N,m}/(\bar{t}_m)\,.$$

Hence the embedding of ideals $(\bar{t}_m) \subset I_{N,m}$ in the ring $R_{N,m}$ gives an exact sequence of relative de Rham complexes

$$0 \longrightarrow \Omega^\bullet_{R_{N,m},(\bar{t}_m)} \longrightarrow \Omega^\bullet_{R_{N,m},I_{N,m}} \longrightarrow \Omega^\bullet_{R_{N,m-1},I_{N,m-1}} \longrightarrow 0\,. \tag{4.7}$$

By the induction hypothesis, we have $H^n_{dR}(R_{N,m-1}, I_{N,m-1}) = 0$ for all $n \geqslant 0$.

Put $S' = S[t_1,\ldots,t_{m-1}]$ and

$$J = (t_1,\ldots,t_m)^N \subset S[t_1,\ldots,t_m] = S'[t_m]\,.$$

Clearly, we have $t_m^N \in J$. Since the ideal J is generated by monomials in $t_1,\ldots,t_m$ with coefficients in S, we see that, in particular, J is generated by monomials in t_m with coefficients in S'. Thus by Lemma 4.10 applied to $J \subset S'[t_m]$, we see that $H^n_{dR}\big(R_{N,m}, (\bar{t}_m)\big) = 0$ for all $n \geqslant 0$.

Using the long exact sequence of cohomology associated with the exact sequence (4.7), we complete the proof. □

As the second part of the following remark claims, given an arbitrary graded split nilpotent extension of S of nilpotency degree N, its relative de Rham cohomology vanish if one requires that more natural numbers than just $N!$ are invertible in S.

Remark 4.12.

(i) Using the same argument as in the proof of Proposition 4.11, one shows the following generalization of both Lemma 4.10 and Proposition 4.11. Let $J \subset S[t_1, \ldots, t_m]$ be an ideal such that $t_1^N, \ldots, t_m^N \in J$ and J is generated by monomials in $t_1, \ldots, t_m$. Put

$$R = S[t_1, \ldots, t_m]/J\,, \qquad I = (\bar{t}_1, \ldots, \bar{t}_m) \subset R\,.$$

Suppose that $N!$ is invertible in S. Then $H^n_{dR}(R, I) = 0$ for all $n \geqslant 0$.

(ii) Let R be a graded S-algebra such that $R_0 = S$ and the ideal $I = \bigoplus_{i \geqslant 1} R_i$ satisfies the condition $I^N = 0$. For simplicity, assume that R is generated as an S-algebra by m homogenous elements of degree one. Then the relative de Rham complex $\Omega^\bullet_{R,I}$ may have non-zero homogenous components of internal degree up to $N + m - 2$ (the component of degree $N+m-1$ vanishes by equalities analogous to (4.6)). For example, the relative de Rham complex $\Omega^\bullet_{R_{N,m}, I_{N,m}}$ has a non-zero component of internal degree $N + m - 2$. Thus, applying the same argument with the action of the Euler field on differential forms by the Lie derivative as in the proof of Lemma 4.10, one shows the vanishing of relative de Rham cohomology under the condition that $(N + m - 2)!$ is invertible in S.

Note that relative de Rham cohomology can be non-trivial for an arbitrary split nilpotent extension of S, even when S is a $\mathbb{Q}$-algebra. Here is a way to construct such examples.

Let $f \in \mathbb{Q}[t_1, \ldots, t_m]$ be a polynomial such that f defines an isolated singularity at the origin or, equivalently, such that the Milnor ring

$$R = \mathbb{Q}[[t_1, \ldots, t_m]]/(\partial_{t_1} f, \ldots, \partial_{t_m} f)$$

is finite-dimensional over $\mathbb{Q}$. The *Milnor number* of f (at the origin) is defined as

$$\mu(f) := \dim_{\mathbb{Q}}(R)\,.$$

Equivalently, we have $\mu(f) = \dim_{\mathbb{Q}} \left(\Omega^m / df \wedge \Omega^{m-1}\right)$, where, for short, we put

$$\Omega^1 := \mathbb{Q}[[t_1, \dots, t_m]]dt_1 \oplus \dots \oplus \mathbb{Q}[[t_1, \dots, t_m]]dt_m \,,$$

$$\Omega^i := \textstyle\bigwedge^i_{\mathbb{Q}[[t_1,\dots,t_m]]} \Omega^1 \,, \qquad i \geqslant 0 \,.$$

Recall that the *Tyurina number* (see [30]) of f is defined as

$$\tau(f) := \dim_{\mathbb{Q}} \left(R/(\bar{f})\right) ,$$

where $\bar{f}$ is the image in R of f. Equivalently, we have $\tau(f) = \dim_{\mathbb{Q}} \left(\Omega^m / (f \cdot \Omega^m + df \wedge \Omega^{m-1})\right)$. Clearly, $\mu(f) \geqslant \tau(f)$.

Suppose that there is a strict inequality

$$\mu(f) > \tau(f) \,, \tag{4.8}$$

or, equivalently, that $\bar{f} \neq 0$. Then $\bar{f}$ gives a non-zero class in $H^0_{dR}(R, I)$, where $I = (\bar{t}_1, \dots, \bar{t}_m) \subset R$. For example, Grauert and Kerner [12, § 1.3] showed explicitly that this holds for $f = t_1^4 + t_1^2 t_2^3 + t_2^5$. A criterion for f to satisfy inequality (4.8) was obtained by Saito [33].

Actually, inequality (4.8) implies that another split nilpotent extension of $\mathbb{Q}$ has non-trivial relative de Rham cohomology in higher degree. Namely, put

$$R' = \mathbb{Q}[[t_1, \dots, t_m]] / \left(f, (t_1, \dots, t_m)^N\right)$$

for a sufficiently large natural number N. There is an equality (cf. Lemma 2.8)

$$\tau(f) = \dim_{\mathbb{Q}} (\Omega^m_{R'}) \,.$$

It was proved independently by Palamodov [29] and Milnor [25, Theor. 7.2] that $\mu(f)$ equals the dimension of the space of vanishing cycles associated with f. Malgrange [23, Théor. 5.1, Théor. 3.7] (see also a nice survey by Mond [26]) gave another proof of this fact and also proved the equality

$$\mu(f) = \dim_{\mathbb{Q}} \left(\Omega^{m-1}_{R'} / d\, \Omega^{m-2}_{R'}\right) .$$

This was generalized later to complete intersections with isolated singularities by Tráng [31] and Greuel [32, Prop. 5.1].

Since the de Rham differential $d \colon \Omega^{m-1} \to \Omega^m$ is surjective, the de Rham differential $d \colon \Omega^{m-1}_{R'} \to \Omega^m_{R'}$ is surjective as well. Hence, we have an exact sequence

$$0 \longrightarrow H^{m-1}_{dR}(R') \longrightarrow \Omega^{m-1}_{R'} / d\, \Omega^{m-2}_{R'} \stackrel{d}{\longrightarrow} \Omega^m_{R'} \longrightarrow 0 \,.$$

Thus, the vanishing $H^i_{dR}(\mathbb{Q}) = 0$, $i > 0$, yields the equality

$$\mu(f) - \tau(f) = \dim_{\mathbb{Q}} H^{m-1}_{dR}(R', I')\,,$$

where $I' = (\bar{t}_1, \ldots, \bar{t}_m) \subset R'$. Therefore inequality (4.8) implies that there is a non-zero class in $H^{m-1}_{dR}(R', I')$. In particular, such examples were found by Reiffen [15, Satz 4, Satz 5] explicitly for $m = 2, 3$ and by Arapura and Kang [1, Exam. 4.4] for $m = 2$ with the help of the Maple subroutines of Rossi and Teraccini [16] to calculate Milnor and Tyurina numbers.

4.4 Finitely freely approximable functors

By $\mathcal{A}b$ denote the category of abelian groups.

Definition 4.13. A functor $F\colon \mathrm{SNil}_N(S) \to \mathcal{A}b$ is *finitely freely approximable* if for any object (R, I) in $\mathrm{SNil}_N(S)$, the natural homomorphism of groups

$$\operatorname*{colim}_{(R',I')\to(R,I)} F(R', I') \longrightarrow F(R, I)$$

is an isomorphism, where the colimit is taken with respect to the category of finite free objects in $\mathrm{SNil}_N(S)$ over (R, I).

Recall that the category of finite free objects in $\mathrm{SNil}_N(S)$ over (R, I) is defined as follows. Objects are arrows $f\colon (R', I') \to (R, I)$ in $\mathrm{SNil}_N(S)$, where (R', I') is a finite free object in $\mathrm{SNil}_N(S)$. Morphisms from $f_1\colon (R'_1, I'_1) \to (R, I)$ to $f_2\colon (R'_2, I'_2) \to (R, I)$ are arrows $\varphi\colon (R'_1, I'_1) \to (R'_2, I'_2)$ in $\mathrm{SNil}_N(S)$ such that $f_1 = f_2 \circ \varphi$.

Note that, in general, the colimit in Definition 4.13 is not filtered. Finitely freely approximable functors are useful because of the following obvious observation.

Lemma 4.14. *Let $\rho\colon F \to G$ be a morphism of functors from $\mathrm{SNil}_N(S)$ to $\mathcal{A}b$. Suppose that F and G are finitely freely approximable and that for any $m \geqslant 0$, the corresponding homomorphism $F(R_{N,m}, I_{N,m}) \to G(R_{N,m}, I_{N,m})$ is an isomorphism. Then the morphism ρ is an isomorphism as well.*

Proof. Indeed, an isomorphism between diagrams induces an isomorphism between their colimits. □

Given a functor $F\colon \mathrm{SNil}_N(S) \to \mathcal{A}b$, define the functor F^{fa} by the formula

$$F^{fa} \;:\; \mathrm{SNil}_N(S) \to \mathcal{A}b\,, \qquad (R,I) \longmapsto \operatorname*{colim}_{(R',I')\to(R,I)} F(R',I')\,,$$

where the colimit is as in Definition 4.13. One checks easily that the functor F^{fa} is finitely freely approximable and that the values of F^{fa} and F coincide on finite free objects in $\mathrm{SNil}_N(S)$. The natural morphism $F^{fa} \to F$ is an isomorphism if and only if F is finitely freely approximable.

Given a morphism of functors $\rho\colon F \to G$, we have a morphism between finitely freely approximable functors $\rho^{fa}\colon F^{fa} \to G^{fa}$. Explicitly, ρ^{fa} is the colimit of ρ over finite free objects in $\mathrm{SNil}_N(S)$. The assignment of F^{fa} to F is the right adjoint functor to the forgetful functor from the category of finitely freely approximable functors to the category of all functors from $\mathrm{SNil}_N(S)$ to $\mathcal{A}b$.

The following results allow to construct finitely freely approximable functors.

Lemma 4.15. *Let $\rho\colon F \to G$ be a morphism of finitely freely approximable functors from $\mathrm{SNil}_N(S)$ to $\mathcal{A}b$. Then the cokernel $\mathrm{Coker}(\rho)$ is a finitely freely approximable functor as well.*

Proof. Indeed, taking (not necessarily filtered) colimits of abelian groups is right exact. □

Proposition 4.16. *Let H be a functor from the category of rings to $\mathcal{A}b$. Given an ideal I in a ring R, we put*

$$F_H(R,I) := \mathrm{Ker}\big(H(R) \to H(R/I)\big)\,.$$

Suppose that the functor H satisfies the following conditions:

(i) for any ring R with a nilpotent ideal $I \subset R$, the corresponding homomorphism $H(R) \to H(R/I)$ is surjective;

(ii) for any ring R with a nilpotent ideal $I \subset R$, the group $F_H(R,I)$ is generated by the images of group homomorphisms

$$F_H\big(R[[t]],(t)\big) \longrightarrow F_H(R,I)$$

induced by homomorphisms of R-algebras $R[[t]] \to R$ that send t to an element in I;

(iii) for any ring R, the natural homomorphism $\operatorname{colim}_{A \subset R} H(A) \longrightarrow H(R)$ *is an isomorphism, where the colimit is taken over finitely generated subrings* $A \subset R$.

Then the functor

$$F_H \,:\, \mathrm{SNil}_N(S) \longrightarrow \mathcal{A}b\,, \qquad (R,I) \longmapsto F_H(R,I)\,,$$

is finitely freely approximable.

Proof. Let (R,I) in $\mathrm{SNil}_N(S)$ be such that R is a finitely generated S-algebra. Put

$$\Gamma = \operatorname*{colim}_{(R',I') \twoheadrightarrow (R,I)} H(R')\,, \tag{4.9}$$

where the colimit is taken with respect to the following category $\mathcal{C}(R,I)$. Objects in $\mathcal{C}(R,I)$ are arrows $f\colon (R',I') \twoheadrightarrow (R,I)$ in $\mathrm{SNil}_N(S)$ such that (R',I') is a finite free object in $\mathrm{SNil}_N(S)$ and f is surjective. Morphisms in $\mathcal{C}(R,I)$ from $f_1\colon (R'_1,I'_1) \twoheadrightarrow (R,I)$ to $f_2\colon (R'_2,I'_2) \twoheadrightarrow (R,I)$ are arrows $\varphi\colon (R'_1,I'_1) \to (R'_2,I'_2)$ in $\mathrm{SNil}_N(S)$ such that $f_1 = f_2 \circ \varphi$. Let us prove that the natural homomorphism of groups

$$\xi \,:\, \Gamma \to H(R)$$

is an isomorphism.

Let $f\colon (R',I') \twoheadrightarrow (R,I)$ be an object in $\mathcal{C}(R,I)$. Put $J = \mathrm{Ker}(f) \subset R'$. The induced map $f\colon H(R') \to H(R)$ is equal to the composition

$$H(R') \longrightarrow \Gamma \stackrel{\xi}{\longrightarrow} H(R)\,, \tag{4.10}$$

where the first arrow is the natural morphism to the colimit. By definition, the homomorphism of rings $f\colon R' \twoheadrightarrow R$ is surjective. Since f is a homomorphism of augmented S-algebras, the ideal J is contained in the augmentation ideal I', whence J is nilpotent. Consequently, by condition (i) applied to $J \subset R'$, the map $f\colon H(R') \to H(R)$ is surjective. Taking into account composition (4.10), we obtain that ξ is surjective as well.

Now we prove injectivity of ξ. One shows directly that Γ coincides with the union of the images $\mathrm{Im}\big(H(R') \to \Gamma\big)$ over all objects $f\colon (R',I') \twoheadrightarrow (R,I)$ in $\mathcal{C}(R,I)$. Thus to prove that ξ is injective, it is enough to show the equality

$$\mathrm{Ker}\,\big(H(R') \to \Gamma\big) = F_H(R',J)$$

for any such object, where, as above, $J = \mathrm{Ker}(f)$. Composition (4.10) yields an embedding $\mathrm{Ker}\big(H(R') \to \Gamma\big) \subset F_H(R', J)$. Let us show that there is also the converse embedding.

Consider an auxiliary object (R'', I'') in $\mathcal{C}(R, I)$ defined as follows:

$$R'' = R'[[t]]/(I', t)^N\,, \qquad I'' = (I', \bar{t}\,)\,,$$

and the morphism to (R, I) is the composition

$$(R'', I'') \xrightarrow{g} (R', I') \xrightarrow{f} (R, I)\,,$$

where g is identical on R' and sends $\bar{t}$ to zero. In particular, g is a morphism in the category $\mathcal{C}(R, I)$. Let a subgroup $\Delta \subset H(R'')$ be the image of the map

$$F_H\big(R'[[t]], (t)\big) \longrightarrow H(R'')$$

induced by the quotient map $R'[[t]] \to R''$. Since the composition

$$R'[[t]] \longrightarrow R'' \xrightarrow{g} R'$$

is identical on R' and sends t to zero, we see that $g(\Delta) = 0$ in $H(R')$.

Fix an element $x \in J$ and define another morphism $h\colon (R'', I'') \to (R', I')$ in $\mathcal{C}(R, I)$ that is identical on R' and sends $\bar{t}$ to x. Since f, g are morphisms in $\mathcal{C}(R, I)$ and $g(\Delta) = 0$, it follows from the definition of the colimit that the subgroup $h(\Delta) \subset H(R')$ is contained in $\mathrm{Ker}\big(H(R') \to \Gamma\big)$. Condition (ii) applied to $J \subset R'$ implies that $F_H(R', J) \subset \mathrm{Ker}\big(H(R') \to \Gamma\big)$. As explained above, this finally proves that ξ is an isomorphism.

Now let (R, I) be an aritrary object in $\mathrm{SNil}_N(S)$. There is a canonical isomorphism

$$\operatorname*{colim}_{(R', I') \to (R, I)} H(R') \xrightarrow{\sim} \operatorname*{colim}_{A \subset R} \Big(\operatorname*{colim}_{(R', I') \twoheadrightarrow (A, A \cap I)} H(R') \Big)\,,$$

where the colimit in the left hand side is as in Definition 4.13, A runs over all finitely generated S-subalgebras in R, and for each such A, we consider in the right hand side the colimit over the category $\mathcal{C}(A, A \cap I)$. Indeed, for each morphism $(R', I') \to (R, I)$ from a finite free object (R', I') in $\mathrm{SNil}_N(S)$, one defines A as the image of R' in R.

By what was shown above, there is an isomorphism

$$\operatorname*{colim}_{A \subset R} \Big(\operatorname*{colim}_{(R', I') \twoheadrightarrow (A, A \cap I)} H(R') \Big) \xrightarrow{\sim} \operatorname*{colim}_{A \subset R} H(A)\,,$$

where the colimit in the right hand side is taken over all finitely generated S-subalgebras $A \subset R$.

Note that any finitely generated subring in R is contained in a finitely generated S-algebra in R and any finitely generated S-subalgebra $A \subset R$ is a union of finitely generated subrings in A. Hence condition (iii) applied to R and to all finitely generated S-subalgebras in R implies that there is an isomorphism

$$\operatorname*{colim}_{A \subset R} H(A) \xrightarrow{\sim} H(R)\,.$$

Altogether this proves that the functor

$$\mathrm{SNil}_N(S) \longrightarrow \mathcal{A}b\,, \qquad (R, I) \longmapsto H(R)\,, \tag{4.11}$$

is finitely freely approximable.

For any (R, I) in $\mathrm{SNil}_N(S)$, the S-algebra structure on R defines an isomorphism $H(R) \simeq F_H(R, I) \oplus H(S)$, which is functorial with respect to (R, I). Thus the functor F_H is finitely freely approximable, being a direct summand of the finitely freely approximable functor defined in formula (4.11). □

4.5 Construction of the Bloch map

As an application of Proposition 4.16, we show that relative Milnor K-groups (see Definition 2.1) and relative modules of differential forms (see Definition 2.7) define finitely freely approximable functors.

Proposition 4.17. *For any $n \geqslant 0$, the functors*

$$K_n^M \;:\; \mathrm{SNil}_N(S) \longrightarrow \mathcal{A}b\,, \qquad (R, I) \longmapsto K_n^M(R, I)\,,$$

$$\Omega^n \;:\; \mathrm{SNil}_N(S) \longrightarrow \mathcal{A}b\,, \qquad (R, I) \longmapsto \Omega^n_{R,I}\,,$$

are finitely freely approximable.

Proof. Condition (i) of Proposition 4.16 is satisfied for these functors trivially. Condition (ii) of Proposition 4.16 is satisfied by Lemma 2.2 and Lemma 2.8. Condition (iii) of Proposition 4.16 is satisfied for Milnor K-groups and modules of differential forms, because they are given by finitely defined generators and relations. We finish the proof applying Proposition 4.16. □

Corollary 4.18.

(i) For any $n \geqslant 0$, the functor

$$\mathrm{SNil}_S \longrightarrow \mathcal{A}b\,, \qquad (R,I) \longmapsto \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\,,$$

is finitely freely approximable.

(ii) Suppose that $N!$ is invertible in S. Then for any object (R,I) in $\mathrm{SNil}_N(S)$, we have an isomorphism

$$(d)^{fa} \;:\; \big(\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\big)^{fa} \simeq \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} \xrightarrow{\sim} \big((\Omega^{n+1}_{R,I})^{cl}\big)^{fa}\,.$$

Proof. (i) This follows directly from Proposition 4.17 and Lemma 4.15.

(ii) By part (i), we have a homomorphism of groups

$$(d)^{fa} \;:\; \big(\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\big)^{fa} \simeq \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} \longrightarrow \big((\Omega^{n+1}_{R,I})^{cl}\big)^{fa}\,.$$

We need to show that this is an isomorphism. By Lemma 4.14, it is enough to show this for $(R_{N,m}, I_{N,m})$, $m \geqslant 0$. In this case, Proposition 4.11 claims that $H^n_{dR}(R_{N,m}, I_{N,m}) = 0$, whence there is an isomorphism

$$d \;:\; \Omega^n_{R_{N,m},I_{N,m}}/d\,\Omega^{n-1}_{R_{N,m},I_{N,m}} \xrightarrow{\sim} (\Omega^{n+1}_{R_{N,m},I_{N,m}})^{cl}\,, \tag{4.12}$$

which finishes the proof. □

In particular, it follows from Corollary 4.18(ii) that the functor $(\Omega^n_{R,I})^{cl}$ is not finitely freely approximable, because there are split nilpotent extensions with non-trivial relative de Rham cohomology (see the discussion at the end of Subsection 4.3).

Now we are ready to construct the Bloch map (see Definition 4.2).

Proof of Theorem 4.6. Let R, I, N, and n be as in the theorem. Put $S = R/I$. Then (R,I) is a split nilpotent extension of S of nilpotency degree N and $N!$ is invertible in S. By Proposition 4.17 and Corollary 4.18, we obtain a homomorphism of groups

$$(d\log)^{fa} \;:\; K^M_{n+1}(R,I)^{fa} \simeq K^M_{n+1}(R,I) \longrightarrow \big((\Omega^n_{R,I})^{cl}\big)^{fa} \simeq \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\,.$$

Let us check that $(d\log)^{fa}$ satisfies the condition of Definition 4.2, that is, that formula (4.2) therein holds. Clearly, it is enough to show this for $(R_{N,m}, I_{N,m})$, where $m \geqslant 0$. In this case, this is implied by the isomorphism (4.12) and formula (4.1) from the proof of Lemma 4.1. Thus $(d\log)^{fa}$ is the Bloch map. □

In other words, we integrate $d\log$ by taking a (co)limit, in the spirit of calculus.

Remark 4.19. Let S be a ring such that $N!$ is invertible in S. Our proof of Theorem 4.6 implies that the restriction of the Bloch map to objects in $\mathrm{SNil}_N(S)$ is the unique collection of homomorphisms $K^M_{n+1}(R,I) \to \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ that are functorial with respect to (R,I) in $\mathrm{SNil}_N(S)$ and such that their composition with d is $d\log$.

It is important to mention that one can prove Theorem 4.6 more directly, avoiding colimits. Nevertheless, the approach with finitely freely approximable functors will allow us further to prove also Theorem 5.1. Still let us sketch an explicit proof of Theorem 4.6.

As above, put $S = R/I$. Also, define a group

$$L_{n+1} = \mathrm{Ker}\left((R^*)^{\otimes(n+1)} \to (S^*)^{\otimes(n+1)}\right).$$

First, one shows that formula (4.2) from Definition 4.2 gives a homomorphism $\widetilde{\mathrm{B}}\colon L_{n+1} \to \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ (actually, this holds in a non-split case as well). For this one uses that

$$\log(1+x)\frac{d(1+y)}{1+y} + \log(1+y)\frac{d(1+x)}{1+x} = d\big(\log(1+x)\log(1+y)\big) \in d\,I$$

for all elements $x, y \in I$ (cf. the end of the proof of Lemma 2.2).

Then one needs to show that the homomorphism $\widetilde{\mathrm{B}}$ vanishes on the intersection $L_{n+1} \cap \mathrm{St}_{n+1}(R) \subset (R^*)^{\otimes(n+1)}$. Since S splits out of R, one obtains an explicit system of generators of this intersection by projecting the Steinberg relations from $(R^*)^{\otimes(n+1)}$ to L_{n+1} with respect to the decomposition

$$(R^*)^{\otimes(n+1)} \simeq L_{n+1} \oplus (S^*)^{\otimes(n+1)}.$$

The vanishing of $\widetilde{\mathrm{B}}$ on these generators is reduced to the case $R = R_{N,m}$, when it holds, because the Bloch map exists in this case by Remark 4.4 and Proposition 4.11.

As an example, consider the case $n = 1$. Then the intersection $L_2 \cap \mathrm{St}_2(R) \subset (R^*)^{\otimes 2}$ is generated by elements of type

$$\zeta(a,x) = (a+x) \otimes (1-a-x) - a \otimes (1-a) \in (R^*)^{\otimes 2},$$

where $a, 1-a \in S^*$ and $x \in I$. Clearly, the element $\zeta(a,x)$ is the image of the analogous element $\zeta(a,\bar t\,) \in \left(R^*_{N,1}\right)^{\otimes 2}$ under the map induced by the homomorphism of S-algebras $R_{N,1} \to R$ that sends $\bar t$ to x. The map $\widetilde{\mathrm{B}}$ vanishes on $\zeta(a,\bar t\,)$, because the Bloch map B exists for $R_{N,1}$ as its relative de Rham cohomology groups vanish by Lemma 4.10.

5 Isomorphism theorem for the Bloch map

Let the pair (R,I) be a split nilpotent extension of the quotient ring R/I as in Theorem 4.6. The isomorphism theorem claims that the Bloch map is an isomorphism for split nilpotent extensions with sufficiently many invertible elements.

Theorem 5.1. *Let $I \subset R$ be a nilpotent ideal and $N \geqslant 1$ be a natural number such that $I^N = 0$. Suppose that the quotient map $R \to R/I$ admits a splitting by a ring homomorphism $R/I \to R$, that $N!$ is invertible in R, and that R is weakly 5-fold stable. Then for any natural number $n \geqslant 0$, the Bloch map is an isomorphism*

$$\mathrm{B}\,:\,K^M_{n+1}(R,I) \xrightarrow{\sim} \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\,.$$

Theorem 5.1 is proved later in this section. Note that Theorem 2.12 presents a special case of this theorem.

Also note that, according to Remark 4.7, Theorem 5.1 also fails in a non-split case. Moreover it is easy to see that there is no functorial isomorphism between $K^M_{n+1}(R,I)$ and $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ when R/I does not necessarily split out of R. Indeed, given an ideal $J \subset R$ contained in I, one has an exact sequence (2.1) from Subsection 2.5 for relative Milnor K-groups, while for relative modules of differential forms one has an exact sequence (2.3) from Subsection 2.3 with, possibly, a non-trivial term $H^{n-1}_{dR}(R',I')$, where $R' = R/J$ and $I' = I/J$. This argument is also used in the proof of Proposition 5.7.

Remark 5.2. An interesting problem is to find an alternative formulation of Theorem 5.1 that would be valid in a non-split case as well. One way might be to replace $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ with the group

$$\mathrm{Im}\big(\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} \to \Omega^n_R/d\,\Omega^{n-1}_R\big) = \mathrm{Ker}\left(\Omega^n_R/d\,\Omega^{n-1}_R \to \Omega^n_{R/I}/d\,\Omega^{n-1}_{R/I}\right).$$

This group coincides with $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ when R/I splits out of R.

A more sophisticated way, which also goes along with Goodwillie's theorem [8], is to consider the groups $K^M_{n+1}(R,I)$ and $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ in Theorem 5.1 as degree zero cohomology of certain (non-positively graded) complexes. Namely, one can show that the group $\Omega^n_R/d\,\Omega^{n-1}_R$ is isomorphic to the degree zero cohomology group of the complex

$$\mathrm{cone}\big(F^{n+1}\mathbb{L}\Omega^\bullet_R \to \mathbb{L}\Omega^\bullet_R\big)[n]\,,$$

where $\mathbb{L}\Omega^\bullet_R$ is the derived de Rham complex of R and $F^\bullet$ is the Hodge filtration on it. Thus a natural substitute for $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ is the degree zero cohomology group of the complex

$$\mathrm{cone}\big(F^{n+1}\mathbb{L}\Omega^\bullet_{R,I} \to \mathbb{L}\Omega^\bullet_{R,I}\big)[n]\,,$$

where

$$\mathbb{L}\Omega^\bullet_{R,I} \simeq \mathrm{cone}\big(\mathbb{L}\Omega^\bullet_R \to \mathbb{L}\Omega^\bullet_{R/I}\big)[-1]$$

is the relative derived de Rham complex. Again, this group coincides with $\Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}$ when R/I splits out of R.

It is not clear what should be a right complex for Milnor K-groups. It seems possible that this might involve a version for commutative simplicial rings of Goncharov's complexes [11], giving sort of derived Milnor K-groups.

5.1 Reduction lemma

The following lemma is our main tool to make reductions when proving that the Bloch map (see Definition 4.2) is an isomorphism.

Lemma 5.3. *Let $J\subset I$ be two nilpotent ideals in a ring R. Put $R'=R/J$, $I'=I/J$. Suppose that for a natural number $n\geqslant 0$, there are equalities*

$$H^n_{dR}(R,I)=H^n_{dR}(R',I')=H^{n-1}_{dR}(R',I')=0\,.$$

Then the following holds:

(i) the Bloch maps

$$\mathrm{B}_{R,I}\;:\;K^M_{n+1}(R,I)\longrightarrow \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\,,$$

$$\mathrm{B}_{R',I'}\;:\;K^M_{n+1}(R',I')\longrightarrow \Omega^n_{R',I'}/d\,\Omega^{n-1}_{R',I'}\,,$$

$$\mathrm{B}_{R,J}\;:\;K^M_{n+1}(R,J)\longrightarrow \Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J}$$

exist;

(*ii*) *if, in addition, any element in R is a sum of invertible elements, then the Bloch map $\mathrm{B}_{R,I}$ is an isomorphism if and only if both $\mathrm{B}_{R',I'}$ and $\mathrm{B}_{R,J}$ are isomorphisms.*

Proof. (*i*) Since $H^n_{dR}(R,I) = H^n_{dR}(R',I') = 0$, by Remark 4.4, the Bloch maps $\mathrm{B}_{R,I}$ and $\mathrm{B}_{R',I'}$ exist. Using also that $H^{n-1}_{dR}(R',I') = 0$ and the exact sequence (2.2) of relative de Rham cohomology from Subsection 2.3, we obtain $H^n_{dR}(R,J) = 0$. Thus, applying Remark 4.4 again, we see that the Bloch map $\mathrm{B}_{R,J}$ exists as well.

(*ii*) The condition $H^{n-1}_{dR}(R',I') = 0$ and the exact sequence (2.3) from Subsection 2.3 imply that we have an exact sequence

$$0 \longrightarrow \Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J} \longrightarrow \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} \longrightarrow \Omega^n_{R',I'}/d\,\Omega^{n-1}_{R',I'} \longrightarrow 0\,.$$

Furthermore, there is a commutative diagram with exact raws

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^M_{n+1}(R,J) & \longrightarrow & K^M_{n+1}(R,I) & \longrightarrow & K^M_{n+1}(R',I') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{B}_{R,J}} & & \downarrow{\scriptstyle \mathrm{B}_{R,I}} & & \downarrow{\scriptstyle \mathrm{B}_{R',I'}} & & \\ 0 & \longrightarrow & \Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J} & \longrightarrow & \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} & \longrightarrow & \Omega^n_{R',I'}/d\,\Omega^{n-1}_{R',I'} & \longrightarrow & 0 \end{array}$$

Since R is generated additively by invertible elements, the same holds for its quotient R'. So, by Lemma 4.3, the Bloch maps in the diagram are surjective. Therefore, there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathrm{B}_{R,J}) \longrightarrow \mathrm{Ker}(\mathrm{B}_{R,I}) \longrightarrow \mathrm{Ker}(\mathrm{B}_{R',I'}) \longrightarrow 0\,,$$

which finishes the proof. □

5.2 Special case of the isomorphism theorem

Let S be a ring such that $N!$ is invertible in it for a natural number $N \geqslant 1$ and S is weakly 5-fold stable (see Definition 2.3).

For short, denote the S-algebra $R_{N,1}$ (see Definition 4.9) just by R_N, that is, put

$$R_N := S[t]/(t^N)\,.$$

As above, $\bar{t}$ denotes the image of t in R_N.

In this subsection, we prove Theorem 5.1 for $(\bar{t}) \subset R_N$. The case $N = 1$ is trivial, so we assume that $N \geqslant 2$.

Define the homomorphism of S-algebras

$$\sigma \ : \ R_2 \longrightarrow R_N \,, \qquad \bar{t} \longmapsto \bar{t}^{\,N-1} \,.$$

Appying Corollary 3.3 of Proposition 3.2, we prove the following useful fact.

Proposition 5.4. *For any natural number $n \geqslant 0$, the homomorphism of groups $\sigma\colon K^M_{n+1}(R_2) \to K^M_{n+1}(R_N)$ restricts to a surjective homomorphism*

$$\sigma \ : \ K^M_{n+1}\big(R_2, (\bar{t}\,)\big) \longrightarrow K^M_{n+1}\big(R_N, (\bar{t}^{\,N-1})\big) \,.$$

Proof. Since the ring S is weakly 5-fold stable, the same holds for the ring R_N, so we may apply Lemma 2.6 to R_N. Combining this with Lemma 2.2, we see that the group $K^M_{n+1}\big(R_N, (\bar{t}^{\,N-1})\big)$ is generated by elements of type $\{1 + a\bar{t}^{\,N-1}, r_1, \ldots, r_n\}$, where $a \in S$ and $r_1, \ldots, r_n \in R_N^*$. Decomposing invertible elements of R_N, we can assume that for each i, $1 \leqslant i \leqslant n$, we have either $r_i \in S^*$ or $r_i = 1 + b\bar{t}^{\,j}$ for some $b \in S$ and $j \geqslant 1$.

If all the elements r_i are from S^*, then the symbol $\{1 + a\bar{t}^{\,N-1}, r_1, \ldots, r_n\}$ is clearly in the image of the map in question. Otherwise, using Lemma 2.6 again, we may assume that $r_1 = 1 + b\bar{t}^{\,j}$, where $b \in S$ and $j \geqslant 1$. Now, by Corollary 3.3, the symbol

$$\{1 + a\bar{t}^{\,N-1}, r_1\} = \{1 + a\bar{t}^{\,N-1}, 1 + b\bar{t}^{\,j}\}$$

vanishes in $K^M_2(R_N)$, because N is invertible in S. Hence the symbol $\{1 + a\bar{t}^{\,N-1}, r_1, \ldots, r_n\}$ vanishes as well, which finishes the proof. □

We will also need the following simple statement.

Lemma 5.5. *For any $n \geqslant 0$, the morphism of S-modules $\sigma\colon \Omega^n_{R_2} \to \Omega^n_{R_N}$ restricts to an isomorphism of S-modules*

$$\sigma \ : \ \Omega^n_{R_2,(\bar{t}\,)} \xrightarrow{\ \sim\ } \Omega^n_{R_N,(\bar{t}^{N-1})} \,.$$

Proof. Let us describe the S-module $\Omega^n_{R_N}$ explicitly. We have an isomorphism

$$\Omega^n_{S[t]} \simeq \Omega^n_S[t] \oplus \Omega^{n-1}_S[t] \wedge dt \,.$$

Also, there is an equality $d(t^N) = Nt^{N-1}dt$. Combining this with the fact that N is invertible in S and with Lemma 2.8 applied to the ring $S[t]$ and the ideal $(t^N) \subset S[t]$, we obtain an isomorphism of S-modules

$$\Omega^n_{R_N}(\Omega^n_S \oplus \Omega^n_S \cdot \bar{t} \oplus \ldots \oplus \Omega^n_S \cdot \bar{t}^{\,N-1}) \oplus (\Omega^{n-1}_S \wedge d\bar{t} \oplus \Omega^{n-1}_S \wedge \bar{t} d\bar{t} \oplus \ldots \oplus \Omega^{n-1}_S \wedge \bar{t}^{\,N-2} d\bar{t}\,) \,.$$

Applying Lemma 2.8 to the ring R_N and the ideal $\bar{t}^{\,N-1}$, we get an isomorphism

$$\Omega^n_{R_N,(\bar{t}^{N-1})} \simeq \Omega^n_S \cdot \bar{t}^{\,N-1} \oplus \Omega^{n-1}_S \wedge \bar{t}^{\,N-2} d\bar{t}\,. \tag{5.1}$$

When $N = 2$, this gives an isomorphism (cf. Example 4.5)

$$\Omega^n_{R_2,(\bar{t}\,)} \simeq \Omega^n_S \cdot \bar{t} \oplus \Omega^{n-1}_S \wedge d\bar{t}\,. \tag{5.2}$$

Clearly, σ induces an isomorphism between the right hand sides of (5.2) and (5.1). □

Now, combining Proposition 5.4, Lemma 5.5, and the main result of [9], we prove the following auxiliary special case of Theorem 5.1.

Proposition 5.6. *For any $n \geqslant 0$, the Bloch map for $(\bar{t}\,) \subset R_N$ is an isomorphism, that is, we have an isomorphism*

$$\mathrm{B} \;:\; K^M_{n+1}\big(R_N, (\bar{t}\,)\big) \xrightarrow{\sim} \Omega^n_{R_N,(\bar{t}\,)}/d\,\Omega^{n-1}_{R_N,(\bar{t}\,)}\,.$$

Proof. The proof is by induction on N. The base of the induction, namely, the case $N = 2$, is [9, Theor. 2.9], which holds for S, because 2 is invertible in it and S is weakly 5-fold stable.

Let us make an induction step from $N - 1$ to N. We will apply Lemma 5.3 to the ideals

$$(\bar{t}^{\,N-1}) \subset (\bar{t}\,) \subset R_N\,.$$

By definition, we have an isomorphism $R_{N-1} \simeq R_N/(\bar{t}^{\,N-1})$. Since $N!$ is invertible in S, by Lemma 4.10, we have the vanishing of relative de Rham cohomology for $(\bar{t}\,) \subset R_N$ and for $(\bar{t}\,) \subset R_{N-1}$. Hence by Lemma 5.3($i$), the Bloch map exists for $(\bar{t}^{\,N-1}) \subset R_N$.

For any $n \geqslant 0$, we have the following commutative diagram:

$$\begin{array}{ccc}
K^M_{n+1}\big(R_2, (\bar{t}\,)\big) & \xrightarrow{\mathrm{B}} & \Omega^n_{R_2,(\bar{t}\,)}/d\,\Omega^{n-1}_{R_2,(\bar{t}\,)} \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\
K^M_{n+1}\big(R_N, (\bar{t}^{\,N-1})\big) & \xrightarrow{\mathrm{B}} & \Omega^n_{R_N,(\bar{t}^{N-1})}/d\,\Omega^{n-1}_{R_N,(\bar{t}^{N-1})}
\end{array}$$

The left vertical map is surjective by Proposition 5.4. The right vertical map is an isomorphism by Lemma 5.5. The top horizontal map is an isomorphism by [9, Theor. 2.9]. Altogether this implies that the left vertical map and the bottom horizontal map are isomorphisms as well. So, the Bloch map is an isomorphism for $(\bar{t}^{N-1}) \subset R_N$ (notice that this is an instance when the Bloch map exists and is an isomorphism in a non-split case).

By the induction hypothesis, the Bloch map is also an isomorphism for $(\bar{t}) \subset R_{N-1}$. Since the ring S is weakly 5-fold stable and, in particular, is weakly 2-fold stable, S is generated additively by invertible elements. Hence by Lemma 5.3(ii), the Bloch map is an isomorphism for $(\bar{t}) \subset R_N$. $\square$

Note that in the proof of Proposition 5.4 (respectively, of Lemma 5.5) we used only that $2N$ (respectively, N) is invertible in S. However, in the proof of Proposition 5.6 we do use the invertibility of $N!$ in S.

5.3 Proof of the isomorphism theorem

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let R, I, N, and n be as in the theorem. Put $S = R/I$. Then (R, I) is a split nilpotent extension of S of nilpotency degree N (see Definition 4.8), we have that $N!$ is invertible in S, and the ring S is weakly 5-fold stable. The Bloch map is a morphism from the functor

$$K^M_{n+1} \,:\, \mathrm{SNil}_N(S) \longrightarrow \mathcal{A}b\,, \qquad (R, I) \longmapsto K^M_{n+1}(R, I)$$

to the functor

$$\Omega^n/d\,\Omega^{n-1} \,:\, \mathrm{SNil}_N(S) \longrightarrow \mathcal{A}b\,, \qquad (R, I) \longmapsto \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\,.$$

Thus the theorem is equivalent to the fact that this morphism of functors is an isomorphism.

By Proposition 4.17 and Corollary 4.18(i), respectively, the functors K^M_{n+1} and $\Omega^n/d\,\Omega^{n-1}$ are finitely freely approximable (see Definition 4.13). Hence by Lemma 4.14, it is enough to prove the theorem for $I_{N,m} \subset R_{N,m}$ (see Definition 4.9), where $m \geqslant 1$.

We proceed by induction on m. The base of the induction $m = 1$ is Proposition 5.6. Let us make an induction step from $m-1$ to m.

Let $J \subset R_{N,m-1}[t_m]/(t_m^N)$ be the kernel of the surjective homomorphism of algebras over $R_{N,m-1}$

$$R_{N,m-1}[t_m]/(t_m^N) \longrightarrow R_{N,m}\,, \qquad t_m \longmapsto \bar{t}_m\,.$$

Explicitly, we have

$$J = (\bar{t}_1, \ldots, \bar{t}_{m-1})^{N-1} \cdot \bar{t}_m + (\bar{t}_1, \ldots, \bar{t}_{m-1})^{N-2} \cdot \bar{t}_m^2 + \ldots (\bar{t}_1, \ldots, \bar{t}_{m-1}) \cdot \bar{t}_m^{N-1} .$$

In particular, $J \subset (\bar{t}_m)$. Let us apply Lemma 5.3 to the ideals

$$J \subset (\bar{t}_m) \subset R_{N,m-1}[t_m]/(t_m^N) .$$

By Lemma 4.10 with S replaced by $R_{N,m-1}$, we have the vanishing of relative de Rham cohomology for $(\bar{t}_m) \subset R_{N,m-1}[t_m]/(t_m^N)$. Furthermore, the preimage of the ideal J in $R_{N,m-1}[t_m]$ contains t_m^N and is generated by monomials in t_m. Hence by Lemma 4.10 again, relative de Rham cohomology groups are trivial for $(\bar{t}_m) \subset R_{N,m}$.

Since $N!$ is invertible in S and the ring S is weakly 5-fold stable, the same holds for the ring $R_{N,m-1}$. Thus by Proposition 5.6 with S replaced by $R_{N,m-1}$, the Bloch map is an isomorphism for $(\bar{t}_m) \subset R_{N,m-1}[t_m]/(t_m^N)$. Hence by Lemma 5.3($ii$), the Bloch map is an isomorphism for $(\bar{t}_m) \subset R_{N,m}$ (and also for $J \subset R_{N,m-1}[t_m]/(t_m^N)$, though we are not using this fact).

Now we apply Lemma 5.3 to the ideals

$$(\bar{t}_m) \subset I_{N,m} \subset R_{N,m} .$$

By definition, we have isomorphisms

$$R_{N,m-1} \simeq R_{N,m}/(\bar{t}_m) , \qquad I_{N,m-1} \simeq I_{N,m}/(\bar{t}_m) .$$

Proposition 4.11 claims the vanishing of relative de Rham cohomology for $I_{N,m} \subset R_{N,m}$ and for $I_{N,m-1} \subset R_{N,m-1}$. By what was shown above, the Bloch map is an isomorphism for $(\bar{t}_m) \subset R_{N,m}$. In addition, by the induction hypothesis, the Bloch map is an isomorphism for $I_{N,m-1} \subset R_{N,m-1}$. Hence by Lemma 5.3($ii$), the Bloch map is an isomorphism for $I_{N,m} \subset R_{N,m}$, which finishes the proof. □

5.4 Non-existence of the Bloch map in a non-split case

Finally, using Theorem 5.1, we show that the Bloch map does not exist in general in a non-split case.

Proposition 5.7. *Let S be a ring such that $N!$ is invertible in S for a natural number $N \geqslant 1$ and S is weakly 5-fold stable. Let (R, I) be a split* *nilpotent extension of S of nilpotency degree N such that $H^{n-1}_{dR}(R,I)=0$ for a natural number $n\geqslant 1$. Let $J\subset R$ be an ideal contained in I such that $H^{n-1}_{dR}(R',I')\neq 0$, where $R'=R/J$, $I'=I/J$. Then there does not exist a Bloch map from $K^M_{n+1}(R,J)$ to $\Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J}$.*

Proof. Assume the converse, that is, that the Bloch map $\mathrm{B}_{R,J}\colon K^M_{n+1}(R,J)\to\Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J}$ does exist. Both (R,I) and (R',I') satisfy the conditions of Theorem 4.6 and Theorem 5.1. Therefore the Bloch maps $\mathrm{B}_{R,I}$ and $\mathrm{B}_{R',I'}$ exist and are isomorphisms. Formula (4.2) from Definition 4.2 implies that the Bloch map is functorial with respect to a ring and an ideal. Hence the exact sequence (2.1) from Subsection 2.5 of relative Milnor K-groups together with the exact sequence (2.3) from Subsection 2.3 give a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^M_{n+1}(R,J) & \longrightarrow & K^M_{n+1}(R,I) & \longrightarrow & K^M_{n+1}(R',I') & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle \mathrm{B}_{R,J}} & & \big\downarrow{\scriptstyle \mathrm{B}_{R,I}} & & \big\downarrow{\scriptstyle \mathrm{B}_{R',I'}} & & \\
H^{n-1}_{dR}(R',I') & \xrightarrow{\alpha} & \Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J} & \longrightarrow & \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I} & \longrightarrow & \Omega^n_{R',I'}/d\,\Omega^{n-1}_{R',I'} & \longrightarrow & 0
\end{array}$$

with exact rows and with $\mathrm{B}_{R,I}$ and $\mathrm{B}_{R',I'}$ being isomorphisms. This implies the vanishing

$$\mathrm{Im}(\mathrm{B}_{R,J})\cap\mathrm{Im}(\alpha)=0 \tag{5.3}$$

of the intersection of subgroups in $\Omega^n_{R,J}/d\,\Omega^{n-1}_{R,J}$.

On the other hand, the map α is injective, because of the condition $H^{n-1}_{dR}(R,I)=0$ and the exact sequence (2.3) extended to the left. Combining this with the condition $H^{n-1}_{dR}(R',I')\neq 0$, we obtain that $\mathrm{Im}(\alpha)\neq 0$. At the same time, by Lemma 4.3, the map $\mathrm{B}_{R,J}$ is surjective. Thus we obtain a contradiction with formula (5.3). □

There are many examples that satisfy the conditions of Proposition 5.7 (see the discussion at the end of Subsection 4.3). For instance, it follows from Proposition 4.11 and [12, § 1.3] that one can take $S=\mathbb{Q}$, $N=6$, $n=1$,

$$R=\mathbb{Q}[t_1,t_2]/(t_1,t_2)^6\,,\qquad I=(t_1,t_2)\,,\qquad J=(\partial_{t_1}f,\partial_{t_2}f)/(t_1,t_2)^6\,,$$

where $f=t_1^4+t_1^2t_2^3+t_2^5\in\mathbb{Q}[t_1,t_2]$.

6 Bloch–Artin–Hasse map

6.1 Artin–Hasse logarithm for δ-rings

From now on, for the sake of simplicity for any element of Ω^n_R, the corresponding images in $\widehat{\Omega}^n_R$ and ${}^D\widehat{\Omega}^n_R$ will be denoted by the same letter. Also, the composition of the homomorphism $d\log\colon R^* \to \Omega^n_R$ with the natural maps $\Omega^n_R \to \widehat{\Omega}^n_R$ and $\Omega^n_R \to {}^D\widehat{\Omega}^n_R$ will be also denoted by $d\log$.

The following map will be called an *Artin–Hasse logarithm* for δ-rings, because it can be considered as a generalization of the classic Artin–Hasse logarithm

$$\log_\delta \;:\; 1 + t\mathbb{Z}_p[[t]]^* \longrightarrow t\mathbb{Z}_p[[t]]\,, \qquad 1+t \longmapsto \sum_{p\nmid i}(-1)^{i-1}\frac{t^i}{i}\,,$$

for the δ-ring $\mathbb{Z}_p[[t]]$, where φ sends t to t^p (see also [34, § 1]).

Proposition 6.1. *Let (R,δ) be a p-adically complete δ-ring. There is a unique group homomorphism $\log_\delta\colon R^* \to R$ that satisfies the equality*

$$d \circ \log_\delta = \Big(1 - \frac{\varphi}{p}\Big) d\log \tag{6.1}$$

between group homomorphisms from R^* to ${}^D\widehat{\Omega}^1_R$ and is functorial with respect to p-adically complete δ-rings.

Note that it is necessary for the ring R from Proposition 6.1 to be p-adically complete in order to define correctly the map $\log_\delta$. Analogously, the group ${}^D\widehat{\Omega}^1_R$ being (derived) p-adically complete is necessary for the equality (6.1) to hold.

In other words, Proposition 6.1 states that for any $r \in R^*$, the form $d\log(r)$ admits an integral after applying the map $1 - \frac{\varphi}{p}$. Thus, a key to the proof is to give meaning to the expression $\big(1 - \frac{\varphi}{p}\big)\log(r)$. Using formally the standard property of the logarithm, we see that

$$\Big(1 - \frac{\varphi}{p}\Big)\log(r) = \frac{(p-\varphi)}{p}\log(r) = \frac{1}{p}\log\Big(\frac{r^p}{\varphi(r)}\Big) = -\frac{1}{p}\log\Big(1 + p\frac{\delta(r)}{r^p}\Big)\,.$$

In order to prove Proposition 6.1, we will need the following simple fact, which is, apparently, well-known. Nevertheless, we decided to provide it with a proof for the sake of completeness.

Lemma 6.2. *Let $S \subset \mathbb{Z}[t_1, t_2, \ldots]$ be a subset of the ring of polynomials in countably many variables. Suppose there exists a collection of integers $a_1, a_2, \ldots \in \mathbb{Z}$ such that for any $f \in S$, the value $f(a_1, a_2, \ldots) \in \mathbb{Z}$ is not divisible by p. Denote by A the ring $\mathbb{Z}[t_1, t_2, \ldots][S^{-1}]^\wedge$. Then the natural map $\mathbb{Z}_p \to H^0({}^D\widehat{\Omega}^\bullet_A)$ is an isomorphism.*

Proof. Since the ring $\mathbb{Z}[t_1, t_2, \ldots][S^{-1}]$ has trivial p-torsion, Lemma 2.23 implies an isomorphism ${}^D\mathbb{Z}[t_1, t_2, \ldots][S^{-1}]^\wedge \simeq \mathbb{Z}[t_1, t_2, \ldots][S^{-1}]^\wedge$. Consider the isomorphism

$$\mathbb{Z}[t_2, t_2, \ldots] \xrightarrow{\sim} \mathbb{Z}[s_1, s_2, \ldots]\,, \qquad t_i \longmapsto s_i - a_i\,,$$

that develops polynomials at the point $(a_1, a_2, \ldots)$. There is a natural embedding of rings

$$A \hookrightarrow E := \mathbb{Z}_p[[s_1, s_2, \ldots]]\,,$$

where the right ring consists of all the infinite linear combinations of (finite) monomials in formal variables $s_1, s_2, \ldots$ with coefficients in $\mathbb{Z}_p$ (here we use the fact that the ring E is p-adically complete).

There is a natural isomorphism $\Omega^1_{\mathbb{Z}[t_1,t_2,\ldots][S^{-1}]} \simeq \bigoplus_{i\geqslant 0} \mathbb{Z}[t_1, t_2, \ldots][S^{-1}]dt_i$. In particular, the group $\Omega^1_{\mathbb{Z}[t_1,t_2,\ldots][S^{-1}]}$ has trivial p-torsion, so by Corollary 2.24, there are isomorphisms

$${}^D\widehat{\Omega}^1_{\mathbb{Z}[t_1,t_2,\ldots][S^{-1}]} \simeq \widehat{\Omega}^1_{\mathbb{Z}[t_1,t_2,\ldots][S^{-1}]}\,, \;\; {}^D\widehat{\Omega}^1_{\mathbb{Z}[t_1,t_2,\ldots][S^{-1}]} \simeq {}^D\widehat{\Omega}^1_A\,,$$

as well as group embeddings

$$\widehat{\Omega}^1_{\mathbb{Z}[t_1,t_2,\ldots][S^{-1}]} \subset \prod_{i\geqslant 0} A\,dt_i \subset \prod_{i\geqslant 0} E\,ds_i\,.$$

Moreover, there is also de Rham differential $d\colon E \to \prod_{i\geqslant 0} E\,ds_i$ defined by taking partial derivatives with respect to variables s_i. This de Rham differential commutes with the de Rham differential $d\colon A \to {}^D\widehat{\Omega}^1_A$.

It follows from what was said above that there is an embedding

$$H^0({}^D\widehat{\Omega}^\bullet_A) \subset \mathrm{Ker}\left(d\colon E \to \prod_{i\geqslant 0} E\,ds_i\right).$$

Finally, it is easy to see that the right hand side is equal to $\mathbb{Z}_p$. $\square$

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. Consider the series

$$f(x) = -\frac{1}{p}\log(1+px) = \sum_{i\geqslant 0}(-1)^i\frac{p^{i-1}}{i}x^i \in \mathbb{Q}[[x]]\,. \tag{6.2}$$

Since $p > 2$, the coefficients of $f(x)$ are p-adic integers and their sequence tends p-adically to zero, that is, $f(x) \in \mathbb{Z}[x]^\wedge$. Since the ring R is p-adically complete, there is a well-defined map

$$\log_\delta \;:\; R^* \longrightarrow R\,, \qquad r \longmapsto f\Big(\frac{\delta(r)}{r^p}\Big)\,. \tag{6.3}$$

Let us show that it satisfies all the properties required. The functionality is obvious. It follows from the standard property of the logarithm series that the equality $f(x+y+pxy) = f(x)+f(y)$ holds in the ring $\mathbb{Q}[[x,y]]$ and hence it holds in the ring $\mathbb{Z}[x,y]^\wedge$ as well. It also follows from the multiplicative property of δ-structure that for all $r, s \in R$, there is an equality in R

$$\frac{\delta(rs)}{(rs)^p} = \frac{\delta(r)}{r^p} + \frac{\delta(s)}{s^p} + p\frac{\delta(r)}{r^p}\frac{\delta(s)}{s^p}\,.$$

Thus, the map $\log_\delta$ is a group homomorphism.

Now we will show that the map $\log_\delta$ satisfies equality (6.1). Note that the element $df(x)$ of the group ${}^D\widehat{\Omega}^1_{\mathbb{Z}[x]^\wedge} \simeq \widehat{\Omega}^1_{\mathbb{Z}[x]^\wedge} \simeq \mathbb{Z}[x]^\wedge dx$ is equal to the form $-\frac{1}{p}d\log(1+px) = -\frac{dx}{1+px}$. Note also that this equality holds in the (derived) p-adic completion of the group $\Omega^1_{\mathbb{Z}[x]^\wedge}$ and does not hold in the non-complete group $\Omega^1_{\mathbb{Z}[x]^\wedge}$ itself. Thus there are equalities in the group ${}^D\widehat{\Omega}^1_R$

$$d\log_\delta(r) = -\frac{d\big(\delta(r)/r^p\big)}{1+p\,\delta(r)/r^p} = \frac{p\,r^{p-1}dr\delta(r) - r^p d\delta(r)}{r^p\varphi(r)} = p\,\frac{\delta(r)}{\varphi(r)}\frac{dr}{r} - \frac{d\delta(r)}{\varphi(r)}\,.$$

On the other hand, according to formula (2.6), there are the equalities in the group Ω^1_R

$$\Big(1-\frac{\varphi}{p}\Big)\frac{dr}{r} = \frac{dr}{r} - \frac{r^{p-1}dr + d\delta(r)}{\varphi(r)} = \frac{\big(r^p + p\delta(r)\big)dr - r^p dr}{r\varphi(r)} - \frac{d\delta(r)}{\varphi(r)} =$$

$$= p\,\frac{\delta(r)}{\varphi(r)}\frac{dr}{r} - \frac{d\delta(r)}{\varphi(r)}\,.$$

This proves equality (6.1).

It remains to prove uniqueness of the functorial homomorphism $\log_\delta\colon R^* \to R$ satisfying equality (6.1) on the category of p-adically complete δ-rings. It is enough to show that there are no non-trivial morphisms of group functors $R^* \to H^0({}^D\widehat{\Omega}^\bullet_R)$. Note that by Lemma 2.17, the p-adically complete δ-ring $R_{\mathbb{G}_m} := \mathbb{Z}[x]_\delta[x^{-1}]^\wedge$ copresents the functor $\mathbb{G}_m\colon R \mapsto R^*$ on the category of p-adically complete δ-rings. Thus, any (not necessary group) morphism from R^* to $H^0({}^D\widehat{\Omega}^\bullet_R)$ is determined uniquely by an element of $H^0({}^D\widehat{\Omega}^\bullet_{R_{\mathbb{G}_m}})$.

It follows from Lemma 6.2 that there is an isomorphism $\mathbb{Z} \simeq H^0({}^D\widehat{\Omega}^\bullet_{R_{\mathbb{G}_m}})$. Thus, any morphism of functors from R^* to $H^0({}^D\widehat{\Omega}^\bullet_R)$ is a constant one, that is, it sends all elements $r \in R^*$ to the same element $c \in \mathbb{Z}_p$. If this morphism of functors is also a morphism of group functors, then $c = 0$. This concludes the proof of uniqueness of the Artin–Hasse logarithm. $\square$

Example 6.3. Let (R, δ) be a p-adically complete δ-ring.

(i) Suppose that $\delta(r) = 0$ for some $r \in R$; these elements are said to have *rank one*, see [2, Remark 2.3]. When R has trivial p-torsion, this is equivalent to $\varphi(r) = r^p$. Then it follows from formulas (6.2) and (6.3) that $\log_\delta(r) = 0$.

(ii) Suppose that the ring R has trivial p-torsion. Then the restriction of the Artin–Hasse logarithm $\log_\delta\colon R^* \to R$ to the subgroup $1 + pR \subset R^*$ is equal to the following composition of group homomorphisms:

$$1 + pR \xrightarrow{\log} pR \xrightarrow{1/p} R \xrightarrow{p-\varphi} R\,,$$

where the first and the second homomorphisms are, in fact, isomorphisms (note that $p \neq 2$).

(iii) Let k be a perfect field of characteristic p and put $R = W(k)$ to be the ring of its Witt vectors. There is a natural δ-structure one the ring $W(k)$ defined by the canonical lifting of the Frobenius homomorphism from k to $W(k)$. Then the Teichmuller representatives define a group decomposition

$$W(k)^* \simeq k^* \times \big(1 + pW(k)\big)\,,$$

where all elements of the subgroup $k^* \subset W(k)^*$ have rank one. It follows from parts (i) and (ii) that the Artin–Hasse logarithm

$\log_\delta\colon W(k)^* \to W(k)$ is equal to the following composition of groups homomorphisms:

$$W(k)^* \longrightarrow 1 + pW(k) \xrightarrow{\log} pW(k) \xrightarrow{1/p} W(k) \xrightarrow{p-\varphi} W(k)\,,$$

where the first homomorphism is the natural projection. Note that the last homomorphism is, in fact, an isomorphism, because the homomorphism $\varphi\colon W(k) \to W(k)$ is an isomorphism, the homomorphism $p\colon W(k) \to W(k)$ is topologically nilpotent, and they commute with each other.

6.2 Bloch–Artin–Hasse map

For an arbitrary ring R, one has the following homomorphism of graded rings:

$$d\log \;:\; \bigoplus_{n\geqslant 0}(R^*)^{\otimes n} \longrightarrow \bigoplus_{n\geqslant 0}({}^D\widehat{\Omega}^n_R)^{cl}\,, \qquad r_1\otimes\ldots\otimes r_n \longmapsto \frac{dr_1}{r_1}\wedge\ldots\wedge\frac{dr_n}{r_n}\,. \tag{6.4}$$

Let (R,δ) be a δ-ring. One can apply the formalism from § 2.7 to the dg-ring ${}^D\widehat{\Omega}^\bullet_R$ equipped with the Hodge filtration $F^n{}^D\widehat{\Omega}^\bullet_R$ and with the collection of morphisms of complexes $\frac{\varphi}{p^n}\colon F^n{}^D\widehat{\Omega}^\bullet_R \to {}^D\widehat{\Omega}^\bullet_R$ (see Proposition 2.19 and Remark 2.25). For simplicity, denote the corresponding cohomology groups $H^n\big(C^\bullet(n)\big)$, $n\geqslant 0$ by $H^n(R,\delta)$ (see formula (2.7)). Thus, we have

$$H^n(R,\delta) \simeq \big\{(\omega,[\eta]) \in ({}^D\widehat{\Omega}^n_R)^{cl} \oplus {}^D\widehat{\Omega}^{n-1}_R/d\,{}^D\widehat{\Omega}^{n-2}_R \;\mid\; (1-\tfrac{\varphi}{p^n})\omega = d\eta\big\}$$

and product of elements $(\omega,[\eta]) \in H^n(R,\delta)$ and $(\omega',[\eta']) \in H^{n'}(R,\delta)$ is defined by the formula

$$(\omega,[\eta])\cdot(\omega',[\eta']) = \big(\omega\wedge\omega', [(-1)^n\omega\wedge\eta' + \eta\wedge\tfrac{\varphi}{p^{n'}}(\omega')]\big)\,. \tag{6.5}$$

Proposition 6.1 can be interpreted as a statement on existence and uniqueness of a lifting of the functorial group homomorphism $d\log\colon R^*\to({}^D\widehat{\Omega}^1_R)^{cl}$ on the category of p-adically complete δ-rings (R,δ) up to a functorial group homomorphism

$$(d\log,\log_\delta) \;:\; R^* \longrightarrow H^1(R,\delta) \simeq \big\{(\omega,r)\in({}^D\widehat{\Omega}^1_R)^{cl}\oplus R \;\mid\; \big(1-\tfrac{\varphi}{p}\big)\omega = dr\big\}$$

on the same category.

Since the graded tensor ring $\bigoplus\limits_{n\geqslant 0}(R^*)^{\otimes n}$ is generated freely by its graded component of degree 1, we obtain the following fact.

Proposition 6.4. *Let (R, δ) be a p-adically complete δ-ring. Then there exists a unique morphism of graded rings*

$$(d\log, \mathrm{B}_\delta) \ : \ \bigoplus_{n\geqslant 0}(R^*)^{\otimes n} \longrightarrow \bigoplus_{n\geqslant 0} H^n(R,\delta) \subset \bigoplus_{n\geqslant 0} \left(({}^D\widehat{\Omega}^n_R)^{cl} \oplus {}^D\widehat{\Omega}^{n-1}_R / d\,{}^D\widehat{\Omega}^{n-2}_R\right)$$

that satisfies the equality

$$d \circ \mathrm{B}_\delta = \Big(1 - \frac{\varphi}{p^n}\Big) d\log \tag{6.6}$$

between group homomorphisms from $(R^*)^{\otimes n}$ to ${}^D\widehat{\Omega}^n_R$ and is functorial with respect to p-adically complete δ-rings.

The map B_δ will be called a *Bloch–Artin–Hasse map*, because it generalizes the notion of the Artin–Hasse logarithm from Proposition 6.1 and also can be considered as a p-adic version of the Bloch map from Section 4.

Using induction on n and formula (6.5), one can see that for any collection of elements $r_1, \ldots, r_n \in R^*$ there is an equality in the group ${}^D\widehat{\Omega}^{n-1}_R / d\,{}^D\widehat{\Omega}^{n-2}_R$ (see [20, § 0.7.1])

$$\mathrm{B}_\delta(r_1 \otimes \ldots \otimes r_n) =$$

$$= \Big[\sum_{i=1}^{n} (-1)^{i-1} \log_\delta(r_i)\, d\log(r_1 \otimes \ldots \otimes r_{i-1}) \wedge \tfrac{\varphi}{p^{n-i}}\big(d\log(r_{i+1} \otimes \ldots \otimes r_n)\big)\Big] .$$

This equality can be rewritten also in a more convenient way:

$$\begin{gathered}\mathrm{B}_\delta(r_1 \otimes \ldots \otimes r_n) = \Big[\log_\delta(r_1) d\log(r_2 \otimes \ldots \otimes r_n) + \\ (-1)^{n-1}\mathrm{B}_\delta(r_2 \otimes \ldots \otimes r_n) d\log(r_1) - \log_\delta(r_1) d\mathrm{B}_\delta(r_2 \otimes \ldots \otimes r_n)\Big].\end{gathered} \tag{6.7}$$

In particular, for all $r, s \in R^*$, there is an equality in the group ${}^D\widehat{\Omega}^1_R / dR$

$$\mathrm{B}_\delta(r \otimes s) = \Big[\log_\delta(r)\, \tfrac{\varphi}{p}\big(d\log(s)\big) - \log_\delta(s)\, d\log(r)\Big] . \tag{6.8}$$

It follows from the equality $\frac{\varphi}{p}\big(d\log(s)\big) = d\log(s) - d\log_\delta(s)$ that the differential form in the brackets can be rewritten in the following way:

$$\log_\delta(r)\, d\log(s) - \log_\delta(s)\, d\log(r) - \log_\delta(r)\, d\log_\delta(s) . \tag{6.9}$$

Note that formula (6.9) coincides with the numerator part in Vostokov's formula for the norm residue symbol for local fields (see [34, formulas (5), (12)]).

Theorem 6.5. *For any p-adically complete δ-ring (R,δ), the group homomorphism $\mathrm{B}_\delta\colon (R^*)^{\otimes n} \to {}^D\widehat{\Omega}^{n-1}_R/d\,{}^D\widehat{\Omega}^{n-2}_R$ sends all Steinberg relations to zero.*

Thus, Theorem 6.5 states that the homomorphism B_δ induces a group homomorphism

$$K^M_n(R) \longrightarrow {}^D\widehat{\Omega}^{n-1}_R/d\,{}^D\widehat{\Omega}^{n-2}_R\,.$$

In turn, by the universal property of the derived p-adic completion, this homomorphism induces a group homomorphism

$${}^D\widehat{K}^M_n(R) \longrightarrow {}^D\widehat{\Omega}^{n-1}_R/d\,{}^D\widehat{\Omega}^{n-2}_R\,.$$

It makes sense to denote the latter homomorphism by the same symbol B_δ and to call it a Bloch–Artin–Hasse map as well as the initial homomorphism.

Theorem 6.5 will be proved in the next section.

Remark 6.6. Let $R \simeq S \oplus I$ be a split nilpotent extension of S such that both rings R and S are p-adically complete with trivial p-torsion. Assume given a δ-structure on R such that $\delta(S) \subset S$ and $\delta(I) \subset I$. The splitting $R \simeq S \oplus I$ one has the following equalities

$${}^D\widehat{K}^M_{n+1}(R,I) := \ker\left({}^D\widehat{K}^M_{n+1}(R) \to {}^D\widehat{K}^M_{n+1}(S)\right),$$

$${}^D\widehat{\Omega}^n_{R,I} := \ker\left({}^D\widehat{\Omega}^n_R \to {}^D\widehat{\Omega}^n_S\right)$$

for any $n \geqslant 0$. Moreover the splitting $R = S \oplus I$ induces the equalities

$$K^M_{n+1}(R) = K^M_{n+1}(S) \oplus K^M_{n+1}(R,I)\,,\ \Omega^n_R/d\Omega^{n-1}_R = \Omega^n_S/\Omega^{n-1}_S \oplus \Omega^n_{R,I}/d\Omega^{n-1}_{R,I}$$

for any $n \geqslant 0$. Since the functor of derived p-adic completion is right exact, applying it to the previous equalities one obtains the equalities

$${}^D\widehat{K}^M_{n+1}(R) = {}^D\widehat{K}^M_{n+1}(S) \oplus {}^D\widehat{K}^M_{n+1}(R,I)\,,\ {}^D\widehat{\Omega}^n_R/d\,{}^D\widehat{\Omega}^{n-1}_R =$$

$$= {}^D\widehat{\Omega}^n_S/d\,{}^D\widehat{\Omega}^{n-1}_S \oplus {}^D\widehat{\Omega}^{n-1}_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}$$

for any $n \geqslant 0$. It follows directly from the definition of B_δ that the composition of its restriction on $K^M_{n+1}(R,I)$ with natural projection ${}^D\widehat{\Omega}^n_R/d\,{}^D\widehat{\Omega}^{n-1}_R \to {}^D\widehat{\Omega}^n_S/\,{}^D\widehat{\Omega}^{n-1}_S$ is zero. Thus for any $n \geqslant 0$ one has the restriction

$$\mathrm{B}_\delta : K^M_{n+1}(R,I) \to {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}\,,$$

which in turn induces the group homomorphism

$$\mathrm{B}_\delta : {}^D\widehat{K}^M_{n+1}(R,I) \to {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}\,.$$

6.3 Proof of Theorem 6.5

In order to prove Theorem 6.5 we will require some supplementary materials. For an arbitrary p-adically complete abelian group G and some non-zero element $g \in G$ define a non-negative integer $v_p(g)$ by the condition $g \in p^{v_p(g)}G \smallsetminus p^{v_p(g)+1}G$. Also, put $v_p(0) := \infty$. By definition, the sequence $\{g_i\}_{i\geqslant 0}$ of elements of G is said to *p-adically tend to zero* if the sequence $\{v_p(g_i)\}_{i\geqslant 0}$ tends to infinity.

Let S be an arbitrary ring and t be a formal variable. Recall that there are canonical isomorphism

$$\Omega_S^{n-1}[t]dt \oplus \Omega_S^n[t] \xrightarrow{\sim} \Omega_{S[t]}^n\,, \qquad n \geqslant 1\,.$$

Thus, an arbitrary element $\eta \in \widehat{\Omega}_{S[t]}^n$ is uniquely decomposed into a sum

$$\eta = \sum_{i\geqslant 0} \xi_i t^i dt + \sum_{i\geqslant 0} \eta_i t^i\,, \tag{6.10}$$

where the sequences $\{\xi_i\}_{i\geqslant 0}$ and $\{\eta_i\}_{i\geqslant 0}$ of elements from $\widehat{\Omega}_S^{n-1}$ and $\widehat{\Omega}_S^n$ respectively, p-adically tend to zero.

Denote by π the natural ring homomorphism $S[t]^\wedge \to \widehat{S}$, $t \mapsto 0$. The induced homomorphisms $\widehat{\Omega}_{S[t]}^n \to \widehat{\Omega}_S^n$, $n \geqslant 1$ will be denoted by the same symbol.

Lemma 6.7. *For an arbitrary closed* 1*-form* $\eta \in \widehat{\Omega}_{S[t]}^1$ *with a decomposition*

$$\eta = \sum_{i\geqslant 0} f_i t^i dt + \sum_{i\geqslant 0} \eta_i t^i\,, \qquad f_i \in \widehat{S}\,, \qquad \eta_i \in \widehat{\Omega}_S^1\,,$$

the form $\eta - \pi(\eta)$ *is exact if and only if the sequence* $\{v_p(f_i) - v_p(i+1)\}_{i\geqslant 0}$ *is non-negative and tends to infinity.*

Proof. Since form η is closed the forms $\pi(\eta) = \eta_0$ and $\eta - \pi(\eta)$ are closed as well. Thus, there is an equality

$$d\big(\eta - \pi(\eta)\big) = \sum_{i\geqslant 0} t^i df_i \wedge dt - \sum_{i\geqslant 1} i t^{i-1}\eta_i \wedge dt + \sum_{i\geqslant 1} t^i d\eta_i = 0 \in \widehat{\Omega}_{S[t]}^2\,.$$

If $n = 2$ it follows from the decomposition (6.10) that there are equalities

$$df_i = (i+1)\eta_{i+1} \in \widehat{\Omega}_S^1\,, \qquad d\eta_{i+1} = 0 \in \widehat{\Omega}_S^2\,, \qquad i \geqslant 0\,.$$

Thus, the form $\eta - \pi(\eta)$ is exact if and only if there exists a series $g = \sum\limits_{i\geqslant 1} g_i t^i$ such that the sequence $\{g_i\}_{i\geqslant 1}$ tends to zero in $\widehat{S}$ and the equalities $f_i = (i+1)g_{i+1}$ are hold for any $i \geqslant 0$ (in this case $\eta = dg$). The existence of such a series g is equivalent to the second condition of the lemma. □

When applying Lemma 6.7 to the proof of Theorem 6.5 the following trivial facts will be used:

Lemma 6.8. *For any positive integer i there is an equality*

$$\lfloor \log_p(i) \rfloor = \max\{v_p(k) \ \mid \ 1 \leqslant k \leqslant i\}\,,$$

where $\lfloor \log_p(i) \rfloor$ denotes the floor of the logarithm of i to the base p.

The proof of Lemma 6.8 is provided directly.

Lemma 6.9. *The sequences*

$$\{i - 1 - \lfloor \log_p(i) \rfloor - v_p(i+1)\}_{i\geqslant 1}\,, \qquad \{i - 1 - \lfloor \log_p(ip-1) \rfloor - v_p(i)\}_{i\geqslant 1}\,,$$

are both non-negative and tend to infinity.

Proof. It follows from Lemma 6.8 that there is an inequality $v_p(i+1) \leqslant \lfloor \log_p(i+1) \rfloor$. It is also easy to see that $\lfloor \log_p(i+1) \rfloor \leqslant \lfloor \log_p(i) \rfloor + 1$. Thus, there is a lower bound

$$i - 1 - \lfloor \log_p(i) \rfloor - v_p(i+1) \geqslant i - 2 - 2\lfloor \log_p(i) \rfloor\,. \tag{6.11}$$

Moreover, according to Lemma 6.8 there is an inequality $v_p(i) \leqslant \lfloor \log_p(i) \rfloor$. There is also an obvious inequality $\lfloor \log_p(ip-1) \rfloor \leqslant \lfloor \log_p(i) \rfloor + 1$. Thus, there is a lower bound

$$i - 1 - \lfloor \log_p(ip-1) \rfloor - v_p(i) \geqslant i - 2 - 2\lfloor \log_p(i) \rfloor\,. \tag{6.12}$$

Furthermore, there are inequalities

$$i - 2 - 2\lfloor \log_p(i) \rfloor \geqslant i - 2 - 2\log_p(i) \geqslant i - 2 - 2\log_3(i) \tag{6.13}$$

that are taken under the assumption that $p \neq 2$. It is obvious that the sequence in the right handside of inequality (6.13) tends to infinity. Thus, we are left to check that for $i \geqslant 1$ the sequences in the left parts of inequalities (6.11) and (6.12) are both greater than zero.

Consider the real function $h(x) := x - 2 - 2\log_3(x)$ for $x > 0$. It is easy to see that if $x \geqslant 2$ then $h'(x) = 1 - \frac{2}{\ln(3)x} > 0$ and also that $h(5) > 0$. Thus, if $x \geqslant 5$ then $h(x) > 0$, and we are left to check that the right handsides of inequalities (6.11) and (6.12) are bigger that zero if $1 \leqslant i \leqslant 4$. It can be done directly under the assumption that $p \neq 2$. □

We are now ready to prove Theorem 6.5.

The proof of Theorem 6.5. From now up until the end of the section all the introduced supplementary rings, as well as their modules of differential forms have trivial p-torsion. Therefore, according to Lemma 2.23 and Corollary 2.24, it does not matter if we consider their classic p-adic completions or derived ones. For the sake of simplicity, we will be considering the classic p-adic completions.

Step 1.

Let us show that it is enough to prove the universal case of the theorem. It is obviously enough to prove the theorem for the case $n = 2$. It follows from Lemma 2.17 that the functor Σ is co-presented in the category of p-adically complete δ-rings by an object

$$R_\Sigma := \mathbb{Z}[x]_\delta[x^{-1}, (1-x)^{-1}]^\wedge . \tag{6.14}$$

Thus, it is enough to prove the theorem for the ring R_Σ and the Steinberg relation $x \otimes (1-x) \in (R_\Sigma^*)^{\otimes 2}$.

Choose the following representative for the element $\mathrm{B}_\delta\big(x \otimes (1-x)\big) \in \widehat{\Omega}^1_{R_\Sigma}/dR_\Sigma$ (see formula (6.9))

$$\omega := \log_\delta(x) d\log(1-x) - \log_\delta(1-x) d\log(x) - \log_\delta(x) d\log_\delta(1-x) \in \widehat{\Omega}^1_{R_\Sigma} . \tag{6.15}$$

It follows from formula (6.6) and the equality $d\log\big(x \otimes (1-x)\big) = 0$ that $d\omega = 0 \in \widehat{\Omega}^2_{R_\Sigma}$, that is, that the form ω is closed.

In order to prove the theorem we will need to show that the form ω is exact in the complex $\widehat{\Omega}^\bullet_{R_\Sigma}$.

Step 2.

let us show that one can get rid of the formal variables $\delta^i x \in R_\Sigma$ for $i \geqslant 2$ and thus move from the bigger ring R_Σ to a smaller one, generated by x and δx, since the form ω involves only these two variables.

Define the rings

$$A := \mathbb{Z}[x^{\pm 1}, (1-x)^{-1}]\,, \qquad B := A[\delta^i x,\, i \geqslant 1]\,.$$

The natural split embedding $A[\delta x] \to B$ induces the split embeddings $\Omega^n_{A[\delta x]} \to \Omega^n_B$, $n \geqslant 0$, and therefore the split embeddings $\widehat{\Omega}^n_{A[\delta x]} \to \widehat{\Omega}^n_B$, $n \geqslant 0$. According to Lemma 2.21, the isomorphism of p-adically complete rings $\widehat{B} \simeq R_\Sigma$ induces the isomorphisms of the groups $\widehat{\Omega}^n_B \simeq \widehat{\Omega}^n_{R_\Sigma}$, $n \geqslant 0$. Thus, there are natural group embeddings

$$\widehat{\Omega}^n_{A[\delta x]} \subset \widehat{\Omega}^n_{R_\Sigma}\,, \qquad n \geqslant 0\,.$$

Furthermore, it follows from formulas (6.2) and (6.3) that both elements $\log_\delta(x)$, $\log_\delta(1-x)$ belong to the subring $A[\delta x]^\wedge$ in R_Σ. Thus, it follows from formula (6.15) that the form $\omega \in \widehat{\Omega}^1_{R_\Sigma}$ belongs to the subgroup $\widehat{\Omega}^1_{A[\delta x]} \simeq \widehat{\Omega}^1_{A[\delta x]^\wedge}$ in $\widehat{\Omega}^1_{R_\Sigma}$ (see Lemma 2.21) and that there is an equality $d\omega = 0$ in the group $\widehat{\Omega}^2_{A[\delta x]}$.

Thereby, it is enough to show that the form ω is exact in the complex $\widehat{\Omega}^\bullet_{A[\delta x]}$.

Step 3.

For the sake of convenience, put $t = \delta x$. Let as describe the form ω from $\widehat{\Omega}^1_{A[\delta x]} = \widehat{\Omega}^1_{A[t]}$ more precisely. In accordance to decomposition (6.10) the form ω can be introduced in the following way:

$$\omega = \sum_{i \geqslant 0} f_i t^i dt + \sum_{i \geqslant 0} \omega_i t^i\,, \qquad f_i \in \widehat{A}\,, \qquad \omega_i \in \widehat{\Omega}^1_A\,.$$

Let us find the explicit values of the coefficients $f_i \in \widehat{A}$.

Note that, according to formula (6.15) for ω, only the component $-\log_\delta(x) d\log_\delta(1-x)$ is involved in the series $\sum\limits_{i \geqslant 0} f_i t^i dt$. According to formula (6.1), there is an equality

$$d\log_\delta(1-x) = \frac{d(1-x)}{1-x} - \frac{\varphi}{p}\Big(\frac{d(1-x)}{1-x}\Big)\,.$$

Again, one can see that there is only the second component of the right handside of this equality that is involved into series $\sum\limits_{i \geqslant 0} f_i t^i dt$:

$$-\frac{\varphi}{p}\Big(\frac{d(1-x)}{1-x}\Big) = \frac{1}{\varphi(1-x)} \cdot \frac{\varphi}{p}(dx) = \frac{x^{p-1}dx + dt}{1 - x^p - pt}\,.$$

Thus, one gets an equality

$$\sum_{i\geqslant 0} f_i t^i dt = -\frac{\log_\delta(x)}{1-x^p-pt}dt\,.$$

Applying formulas (6.2) and (6.3) for $\log_\delta(x)$, one can see that the right handside of the last equality is equal to the product of the following p-adically convergent series:

$$-\log_\delta(x) = \frac{1}{p}\log\Big(1+\frac{pt}{x^p}\Big) = \sum_{k\geqslant 1}(-1)^{k-1}\frac{p^{k-1}t^k}{k\,x^{pk}}\,, \qquad (6.16)$$

$$\frac{1}{1-x^p-pt} = \frac{1}{1-x^p}\Big(1-\frac{pt}{1-x^p}\Big)^{-1} = \frac{1}{1-x^p}\cdot\sum_{m\geqslant 0}\frac{p^m x_1^m}{(1-x^p)^m}\,. \qquad (6.17)$$

In order to provide the correctness of the expressions above, one can note that $x, 1-x\in A^*$ and $1-x^p\in \widehat{A}^*$, since one has $1-x^p=(1-x)^p(1+pa)$ for some $a\in A$.

By multiplying series (6.16) and (6.17), one obtains that $f_0=0$ and the following equalities hold

$$f_i = \frac{1}{1-x^p}\cdot\sum_{k=1}^{i}(-1)^{k-1}\frac{p^{i-1}}{k}\frac{1}{x^{pk}(1-x^p)^{i-k}}\,, \qquad i\geqslant 1\,. \qquad (6.18)$$

Thus, we had found the explicit view of the series $\sum_{i\geqslant 0} f_i t^i \in A[t]^\wedge$.

Step 4.

Using Lemma 6.7 one can reduce the question about the form ω being exact to the question about $\pi(\omega)$ being exact, where $\pi\colon A[t]^\wedge \to \widehat{A}$ is a natural homomorphism of $\widehat{A}$-algebras that sends t to 0.

Let us check that the series $\sum_{i\geqslant 0} f_i t^i$ from formula (6.18) satisfies the requirements of Lemma 6.7. Indeed, applying Lemma 6.8, one can see that

$$v_p(f_i) - v_p(i+1) \geqslant i-1-\lfloor\log_p(i)\rfloor - v_p(i+1)\,, \qquad i\geqslant 1\,.$$

According to Lemma 6.9 the sequence in the right part of this equality is non-negative and tends to infinity. Applying Lemma 6.7, one obtains the exactness of the form $\omega-\pi(\omega)$ in the complex $\widehat{\Omega}^\bullet_{A[t]}$.

Thus, in order to prove the exactness of ω in the complex $\widehat{\Omega}^\bullet_{A[t]}$ in is enough to prove the exactness of $\pi(\omega)$ in the complex $\widehat{\Omega}^\bullet_A$, which is a direct component of the complex $\widehat{\Omega}^\bullet_{A[t]}$.

Step 5.

Note that one can interpret the previous condition as a case of Theorem **??** for the ring $\widehat{A}$ with a δ-structure, uniquely defined by the condition $\delta(x) = 0$, that is, $\varphi(x) = x^p$.

It is easy to see that the ring homomorphism $\pi\colon A[t]^\wedge \to \widehat{A}$ can be uniquely extended to a morphism of δ-rings $R_\Sigma \to \widehat{A}$ such that $x_i \mapsto 0$, $i \geqslant 1$. Therefore, the equivalence class of $\pi(\omega)$ in $\widehat{\Omega}^1_A/d\widehat{A}$ coincides with $\mathrm{B}_\delta\big(x \otimes (1-x)\big)$.

Moreover, it follows from formulas (6.2) and (6.3)that $\log_\delta(x) = 0$ in $\widehat{A}$. Thus, it follows from formula (6.9) that the element $\mathrm{B}_\delta\big(x\otimes(1-x)\big) \in \widehat{\Omega}^1_A/d\widehat{A}$ coincides with the equivalence class of the form

$$\xi := -\log_\delta(1-x)\frac{dx}{x}\,. \tag{6.19}$$

Note that this statement can also be deduced from formula (6.15), by using the fact that $\xi = \pi(\omega)$.

Therefore, we need to proof the exactness of the form ξ in the complex $\widehat{\Omega}^\bullet_A$.

Step 6.

It turns out that it is more convenient to express the form ξ through the coordinate $y := (1-x)^{-1}$. Namely, note that since $1 - y = -\frac{x}{1-x}$, one has $A = \mathbb{Z}[y^{\pm 1}, (1-y)^{-1}]$. Therefore, one has $\xi \in \mathbb{Z}[y^{\pm 1}, (1-y)^{-1}]^\wedge dy$. Let us show that ξ is, in fact, an element of $\mathbb{Z}[y]^\wedge dy$, and it is also exact in the complex $\widehat{\Omega}^\bullet_{\mathbb{Z}[y]}$.

Ir follows from the equality $x = \frac{y-1}{y}$ that there is an equality

$$\frac{dx}{x} = \frac{dy}{y(y-1)}\,.$$

Moreover, applying formulas (6.2) and (6.3) in order to calculate $\log_\delta(1-x)$, one obtains the equality

$$-\log_\delta(1-x) = \frac{1}{p}\log\left(1 + p\frac{\delta(1-x)}{(1-x)^p}\right) = \frac{1}{p}\log\left(\frac{\varphi(1-x)}{(1-x)^p}\right) =$$

$$= \frac{1}{p} \log\Big(\frac{1-x^p}{(1-x)^p}\Big) = \frac{1}{p}\log\Big(\frac{1-\big((y-1)/y\big)^p}{y^{-p}}\Big) = \frac{1}{p}\log\big(1+pg(y)\big)\,,$$

where the polynomial $g(y) \in \mathbb{Z}[y]$ is defined by formula

$$g(y) = \big(y^p - 1 - (y-1)^p\big)/p\,.$$

One can see that according to formula (6.19) there is an equality

$$\xi = \frac{1}{p}\log\big(1+pg(y)\big)\frac{dy}{y(y-1)} = \sum_{i\geqslant 1}(-1)^{i-1}\frac{p^{i-1}}{i}\frac{g^i}{y(y-1)}dy\,.$$

Note that $g(0) = g(1) = 0$, and thus the rational function $g^i/\big(y(y-1)\big)$ is, in fact, a polynomial from $\mathbb{Z}[y]$. Therefore, we have shown that $\xi \in \mathbb{Z}[y]^\wedge dy$.

All that is left to do is to prove the exactness of ξ in $\widehat{\Omega}^\bullet_{\mathbb{Z}[y]}$. Note that the monomials of the polynomial g, are of the degree from 1 to $p-1$ and thus, the monomials of the polynomial $g^i/\big(y(y-1)\big)$ are of the degree from $i-1$ to $i(p-1)-2$. Therefore, in order to prove that the form ξ is exact, it is enough to show that for $i-1 \leqslant j \leqslant i(p-1)-2$ the numbers $(-1)^{i-1}\frac{p^{i-1}}{i(j+1)}$ are all p-adic integers and tend to zero when i tends to infinity (this can also be considered as a case of Lemma 6.7). According to Lemma 6.8, it is enough to show that the sequence $\{i-1-v_p(i)-\lfloor \log_p(i(p-1)-1)\rfloor\}_{i\geqslant 1}$ is non-negative and tends to infinity. This fact follows from Lemma 6.9, since one obviously has $\lfloor \log_p(i(p-1)-1)\rfloor \leqslant \lfloor \log_p(ip-1)\rfloor$. □

The fact from Step 6 of the proof of Theorem 6.5 about ξ belonging to $\mathbb{Z}[y]^\wedge dy$ can be illustrated geometrically in the following way.

Divide the projective line $\mathbb{P}^1$ with a coordinate x, considered as rigid analytic space over $\mathbb{Z}_p$, into four disjoint opened analytic subsets:

$$U_1 := \{|x|_p < 1\}\,, \quad U_2 := \{|x|_p = 1\,, |1-x|_p < 1\}\,,$$

$$U_3 := \{|x|_p = 1,\ |1-x|_p = 1\}\,, \qquad U_4 := \{|x|_p > 1\}\,.$$

It follows from the p-adic norm being non-archimedean that for the sheaf of analytic functions $\mathcal{O}$ the following equalities hold

$$\mathcal{O}(U_1 \cup U_3) = \mathcal{O}\big(\{|1-x|_p = 1\}\big) = \mathbb{Z}[(1-x)^{\pm 1}]^\wedge\,,$$

$$\mathcal{O}(U_1 \cup U_3 \cup U_4) = \mathcal{O}\big(\{|(1-x)^{-1}|_p \leqslant 1\}\big) = \mathbb{Z}[(1-x)^{-1}]^\wedge = \mathbb{Z}[y]^\wedge\,.$$

It follows from formulas (6.2) and (6.3) that the function $\log_\delta(1-x)$ belongs to the ring $\mathbb{Z}[x,(1-x)^{-1}]^\wedge = \mathbb{Z}[(1-x)^{\pm 1}]^\wedge$, that is, it is an analytic function on $U_1 \cup U_3$.

It turns out that the function $\log_\delta(1-x)$ can be analytically extended onto $U_1 \cup U_3 \cup U_4$ and has zeroes at points $x = 0$ and $x = \infty$. Indeed, consider an involution map $\iota\colon x \mapsto x^{-1}$ on $\mathbb{P}^1$. This involution map mutually transforms U_1 into U_4, and acts as an automorphism on U_2 and U_3. The restriction of the function $\log_\delta(1-x)$ on U_3 is ι-invariant, since because of equalities $\varphi(-x) = (-x)^p$ and $\delta(-x) = 0$ one has $\log_\delta(1-x^{-1}) = \log_\delta(1-x) - \log_\delta(-x)$ and $\log_\delta(-x) = 0$. Therefore, the function $\log_\delta(1-x)$ can be analytically extended to a ι-invariant function on $U_1 \cup U_3 \cup U_4$. One obviously has $\log_\delta(1-x)(0) = \log_\delta(1) = 0$, so because of the ι-invariance one also has $\log_\delta(1-x)(\infty) = 0$.

As we can see the form $\xi = -\log_\delta(1-x)\frac{dx}{x}$ is also defined on $U_1 \cup U_3 \cup U_4$ without having any poles on it, that is, one has $\xi \in \mathbb{Z}[y]^\wedge dy$.

Remark 6.10. All major statements of this section, except for Remark 6.6 stay true for the case of classic p-adic completions of modules Ω^n_R.

7 Isomorphism theorem for the Bloch–Artin–Hasse map

7.1 Isomorphism property for the Artin–Hasse logarithm

Let p be a prime number not equal to 2. Let $I \subset R$ be a nilpotent ideal of a p-adically complete δ-ring R such that $\delta(I) \subset I^2$. Then the corresponding restriction of $\mathrm{B}_\delta = \log_\delta$ takes $1+I$ to I.

Consider the following finite filtrations

$$1+I \supset 1+I^2 \supset \ldots \supset 1+I^m \supset \ldots,\; I \supset I^2 \supset \ldots \supset I^m \supset \ldots$$

Lemma 7.1.

(i) For any $x \in I$ and $m \geqslant 1$ the homomorphism $1+x \mapsto \log_\delta(1+x)$ maps $1+I^m$ to I^m.

(ii) For any $m \geqslant 1$, $x \in I^m$ the corresponding quotient homomorphism $1+I^m/1+I^{m+1} \to I^m/I^{m+1}$ sends the class $[1+x]$ to the class $[x]$.

Proof. Given the fact that $\log_\delta(1+x) = -\frac{1}{p}\log(1+p\frac{\delta(1+x)}{(1+x)^p})$, for the first part it is enough to show that if $x \in I^m$ then $\delta(1+x) \in I^m$. Since $\delta(1+x) = \delta(x) + F_p(x,1)$ where $F_p(x,1) \in I^m$ it is enough to show that $\delta(x) \in I^{m+1}$. This can easily be done through the induction by m since $\delta(I) \subset I^2$.

For the second part note that $F_p(x,1) \in x + I^{m+1}$, hence $\delta(1+x) \in x + I^{m+1}$. Thus, one has the image $\Big[-\frac{1}{p}\log(1+p\frac{\delta(1+x)}{(1+x)^p})\Big]$ of the class $[1+x]$ belonging to $-x + I^{m+1}$. This concludes the proof.

□

As a corollary we obtain the following statement (compare it to Theorem 5.1 for the case $n = 0$)

Corollary 7.2. *The homomorphism* $\log_\delta : 1 + I \to I$ *is an isomorphism.*

7.2 Isomorphism property for the Bloch–Artin–Hasse map

Let $R = S \oplus I$ be a split nilpotent extension of S such that both rings R and S are p-adically complete with trivial p-torsion and $I^N = 0$ for some $N \in \mathbb{N}$. Suppose that there is a δ-structure on R with $\delta(S) \subset S$ and $\delta(I) \subset I$. As usual, denote the corresponding Frobenius map by φ.

By Remark 6.6, the map B_δ induces a group homomorphism

$$\mathrm{B}_\delta : {}^D\widehat{K}^M_{n+1}(R,I) \to {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}\,.$$

There are reasons to believe that if enlisted conditions stay true and, in addition, S is weakly 5-fold stable then this homomorphism is actually an isomorphism (compare this to Theorem 5.1). In this section we will introduce the basic case when this assumption turns out to be true, although for now only for the case $n = 1$. Note that unlike Remark 6.6 there is an additional condition $\delta(I) \subset I^2$ that already turned out to be crucial in the case of Proposition 7.2.

Remark 7.3. Given that for any $k > 0$ there is an inclusion $\delta(I^k) \subset I^{k+1}$, it follows from Lemma 2.13 that for any natural k the quotient ring R/I^k admits a unique δ-structure compatible with the one on R.

Let S be an arbitrary δ-ring. Put

$$R_N = S[t]/(t^N)$$

and by I_N denote the nilpotent ideal $(\bar{t}) \subset R_N$ as in Subsection 5.2. Assume given a δ-structure on R_N that coincides with the given δ-structure on S and satisfies the condition $\delta(I_N) \subset I_N^2$. Note that giving such a δ-structure is equivalent to specifying an arbitrary element $\delta(\bar{t}^2) \in I^2$.

Theorem 7.4. *Suppose that S is a p-adically complete weakly 5-fold stable δ-ring with trivial p-torsion and let R_N and a δ-structure on R_N be as above. Then for any $N \in \mathbb{N}$ the homomorphism*

$$\mathrm{B}_\delta \ : \ {}^D\widehat{K}_2^M(R_N, I_N) \longrightarrow {}^D\widehat{\Omega}^1_{R_N, I_N}/d\, I_N$$

is an isomorphism.

7.3 Proof of Theorem 7.4

As in case of Theorem 5.1 the proof will be carried by induction on N. In order to prove the base case $N = 2$, we will need the following observation:

Let $R = S \oplus I$ be a split nilpotent extension of S as in first paragraph of subsection 7.2. Suppose that $N = 2$. Then, according to the properties of δ-structure the map $\delta : I \mapsto I$ is additive and for any $x \in I$ there is an equality $\varphi(x) = p\delta(x)$.

Moreover, since $N < p$ all numbers from 1 to N are invertible, so one can define a group homomorphism

$$\log : 1 + I \longrightarrow I\,, 1 + x \mapsto \sum_{i=1}^{N-1} (-1)^{i+1} \frac{x^i}{i}\,.$$

Lemma 7.5. *The homomorphism* $\log_\delta : 1 + I \to I$ *coincides with the composition* $(1 - \delta) \circ \log$.

Proof. The equality becomes trivial after multiplying it by p. Hence the proof directly follows from R having trivial p-torsion. □

The following proposition proves Theorem 7.4 for $N = 2$.

Proposition 7.6. *If S is weakly* 5*-fold stable then the homomorphism* B_δ : $K^M_{n+1}(R,I) \to {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}$ *coincides with the composition*

$$K^M_{n+1}(R,I) \xrightarrow{\mathrm{B}} \Omega^n_{R,I}/d\Omega^{n-1}_{R,I} \longrightarrow {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}\,, \tag{7.1}$$

where the first arrow is Bloch map B *and the second arrow is derived* p*-adic completion. In particular, homomorphism* B_δ : ${}^D\widehat{K}^M_{n+1}(R,I) \to {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}$ *is an isomorphism.*

Proof. Since all the conditions of Theorem 5.1 are met, the homomorphism B is well defined and is an isomorphism. Given that $I^2 = 0$, the restriction of δ on ideal I is zero. In particular, according to Lemma 7.5, there is an equality $\log_\delta|_{1+I} = \log|_{1+I}$. Now for any $x \in I_N$ and $r_1, \dots, r_n \in R^*_N$ applying B_δ to the element $\{1+x, r_1, \dots, r_n\}$ and using formula (6.7) one will have an equation

$$\mathrm{B}_\delta\big(\{1+x, r_1, \dots, r_n\}\big) =$$

$$= \Big[\log(1+x)d\log\big(\{r_1, \dots r_n\}\big) + (-1)^n\mathrm{B}_\delta\big(\{r_1, \dots, r_n\}\big)d\log(1+x) -$$

$$- \log(1+x)d\mathrm{B}_\delta\big(\{r_1, \dots r_n\}\big)\Big] = \Big[\log(1+x)d\log\big(\{r_1, \dots, r_n\}\big)\Big]\,,$$

where its right handside coincides with the image of $\mathrm{B}\big(\{1+x, r_1, \dots, r_n\}\big)$ in ${}^d\widehat{\Omega}^n_{R_N,I_N}/d\,{}^d\widehat{\Omega}^{n-1}_{R_N,I_N}$. According to Lemma 2.6 the elements of such type generate the group $K^M_{n+1}(R,I)$. This finishes the proof. □

Remark 7.7. Note that the statement of Proposition 7.6 is much more general than Theorem 7.4 for $N = 2$.

Now consider an arbitrary natural N. Denote by J_N the ideal in R_N generated by $\bar{t}^{N-1}$. The quotient R_N/J_N is naturally isomorphic to R_{N-1} with I_N/J_N being isomorphic to the ideal $I_{N-1} \in R_N$. Moreover, according to Remark 7.3 there is a unique δ-structure on R_{N-1} compatible with δ-structure on R_N, and it is also easy to see that $\delta I_{N-1} \subset I^2_{N-1}$. According to formula (2.1) there is an exact sequence

$$0 \longrightarrow K^M_n(R_N, J_N) \longrightarrow K^M_n(R_N, I_N) \longrightarrow K^M_n(R_{N-1}, I_{N-1}) \longrightarrow 0\,.$$

Moving to differential forms, according to formula (2.3) there is an exact sequence

$$\ldots \to H^0_{dR}(R_{N-1}, I_{N-1}) \to \Omega^1_{R_N,J_N}/dJ_N \to \Omega^1_{R_N,I_N}/dI_N \to \\ \Omega^1_{R_{N-1},I_{N-1}}/dI_{N-1} \to 0\,.$$

For any $M \in \mathbb{N}$ one has the following equalities:

$$\Omega^1_{R_M} = \Omega^1_{S[t]}/\big(t^M\Omega^1_{S[t]} + S[t]dt^M\big) = \\ = \bigoplus_{i=0}^{M-1}\Omega^1_S\cdot\bar{t}^i \oplus \bigoplus_{j=0}^{M-2} S\cdot\bar{t}^j d\bar{t} \oplus S/nS\cdot\bar{t}^{M-1}d\bar{t}\,,$$

$$\Omega^1_{R_M,I_M} = \bigoplus_{i=1}^{M-1}\Omega^1_S\cdot\bar{t}^i \oplus \bigoplus_{j=0}^{M-2} S\cdot\bar{t}^j d\bar{t} \oplus S/nS\cdot\bar{t}^{M-1}d\bar{t} = \\ = \bigoplus_{i=1}^{M-1}(\Omega^1_S\cdot\bar{t}^i + S\cdot\bar{t}^{i-1}d\bar{t}) \oplus S/nS\cdot\bar{t}^{M-1}d\bar{t}\,,$$

$$dI_M = \Big\{d\Big(\sum_{i=1}^{M-1} s_i\cdot\bar{t}^i\Big)\Big| s_i \in S\Big\} = \Big\{\sum_{i=1}^{M-1}(ds_i\cdot\bar{t}^i + is_i\cdot\bar{t}^{i-1}d\bar{t})\Big| s_i \in S\Big\}\,.$$

In particular, one has $H^0_{dR}(R_{N-1}, I_{N-1}) = \ker\big(I_{N-1} \to \Omega^1_{R_{N-1},I_{N-1}}\big) = 0$, which means that there is an exact sequence

$$0 \to \Omega^1_{R_N,J_N}/dJ_N \to \Omega^1_{R_N,I_N}/dI_N \to \Omega^1_{R_{N-1},I_{N-1}}/dI_{N-1} \to 0\,, \tag{7.2}$$

For simplicity, from now on we will be denoting the quotient $(\Omega^1_S \oplus S)/\{ds + is|s \in S\}$ by Ψ_i. Using prior equalities one can describe the elements of sequence (7.2) in the following way

$$\Omega^1_{R_N,I_N}/dI_N \simeq \bigoplus_{i=1}^{N-1}\Psi_i \oplus S/NS\,, \tag{7.3}$$

$$\Omega^1_{R_{N-1},I_{N-1}}/dI_{N-1} \simeq \bigoplus_{i=1}^{N-2}\Psi_i \oplus S/(N-1)S\,, \tag{7.4}$$

$$\Omega^1_{R_N,J_N}/dJ_N = \ker\big(\Omega^1_{R_N,I_N}/dI_N \to \Omega^1_{R_{N-1},I_{N-1}}/dI_{N-1}\big) = \\ = (\Omega^1_S \oplus (N-1)S)/\{ds + (N-1)s|s \in S\} \oplus S/NS \simeq \Psi_1 \oplus S/NS\,. \tag{7.5}$$

Remark 7.8. Note that there is an exact sequence

$$0 \to \Omega^1_S \to (\Omega^1_S \oplus S)/\{ds + is|s \in S\} \to S/iS \to 0\,, \tag{7.6}$$

where the second arrow sends an element $w \in \Omega^1_S$ to the class $\overline{(w,0)}$ and the third arrow sends a class $\overline{(w,s)}$ to $\overline{s} \in S/iS$. In particular if $i \in S$ is invertible then there is an isomorphism $\Psi_i \simeq \Omega^1_S$.

Lemma 7.9. *Derived p-adic completion* ${}^D\widehat{\Psi}_i$ *of the group* Ψ_i *is isomorphic to the quotient group* $({}^D\widehat{\Omega}^1_S \oplus S)/\{ds + is | s \in S\}$. *In particular, the sequence* (7.6) *stays exact after derived p-adic completion of its elements.*

Proof. Given that Ψ_n is a cokernel of the map $S \to \Omega^1_S \oplus S$ that sends $s \in S$ to a pair (ds, is) and the facts that derived p-adic completion is right exact and S is already p-adically complete one obtains the first statement. Now it is enough to prove that the induced map ${}^D\widehat{\Omega}^1_S \to {}^D\widehat{\Psi}_i$ is still a monomorphism. But the map

$$ {}^D\widehat{\Omega}^1 \to ({}^D\widehat{\Omega}^1_S \oplus S)/\{ds + is | s \in S\}\,,\ w \mapsto \overline{(w,0)} $$

is obviously a monomorphism. □

Corollary 7.10. *The sequence* (7.2) *stays exact after its elements being derived p-adic completed.*

Proof. Decompose the sequence (7.2) into direct sum of three components

$$
\begin{array}{c}
0 \to S/NS \xrightarrow{id} S/NS \longrightarrow 0 \longrightarrow 0 \\
\oplus \\
0 \to 0 \longrightarrow \bigoplus_{i=1}^{N-2} \Psi_i \xrightarrow{id} \bigoplus_{i=1}^{N-2} \Psi_i \to 0 \\
\oplus \\
0 \to \Omega^1_S \to (\Omega^1_S \oplus S)/\{ds + (N-1)s | s \in S\} \to S/(N-1)S \to 0\,.
\end{array}
$$

The first and the second sequences will obviously stay exact after derived p-adic completion. For the third secuence it follows from lemma 7.3. □

Remark 7.11. Note that derived p-adically completion of sequence (7.2) will be equal to

$$ 0 \to {}^D\widehat{\Omega}^1_{R_N,J_N}/dJ_N \to {}^D\widehat{\Omega}^1_{R_N,I_N}/dI_N \to {}^D\widehat{\Omega}^1_{R_{N-1},I_{N-1}}/dI_{N-1} \to 0 \qquad (7.7) $$

because of derived p-adic completion being right exact and groups I_N, J_N and I_{N-1} being p-adically complete.

Thus there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & {}^D\widehat{K}_2^M(R_N,J_N) & \longrightarrow & {}^D\widehat{K}_2^M(R_N,I_N) & \longrightarrow & {}^D\widehat{K}_2^M(R_{N-1},I_{N-1}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \widetilde{\mathrm{B}}_\delta} & & \Big\downarrow{\scriptstyle \mathrm{B}_\delta} & & \Big\downarrow{\scriptstyle \mathrm{B}_\delta} & & \\
0 & \longrightarrow & {}^D\widehat{\Omega}^1_{R_N,J_N}/dJ_N & \longrightarrow & {}^D\widehat{\Omega}^1_{R_N,I_N}/dI_N & \longrightarrow & {}^D\widehat{\Omega}^1_{R_{N-1},I_{N-1}}/dI_{N-1} & \longrightarrow & 0
\end{array}$$

where the homomorphism $\widetilde{\mathrm{B}}_\delta$ is induced by exactness of the second line.

Lemma 7.12. *The group $K_2^M(R_N,J_N)$ is generated by set of elements $\langle\{1+a\bar{t}^{N-1},b\}|a\in S,b\in S^*\rangle$ together with the set of elements $\langle\{1+a\bar{t}^{N-1},1+b\bar{t}\}|a,b\in S\rangle$*

Proof. Let $j\geqslant 2$. Then $N-1+j>N$ and according to the Corollary 3.3(i), it means that for any a, $b\in S$ there is equality $\{1+a\bar{t}^N,1+b\bar{t}^j\}=0$ in $K_2^M(R_N)$. Since S is weakly 5-fold stable, this together with Lemma 2.6 concludes the proof. □

Lemma 7.13. *For any a, $b\in S$ there is an equality*

$$\{1+at^{N-1},1+bt\}=\{1+t^{N-1},1+abt\}\in K_2^M(R_N)\,.$$

Proof. Since S is weakly 2-fold stable it is enough to consider a case when a and b are invertible. Suppose that $N|p$ then $N+1$ is invertible in S and according to Proposition 3.2 one has equalities

$$K_2^M(S[[t]])/V_{N+1}=K_2^M(S[[t]])/W_{N+1}=K_2^M(R_{N+1})\,.$$

Hence, according to Corollary 3.9 one has an equality

$$\{1+a\bar{t}^{N-1},1+b\bar{t}\}-\{1+\bar{t}^{N-1},1+ab\bar{t}\}=-\phi_N(bda)\in K_2^M(R_{N+1})\,,$$

where the right handside is equal to the symbol $-\{1+abt^N,a\}$. Applying natural homomorphism $R_{N+1}\twoheadrightarrow R_N$ one obtains the required equality.

Now if $N\nmid p$ then $V_N=W_N$ and according to Corollary 3.3 both handsides of the equality are equal to zero. □

According to the Remark 7.8 and Lemma 7.3 one has ${}^D\widehat{\Omega}^1_{R_N,J_N}/dJ_N\simeq {}^D\widehat{\Omega}^1_S\oplus S/NS$. There is a homomorphism from ${}^D\widehat{\Omega}^1_S$ to ${}^D\widehat{K}^M_{n+1}(R_N,J_N)$ that is defined by the composition

$$\Omega^1_{R_N}\longrightarrow K_2^M(R_2,I_2)\longrightarrow K_2^M(R_N,J_N)\,,$$

where the first map is inverse to the well-known isomorphism $K_2^M(R_2, I_2) \xrightarrow{\sim} \Omega^1_S$ and the second map is defined by a homomorphism $R_2 \to R_N$, $\bar{t} \mapsto \bar{t}^{N-1}$. There is also a homomorphism from S/NS to ${}^D\widehat{K}^M_{n+1}(R_N, J_N)$ that sends $\overline{s} \in S/NS$ to the image of $\{1+\bar{t}^{N-1}, 1+s\bar{t}\}$ in ${}^D\widehat{K}^M_2(R_N, J_N)$. Indeed suppose that $s = Nc$ for some $c \in S$. Since S is uniquely 2-fold stable one can assume that c is invertible. Then according to the Proposition 3.7 one have the equality

$$\{1+\bar{t}^{N-1}, 1+Nc\bar{t}\} = (N-1+1)\{1+\bar{t}^{N-1}, 1+c\bar{t}\} =$$
$$= \phi_N((N-1)dc) \in K_2^M(S[[t]])/V_{N+1}\,.$$

According to the formula (3.5) the right handside of this equality is equal to the element $[\{1+(N-1)ct^N, c\}] \in K_2^M(S[[t]])/V_{N+1}$. Since either N or $N+1$ are invertible in S, according to the Proposition 3.2 either $V_N = W_N$ or $V_{N+1} = W_{N+1}$. Hence one has an inclusion $V_{N+1} \subset W_N$ and the corresponding homomorphism $K_2^M(S[[t]])/V_{N+1} \twoheadrightarrow K_2^M(S[[t]])/W_N \simeq K_2^M(R_N)$ obviously sends $[\{1+(N-1)ct^N, c\}]$ to zero.

Combining these two homomorphisms one obtains homomorphism

$$\Phi : {}^D\widehat{\Omega}^1_{R_N,J_N}/dJ_N \simeq {}^D\widehat{\Omega}^1_S \oplus S/NS \longrightarrow {}^D\widehat{K}^M_{n+1}(R_N, J_N)\,.$$

Proposition 7.14. *Homomorphisms* Φ *and* $\widetilde{\mathrm{B}}_\delta$ *are mutually inverse. In particular* $\widetilde{\mathrm{B}}_\delta$ *is an isomorphism.*

Proof. Let us first check that $\widetilde{\mathrm{B}}_\delta \circ \Phi = 1$ on ${}^D\widehat{\Omega}^1_S \oplus S/NS$. For the component ${}^D\widehat{\Omega}^1_S$ it follows from the equality (3.5) and the commutativity of the diagram

$$\begin{array}{ccc} {}^D\widehat{K}^M_2\big(R_2, I_2\big) & \xrightarrow{\mathrm{B}_\delta = {}^D\widehat{\mathrm{B}}} & {}^D\widehat{\Omega}^1_{R_2,I_2}/d\,I_2 \simeq {}^D\widehat{\Omega}^1_S \\ \Big\downarrow{\scriptstyle \bar{t}\to\bar{t}^{N-1}} & & \Big\downarrow{\scriptstyle \bar{t}\to\bar{t}^{N-1}} \\ {}^D\widehat{K}^M_2\big(R_N, J_N\big) & \xrightarrow{\widetilde{\mathrm{B}}_\delta} & {}^D\widehat{\Omega}^1_{R_N,J_N}/d\,J_N \end{array}$$

where the top horizontal map B_δ coincide with the derived p-adic completion of the Bloch map B and is an isomorphism according to the Proposition 7.6.

For the second component, according to formula (6.9) for any a, $b \in S$ we have

$$\mathrm{B}_\delta\big(\{1+a\bar{t}^{N-1}, 1+b\bar{t}\}\big) =$$
$$= \Big[\log_\delta(1+a\bar{t}^{N-1})\, d\log(1+b\bar{t}) - \log_\delta(1+b\bar{t})\, d\log(1+a\bar{t}^{N-1}) -$$

$$- \log_\delta(1 + a\bar{t}^{N-1})\, d\log_\delta(1 + b\bar{t})\Big]$$

Given the fact that $\delta(a\bar{t}^{N-1}) = 0$ one can easily see that $\log_\delta(1 + a\bar{t}^{N-1}) = a\bar{t}^{N-1}$ and $d\log(1 + a\bar{t}^{N-1}) = d\log_\delta(1 + a\bar{t}^{N-1})$. Therefore there is an equality

$$\mathrm{B}_\delta\big(\{1 + a\bar{t}^{N-1}, 1 + b\bar{t}\}\big) = \Big[\log_\delta(1 + a\bar{t}^{N-1})\, d\log(1 + b\bar{t})\Big].$$

from which it easily follows that the image of $\{1 + a\bar{t}^{N-1}, 1 + b\bar{t}\}$ in ${}^D\widehat{\Omega}^1_{R_N,J_N}/dJ_N \simeq {}^D\widehat{\Omega}^1_S \oplus S/NS$ coincides with a pair $(0, \overline{ab})$. This concludes the first checking.

Let us now check that $\Phi \circ \widetilde{\mathrm{B}}_\delta = 1$ on ${}^D\widehat{K}^M_{n+1}(R_N, J_N)$. According to Lemma 7.12 it is enough to check this equality on elements of the type $\{1 + a\bar{t}^{N-1}, b\}$ where $a \in S$, $b \in S^*$ and on elements of the type $\{1 + a\bar{t}^{N-1}, 1 + b\bar{t}\}$ where a, $b \in S$. For the elements of the first type the proof again follows from the commutativity of the above diagram. For the elements of the second type one has an equality

$$\Phi \circ \widetilde{\mathrm{B}}_\delta\big(\{1 + a\bar{t}^{N-1}, 1 + b\bar{t}\}\big) = \{1 + \bar{t}^{N-1}, 1 + ab\bar{t}\},$$

for any a, $b \in S$. Therefore the proof follows from Corollary 3.9. □

We are now ready to prove Theorem 7.4.

The proof of Theorem 7.4. The proof is by induction on N. The base case is $N = 2$ has already been proven. For an arbitrary N consider the commutative diagram on the page 76. One has right vertical map being an isomorphism because of an induction hypothesis and the left vertical map being an isomorphism because of the Proposition 7.14. Therefore the middle vertical map is an isomorphism as well. □

References

[1] D. Arapura, S.-J. Kang, "Kähler–de Rham cohomology and Chern classes", *Comm. Algebra*, **39**:4, 1153–1167, (2011)

[2] B. Bhatt, P. Scholze, "Prisms and prismatic cohomology", preprint (2019) arXiv:1905.08229.

[3] S. Bloch "On the tangent space to Quillen K-theory", *Lecture Notes in Math.*, **341**, 205–210, (1973)

[4] S. Bloch, "K_2 of Artinian $\mathbb{Q}$-algebras, with application to algebraic cycles", *Comm. Algebra*, **3** (1975), 405–428.

[5] S. Bloch, "Higher regulators, algebraic K-theory, and zeta functions of elliptic curves", *CRM Monograph Series*, **11**, American Mathematical Society (2000).

[6] S. Bloch, "Applications of the dilogarithm function in algebraic K-theory and algebraic geometry", *Proceedings of the International Symposium on Algebraic Geometry (Kyoto, 1977)*, (M. Nagata, ed.), Kinokuniya, Tokyo (1978), 103–114.

[7] A. Buium, "Arithmetic analogues of derivations", *Journal of Algebra*, **198** (1997), 290–299.

[8] T. G. Goodwillie, "Relative algebraic K-theory and cyclic homology", *Ann. of Math. (2)*, **124**:2, 347–402, (1986)

[9] S. O. Gorchinskiy, D. V. Osipov, "Tangent space to Milnor K-groups of rings", *Proceedings of the Steklov Institute of Mathematics*, **290**, 26–34 (2015)

[10] S. O. Gorchinskiy, D. N. Tyurin, "Relative Milnor K-groups and differential forms of split nilpotent extensions", *Izv. Math.*, **82**:5 (2018), 880–913.

[11] A. B. Goncharov, "Geometry of configurations, polylogarithms, and motivic cohomology", *Adv. Math.*, **114**:2, 197–318, (1995)

[12] H. Grauert, H. Kerner, "Deformationen von Singularitäten komplexer Räume", *ath. Ann.*, **153**, 236–260, (1964)

[13] T. tom Dieck, "The Artin–Hasse logarithm for λ-rings", *Algebraic topology (Arcata, CA, 1986), Lecture Notes in Math.*, **1370** (1989), 409–415.

[14] B. Dribus, "A Goodwillie-type theorem for Milnor K-Theory", preprint arXiv:1402.2222.

[15] H.-J. Reiffen, "Das Lemma von Poincaré für holomorphe Differentialformen auf komplexen Räumen", *Math. Z.*, **101**, 269–284, (1967)

[16] M. Rossi, L. Terracini, "Maple subroutines for computing Milnor and Tyurina numbers of hypersurface singularities with application to Arnol'd adjacencies", *Rend. Semin. Mat. Univ. Politec. Torino*, **73**:3-4, 269–316, (2015)

[17] A. Joyal, "δ-anneaux et vecteurs de Witt", *C. R. Math. Rep. Acad. Sci. Canada*, **7**:3 (1985), 177–182.

[18] W. van der Kallen, "The K_2 of rings with many units", *Ann. Sci. École Norm. Sup. (4)*, **10**:4 (1977), 473–515.

[19] K. Kato, "On p-adic vanishing cycles", *Adv. Stud. Pure Math.*, **10** (1987), 207–251.

[20] K. Kato, "The explicit reciprocity law and the cohomology of Fontaine–Messing", *Bulletin de la S.M.F.*, **119**:4 (1991), 397–441.

[21] M. Kerz, "The Gersten conjecture for Milnor K-theory", *Invent. Math.*, **175**:1 (2009), 1–33.

[22] H. Maazen, J. Stienstra, "A presentation for K_2 of split radical pairs", *J. Pure Appl. Algebra*, **10**:3 (1977), 271–294.

[23] B. Malgrange, "Intégrales asymptotiques et monodromie",*Ann. Sci. École Norm. Sup. (4)*, **7**, 405–430, (1974),

[24] H. Matsumura,"Commutative ring theory",*Cambridge Studies in Advanced Mathematics***8** (1989)

[25] J. Milnor, "Singular points of complex hypersurfaces", *Annals of Mathematics Studies, No. 61*, Princeton University Press, University of Tokyo Press, (1968)

[26] D. Mond, "From the index of a differential operator to the Milnor number of a singularity", *Differential algebra, complex analysis and orthogonal polynomials, Contemp. Math.*, **509**, 129–141, (2010),

[27] M. Morrow, "K_2 of localisations of local rings", *J. Algebra*, **399** (2014), 190–204.

[28] Yu. P. Nesterenko, A. A. Suslin, “Homology of the general linear group over a local ring, and Milnor’s K-theory”, *Math. USSR-Izv.*, **34**:1 (1990), 121–146.

[29] V. P. Palamodov, “Multiplicity of holomorphic mappings”, *Funct. Anal. Its Appl.*, **1**:3, 218–226, (1967)

[30] G. N. Tjurina, “Locally semi-universal flat deformations of isolated singularities of complex spaces”, *Izv. Akad. Nauk SSSR Ser. Mat.*, **33**:5, 967–999 (1969)

[31] Lê Dũng Tráng, “Calculation of Milnor number of isolated singularity of complete intersection”, *Funct. Anal. Its Appl.*, **8**:2, 127–131, (1974)

[32] G.-M. Greuel, “Der Gauss–Manin–Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten”, *Math. Ann.*, **214**, 235–266, (1975)

[33] K. Saito, “Quasihomogene isolierte Singularitäten von Hyperflächen” *Invent. Math.*, **14**, 123–142, (1971)

[34] S. V. Vostokov, “An explicit form of the reciprocity law”, *Math. USSR-Izv.*, **13**:3 (1979), 557–588.