The Top Homology Group of the Genus 3 Torelli Group

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Abstract

The Torelli group of a genus g oriented surface Σ_g is the subgroup $\mathcal{I}_g$ of the mapping class group $\mathrm{Mod}(\Sigma_g)$ consisting of all mapping classes that act trivially on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$. The quotient group $\mathrm{Mod}(\Sigma_g)/\mathcal{I}_g$ is isomorphic to the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. The cohomological dimension of the group $\mathcal{I}_g$ equals to $3g-5$. The main goal of the present paper is to compute the top homology group of the Torelli group in the case $g = 3$ as $\mathrm{Sp}(6, \mathbb{Z})$-module. We prove an isomorphism

$$\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z}) \cong \mathrm{Ind}_{S_3 \ltimes \mathrm{SL}(2,\mathbb{Z})^{\times 3}}^{\mathrm{Sp}(6,\mathbb{Z})} \mathcal{Z},$$

where $\mathcal{Z}$ is the quotient of $\mathbb{Z}^3$ by its diagonal subgroup $\mathbb{Z}$ with the natural action of the permutation group S_3 (the action of $\mathrm{SL}(2, \mathbb{Z})^{\times 3}$ is trivial). We also construct an explicit set of generators and relations for the group $\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z})$.

Contents

1 Introduction

Let Σ_g be a compact oriented genus g surface. Let $\mathrm{Mod}(\Sigma_g)$ be the *mapping class group* of Σ_g, defined by $\mathrm{Mod}(\Sigma_g) = \pi_0(\mathrm{Homeo}^+(\Sigma_g))$, where $\mathrm{Homeo}^+(\Sigma_g)$ is the group of orientation-preserving homeomorphisms of Σ_g. The group $\mathrm{Mod}(\Sigma_g)$ acts on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$ and preserves the algebraic intersection form, so we have a representation $\mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$, which is well-known to be surjective. The kernel $\mathcal{I}_g$ of this representation is known as the *Torelli group*. This can be written as the short exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

The *extended Torelli group* $\hat{\mathcal{I}}_g$ is the preimage of the center $\{\pm 1\}$ of $\mathrm{Sp}(2g, \mathbb{Z})$. So we have the extension

$$1 \to \mathcal{I}_g \to \hat{\mathcal{I}}_g \to \{\pm 1\} \to 1.$$

If $g = 1$ we have $\mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}(2, \mathbb{Z})$, so the group $\mathcal{I}_1$ is trivial. McCullough and Miller [11] proved, that the group $\mathcal{I}_2$ is not finitely generated. Mess [12] showed, that in fact $\mathcal{I}_2$ is an infinitely generated free group. Johnson [8] proved, that $\mathcal{I}_g$ is finitely generated if $g \geq 3$. However, it is an open problem if the group $\mathcal{I}_g$ is finitely presented for some $g \geq 3$.

The Torelli groups have one more interpretation. Consider the *Torelli space* $\mathcal{T}_g$, i.e. the module space of smooth complex curves with fixed symplectic basis in the first homology. It is well known that $\mathcal{T}_g$ is of type $K(\mathcal{I}_g, 1)$, hence we have $\mathrm{H}^*(\mathcal{I}_g, \mathbb{Z}) \cong \mathrm{H}^*(\mathcal{T}_g, \mathbb{Z})$. As the consequence, we have that $\mathrm{H}^*(\mathcal{I}_g, \mathbb{Z})$ is precisely the set of characteristic classes of homologically trivial surface bundles.

The natural problem is to study (co)homology of the groups $\mathcal{I}_g$ in the case $g \geq 3$. Bestvina, Bux and Margalit [1] in 2007 constructed the contractible complex of cycles $\mathcal{B}_g$, on which the Torelli group $\mathcal{I}_g$ acts cellularly. Using the spectral sequence associated with this action, they showed, that the group $\mathcal{I}_g$ has cohomological dimension $3g - 5$ and that the top homology group $\mathrm{H}_{3g-5}(\mathcal{I}_g, \mathbb{Z})$ is not finitely generated. Gaifullin [3] in 2019 proved that for $2g-3 \leq k \leq 3g-5$ the homology group $\mathrm{H}_k(\mathcal{I}_g, \mathbb{Z})$ contains a free abelian subgroup of an infinite rank.

Let us fix a complex structure of Σ_g so that it becomes a hyperelliptic smooth complex curve. Denote by $\iota \in \mathrm{Mod}(\Sigma_g)$ the corresponding *hyperelliptic involution*. We have $\iota^2 = \mathrm{id}$ and ι acts on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$ as (-1). By definition ι is an element of $\hat{\mathcal{I}}_g$, not belonging to $\mathcal{I}_g$. Since $\hat{\mathcal{I}}_g/\mathcal{I}_g \cong \mathbb{Z}/2\mathbb{Z}$, there is the natural action of the group $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{H}_*(\mathcal{I}_g, \mathbb{Z})$. This action coincides with the action of the hyperelliptic involution and does not depend on the choice of (hyperelliptic) complex structure on Σ_g. If 2 is invertible in the coefficient ring R, then we obtain the splitting

$$\mathrm{H}_*(\mathcal{I}_g, R) = \mathrm{H}_*^+(\mathcal{I}_g, R) \oplus \mathrm{H}_*^-(\mathcal{I}_g, R),$$

where a hyperelliptic involution acts trivially on $\mathrm{H}_*^+(\mathcal{I}_g, R)$ and acts as (-1) on $\mathrm{H}_*^-(\mathcal{I}_g, R)$.

In the case $g = 3$ there are some special results on the structure of $H_*(\mathcal{I}_3, \mathbb{Z})$. Hain [9] computed explicitly the groups $\mathrm{H}_*^+(\mathcal{I}_3, \mathbb{Z}[1/2])$ as $\mathrm{Sp}(6, \mathbb{Z})$-modules. In particular, he proved that

$$\mathrm{H}_4^+(\mathcal{I}_3, \mathbb{Z}[1/2]) \cong \mathrm{Ind}_{S_3 \ltimes \mathrm{SL}(2,\mathbb{Z})^{\times 3}}^{\mathrm{Sp}(6,\mathbb{Z})} \mathcal{Z} \otimes \mathbb{Z}[1/2], \tag{1}$$

where $\mathcal{Z}$ is the quotient of $\mathbb{Z}^3$ by its diagonal subgroup with the natural action of the permutation group S_3 (the action of $\mathrm{SL}(2, \mathbb{Z})^{\times 3}$ is trivial). Hain's approach was to use stratified Morse theory for the image of the period map $\mathcal{T}_3 \to \mathfrak{h}_3$, where $\mathfrak{h}_3$ is the upper Siegel half-space.

In the present paper we study the structure of the whole group $H_4(\mathcal{I}_3, \mathbb{Z})$ as $\mathrm{Sp}(6, \mathbb{Z})$-module. The main result is as follows.

Theorem 1.1. *There is an isomorphism of* $\mathrm{Sp}(6, \mathbb{Z})$*-modules*

$$\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z}) = \mathrm{H}_4(\mathcal{T}_3, \mathbb{Z}) \cong \mathrm{Ind}_{S_3 \ltimes \mathrm{SL}(2,\mathbb{Z})^{\times 3}}^{\mathrm{Sp}(6,\mathbb{Z})} \mathcal{Z},$$

where $\mathcal{Z}$ is the quotient of $\mathbb{Z}^3$ by its diagonal subgroup $\mathbb{Z}$ with the natural action of the permutation group S_3 (the action of $\mathrm{SL}(2, \mathbb{Z})^{\times 3}$ *is trivial).*

Corollary 1.2. *The hyperelliptic involution acts trivially on* $\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z})$*. In particular, we have* $\mathrm{H}_4^-(\mathcal{I}_3, \mathbb{Z}[1/2]) = 0$.

The explicit construction of the generators and relations for the group $\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z})$ is given in Section 3.

Let us remark that Theorem 1.1 agrees with Hain's results. Our main approach is based on the spectral sequence for the action of the Torelli group on the complex of cycles. The proof is independent on Hain's results.

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2 Preliminaries

2.1 The Torelli group of a surface with puncture

Let Σ be an oriented surface, possibly with punctures and boundary components. We do not assume that Σ is connected. However, we require $\mathrm{H}_*(\Sigma, \mathbb{Q})$ be a finite dimensional vector space. The mapping class group of Σ is defined as $\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Homeo}^+(\Sigma, \partial\Sigma))$, where $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ is the group of orientation-preserving homeomorphisms of Σ that act as the identity on $\partial\Sigma$ and preserve the connected components of Σ. By $\mathrm{PMod}(\Sigma) \subseteq \mathrm{Mod}(\Sigma)$ we denote the *pure mapping class group* of Σ, i.e. the subgroup that acts trivially on punctures. We have the exact sequence

$$1 \to \mathrm{PMod}(\Sigma_{g,n}^b) \to \mathrm{Mod}(\Sigma_{g,n}^b) \to S_n \to 1, \tag{2}$$

where by $\Sigma_{g,n}^b$ we denote the connected genus g surface with n punctures and b boundary components. In the case $\Sigma = \Sigma_{g,1}$ we can also define the Torelli group $\mathcal{I}_{g,1}$ as the kernel of the action of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathrm{H}_1(\Sigma_{g,1}, \mathbb{Z})$.

2.2 Hochschild-Serre spectral sequence

Given a short exact sequence of groups

$$1 \to Q \to G \to P \to 1$$

there is an associated Hochschild-Serre spectral sequence. The second page is given by

$$E_{p,q}^2 \cong \mathrm{H}_p(P, H_q(Q, \mathbb{Z})) \Rightarrow \mathrm{H}_{p+q}(G, \mathbb{Z}), \tag{3}$$

where the coefficients are local: P acts on Q by conjugations. The group $\bigoplus_{p+q=n} E_{p,q}^\infty$ is the adjoint graded group for certain filtration in the homology group $\mathrm{H}_{p+q}(G, \mathbb{Z})$.

The following fact immediately follow from the existence of the Hochschild-Serre spectral sequence.

Fact 2.1. Consider a short exact sequence of groups

$$1 \to Q \to G \to P \to 1$$

with $\mathrm{cd}(P) = p < \infty$ and $\mathrm{cd}(Q) = q < \infty$. Then

$$\mathrm{H}_{p+q}(G, \mathbb{Z}) \cong \mathrm{H}_p(P, H_q(Q, \mathbb{Z})),$$

where the coefficients are local: P acts on Q by conjugations.

Recall that for n pairwise commuting elements $h_1, \ldots, h_n$ of the group G one can construct an *abelian cycle* $\mathcal{A}(h_1, \ldots, h_n) \in \mathrm{H}_n(G, \mathbb{Z})$ defined by in the following way. Consider the homomorphism $\phi : \mathbb{Z}^n \to G$ that maps the generator of the i-th factor to the h_i. Then $\mathcal{A}(h_1, \ldots, h_n) = \phi_*(\mu_n)$, where μ_n is the standard generator of $\mathrm{H}_n(\mathbb{Z}^n, \mathbb{Z})$. Fact 2.1 implies the following results.

Fact 2.2. Consider a central extension

$$1 \to \mathbb{Z} \to G \to P \to 1$$

with $\mathrm{cd}(P) = p < \infty$ and $\mathbb{Z} = \langle a \rangle$. Then

$$\mathrm{H}_p(P, \mathbb{Z}) \cong \mathrm{H}_{p+1}(G, \mathbb{Z}). \tag{4}$$

Moreover, for p pairwise commuting elements $h_1, \ldots, h_p$ of the group P, the isomorphism (4) maps the abelian cycle $\mathcal{A}(h_1, \ldots, h_p)$ to the abelian cycle $\mathcal{A}(h_1, \ldots, h_p, a)$.

Fact 2.3. Consider a short exact sequence

$$1 \longrightarrow R_1 \xrightarrow{\iota} G \xrightarrow{p} R_2 \longrightarrow 1$$

where R_1 and R_2 are free groups. Assume that the action of R_2 on $\mathrm{H}_*(R_1,\mathbb{Z})$ is trivial. Then

$$\mathrm{H}_2(G,\mathbb{Z}) \cong \mathrm{H}_1(R_2,\mathrm{H}_1(R_1,\mathbb{Z})) \cong \mathrm{H}_1(R_2,\mathbb{Z}) \otimes \mathrm{H}_1(R_1,\mathbb{Z}). \tag{5}$$

Let $f \in R_1$ and $g \in G$ such that $[\iota(f), g] = 1$. Then the isomorphism (5) maps the abelian cycle $\mathcal{A}(g,\iota(f))$ to $[p(g)] \otimes [f] \in \mathrm{H}_1(R_2,\mathbb{Z}) \otimes \mathrm{H}_1(R_1,\mathbb{Z})$.

2.3 Complex of cycles

Bestvina, Bux, and Margalit [1] constructed a contractible CW-complex $\mathcal{B}_g$ called *complex of cycles* on which the Torelli group acts without rotations. "Without rotations" means that if an element $h \in \mathcal{I}_g$ stabilizes a cell σ, then h stabilizes σ pointwise. Let us recall the construction of $\mathcal{B}_g$. More details can be found in [3] and [5].

Let us denote by $\mathcal{C}$ the set of all isotopy classes of oriented non-separating simple closed curves on Σ_g. Fix any element $0 \neq x \in \mathrm{H}_1(\Sigma_g,\mathbb{Z})$. The construction of $\mathcal{B}_g = \mathcal{B}_g(x)$ depends on the choice of the homology class x, however the CW-complexes $\mathcal{B}_g(x)$ are pairwise homeomorphic for different x.

Basis 1-cycle for the homology class x is a formal linear combination $\gamma = \sum_i^n k_i\gamma_i$ where $\gamma_i \in \mathcal{C}$ and $k_i \in \mathbb{N}$ satisfying the following properties:

(1) the homology classes $[\gamma_1], \dots, [\gamma_n]$ are linearly independent,

(2) $\sum_i^n k_i[\gamma_i] = x$,

(3) we can choose pairwise disjoint representatives of the isotopy classes $\gamma_1, \dots, \gamma_n$.

The multicurve $\gamma_1 \cup \dots \cup \gamma_n$ is called the *support* of γ.

Let us denote by $\mathcal{M}(x)$ the set of oriented multicurves $M = \gamma_1 \cup \dots \cup \gamma_s$ (for arbitrary s) satisfying the following properties:

(i) no nontrivial linear combination of the homology classes $[\gamma_1], \dots, [\gamma_s]$ with nonnegative coefficients equals zero,

(ii) for each $1 \leq i \leq s$ there exists a basis 1-cycle for x supported in M and containing γ_i.

For each $M \in \mathcal{M}(x)$ let us denote by $P_M \subset \mathbb{R}_+^{\mathcal{C}}$ the convex hull of the basis 1-cycles supported in M. Obviously P_M is a convex polytope. By definition complex of cycles is the regular CW-complex given by $\mathcal{B}_g(x) = \cup_{M \in \mathcal{M}(x)} P_M$.

1-cycle for the homology class $x \in \mathrm{H}_1(\Sigma_g,\mathbb{Z})$ is a formal linear combination $\gamma = \sum_i^n k_i\gamma_i$ where $\gamma_i \in \mathcal{C}$ and $k_i \in \mathbb{R}_+$ satisfying the properties (2) and (3). The multicurve $\gamma_1 \cup \dots \cup \gamma_n$ is called the *support* of γ. By definition the set of 1-cycles for x is precisely the set points of $\mathcal{B}_g(x)$. Therefore, an oriented multicurve M belongs to $\mathcal{M}(x)$ if and only if it is the support of some 1-cycle γ for x. Moreover, for each $M \in \mathcal{M}(x)$, the set of vertices of P_M is precisely the set of basis 1-cycles for x supported in M. Bestvina, Bux, and Margalit [1, Lemma 2.1] showed that

$$\dim P_M = |M| - \mathrm{rk} M = |\Sigma_g \setminus M| - 1, \tag{6}$$

where $|M|$ is the number of components of M, $|\Sigma_g \setminus M|$ is the number of connected components of $|\Sigma_g \setminus M|$, and $\mathrm{rk} M$ is the rank of the subgroup of $\mathrm{H}_1(\Sigma_g,\mathbb{Z})$ spanned by the homology classes of the components of M. Consequently, we have $\dim \mathcal{B}_g = 2g-3$. By $\mathcal{M}_p(x) \subseteq \mathcal{M}(x)$ we denote the set of multicurves corresponding to the cells of dimension p.

Theorem 2.4. *[1, Theorem E] Let $g \geq 2$ and $0 \neq x \in \mathrm{H}_1(\Sigma_g, \mathbb{Z})$. Then $\mathcal{B}_g(x)$ is contractible.*

2.4 One more spectral sequence

Suppose that a group G acts cellularly without rotations on a contractible CW-complex X. Let $C_*(X,\mathbb{Z})$ be the cellular chain complex of X and $\mathcal{R}_*$ be a projective resolution for $\mathbb{Z}$ over $\mathbb{Z}G$. Consider the double complex $B_{p,q} = C_p(X,\mathbb{Z}) \otimes_G \mathcal{R}_q$ with the filtration by columns. The corresponding spectral sequence (see (7.7) in [2, Section VII.7]) has the form

$$E^1_{p,q} \cong \bigoplus_{\sigma \in \mathcal{X}_p} \mathrm{H}_q(\mathrm{Stab}_G(\sigma)) \Rightarrow \mathrm{H}_{p+q}(G,\mathbb{Z}), \qquad (7)$$

where $\mathcal{X}_p$ is a set containing one representative from each G-orbit of p-cells of X. The group $\bigoplus_{p+q=n} E^\infty_{p,q}$ is the adjoint graded group for certain filtration in the homology group $H_{p+q}(G,\mathbb{Z})$. Note that for an arbitrary CW-complex X the spectral sequence (7) converges to the equivariant homology $\mathrm{H}^G_{p+q}(X,\mathbb{Z})$. So for a contractable CW-complex X we have $\mathrm{H}^G_{p+q}(X,\mathbb{Z}) \cong \mathrm{H}_{p+q}(G,\mathbb{Z})$.

Now let $E^*_{*,*}$ be the spectral sequence (7) for the action on $\mathcal{I}_g$ on $\mathcal{B}_g(x)$ for some primitive element $0 \neq x \in \mathrm{H}_1(\Sigma_g,\mathbb{Z})$. The fact that $\mathcal{I}_g$ acts on $\mathcal{B}_g(x)$ without rotations follows from the result of Ivanov [7, Theorem 1.2]: if an element $h \in \mathcal{I}_g$ stabilises some multicurve M the h stabilises each component of M. We have the spectral sequence

$$E^1_{p,q} \cong \bigoplus_{M \in \mathcal{M}_p(x)/\mathcal{I}_g} \mathrm{H}_q(\mathrm{Stab}_{\mathcal{I}_g}(M)) \Rightarrow \mathrm{H}_{p+q}(\mathcal{I}_g,\mathbb{Z}). \qquad (8)$$

(For a group G acting on a set X we denote by X/G any set containing one representative from each G-orbit in the set X.) Let us introduce some notation. Let $P \subseteq \mathcal{B}_g(x)$ be a p-cell and $h \in \mathrm{H}_q(\mathcal{I}_g,\mathbb{Z})$ be a homology class. By $P \otimes h \in E^1_{p,q}$ we denote the element that maps to $h \in \mathrm{H}_q(\mathrm{Stab}_{\mathcal{I}_g}(P),\mathbb{Z})$ under the isomorphism (8). This notation is convenient because the term $E^0_{p,q}$ is defined as the tensor product. The differential $d^1_{p,q}$ has the form

$$d^1_{p,q}(P \otimes h) = \partial P \otimes h \in E^1_{p-1,q},$$

where h in the right hand side denotes the images of h under the mappings induced by the inclusions $\mathrm{Stab}_{\mathcal{I}_g}(P) \hookrightarrow \mathrm{Stab}_{\mathcal{I}_g}(Q)$ for all cells $Q \subseteq \partial P$.

Bestvina, Bux and Margalit proved [1, Proposition 6.2] that for each cell $\sigma \in \mathcal{B}_g(x)$ we have

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_{\mathcal{I}_g}(\sigma)) \leq 3g - 5. \qquad (9)$$

Formulas (8) and (9) immediately imply the following fact.

Corollary 2.5. *Let $E^*_{*,*}$ be the spectral sequence (8). Then $E^1_{p,q} = 0$ for $p + q > 3g - 5$.*

2.5 Stabilisers of multicurves

We denote by T_γ the left Dehn twist about a curve γ. Let M be a multicurve on Σ_g. We denote by $\mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(M) \subseteq \mathrm{Mod}(\Sigma_g)$ the subgroup consisting of all mapping classes that stabilize every component of M and preserve the orientation of every component of M,. Then there is the following Birman-Lubotzky–McCarthy exact sequence (see [10, Lemma 2.1])

$$1 \to G(M) \to \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(M) \to \mathrm{PMod}(\Sigma_g \setminus M) \to 1, \qquad (10)$$

where $G(M)$ is the group generated by Dehn twists about the components of M.

There is the a version of the sequence (10) for the Torelli group, see [1, Section 6.2]. Let M be a multicurve on Σ_g without separating components. Then we have the exact sequence

$$1 \to \mathbb{Z}^{BP(M)} \to \mathrm{Stab}_{\mathcal{I}_g}(M) \to \mathrm{PMod}(\Sigma_g \setminus M), \tag{11}$$

where $BP(M)$ is the number of curves of M minus the number of distinct homology classes represented by the curves of M. The group $\mathbb{Z}^{BP(M)}$ is generated by the twists about bounding pairs contained in M. Recall that a twist about bounding pair is a map $T_{\delta_2}^{-1}T_{\delta_1}$, where δ and δ' are disjoint curves representing the same homology class. Moreover, we have the following inequality ([1, Lemma 6.13]).

$$\mathrm{cd}(\mathrm{Stab}_{\mathcal{I}_g} M) \leq 3g - 3 - P(M) - |M| + BP(M), \tag{12}$$

where $P(M)$ is the number of components of $\Sigma_g \setminus M$ of positive genus.

Also, we will need the following proposition.

Proposition 2.6. [6, Theorem 4.6] **(Birman exact sequence).** Let Σ be a surface with $\chi(\Sigma) < 0$. Denote by Σ' the surface obtained from Σ by removing a point x in the interior of Σ. Then the following sequence is exact

$$1 \to \pi_1(\Sigma, x) \to \mathrm{PMod}(\Sigma') \to \mathrm{PMod}(\Sigma) \to 1. \tag{13}$$

Here the map $\mathrm{PMod}(\Sigma') \to \mathrm{PMod}(\Sigma)$ is obtained by "forgetting" the puncture x, and the image of an element of $\pi_1(\Sigma, x)$ in $\mathrm{PMod}(\Sigma')$ is realized by "pushing" the puncture x around that element of $\pi_1(\Sigma, x)$. There is a version of the Birman exact sequence for the Torelli group [6, Proposition 6.13]:

$$1 \to \pi_1(\Sigma_g, \mathrm{pt}) \to \mathcal{I}_{g,1} \to \mathcal{I}_g \to 1. \tag{14}$$

3 Strategy of the proof

From now throughout the paper we fix $g = 3$ and put $\Sigma = \Sigma_3$, $\mathcal{I} = \mathcal{I}_3$, $\mathcal{B}(x) = \mathcal{B}_3(x)$. We denote by $\mathcal{I}_M$ the stabiliser of a (multi)curve M in $\mathcal{I}$. Sometimes we denote the stabiliser of a curve γ by $\mathcal{I}^{(\gamma)}$, i.e. $\mathcal{I}_\gamma$ and $\mathcal{I}^{(\gamma)}$ denote the same group. Also we set $\mathrm{H} = \mathrm{H}_1(\Sigma, \mathbb{Z})$. We say that a subgroup $U \subset \mathrm{H}$ is *symplectic* if $\mathrm{rk} U = 2$ and the restriction of the intersection form on U has determinant 1. We say that (V_1, V_2, V_3) is a (ordered) *splitting* of H, if $\mathrm{H} = V_1 \oplus V_2 \oplus V_3$ where the subgroups V_1, V_2 and V_3 are symplectic. For $U \subseteq \mathrm{H}$ we denote by $U^\perp \subseteq \mathrm{H}$ the orthogonal subgroup with respect to the intersection form.

For a separating curve θ we denote by X_θ the one-punctured torus bounded by θ. Also we denote $\mathrm{H}_\theta = \mathrm{H}_1(X_\theta, \mathbb{Z}) \subset \mathrm{H}$. For a multicurve M we say that a symplectic subgroup $U \subset \mathrm{H}$ is *admissible*, if $U = \mathrm{H}_\theta$ for some separating curve θ disjoint from M. Similarly, we say that a splitting $V = (V_1, V_2, V_3)$ of H is *admissible*, if V is given by two separating curves disjoint from M.

3.1 The construction of s-classes

Now let us describe explicitly the set of generators of the group $\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z})$, which we will call *s-classes*. For a splitting (V_1, V_2, V_3) let us now define the correspondent s-class $s(V_1, V_2, V_3) \in \mathrm{H}_4(\mathcal{I}_3, \mathbb{Z})$ in the following way.

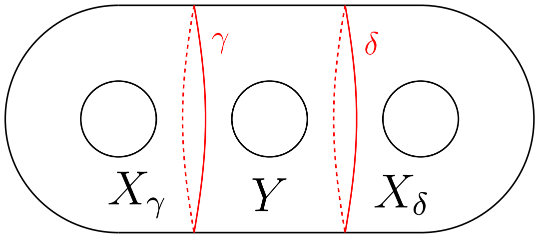


Figure 1:

Consider any two disjoint separating curves γ and δ on Σ_3 bounding punctured tori X_γ and X_δ respectively (see Fig. 1), such that $\mathrm{H}_\gamma = \mathrm{H}_1(X_\gamma, \mathbb{Z}) = V_1$ and $\mathrm{H}_\delta = \mathrm{H}_1(X_\delta, \mathbb{Z}) = V_3$ (this implies that $\mathrm{H}_1(Y, \mathbb{Z}) = V_2$, where Y is the third connected component of $\Sigma_3 \setminus \{\gamma, \delta\}$).

Consider the group $\mathcal{I}_3^{(\gamma)} = \mathrm{Stab}_{\mathcal{I}_3}\gamma$. The exact sequence (10) implies that we have

$$1 \to \langle T_\gamma \rangle \to \mathcal{I}^{(\gamma)} \to \mathrm{PMod}(\Sigma_{1,1}) \times \mathrm{PMod}(\Sigma_{2,1}). \tag{15}$$

Since the group $\mathcal{I}_{1,1}$ is trivial, one can easily compute the image of $\mathcal{I}^{(\gamma)}$ is precisely $\mathcal{I}_{2,1}$.

$$1 \to \langle T_\gamma \rangle \to \mathcal{I}^{(\gamma)} \to \mathcal{I}_{2,1} \to 1. \tag{16}$$

Also let us consider the exact sequence (14) in the case $g = 2$:

$$1 \to \pi_1(\Sigma_2, \mathrm{pt}) \to \mathcal{I}_{2,1} \to \mathcal{I}_2 \to 1. \tag{17}$$

The groups $\pi_1(\Sigma_g, \mathrm{pt})$ and $\mathcal{I}_2$ are generated by Dehn twists and bounding pair maps disjoint from γ, therefore $\langle \mathcal{T}_\gamma \rangle$ belongs to the center of $\mathcal{I}^{(\gamma)}$. Consequently, $\mathcal{I}_{2,1}$ acts trivially on $\mathrm{H}_1(\langle T_\gamma \rangle, \mathbb{Z})$. Hence Fact 2.1 applied to the exact sequence (16) implies an isomorphism

$$\alpha^{(\gamma)} : \mathrm{H}_3(\mathcal{I}_{2,1}, \mathbb{Z}) \cong \mathrm{H}_4(\mathcal{I}_3^{(\gamma)}, \mathbb{Z}).$$

The group $\mathcal{I}_2$ acts trivially on $\mathrm{H}_2(\pi_1(\Sigma_2, \mathrm{pt}), \mathbb{Z}) = \mathrm{H}_2(\Sigma_2, \mathbb{Z}) \cong \mathbb{Z}$. Hence Fact 2.1 applied to the exact sequence (17) implies an isomorphism

$$\beta^{(\gamma)} : \mathrm{H}_1(\mathcal{I}_2, \mathbb{Z}) \cong \mathrm{H}_3(\mathcal{I}_{2,1}, \mathbb{Z}).$$

Therefore we have

$$\alpha^{(\gamma)} \circ \beta^{(\gamma)} : \mathrm{H}_1(\mathcal{I}_2, \mathbb{Z}) \cong \mathrm{H}_4(\mathcal{I}_3^{(\gamma)}, \mathbb{Z}).$$

Denote

$$s(V_2, V_3) = \alpha^{(\gamma)} \circ \beta^{(\gamma)}([\mathcal{T}_\delta]) \in \mathrm{H}_4(\mathcal{I}_3^{(\gamma)}, \mathbb{Z}),$$

we call these homology classes *primary s-classes.*

Mess [12] proved that the group $\mathcal{I}_2$ is freely generated by an infinite number of Dehn twists about separating curves. Moreover, $\mathcal{I}_2$ has one generator for each splitting of the group $\mathrm{H}_1(\Sigma_2, \mathbb{Z})$ into direct of two symplectic subgroups. Hence we have the following result.

Proposition 3.1. Let γ be a separating curve on Σ bounding one-puncture torus with the first homology group $V_1 \subset \mathrm{H}$. Then $\mathrm{H}_4(\mathcal{I}_3^{(\gamma)}, \mathbb{Z})$ is a free abelian group with the basis consisting of the elements $s(V_2, V_3)$ for all splittings of H of the form (V_1, V_2, V_3).

Now let as define the s-class as follows

$$s(V_1, V_2, V_3) = \iota_*^{(\gamma)} s(V_2, V_3) \in \mathrm{H}_4(\mathcal{I}_3, \mathbb{Z}),$$

where by $\iota^{(\gamma)}$ we denote the inclusion $\iota^{(\gamma)} : \mathcal{I}^{(\gamma)} \hookrightarrow \mathcal{I}$.

Note that homology class $s(V_1, V_2, V_3) \in \mathrm{H}_4(\mathcal{I}, \mathbb{Z})$ does not depend on the choice of γ and δ. This follows from the fact that all such pairs of curves are $\mathcal{I}$-equivalent.

Proposition 3.2. For a splitting (V_1, V_2, V_3) of H we have

$$s(V_1, V_2, V_3) = s(V_1, V_3, V_2). \tag{18}$$

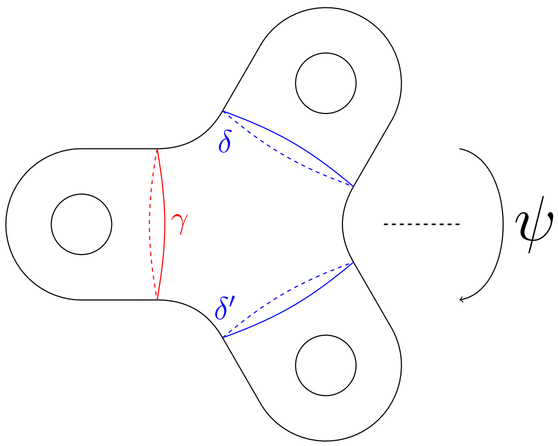


Figure 2:

Proof. Is suffices to prove that $s(V_2, V_3) = s(V_3, V_2)$. Consider the surface Σ as shown in Fig. 2. Let $\psi \in \mathrm{Mod}(\Sigma)$ be the "half Dehn twist" about γ, that is identical on the left handle and replaces the two right handles. We have $\psi(\delta) = \delta'$, so $\psi(s(V_2, V_3)) = s(V_3, V_2)$. Consider the action of ψ on the Hochschild-Serre spectral sequence of the extension (17). Since ψ preserves the orientation, it acts trivially on $\mathrm{H}_2(\pi_1(\Sigma_2, \mathrm{pt}), \mathbb{Z})$. We have $[T_\delta] = [T_{\delta'}] \in \mathrm{H}_1(\mathcal{I}_2, \mathbb{Z})$, so ψ acts trivially on the homology class $[T_\delta] \in \mathrm{H}_1(\mathcal{I}_2, \mathbb{Z})$. Therefore, ψ acts trivially on $s(V_2, V_3)$, so $s(V_2, V_3) = \psi(s(V_2, V_3)) = s(V_3, V_2)$. ☐

3.2 The spectral sequences for the complex of cycles

Now let $E^*_{*,*}$ be the spectral sequence (7) for the action on $\mathcal{I}$ on $\mathcal{B}(x)$ for some primitive element $0 \neq x \in \mathrm{H}$. By Corollary 2.5 we have $E^1_{p,q} = 0$ for $p + q > 4$. Since $\dim \mathcal{B}(x) = 3$ it follows that $E^1_{4,0} = 0$. Hence all nonzero terms of the page E^1 are shown on the left in Fig. 3.

The following two propositions imply that all nonzero terms of the page E^2 are shown on the right in Fig. 3.

Proposition 3.3. The differential $d^1_{3,1} : E^1_{3,1} \to E^1_{2,1}$ is injective.

Proposition 3.4. The differential $d^1_{2,2} : E^1_{2,2} \to E^1_{2,1}$ is injective.

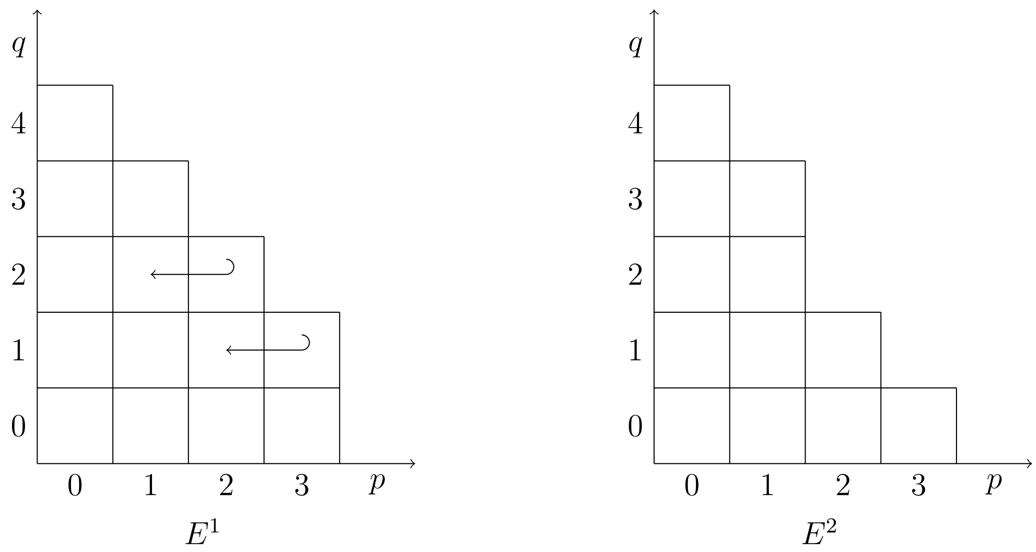


Figure 3: The pages E^1 and E^2 and the differentials $d^1_{3,1}$ and $d^1_{2,2}$.

Proposition 3.3 is not hard. However, the proof of Proposition 3.4 is quite complicated, and is one of the central results of the whole work.

Corollary 3.5. *We have the exact sequence*

$$1 \to E^1_{0,4} \to \mathrm{H}_4(\mathcal{I}, \mathbb{Z}) \to E^2_{1,3} \to 1.$$

Proof. All differentials $d^2, d^3, \ldots$ from and to the group $E^2_{1,3}$ are trival, so $E^\infty_{1,3} = E^2_{1,3}$. Also we obviously have $E^\infty_{0,4} = E^1_{0,4}$. Propositions 3.3 and 3.4 imply that the page E^2 has the form shown in Fig. 3. Therefore the page E^∞ has the same form. The exact sequence

$$1 \to E^\infty_{0,4} \to \mathrm{H}_4(\mathcal{I}, \mathbb{Z}) \to E^\infty_{1,3} \to 1$$

implies the result. □

3.3 The morphism of the spectral sequences

For each separating curve γ on Σ denote by $E^{(\gamma)*}_{*,*}$ the spectral sequence (7) for the action of $\mathcal{I}^{(\gamma)}$ on $\mathcal{B}(x)$. Denote by $j^{(\gamma)*}_{*,*} : E^{(\gamma)*}_{*,*} \to E^*_{*,*}$ the morphism of the spectral sequences induced by the inclusion $\iota^{(\gamma)} : \mathcal{I}^{(\gamma)} \hookrightarrow \mathcal{I}$.

Consider the morphism

$$J^*_{*,*} : \bigoplus_\gamma E^{(\gamma)*}_{*,*} \to E^*_{*,*}, \tag{19}$$

where $J^*_{*,*}$ is induces by $j^{(\gamma)*}_{*,*}$, and denote $\widehat{E}^*_{*,*} = \bigoplus_\gamma E^{(\gamma)*}_{*,*}$, where the sums are over all separating curves γ on Σ. The following two propositions describe some properties of the morphism $J^*_{*,*} : \widehat{E}^*_{*,*} \to E^*_{*,*}$.

Proposition 3.6. The map $J^2_{1,3} : \widehat{E}^2_{1,3} \to E^2_{1,3}$ is surjective.

Proposition 3.7. The map $J^1_{0,4} : \widehat{E}^1_{0,4} \to E^1_{0,4}$ is surjective.

Corollary 3.8. *The set of all s-classes generates the group* $\mathrm{H}_4(\mathcal{I}_3, \mathbb{Z})$*.*

Proof. Let γ be a separating curve on Σ_g. Proposition 3.1 implies that $\mathrm{H}_4(\mathcal{I}_3^{(\gamma)},\mathbb{Z})$ is generated by primary s-classes. Therefore the group $\widehat{E}^1_{0,4}$ is generated by linear combinations of primary s-classes and the group $\widehat{E}^2_{1,3} \subseteq \mathrm{H}_4(\mathcal{I}_3^{(\gamma)},\mathbb{Z})/\widehat{E}^1_{0,4}$ is generated by cosets of linear combinations of primary s-classes. Hence Propositions 3.7 implies that $E^1_{0,4}$ is generated by linear combinations of s-classes. Proposition 3.6 implies that $E^2_{1,3} = \mathrm{H}_4(\mathcal{I}_3,\mathbb{Z})/E^1_{0,4}$ are generated by cosets of linear combinations of s-classes. Corollary 3.5 concludes the proof. □

Using the morphism (19) we will also prove the following result.

Theorem 3.9. *For any splitting* (V_1, V_2, V_3) *we have*

$$s(V_1,V_2,V_3) + s(V_2,V_3,V_1) + s(V_3,V_1,V_2) = 0. \tag{20}$$

Any linear relation between s-classes follows from (18) and (20).

Now let us prove the main result.

Proof of Theorem 1.1. Obviously the action of $\mathrm{Sp}(6,\mathbb{Z})$ on the s-classes has the form $h\cdot s(V_1,V_2,V_3) = s(hV_1,hV_2,hV_3)$. The stabiliser in $\mathrm{Sp}(6,\mathbb{Z})$ of an unordered splitting $\mathrm{H} = V_1 \oplus V_2 \oplus V_3$ is isomorphic to $S_3 \ltimes \mathrm{SL}(2,\mathbb{Z})^{\times 3}$. The $S_3 \ltimes \mathrm{SL}(2,\mathbb{Z})^{\times 3}$-module, formally generated by the six s-classes corresponding to six permutations of V_1, V_2, V_3 in the splitting (V_1,V_2,V_3), satisfying the relations (18) and (20), is isomorphic to $\mathcal{Z}$. Therefore Corollary 3.8 together with the last statement of Theorem 3.9 imply that

$$\mathrm{H}_4(\mathcal{I}_3,\mathbb{Z}) \cong \mathcal{Z} \otimes_{S_3\ltimes\mathrm{SL}(2,\mathbb{Z})^{\times 3}} \mathbb{Z}[\mathrm{Sp}(6,\mathbb{Z})] = \mathrm{Ind}^{\mathrm{Sp}(6,\mathbb{Z})}_{S_3\ltimes\mathrm{SL}(2,\mathbb{Z})^{\times 3}}\mathcal{Z}.$$

This concludes the proof. □

4 Proof of Proposition 3.3

In order to prove Proposition 3.3 we need to compute the group $E^1_{3,1}$ explicitly. If we forget about orientation, and consider multicurves up to the action of the whole group $\mathrm{Mod}(\Sigma)$, then there are only two types of multicurves in $\mathcal{M}_3(x)$, see Fig. 4.

The multicurve M_3'' does not contain bounding pairs. Since the mapping class group of a three punctured sphere is trivial, formula (11) implies that the group $\mathcal{I}_{M_3''})$ is also trivial, so we have $\mathrm{H}_1(\mathcal{I}_{M_3''},\mathbb{Z}) = 0$. Consider the multicurve M_3'. The exact sequence (11) implies that $\mathcal{I}_{M_3'} = \langle T_{\gamma_1}T_{\gamma_2}^{-1}\rangle \cong \mathbb{Z}$. Denote by $\mathcal{M}_3'(x) \subset \mathcal{M}_3(x)$ the subset of multicurves belonging to the $\mathrm{Mod}(\Sigma)$-orbit of M_3'. We have the following result.

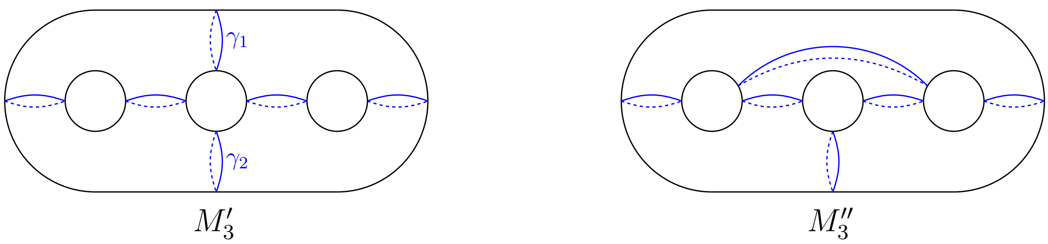


Figure 4: The multicurves M_3' and M_3''.

Proposition 4.1. The elements $P_{M_3'} \otimes [T_{\gamma_1}T_{\gamma_2}^{-1}]$, form a basis of the free abelian group $E^1_{3,1}$. Here M_3' runs over the set $\mathcal{M}_3'(x)/\mathcal{I}$, and the curves γ_1, γ_2 are shown in Fig. 4.

We also need some information about the group $E^1_{2,1}$. Consider the multicurve M_2' shown in blue in Fig. 5. Denote by $\mathcal{M}_2'(x) \subset \mathcal{M}_2(x)$ the subset of multicurves belonging to the $\mathrm{Mod}(\Sigma)$-orbit of M_2'.

Proposition 4.2. [5, Proposition 6.11] The stabiliser $\mathcal{I}_{M_2'}$ is a free group with an infinite number of generators

$$T_\delta^k T_{\gamma_1} T_{\gamma_2}^{-1} T_\delta^{-k}, \quad k \in \mathbb{Z}.$$

Corollary 4.3. *The elements $P_{M_2'} \otimes [T_\delta^k T_{\gamma_1} T_{\gamma_2}^{-1} T_\delta^{-k}]$ generate a free abelian subgroup in $E^1_{2,1}$ (here M_2' runs over the set $\mathcal{M}_2'(x)/\mathcal{I}$, $k \in \mathbb{Z}$, and the curves γ_1, γ_2, δ are shown in Fig. 4).*

Denote the subgroup constructed in Corollary 4.3 by $Q \subset E^1_{2,1}$. Let $p_Q : E^1_{2,1} \to Q$ be the projection corresponding to the inclusion $\mathcal{M}_2'(x)/\mathcal{I} \hookrightarrow \mathcal{M}_2(x)/\mathcal{I}$.

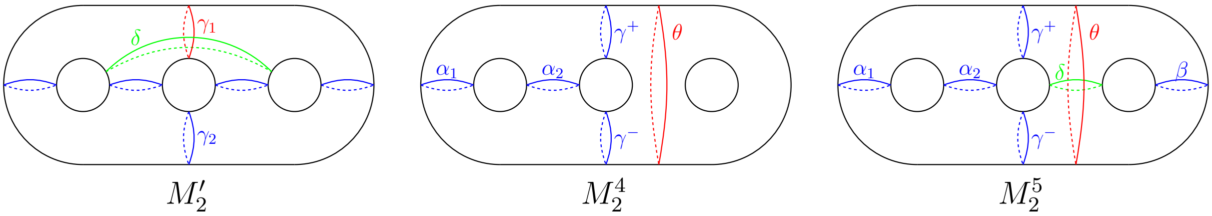


Figure 5: The multicurves M_2', M_2^4 and M_2^5.

Proof of Proposition 3.3. It suffices to prove that the map $p_Q \circ d^1_{3,1} : E^1_{3,1} \to Q$ is injective. Consider a basis element $P_{M_3'} \otimes [T_{\gamma_1} T_{\gamma_2}^{-1}]$ from Proposition 4.1. We have

$$Q \circ d^1_{3,1}(P_{M_3'} \otimes [T_{\gamma_1} T_{\gamma_2}^{-1}]) = P_{M_3' \setminus \gamma_2} \otimes [T_{\gamma_1} T_{\gamma_2}^{-1}] - P_{M_3' \setminus \gamma_1} \otimes [T_{\gamma_1} T_{\gamma_2}^{-1}], \tag{21}$$

the sign here depends on the orientation on $P_{M_3'}$. Therefore the image under the mapping $Q \circ d^1_{3,1}$ of each basis element of $E^1_{3,1}$ is the difference of some two basis elements of Q (see Corollary 4.3). Let us show that these pairs of basis elements of Q do not intersect with each other. This follows from the fact that all curves $T_\delta^k T_{\gamma_1} T_\delta^{-k}$ (see Fig. 5) belong to pairwise disjoint $\mathcal{I}$-orbits. This concludes the proof. □

5 Proof of Proposition 3.4

5.1 The term $E^1_{2,2}$

In order to prove Proposition 3.4 we need to compute the group $E^1_{2,2}$ explicitly. Similar to the previous section, we forget about orientation, and consider multicurves up to the action of the whole group $\mathrm{Mod}(\Sigma)$. In this case we have three possibilities. The corresponding representatives M_2', M_2^4 and M_2^5 are shown in blue in Fig. 5. By Proposition 4.2 the group $\mathcal{I}_{M_2'}$ is free, so $\mathrm{H}_2(\mathcal{I}_{M_2'}, \mathbb{Z}) = 0$. Denote by $\mathcal{M}_2^4(x) \subset \mathcal{M}_2(x)$ and $\mathcal{M}_2^5(x) \subset \mathcal{M}_2(x)$ the subsets of multicurves belonging to the $\mathrm{Mod}(\Sigma)$-orbit of M_2^4 and M_2^5 respectively.

Proposition 5.1. There is an isomorphism $\mathcal{I}_{M_2^4} \cong \mathbb{Z} \times F_\infty$. Here $\mathbb{Z} = \left\langle T_{\gamma^+} T_{\gamma^-}^{-1} \right\rangle$ and F_∞ is a free group generated by the infinite number of Dehn twists T_θ. Here H_θ runs over the set of admissible symplectic subgroups for M_2^4.

Remark 5.2. We do not claim that one can take *any* collection of Dehn twists T_θ satisfying the above conditions. We do not need an explicit set of generators here. The same can be said about Proposition 5.3.

Proof. Since the group $\mathrm{PMod}(\Sigma_{0,3})$ is trivial, the exact sequence (13) has the form

$$1 \to \left\langle T_{\gamma^+} T_{\gamma^-}^{-1} \right\rangle \to \mathcal{I}_{M_2^4} \to \mathrm{PMod}(\Sigma_{1,2}).$$

The image K of $\mathcal{I}_{M_2^4}$ in $\mathrm{PMod}(\Sigma_{1,2})$ is contained in the kernel J of the obvious homomorphism $\mathrm{PMod}(\Sigma_{1,2}) \to \mathrm{PMod}(\Sigma_{1,1}) = \mathrm{SL}(2,\mathbb{Z})$. The exact sequence (13) implies that $J = \pi_1(\Sigma_{1,1}, \mathrm{pt})$. Hence we have $K \subseteq \pi_1(\Sigma_{1,1}, \mathrm{pt})$.

Let us consider this group separately. The subsurface $\Sigma_{1,2}$ bounded by γ^+ and γ^- is shown in Fig. 6. The group $\pi_1(\Sigma_{1,1}, \mathrm{pt})$ is a free group with two generators $u = T_\epsilon T_{\epsilon'}^{-1}$ and $v = T_\delta T_{\delta'}^{-1}$. We claim that $K = [\pi_1(\Sigma_{1,1}, \mathrm{pt}), \pi_1(\Sigma_{1,1}, \mathrm{pt})]$.

First, we show that $K \subseteq [\pi_1(\Sigma_{1,1}, \mathrm{pt}), \pi_1(\Sigma_{1,1}, \mathrm{pt})]$. Consider the action of $\pi_1(\Sigma_{1,1}, \mathrm{pt})$ on H. Direct computation show that after the choice of suitable orientation on the curves we have

$$u : [\epsilon] \mapsto [\epsilon], \quad [\delta] \mapsto [\delta] + [\gamma^+], \quad [\gamma^+] \mapsto [\gamma^+],$$
$$v : [\epsilon] \mapsto [\epsilon] + [\gamma^+], \quad [\delta] \mapsto [\delta], \quad [\gamma^+] \mapsto [\gamma^+].$$

The result follows.

Now let us show that $K \supseteq [\pi_1(\Sigma_{1,1}, \mathrm{pt}), \pi_1(\Sigma_{1,1}, \mathrm{pt})]$. The group $[\pi_1(\Sigma_{1,1}, \mathrm{pt}), \pi_1(\Sigma_{1,1}, \mathrm{pt})]$ is freely generated by the commutators $\{[u^k, v^l] \mid k, l \in \mathbb{Z} \setminus \{0\}\}$. Each of these elements is represented (modulo $\mathbb{Z} = \left\langle T_{\gamma^+} T_{\gamma^-}^{-1} \right\rangle$) by the some Dehn twist T_θ, where θ is a separating curve. Such Dehn twists obviosly belong to K.

Therefore K is freely generated by the commutators $\{[u^k, v^l] \mid k, l \in \mathbb{Z} \setminus \{0\}\}$ and each of these elements is some Dehn twist T_θ about a separating curve θ. The pair of integers (k, l) uniquely determines the subgroup $\mathrm{H}_\theta^\perp \subset \mathrm{H}$. Hence H_θ is also uniquely determined by (k, l). Obviously this correspondence is a bijection between the set $\{(k, l) \mid k, l \in \mathbb{Z} \setminus \{0\}\}$ and the set of admissible symplectic subgroups for M_2^4. □

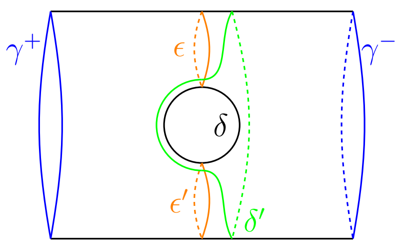


Figure 6: The subsurface bounded by γ^+ and γ^-.

Proposition 5.3. There is an isomorphism $\mathcal{I}_{M_2^5} \cong \mathbb{Z} \times F_\infty$. Here $\mathbb{Z} = \left\langle T_{\gamma^+} T_{\gamma^-}^{-1} \right\rangle$ and F_∞ is a free group generated by the infinite number of Dehn twists T_θ. Here H_θ runs over the set of admissible symplectic subgroups for M_2^5.

Proof. Since the group $\mathrm{PMod}(\Sigma_{0,3})$ is trivial, the exact sequence (13) has the form

$$1 \to \left\langle T_{\gamma^+} T_{\gamma^-}^{-1} \right\rangle \to \mathcal{I}_{M_2^5} \to \mathrm{PMod}(\Sigma_{0,4}).$$

The group $\mathrm{PMod}(\Sigma_{0,4})$ is freely generated by T_θ and T_δ (see Fig. 5). The image of $\mathcal{I}_{M_2^5}$ in $\mathrm{PMod}(\Sigma_{0,4})$ coincides with the kernel of the homomorphism $\mathrm{PMod}(\Sigma_{0,4}) \to \mathbb{Z}$, which sends T_θ to 0 and T_δ to 1. This kernel is freely generated by the infinite number of elements

$$T_{T_\delta^k \theta} = T_\delta^k T_\theta T_\delta^{-k}, \quad k \in \mathbb{Z}.$$

One can easily check that this set coincides with the set of generators from Proposition 5.3. □

Consider the multicurve M_2^i, $i = 4, 5$. Let $U \subset \mathrm{H}$ be an admissible symplectic subgroup for $\mathcal{M}_2^i$. Define the homology class

$$\mathcal{A}_U = \mathcal{A}(T_{\gamma^+} T_{\gamma^-}^{-1}, T_\theta) \in \mathrm{H}_2(\mathcal{I}_{\mathcal{M}_2^i}, \mathbb{Z}),$$

where θ is a separating curve disjoint from M_2^i such that $\mathrm{H}_\theta = U$. Note that the homology class $\mathcal{A}_U$ does not depend on the choice of θ.

Corollary 5.4. *The homology classes $\mathcal{A}_U$ form a basis of the free abelian group $\mathrm{H}_2(\mathcal{I}_{\mathcal{M}_2^i}, \mathbb{Z})$. Here $U \subset \mathrm{H}$ runs over all admissible symplectic subgroups for M_2^4.*

Corollary 5.4 immediately imply the following result.

Corollary 5.5. *The elements $P_{M_2^4} \otimes \mathcal{A}_{U^4}$ and $P_{M_2^5} \otimes \mathcal{A}_{U^5}$ form a basis of the free abelian group $E^1_{2,2}$. Here $M_2^4 \in \mathcal{M}_2^4/\mathcal{I}$, $M_2^5 \in \mathcal{M}_2^5/\mathcal{I}$, U^4 and U^5 run over the sets of admissible symplectic subgroups for $P_{M_2^4}$ and $P_{M_2^5}$ respectively.*

5.2 The term $E^1_{1,2}$

In order to prove Proposition 3.4 we also need some information about the group $E^1_{1,2}$. Consider the multicurves M_1^2, M_1^3 and M_1^4 shown in blue in Fig. 7. Denote by $\mathcal{M}_1^2(x) \subset \mathcal{M}_1(x)$, $\mathcal{M}_1^3(x) \subset \mathcal{M}_1(x)$ and $\mathcal{M}_1^4(x) \subset \mathcal{M}_1(x)$ the subset of multicurves belonging to the $\mathrm{Mod}(\Sigma)$-orbit of M_1^2, M_1^3 and M_1^4 respectively.

Proposition 5.6. Let $i = 1, 2, 3$. We have an isomorphism $\mathcal{I}_{M_1^i} \cong F_\infty \times \mathbb{Z} \times F_\infty$. Here $\mathbb{Z} = \left\langle T_{\gamma^+} T_{\gamma^-}^{-1} \right\rangle$. The first copy of F_∞ is a free group generated by an infinite number of Dehn twists T_{θ_1}; the second copy of F_∞ is a free group generated by an infinite number of Dehn twists T_{θ_2}. Here H_{θ_1} (H_{θ_2}) runs over all admissible symplectic subgroups for M_1^i belongs to the left (right) hand side of Σ (see Fig 7).

Proof. The proof is similar to the proofs of Propositions 5.1 and 5.3. □

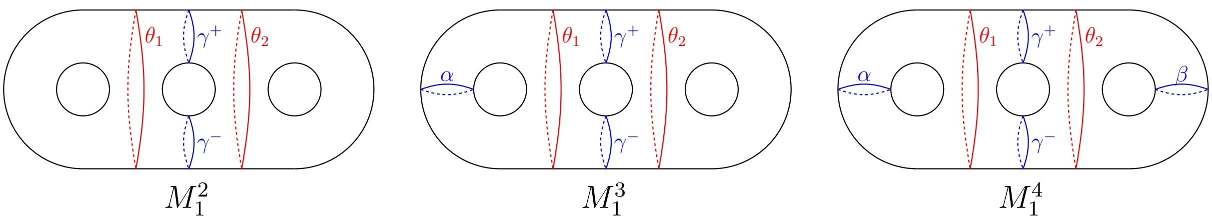


Figure 7: The multicurves M_1^2, M_1^3 and M_1^4.

Consider the multicurve M_1^i, $i = 3, 4, 5$. Let $U \subset \mathrm{H}$ be an admissible symplectic subgroup for $\mathcal{M}_1^i$. Define the homology class

$$\mathcal{A}_U = \mathcal{A}(T_{\gamma^+}T_{\gamma^-}^{-1}, T_\theta) \in \mathrm{H}_2(\mathcal{I}_{\mathcal{M}_1^i}, \mathbb{Z}),$$

where θ is a separating curve disjoint from M_1^i such that $\mathrm{H}_\theta = U$.

Corollary 5.7. *The homology classes $\mathcal{A}_U$ generate a free abelian subgroup in $\mathrm{H}_2(\mathcal{I}_{M_1^i}, \mathbb{Z})$. Here $U \subset \mathrm{H}$ and runs over the set of all admissible symplectic subgroups for M_1^i.*

Corollary 5.8. *The elements $P_{M_1^i} \otimes \mathcal{A}_{U^i}$ form a basis of a free abelian subgroup in $E^1_{1,2}$. Here $i = 2, 3, 4$, $M_1^i \in \mathcal{M}_1^i/\mathcal{I}$, $U^i \subset \mathrm{H}$ runs over the set of all admissible symplectic subgroups for M_1^i.*

5.3 Homomorphisms $\nu_{\gamma,\mathcal{W}}$

Let γ be a nonseparating curve on Σ and let $\mathcal{W} = (W_1, W_2)$ be an orthogonal splitting of the group $\mathrm{H}_\gamma = [\gamma]^\perp/[\gamma]$. Using the results of Mess [12], Gaifullin [5] constructed a homomorphism $\nu_{\gamma,\mathcal{W}} : \mathcal{I}_\gamma \to \mathbb{Z}$ with the following properties.

Proposition 5.9. [5, Proposition 6.4]

(a) Suppose that δ is a separating curve on Σ disjoint from γ. Then $\nu_{\gamma,\mathcal{W}}(T_\delta) = 1$ is δ yields the splitting $\mathcal{W}$ for H_γ, and $\nu_{\gamma,\mathcal{W}}(T_\delta) = 0$ otherwise.

(b) Suppose the $\{\gamma, \gamma'\}$ is a bounding pair. Then $\nu_{\gamma,\mathcal{W}}(T_\gamma T_{\gamma'}^{-1}) = -1$ is $\{\gamma, \gamma'\}$ yields the splitting $\mathcal{W}$ for H_γ, and $\nu_{\gamma,\mathcal{W}}(T_\gamma T_{\gamma'}^{-1}) = 0$ otherwise.

(c) Suppose that $\{\delta, \delta'\}$ is a bounding pair such that δ and δ' are disjoint from γ and neither δ nor δ' is homotopic to γ. Then $\nu_{\gamma,\mathcal{W}}(T_\delta T_{\delta'}^{-1}) = 0$.

Now consider the multicurves N_3^1 and N_4^1 shown in blue in Fig. 8. Denote by $\mathcal{N}_1^3(x) \subset \mathcal{M}_1(x)$ and $\mathcal{N}_1^4(x) \subset \mathcal{M}_1(x)$ the subset of multicurves belonging to the $\mathrm{Mod}(\Sigma)$-orbit of N_1^3 and N_1^4 respectively.

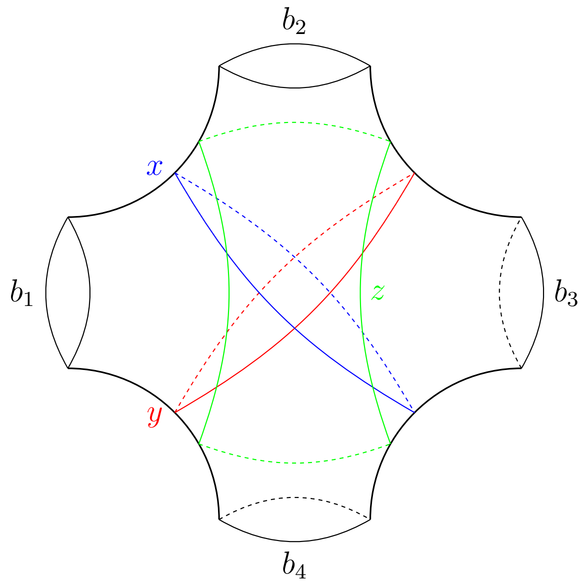
Figure 8: The multicurves N_1^3 and N_1^4.

Let us recall about the following useful relation in the mapping class group.

Proposition 5.10 (Lantern relation). [6, Proposition 5.1] Let the curves b_1, b_2, b_3, b_4 bound a sphere with 4 punctures of the surface Σ_g and let the curves x, y, z are as shown in Fig. 9. Then we have the relation

$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}.$$

Let $i = 3, 4$ and let $U \subset \mathrm{H}$ be an admissible subgroup for N_1^i given by a separating curve θ. Consider the subgroup $\mathcal{O}_U^i \subseteq \mathrm{H}_2(\mathcal{I}_{N_1^i}, \mathbb{Z})$ generated by the abelian cycles $\{\mathcal{A}_U^j = \mathcal{A}(T_{\alpha_j} T_{\alpha_j'}^{-1}, T_\theta)\}$, where $j = 1, 2, 3$ and α_j' is a curve disjoint from $N_1^i \cup \theta$ such that $\{\alpha_j, \alpha_j'\}$ is a bounding pair. Note that $\mathcal{A}_U^j$ does not depend on the choice of α_j'.



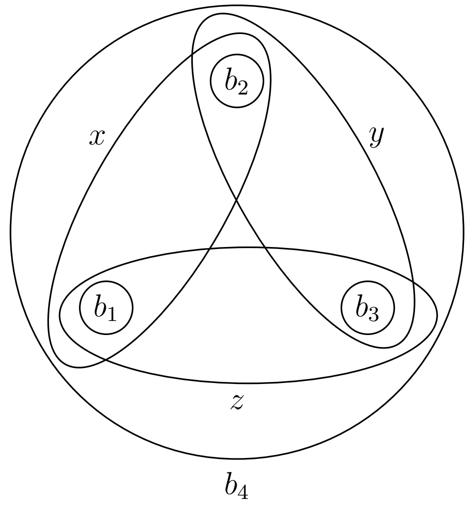


Figure 9: Two views of the Lantern relation.

Lemma 5.11. Let $i = 3, 4$. The abelian group $\mathcal{O}_U^i \cong \mathbb{Z}^2$ has a presentation with three generators

$$\mathcal{A}(T_{\alpha_1}T_{\alpha_1'}^{-1}, T_\theta),\ \mathcal{A}(T_{\alpha_2}T_{\alpha_2'}^{-1}, T_\theta),\ \mathcal{A}(T_{\alpha_3}T_{\alpha_3'}^{-1}, T_\theta), \tag{22}$$

and one relation

$$\mathcal{A}(T_{\alpha_1}T_{\alpha_1'}^{-1}, T_\theta) + \mathcal{A}(T_{\alpha_2}T_{\alpha_2'}^{-1}, T_\theta) + \mathcal{A}(T_{\alpha_3}T_{\alpha_3'}^{-1}, T_\theta) = 0. \tag{23}$$

Proof. First, let us prove the relation (23). We can assume that the curves $\theta, \alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3'$ are shown in Fig. 10. The Lantern relation implies

$$T_{\alpha_3'}T_{\alpha_1'}T_{\alpha_2'} = T_{\alpha_1}T_{\alpha_2}T_{\alpha_3}T_\theta.$$

We can rewrite this equation as follows.

$$(T_{\alpha_2}T_{\alpha_2}^{-1})(T_{\alpha_1}T_{\alpha_1}^{-1})(T_{\alpha_3}T_{\alpha_3}^{-1}) = T_\theta^{-1}.$$

Hence

$$\mathcal{A}(T_{\alpha_1}T_{\alpha_1'}^{-1}, T_\theta) + \mathcal{A}(T_{\alpha_2}T_{\alpha_2'}^{-1}, T_\theta) + \mathcal{A}(T_{\alpha_3}T_{\alpha_3'}^{-1}, T_\theta) =$$

$$= \mathcal{A}((T_{\alpha_2}T_{\alpha_2}^{-1})(T_{\alpha_1}T_{\alpha_1}^{-1})(T_{\alpha_3}T_{\alpha_3}^{-1}), T_\theta) = \mathcal{A}(T_\theta^{-1}, T_\theta) = 0.$$

Now let show that all relation among the three elements (22) follow from (23). Consider the homomorphisms $\nu_{\alpha_1,\mathcal{W}}$ and $\nu_{\alpha_2,\mathcal{W}}$, where $\mathcal{W} = (U, U^{\perp_{\mathrm{H}_\gamma}})$. Their restrictions on $\mathcal{I}_{N_1^3}$ represent cohomology classes $[\nu_{\alpha_1,\mathcal{W}}], [\nu_{\alpha_2,\mathcal{W}}] \in \mathrm{H}^1(\mathcal{I}_{N_1^3}, \mathbb{Z})$. Let us compute the value of their product on the abelian cycle $\mathcal{A}(T_{\alpha_1}T_{\alpha_1'}^{-1}, T_\theta)$. By Proposition 5.9 we have

$$\left\langle [\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}], \mathcal{A}(T_{\alpha_1}T_{\alpha_1'}^{-1}, T_\theta) \right\rangle = -\det \begin{pmatrix} \nu_{\alpha_1,\mathcal{W}}(T_{\alpha_1}T_{\alpha_1'}^{-1}) & \nu_{\alpha_1,\mathcal{W}}(T_\theta) \\ \nu_{\alpha_2,\mathcal{W}}(T_{\alpha_1}T_{\alpha_1'}^{-1}) & \nu_{\alpha_2,\mathcal{W}}(T_\theta) \end{pmatrix} =$$

$$= -\det \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \neq 0.$$

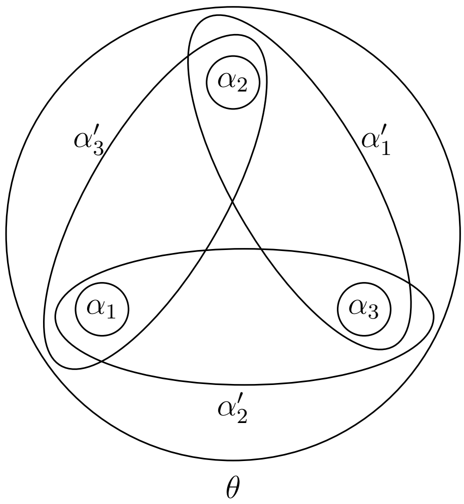


Figure 10: The curves $\theta, \alpha_1, \alpha_1', \alpha_2, \alpha_2', \alpha_3, \alpha_3'$.

Similarly,

$$\left\langle [\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}], \mathcal{A}(T_{\alpha_2}T_{\alpha_2'}^{-1}, T_\theta) \right\rangle = -1.$$

Moreover, we have

$$\left\langle [\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}], \mathcal{A}(T_{\alpha_3}T_{\alpha_3'}^{-1}, T_\theta) \right\rangle = -\det \begin{pmatrix} \nu_{\alpha_1,\mathcal{W}}(T_{\alpha_3}T_{\alpha_3'}^{-1}) & \nu_{\alpha_1,\mathcal{W}}(T_\theta) \\ \nu_{\alpha_2,\mathcal{W}}(T_{\alpha_3}T_{\alpha_3'}^{-1}) & \nu_{\alpha_2,\mathcal{W}}(T_\theta) \end{pmatrix} =$$

$$= -\det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0.$$

Consider homology class

$$h = \sum_{j=1}^{3} \lambda_j \mathcal{A}(T_{\alpha_j}T_{\alpha_j'}^{-1}, T_\theta) \in \mathcal{O}_U^i.$$

If $h = 0$, then

$$\langle [\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}], h\rangle = \langle [\nu_{\alpha_2,\mathcal{W}}] \smile [\nu_{\alpha_3,\mathcal{W}}], h\rangle = \langle [\nu_{\alpha_3,\mathcal{W}}] \smile [\nu_{\alpha_1,\mathcal{W}}], h\rangle = 0,$$

that is

$$\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \lambda_3 - \lambda_1 = 0.$$

Hence $\lambda_1 = \lambda_2 = \lambda_3$. This concludes the proof. □

Lemma 5.12. Let $i = 3, 4$. The inclusions $\mathcal{O}_U^i \hookrightarrow \mathrm{H}_2(\mathcal{I}_{N_1^i}, \mathbb{Z})$ induce the injective homomorphism

$$\bigoplus_U \mathcal{O}_U^i \hookrightarrow \mathrm{H}_2(\mathcal{I}_{N_1^i}, \mathbb{Z}),$$

where the sum is over all admissible subgroup for N_1^i.

Proof. Assume the converse. By Lemma 5.11 we can assume that there is a nontrivial linear dependence

$$\sum_{U \in S} \left(\lambda_{U,2}\mathcal{A}_U^2 + \lambda_{U,3}\mathcal{A}_U^3\right) = 0, \tag{24}$$

where S is a finite set and $\lambda_{U,2}, \lambda_{U,3} \in \mathbb{Z}$. Let $U' \in S$. Let us compute the value of the cohomology class $[\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}] \in \mathrm{H}^2(\mathcal{I}_{N_1^i}, \mathbb{Z})$ on the both sides of (24), where $W = (U', U'^{\perp})$ are the splitting of $\alpha_1^{\perp}/\alpha_1$ and $\alpha_2^{\perp}/\alpha_2$. We have

$$\sum_{U \in S} \lambda_{U,2} \left\langle [\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}], \mathcal{A}_U^2 \right\rangle = 0, \tag{25}$$

Let $U \in S$ is given by a separating curve θ disjoint from N_1^3 and $\mathcal{A}_U^2 = \mathcal{A}(T_{\alpha_2}T_{\alpha_2'}^{-1}, T_\theta)$. Then

$$\left\langle [\nu_{\alpha_1,\mathcal{W}}] \smile [\nu_{\alpha_2,\mathcal{W}}], \mathcal{A}_U^2 \right\rangle = -\det \begin{pmatrix} \nu_{\alpha_1,\mathcal{W}}(T_{\alpha_2}T_{\alpha_2'}^{-1}) & \nu_{\alpha_1,\mathcal{W}}(T_\theta) \\ \nu_{\alpha_2,\mathcal{W}}(T_{\alpha_2}T_{\alpha_2'}^{-1}) & \nu_{\alpha_2,\mathcal{W}}(T_\theta) \end{pmatrix}. \tag{26}$$

Since $\nu_{\alpha_1,\mathcal{W}}(T_{\alpha_2}T_{\alpha_2'}^{-1}) = 0$, we have that (26) is nonzero if and only if $\nu_{\alpha_1,\mathcal{W}}(T_\theta)$ and $\nu_{\alpha_2,\mathcal{W}}(T_{\alpha_2}T_{\alpha_2'}^{-1})$. Therefore U is admissible for $N_1^i \cup \alpha_1'$ and $N_1^i \cup \alpha_2'$. This condition defines U uniquely, hence $U = U'$. Therefore (25) implies $\lambda_{U',2} = 0$. Similarly $\lambda_{U',3} = 0$ for any $U' \in S$. This concludes the proof. □

5.4 The differential $d_{2,2}^1$

Proof of Proposition 3.4. In Corollary 5.5 we constructed a basis of the free abelian group $E_{2,2}^1$. The differential $d_{2,2}^1 : E_{2,2}^1 \to E_{1,2}^1$ has the following form.

$$d_{2,2}^1 \left(P_{M_2^i} \otimes \mathcal{A}_{U^i} \right) = \partial P_{M_2^i} \otimes \mathcal{A}_{U^i},$$

where $i = 4, 5$. By Corollary 5.8 and Lemma 5.12, it suffices to prove that the image of any nontrivial linear combination of basis elements corresponding to some fixed U^i (and some fixed $i = 4, 5$) is nonzero. Let us consider the case $i = 4$.

We prove by a contradiction. Let us fix some symplectic subgroup $U \subset \mathrm{H}$ and assume that we have

$$d_{2,2}^1 \left(\sum_{M \in S} \lambda_M P_M \otimes \mathcal{A}_U \right) = 0, \tag{27}$$

where $M \in \mathcal{M}_2^4/\mathcal{I}$ are such that U s admissible for M and S is a finite set. Let us remark that for each $M \in S$ and for each edge $e \subset \partial M$ the restriction of the homology class $\mathcal{A}_U$ from $\mathcal{I}_M$onto $\mathcal{I}_e$ is nonzero. Therefore (27) implies the following useful observation.

Proposition 5.13. If e is an edge such that $e \subset \partial P_M$ for some $M \in S$, then there exists $M \neq M' \in S$ such that $e \subset \partial P_{M'}$.

Denote by $\Gamma \subset \mathcal{B}(x)/\mathcal{I}$ the subcomplex given by the closure of the union of all cells corresponding to the multicurves M from the sets $\mathcal{M}_2^4$ such that U is admissible for M.

For a homology class $y \in U^{\perp}$ choose a bounding pair $\{\gamma^+, \gamma^-\}$ on Σ such that $[\gamma^+] = [\gamma^-] = y$ and U is admissible for $\gamma^+ \cup \gamma^-$. Any multicurve M from the set $\mathcal{M}_2^4/\mathcal{I}$ containing γ^+ and γ^- (see Fig. 5) is uniquely determined (up to $\mathcal{I}$-equivalence) by the homology classes $[\alpha_1], [\alpha_2]$. Denote by $\Gamma_y \subset \Gamma$ the subcomplex spanned by these cells.

Let $\alpha_1 = \alpha$ be a nonseparating curve such that $x = m[\alpha] + ny$ with $m, n > 0$. There are three possible situation: $[\alpha_2] = [\alpha] \pm y$ and $[\alpha_2] = y - [\alpha]$. Denote $l = \lfloor \frac{n}{m} \rfloor$. There are five possible types of edges in Γ_y. We denote them as follows.

• d_k is the image in $\mathcal{B}(x)/\mathcal{I}$ of an edge in $\mathcal{B}(x)$ corresponding to a multicurve containing γ^+, γ^- and a curve of the homology class $[\alpha] + ky$; $k \in \mathbb{Z}$, $k \leq l$.

• c_k^+ is the image in $\mathcal{B}(x)/\mathcal{I}$ of an edge in $\mathcal{B}(x)$ corresponding to a multicurve containing γ^+ and curves of the homology classes $[\alpha]+ky, [\alpha]+(k+1)y$; $k\in\mathbb{Z}$, $k\le l$.

• c_k^- is the image in $\mathcal{B}(x)/\mathcal{I}$ of an edge in $\mathcal{B}(x)$ corresponding to a multicurve containing γ^- and curves of the homology classes $[\alpha]+ky, [\alpha]+(k+1)y$; $k\in\mathbb{Z}$, $k\le l$.

• e_k^+ is the image in $\mathcal{B}(x)/\mathcal{I}$ of an edge in $\mathcal{B}(x)$ corresponding to a multicurve containing γ^+ and curves of the homology classes $[\alpha]+ky, -[\alpha]-(l-1)y$; $k\in\mathbb{Z}$, $k\le l$.

• e_k^- is the image in $\mathcal{B}(x)/\mathcal{I}$ of an edge in $\mathcal{B}(x)$ corresponding to a multicurve containing γ^- and curves of the homology classes $[\alpha]+ky, -[\alpha]-(l-1)y$; $k\in\mathbb{Z}$, $k\le l$.

Note that the edges c_k^+ and c_k^- (e_k^+ and e_k^-) are different. Indeed, in these cases the 3-punctured sphere in on opposite sides of the curve of the homology class y (recall that we consider oriented multicurves). However, the edges c_k^+ and c_k^- (e_k^+ and e_k^-) have common endpoints (but these edges are not loops). Consequently, each edge d_k is a loop in Γ_y.

Therefore Γ_y is obtained from the complex Γ_y' shown in Fig 11 by gluing together the pairs of endpoints of the edges d_k, where $k\in\mathbb{Z}$ and $k\le l$. We are interested only on the structure of 2- and 3-cells of Γ_y, so it is convenient to work with Γ_y'.

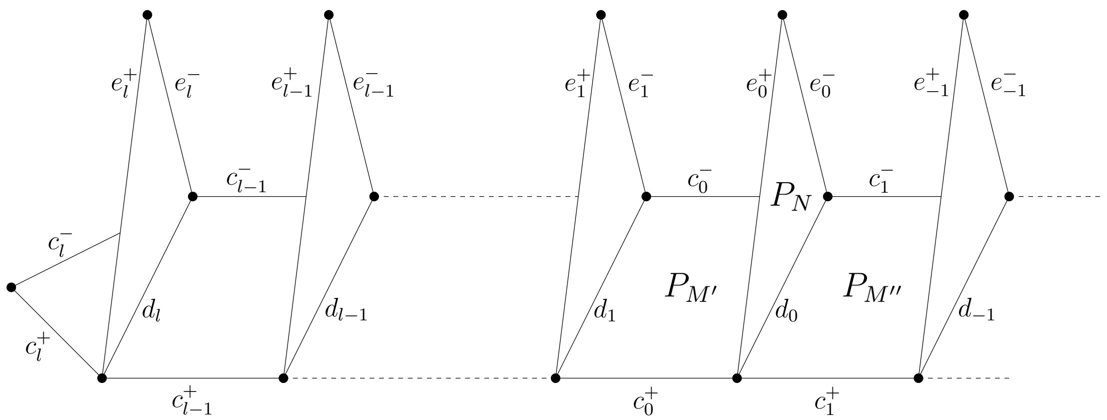


Figure 11: The complex Γ_y'.

Obviously we have

$$\Gamma=\bigcup_{y\in U^\perp}\Gamma_y,$$

and for $y\ne z$ the intersection $\Gamma_y\cap\Gamma_z$ consists of some boundary edges of Γ_y and Γ_z.

Let σ be a cell of $\mathcal{B}$. Define the number $\Psi(\sigma)\in\mathbb{N}$ as follows. Each vertex v of σ corresponds to some multicurve $\gamma_1\cup\cdots\cup\gamma_t$ such that the homology class x can be uniquely represented as $x=\sum_{j=1}^t k_j[\gamma_j]$. Let us define $\psi(v)=\sum_{j=1}^t k_j$ and

$$\Psi(\sigma)=\max_{v\in\sigma}\psi(v),$$

where maximum is taken over all vertices v of σ.

Each subcomplex Γ_y has 'horizontal' 2-cells (all rectangles and the leftmost triangle) and 'vertical' 2-cells (all other triangles) as shown in Fig. 11. The edges of the horizontal 2-cells are also said to be 'horizontal' (that is, c_k^+, c_k^- and d_k are horizontal and e_k^+, e_k^- are vertical). Let us remark that if (27) holds, then at least one of the cells $\{P_M, \mid M\in S\}$ is horizontal. This follows from the fact that the boundary of each vertical cell contains a horizontal edge that is not contained in the boundary of any other vertical 2-cells.

Let $M' \in S$ be a multicurve such that $P_{M'}$ is horizontal and $\Psi(P_{M'}) \geq \Psi(P_M)$ for any horizontal cell P_M, where $M \in S$. Without loss of generality we can assume that $M' = M_2^4$ shown in Fig 5. Let $\alpha_1 = \alpha$ and $[\alpha_2] = [\alpha] + [\gamma^+]$, where $x = m[\alpha] + n[\gamma^+]$. The cell $P_{M'}$ is shown in Fig. 11 (the case when $P_{M'}$ triangular horizontal 2-cell is completely similar). We have $\Psi(P_{M'}) = m + n$.

Let β be a curve disjoint from γ^+, γ^- and α such that $[\beta] = [\alpha] - [\gamma^+]$. Consider the multicurve $M'' = \gamma^+ \cup \gamma^- \cup \alpha \cup \beta$. The corresponding cell $P_{M''}$ is the rectangular 2-cell located on right hand side of $P_{M'}$ on Fig 11. We have

$$\Psi(P_{M''}) = 2m + n > m + n = \Psi(P_{M'}),$$

hence $M'' \notin S$. Consider the multicurve $N = \gamma^+ \cup \gamma^- \cup \alpha \cup \beta$. The cell P_N is the triangular 2-cell located on the top right of $P_{M'}$, see Fig 11. The edge $d_0 = P_{\gamma^+ \cup \gamma^- \cup \alpha}$ belongs to the boundary of exactly three 2-cells: $P_{M'}$, $P_{M''}$ and P_N. Since $M' \in S$ and $M'' \notin S$, Proposition 5.13 implies that $N \in S$.

Consider the edge $e_0^+ = P_{\gamma^+ \cup \alpha \cup \beta}$. Obviously $e_0^+ \subset \partial P_N$. By Proposition 5.13 there exist $L \in S$, such that $L \neq N$ and $e_0^+ \subset \partial P_L$. Therefore the homology classes of the components of L are either $\{[\alpha], [\alpha], y, [\alpha] - y\}$ or $\{y - [\alpha], y - [\alpha], [\alpha], y\}$. Both of the corresponding cells are horizontal (the first cell is rectangular, the second one is triangular). In the first case we have

$$\Psi(P_L) = 2m + n > m + n = \Psi(P_{M'}).$$

In the second case we have

$$\Psi(P_L) = m + 2n > m + n = \Psi(P_{M'}).$$

Therefore we come to a contradiction in the case $i = 4$.

Now consider the case $i = 5$. The strategy is the same as for $i = 4$. Let us remark that in (27) we can assume that all multicurves $M \in \mathcal{M}_2^5/\mathcal{I}$ have the same component β (see Fig. 5) and the expansion of x has the same coefficient m at $[\beta]$. Then the previous argument works with replacing x by $x - m[\beta]$. $\square$

6 Proof of Proposition 3.6

6.1 The term $E_{1,3}^1$

In order to prove Proposition 3.6 we need to compute the group $E_{1,3}^1$ explicitly. If we forget about orientation, there are six distinct $\mathrm{Mod}(\Sigma_g)$-orbits in the set $\mathcal{M}_1(x)$. The representatives are shown in blue in Fig. 7, 8 and 12. Formula (12) applied the multicurves N_1^3, N_1^4 and M_1' implies

$$\mathrm{cd}(\mathcal{I}_{N_1^3}) \leq 6 - 1 - 3 + 0 = 2,$$
$$\mathrm{cd}(\mathcal{I}_{N_1^4}) \leq 6 - 0 - 4 + 0 = 2,$$
$$\mathrm{cd}(\mathcal{I}_{M_1'}) \leq 6 - 0 - 4 + 0 = 2.$$

Hence we have $\mathrm{H}_3(\mathcal{I}_{N_1^3}, \mathbb{Z}) = \mathrm{H}_3(\mathcal{I}_{N_1^4}, \mathbb{Z}) = \mathrm{H}_3(\mathcal{I}_{M_1'}, \mathbb{Z}) = 0$.

Now consider the multicurves M_1^i for $i = 2, 3, 4$ (see Fig. 7). Let $U_1, U_2 \subset \mathrm{H}$ be admissible for symplectic subgroup for M_1^i such that $U_1 \perp U_2$. Define the homology class

$$\mathcal{A}_{U_1,U_2} = \mathcal{A}(T_{\theta_1}, T_{\gamma^+}T_{\gamma^-}^{-1}, T_{\theta_2}) \in \mathrm{H}_3(\mathcal{I}_{\mathcal{M}_1^i}, \mathbb{Z}),$$

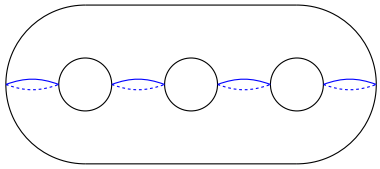


Figure 12: The multicurve M_1'.

where θ_1 and θ_2 are separating curves disjoint from $\mathcal{M}_1^i$ such that $\mathrm{H}_{\theta_1} = U_1$ and $\mathrm{H}_{\theta_2} = U_2$. In order to distinguish the curves γ^+ and γ^- let us agree that θ_1 is on the right hand side with respect to γ^+ (recall cells of $\mathcal{B}$ correspond to oriented multicurves). Obviously we have $\mathcal{A}_{U_1,U_2} = -\mathcal{A}_{U_2,U_1}$. Proposition (5.6) immediately implies the following result.

Proposition 6.1. The homology classes $\mathcal{A}_{U_1,U_2}$ form a basis of the free abelian group $\mathrm{H}_3(\mathcal{I}_{\mathcal{M}_1^i}, \mathbb{Z})$. Here $U_1, U_2 \subset \mathrm{H}$ run over all unordered pairs orthogonal admissible symplectic subgroups for M_1^i.

Corollary 6.2. *The elements*

$$P_{M_1^i} \otimes \mathcal{A}_{U_1^i,U_2^i}, \quad i = 2,3,4$$

form a basis of the free abelian group $E^1_{1,3}$*. Here* $M_1^i \in \mathcal{M}_1^i/\mathcal{I}$*, and* $U_1^i, U_2^i \subset \mathrm{H}$ *run over all unordered pairs orthogonal admissible symplectic subgroups for* M_1^i*.*

6.2 The differential $d^1_{1,3}$

Our next goal is to compute the images under the differential $d^1_{1,3}$ of the elements $P_{M_1^i} \otimes \mathcal{A}_{U_1^i,U_2^i} \in E^1_{1,3}$ (see Corollary 6.2). First we need to prove some general results.

Consider the surface $\Sigma_{2,1} = \Sigma \setminus \overline{X_{\theta_1}}$. We have the exact sequences

$$1 \longrightarrow \langle T_{\theta_1} \rangle \longrightarrow \mathcal{I}^{(\theta_1)} \xrightarrow{p} \mathcal{I}_{2,1} \longrightarrow 1.$$

$$1 \longrightarrow \pi_1(\Sigma_2, \mathrm{pt}) \longrightarrow \mathcal{I}_{2,1} \xrightarrow{q} \mathcal{I}_2 \longrightarrow 1.$$

Consider the subgroups $Q = q^{-1}(\langle T_{\theta_3} \rangle) \subset \mathcal{I}_{2,1}$ and $G = p^{-1}(Q) \subset \mathcal{I}^{(\theta_1)}$. Let $\overline{G} \subseteq \mathcal{I}$ be a subgroup such that $G \subseteq \overline{G}$. Denote $\overline{Q} = p(\overline{G})$. Suppose that $\overline{\mathcal{B}}(x) \subseteq \mathcal{B}(x)$ is a subcomplex such that the action of $\overline{G}$ on $\overline{\mathcal{B}}(x)$ is well-defined. Denote by $\overline{E}^*_{*,*}$ the spectral sequence (7) for the action of $\overline{G}$ on $\overline{\mathcal{B}}(x)$.

Let $\theta_1, \theta_2, \theta_3$ be pairwise disjoint separating curves on Σ_g. Consider the multicurves $M_1^2 = \alpha_2 \cup \alpha_2'$, $M_1^3 = \alpha_1 \cup \alpha_2 \cup \alpha_2'$ and $M_1^4 = \alpha_1 \cup \alpha_2 \cup \alpha_2' \cup \alpha_3$ shown in blue in Fig. 13. Consider the homology class $\mathcal{A}(T_{\theta_1}, T_{\theta_2}, T_{\theta_3}) \in \mathrm{H}_3(\overline{G}_{M_1^i \setminus \alpha_2'}, \mathbb{Z})$. Assume that the cells $P_{M_1^i} \subset \mathcal{B}(x)$ are well-defined and belong to $\overline{\mathcal{B}}(x)$.

Lemma 6.3. Suppose that $i = 2,3,4$. Then

$$\overline{d}^1_{1,3}(P_{M_1^i} \otimes \mathcal{A}(T_{\theta_1}, T_{\alpha_2}T^{-1}_{\alpha_2'}, T_{\theta_3})) = P_{M_1^i \setminus \alpha_2'} \otimes \mathcal{A}(T_{\theta_1}, T_{\theta_2}, T_{\theta_3}).$$

Proof. Consider the curves δ_1 and δ_2 shown in Fig. 13. Note that $T_{\delta_2}T^{-1}_{\delta_1} \in G \subseteq \overline{G}$. A direct computation show that $T_{\delta_2}T^{-1}_{\delta_1}(\alpha_2') = \alpha_2$ and $T_{\delta_2}T^{-1}_{\delta_1}(\alpha_2) = \beta$. Also by Lantern relation we have

$$T_{\theta_2}T_{\alpha_2'}T_\beta = T_{\theta_1}T^2_{\alpha_2}T_{\theta_3},$$

that is,
$$T_{\theta_1}^{-1}T_{\theta_2}T_{\theta_3}^{-1} = T_{\alpha_2'}^{-1}T_{\alpha_2}^2T_{\beta}^{-1}. \tag{28}$$
Therefore we have

$$\overline{d}^1_{1,3}(P_{M_i^2}\otimes\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3}) = P_{M_1^i\setminus\alpha_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3}) - P_{M_1^i\setminus\alpha_2}\otimes\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3}) =$$
$$= P_{M_1^i\setminus\alpha_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3}) - \Big((T_{\delta_2}T_{\delta_1}^{-1})\cdot P_{M_1^i\setminus\alpha_2}\Big)\otimes\Big((T_{\delta_2}T_{\delta_1}^{-1})\cdot\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3})\Big) =$$
$$= P_{M_1^i\setminus\alpha_2'}\otimes\Big(\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3}) - \mathcal{A}(T_{\theta_1},T_{\beta}T_{\alpha_2}^{-1},T_{\theta_3})\Big) =$$
$$= P_{M_1^i\setminus\alpha_2'}\otimes\Big(\mathcal{A}(T_{\theta_1},T_{\alpha_2'}^{-1}T_{\alpha_2},T_{\theta_3}) + \mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\beta}^{-1},T_{\theta_3})\Big) =$$
$$= P_{M_1^i\setminus\alpha_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\alpha_2'}^{-1}T_{\alpha_3}^2T_{\beta}^{-1},T_{\theta_3}) = P_{M_1^i\setminus\alpha_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\theta_1}^{-1}T_{\theta_2}T_{\theta_3}^{-1},T_{\theta_3}) =$$
$$= P_{M_1^i\setminus\alpha_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\theta_2},T_{\theta_3}).$$

□

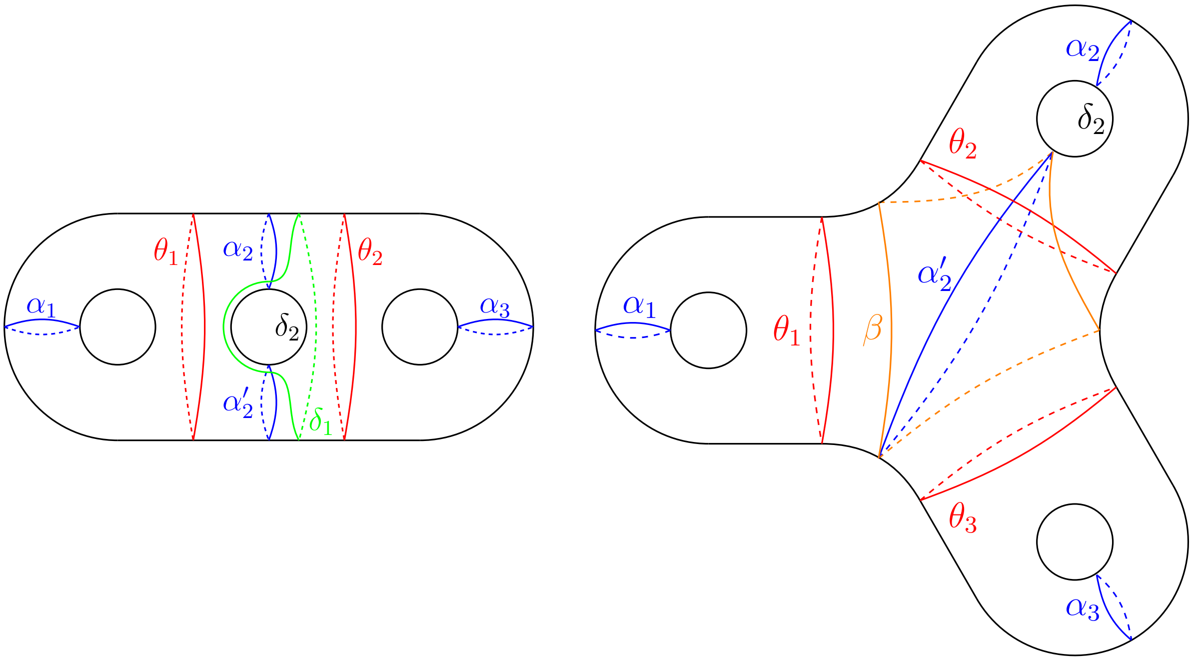


Figure 13: The multicurves $M_1^2 = \alpha_2\cup\alpha_2'$, $M_1^3 = \alpha_1\cup\alpha_2\cup\alpha_2'$ and $M_1^4 = \alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3$.

Lemma 6.4. Let $i = 2, 3$. Then the abelian cycle $\mathcal{A}(T_{\theta_1},T_{\theta_2},T_{\theta_3}) \in \mathrm{H}_3(\overline{G}_{M_1^i\setminus\alpha_2'},\mathbb{Z})$ is zero.

Proof. It suffices to prove that the abelian cycle $\mathcal{A}(T_{\theta_1},T_{\theta_2},T_{\theta_3}) \in \mathrm{H}_3(G_{\theta_1\cup\alpha_1\cup\alpha_2},\mathbb{Z})$ is zero. We have $\mathcal{A}(T_{\theta_1},T_{\theta_2},T_{\theta_3}) = \mathcal{A}(T_{\theta_2}T_{\theta_3}^{-1},T_{\theta_3},T_{\theta_1})$.

Let $\Sigma_{2,1}\subset\Sigma$ be the genus 2 subsurface bounded by θ_1 and let $\mathcal{I}_{2,1}$ be its Torelli group. We have the exact sequence
$$1\to\langle T_{\theta_1}\rangle\to\overline{G}_{\theta_1\cup\alpha_1\cup\alpha_2}\to\overline{Q}_{\alpha_2}\to 1,$$
where $\overline{Q}_{\alpha_2} = \mathrm{Stab}_{\overline{Q}}(\alpha_2)$.

Fact 2.2 implies that there is an isomorphism

$$\mathrm{H}_3(\overline{G}_{\theta_1\cup\alpha_1\cup\alpha_2},\mathbb{Z})\cong\mathrm{H}_2(\overline{Q}_{\alpha_2},\mathbb{Z}),$$

that maps $\mathcal{A}(T_{\theta_3}T_{\theta_2}^{-1},T_{\theta_2},T_{\theta_1})$ to $\mathcal{A}(T_{\theta_3}T_{\theta_2}^{-1},T_{\theta_2})$. Hence it suffices to show that the homology class $\mathcal{A}(T_{\theta_2},T_{\theta_3}T_{\theta_2}^{-1})\in\mathrm{H}_2(\overline{Q}_{\alpha_2},\mathbb{Z})$ is zero.

Let S_2 be the genus 2 surface given by capping the puncture on $\Sigma_{2,1}$. By our assumptions on $\overline{G}$ we have the exact sequence

$$1\to\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt})\to\overline{Q}_{\alpha_2}\to J\to 1,$$

where $J\subseteq\mathcal{I}_2^{(\alpha_2)}$. The groups $\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt})$ and $\mathcal{I}_2^{(\alpha_2)}$ are free, so J is also free. The action of $\mathcal{I}_2^{(\alpha_2)}$ on $\mathrm{H}_*(\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt}),\mathbb{Z})$ is trivial. Note that mapping class $T_{\theta_3}T_{\theta_2}^{-1}$ is 'pushing' the puncture along some loop on $\Sigma_2\setminus\alpha_2$, so $T_{\theta_3}T_{\theta_2}^{-1}\in\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt})$. Therefore Fact 2.3 implies that there is an isomorphism

$$\mathrm{H}_2(\overline{Q}_{\alpha_2},\mathbb{Z})\cong\mathrm{H}_1(J,\mathbb{Z})\otimes\mathrm{H}_1(\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt}),\mathbb{Z}),$$

that maps the abelian cycle $\mathcal{A}(T_{\theta_2},T_{\theta_3}T_{\theta_2}^{-1})$ to $[T_{\theta_2}]\otimes[T_{\theta_3}T_{\theta_2}^{-1}]$. One can easily check the $T_{\theta_3}T_{\theta_2}^{-1}$ belongs to the commutator subgroup of $\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt})$, therefore the homology class $[T_{\theta_3}T_{\theta_2}^{-1}]\in\mathrm{H}_1(\pi_1(\Sigma_2\setminus\alpha_2,\mathrm{pt}),\mathbb{Z})$ is trivial. Therefore $[T_{\theta_2}]\otimes[T_{\theta_3}T_{\theta_2}^{-1}]=0$. This concludes the proof. □

It turns out that in the case $i=4$ the abelian cycle $\mathcal{A}(T_{\theta_1},T_{\theta_3},T_{\theta_2})\in\mathrm{H}_3(\mathcal{I}_{M_1^i},\mathbb{Z})$ is nonzero. Now let us consider this case in more detail. Consider the multicurve $N=\alpha_1\cup\alpha_2\cup\alpha_3$ shown in Fig. 13. Fore each admissible symplectic splitting $\mathrm{H}=U_1\oplus U_2\oplus U_3$ we can take disjoint admissible separating curves θ_j with $\mathrm{H}_{\theta_j}=U_j$, $j=1,2,3$. Let us denote

$$\mathcal{A}_{U_1,U_2,U_3}=\mathcal{A}(T_{\theta_1},T_{\theta_2},T_{\theta_3})\in\mathrm{H}_3(\mathcal{I}_N,\mathbb{Z}).$$

The main goal of the next two subsections is to prove the following lemma.

Lemma 6.5. The abelian cycles $\mathcal{A}_{U_1,U_2,U_3}$ is a basis of a free abelian subgroup in $\mathrm{H}_3(\mathcal{I}_N,\mathbb{Z})$. Here $\{U_1,U_2,U_3\}$ runs over the set of all unordered admissible symplectic splittings of H.

In order to prove Lemma 6.5 we need to construct new *CW*-complex called *complex of relative cycles*. The idea is to introduce some analogue of $\mathcal{B}_g$ that makes sense for a sphere (i.e. $g=0$ case) with punctures.

6.3 Complex of relative cycles

Recall that by $\Sigma_{0,2g}$ we denote a sphere with $2g$ punctures. In order to construct the complex of relative cycles $\mathcal{B}_{0,2g}$ we need some additional structure. Let us split the punctures into two disjoint sets: $P=\{p_1,\dots,p_g\}$ and $Q=\{q_1,\dots,q_g\}$.

By *arc* on $\Sigma_{0,2g}$ we mean an embedded oriented curve with endpoints at punctures. By *multiarc* we mean a disjoint union of arcs (common endpoints are allowed). We always consider arcs and multiarcs up to an isotopy.

Denote by $\mathcal{D}$ the set of isotopy classes of arcs starting at some point from P and finishing at some point from Q. *Relative basis 1-cycle* is a formal sum $\gamma=\gamma_1+\dots+\gamma_g$ where $\gamma_i\in\mathcal{D}$ such that

(1) in the group of singular 1-chains we have $\partial(\sum_{i=1}^g\gamma_i)=\sum_{i=1}^g(q_i-p_i)$.

(2) we can choose pairwise disjoint representatives of the isotopy classes $\gamma_1, \dots, \gamma_g$,

The set $\gamma_1 \cup \dots \cup \gamma_g$ is called the *support* of γ. Denote by $\mathcal{L}$ the set multiarcs $L = \gamma_1 \cup \dots \cup \gamma_s$ (for arbitrary s) satisfying the following property:

(i) fore each $1 \le i \le s$ there exists a relative basis 1-cycle supported in L and contains γ_i.

For each $L \in \mathcal{L}$ let us denote by $P_L \subset \mathbb{R}_+^{\mathcal{D}}$ the convex hole of the relative basis 1-cycles supported in L. Obviously P_L is a convex polytope. By definition complex of relative cycles is the regular CW-complex given by $\mathcal{B}_{0,2g} = \cup_{L\in\mathcal{L}} P_L$. Denote by $\mathcal{L}_0 \subseteq \mathcal{L}$ the set of supports of all relative basis 1-cycles. Obviously $\{P_L \mid L \in \mathcal{L}_0\}$ is the set of 0-cells of $\mathcal{B}_{0,2g}$.

Remark 6.6. By construction $\mathcal{B}_{0,2g}$ is the subset of $\mathbb{R}^{\mathcal{D}}$ consisting of the points (formal sums) $\sum_{i=1}^n k_i\gamma_i$ where $\gamma_i \in \mathcal{D}$ and $k_i \in \mathbb{R}_{\ge 0}$ satisfying the following conditions:

(1) in the group of singular 1-chains we have $\partial(\sum_{i=1}^n k_i\gamma_i) = \sum_{i=1}^g (q_i - p_i)$.

(2) we can choose pairwise disjoint representatives of the isotopy classes $\gamma_1, \dots, \gamma_n$.

Theorem 6.7. *Let $g \ge 1$. Then $\mathcal{B}_{0,2g}$ is contractible.*

In our proof we follow ideas of [1, Section 5]. Let us define the auxiliary complex $\widetilde{\mathcal{B}}_{0,2g}$. Denote by $\widetilde{\mathcal{D}}$ the union of $\mathcal{D}$ and the set consisting of the isotopy classes of all oriented simple closed curves on $\Sigma_{0,2g}$ (including contractible curves). Let us define $\widetilde{\mathcal{B}}_{0,2g}$ as the subset in $\mathbb{R}_{\ge 0}^{\widetilde{\mathcal{D}}}$ consisting of the points (formal sums) $\sum_{i=1}^n k_i\gamma_i$ where $\gamma_i \in \widetilde{D}$ and $k_i \in \mathbb{R}_{\ge 0}$ satisfying the following conditions:

(1) in the group of singular 1-chains we have $\partial(\sum_{i=1}^n k_i\gamma_i) = \sum_{i=1}^g (q_i - p_i)$.

(2) we can choose pairwise disjoint representatives of the isotopy classes $\gamma_1, \dots, \gamma_n$.

Remark 6.6 implies that $\mathcal{B}_{0,2g} \subseteq \widetilde{\mathcal{B}}_{0,2g}$. Denote by Drain : $\widetilde{\mathcal{B}}_{0,2g} \to \mathcal{B}_{0,2g}$ the retraction induced by the natural projection $\mathbb{R}_{\ge 0}^{\widetilde{\mathcal{D}}} \to \mathbb{R}_{\ge 0}^{\mathcal{D}}$.

Let d and d' be two points of $\mathcal{B}_{0,2g} \subseteq \mathbb{R}_{\ge 0}^{\mathcal{D}}$ and $t \in [0,1]$. In general the point $c = td + (1-t)d' \in \mathbb{R}_{\ge 0}^{\mathcal{D}}$ does not belong to $\mathcal{B}_{0,2g}$. We now explain how to do surgery to convert c into the point Surger$(c) \in \widetilde{\mathcal{B}}_{0,2g} \subseteq \mathbb{R}_{\ge 0}^{\widetilde{\mathcal{D}}}$, which is canonical up to isotopy.

Let $c = \sum_{i=1}^n k_ic_i$ where $c_i \in \widetilde{\mathcal{D}}$ and $k_i \in \mathbb{R}_+$. In the group of singular 1-chains we have $\partial(\sum_{i=1}^n k_ic_i) = \sum_{i=1}^g (q_i - p_i)$. Now it is convenient to assume that $p_1, \dots, p_g, q_1, \dots, q_g$ are not punctures but closed discs. We thicken each c_i to a rectangle $R_i = [0,1] \times [0, k_i]$ of width k_i with coordinates $x_i \in [0,1]$ and $t_i \in [0, k_i]$ such that the curves $t_i = const$ are transversal to each other. We assume that the sides of R_i given by $x = 0$ and $x = 1$ are subsets of ∂p_a and ∂q_b respectively, where $\partial c_i = q_b - p_a$. Also we assume that t_i is a smooth function in some neighbourhood of R_i and $dt_i = 0$ outside R_i.

For a path $\alpha : [0,1] \to \Sigma_{0,2g}$ define $\mu_i(\alpha) = \int_\alpha dt_i$ and $\mu(\alpha) = \sum_{i=1}^g \mu_i(\alpha)$. Let us fix an arbitrary point $x_0 \in \Sigma_{0,2g}$. For each point $x \in \Sigma_{0,2g}$ choose any path α_x from x_0 to x. Consider the map $\phi : \Sigma_{0,2g} \to S^1 = \mathbb{R}/\mathbb{Z}$ given by $\phi(x) = \{\mu(\alpha_x)\}$ (here by $\{y\}$ we denote the fractional part of y).

Let us check that the map ϕ is well defined. Obviously, $\phi(x) = \{\mu(\alpha_x)\}$ depends only on homotopy class of α_x. Therefore it suffices to check that $\mu(\partial p_i) \in \mathbb{Z}$ and $\mu(\partial q_i) \in \mathbb{Z}$ for all i. This follows from the fact that $\partial(\sum_{i=1}^n k_ic_i) = \sum_{i=1}^g (q_i - p_i)$.

The set of zeros of $d\phi$ is precisely $\Sigma_{0,2g} \backslash \cup_{i=1}^g R_i$, that is finite disjoint union of connected open sets. Therefore the map ϕ has a finite number of critical values separating S^1 into a finite number of intervals $w_1, \dots, w_l$. For any $1 \le j \le l$ take any point $y_j \in w_j$. The preimage $\eta_j = \phi^{-1}(y_j) \subset \Sigma_{0,2g}$ is a smooth 1-dimensional submanifold. Moreover,

$\eta_1, \ldots, \eta_l$ are pairwise disjoint. Define Surger(c) as the formal sum $\sum_{j=1}^{l} |w_j|\eta_j \in \mathbb{R}_{\geq 0}^{\widetilde{\mathcal{D}}}$. Since for all i the restrictions $\phi|_{\partial p_i}$ and $\phi|_{\partial q_i}$ have degrees -1 and 1 respectively we have Surger$(c) \in \widetilde{\mathcal{B}}_{0,2g}$.

Proof of Theorem 6.7. Take a point $c \in \mathcal{B}_{0,2g}$. Then the map

$$d \mapsto \text{Drain}(\text{Surger}(tc + (1-t)d))$$

is a deformation retraction from $\mathcal{B}_{0,2g}$ to the point c. □

6.4 Stabilizer dimensions

Proposition 6.8. The group $\text{PMod}(\Sigma_{0,2g})$ acts on $\mathcal{B}_{0,2g}$ without rotations.

Proof. Assume the converse and consider an element $[\phi] \in \text{PMod}(\Sigma_{0,2g})$ and a cell corresponding to a multiarc $\gamma = \gamma_1 \cup \cdots \cup \gamma_s$ such that $\phi(\gamma_i) = \gamma_{\pi(i)}$ for some nontrivial permutation π. Without loss of generality can assume that there exist arcs $\gamma_1, \gamma_2, \gamma_3$ from $p \in P$ to $q \in Q$ such that $\phi(\gamma_1) = \gamma_2$ and $\phi(\gamma_2) = \gamma_3$ (possibly $\gamma_1 = \gamma_3$). Denote by $W_1 \subset \Sigma_{0,2g}$ and $W_2 \subset \Sigma_{0,2g}$ the subsurfaces bounded by the loops $\gamma_1 \circ \overline{\gamma}_2$ and $\gamma_2 \circ \overline{\gamma}_3$ respectively (by $\overline{\gamma}_i$ we denote the arc γ_i with opposite direction).

By construction of $\mathcal{B}_{0,2g}$ we have that γ_1 is not isotopic to γ_2, so W_1 contains some nonempty set of punctures $\varnothing \neq Z_1 \subset P \sqcup Q$. Define $\varnothing \neq Z_2 \subset P \sqcup Q$ in the similar way. Since γ_2 separates W_1 and W_2 we have that $Z_1 \neq Z_2$. The map f preserves the orientation, therefore $f(W_1) = W_2$ and so $f(Z_1) = Z_2$. However, $f \in \text{PMod}(\Sigma_{0,2g})$ preserves the punctures, a contradiction. □

Theorem 6.9. *Let σ be a cell of $\mathcal{B}_{0,2g}$. Then*

$$\dim(\sigma) + \text{cd}(\text{Stab}_{\text{PMod}(\Sigma_{0,2g})}(\sigma)) \leq 2g - 3. \tag{29}$$

Proof. The cell σ is given by some multiarc $\gamma_1 \cup \cdots \cup \gamma_E$. Consider the planar graph Υ on the sphere with the vertices $p_1, \ldots, p_g, q_1, \ldots, q_g$ and the edges $\gamma_1, \ldots, \gamma_E$. It is convenient for us to denote number of vertices by $V = 2g$. Also let us denote by C number of the connected components of Υ and by F number of its faces (i.e. number of connected components of $\Sigma_{0,2g} \setminus \Upsilon$).

Lemma 6.10.

$$\dim(\sigma) = \dim(\text{H}_1(\Upsilon, \mathbb{R})) = E - V + C. \tag{30}$$

Proof. The condition $\partial(\sum_{i=1}^{E} k_i\gamma_i) = \sum_{i=1}^{g}(q_i - p_i)$ is a nonhomogeneous system of linear equation in $\mathbb{R}^E$. The affine space of its solutions has the same dimension as the space of solutions of the homogeneous system $\partial(\sum_{i=1}^{E} k_i\gamma_i) = 0$. This space is precisely $\text{H}_1(\Upsilon, \mathbb{R})$. The cell σ is given by the intersection of this affine space with $\mathbb{R}_{\geq 0}^{E}$. Condition (i) in the construction of $\mathcal{B}_{0,2g}$ implies that σ contains a point from the interior of $\mathbb{R}_{\geq 0}^{E}$, therefore we have $\dim(\sigma) \leq \dim(\text{H}_1(\Upsilon, \mathbb{R}))$. The second equality in formula (30) is trivial. □

Denote by $Y_1, \ldots, Y_F$ the connected components of $\Sigma_{0,2g} \setminus \Upsilon$. We have $Y_i \cong \Sigma_{0,k_i}$ for some k_i.

Proposition 6.11. $\text{Stab}_{\text{PMod}(\Sigma_{0,2g})}(\sigma) \cong \text{Mod}(\Sigma_0^{k_1}) \times \cdots \times \text{Mod}(\Sigma_0^{k_F})$.

Proof. Denote by $\overline{Y}_i$ the closure of $Y_i = \Sigma_{0,k_i}$ in the sphere. Let $\widetilde{Y}_i \cong \Sigma_0^{k_i}$ be the compactification of Y_i given by replacing each puncture by a boundary component. Let $p_i : \widetilde{Y}_i \to \overline{Y}_i$ be the natural projection. Hence we have the corresponding mapping $\Phi_i : \mathrm{Mod}(\widetilde{Y}_i) \to \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$. It suffuses to prove that the obvious mapping

$$\Phi : \mathrm{Mod}(\widetilde{Y}_1) \times \cdots \times \mathrm{Mod}(\widetilde{Y}_F) \to \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$$

is an isomorphism. We use the Alexander method (see [6, Proposition 2.8]). In the proof we need to distinguish between mapping classes and their representatives. The mapping class of a homeomorphism ψ is denoted by $[\psi]$.

First we prove the surjectivity of Φ. Let $[\psi] \in \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$. Then $\psi(\delta)$ is isotopic to δ for each arc δ of Υ. All such arcs are disjoint, so the Alexander method implies that there is a representative $\psi' \in [\psi]$ such that $\psi(\delta) = \delta$ for each arc δ of Υ. Denote $\phi'_i = \psi'|_{\overline{Y}_i}$. Since ϕ'_i is identical on $\partial\overline{Y}_i$ there exist $\phi_i \in \mathrm{Homeo}^+(\widetilde{Y}_i)$ such that $p_i \circ \phi_i = \phi'_i \circ p_i$. Hence we have $\Phi([\phi_1], \dots, [\phi_F]) = [\psi]$.

Now we prove that Φ is injective. Let $\Phi([\psi_1], \dots, [\psi_F]) = [\mathrm{id}]$. Since for each i the mapping $\psi_i|_{\partial\widetilde{Y}_i}$ is identical, there exists $\psi'_i \in \mathrm{Homeo}^+(\overline{Y}_i)$ such that $p_i \circ \psi_i = \psi'_i \circ p_i$. Consider the mapping $\psi' \in \mathrm{Homeo}^+(\Sigma_{0,2g})$ such that $\psi'|_{\overline{Y}_i} = \psi'_i$ for all i. By assumption ψ' is isotopic to the identity map.

Let Υ' be a planar graph the sphere obtained from Υ by adding some number of arcs such that each face of Υ' is a disk. Let us show that there is an isotopy F with $F(0) = \psi'$ such that F is identical on Υ and maps $F_1(\psi'(\delta)) = \delta$ for each arc δ of Υ'. We prove by induction on the number of arcs in $\Upsilon' \setminus \Upsilon$. Hence it suffices to consider the case $\Upsilon' = \Upsilon \cup \{\gamma\}$. We can assume that $\psi'(\gamma)$ is transversal to Υ. If $\psi'(\gamma)$ is disjoint from γ then these two arcs bound a disk on the sphere. This disk is disjoint from Υ since $\psi(\gamma)$ and γ are disjoint from Υ. Hence in this case such un isotopy exists. If $\psi'(\gamma)$ and γ intersect, they form a bigon (see [6, Proposition 1.7]) that is also disjoint from Υ. Hence we can increase the number of intersection points of γ and $\psi'(\gamma)$.

Denote $\phi' = F(1)$, $\phi'_i = \phi'|_{\overline{Y}_i}$ and $F'_i = F'|_{\overline{Y}_i}$. There exist the mappings $\phi_i \in \mathrm{Homeo}(\widetilde{Y}_i)$ and the isotopies F_i of $\widetilde{Y}_i$ such that $p_i \circ \phi_i = \phi'_i \circ p_i$ and $p_i \circ F_i = F'_i \circ p_i$. Therefore F_i is an isotopy between ψ_i and ϕ_i. By construction ϕ_i is identical on a collection of arcs that fill $\widetilde{Y}_i$ (*fill* mean that each connected component of the complement to this collection is a disk). Hence the Alexander method implies that ϕ_i is isotopic to identity for each i. Therefore ψ_i is also isotopic to identity. This concludes the proof. □

Denote by PB_n the pure braid group on n strands. For $k \geq 2$ we have $\mathrm{Mod}(\Sigma_0^k) \cong \mathrm{PB}_{k-1} \times \mathbb{Z}^{k-1}$. Since $\mathrm{cd}(\mathrm{PB}_{k-1}) = k-2$ we have $\mathrm{cd}(\mathrm{PB}_{k-1} \times \mathbb{Z}^{k-1}) = 2k-3$. In the case $k = 1$ we have $\mathrm{cd}(\mathrm{Mod}(\Sigma_0^1)) = 0$. Denote by D the number of Y_i that are homeomorphic to the disk. Proposition 6.11 immediately implies the following result.

Corollary 6.12. $\mathrm{cd}(\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma)) = \sum_{i=1}^{F}(2k_i - 3) + D$.

Let us finish the proof of Theorem 6.9. By Lemma 6.10 and Corollary 6.12 we have

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma)) = E - V + C + \sum_{i=1}^{F}(2k_i - 3) + D =$$

$$= E - V + C + D - 3F + 2\sum_{i=1}^{F} k_i.$$

Let $\Theta_1, \dots, \Theta_C$ we the connected components of Υ. Note that

$$\sum_{i=1}^{F} k_i = |\{(Y_i, \Theta_j) \;|Y_i \text{ adjacent to } \Theta_j\}| = \sum_{j=1}^{C} (\dim(H_1(\Theta, \mathbb{R})) + 1) =$$

$$= \dim(H_1(\Upsilon, \mathbb{R})) + C = E - V + 2C.$$

Therefore, we have

$$E - V + C + D - 3F + 2\sum_{i=1}^{F} k_i = E - V + C + D - 3F + 2(E - V + 2C) =$$

$$= 3E - 3V + 5C - 3F + D = 2C + D - 3(V - E + F - C).$$

We will use the following simple fact.

Fact 6.13. For each planar graph on the sphere we have $V - E + F - C = 1$.

Lemma 6.14. $2C + D \leq 2g$.

Proof. If $F = 1$ then it is obvious. From now let us assume that $F \geq 2$. Since $D \leq F$ and $V = 2g$ it suffices to check that

$$2C + F \leq V. \tag{31}$$

Assume that Υ has a connected component with only one edge. If we remove this component, then the both sides of inequality (31) decrease by 2 and the remaining graph will represent some 0-cell of $\mathcal{B}_{0,2g-2}$. Therefore we can assume that Υ contains no such a connected component. We also can assume that $F \geq 2$ since this operation does not change F.

Note that Υ is a bipartite graph and does not contain isotopic edges. This implies that if foe some i, j we have Y_i adjacent to Θ_j then Y_i adjacent to at least 4 edges of Θ_j. Then we have

$$E \geq 2\sum_{i=1}^{F} k_i = 2E - 2V + 4C = 2C + 2F - 2.$$

The last inequality follows from Fact 6.13. Since $C \geq 1$ we have

$$E \geq 2C + 2F - 2 \geq 2F + C - 1.$$

We can rewrite this as follows.

$$2C + F \leq 1 + C - F + E. \tag{32}$$

Fact 6.13 implies that the right hand side of (32) equals V. Therefore the inequality (31) holds. □

Lemma 6.14 and Fact 6.13 imply that

$$2C + D - 3(V - E + F - C) \leq 2g - 3.$$

This completes the proof of Theorem 6.9. □

Let $K \subseteq \mathrm{PMod}(\Sigma_{0,2g})$ be a subgroup. Denote by $\hat{E}^*_{*,*}$ the spectral sequence (7) for the action of K on $\mathcal{B}_{0,2g}$. Theorem 6.9 implies that for any cell σ of $\mathcal{B}_{0,2g}$ we have

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_K(\sigma)) \leq 2g-3. \tag{33}$$

This immediately implies $\hat{E}^1_{p,q} = 0$ for $p+q > 2g-3$. Hence all differentials $\hat{d}^1, \hat{d}^2, \dots$ to the group $\hat{E}^1_{0,2g-3}$ are trivial, so $\hat{E}^1_{0,2g-3} = \hat{E}^\infty_{0,2g-3}$. Therefore we have the following result.

Proposition 6.15. Let $\mathfrak{L} \subseteq \mathcal{L}_0$ be a subset consisting of mutiarcs from pairwise disjoint K-orbits. Then the inclusions $\mathrm{Stab}_K(L) \subseteq K$, $L \in \mathfrak{L}$ induce the injective homomorhpism

$$\bigoplus_{L\in\mathfrak{L}} \mathrm{H}_{2g-3}(\mathrm{Stab}_K(L),\mathbb{Z}) \hookrightarrow \mathrm{H}_{2g-3}(K),\mathbb{Z}). \tag{34}$$

Proof of Lemma 6.5. Let $N = \alpha_1\cup\alpha_2\cup\alpha_3$ as in Fig. 13. Consider an admissible symplectic splitting $\mathcal{U} = \{U_1, U_2, U_3\}$ and let θ_j be separating curves such that $\mathrm{H}_{\theta_j} = U_j$, $j = 1, 2, 3$. Denote by X_{θ_j} the one-punctured tori bounded by θ_j. Let β'_j be a curve on X_{θ_j} such that the algebraic intersection number of α_j and β'_j is one. Denote $B'_{\mathcal{U}} = \beta'_1 \cup \beta'_2 \cup \beta'_3$. The homology class $\mathcal{A}_{U_1,U_2,U_3} \in \mathrm{H}_3(\mathrm{Stab}_{I_N}(B'_{\mathcal{U}}),\mathbb{Z})$ is well defined.

Consider the surface $\Sigma_{0,6} = \Sigma \setminus N$; we have the inclusion $\mathcal{I}_N \hookrightarrow \mathrm{PMod}(\Sigma_{0,6})$. Denote $\beta_j = \beta'_j \cap \Sigma_{0,6}$ and consider the multiarc $B_{\mathcal{U}} = \beta_1 \cup \beta_2 \cup \beta_3$. Obviously $\mathrm{Stab}_{I_N}(B'_{\mathcal{U}}) = \mathrm{Stab}_{I_N}(B_{\mathcal{U}})$. One can easily check that $\mathrm{Stab}_{I_N}(B_{\mathcal{U}}) \cong \mathbb{Z}^3$ and the homology class $\mathcal{A}_{U_1,U_2,U_3} \in \mathrm{H}_3(\mathrm{Stab}_{I_N}(B_{\mathcal{U}}),\mathbb{Z})$ is nonzero.

If $\mathcal{U} \neq \mathcal{U}'$ are two admissible symplectic splittings for N it follows that $B_{\mathcal{U}}$ and $B_{\mathcal{U}'}$ belong to different $\mathcal{I}_N$-orbits. Let $\mathfrak{L} = \{\mathcal{B}_{\mathcal{U}}\}$, where $\mathcal{U}$ runs over the set of all admissible symplectic splittings for N. Proposition 6.15 implies the result. □

6.5 The term $E^2_{1,3}$

Consider an unordered symplectic splitting $\mathcal{U} = \{U_1, U_2, U_3\}$ of H. There is the unique decomposition $x = x_1 + x_2 + x_3$, where $x_j \in U_j$. Renumbering U_1, U_2, and U_3 we may achieve that there are three possible cases: $x_1 \neq 0$ and $x_2 = x_3 = 0$, $x_1, x_2 \neq 0$ and $x_3 = 0$, $x_1, x_2, x_3 \neq 0$. We say that the splitting $\mathcal{U}$ *is of type (i), (ii) or (iii)* w.r.t x, respectively.

For $j = 1, 2, 3$ let $x_j = k_j a_j$, where a_j is a primitive homology class and $k_j \in \mathbb{N}$ (if $x_j = 0$ we put $a_j = 0$). Consider a bounding pair α_j, α'_j, with $[\alpha_j] = [\alpha'_j] = a_j$ and the splitting $\mathcal{U}$ is admissible for $\alpha_j \cup \alpha'_j$ (if $a_j = 0$ we define $\alpha_j = \alpha'_j = \varnothing$). All such bounding pairs are $\mathcal{I}$-equivalent.

If $\mathcal{U}$ is of type (i) w.r.t x, we consider the element

$$P_{\alpha_1\cup\alpha'_1} \otimes \mathcal{A}_{U_2,U_3} \in E^1_{1,3}. \tag{35}$$

If $\mathcal{U}$ is of type (ii) w.r.t x, we consider the elements

$$P_{\alpha_1\cup\alpha'_1\cup\alpha_2} \otimes \mathcal{A}_{U_2,U_3}, P_{\alpha_1\cup\alpha_2\cup\alpha'_2} \otimes \mathcal{A}_{U_1,U_3} \in E^1_{1,3}. \tag{36}$$

If $\mathcal{U}$ is of type (iii) w.r.t x, we consider the elements

$$P_{\alpha_1\cup\alpha'_1\cup\alpha_2\cup\alpha_3} \otimes \mathcal{A}_{U_2,U_3}, P_{\alpha_1\cup\alpha_2\cup\alpha'_2\cup\alpha_3} \otimes \mathcal{A}_{U_3,U_1}, P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha'_3} \otimes \mathcal{A}_{U_1,U_2} \in E^1_{1,3}. \tag{37}$$

Note that the cells mentioned above are uniquely determined (up to $\mathcal{I}$-equivalence) by $\mathcal{U}$. Corollary 6.2 implies that the elements (35), (36) and (37), where $\mathcal{U}$ runs over the set of all splittings of H, form a basis of the free abelian group $E^1_{1,3}$.

Lemmas 6.3 and 6.4 (we take $\overline{G}=\mathcal{I}$ and $\overline{\mathcal{B}}(x)=\mathcal{B}(x)$) imply that the images of the elements (35) and (36) differential $d^1_{1,3}$ are zero. By Lemma 6.5, the image of each of three elements (37) is

$$P_{\alpha_1\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}_{U_1,U_3,U_2}\in E^1_{0,3},$$

and these classes are linearly independent for different $\mathcal{U}$. Therefore we have the following result.

Proposition 6.16. The free abelian group $E^2_{1,3}=\ker d^1_{1,3}$ has a basis consisting of the following elements:

$$P_{\alpha_1\cup\alpha_1'}\otimes\mathcal{A}_{U_2,U_3},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (i) w.r.t x,

$$P_{\alpha_1\cup\alpha_1'\cup\alpha_2}\otimes\mathcal{A}_{U_2,U_3},\,P_{\alpha_1\cup\alpha_2\cup\alpha_2'}\otimes\mathcal{A}_{U_1,U_3},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (ii) w.r.t x,

$$P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}_{U_2,U_3}-P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}_{U_3,U_1},\,P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}_{U_3,U_1}-P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'}\otimes\mathcal{A}_{U_1,U_2},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (iii) w.r.t x.

Proof of Proposition 3.6. Recall that for a separating curve γ on Σ denote by $E^{(\gamma)*}_{*,*}$ the spectral sequence (7) for the action of $\mathcal{I}^{(\gamma)}$ on $\mathcal{B}(x)$. By $j^{(\gamma)*}_{*,*}:E^{(\gamma)*}_{*,*}\to E^*_{*,*}$ we denote the morphism of the spectral sequences induced by the inclusion $\iota^{(\gamma)}:\mathcal{I}^{(\gamma)}\hookrightarrow\mathcal{I}$.

We have the morphism of the spectral sequences

$$J^*_{*,*}:\widehat{E}^*_{*,*}\to E^*_{*,*}.$$

Here $J^*_{*,*}=\bigoplus_\gamma j^{(\gamma)*}_{*,*}$ and $\widehat{E}^*_{*,*}=\bigoplus_\gamma E^{(\gamma)*}_{*,*}$, where the sums are over all separating curves γ on Σ. Our goal is to prove the surjectivity of $J^2_{1,3}$. In Proposition 6.16 we constructed a basis of the free abelian group $E^2_{1,3}$. Let us show that each of this elements belongs to the image of $J^2_{1,3}$. Let $\mathcal{U}=(U_1,U_2,U_3)$ be a symplectic splitting of H. As before, let $\theta_1,\theta_2,\theta_3$ be separating curves disjoint from α_j and α_j' such that $\mathrm{H}_{\theta_j}=U_j$ ($j=1,2,3$). There are three possible cases.

(i) *$\mathcal{U}$ is of type (i) w.r.t x.* We need to check that $P_{\alpha_1\cup\alpha_1'}\otimes\mathcal{A}_{U_2,U_3}$ belongs to the image of $J^2_{1,3}$. Consider the group $\mathcal{I}^{(\theta_2)}$ and the homology class

$$\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha_1'},T_{\theta_3})\in\mathrm{H}_3(\mathcal{I}^{(\theta_2)}_{\alpha_1\cup\alpha_1'},\mathbb{Z}).$$

Let us consider the element

$$P_{\alpha_1\cup\alpha_1'}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha_1'},T_{\theta_3})\in E^{(\theta_2)1}_{3,1}.$$

Lemmas 6.3 and 6.4 (we take $\overline{G}=\mathcal{I}^{(\theta_2)}$ and $\overline{\mathcal{B}}(x)=\mathcal{B}_\alpha(x)$) imply that

$$d^{(\theta_2)1}_{1,3}\left(P_{\alpha_1\cup\alpha_1'}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha_1'},T_{\theta_3})\right)=0.$$

Hence

$$P_{\alpha_1\cup\alpha_1'}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha_1'},T_{\theta_3})\in E^{(\theta_2)2}_{3,1}.$$

Obviously we have

$$J^2_{1,3}\left(P_{\alpha_1\cup\alpha'_1}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\right)=P_{\alpha_1\cup\alpha'_1}\otimes\mathcal{A}_{U_2,U_3}.$$

(ii) $\mathcal{U}$ *is of type (ii) w.r.t* x. We need to check that $P_{\alpha_1\cup\alpha'_1\cup\alpha_2}\otimes\mathcal{A}_{U_2,U_3}$ belongs to the image of $J^2_{1,3}$ (the case of $P_{\alpha_1\cup\alpha_2\cup\alpha'_2}\otimes\mathcal{A}_{U_1,U_3}$ is similar). Consider the group $\mathcal{I}^{(\theta_2)}$ and the homology class

$$\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\in \mathrm{H}_3(\mathcal{I}^{(\theta_2)}_{\alpha_1\cup\alpha'_1\cup\alpha_2},\mathbb{Z}).$$

Let us consider the element

$$P_{\alpha_1\cup\alpha'_1\cup\alpha_2}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\in E^{(\theta_2)1}_{3,1}.$$

Lemmas 6.3 and 6.4 imply that

$$d^{(\theta_2)1}_{1,3}\left(P_{\alpha_1\cup\alpha'_1\cup\alpha_2}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\right)=0.$$

Hence

$$P_{\alpha_1\cup\alpha'_1\cup\alpha_2}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\in E^{(\theta_2)2}_{3,1}.$$

Obviously we have

$$J^2_{1,3}\left(P_{\alpha_1\cup\alpha'_1\cup\alpha_2}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\right)=P_{\alpha_1\cup\alpha'_1\cup\alpha_2}\otimes\mathcal{A}_{U_2,U_3}.$$

(iii) $\mathcal{U}$ *is of type (iii) w.r.t* x. We need to check that

$$P_{\alpha_1\cup\alpha'_1\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}_{U_2,U_3}-P_{\alpha_1\cup\alpha_2\cup\alpha'_2\cup\alpha_3}\otimes\mathcal{A}_{U_3,U_1}$$

belongs to the image of $J^2_{1,3}$ (the case of

$$P_{\alpha_1\cup\alpha_2\cup\alpha'_2\cup\alpha_3}\otimes\mathcal{A}_{U_3,U_1}-P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha'_3}\otimes\mathcal{A}_{U_1,U_2}$$

is similar). Consider the group $\mathcal{I}^{(\theta_3)}$ and the homology classes

$$\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3}),\in \mathrm{H}_3(\mathcal{I}^{(\theta_2)}_{\alpha_1\cup\alpha'_1\cup\alpha_2\cup\alpha_3},\mathbb{Z})$$

and

$$\mathcal{A}(T_{\theta_3},T_{\alpha_2}T^{-1}_{\alpha'_2},T_{\theta_1})\in \mathrm{H}_3(\mathcal{I}^{(\theta_2)}_{\alpha_1\cup\alpha_2\cup\alpha'_2\cup\alpha_3},\mathbb{Z}).$$

Let us consider the elements

$$P_{\alpha_1\cup\alpha'_1\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\in E^{(\theta_3)1}_{3,1}$$

and

$$P_{\alpha_1\cup\alpha_2\cup\alpha'_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_3},T_{\alpha_2}T^{-1}_{\alpha'_2},T_{\theta_1})\in E^{(\theta_3)1}_{3,1}.$$

Lemma 6.3 implies that

$$d^{(\theta_3)1}_{1,3}\left(P_{\alpha_1\cup\alpha'_1\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T^{-1}_{\alpha'_1},T_{\theta_3})\right)=P_{\alpha_1\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\theta_3},T_{\theta_1})$$

and

$$d^{(\theta_3)1}_{1,3}\left(P_{\alpha_1\cup\alpha_2\cup\alpha'_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_3},T_{\alpha_2}T^{-1}_{\alpha'_2},T_{\theta_1})\right)=P_{\alpha_1\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\theta_3},T_{\theta_1}).$$

Hence

$$d_{1,3}^{(\theta_3)1}\left(P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T_{\alpha_1'}^{-1},T_{\theta_3})-P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_3},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_1})\right)=0.$$

Therefore

$$\left(P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T_{\alpha_1'}^{-1},T_{\theta_3})-P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_3},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_1})\right)\in E_{3,1}^{(\theta_3)2}.$$

Obviously we have

$$J_{1,3}^2\left(P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_1}T_{\alpha_1'}^{-1},T_{\theta_3})-P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_3},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_1})\right)=$$

$$=P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3}\otimes\mathcal{A}_{U_2,U_3}-P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}_{U_3,U_1}.$$

This concludes the proof. □

7 Proof of Proposition 3.7

7.1 Auxiliary complex of cycles

Let us describe the structure of the group $E_{0,4}^1$. If we forget about orientation, there are three distinct $\mathrm{Mod}(\Sigma_g)$-orbits in the set $\mathcal{M}_0(x)$. Their representatives are $M_0^1=\alpha_1$, $M_0^2=\alpha_1\cup\alpha_2$ and $M_0^3=\alpha_1\cup\alpha_2\cup\alpha_3$, where the curves α_1, α_2 and α_3 are shown in Fig. 13. Formula (12) applied the multicurves M_0^2 and M_0^3 implies

$$\mathrm{cd}(\mathcal{I}_{M_0^2})\le 6-1-2+0=3,$$

$$\mathrm{cd}(\mathcal{I}_{M_0^3})\le 6-0-3+0=3,$$

Hence we have $\mathrm{H}_4(\mathcal{I}_{M_0^2},\mathbb{Z})=\mathrm{H}_4(\mathcal{I}_{M_0^3},\mathbb{Z})=0$.

Let $\alpha=\alpha_1$ be a curve with $[\alpha]=x$. Since any two homological curves on Σ are $\mathcal{I}$-equivalent, we have in isomorphism

$$\mathrm{H}_4(\mathcal{I}^{(\alpha)},\mathbb{Z})\cong E_{0,4}^1,$$

given by $h\mapsto P_\alpha\otimes h$, where $h\in\mathrm{H}_4(\mathcal{I}^{(\alpha)},\mathbb{Z})$.

Consider any nonzero homology class $y\in\mathrm{H}_1(\Sigma,\mathbb{Z})$ with $y\ne x$ and $x\cdot y=0$. Now we need to introduce auxiliary complex $\mathcal{B}_\alpha(y)\subset\mathcal{B}(y)$. By definition $\mathcal{B}_\alpha(y)$ consists of those cells $P_M\subset\mathcal{B}(y)$, for which we can choose a representative of M disjoint from α. Obviously $\mathcal{B}_\alpha(y)\subset\mathcal{B}(y)$ is a subcomplex and the group $\mathcal{I}^{(\alpha)}$ acts on $\mathcal{B}_\alpha(y)$ cellularly and without rotations. Moreover, for each cell $\sigma\in\mathcal{B}\alpha(y)$ we have

$$\dim(\sigma)+\mathrm{cd}(\mathrm{Stab}_{\mathcal{I}^{(\alpha)}}(\sigma))\le 4. \tag{38}$$

Proposition 7.1. $\mathcal{B}_\alpha(y)$ is contractible.

Proof. The surgical proof of contractibility of $\mathcal{B}(y)$ [1, Section 5] works with no modification. □

Denote by $\mathcal{M}_{\alpha,p}(y)$ the set of multicurves corresponding to the cells of $\mathcal{B}_\alpha(y)$ of dimension p.

Now let $\widetilde{E}^*_{*,*}$ be the spectral sequence (7) for the action on $\mathcal{I}^{(\alpha)}$ on $\mathcal{B}_\alpha(y)$. This spectral sequence has the form

$$\widetilde{E}^1_{p,q} \cong \bigoplus_{M \in \mathcal{M}_{\alpha,p}(y)/\mathcal{I}^{(\alpha)}} \mathrm{H}_q(\mathrm{Stab}_{\mathcal{I}^{(\alpha)}}(M)) \Rightarrow \mathrm{H}_{p+q}(\mathcal{I}^{(\alpha)}, \mathbb{Z}), \tag{39}$$

where by $\mathcal{M}_{\alpha,p}(y)/\mathcal{I}^{(\alpha)}$ we denote the set containing one representative from each $\mathcal{I}^{(\alpha)}$-orbit in the set $\mathcal{M}_{\alpha,p}(y)$.

Corollary 7.2. *Let $\widetilde{E}^*_{*,*}$ be the spectral sequence (39). Then $\widetilde{E}^1_{p,q} = 0$ for $p+q > 4$.*

Lemma 7.3. We have $\widetilde{E}^1_{0,4} = 0$.

Proof. Using formula (12) one can easily check that for any multicurve M from the set $\mathcal{M}_{\alpha,0}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$, we have $\mathrm{cd}(\mathcal{I}^{(\alpha)}_M) < 4$. This implies the result. □

Lemma 7.4. The differential $\widetilde{d}^1_{3,1} : \widetilde{E}^1_{3,1} \to \widetilde{E}^1_{2,1}$ is injective, hence $\widetilde{E}^2_{3,1} = \widetilde{E}^\infty_{3,1} = 0$.

Proof. Let us consider the multicurves M from the set $\mathcal{M}_{\alpha,3}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$, such that $\mathrm{cd}(\mathcal{I}^{(\alpha)}_M) = 1$. Using formula (12) one can easily check that all the representatives are shown in Fig. 14. The multicurve M is shown in green, the curve α is shown in blue. The remaining part of the proof is similar to the proof of Proposition 3.3. □

Figure 14: The multicurves M from the set $\mathcal{M}_{\alpha,3}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$ with $\mathrm{cd}(\mathcal{I}^{(\alpha)}_M) = 1$.

Lemma 7.5. The differential $\widetilde{d}^1_{2,2} : \widetilde{E}^1_{2,2} \to \widetilde{E}^1_{1,2}$ is injective, hence $\widetilde{E}^2_{2,2} = \widetilde{E}^\infty_{2,2} = 0$.

Proof. Let us consider the multicurves M from the set $\mathcal{M}_{\alpha,2}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$, such that $\mathrm{cd}(\mathcal{I}^{(\alpha)}_M) = 2$. Using formula (12) one can easily check that all the representatives are shown in Fig. 15. The multicurve M is shown in green, the curve α is shown in blue.

Let M be one of the multicurves shown in Fig. 15. We have the inclusion

$$\mathrm{H}_2(\mathcal{I}^{(\alpha)}_M, \mathbb{Z}) \hookrightarrow \mathrm{H}_2(\mathcal{I}_M, \mathbb{Z})$$

(these homology groups are already computed in Section 5).

Let us show that if $M, M' \in \mathcal{M}_{\alpha,2}(y)$ are $\mathcal{I}$-equivalent and $\mathrm{cd}(\mathcal{I}^{(\alpha)}_M) = \mathrm{cd}(\mathcal{I}^{(\alpha)}_{M'}) = 2$, it follows that M and M' are $\mathcal{I}^{(\alpha)}$-equivalent. Indeed, let $M = h(M')$ for some $h \in \mathcal{I}$. Then $\beta = h(\alpha)$ is disjoint from M and $[\beta] = [\alpha]$. We may assume that M is one of the green curves shown in Fig. 15. In the cases (b), (e) and (f) the only possibility is $\beta = \alpha$, so $h \in \mathcal{I}^{(\alpha)}$. In the cases (a) and (d) the curve β coincides either with α or with the another component of M, that is homological to α. But the last case is impossible because $\mathcal{I}$ acts on $\mathcal{B}$ without rotations. So we also have $h \in \mathcal{I}^{(\alpha)}$. Let us consider the case (c). Obviously there is an element $f \in \mathrm{Stab}_{\mathrm{Mod}_3}(M)$ such that $f(\beta) = \alpha$. Then there exists $k \in \mathbb{Z}$ such

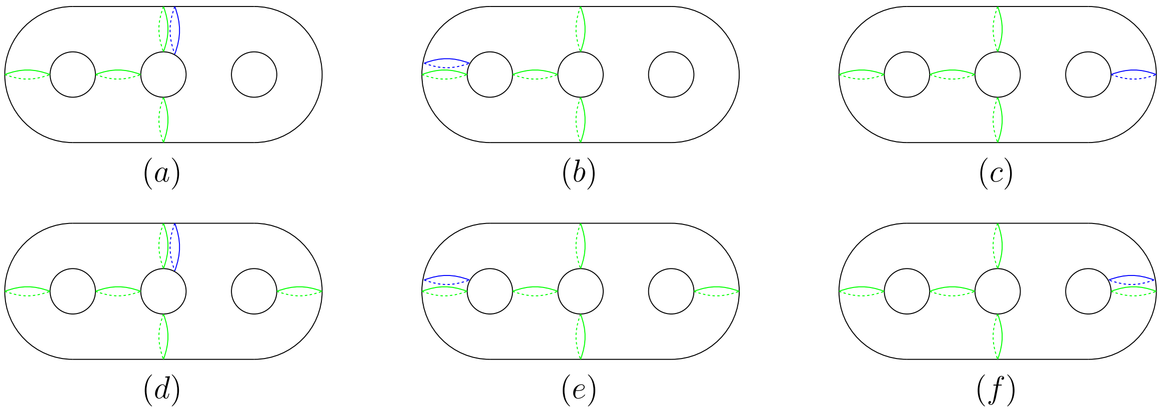


Figure 15: The multicurves M from the set $\mathcal{M}_{\alpha,2}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$ with $\mathrm{cd}(\mathcal{I}_M^{(\alpha)}) = 2$.

that $T_\alpha^k \circ f \in \mathcal{I}$. Therefore we have $T_\alpha^k \circ f \circ h \in \mathcal{I}^{(\alpha)}$ and $T_\alpha^k \circ f \circ h(M') = M$, so M and M' are $\mathcal{I}^{(\alpha)}$-equivalent.

Therefore, the injectivity of the differential $\widetilde{d}^1_{2,2}$ immediately follows from the injectivity of the differential $d^1_{2,2}$ (see Proposition 3.4). □

Lemmas 7.3, 7.4 and 7.5 imply that

$$\widetilde{E}^\infty_{0,4} = \widetilde{E}^\infty_{2,2} = \widetilde{E}^\infty_{3,1} = 0.$$

Corollary 7.6. *We have an isomorphism*

$$\mathrm{H}_4(\mathcal{I}^{(\alpha)}, \mathbb{Z}) \cong \widetilde{E}^\infty_{1,3} \cong \widetilde{E}^2_{1,3}.$$

7.2 The term $\widetilde{E}^2_{1,3}$

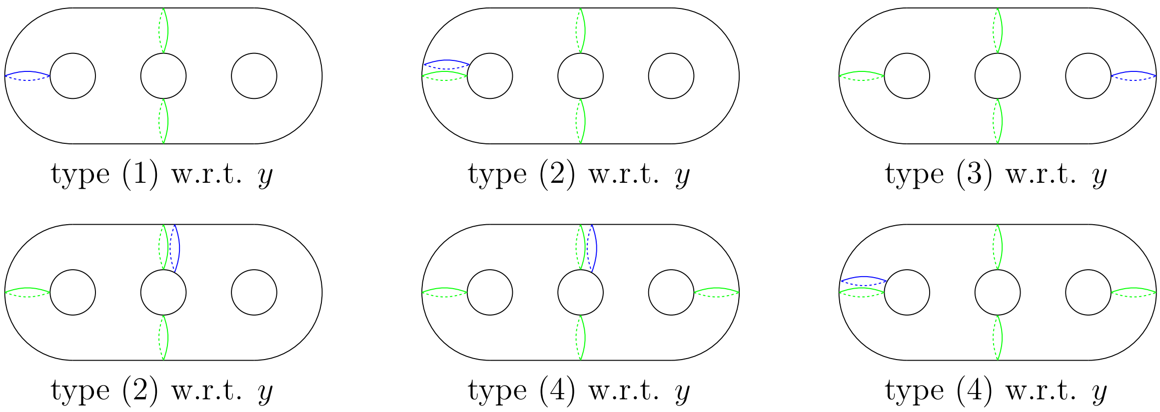


Figure 16: The multicurves M from the set $\mathcal{M}_{1,\alpha}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$ with $\mathrm{cd}(\mathcal{I}_M^{(\alpha)}) = 3$.

Let us compute the group $\widetilde{E}^2_{1,3}$ explicitly. We forget about the orientation and consider the multicurves M from the set $\mathcal{M}_{\alpha,3}(y)/\mathrm{Stab}_{\mathrm{Mod}(\Sigma)}(\alpha)$, such that $\mathrm{cd}(\mathcal{I}_M^{(\alpha)}) = 3$. Using formula (12) one can easily check that all the representatives are shown in Fig. 16. The multicurve M is shown in green, the curve α is shown in blue.

Now let us describe a basis of the free abelian group $\widetilde{E}^1_{1,3}$. Consider an unordered symplectic splitting $\mathcal{U} = \{U_1, U_2, U_3\}$ of H such that $x \in U_1$ (that is, $\mathcal{U}$ is of type (i) w.r.t x). There is the unique decomposition $y = y_1 + y_2 + y_3$, where $y_j \in U_j$. We assume that there are four possible cases:

(1) $y_2 \neq 0$ and $y_1 = y_3 = 0$,

(2) $y_1, y_2 \neq 0$ and $y_3 = 0$,

(3) $y_2, y_3 \neq 0$ and $y_1 = 0$,

(4) $y_1, y_2, y_3 \neq 0$.

We say that $\mathcal{U}$ *is of type* (1) - (4) w.r.t y, respectively.

Suppose that $j = 1, 2, 3$. If $y_i \neq 0$, then put $y_j = k_j b_j$, where b_j is a primitive homology class and $k_j \in \mathbb{N}$. Consider a bounding pair β_j, β'_j disjoint from α, with $[\beta_j] = [\beta'_j] = b_j$ and the splitting $\mathcal{U}$ is admissible for $\beta_j \cup \beta'_j$ (if $b_j = 0$ we define $\beta_j = \beta'_j = \varnothing$). We also always take $\beta_1 = \alpha$ (if β_1 exists). All such bouning pairs are $\mathcal{I}^{(\alpha)}$-equivalent.

If $\mathcal{U}$ is of type (1) w.r.t y, we consider the element

$$P_{\beta_2 \cup \beta'_2} \otimes \mathcal{A}_{U_1, U_3} \in \widetilde{E}^1_{1,3}. \tag{40}$$

If $\mathcal{U}$ is of type (2) w.r.t y, we consider the elements

$$P_{\beta_1 \cup \beta'_1 \cup \beta_2} \otimes \mathcal{A}_{U_2, U_3} \in \widetilde{E}^1_{1,3}. \tag{41}$$

$$P_{\beta_1 \cup \beta_2 \cup \beta'_2} \otimes \mathcal{A}_{U_1, U_3} \in \widetilde{E}^1_{1,3}, \tag{42}$$

If $\mathcal{U}$ is of type (3) w.r.t y, we consider the elements

$$P_{\beta_2 \cup \beta'_2 \cup \beta_3} \otimes \mathcal{A}_{U_3, U_1}, \qquad P_{\beta_2 \cup \beta_3 \cup \beta'_3} \otimes \mathcal{A}_{U_1, U_2} \in \widetilde{E}^1_{1,3}. \tag{43}$$

If $\mathcal{U}$ is of type (4) w.r.t y, we consider the elements

$$P_{\beta_1 \cup \beta'_1 \cup \beta_2 \cup \beta_3} \otimes \mathcal{A}_{U_2, U_3} \in \widetilde{E}^1_{1,3}, \tag{44}$$

$$P_{\beta_1 \cup \beta_2 \cup \beta'_2 \cup \beta_3} \otimes \mathcal{A}_{U_3, U_1}, \qquad P_{\beta_1 \cup \beta_2 \cup \beta_3 \cup \beta'_3} \otimes \mathcal{A}_{U_1, U_2} \in \widetilde{E}^1_{1,3}. \tag{45}$$

Note that the cells mentioned above are uniquely determined (up to $\mathcal{I}^{(\alpha)}$-equivalence) by $\mathcal{U}$.

Corollary 6.2 implies that the elements (40), (41), (41), (43) and (44), where $\mathcal{U}$ runs over the set of all splittings of H, form a basis of the free abelian group $\widetilde{E}^1_{1,3}$.

Lemmas 6.3 and 6.4 (we take $\overline{G} = \mathcal{I}^{(\alpha)}$ and $\overline{\mathcal{B}}(x) = \mathcal{B}_\alpha(x)$) imply that the images of the elements (40) and (42) differential $\widetilde{d}^1_{1,3}$ are zero. By Lemma (6.5), the image of (41) is

$$P_{\beta_1 \cup \beta_2} \otimes \mathcal{A}_{U_2, U_3} - P_{\beta'_1 \cup \beta_2} \otimes \mathcal{A}_{U_2, U_3} \in \widetilde{E}^1_{0,3}.$$

The image of the elements (43) is

$$P_{\beta_2 \cup \beta_3} \otimes \mathcal{A}_{U_1, U_3, U_2} \in \widetilde{E}^1_{0,3};$$

note that by Lemma 6.5 this element is nonzero because $\mathcal{I}^{(\alpha)}_{\beta_2 \cup \beta_3} = \mathcal{I}_{\beta_1 \cup \beta_2 \cup \beta_3}$. The image of (44) is

$$P_{\beta_1 \cup \beta_2 \cup \beta_3} \otimes \mathcal{A}_{U_2, U_3} - P_{\beta'_1 \cup \beta_2 \cup \beta_3} \otimes \mathcal{A}_{U_2, U_3} \in \widetilde{E}^1_{0,3}.$$

The image of the elements (45) is

$$P_{\beta_1 \cup \beta_2 \cup \beta_3} \otimes \mathcal{A}_{U_1, U_3, U_2} \in \widetilde{E}^1_{0,3}.$$

By Lemma 6.5 these elements are linearly independent for different $\mathcal{U}$. Therefore we have the following result.

Proposition 7.7. The free abelian group $\widetilde{E}^2_{1,3} = \ker \widetilde{d}^1_{1,3}$ has a basis consisting of the following elements:

$$P_{\beta_2\cup\beta_2'} \otimes \mathcal{A}_{U_1,U_3},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (1) w.r.t y,

$$P_{\beta_1\cup\beta_2\cup\beta_2'} \otimes \mathcal{A}_{U_1,U_3},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (2) w.r.t y,

$$P_{\beta_2\cup\beta_2'\cup\beta_3} \otimes \mathcal{A}_{U_3,U_1} - P_{\beta_2\cup\beta_3\cup\beta_3'} \otimes \mathcal{A}_{U_1,U_2},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (3) w.r.t y,

$$P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3} \otimes \mathcal{A}_{U_3,U_1} - P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'} \otimes \mathcal{A}_{U_1,U_2},$$

where $\mathcal{U}$ runs over the set of all splittings of H of type (4) w.r.t y.

7.3 The term $E^1_{0,4}$

Proof of Proposition 3.7. We need to prove that the map $J^1_{0,4} : \widehat{E}^1_{0,4} \to E^1_{0,4}$ is surjective. We have in isomorphism

$$E^1_{0,4} \cong \mathrm{H}_4(\mathcal{I}_\alpha, \mathbb{Z}).$$

We need to prove that the map

$$\bigoplus_\gamma \mathrm{H}_4(\mathcal{I}_\alpha^{(\gamma)}, \mathbb{Z}) \to \mathrm{H}_4(\mathcal{I}_\alpha, \mathbb{Z}),$$

where the sum is over all separating curves γ disjoint from α, is surjective. For each such γ let us denote by $\widetilde{E}^{(\gamma)*}_{*,*}$ the spectral sequence (7) for the action of $\mathcal{I}^{(\gamma)}$ on $\mathcal{B}_\alpha(y)$.

Denote by $\widetilde{j}^{(\gamma)*}_{*,*} : \widetilde{E}^{(\gamma)*}_{*,*} \to \widetilde{E}^*_{*,*}$ the morphism of the spectral sequences induced by the inclusion $\widetilde{\iota}^{(\gamma)} : \mathcal{I}_\alpha^{(\gamma)} \hookrightarrow \mathcal{I}_\alpha$. Consider the morphism

$$\bigoplus_\gamma \widetilde{j}^{(\gamma)*}_{*,*} : \bigoplus_\gamma \widetilde{E}^{(\gamma)*}_{*,*} \to \widetilde{E}^*_{*,*} \tag{46}$$

and denote $\widetilde{J}^*_{*,*} = \bigoplus_\gamma \widetilde{j}^{(\gamma)*}_{*,*}$, where the sums are over all separating curves γ on Σ disjoint from α. Corollary 7.6 implies that it suffices to prove that the map $\widetilde{J}^2_{1,3}$ is surjective.

Proposition 7.7 we constructed a basis of the free abelian group $\widetilde{E}^2_{1,3}$. Let us show that each of this elements belongs to the image of $\widetilde{J}^2_{1,3}$. Let $\mathcal{U} = (U_1, U_2, U_3)$ be a symplectic splitting of H. Let $\theta_1, \theta_2, \theta_3$ be separating curves disjoint from α such that $\mathrm{H}_{\theta_j} = U_j$ ($j = 1, 2, 3$). Let β_j, β_j' be as before such that all these curves are pairwise disjoint except the three pairs $(\theta_j,\ \beta_j')$. There are four possible cases.

(1) *$\mathcal{U}$ is of type (1) w.r.t y.* We need to check that $P_{\beta_2\cup\beta_2'} \otimes \mathcal{A}_{U_1,U_3}$ belongs to the image of $\widetilde{J}^2_{1,3}$. Consider the group $\mathcal{I}_\alpha^{(\theta_1)}$ and the homology class

$$\mathcal{A}(T_{\theta_1}, T_{\beta_2}T^{-1}_{\beta_2'}, T_{\theta_3}) \in \mathrm{H}_3(\mathcal{I}^{(\theta_1)}_{\alpha\cup\beta_2\cup\beta_2'}, \mathbb{Z}).$$

Let us consider the element

$$P_{\beta_2\cup\beta_2'} \otimes \mathcal{A}(T_{\theta_1}, T_{\beta_2}T^{-1}_{\beta_2'}, T_{\theta_3}) \in \widetilde{E}^{(\theta_2)1}_{3,1}.$$

The arguments similar to Lemmas 6.3 and 6.4 imply that

$$\widetilde{d}^{(\theta_2)1}_{1,3}\left(P_{\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\right)=0.$$

Hence

$$P_{\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\in\widetilde{E}^{(\theta_2)2}_{3,1}.$$

Obviously we have

$$\widetilde{J}^2_{1,3}\left(P_{\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\right)=P_{\beta_2\cup\beta_2'}\otimes\mathcal{A}_{U_1,U_3}.$$

(2) $\mathcal{U}$ *is of type (2) w.r.t* y. We need to check that $P_{\beta_1\cup\beta_2\cup\beta_2'}\otimes\mathcal{A}_{U_1,U_3}$ belongs to the image of $\widetilde{J}^2_{1,3}$. Consider the group $\mathcal{I}^{(\theta_1)}_\alpha$ and the homology class

$$\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\in\mathrm{H}_3(\mathcal{I}^{(\theta_1)}_{\alpha\cup\beta_1\cup\beta_2\cup\beta_2'},\mathbb{Z}).$$

Let us consider the element

$$P_{\beta_1\cup\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\in\widetilde{E}^{(\theta_1)1}_{3,1}.$$

The arguments similar to Lemmas 6.3 and 6.4 imply that

$$\widetilde{d}^{(\theta_2)1}_{1,3}\left(P_{\beta_1\cup\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\right)=0.$$

Hence

$$P_{\beta_1\cup\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\in\widetilde{E}^{(\theta_1)2}_{3,1}.$$

Obviously we have

$$\widetilde{J}^2_{1,3}\left(P_{\beta_1\cup\beta_2\cup\beta_2'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_3})\right)=P_{\beta_1\cup\beta_2\cup\beta_2'}\otimes\mathcal{A}_{U_1,U_3}.$$

(3) $\mathcal{U}$ *is of type (3) w.r.t* y. This case is similar to the case (4) with removed β_1 everywhere.

(4) $\mathcal{U}$ *is of type (4) w.r.t* y. We need to check that

$$P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}_{U_3,U_1}-P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}_{U_1,U_2}$$

belongs to the image of $\widetilde{J}^2_{1,3}$. Consider the group $\mathcal{I}^{(\theta_1)}_\alpha$ and the homology classes

$$\mathcal{A}(T_{\theta_3},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_1}),\in\mathrm{H}_3(\mathcal{I}^{(\theta_1)}_{\alpha\cup\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3'},\mathbb{Z})$$

and

$$\mathcal{A}(T_{\theta_1},T_{\beta_3}T^{-1}_{\beta_3'},T_{\theta_2})\in\mathrm{H}_3(\mathcal{I}^{(\theta_1)}_{\alpha\cup\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'},\mathbb{Z}).$$

Let us consider the elements

$$P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\beta_2}T^{-1}_{\beta_2'},T_{\theta_1})\in\widetilde{E}^{(\theta_1)1}_{3,1}$$

and

$$P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_3}T^{-1}_{\beta_3'},T_{\theta_2})\in\widetilde{E}^{(\theta_1)1}_{3,1}.$$

The arguments similar to Lemmas 6.3 and 6.5 imply that

$$\widetilde{d}_{1,3}^{(\theta_1)1}\left(P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\beta_2}T_{\beta_2'}^{-1},T_{\theta_1})\right)=P_{\beta_1\cup\beta_2\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\theta_2},T_{\theta_1})$$

and

$$\widetilde{d}_{1,3}^{(\theta_1)1}\left(P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_3}T_{\beta_3'}^{-1},T_{\theta_2})\right)=P_{\beta_1\cup\beta_2\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\theta_2},T_{\theta_1}).$$

Hence

$$\widetilde{d}_{1,3}^{(\theta_1)1}\left(P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\beta_2}T_{\beta_2'}^{-1},T_{\theta_1})-P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_3}T_{\beta_3'}^{-1},T_{\theta_2})\right)=0.$$

Therefore

$$\left(P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\beta_2}T_{\beta_2'}^{-1},T_{\theta_1})-P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_3}T_{\beta_3'}^{-1},T_{\theta_2})\right)\in\widetilde{E}_{3,1}^{(\theta_1)2}.$$

Obviously we have

$$\widetilde{J}_{1,3}^{2}\left(P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}(T_{\theta_3},T_{\beta_2}T_{\beta_2'}^{-1},T_{\theta_1})-P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}(T_{\theta_1},T_{\beta_3}T_{\beta_3'}^{-1},T_{\theta_2})\right)=$$

$$=P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3}\otimes\mathcal{A}_{U_3,U_1}-P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'}\otimes\mathcal{A}_{U_1,U_2}.$$

This concludes the proof. □

8 Proof of Theorem 3.9

8.1 Alternative construction of s-classes

Let $\mathcal{U}=\{U_1,U_2,U_3\}$ be a symplectic splitting of H and let θ_j be a separating curve on Σ with $\mathrm{H}_{\theta_j}=U_j$, where $j=1,2,3$. In Section 3 we have defined the homology class $s(U_2,U_3)\in\mathrm{H}_4(\mathcal{I}^{(\theta_1)},\mathbb{Z})$. Let $0\neq x\in\mathrm{H}$ be a homology class such that $\mathcal{U}$ is of type (iii) w.r.t x and let the curves α_j,α_j' be as in Subsection 6.5. Recall that we have the natural inclusions

$$E_{0,4}^{(\theta_1)1}=E_{0,4}^{(\theta_1)\infty}\hookrightarrow\mathrm{H}_4(\mathcal{I}^{(\theta_1)},\mathbb{Z}).\qquad(47)$$

and

$$E_{1,3}^{(\theta_1)2}=E_{1,3}^{(\theta_1)\infty}\hookrightarrow\mathrm{H}_4(\mathcal{I}^{(\theta_1)},\mathbb{Z})/E_{0,4}^{(\theta_1)1}.\qquad(48)$$

Lemma 8.1. The element

$$\left(P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3}\otimes\mathcal{A}(T_{\theta_1},T_{\alpha_2}T_{\alpha_2'}^{-1},T_{\theta_3})-P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'}\otimes\mathcal{A}(T_{\theta_2},T_{\alpha_3}T_{\alpha_3'}^{-1},T_{\theta_1})\right)\in E_{1,3}^{(\theta_1)2}$$

maps to the coset containing $\pm s(U_2,U_3)$ under the map (48).

Proof. Consider the surface $\Sigma_{2,1}=\Sigma\setminus\overline{X_{\theta_1}}$. We have the exact sequences

$$1\longrightarrow\langle T_{\theta_1}\rangle\longrightarrow\mathcal{I}^{(\theta_1)}\xrightarrow{\ p\ }\mathcal{I}_{2,1}\longrightarrow 1.$$

$$1\longrightarrow\pi_1(\Sigma_2,\mathrm{pt})\longrightarrow\mathcal{I}_{2,1}\xrightarrow{\ q\ }\mathcal{I}_2\longrightarrow 1.$$

Consider the subgroups $Q=q^{-1}(\langle T_{\theta_3}\rangle)\subset\mathcal{I}_{2,1}$ and $G=p^{-1}(Q)\subset\mathcal{I}^{(\theta_1)}$. Hochschild-Serre spectral sequence implies that we have

$$\mathrm{H}_4(G,\mathbb{Z})=\langle s(U_2,U_3)\rangle\cong\mathbb{Z}.$$

Let $\mathcal{E}^{(\theta_1)*}_{*,*}$ be the spectral sequence for the action of G on $\mathcal{B}(x)$. By Lemma 6.3 (we take $\overline{G} = G$ and $\overline{\mathcal{B}}(x) = \mathcal{B}(x)$) the element

$$\Big(P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_1}, T_{\alpha_2}T^{-1}_{\alpha_2'}, T_{\theta_3}) - P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'} \otimes \mathcal{A}(T_{\theta_2}, T_{\alpha_3}T^{-1}_{\alpha_3'}, T_{\theta_1})\Big) \tag{49}$$

has an infinite order and lies in the kernel of $d^{(\theta_1)*}_{1,3} : \mathcal{E}^{(\theta_1)1}_{1,3} \to \mathcal{E}^{(\theta_1)1}_{0,3}$. So (49) belongs to the group $\mathcal{E}^{(\theta_1),2}_{1,3}$. Hence $\mathcal{E}^{(\theta_1)2}_{1,3} \subseteq \mathrm{H}_4(G,\mathbb{Z})/\mathcal{E}^{(\theta_1)1}_{0,4}$ contains a subgroup $\mathbb{Z}$ generated by the element (49). Since $\mathrm{H}_4(G,\mathbb{Z}) \cong \mathbb{Z}$ it follows that the group $\mathcal{E}^{(\theta_1)1}_{0,4}$ is zero.

Let us show that the groups $\mathcal{E}^{(\theta_1)\infty}_{2,2}$ and $\mathcal{E}^{(\theta_1)\infty}_{3,1}$ are free. For a 1-cell $\sigma \in \mathcal{B}(x)$ we have $\mathrm{Stab}_G(\sigma) \subseteq \mathrm{Stab}_{\mathcal{I}}(\sigma)$. By Proposition 4.2 the group $\mathrm{Stab}_{\mathcal{I}}(\sigma)$ is free. Therefore $\mathrm{H}_1(\mathrm{Stab}_G(\sigma),\mathbb{Z})$ is a free abelian group. For a 1-cell $\sigma \in \mathcal{B}(x)$ we also have $\mathrm{Stab}_G(\sigma) \subseteq \mathrm{Stab}_{\mathcal{I}}(\sigma)$. If these groups are not trivial, Propositions 5.1 and 5.3 imply that $\mathrm{Stab}_{\mathcal{I}}(\sigma) \cong \mathbb{Z} \times F_\infty$. So if suffices to prove the following lemma.

Lemma 8.2. Let F be a free group. Suppose that $K \subseteq \mathbb{Z} \times F$ is a subgroup. Then $\mathrm{H}_2(K,\mathbb{Z})$ is a free abelian group.

Proof. Consider the projection $p : \mathbb{Z} \times F \twoheadrightarrow F$. Then $\mathrm{Im} p|_K \subseteq F$ is a free group. We have $\ker p|_K = K \cap \mathbb{Z}$ that is either trivial or isomorphic to $\mathbb{Z}$. In the first case K is a free group. In the second case we have that $K \cong (K \cap \mathbb{Z}) \times \mathrm{Im} p|_K \cong \mathbb{Z} \times \mathrm{Im} p|_K$. This immediately implies the result. □

Hence the groups $\mathcal{E}^{(\theta_1)\infty}_{2,2}$ and $\mathcal{E}^{(\theta_1)\infty}_{3,1}$ are free. Since $\mathrm{H}_4(G,\mathbb{Z}) \cong \mathbb{Z}$ it follows that they are zero. Consequently, we have an isomorphism

$$\Big\langle P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_1}, T_{\alpha_2}T^{-1}_{\alpha_2'}, T_{\theta_3}) - P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'} \otimes \mathcal{A}(T_{\theta_2}, T_{\alpha_3}T^{-1}_{\alpha_3'}, T_{\theta_1})\Big\rangle \cong$$

$$\cong \langle s(U_2, U_3)\rangle = \mathrm{H}_4(G,\mathbb{Z})$$

Therefore this isomorphism maps

$$\Big(P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_1}, T_{\alpha_2}T^{-1}_{\alpha_2'}, T_{\theta_3}) - P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'} \otimes \mathcal{A}(T_{\theta_2}, T_{\alpha_3}T^{-1}_{\alpha_3'}, T_{\theta_1})\Big)$$

to $\pm s(U_2, U_3)$.

Consider the morphism $\mathcal{E}^{(\theta_1)*}_{*,*} \to E^{(\theta_1)*}_{*,*}$ induced by the inclusion $G \hookrightarrow \mathcal{I}^{\theta_1}$. The result follows by functoriality. □

Without loss of generality can assume that in Lemma 8.1 we have the sign $'+'$.

Lemma 8.3. The group $E^1_{0,4}$ is generated by elements of the form $s(U_1, U_2, U_3)$, such that $\mathcal{U} = (U_1, U_2, U_3)$ is of type (i) w.r.t x.

Proof. Recall that

$$E^1_{0,4} \cong \mathrm{H}_4(\mathcal{I}_\alpha, \mathbb{Z})$$

and

$$\Phi : \mathrm{H}_4(\mathcal{I}_\alpha, \mathbb{Z}) \cong \widetilde{E}^2_{1,3}.$$

Consider the preimages under the mapping $\widetilde{J}^2_{1,3}$ of the basis elements of $\widetilde{E}^2_{1,3}$ (see Proposition 7.7) constructed in the proof of Proposition 3.6. The arguments similar to the proof of Lemma 8.1 show that $\pm\Phi(s(U_2, U_3))$ equals

$$P_{\beta_2\cup\beta_2'} \otimes \mathcal{A}(T_{\theta_1}, T_{\beta_2}T^{-1}_{\beta_2'}, T_{\theta_3}),$$

if $\mathcal{U}$ is of type (1) w.r.t y;

$$P_{\beta_1\cup\beta_2\cup\beta_2'} \otimes \mathcal{A}(T_{\theta_1}, T_{\beta_2}T_{\beta_2'}^{-1}, T_{\theta_3}),$$

if $\mathcal{U}$ is of type (2) w.r.t y;

$$\left(P_{\beta_2\cup\beta_2'\cup\beta_3} \otimes \mathcal{A}(T_{\theta_3}, T_{\beta_2}T_{\beta_2'}^{-1}, T_{\theta_1}) - P_{\beta_2\cup\beta_3\cup\beta_3'} \otimes \mathcal{A}(T_{\theta_1}, T_{\beta_3}T_{\beta_3'}^{-1}, T_{\theta_2})\right),$$

if $\mathcal{U}$ is of type (3) w.r.t y;

$$\left(P_{\beta_1\cup\beta_2\cup\beta_2'\cup\beta_3} \otimes \mathcal{A}(T_{\theta_3}, T_{\beta_2}T_{\beta_2'}^{-1}, T_{\theta_1}) - P_{\beta_1\cup\beta_2\cup\beta_3\cup\beta_3'} \otimes \mathcal{A}(T_{\theta_1}, T_{\beta_3}T_{\beta_3'}^{-1}, T_{\theta_2})\right),$$

if $\mathcal{U}$ is of type (4) w.r.t y. Since $x = [\alpha] \in U_1$, in each of these cases $\mathcal{U}$ is of type (i) w.r.t x. □

8.2 Proof of linear independence

Denote by $S_{\mathcal{U}} \subseteq \mathrm{H}_4(\mathcal{I}, \mathbb{Z})$ the subgroup by three homology classes

$$s(U_1, U_2, U_3), s(U_2, U_3, U_1), s(U_3, U_1, U_2) \in \mathrm{H}_4(\mathcal{I}, \mathbb{Z}).$$

Lemma 8.4. (a)The inclusions $S_{\mathcal{U}} \hookrightarrow \mathrm{H}_4(\mathcal{I}, \mathbb{Z})$ induce an injective homomorphism

$$\bigoplus_{\mathcal{U}} S_{\mathcal{U}} \hookrightarrow \mathrm{H}_4(\mathcal{I}, \mathbb{Z}),$$

where the sum is over all unordered symplectic splittings $\mathcal{U}$ of H.

(b)The homology classes

$$s(U_1, U_2, U_3), s(U_2, U_3, U_1) \in \mathrm{H}_4(\mathcal{I}, \mathbb{Z}),$$

where $\mathcal{U}$ rans over the set of all unordered symplectic splittings of H, are linearly independent.

Proof. It suffices to prove that for a finite set of splittings $\{\mathcal{U}^1, \dots, \mathcal{U}^k\}$ the map

$$\bigoplus_{1}^{r} S_{\mathcal{U}^k} \to \mathrm{H}_4(\mathcal{I}, \mathbb{Z})$$

is injective.

The following straightforward result is proved in [4].

Proposition 8.5. [4, Lemma 4.5] There is a homology class $x \in \mathrm{H}$ such that the splittings $\mathcal{U}^1, \dots, \mathcal{U}^k$ are of type (iii) w.r.t x.

Take any $x \in \mathrm{H}$ satisfying the conditions of Proposition 8.5. It suffices to prove that the map

$$\bigoplus_{1}^{r} S_{\mathcal{U}^k} \to \mathrm{H}_4(\mathcal{I}, \mathbb{Z})/E^1_{0,4} \cong E^2_{1,3} \tag{50}$$

is injective.

By Lemma 8.1 the image of $S_{\mathcal{U}}$ ($\mathcal{U} \in \{\mathcal{U}^1, \dots, \mathcal{U}^k\}$) in $E^2_{1,3}$ under the mapping (50) is contained in the linear span of the elements

$$P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3} \otimes \mathcal{A}_{U_2,U_3} - P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}_{U_3,U_1}, P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}_{U_3,U_1} - P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'} \otimes \mathcal{A}_{U_1,U_2}.$$

By Proposition 6.16 these elements are linearly independent for different $\mathcal{U}$. This implies the result. □

8.3 Relations between s-classes

Proof of Theorem 3.9. Lemma 8.4 and Proposition 3.2 imply that it suffices to prove that for any splitting (U_1, U_2, U_3) we have

$$s(U_1, U_2, U_3) + s(U_2, U_3, U_1) + s(U_3, U_1, U_2) = 0. \tag{51}$$

By Lemma 8.1 the images of $s(U_1, U_2, U_3)$, $s(U_2, U_3, U_1)$, $s(U_3, U_1, U_2)$ in the group $\mathrm{H}_4(\mathcal{I}, \mathbb{Z})/E^1_{0,4}$ are

$$\left(P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_1}, T_{\alpha_2}T^{-1}_{\alpha_2'}, T_{\theta_3}) - P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'} \otimes \mathcal{A}(T_{\theta_2}, T_{\alpha_3}T^{-1}_{\alpha_3'}, T_{\theta_1})\right),$$

$$\left(P_{\alpha_1\cup\alpha_2\cup\alpha_3\cup\alpha_3'} \otimes \mathcal{A}(T_{\theta_2}, T_{\alpha_3}T^{-1}_{\alpha_3'}, T_{\theta_1}) - P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_3}, T_{\alpha_1}T^{-1}_{\alpha_1'}, T_{\theta_2})\right),$$

$$\left(P_{\alpha_1\cup\alpha_1'\cup\alpha_2\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_3}, T_{\alpha_1}T^{-1}_{\alpha_1'}, T_{\theta_2}) - P_{\alpha_1\cup\alpha_2\cup\alpha_2'\cup\alpha_3} \otimes \mathcal{A}(T_{\theta_1}, T_{\alpha_2}T^{-1}_{\alpha_2'}, T_{\theta_3})\right),$$

respectively.

Therefore the image of

$$s(U_1, U_2, U_3) + s(U_2, U_3, U_1) + s(U_3, U_1, U_2)$$

in $\mathrm{H}_4(\mathcal{I}, \mathbb{Z})/E^1_{0,4}$ is zero. Hence we have

$$s(U_1, U_2, U_3) + s(U_2, U_3, U_1) + s(U_3, U_1, U_2) \in E^1_{0,4}.$$

By Lemma 8.3 the group $E^1_{0,4}$ is generated by elements of subgroups $S_{\mathcal{U}'}$, where $\mathcal{U}'$ is of type (i) w.r.t x. Since $\mathcal{U}$ is of type (iii) w.r.t x, Lemma 8.4 (a) implies (51). ☐

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