CHOW DILOGARITHM AND RECIPROCITY LAWS

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Abstract. We prove a conjecture of A. Goncharov concerning so-called strong reciprocity laws. The main idea of the proof is the construction of the norm map on these strong reciprocity laws. This construction is similar to the construction of the norm map on Milnor K-theory. As an application, we express Chow dilogarithm in terms of Bloch-Wigner dilogarithm. Also we obtain a new reciprocity law for four rational functions on arbitrary proper surface with values in the pre-Bloch group.

1. Introduction

1.1. **Summary.** Fix some algebraically closed field k of characteristic zero. We recall that for $x\in X(k)$ and $f,g\in k(X)^\times$ *the tame-symbol of unctions f and g at x* is defined by the following formula:

$$(f,g)_x=(-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)}\frac{g^{\mathrm{ord}_x(f)}}{f^{\mathrm{ord}_x(g)}}.$$

We have interchanged f and g in this formula for convenience with the further definitions.

The famous Weil reciprocity law states that for any $f,g\in k(X)^\times$ the following product is equal to zero:

$$\prod_{x\in X(k)}(f,g)_x=1.$$

The tame symbol map induces a well-defined map $\Lambda^2k(X)^\times\to k^\times$ which we denote by $\partial_x^{(2)}$. In a similar way one can define the map $\Lambda^3k(X)^\times\to\Lambda^2k^\times$ which we denote by $\partial_x^{(3)}$. Unlike the previous case, the total residue map

$$\sum_{x\in X(k)}\partial_x^{(3)}\colon\Lambda^3k(X)^\times\to\Lambda^2k^\times$$

is not equal to zero. A. Suslin [13] proved that the image of this map is generated by the elements of the form $c\wedge(1-c), c\in k^\times$. This result is called *Suslin reciprocity law.* Denote the element $c\wedge(1-c)$ by $\delta_2(c)$. Between elements of the form $\delta_2(c)$ there are a lot of relations. The following proposition is true:

Proposition 1.1. *For any $x,y\in k\backslash\{0,1\}, x\neq y$ the following formula holds:*

$$\delta_2(x)+\delta_2(y/x)+\delta_2\left((1-x)/(1-y)\right)=\delta_2(y)+\delta_2\left(\frac{1-x^{-1}}{1-y^{-1}}\right),$$

This proposition motivates the following definition:

Definition 1.2 (the pre-Bloch group)**.** For a field F denote by $B_2(F)$ the *pre-Bloch* group of F. It is an abelian group generated by the elements $\{x\}_2, x\in F^\times$ modulo the following relations:

$$\{x\}_2-\{y\}_2+\{y/x\}_2-\left\{\frac{1-x^{-1}}{1-y^{-1}}\right\}_2+\{(1-x)/(1-y)\}_2=0,\quad\{1\}_2=0, \tag{1.1}$$

where $x,y\in F\backslash\{0,1\}, x\neq y$.

Suslin reciprocity law implies that there is a map $(\Lambda^3 k(X)^\times)\otimes\mathbb{Q} \dashrightarrow B_2(k)\otimes\mathbb{Q}$ making the following diagramm commutative:

$$\begin{tikzcd} & (\Lambda^3 k(X)^\times)\otimes\mathbb{Q} \arrow[dl, dashed] \arrow[d, "\sum\limits_{x\in X(k)} \partial_x^{(3)}"] \\ B_2(k)\otimes\mathbb{Q} \arrow[r, "\delta_2"] & (\Lambda^2 k^\times)\otimes\mathbb{Q}. \end{tikzcd} \tag{1.2}$$

The dotted map is called *a strong reciprocity law*, if it satisfies some natural properties which will be explained bellow. Main result of this paper is that on any smooth projective curve over k one can choose a strong reciprocity law functorial under arbitrary non-constant morphism of curves. This statement is a solution of a conjecture formulated by A. Goncharov in [4].

1.2. **The organisation of the paper.** The paper is organised as follows. In Section 2 we give some basic definitions and in Section 3 we present our main results. Section 4 has three subsection. In the first subsection we prove some basic properties of strong reciprocity laws. In the second subsection we give the definition of strictly regular elements and prove for them some version of Parshin reciprocity law. In the third subsection we prove some analogue of Bass and Tate exact sequence for Milnor K-theory before taking the quotient by Steinberg elements.

Section 5 takes the most of this paper. In this section, using the results from the previous two sections, we construct the functorial norm map on strong reciprocity laws. It has three subsection. In the first subsection we gives the definition of a system of strong reciprocity laws. In the second subsection we prove our key result stating that systems of strong reciprocity laws on the field $F(t)$ are in natural bijection with strong reciprocity laws on the field F. As an application of this result in the third subsection we construct the norm map on strong reciprocity laws. Finally, in the Section 6 we proves our main results.

1.3. **Acknowledgment.** The author is grateful to A. Levin and D. Rudenko for setting the problem and stimulating discussion. I also thank S. Gorchinskiy for his interest in this paper.

2. Definitions

2.1. Truncated polylogarithmic complexes.

Definition 2.1. For $n\geq 2$ define the following complex $\Gamma_2(F,n)$ placed in degrees $[1,2]$:

$$B_2(F)\otimes\Lambda^{n-2}F^\times \xrightarrow{\delta_n} \Lambda^n F^\times.$$

The differential is defined by the formula: $\delta_n(\{\xi_1\}_2\wedge\xi_3\wedge\dots\wedge\xi_n)=\xi_1\wedge(1-\xi_1)\wedge\xi_3\wedge\dots\wedge\xi_n$.

Up to shift these complexes coincide with the stupid truncation of the polylogarithmic complexes defined by A. Goncharov in [3, Section 9], see also [12].

Let (F,ν) be a discrete valuation field. Denote $\mathcal{O}_\nu=\{x\in F|\nu(x)\geq 0\}, m_\nu=\{x\in F|\nu(x)>0\}$ and $\overline{F}_\nu=\mathcal{O}_\nu/m_\nu$. We recall that an element $a\in F^\times$ is called *an uniformiser* if $\nu(a)=1$ and *a unit* if $\nu(a)=0$. For $u\in\mathcal{O}_\nu$ denote by $\overline{u}$ its residue class in $\overline{F}_\nu$.

Proposition 2.2 (Definition of the tame-symbol map)**.** *Let (F,ν) be a discrete valuation field and $n \geq 3$. There is a unique morphism of complexes $\partial_\nu^{(n)}\colon \Gamma_2(F,n) \to \Gamma(\overline{F}_\nu, n-1)$:*

$$\begin{tikzcd} B_2(F)\otimes\Lambda^{n-2}F^\times \arrow[r, "\delta_n"] \arrow[d, "\partial_\nu^{(n)}"] & \Lambda^n F^\times \arrow[d, "\partial_\nu^{(n)}"] \\ B_2(\overline{F}_\nu)\otimes\Lambda^{n-3}\overline{F}_\nu^\times \arrow[r, "\delta_{n-1}"] & \Lambda^{n-1}\overline{F}_\nu^\times, \end{tikzcd}$$

satisfying the following conditions:

(1) *For any units $u_1,\dots u_n$ we have $\partial_\nu^{(n)}(u_1\wedge\dots\wedge u_n)=0$.*
(2) *For any uniformiser π and units $u_2,\dots u_n\in F$ we have $\partial_\nu^{(n)}(\pi\wedge u_2\wedge\dots\wedge u_n) = \overline{u_2}\wedge\dots\overline{u_n}$.*
(3) *For any $a\in F\backslash\{0,1\}$ with $\nu(a)\neq 0$ and any $b\in\Lambda^{n-2}F^\times$ we have $\partial_\nu^{(n)}(\{u\}_2\otimes b) = 0$.*
(4) *For any unit u and $b\in\Lambda^{n-2}F^\times$ we have $\partial_\nu^{(n)}(\{u\}_2\otimes b)=\{\overline{a}\}\otimes\partial_\nu^{(n-2)}(b)$.*

The proof of this proposition can be found in [3, Section 14; 12, Subsection 2.1].

2.2. **The category $\mathbf{Fields}_d$.** We recall that we have fixed some algebraically closed field k of characteristic zero. Denote by $\mathbf{Fields}_d$ the category of finitely generated extensions of k of transcendent degree d. Any morphism in this category is a finite extension. For $F\in\mathbf{Fields}_d$, denote by $\mathrm{dval}(F)$ the set of discrete valuations given by a Cartier divisor on some birational model of F. When $F\in\mathbf{Fields}_1$ this set is equal to the set of all 1-dimensional valuations that are trivial on k. We denote this set by $\mathrm{val}(F)$.

Let $j\colon K\hookrightarrow F$ be an extension from $\mathbf{Fields}_d$ and $\nu\in\mathrm{dval}(K)$. Denote by $\mathrm{ext}(\nu,F)$ the set of extensions of the valuation ν to F. Let $\nu'\in\mathrm{ext}(\nu,F)$. Denote by $j_{\nu'|\nu}$ the natural embedding $\overline{K}_\nu\hookrightarrow\overline{F}_{\nu'}$. *The inertia degree* $f_{\nu'|\nu}$ is defined as $\deg j_{\nu'|\nu}$. *The ramification index* $e_{\nu'|\nu}$ is defined by the formula $\pi_K = u\pi_F^{e_{\nu'|\nu}}$, where π_K,π_F are uniformisers of K,F and u is some unit. By [9, Chapter II, §8] the set $\mathrm{ext}(\nu,F)$ is finite and moreover the following formula holds:

$$\sum_{\nu'\in\mathrm{ext}(\nu,F)} e_{\nu'|\nu}f_{\nu'|\nu} = [F:K]. \tag{2.1}$$

By Theorem of O. Zariski [14, Chapter VI, §14, Theorem 31] a discrete valuation on F is divisorial if and only if the corresponding residue field is finitely generated and has transcendence degree 1. It implies that for any $\nu'\in\mathrm{ext}(\nu,F)$ we have $\nu'\in\mathrm{dval}(F)$.

For any $n\geq 0$ there is the natural map $j_*\colon\Lambda^nK^\times\to\Lambda^nF^\times$, given by the formula $j_*(a)=a$. It is easy to see that for any $\nu'\in\mathrm{ext}(\nu,F)$ the following formula holds:

$$\partial_{\nu'}^{(n)} j_*(a) = e_{\nu'|\nu}\cdot(j_{\nu'|\nu})_*(\partial_\nu^{(n)}(a)). \tag{2.2}$$

2.3. **Strong reciprocity laws.**

Definition 2.3 (Strong reciprocity law)**.** Let $F\in\mathbf{Fields}_1$. *A strong reciprocity law* on the field F is a map $h\colon\Lambda^3F^\times\to B_2(k)$ satisfying the following conditions:

(1) The following diagram is commutative:

$$\begin{tikzcd} B_2(F)\otimes F^\times \arrow[r, "\delta_3"] \arrow[d, "\sum\limits_{\nu\in \mathrm{val}(F)} \partial_\nu^{(3)}"'] & \Lambda^3 F^\times \arrow[d, "\sum\limits_{\nu\in \mathrm{val}(F)} \partial_\nu^{(3)}"] \arrow[dl, dashed, "h"'] \\ B_2(k) \arrow[r, "\delta_2"] & \Lambda^2(k^\times) \end{tikzcd} \tag{2.3}$$

(2) The map h vanishes on the image of the multiplication map $\Lambda^2 F^\times \otimes k^\times \to \Lambda^3 F^\times$.

The set of all strong reciprocity laws has a structure of affine space over $\mathbb{Q}$. The corresponding vector space is the set of all $\mathbb{Q}$-linear maps $\Lambda^3 F^\times \to \ker(\delta_2)$ vanishing on the image of the maps δ_3 and $\Lambda^2 F^\times \otimes k^\times \to \Lambda^3 F^\times$.

Denote by **Set** the category of sets. Define a contravariant functor

$$\mathrm{SRL}\colon \mathbf{Fields}_1 \to \mathbf{Set}$$

as follows. For any $F \in \mathbf{Fields}_1$ the set $\mathrm{SRL}(F)$ is equal to the set of all strong reciprocity laws on F. If $j\colon K \hookrightarrow F$ then $\mathrm{SRL}(j)(h_F)$ is defined by the formula $h_K := \dfrac{1}{\deg j} h_F(j_*(a))$. In Section 4.1 we will show that in this way we indeed get a functor.

2.4. **Conventions.** Everywhere we work over $\mathbb{Q}$. This means that any abelian group is supposed to be tensored by $\mathbb{Q}$. For example when we write $\Lambda^2 k^\times$, this actually means $(\Lambda^2 k^\times)\otimes \mathbb{Q}$. All exterior powers and tensor products are over $\mathbb{Q}$.

If C is a chain complex denote by C_d the elements lying in degree d. The symbol δ_n means the differential in the truncated polylogarithmic complex $\Gamma_2(F,n)$. Although it depends on the field F we will omit the corresponding sign from the notation. In the same way, when (F,ν) is a discrete valuation field we denote by $\partial_\nu^{(n)}$ the tame-symbol map $\Gamma_2(F,n) \to \Gamma_2(\overline{F}_\nu, n-1)$.

3. Main results

The following result is a solution of Conjecture 6.2 from [4]:

Theorem 3.1. *On any field $F \in \mathbf{Fields}_1$ one can choose a strong reciprocity law $\mathcal{H}_F$ such that for any embedding $j\colon F_1 \to F_2$ we have $SRL(j)(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$. Such a collection of strong reciprocity laws is unique.*

Remark 3.2. One of the main results of [12] states that for any field $F \in \mathbf{Fields}_1$ there is a map $\Lambda^3 F^\times \to B_2(k)$ satisfying all but the second condition of Definition 2.3. It is not clear why this map can be chosen functorial.

In particular the proof of Corollary 1.5 from loc. cit. is not correct, because it relies on remark after Conjecture 6.2 from [4], which uses functorial property.

Remark 3.3. In [4, Section 6] A. Goncharov formulated his conjecture for some quotient $\mathcal{B}_2(k)$ of the group $B_2(k)$. In this setting he proved that for any elliptic curve E over k there is a $\mathcal{B}_2(k)$-valued strong reciprocity law on $k(E)$. From the proof of Theorem 5.3 it is not difficult to show that his map coincides with $i \circ \mathcal{H}_{k(E)}$, where i is the natural map $B_2(k) \to \mathcal{B}_2(k)$.

3.1. **Chow dilogarithm.** The definition of Chow dilogarithm can be found in Section 6 of [4]. This function associate to any smooth projective curve X over $\mathbb{C}$ and three non-zero rational functions f_1, f_2, f_3 on X the value $\mathcal{P}_2(X; f_1, f_2, f_3) \in \mathbb{R}$. Remark after Conjecture 6.2 in loc. cit. implies that Theorem 3.1 has the following corollary:

Corollary 3.4. *For any smooth projective curve X over $\mathbb{C}$ and three non-zero rational functions f_1, f_2, f_3 on X the following formula holds:*

$$\mathcal{P}_2(X; f_1, f_2, f_3) = \widetilde{\mathcal{L}}_2(\mathcal{H}_{\mathbb{C}(X)}(f_1 \wedge f_2 \wedge f_3)).$$

Here $\widetilde{\mathcal{L}}_2 \colon B_2(\mathbb{C}) \to \mathbb{R}$ is a map given on the generators $\{x\}_2$ by the formula

$$\widetilde{\mathcal{L}}_2(\{x\}_2) = \mathcal{L}_2(x),$$

where $\mathcal{L}_2$ is Bloch-Wigner dilogarithm.

3.2. **Two-dimensional reciprocity law.** For a field $L \in \mathbf{Fields}_2$ denote by $\mathrm{dval}(L)$ the set of all divisorial valuations of L. A valuation is called divisorial if it is given by an irreducible Cartier divisor on some smooth model of L. For a valuation $\nu \in \mathrm{dval}(L)$ denote by $\partial_\nu^{(4)}$ the corresponding tame-symbol map $\Lambda^4 L^\times \to \Lambda^3 \overline{L}_\nu^\times$. From the proof of Theorem 3.1 we get the following corollary:

Corollary 3.5. *Let $L \in \mathbf{Fields}_2$. For any $b \in \Lambda^4 L^\times$ and all but finite number $\nu \in \mathrm{dval}(L)$ we have $\mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b) = 0$. Moreover the following sum is equal to zero:*

$$\sum_{\nu \in \mathrm{dval}(L)} \mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b) = 0.$$

This corollary is a natural generalisation of Weil reciprocity law to algebraic surfaces.

3.3. **The norm map.** The proof of Theorem 3.1 takes most of this paper. Uniqueness is easy. It is non-trivial to prove that such a family of strong reciprocity laws exists. On the field $k(t)$ there is a unique strong reciprocity law, which we denote by $\mathcal{H}_{k(t)}$. To construct the strong reciprocity law $\mathcal{H}_F$, for any embeddings of fields $j \colon F_1 \hookrightarrow F_2$ we define the canonical norm map $N_{F_2/F_1} \colon \mathrm{SRL}(F_1) \to \mathrm{SRL}(F_2)$. We will prove the following theorem:

Theorem 3.6. *The map N satisfies the following properties:*

1. *Let $j \colon F_1 \hookrightarrow F_2$ be an embedding. We have $SRL(j) \circ N_{F_2/F_1} = id$.*
2. *If $F_1 \subset F_2 \subset F_3$ is a tower of extension from $\mathbf{Fields}_1$ then $N_{F_3/F_1} = N_{F_3/F_2} \circ N_{F_2/F_1}$.*
3. *Let $F \in \mathbf{Fields}_1$. For any $a \in F \backslash k$ we have a finite extension $k(a) \subset F$. The element $\mathcal{H}_F := N_{F/k(a)}(\mathcal{H}_{k(a)}) \in \mathrm{SRL}(F)$ does not depend on a.*

Existence in Theorem 3.1 follows immediately from the above theorem. The proof of Theorem 3.6 is similar to the construction of the norm map on Milnor K-theory [1,6,8,13].

4. The preliminary results

4.1. **Strong reciprocity laws.**

Proposition 4.1. SRL *is indeed a functor.*

Proof. If j_1, j_2 are some embeddings from $\mathbf{Fields}_1$ then the formula $\mathrm{SRL}(j_2 \circ j_1) = \mathrm{SRL}(j_1) \circ \mathrm{SRL}(j_2)$ follows from the fact that the ramification index is multiplicative. So it is enough to show that for any embedding $j \colon K \hookrightarrow F$ and $h_F \in SRL(F)$ the map $h_K := \mathrm{SRL}(j)(h_F) \colon \Lambda^3 K^\times \to B_2(k)$ is a strong reciprocity laws on K.

The statement that h_K is zero on the image of the map $K^\times \otimes \Lambda^2 k^\times \to \Lambda^3 K^\times$ is obvious. Let us prove that diagram 2.3 is commutative.

For any $\nu \in \mathrm{val}(K)$ and any $\nu' \in \mathrm{ext}(\nu, F)$ we have $f_{\nu'|\nu} = 1$. So Formula (2.1) is simplified to $\sum\limits_{\nu' \in \mathrm{ext}(\nu,F)} e_{\nu'|\nu} = [F : K]$. Since in our case $\overline{K}_\nu \cong \overline{F}_{\nu'} \cong k$, the formula (2.2) takes the form $e_{\nu'|\nu}\partial_\nu^{(3)}(a) = \partial_{\nu'}^{(3)} j_*(a)$.

For any $a \in \Lambda^3 K^\times$, we have:

$$\delta_2(h_K(a)) = \frac{1}{[F:K]}\delta_2(h_F(j_*(a))) = \frac{1}{[F:K]}\sum_{\nu' \in \mathrm{val}(F)} \partial_{\nu'}^{(3)} j_*(a) =$$
$$= \frac{1}{[F:K]}\sum_{\nu \in \mathrm{val}(K)}\sum_{\nu' \in \mathrm{ext}(\nu,F)} \partial_{\nu'}^{(3)} j_*(a) =$$
$$= \frac{1}{[F:K]}\sum_{\nu \in \mathrm{val}(K)}\sum_{\nu' \in \mathrm{ext}(\nu,F)} e_{\nu'|\nu}\partial_\nu^{(3)}(a) = \sum_{\nu \in \mathrm{val}(K)} \partial_\nu^{(3)}(a).$$

Here in the fourth equality we have used the formula $\partial_{\nu'}^{(3)}(j_*(a)) = e_{\nu'|\nu}\partial_\nu^{(3)}(a)$ and in the last formula we have used the formula $\sum\limits_{\nu' \in \mathrm{ext}(\nu,F)} e_{\nu'|\nu} = [F : K]$. So the lower right triangle is commutative. The commutativity of the upper left triangle is similar. □

Proposition 4.2. *On the field $k(t)$ there is a unique reciprocity law. We will denote it by $\mathcal{H}_{k(t)}$*

Proof. Elementary calculation shows that the group $\Lambda^3 k(t)^\times$ is generated by the image of the group $k(t)^\times \otimes \Lambda^2 k^\times$ and by the image of δ_3. Uniqueness follows from this statement. Existence was proved in [3, Theorem 6.5]. We remark that although the proof of Proposition 6.6 from [3] uses rigidity argument, this proposition can be easily deduced from [2] where it was proved that $B_2(k(t))$ is generated by elements of the form $\{at + b\}_2\,, a, b \in k$. □

4.2. **Parshin reciprocity laws.**

Definition 4.3. Let X be a smooth algebraic variety of dimension n and $x \in X$. A Cartier divisor D on X is called *supported on a simple normal crossing divisor* if there is some open affine neighborhood of the point x such that D is cut out by the function $\prod\limits_{i=1}^n x_i^{n_i}$ where $n_i \geq 0$ and x_i is a regular system of parameters at x.

We have the following statement [7]:

Theorem 4.4. *Let X be a variety over an algebraically closed field of characteristic zero and D an effective Weil divisor on X. There is a birational morphism $f\colon \widetilde{X} \to X$, such that $\widetilde{X}$ is smooth and $f^*(D)$ is supported on a simple crossing divisor at all points of $\widetilde{X}$.*

Definition 4.5 (Strictly regular element)**.** Let $\xi \in k(X)$. Write $D_0(\xi), D_\infty(\xi)$ for the divisors of zeros and poles of ξ and let $|\xi| = D_0(\xi) + D_\infty(\xi)$. An element $\Gamma_2(k(X), 4)_1$ (resp. $\Gamma_2(k(X), 4)_2$) is called strictly regular at $x \in X$ if it can be represented as a linear combination of elements of the form $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$ (resp. $\{\xi_1\}_2 \otimes \xi_3 \wedge \xi_4$) such that all the divisors $|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|$ (resp. $|\xi_1| + |\xi_3| + |\xi_4|$) are supported on a simple crossing divisor at x.

Theorem 4.4 has the following corollary:

Corollary 4.6. *Let S be a smooth surface and $j \in \{1, 2\}$. For any element $a \in \Gamma_2(k(S), 4)_j$ there is a birational morphism $p\colon \widetilde{S} \to S$ such that the element $p^*(a)$ is strictly regular at all points.*

The following lemma characterises strictly regular elements:

Lemma 4.7. *Let S be a smooth algebraic surface and $x \in S$.*

(1) *The subgroup of strictly regular elements of $\Gamma_2(k(S), 4)_1$ is generated by elements of the following form:*
   (a) $\{\pi_1^n \pi_2^m \xi_1\} \otimes \pi_1 \wedge \pi_2$.
   (b) $\{\pi_1^n \pi_2^m \xi_1\} \otimes \pi_1 \wedge \xi_4$.
   (c) $\{\pi_1^n \pi_2^m \xi_1\} \otimes \xi_3 \wedge \xi_4$.
   Here all the functions ξ_i take non-zero values at x and π_i is a regular system of parameters.
(2) *The subgroup of strictly regular elements of $\Gamma_2(k(S), 4)_2$ is generated by elements of the following form:*
   (a) $\pi_1 \wedge \pi_2 \wedge \xi_3 \wedge \xi_4$.
   (b) $\pi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$.
   (c) $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$.
   The functions ξ_i, π_i satisfy the same conditions as in item (1).

Proof. Follows from the fact that if π_1, π_2 is a regular system of parameters at x, then any function $f \in k(S)$ can be written in the form $\pi_1^{n_1} \pi_2^{n_2} \xi$ where $n_i \in \mathbb{Z}$ and ξ is a regular function at x such that $\xi(x) \neq 0$. □

The following result is a version of the classical Parshin reciprocity law for strictly regular elements (see [5, 10, 11]).

Theorem 4.8 (Strict Parshin reciprocity law)**.** *Let S be a surface smooth at some point $x \in S$ and $j \in \{1, 2\}$. For any strictly regular element b of the group $\Gamma_2(k(S), 4)_j$ at x the following sum is equal to zero:*

$$\sum_{\substack{C \subset S \\ C \ni x}} \partial^{(3)}_{\nu_{x,C}} \partial^{(4)}_{\nu_C}(b) = 0. \tag{4.1}$$

Here the sum is taken over all irreducible curves $C \subset S$ containing x that are smooth at this point, ν_C is the valuation corresponding to C and $\nu_{x,C}$ is a valuation of the residue field $\overline{k(S)}_{\nu_C}$ corresponding to $x \in C$.

Proof. It is enough to prove this theorem for any of the generators from Lemma 4.7. We will only consider the most interesting case $(1), (a)$. We can assume that S is a smooth surface, $x \in S$ and π_1, π_2 is a system of regular parameters at x. Passing to some open affine neighborhood of the point x, we can assume that the following conditions are satisfied:

(1) ξ_1 is invertible and
(2) For any $i \in \{1, 2\}$ the divisor of the function π_i is equal to some irreducible curve C_i passing through x. In particular, these functions are regular.

In general if X is a subvariety of algebraic variety Y and f is a regular function on Y we denote its restriction to X by $f|_X$. Let $b = \{\pi_1^n \pi_2^m \xi_1\} \otimes \pi_1 \wedge \pi_2$. Obviously the only curves on S satisfying $\partial^{(4)}_{\nu_C}(b) \neq 0$ are C_1 and C_2. Consider the following cases:

Case $n, m \neq 0$**:** In this case both of the tame symbols $\partial^{(4)}_{\nu_{C_1}}(b)$ and $\partial^{(4)}_{\nu_{C_2}}(b)$ vanish and the statement is obvious.

Case $n \neq 0, m = 0$ **or** $m \neq 0, n = 0$**:** Consider, say, the first case. Obviously,

$$\partial^{(4)}_{\nu_{C_1}}(b) = 0.$$

So it is enough to prove that $\partial^{(3)}_{\nu_x,C_2}\partial^{(4)}_{\nu_{C_2}}(b)=0$. This follows the following formula:

$$\mathrm{ord}_x((\pi_1^n\xi_1)|_{C_2})=n\neq 0.$$

Case $n=m=0$**:** In this case the statement follows from the following formula:

$$\partial^{(3)}_{\nu_x,C_1}\partial^{(4)}_{\nu_{C_1}}(b)=-\partial^{(3)}_{\nu_x,C_2}\partial^{(4)}_{\nu_{C_2}}(b)=\xi_1(x).$$

□

We have the following corollary:

Corollary 4.9. *Let* $L\in\mathbf{Fields}_2$ *and* $j\in\{1,2\}$*. For any* $b\in\Gamma_2(L,4)_j$ *and all but finite* $\mu\in\mathrm{dval}(L)$ *the following sum is zero:*

$$\sum_{\mu'\in\mathrm{val}(\overline{L}_\mu)}\partial^{(3)}_{\mu'}\partial^{(4)}_{\mu}(b)=0.$$

Moreover the following sum is zero:

$$\sum_{\mu\in\mathrm{dval}(L)}\sum_{\mu'\in\mathrm{val}(\overline{L}_\nu)}\partial^{(3)}_{\mu'}\partial^{(4)}_{\mu}(b)=0.$$

This corollary can be interpreted as the statement that the composition of the vertical arrows in the following diagram is zero:

(4.2)
$$\begin{tikzcd}
\Gamma_2(L,4)_1 \arrow[r, "\delta_4"'] \arrow[d, "(\partial^{(4)}_\mu)"] & \Gamma_2(L,4)_2 \arrow[d, "(\partial^{(4)}_\mu)"] \\
\bigoplus\limits_{\mu\in\mathrm{dval}(L)}\Gamma_2(\overline{L}_\mu,3)_1 \arrow[r, "\delta_3"] \arrow[d, "\sum\partial^{(3)}_{\mu'}"] & \bigoplus\limits_{\mu\in\mathrm{dval}(L)}\Gamma_2(\overline{L}_\mu,3)_2 \arrow[d, "\sum\partial^{(3)}_{\mu'}"] \\
\Gamma_2(k,2)_1 \arrow[r, "\delta_2"'] & \Gamma_2(k,2)_2
\end{tikzcd}$$

Proof. Let S be an algebraic surface with $k(S)\cong L$. Denote by $\mathrm{dval}(L)_S$ the subset of divisorial valuations coming from divisors on S. Choose S in such a way that b would be strictly regular at all points of S. Theorem 4.8 implies the following formula:

$$\sum_{\mu\in\mathrm{dval}(L)_S}\sum_{\mu'\in\mathrm{val}(\overline{L}_\mu)}\partial^{(3)}_{\mu'}\partial^{(4)}_{\mu}(b)=0.$$

It remains to prove that for any $\mu\in\mathrm{dval}(L)\setminus\mathrm{dval}(L)_S$ the following sum vanishes:

$$\sum_{\mu'\in\mathrm{val}(\overline{L}_\mu)}\partial^{(3)}_{\mu'}\partial^{(4)}_{\mu}(b)=0.$$

There is a birational morphism $p\colon\widetilde{S}\to S$ such that μ is given by a divisor on $\widetilde{S}$ contracted under p. The morphism p is a sequence of blow-ups $p_m\circ\cdots\circ p_1, p_i\colon S_i\to S_{i-1}, S_m=\widetilde{S}, S_0=S$. Let $D_i\subset S_i$ be the corresponding exceptional curve. Denote by ν_i the corresponding valuation. It is enough to show that for any i the following formula holds:

$$\sum_{\mu'\in\mathrm{val}(\overline{L}_{\mu_i})}\partial^{(3)}_{\mu'}\partial^{(4)}_{\mu_i}(b)=0.$$

This formula follows from Theorem 4.8 for the element b and the surfaces S_i and S_{i-1}. □

4.3. **Results of D.Rudenko.** Results of this section in a different form are contained in [12].

Let $F \in \mathbf{Fields}_1$. A valuation $\nu \in \mathrm{dval}(F(t))$ is called *general* if it corresponds to some irreducible polynomial over F. The set of general valuations are in bijection with the set of all closed points on the affine line over F, which we denote by $\mathbb{A}^1_{F,(0)}$. A valuation is called *special* if it is not general. Denote the set of general (resp. special) valuations by $\mathrm{dval}(F(t))_{gen}$ (resp. $\mathrm{dval}(F(t))_{sp}$). Denote by $\Gamma_2^{sp}(F(t),4)_1$ the subgroup of $\Gamma_2(F(t),4)_1$ lying in the kernel of all the maps $\partial_\nu^{(3)}$ where $\nu \in \mathrm{dval}(F(t))_{gen}$.

The following theorem is the main result of this subsection:

Theorem 4.10. *The following sequence is exact:*

$$\Lambda^4 F^\times \oplus \Gamma_2^{sp}(F(t),4)_1 \to \Lambda^4 F(t)^\times \to \bigoplus_{\nu \in \mathrm{dval}(F(t))_{gen}} \Lambda^3 \overline{F(t)}_\nu^\times \to 0. \tag{4.3}$$

Here the first component of the first map is the natural embedding, the second component is induced by δ_4 and the second map is given by $(\partial_\nu^{(4)})_{\nu \in \mathrm{dval}(F(t))_{sp}}$.

For a point $p \in \mathbb{A}^1_{F,(0)}$ denote by f_p the corresponding monic irreducible polynomial over F. By definition, $\deg p = \deg f_p$. Denote by $F(p)$ the residue field of point $p \in \mathbb{A}^1_{F,(0)}$ and by ν_p the corresponding valuation.

Lemma 4.11. *Let $m \geq 3$ be an integer. The following map is surjective:*

$$\Gamma_2(F(t),m) \xrightarrow{\left(\partial_{\nu_P}^{(m)}\right)} \bigoplus_{p \in \mathbb{A}^1_{F,(0)}} \Gamma_2(F(p),m-1).$$

The proof of this lemma is completely similar to the proof of surjectivity in the Bass-Tate exact sequence for Milnor K-theory [1,8].

Proof. For simplicity, we will only consider the case $m = 4$. The general case is completely similar. Denote by $\Gamma_2(F(t),4)_{\leq d}$ the set of elements lying in the kernels of all the maps $\partial_{\nu_P}^{(4)}$ with $\deg P > d$. It is enough to prove that for any $d \geq 0$ the following map is surjective:

$$\Gamma_2(F(t),4)_{\leq d} \to \bigoplus_{\substack{p \in \mathbb{A}^1_{F,(0)} \\ \deg p \leq d}} \Gamma_2(F(p),3).$$

The proof is by induction on d. The case $d = -1$ is trivial. Let us prove the inductive step. It is enough to show that for any $a \in \Gamma_2(F(p),3)_j, j \in \{1,2\}$ there is an element $\widetilde{a} \in \Gamma_2(F(x),4)_j$ with the following properties:

(1) for any $p' \neq p$ with $\deg p' \geq \deg p$ we have
$$\partial_{\nu_{p'}}^{(4)}(\widetilde{a}) = 0$$
(2) $\partial_{\nu_p}^{(4)}(\widetilde{a}) = a$.

For an element $\xi \in F(p)$ there is a unique polynomial $l_p(\xi)$ of degree $< \deg p$ such that the image of $l_p(\xi)$ under the natural projection $F[x] \to F[x]/f_p \cong F(p)$ is equal to ξ.

The following formulas for $\widetilde{a}$ are taken from [12, Section 5.2].

Case $j = 1$**:** Choose a representation $a = \sum_\alpha n_\alpha \cdot (\{\xi_1^\alpha\}_2 \otimes \xi_3^\alpha)$. Define the element $\widetilde{a}$ by the formula

$$\widetilde{a} = \sum_\alpha n_\alpha \cdot (\{l_P(\xi_1^\alpha)\}_2 \otimes f_P \wedge l_P(\xi_3^\alpha)).$$

Case $j=2$**:** Choose a representation $a=\sum\limits_{\alpha}n_\alpha\cdot(\xi_1^\alpha\wedge\xi_2^\alpha\wedge\xi_3^\alpha)$. The element $\widetilde{a}$ is defined by the formula

$$\widetilde{a}=\sum_{\alpha}n_\alpha\cdot(f_P\wedge l_P(\xi_1^\alpha)\wedge l_P(\xi_2^\alpha)\wedge l_P(\xi_3^\alpha)).$$

It is easy to see that these elements satisfy the conditions stated above. □

Proposition 4.12. *The following sequence is exact for $j=2$ and exact in the third term for $j=1$:*

$$\begin{aligned}&0\to H^j(\Gamma_2(F,4))\to H^j(\Gamma_2(F(t),4))\xrightarrow{(\partial_\nu^{(4)})}\\ &\xrightarrow{(\partial_\nu^{(4)})}\bigoplus_{p\in\mathbb{A}^1_{F,(0)}}H^j(\Gamma_2(F(p),3))\to 0.\end{aligned}\tag{4.4}$$

Proof. The case $j=2$ was proven in [8] (see also [1]). The exactness in the last term for $j=1$ is a particular case of the main result of [12]. □

The proof of Theorem 4.10. We need to prove that the following sequence is exact:

$$\Lambda^4F^\times\oplus\Gamma_2^{sp}(F(t)),4)_1\to\Lambda^4F(t)^\times\to\bigoplus_{p\in\mathbb{A}^1_{F,(0)}}\Lambda^3F(p)^\times\to 0.\tag{4.5}$$

Consider the following double complex:

$$\begin{tikzcd}
\Gamma_2(F,4)_1 \arrow[r, "\delta_4"] \arrow[d] & \Gamma_2(F,4)_2 \arrow[d]\\
\Gamma_2(F(t),4)_1 \arrow[r, "\delta_4"] \arrow[d] & \Gamma_2(F(t),4)_2 \arrow[d]\\
\bigoplus\limits_{p\in\mathbb{A}^1_{F,(0)}}\Gamma_2(F(p),3)_1 \arrow[r, "\bigoplus\delta_3"] & \bigoplus\limits_{p\in\mathbb{A}^1_{F,(0)}}\Gamma_2(F(p),3)_2
\end{tikzcd}$$

Denote by Tot the total complex placed in degrees $[1,4]$. Using Proposition 4.12, the spectral sequence argument shows that Tot has no cohomology in degree 3. It follows that the sequence (4.5) is exact in the second term. The exactness in the third term follows from Lemma 4.11 for $m=4$. □

5. The proof of Theorem 3.6

5.1. **Systems of strong reciprocity laws.** Let $L\in\mathbf{Fields}_2$. *A pre-system of strong reciprocity laws* σ on L is a choice of a strong reciprocity law σ_ν on the field $\overline{L}_\nu$ for any $\nu\in\operatorname{dval}(L)$.

We have the following lemma:

Lemma 5.1. *Let $L\in\mathbf{Fields}_2$. For any $b\in\Lambda^4L^\times$ and all but finite number of $\nu\in\operatorname{dval}(L)$ the element $\partial_\nu^{(4)}(b)$ belongs to the image of the map $\overline{L}_\nu^\times\otimes\Lambda^2k^\times\to\Lambda^3\overline{L}_\nu^\times$.*

In particular, if σ is a pre-system of strong reciprocity laws on L then for any $b\in\Lambda^4L^\times$ and all but finite number of $\nu\in\operatorname{dval}(L)$ we have $\sigma_\nu\partial_\nu^{(4)}(b)=0$.

Proof. Choose a smooth proper algebraic surface S such that $L=k(S)$ and b is strictly regular at all points of S. Let $\nu\in\operatorname{dval}(L)\setminus\operatorname{dval}(L)_S$. We claim that $\partial_\nu^{(3)}(b)$ lies in the subgroup indicated by the lemma. Let $p\colon\widetilde{S}\to S$ be a birational morphism such that ν

correspond to some divisor D on S. The divisor D is contracted under p. We use Lemma 4.7. The restriction of all the functions ξ_i to D lie in k. Now the statement follows from the definition of the tame symbol. $\square$

It follows from the previous lemma that for any pre-system of strong reciprocity laws on L and $b \in \Lambda^4 L^\times$, the following sum is well defined:

$$\sum_{\nu \in \mathrm{dval}(L)} \sigma_\nu \partial_\nu^{(4)}(b).$$

Definition 5.2 (System of strong reciprocity laws)**.** A pre-system of strong reciprocity laws is called a system of strong reciprocity laws if this sum is zero for any $b \in \Lambda^4 L^\times$.

Define a functor

$$\mathrm{SOSRL}\colon \mathbf{Fields}_2 \to \mathbf{Set}.$$

On objects it is equal to the set of all systems of strong reciprocity laws. On morphism it is defined as follows. Assume that $j\colon L \hookrightarrow M$ be an embedding of fields and σ is a system of strong reciprocity laws on M. Define a system of strong reciprocity laws $\mathrm{SOSRL}(j)(\sigma)$ on L by the following formula:

$$\mathrm{SOSRL}(j)(\sigma)_\nu = \frac{1}{[M:L]} \sum_{\nu' \in \mathrm{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} \, \mathrm{SRL}(j_{\nu'|\nu})(\sigma_{\nu'}).$$

We recall that $e_{\nu'|\nu}$, $f_{\nu'|\nu}$ and $j_{\nu'|\nu}$ was defined in Subsection 2.2. The proof of the fact that SOSRL is indeed a functor is similar to the proof of Proposition 4.1.

Denote by

$$Rf\colon \mathbf{Fields}_1 \to \mathbf{Fields}_2$$

a functor given by the formula $F \mapsto F(t)$.

Define the natural transformation $res\colon \mathrm{SOSRL} \circ Rf \to \mathrm{SRL}$ as follows. We need to define a map $res_F\colon \mathrm{SOSRL}(F(t)) \to \mathrm{SRL}(F)$ for any $F \in \mathbf{Fields}_1$. We set $res_F(\sigma) = \sigma_{\nu_\infty}$, where ν_∞ is the valuation corresponding to the point $\infty \in \mathbb{P}^1_F$. It is not difficult to show that res is indeed a natural transformation.

Here is the main result of this subsection:

Theorem 5.3. *The natural transformation* $res\colon \mathrm{SOSRL} \circ Rf \to \mathrm{SRL}$ *is an isomorphism of functors.*

5.2. **Proof of Theorem 5.3.**

Lemma 5.4. *Let* $F \in \mathbf{Fields}_1$*. For any strong reciprocity law* h *on the field* F*, there is a system of strong reciprocity laws* σ *on the field* $F(t)$ *such that* $res(\sigma) = h$*.*

Set $L = F(t)$. We need to show that for any strong reciprocity law h on F there is a strong reciprocity law σ on L satisfying $res_F(\sigma) = h$. First, for any $\nu \in \mathrm{dval}(L)$ we construct a map $\sigma_\nu\colon \Lambda^3 \overline{L}_\nu^\times \to B_2(k)$. Then we will show that in this way we get a system of strong reciprocity laws.

Let ν be special. If $\nu = \nu_\infty$ then define $\sigma_{\nu_\infty} = h$ (here we have used the identification of $\overline{L}_{\nu_\infty}$ with F). In the other case we have $\overline{F(t)}_\nu \simeq k(t)$. In this case define σ_ν to be a unique strong reciprocity law from Proposition 4.2.

We have defined σ_ν for any $\nu \in \mathrm{dval}(L)_{sp}$. Define the map $H\colon \Lambda^4 L^\times \to B_2(k)$ by the following formula:

$$H(b) = -\sum_{\nu \in \mathrm{dval}(L)_{sp}} \sigma_\nu \partial_\nu^{(4)}(b).$$

This sum is well defined by lemma 5.1.

Lemma 5.5. *The map H is zero on the image of the map*

$$\Lambda^4 F^\times \oplus \Gamma_2^{sp}(L,4)_1 \to \Lambda^4 L^\times.$$

Here the first map is the natural embedding and the second map is induced by δ_4.

Proof of lemma 5.5. (1) Direct computation shows that for any $a \in \Lambda^4 F^\times$ and any $\nu \in \mathrm{dval}(L)$ the element $\partial_\nu^{(4)}(a)$ lies in the subgroup $\Lambda^3 k^\times \subset \Lambda^3 \overline{L}_\nu^\times$. It follows that the map H vanishes on the image of the group $\Lambda^4 F^\times$.

(2) Let us prove that H is zero on the image of the group $\Gamma_2^{sp}(L,4)_1$. Consider the following commutative diagram:

$$\begin{tikzcd}
\Gamma_2(L,4)_1 \arrow[r, "\delta_4"] \arrow[d, "(\partial_\nu^{(4)})"] & \Gamma_2(L,4)_2 \arrow[d, "(\partial_\nu^{(4)})"] \\
\bigoplus_{\nu\in\mathrm{dval}(L)_{sp}} \Gamma_2(\overline{L}_\nu,3)_1 \arrow[r, "\delta_3"] \arrow[d, "\sum \partial_{\nu'}^{(3)}"] & \bigoplus_{\nu\in\mathrm{dval}(L)_{sp}} \Gamma_2(\overline{L}_\nu,3)_2 \arrow[d, "\sum \partial_{\nu'}^{(3)}"] \arrow[dl, "\sum \sigma_\nu"'] \\
\Gamma_2(k,2)_1 \arrow[r, "\delta_2"] & \Gamma_2(k,2)_2
\end{tikzcd} \tag{5.1}$$

For any $b \in \Gamma_2^{sp}(L,4)_1$ we get

$$\sum_{\nu\in\mathrm{dval}(L)_{sp}} \sigma_\nu \partial_\nu^{(4)} \delta_4(b) = \sum_{\nu\in\mathrm{dval}(L)_{sp}} \sum_{\nu'\in\mathrm{val}(\overline{L}_\nu)} \partial_{\nu'}^{(3)} \partial_\nu^{(4)}(b).$$

So by definition of H, we obtain:

$$\begin{aligned} H(\delta_4(b)) &= -\sum_{\nu\in\mathrm{dval}(L)_{sp}} \sigma_\nu \partial_\nu^{(4)} \delta_4(b) = -\sum_{\nu\in\mathrm{dval}(L)_{sp}} \sum_{\nu'\in\mathrm{val}(L_\nu)} \partial_{\nu'}^{(3)} \partial_\nu^{(4)}(b) = \\ &= -\sum_{\nu\in\mathrm{dval}(L)} \sum_{\nu'\in\mathrm{val}(L_\nu)} \partial_{\nu'}^{(3)} \partial_\nu^{(4)}(b) = 0. \end{aligned} \tag{5.2}$$

Here the third equality holds because the element b lies in the group $\Gamma_2^{sp}(L,4)_1$ and has residues only to special valuations. The fourth equality follows from Corollary 4.9.

□

Therefore thanks to Theorem 4.10 we get a well-defined map

$$\bigoplus_{\nu\in\mathrm{dval}(L)_{gen}} \Lambda^3 \overline{L}_\nu^\times \to B_2(k).$$

Define σ_ν for any $\nu \in \mathrm{dval}(L)_{gen}$ to be the restriction of this map to the subgroup $\Lambda^3 \overline{L}_\nu^\times$.

For $j \in \{1,2\}$, $\nu \in \mathrm{dval}(L)_{gen}$ and $a \in \Gamma_2(\overline{L}_\nu,3)_j$ denote by $\mathcal{L}(a)$ the set of elements $b \in \Gamma_2(L,4)_j$ such that $\partial_\nu^{(4)}(b) = a$ and $\partial_{\nu'}^{(4)} = 0$ for $\nu' \neq \nu, \nu' \in \mathrm{dval}(L)_{gen}$. This set is non-empty by Lemma 4.11.

Lemma 5.6. *Let $\nu \in \mathrm{dval}(L)_{gen}$ and $a \in \Lambda^3 \overline{L}_\nu^\times$. For any $b \in \mathcal{L}(a)$ the element $\sigma_\nu(a)$ is equal to $H(b)$.*

Proof. Follows from the definition of σ_ν. □

Lemma 5.7. *Let $j \in \{1,2\}$, $\nu \in \mathrm{dval}(L), a \in \Gamma_2(\overline{L}_\nu)_j$ and $b \in \mathcal{L}(a)$. The following formula holds:*

$$\sum_{\mu' \in \mathrm{val}(\overline{L}_\nu))} \partial_{\mu'}^{(3)}(a) = -\sum_{\mu \in \mathrm{dval}(L)_{sp}} \sum_{\mu' \in \mathrm{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b)$$

Proof. Corollary 4.9 implies the following formula:

$$\sum_{\mu \in \mathrm{dval}(L)} \sum_{\mu' \in \mathrm{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) = 0.$$

From the other side:

$$\sum_{\mu \in \mathrm{dval}(L)} \sum_{\mu' \in \mathrm{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) =$$
$$= \sum_{\mu' \in \mathrm{val}(\overline{L}_\nu))} \partial_{\mu'}^{(3)}(a) + \sum_{\mu \in \mathrm{dval}(L)_{sp}} \sum_{\mu' \in \mathrm{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b).$$

The statement of the lemma follows. □

Lemma 5.8. *For any $\sigma \in \mathrm{dval}(L)$ σ_ν is a strong reciprocity law.*

Proof. Let us show the following diagram is commutative:

$$\begin{tikzcd}
B_2(\overline{L}_\nu) \otimes \overline{L}_\nu^\times \arrow[r, "\delta_3"] \arrow[d, "\sum\limits_{\mu' \in \mathrm{val}(\overline{L}_\nu)} \partial_{\mu'}^{(3)}"'] & \Lambda^3 \overline{L}_\nu^\times \arrow[d, "\sum\limits_{\mu' \in \mathrm{val}(\overline{L}_\nu)} \partial_{\mu'}^{(3)}"] \arrow[dl, dashed, "\sigma_\nu"'] \\
B_2(k) \arrow[r, "\delta_2"] & \Lambda^2(k^\times)
\end{tikzcd}$$

The lower right triangle: Let $a \in \Lambda^3 \overline{L}_\nu^\times$. Choose some $b \in \mathcal{L}(a)$. By Lemma 5.7, we have:

$$\sum_{\mu' \in \mathrm{dval}(\overline{L}_\nu))} \partial_{\mu'}^{(3)}(a) = -\sum_{\mu \in \mathrm{dval}(L)_{sp}} \sum_{\mu' \in \mathrm{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) =$$
$$= -\sum_{\mu \in \mathrm{dval}(L)_{sp}} \delta_2 \sigma_\mu \partial_\mu^{(4)}(b) = -\delta_2 H(b) = \delta_2 \sigma_\nu(a).$$

Here the last formula follows from Lemma 5.6.

The upper left triangle: Let $a_1 \in \Gamma_2(\overline{L}_\nu, 3)_1$. Choose $b_1 \in \mathcal{L}(a_1)$. We have $\delta_4(b_1) \in \mathcal{L}(\delta_3(a_1))$. By Lemma 5.7, we have:

$$\sum_{\mu' \in \mathrm{val}(\overline{L}_\nu))} \partial_{\mu'}^{(3)}(a_1) = -\sum_{\mu \in \mathrm{dval}(L)_{sp}} \sum_{\mu' \in \mathrm{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b_1) =$$
$$= -\sum_{\mu \in \mathrm{dval}(L)_{sp}} \sigma_\mu \delta_3 \partial_\mu^{(4)}(b_1) = -\sum_{\mu \in \mathrm{dval}(L)_{sp}} \sigma_\mu \partial_\mu^{(4)} \delta_4(b_1) =$$
$$= H(\delta_4(b_1)) = \sigma_\nu \delta_3(a_1).$$

Here the second equality holds because for any $\mu \in \mathrm{val}(L)_{sp}$ the map σ_μ is a strong reciprocity law, the third equality holds because $\partial_\mu^{(4)}$ is a morphism of complexes and the fifth equality holds by Lemma 5.6.

To prove that σ_ν is a strong reciprocity law it remains to show that it vanishes on elements of the form $a\wedge c, a \in \Lambda^2\overline{L}_\nu^\times, c \in k^\times$. Lemma 4.11 for $m = 3$ in the degree 2 implies that there is $b \in \Lambda^3 L^\times$ such that $\partial_\nu^{(3)}(b) = a$ and $\partial_{\nu'}^{(3)}(b) = 0$ for any $\nu \neq \nu, \nu' \in \operatorname{dval}(L)_{gen}$. It follows that $b \wedge c \in \mathcal{L}(a \wedge c)$. Now the statement follows from the fact that for any $\mu \in \operatorname{dval}(L)$, we have $\partial_\mu^{(4)}(b \wedge c) = \partial_\mu^{(4)}(b) \wedge c$ and so for any $\mu \in \operatorname{dval}(L)_{sp}$, we have $\sigma_\mu \partial_\mu^{(4)}(b \wedge c) = 0$, since σ_μ is a strong reciprocity law. Now the statement follows from the definition of σ_ν. □

Proof of Lemma 5.4. Let us prove that σ is a system of strong reciprocity laws. We need to show that for any $b \in \Lambda^4 L^\times$ the following formula holds:

$$\sum_{\nu \in \operatorname{dval}(L)} \sigma_\nu \partial_\nu^{(4)}(b) = 0. \tag{5.3}$$

By Theorem 4.10 the group $\Lambda^4 L^\times$ is generated by the subsets $\mathcal{L}(a), a \in \Lambda^3 \overline{L}_\nu, \nu \in \operatorname{dval}(L)_{gen}$. So we may assume that $b \in \mathcal{L}(a)$ for some $a \in \Lambda^3 \overline{L}_\nu$. In this case formula (5.3) follows from the definition of σ_ν. □

Proof of Theorem 5.3 . To prove that res is an isomorphism of functors, we need to show that for any $F \in \mathbf{Fields}_1$ the map

$$res_F \colon \operatorname{SOSRL}(F(t)) \to \operatorname{SRL}(F)$$

is a bijection. By Lemma 5.4 this map is surjective. So we need to show that it is injective.

Assume that h is a system of strong reciprocity laws on F and σ, σ' be two systems of strong reciprocity laws on L satisfying $res(\sigma) = res(\sigma') = h$. We need to show that $\sigma = \sigma'$.

By definition $\sigma_{\nu_\infty} = \sigma'_{\nu_\infty} = h$. If ν is special valuation different from ν_∞ then $\overline{L}_\nu \cong k(t)$ and so by lemma 4.2 there is a unique strong reciprocity low on $\overline{L}_\nu$. We conclude that when ν is special $\sigma_\nu = \sigma'_\nu$.

Let $\nu \in$ be an arbitrary element of $\operatorname{dval}(L)_{gen}$. It remains to show that for any $a \in \Lambda^3 \overline{L}_\nu^\times$ we have $\sigma_\nu(a) = \sigma'_\nu(a)$.

Choose some $b \in \mathcal{L}(a)$. By definition of b and the fact that σ is a system of strong reciprocity laws we have:

$$\sigma_\nu(a) + \sum_{\mu \in \operatorname{dval}(L)_{sp}} \sigma_\mu \partial_\mu^{(4)}(b) = 0.$$

So

$$\sigma_\nu(a) = -\sum_{\mu \in \operatorname{dval}(L)_{sp}} \sigma_\mu \partial_\mu^{(4)}(b) = 0.$$

In the same way

$$\sigma'_\nu(a) = -\sum_{\mu \in \operatorname{dval}(L)_{sp}} \sigma'_\mu \partial_\mu^{(4)}(b) = 0.$$

The right hand side of the last two formulas coincide because σ and σ' coincide on special valuations. We conclude that $\sigma_\nu(a) = \sigma'_\nu(a)$ as well. □

5.3. **The norm map.** In this subsection we will use Theorem 5.3 to construct the norm map on strong reciprocity laws. We follow ideas from [13, §1] (see also [1, 6, 8]).

Since res is an isomorphism of functors, it has an inverse. Denote it by $\mathcal{N}$. Let $F \in \mathbf{Fields}_1$ and $\nu \in \operatorname{dval}(F(t))$. Define the norm map $N_\nu \colon \operatorname{SRL}(F) \to \operatorname{SRL}(\overline{F(t)}_\nu)$ by the formula $N_\nu(h) = \mathcal{N}_F(h)_\nu$.

Let $j\colon F \hookrightarrow K$ be an extension of some fields from $\mathbf{Fields}_1$. Let a be some generator of K over F. Consider a map $F[t] \to K$ given by the formula $p(t) \mapsto p(a)$. The kernel of this map is an irreducible polynomial p_a over F. Denote by ν_a the corresponding valuation. The residue field $\overline{F(t)}_{\nu_a}$ is canonically isomorphic to K. So we get a map $N_{\nu_a}\colon \mathrm{SRL}(F) \to \mathrm{SRL}(K)$ which we denote by $N_{K/F,a}$. We will prove that this map does not depend on a.

As a corollary from Theorem 5.3 we get the following lemma:

Lemma 5.9. (1) *Let $j\colon F \hookrightarrow K$ be an extension of two fields from* $\mathbf{Fields}_1$, *$\nu \in \mathrm{dval}(F(t))$ and $n = [K : F]$. The following diagram is commutative:*

$$
\begin{tikzcd}
\mathrm{SRL}(F) \arrow[rr, "N_\nu"] & & \mathrm{SRL}(\overline{F(t)}_\nu) \\
\mathrm{SRL}(K) \arrow[u, "\mathrm{SRL}(j)"] \arrow[r, "(N_{\nu'})"] & \bigoplus\limits_{\nu' \in ext(\nu, K(t))} \mathrm{SRL}(\overline{K(t)_{\nu'}}) \arrow[ur, "\sum\limits_{\nu' \in ext(\nu,K(t))} \frac{e_{\nu'|\nu} f_{\nu'|\nu}}{n} \mathrm{SRL}(j_{\nu'|\nu})"'] &
\end{tikzcd}
\tag{5.4}
$$

(2) *Let $j\colon F_1 \subset K, F_1 \subset F_2$, be an extensions and $F_2 \otimes_{F_1} K = \bigoplus\limits_{i=1}^{m} F_{2,i}$. Denote by j_i the natural embedding $F_2 \hookrightarrow F_{2,i}$. Let $n = [K : F_1]$ and $n_i = [F_{2,i} : F_2]$. Let a be generator of F_2 over F_1. Denote by a_i the corresponding generators of $F_{2,i}$ over K. The following diagram is commutative:*

$$
\begin{tikzcd}
\mathrm{SRL}(F_1) \arrow[r, "N_{F_2/F_1,a}"] & \mathrm{SRL}(F_2) \\
\mathrm{SRL}(K) \arrow[u, "\mathrm{SRL}(j)"] \arrow[r, "(N_{F_{2,i}/K,a_i})"] & \bigoplus\limits_{i=1}^{k} \mathrm{SRL}(F_{2,i}) \arrow[u, "\sum\limits_{i=1}^{m} \frac{n_i}{n} \mathrm{SRL}(j_i)"']
\end{tikzcd}
\tag{5.5}
$$

Proof. (1) Denote by j' the embedding $F(t) \hookrightarrow K(t)$. Since $\mathcal{N}$ is a natural transformation of functors, the following diagram is commutative:

$$
\begin{tikzcd}
\mathrm{SRL}(F) \arrow[r, "\mathcal{N}_F"] & \mathrm{SOSRL}(F(t)) \\
\mathrm{SRL}(K) \arrow[u, "\mathrm{SRL}(j)"] \arrow[r, "\mathcal{N}_K"] & \mathrm{SOSRL}(K(t)) \arrow[u, "\mathrm{SOSRL}(j')"']
\end{tikzcd}
$$

So for any $h_K \in \mathrm{SRL}(K)$ and any $\nu \in \mathrm{dval}(F(t))$, we get the following identity:

$$\mathcal{N}_F(\mathrm{SRL}(j)(h_K))_\nu = \mathrm{SOSRL}(j')(\mathcal{N}_K(h_K))_\nu.$$

Writing out, we get:

$$\mathcal{N}_F(\mathrm{SRL}(j)(h_K))_\nu = N_\nu(\mathrm{SRL}(j)(h_K)),$$

$$\mathrm{SOSRL}(j')(\mathcal{N}_K(h_K))_\nu = \sum_{\nu' \in \mathrm{ext}(\nu, K(t))} \frac{e_{\nu'|\nu} f_{\nu'|\nu}}{n} \mathrm{SRL}(j_{\nu'|\nu})(\mathcal{N}_K(h_K)_{\nu'}) =$$

$$= \sum_{\nu' \in \mathrm{ext}(\nu, K(t))} \frac{e_{\nu'|\nu} f_{\nu'|\nu}}{n} \mathrm{SRL}(j_{\nu'|\nu})(N_{\nu'}(h_K)).$$

Here in the second formula we have used the definition of the functor SOSRL. The statement follows.

(2) Let p_a be the minimal polynomial of a over F_1. We apply the previous statement for $F = F_1, K = K$ and ν corresponding to p_a. Let $p_a = \prod_i^l p_{a,i}$ be the decomposition of P in the field $K(t)$. The set $\mathrm{ext}(\nu, K(t))$ are in bijection with the irreducible factors of p_a in $K(t)$. Denote by $\nu_i \in \mathrm{ext}(\nu, K(t))$ the valuation corresponding to $p_{a,i}$. We have $F_{2,i} \cong \overline{K(t)}_{\nu_i}$. The embeddings j_i correspond to the embeddings $j_{\nu_i|\nu}$. Since the polynomial p_a is separable we have $f_{\nu'|\nu} = 1$ and so $e_{\nu'|\nu} f_{\nu'|\nu} = [F_{2,i} : F_2]$. So the diagram (5.4) can be identified with (5.5).

□

Lemma 5.10. *We have* $\mathrm{SRL}(j) \circ N_{F_2/F_1,a} = id$. *In particular if* $F_1 = F_2$ *then* $N_{F_2/F_1,a}$ *is identical and for any* $j\colon F_1 \hookrightarrow F_2$ *the map* $\mathrm{SRL}(j)$ *is surjective.*

Proof. Let $h \in \mathrm{SRL}(F_1)$, $\sigma = \mathcal{N}_{F_1}(h)$ and $x \in \Lambda^3 F_1^\times$. Consider the element $b = p_a \wedge x \in \Lambda^4 F_1(t)^\times$, where p_a is the minimal polynomial of a over F_1. Since σ is a structure of strong reciprocity laws we have:

$$\sum_{\nu \in \mathrm{dval}(F(t))_{gen}} \sigma_\nu \partial_\nu^{(4)}(b) + \sum_{\nu \in \mathrm{dval}(F(t))_{sp}} \sigma_\nu \partial_\nu^{(4)}(b) = 0$$

We have $\partial_{\nu_{p_a}}^{(4)}(b) = x$ and this this is the only general valuation such that $\partial_\nu^{(4)}(b) \neq 0$. So by the definition of the norm the first term is equal to $N_{F_2/F_1,a}(h)(x)$. From the other side it is easy to see that there is only one special valuation ν such that $\sigma_\nu \partial_\nu^{(4)}(b) \neq 0$ namely ν_∞. We have $\partial_{\nu_\infty}^{(4)}(b) = -na$. The statement follows. □

Proposition 5.11. *The map* $N_{F_2/F_1,a}$ *does not depend on* a. *Denote* $N_{F_2/F_1,a}$ *simply by* N_{F_2/F_1}.

Proof. We need to show that $N_{F_2/F_1,a}$ does not depend on a. Let $j\colon F_1 \hookrightarrow K$ be a field extension of F_1 satisfying $F_2 \otimes_{F_1} K \cong K^{\oplus [F_2:F_1]}$. We apply the second statement of Lemma 5.9. By definition of K for any n we have $F_{2,i} \cong K$. By item (1) of this theorem the maps $N_{F_{2,i}/K,a_i}$ are identical. We conclude that in the diagram from Lemma 5.9 all the maps except maybe $N_{F_2/F_1,a}$ do not depend on a. So the map $N_{F_2/F_1,a}$ does not depend on a on the image of $\mathrm{SRL}(j)$. By the previous lemma this image coincides with $\mathrm{SRL}(F_1)$. □

Proof of Theorem 3.6. (1) Follows from Lemma 5.10.

(2) Choose a field extension $j\colon F_1 \hookrightarrow K$. Denote $F_2 \otimes_{F_1} K \cong \bigoplus_{i=1}^{n_2} F_{2,i}$ and $F_3 \otimes_{F_2} F_{2,i} \cong \bigoplus_{s=1}^{n_{3,i}} F_{3,i,s}$. By associativity of tensor product we have $F_3 \otimes_{F_1} K \cong \bigoplus_{i,s} F_{3,i,s}$ Denote by $j_{i,s}$ the natural embeddings $F_3 \hookrightarrow F_{3,i,s}$. Let $n_{i,s} = [F_{3,i,s} : F_3]$. Let n be the degree of F_3 over F_1. Repeated application of Lemma 5.9 together with (2) of this theorem shows that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{SRL}(F_1) \arrow[r, "N_{F_3/F_2} \circ N_{F_2/F_1}"] & \mathrm{SRL}(F_3) \\
\mathrm{SRL}(K) \arrow[u, "\mathrm{SRL}(j)"] \arrow[r, "(N_{F_{3,i,s}/F_{2,i}} \circ N_{F_{2,i}/K})"] & \bigoplus_{i,s} \mathrm{SRL}(F_{3,i,s}) \arrow[u, "\sum_{i,s} \frac{n_{i,s}}{n} \mathrm{SRL}(j_{i,s})"']
\end{tikzcd}$$

Choose K such that $F_3 \otimes_{F_1} K \cong K^{\oplus [F_3:F_1]}$. It follows that $F_{3,i,s} \cong F_{2,i} \cong K$. So the bottom maps in the above diagram are identical. Let us compare this diagram with diagram (5.5) for $F_2 = F_3$. We see that the left, right and bottom maps are the same. Since $\mathrm{SRL}(j)$ is surjective, the statement of the proposition follows.

(3) Let $a, b \in F\backslash k$ be two functions generating F over k. We first prove this statement for the functions satisfying these conditions.

Let $p_b \in k(a)[t]$ be the minimal polynomial of $b \in F$ over $k(a)$. Multiplying p_a on some rational function of a we can assume that p_a lies in $k[a,t]$ and that p_a is irreducible as a polynomial of two variables. Denote this polynomial of two variables by $h(a,t)$. It is easy to see that the polynomial $h'(b,t)$ given by the formula $h'(b,t) = h(t,b)$ is a minimal polynomial of a over $k(b)$.

Let $A = k[x][y]$ and $L = k(x)(y)$. If we consider the polynomials h, h' as polynomials of x, y they gives two elements $\nu, \nu' \in \mathrm{dval}(L)$.

By Theorem 5.3 and Theorem 4.2 on the field L there is a unique system of strong reciprocity laws. Denote it by σ. Denote by λ an automorphism of L interchanging x and y. Since σ is a unique system of strong reciprocity laws on L, it is invariant under λ. Since λ interchanges ν and ν' it induces a map $\overline{\lambda}\colon \overline{L}_\nu \to \overline{L}_{\nu'}$. Because σ is invariant under λ we have $\mathrm{SRL}(\overline{\lambda})(\sigma_{\nu'}) = \sigma_\nu$.

A map $A \to F$ given by the formula $x \mapsto a, y \mapsto b$ induces an isomorphism $\theta\colon \overline{L}_\nu \to F$. In the same way a map $A \to F$ given by the formula $x \mapsto b, y \mapsto a$ induces an isomorphism of $\theta\colon \overline{L}_{\nu'} \to F$. Since $\theta = \theta' \circ \overline{\lambda}$, we have

$$\mathrm{SRL}(\theta^{-1})(\sigma_\nu) = \mathrm{SRL}(\lambda^{-1} \circ \theta'^{-1})(\sigma_\nu) =$$
$$= \mathrm{SRL}(\theta'^{-1}) \circ \mathrm{SRL}(\overline{\lambda}^{-1})(\sigma_\nu) = \mathrm{SRL}(\theta'^{-1})(\sigma_{\nu'}).$$

Here in the last formula we have used the formula $\mathrm{SRL}(\overline{\lambda})(\sigma_{\nu'}) = \sigma_\nu$. By definition $N_{F/k(a),b} = \mathrm{SRL}(\theta^{-1})(\sigma_\nu)$ and $N_{F/k(b),a} = \mathrm{SRL}(\theta'^{-1})(\sigma_{\nu'})$. So we have proved that $N_{F/k(a),b}(\mathcal{H}_{k(a)}) = N_{F/k(b),a}(\mathcal{H}_{k(b)})$.

By Proposition 5.11 we have

$$N_{F/k(a),b} = N_{F/k(a)}, \quad N_{F/k(b),a} = N_{F/k(b)}.$$

So we have proved that for any $a, b \in F\backslash k$ generating F over k we have $N_{F/k(a)}(\mathcal{H}_{k(a)}) = N_{F/k(b)}(\mathcal{H}_{k(b)})$. Now the statement follows from the following fact: for any $a, b \in F\backslash k$, there is $c \in F\backslash k$ such that the pairs $(a,c), (b,c)$ generate F over k.

□

6. The proof of Theorem 3.1 and Corollary 3.5

Proof of Theorem 3.1. **Existence:** Let $F \in \mathbf{Fields}_1$. Choose some embedding $j\colon k(t) \hookrightarrow F$. Define the element $\mathcal{H}_F$ by the formula

$$\mathcal{H}_F := N_{F/k(t)}(\mathcal{H}_{k(t)}).$$

By the fourth statement of Theorem 3.6 this element does not depend on j.

We need to show that if $j'\colon F_1 \subset F_2$ is an embedding, then

$$\mathrm{SRL}(j')(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}.$$

It follows from the first and the third statement of the same theorem:

$$\mathrm{SRL}(j')(\mathcal{H}_{F_2}) = \mathrm{SRL}(j')N_{F_2/k(t)}(\mathcal{H}_{k(t)}) = \mathrm{SRL}(j')N_{F_2/F_1}N_{F_1/k(t)}(\mathcal{H}_{k(t)}) =$$
$$= (\mathrm{SRL}(j') \circ N_{F_2/F_1})(N_{F_1/k(t)}\mathcal{H}_{k(t)}) = \mathcal{H}_{F_1}.$$

Uniqueness: Let $\mathcal{H}_F, \mathcal{H}'_F, F \in \mathbf{Fields}_1$ be two family of strong reciprocity laws such that for any $j\colon F_1 \hookrightarrow F_2$ we have $\mathrm{SRL}(j)(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$ and $\mathrm{SRL}(j)(\mathcal{H}'_{F_2}) = \mathcal{H}'_{F_1}$. We need to show that $\mathcal{H}_F = \mathcal{H}'_F$ for any $F \in \mathbf{Fields}_1$. First of all it is true when $F = k(t)$. Let F be any field. There is a field $F' \in \mathbf{Fields}_1$ together with two embeddings $F \subset F', k(t) \subset F'$ such that $F'/k(t)$ is Galois. It is enough to prove the statement for F'. Denote by G the Galois group of F' over $k(t)$. Since $\mathcal{H}_{F'}$ and $\mathcal{H}'_{F'}$ are invariant under the group G, it is enough to prove that they equal on the subgroup $(\Lambda^3 F'^{\times})^G$. We know that $\left(K_3^M(F')\right)^G = K_3^M(k(t))$. It follows that $(\Lambda^3 F'^{\times})^G$ is generated by the Steinberg elements and the elements coming from $k(t)$. On the Steinberg elements $\mathcal{H}_{F'}$ and $\mathcal{H}'_{F'}$ coincide because they are strong reciprocity laws. On the elements coming from $k(t)$ they coincide because $\mathcal{H}_{k(t)} = \mathcal{H}'_{k(t)}$.

□

Let $L \in \mathbf{Fields}_2$. Define the map $H_L \colon \Lambda^4 L^{\times} \to B_2(k)$ by the formula:

$$H_L(b) = \sum_{\nu \in \mathrm{dval}(L)} \mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b).$$

This formula is well defined by Lemma 5.1. The following lemma is corollary from Theorem 3.1:

Lemma 6.1. *If $j \colon L \hookrightarrow M$ is an extension of some fields from* $\mathbf{Fields}_2$ *then for any $b \in \Lambda^4 L^{\times}$ we have $H_L(b) = \dfrac{1}{[M:L]} H_M(j_*(b))$.*

Proof. Let $\nu \in \mathrm{dval}(L)$. It is enough to show the following formula:

$$\mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b) = \frac{1}{[M:L]} \sum_{\nu' \in \mathrm{ext}(\nu, M)} \mathcal{H}_{\overline{M}_{\nu'}} \partial_{\nu'}^{(4)}(j_*(b)) \tag{6.1}$$

We have

$$\partial_{\nu'}^{(4)}(j_*(b)) = e_{\nu'|\nu} j_{\nu'|\nu} \cdot \partial_\nu^{(4)}(b).$$

Theorem 3.1 implies that

$$\mathcal{H}_{\overline{M}_{\nu'}} j_{\nu'|\nu} \partial_\nu^{(4)}(b) = f_{\nu'|\nu} \mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b).$$

So

$$\begin{aligned} &\frac{1}{[M:L]} \sum_{\nu' \in \mathrm{ext}(\nu, M)} \mathcal{H}_{\overline{M}_{\nu'}} \partial_{\nu'}^{(4)} j_*(b) = \\ &= \frac{1}{[M:L]} \sum_{\nu' \in \mathrm{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} \mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b) = \\ &= \mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b) \frac{1}{[M:L]} \sum_{\nu' \in \mathrm{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} = H_{\overline{L}_\nu} \partial_\nu^{(4)}(b). \end{aligned} \tag{6.2}$$

The last equality follows from the formula $\sum\limits_{\nu' \in \mathrm{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} = [M:L]$. □

Proof of Corollary 3.5. We need to show that $H_L = 0$ for any $L \in \mathbf{Fields}_2$. Let us prove that it is true when $L = k(x)(y)$. By Theorem 5.3 and Proposition 4.2 in this case on L there is a unique system σ of strong reciprocity laws. It is enough to show that for any $\nu \in \mathrm{dval}(L)$ we have $\sigma_\nu = \mathcal{H}_{\overline{L}_\nu}$. When ν is special it is true by Proposition 4.2. When ν is general it follows from the definition of $\mathcal{H}_{\overline{L}_\nu}$.

Let us prove the statement for an arbitrary L. There are finite extensions

$$j \colon k(x)(y) \hookrightarrow L', j' \colon L \hookrightarrow L'$$

such that j is a Galois extension. Lemma 6.1 shows that it is enough to prove the statement for L'. Denote the Galois group of j by G. Since $H_{L'}$ is invariant under G it is enough to prove that $H_{L'}$ is zero on the subgroup $(\Lambda^4 L'^\times)^G$. Since $\left(K_4^M(L')\right)^G = K_4^M(k(x)(y))$, $(\Lambda^4 L'^\times)^G$ is generated by the Steinberg elements and by the elements coming from $k(x)(y)$. Vanishing of $H_{L'}$ on the elements coming from $k(x)(y)$ follows from Lemma 6.1 together with the formula $H_{k(x)(y)} = 0$. Let us prove that $H_{L'}$ is zero on the Steinberg elements. For any $b \in B_2(L', 4)_1$ we have

$$H_{L'}(\delta_4(b)) = \sum_{\nu \in \mathrm{dval}(L')} \mathcal{H}_{\overline{L'}_\nu} \partial_\nu^{(4)} \delta_4(b) = \sum_{\nu \in \mathrm{dval}(L')} \mathcal{H}_{\overline{L'}_\nu} \delta_3 \partial_\nu^{(4)}(b) =$$
$$= \sum_{\nu \in \mathrm{dval}(L')} \sum_{\nu' \in \mathrm{val}(\overline{L'}_\nu)} \partial_{\nu'}^{(3)} \partial_\nu^{(4)}(b) = 0.$$

Here the second formula is true because $\partial_\nu^{(4)}$ is morphism of complexes, the third formula is true because $\mathcal{H}_{\overline{L'}_\nu}$ is a strong reciprocity law and the fourth formula follows from Corollary 4.9. $\square$

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