Combinatorial Properties of Compact Hyperbolic Coxeter Polytopes

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Abstract

This paper shows that all compact hyperbolic Coxeter polytope that is a product of some number of simplices are already described in articles by other authors. We also consider compact hyperbolic Coxeter polytopes whose every Lannér subdiagram has order 2. We show that for such polytopes Vinberg's upper bound on dimension might be improved to 22.

1 Introduction

A convex polytope is called a *Coxeter polytope* if its dihedral angles are all integer submultiples of π. Compact Coxeter polytopes in $\mathbb{S}^d$ and $\mathbb{E}^d$ were classified by Coxeter in [Cox34]. Vinberg initiated the study of such polytopes in $\mathbb{H}^d$ and proved in [Vin84] that there are no compact Coxeter polytopes in $\mathbb{H}^{\geq 30}$. Examples are known only in $\mathbb{H}^{\leq 8}$, the unique known example in $\mathbb{H}^8$ and both known examples in $\mathbb{H}^7$ are due to Bugaenko ([Bug92]).

Thus, there are very hard long-standing open problems of constructing new hyperbolic Coxeter polytopes, especially higher-dimensional ones, and of classification of hyperbolic reflection groups.

Generally speaking, there are two different approaches to both problems: classification of finite-volume Coxeter polytopes of some certain combinatorial properties (see [Kap74; Ess96; Tum07; FT08; FT09; JT18]) and the theory of arithmetic hyperbolic reflection groups (see [Vin72; Bel16; Bog17]).

We provide the following theorem which combine several results in the direction of combinatorics of compact Coxeter polytopes.

Theorem A. *All compact hyperbolic Coxeter products of simplices are listed in [Lan50] (simplices), [Poi82] (2-dimensional cubes), [JT18] (cubes), [Kap74] (simplicial prisms), [Ess96] (products of two simplices), [Tum07] (products of three simplices).*

The next result is an attempt to generalize the improvements of the Vinberg's estimation in special cases. It known that there are no compact right-angled polytopes in $\mathbb{H}^{\geq 5}$ ([PV05]) and that there are no Coxeter cubes in $\mathbb{H}^{\geq 6}$ ([JT18]). Coxeter diagram of every such polytope contain no Lannér subdiagram of order ≥ 3.

Theorem B. *Let P be a compact hyperbolic Coxeter polytope whose every Lannér subdiagram has order 2. Then $\dim P \le 22$.*

2 Coxeter polytopes

2.1 Coxeter diagrams

The *Coxeter diagram* $S(P)$ of a Coxeter polytope P with facets $f_1, \dots f_n$ is a graph, whose vertex v_i corresponds to the facet f_i and the edge $v_i v_j$ is:

- simple edge with integer label $m \ge 2$ or $(m-2)$-fold edge if $\angle f_i f_j = \frac{\pi}{m}$;
- bold edge if f_i is parallel to f_j;
- dotted edge with real label $\cosh(\rho) > 1$ if f_i and f_j diverge and the distance between the facets is equal to ρ.

The *Gram matrix* G(P) $= \{g_{ij}\}$ of a Coxeter polytope P with facets $f_1, \dots f_n$ is a symmetric $n \times n$ matrix with ones on the diagonal such that

$$g_{ij} = \begin{cases} -\cos(\frac{\pi}{m}), & \text{if } \angle f_i f_j = \frac{\pi}{m}; \\ -1, & \text{if } f_i \text{ is parallel to } f_j; \\ -\cosh(\rho_{ij}), & \text{if } f_i \text{ and } f_j \text{ diverge and lie at distance } \rho_{ij}. \end{cases}$$

The Gram matrix $G(S)$ of a Coxeter diagram S have the same definition.

We say that a Coxeter diagram have some property if its Gram matrix have the same property (e.g., positive definiteness). A Coxeter diagram have the same determinant and signature as its Gram matrix.

2.2 Spherical and Euclidean Coxeter polytopes

A *product* of Coxeter diagrams S_1 and S_2 is a Coxeter diagram whose set of vertices is a disjoint union of sets of vertices of S_1 and S_2 and set of edges is a union of sets of edges of S_1 and S_2 (informally speaking, the product the schemes is two schemes drawn side by side). The Gram matrix of such Coxeter diagram is equal to $G(S_1) \oplus G(S_2)$ up to simultaneous permutation of rows and columns. A Coxeter diagram is *connected* if its is not a product of some other Coxeter diagrams.

A *subdiagram* of Coxeter diagram is an induced subgraph of the Coxeter diagram. A Coxeter diagram is said to be *elliptic* if it is positive definite. A connected Coxeter diagram is said to be parabolic if it is degenerate and its any subdiagram is elliptic. A *parabolic* Coxeter diagram is a product of connected parabolic Coxeter diagrams. It is obvious that any elliptic or parabolic diagram is a product of some connected elliptic or parabolic diagrams respectively.

Theorem 2.1 ([Cox34])**.** *There is a spherical Coxeter polytope with a given Coxeter diagram S if and only if S is elliptic. There is an Euclidean Coxeter polytope with a given Coxeter diagram S if and only if S is parabolic. Connected elliptic and parabolic diagrams are listed in Table 1 and Table 2.*

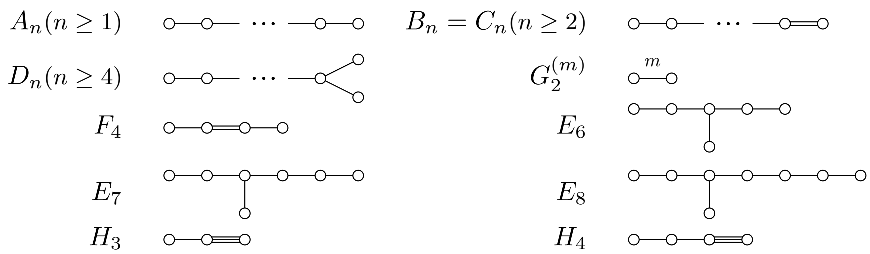


Table 1: Connected elliptic Coxeter diagrams

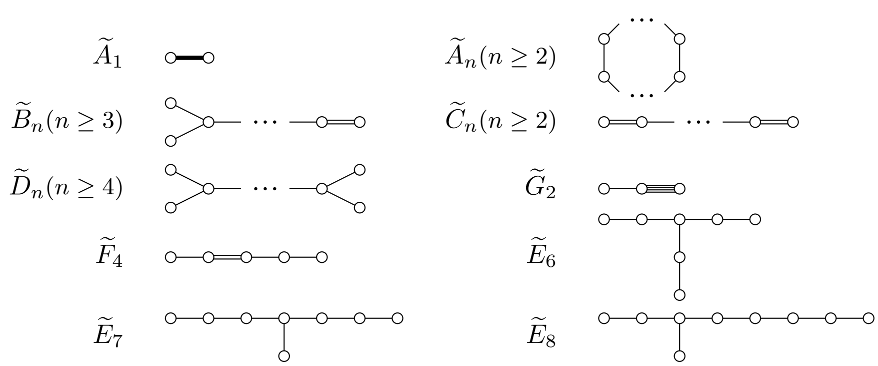


Table 2: Connected parabolic Coxeter diagrams

Corollary 2.2. *Every elliptic diagram contain no cycle. Every vertex of an elliptic diagram is connected with at most three other vertices.*

2.3 Hyperbolic Coxeter Polytopes

A Coxeter diagram is said to be *hyperbolic* if its negative inertia index is equal to 1. A Coxeter diagram S is called a *Lannér diagram* if any subdiagram of S is elliptic, and the diagram S is neither elliptic nor parabolic. All Lannér diagram were classified by Lannér in [Lan50]. They are listed in Table 3.

For polytope P we denote partially ordered set of its faces with $\mathcal{F}(P)$. For Coxeter diagram S we denote partially ordered set of its elliptic subdiagrams with $\mathcal{F}(S)$.

Proposition 2.3 ([Vin85, Theorem 3.1])**.** *Let $P \subset \mathbb{H}^d$ be a compact hyperbolic Coxeter polytope. Partially ordered sets $\mathcal{F}(S(P))$ and $\mathcal{F}(P)$ are isomorphic.*

Proposition 2.4 ([Vin84, Proposition 3.2])**.** *Let $P \subset \mathbb{H}^d$ be a compact hyperbolic Coxeter polytope. Its Coxeter diagram $S(P)$ contain no parabolic subdiagrams.*

Proposition 2.5 ([Vin84])**.** *A Coxeter diagram S is a Coxeter diagram of any compact Coxeter polytope if and only if it is hyperbolic, contain no parabolic*

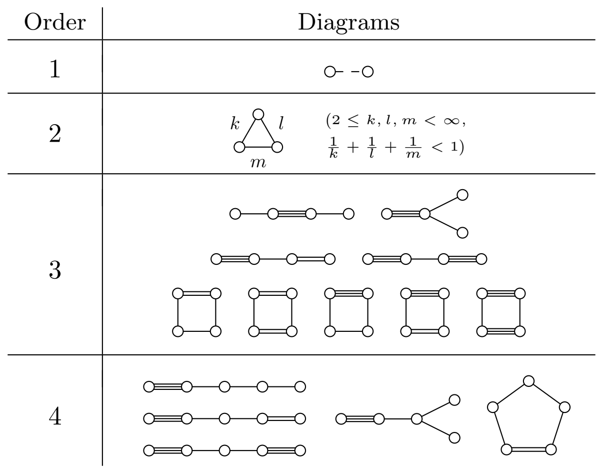


Table 3: Lannér diagrams

subdiagrams, and there is a polytope $P \subset \mathbb{E}^d$ such that $\mathcal{F}(P)$ and $\mathcal{F}(S)$ are isomorphic.

Proposition 2.6. *A Coxeter diagram S is a diagram of some compact hyperbolic polytope*

Corollary 2.7. *A polytope $P \subset \mathbb{H}^d$ is a compact simplex if and only if $S(P)$ is a Lannér diagram.*

Theorem 2.8 ([Vin84, Theorem 1])**.** *There are no compact Coxeter polytopes in $\mathbb{H}^{\geq 30}$.*

2.4 Superhyperbolic diagrams

A Coxeter diagram is said to be *superhyperbolic* if its negative inertia index is greater that 1. A *local determinant* of S on its subdiagram T is

$$\det(S,T) = \frac{\det(S)}{\det(S \setminus T)}.$$

Usually we will mark the vertices of the subdiagram T with $\vee$.

We denote by $p(\gamma)$ the product of absolute values of elements of Gram matrix that correspond to edges of a cycle γ. The following proposition is very useful for calculating of determinants.

Proposition 2.9 ([Vin84, Proposition 11])**.** *A determinant of a Coxeter diagram S is equal to the sum of the products*

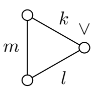
$$(-1)^k \cdot p(\gamma_1) \dots p(\gamma_k)$$

over all sets $\{\gamma_1, \dots, \gamma_k\}$ of disjoint cycles of positive length.

Proposition 2.10 ([Vin84, Proposition 13]). *If a Coxeter diagram S is spanned by two disjoint subdiagrams S_1 and S_2 joined by a unique edge v_1v_2 of weight a, then*

$$\det(S, \langle v_1, v_2\rangle) = \det(S_1, v_1)\det(S_2, v_2) - a^2.$$

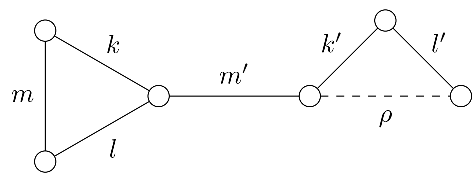
Proposition 2.11 ([Vin84, Table 2]).

det () $= -d(k, l, m)$,

where

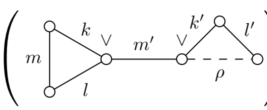
$$d(k,l,m) = \frac{\left(\cos\frac{\pi}{k}\right)^2 + \left(\cos\frac{\pi}{l}\right)^2 + 2\cos\frac{\pi}{k}\cos\frac{\pi}{l}\cos\frac{\pi}{m}}{\left(\sin\frac{\pi}{m}\right)^2} - 1.$$

Note that the signature of the following diagram

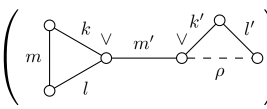
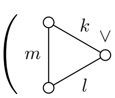
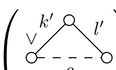


(1)

is either $(4, 1, 1)$, or $(5, 1, 0)$, or $(4, 2, 0)$ since this diagram contains a Lannér subdiagram and an elliptic diagram of order 4. So the diagram is hyperbolic if and only if

det () ≤ 0

but

det () = det () det () $- \left(\cos\frac{\pi}{m'}\right)^2$

$$= d(k,l,m)\frac{\left(\cos\frac{\pi}{l'}\right)^2 + \left(\cos\frac{\pi}{k'}\right)^2 + \rho^2 + 2\rho\cos\frac{\pi}{k'}\cos\frac{\pi}{l'} - 1}{\left(\sin\frac{\pi}{l'}\right)^2} - \left(\cos\frac{\pi}{m'}\right)^2.$$

If $d(k, l, m) \neq 0$ then the last inequality is equivalent to the following:

$$\rho^2 + 2\rho\cos\frac{\pi}{k'}\cos\frac{\pi}{l'} + \left(\cos\frac{\pi}{l'}\right)^2 + \left(\cos\frac{\pi}{k'}\right)^2 - 1 - \frac{\left(\sin\frac{\pi}{l'}\right)^2\left(\cos\frac{\pi}{m'}\right)^2}{d(k,l,m)} \leq 0.$$

Consider the left part of this inequality as quadratic function in ρ. One of the zeros of this function is not greater than 1. So there is a $\rho > 1$ satisfying the inequality if and only if for $\rho = 1$ the strict inequality holds:

$$D(k,l,m,k',l',m') = \left(\cos\frac{\pi}{l'} + \cos\frac{\pi}{k'}\right)^2 - \frac{\left(\sin\frac{\pi}{l'}\right)^2\left(\cos\frac{\pi}{m'}\right)^2}{d(k,l,m)} < 0.$$

This proves the following lemma.

Lemma 2.12. *Let $d(k,l,m) \neq 0$. The Coxeter diagram (1) is superhyperbolic for any $\rho > 1$ if and only if*

$$D(k,l,m,k',l',m') \geq 0.$$

Direct calculations show that if $d(k,l,m) > 0$ then the function D is increasing in k, l, m, k', l', and decreasing in m'.

3 Product of simplices

Let Σ_1 and Σ_2 be sets of Coxeter diagram. We denote by $\Sigma_1 \times_k \Sigma_2$ the set of all Coxeter diagrams S that spanned by subdiagrams $S_1 \in \Sigma_1$ and $S_2 \in \Sigma_2$ whose intersection consist of k vertices and every Lannér or parabolic subdiagram is contained in either S_1 or S_2.

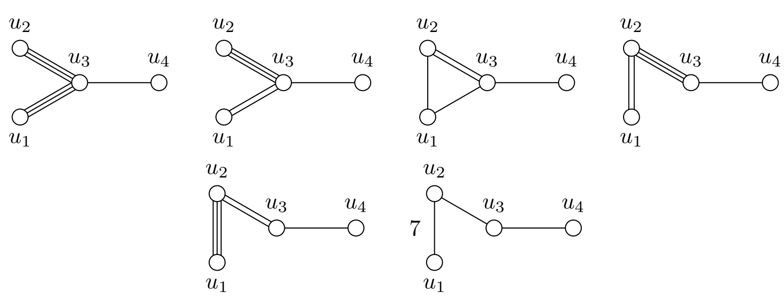
Denote by $\mathcal{L}_k$ the set of all Lannér diagrams of order k and by Δ_k k-dimensional simplex. Suppose that $\mathcal{F}(P)$ and $\mathcal{F}(\Delta_{k_1-1} \times \cdots \times \Delta_{k_n-1})$ are isomorphic for some compact hyperbolic Coxeter polytope P. Then by Vinberg's theorem $S(P) \in \mathcal{L}_{k_1} \times_0 \cdots \times_0 \mathcal{L}_{k_n}$. Without loss of generality $k_1 \geq \cdots \geq k_n$. We may also assume that $k_1 \geq 3$, $k_n = 2$, and $n \geq 4$.

3.1 Case $k_1 \geq 4$

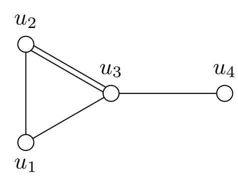
Direct calculations show that no Lannér diagram of order 4 or 5 can be expanded with three vertices without forming a new Lannér or parabolic subdiagram. Therefore $k_1 \leq 3$.

3.2 Case $k_1 = k_2 = 3$

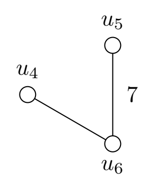
Consider Lannér subdiagrams $L_1 = \langle u_1, u_2, u_3 \rangle$ and $L_2 = \langle u_4, u_5, u_6 \rangle$ of order 3. In is shown in [Tum07, Lemma 4.10] that if $|\det(L_1, u_3)| \leq |\det(L_2, u_4)|$ then the subdiagram $\langle L_1, u_4 \rangle$ is one of the following.



The diagram L_1 might be expanded with two vertices, so subdiagram $\langle L_1, u_4 \rangle$ is the following, or a new Lannér or parabolic subdiagram is forming.



So, $|\det(L_1, u_3)| = \frac{\sqrt{2}}{3}$ which means that $|\det(L_2, u_4)| \leq \frac{3}{4\sqrt{2}}$ according to Proposition 2.10. The multiplicity of the edges u_4u_5 and u_4u_6 do not exceed one. There is the only Lannér diagram of order 3 with such properties which is shown below.



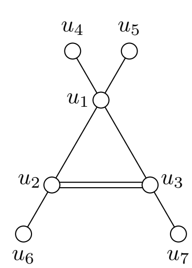
This diagram is no suitable since the diagram can not be expanded with three vertices without forming a new Lannér or parabolic subdiagram.

3.3 Case $k_2 = 2$

Lannér diagram of order 3 can not be expanded with five vertices which means that $n \leq 5$. Let us denote a multiplicity of an edge uv with $[u, v]$.

Lemma 3.1. *$n \leq 4$.*

Proof. Suppose that $n = 5$. Denote Lannér subdiagrams with $L_1 = \langle u_1, u_2, u_3\rangle$, $L_2 = \langle u_4, u_8\rangle$, $L_3 = \langle u_5, u_9\rangle$, $L_4 = \langle u_6, u_{10}\rangle$, and $L_5 = \langle u_7, u_{11}\rangle$. Without loss of generality the vertices u_4, u_5, u_6, and u_7 are connected to the subdiagram L_1. The subdiagram $\langle L_1, u_4, u_5, u_6, u_7\rangle$ should have the following form.



It is easy to check that

$$[u_6, u_4] = [u_6, u_5] = [u_6, u_8] = [u_6, u_9] =$$
$$[u_7, u_4] = [u_7, u_5] = [u_7, u_8] = [u_7, u_9] = 0.$$

This means that the vertices u_{10} and u_{11} should be connected to the subdiagrams $\langle u_4, u_8\rangle$ and $\langle u_5, u_9\rangle$. There are two cases:

1. $[u_{10}, u_{11}] \geq 1$. Then without loss of generality we may assume that
$$[u_{10}, u_8] = [u_{10}, u_5] = [u_{11}, u_4] = [u_{11}, u_9] \geq 1$$
and
$$[u_{11}, u_8] = [u_{11}, u_5] = [u_{10}, u_4] = [u_{10}, u_9] = 0.$$
Then
$$[u_4, u_5] = [u_4, u_9] = [u_8, u_5] = [u_8, u_9] = 0,$$
which means that the subdiagram $\langle L_2, L_3 \rangle$ is not connected.

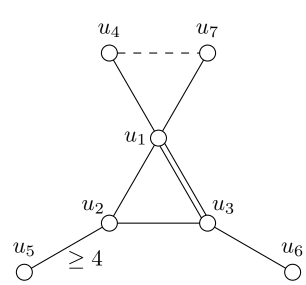
2. $[u_{10}, u_{11}] = 0$. Then without loss of generality we may assume that $[u_6, u_{11}] = 1$. In this case the subdiagram L_5 might be connected with L_2 and L_3 only if
$$[u_{11}, u_8] = [u_{11}, u_9] \geq 1.$$
Then
$$[u_4, u_5] = [u_4, u_9] = [u_8, u_5] = [u_8, u_9] = 0$$
ant the subdiagram $\langle L_2, L_3 \rangle$ is not connected.

□

Thus, only the case of the product of a triangle on a 3-dimensional cube left. Denote Lannér subdiagrams with $L_1 = \langle u_1, u_2, u_3 \rangle$, $L_2 = \langle u_4, u_7 \rangle$, $L_3 = \langle u_5, u_8 \rangle$, $L_4 = \langle u_6, u_9 \rangle$. We suppose that the subdiagrams $\langle L_1, u_4 \rangle$, $\langle L_1, u_5 \rangle$, and $\langle L_1, u_6 \rangle$ are connected. If the subdiagram $\langle L_1, u_4, u_5, u_6 \rangle$ contains the only Lannér subdiagram then all edges of the subdiagram L_1 must have a positive multiplicity. This means that any vertex of the subdiagrams L_2, L_3, and L_4 is connected to L_1 by at most one edge. Denote the multiplicity of such an edge with $[u, L_1]$. If $[u_7, L_1] \geq 1$ and $[u_8, L_1] \geq 1$ then L_2 and L_3 are not connected. Thus, we may assume that $[u_8, L_1] = [u_9, L_1] = 0$, $[u_4, L_1] \geq [u_7, L_1]$, and $[u_5, L_1] \geq [u_6, L_1] = 1$.

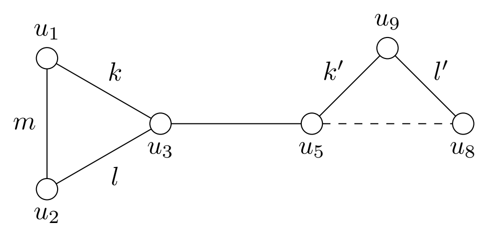
Lemma 3.2. *If* $[u_5, L_1] \geq 2$ *then* $[u_7, L_1] = 0$.

Proof. Assume that $[u_7, L_1] \geq 1$. Then the subdiagram $\langle L_1, L_2, u_5, u_6 \rangle$ has the following form.

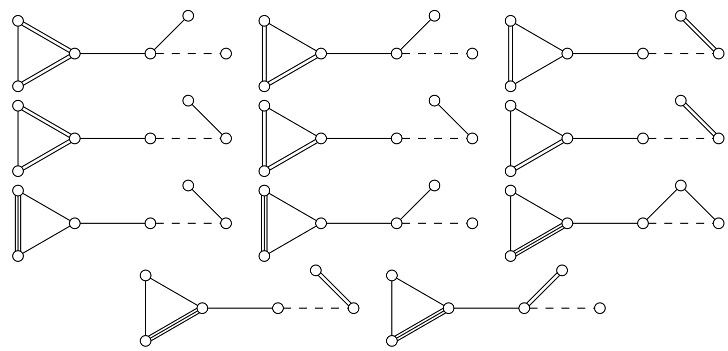


Then $[u_5, u_4] = [u_5, u_7] = [u_8, u_4] = [u_8, u_7] = 0$ and the subdiagrams L_2 and L_3 are not connected. □

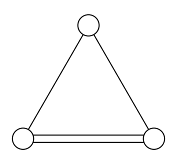
We may suppose that $[u_5, L_1] = 1$ since otherwise we can swap L_2 and L_3. The vertex u_8 is connected to L_4 or the vertex u_9 is connected to L_3. Without loss of generality u_9 is connected to L_3. The subdiagram $\langle L_1, L_3, u_9\rangle$ has the following form for $k' \geq 3$ or $l' \geq 3$.



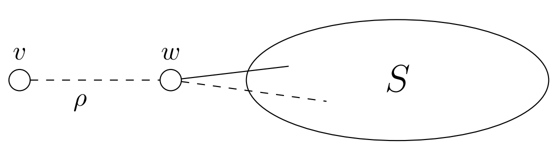
Some superhyperbolic diagrams of this form are listed below.



Lemma 2.12, monotonicity of the function D, and a simple check imply that the subdiagram L_1 have the following form.



Lemma 3.3. *If a diagram S contains hyperbolic subdiagram, a vertex v is connected only with a vertex w by a dotted edge, and* $\det(\langle w, S\rangle) - \det(S) > 0$ *then the diagram $\langle v, w, S\rangle$ is superhyperbolic for any admissible labels on dotted edges.*



Proof. If ρ is a label on dotted edge between v and w then

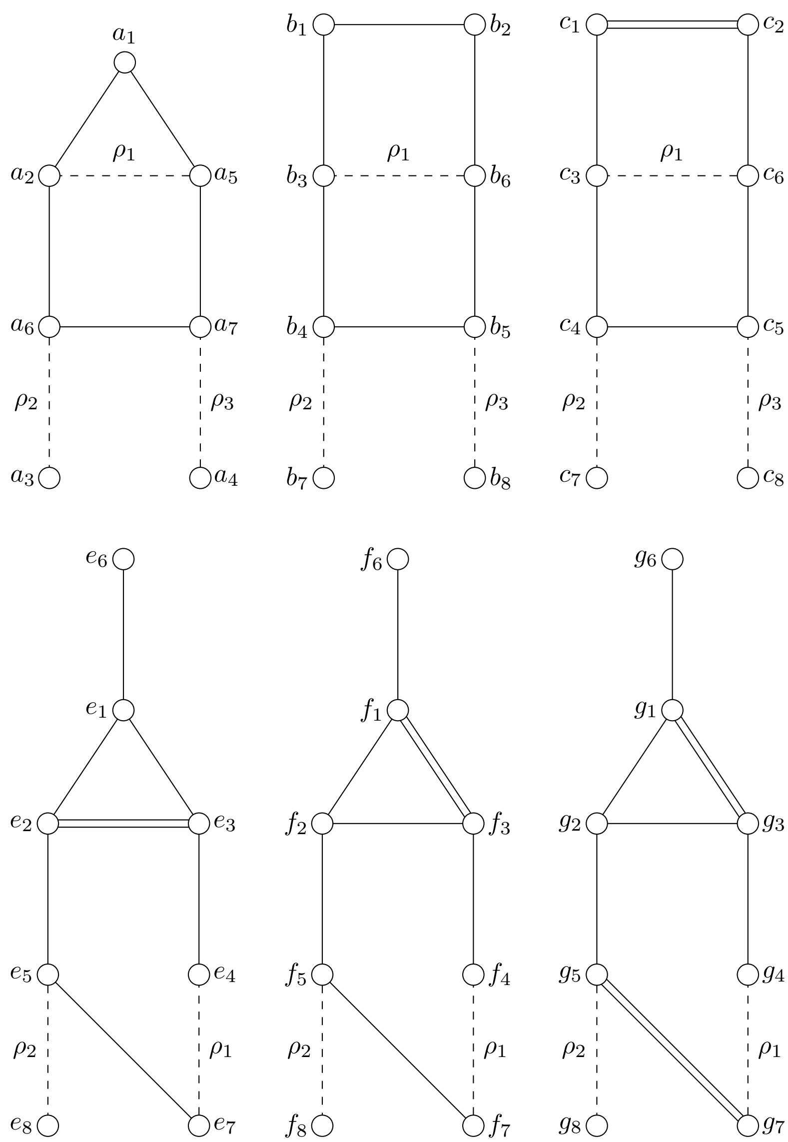
$$\det(\langle v, w, S\rangle) = \det(\langle w, S\rangle) - \rho^2 \det(S).$$

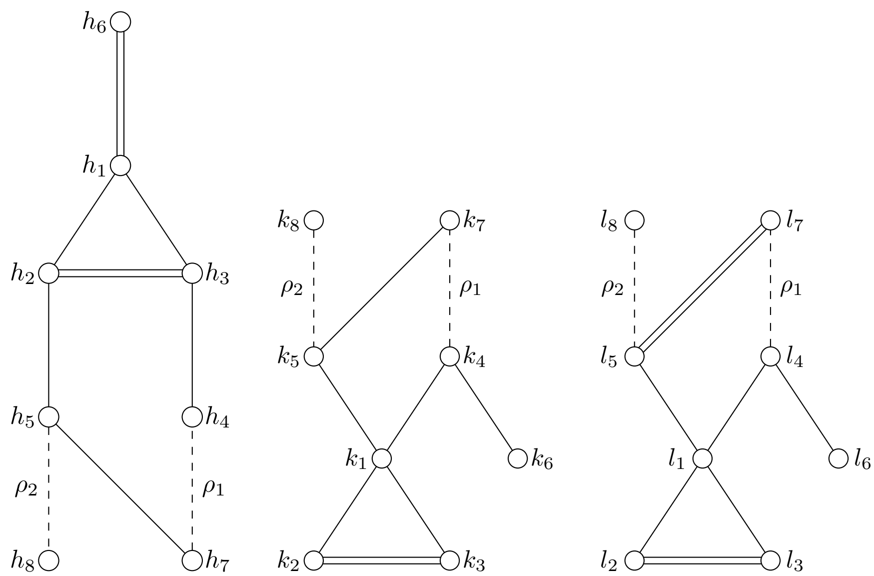
Suppose that the diagram $\langle v, w, S\rangle$ is hyperbolic. If $\det(S) < 0$ then

$$\rho \le \sqrt{\frac{\det(\langle w, S\rangle)}{\det(S)}} = \sqrt{1 + \frac{\det(\langle w, S\rangle) - \det(S)}{\det(S)}} \le 1.$$

This means that $\det(S) = 0$ and $\det(\langle w, S\rangle) > 0$ which implies the superhyperbolicity of the diagram $\langle w, S\rangle$. □

Corollary 3.4. *The diagrams below are superhyperbolic for any $\rho_1, \rho_2, \rho_3 > 1$.*





Proof.

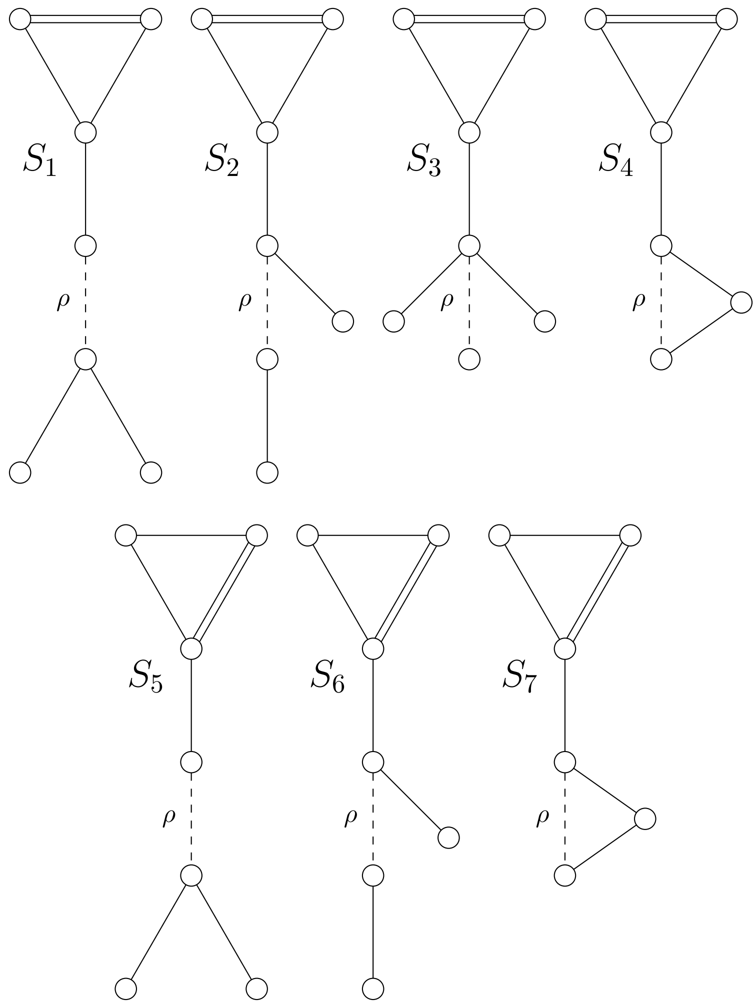
$$\det(\langle a_7, A\rangle) - \det(A) = \frac{1}{16}\big(3\rho_2^2 + 4\rho_1^2 - 2\rho_1 - 5\big) > 0,$$
$$\det(\langle b_5, B\rangle) - \det(B) = \frac{1}{16}\big(2\rho_2^2 + 3\rho_1^2 - 2\rho_1 - 3\big) > 0,$$
$$\det(\langle c_5, C\rangle) - \det(C) = \frac{1}{64}\big(4\rho_2^2 + 8\rho_1^2 - 4(2-\sqrt{2})\rho_1 - 2\sqrt{2} - 3\big) > 0,$$
$$\det(\langle e_5, E\rangle) - \det(E) = \frac{1}{64}\big(8\rho_1^2 - 2(2+3\sqrt{2})\rho_1 + 4\sqrt{2} - 1\big) > 0,$$
$$\det(\langle f_5, F\rangle) - \det(F) = \frac{1}{32}\big(2\rho_1^2 - (3+2\sqrt{2})\rho_1 + 2\sqrt{2} + 2\big) > 0,$$
$$\det(\langle g_5, G\rangle) - \det(G) = \frac{1}{64}\big(4\rho_1^2 - 2(4+3\sqrt{2})\rho_1 + 8\sqrt{2} + 9\big) > 0,$$
$$\det(\langle h_5, H\rangle) - \det(H) = \frac{1}{64}\big(4\rho_1^2 - 4(1+\sqrt{2})\rho_1 + 4\sqrt{2} + 3\big) > 0,$$
$$\det(\langle k_5, K\rangle) - \det(K) = \frac{1}{64}\big(8\rho_1^2 - 8\rho_1 + 3\sqrt{2} - 4\big) > 0,$$
$$\det(\langle l_5, L\rangle) - \det(L) = \frac{1}{32}\big(4\rho_1^2 - 4\sqrt{2}\rho_1 + 3\sqrt{2} - 1\big) > 0,$$

where

$$A = \langle a_1, a_2, a_3, a_5, a_6\rangle, \qquad G = \langle g_1, g_2, g_3, g_4, g_6, g_7\rangle,$$
$$B = \langle b_1, b_2, b_3, b_4, b_6, b_7\rangle, \qquad H = \langle h_1, h_2, h_3, h_4, h_6, h_7\rangle,$$
$$C = \langle c_1, c_2, c_3, c_4, c_6, c_7\rangle, \qquad K = \langle k_1, k_2, k_3, k_4, k_6, k_7\rangle,$$
$$E = \langle e_1, e_2, e_3, e_4, e_6, e_7\rangle, \qquad L = \langle l_1, l_2, l_3, l_4, l_6, l_7\rangle,$$
$$F = \langle f_1, f_2, f_3, f_4, f_6, f_7\rangle.$$

□

Lemma 3.5. *The diagrams below are superhyperbolic for any $\rho > 1$.*

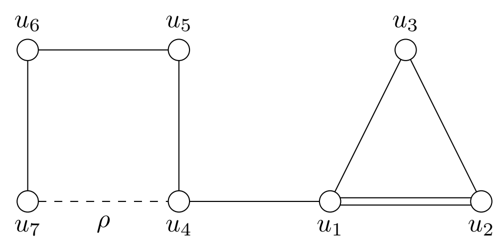


Proof.

$$\det(S_1) = \frac{1}{16}\Big(4\sqrt{2}\rho^2 - 2\sqrt{2} - 1\Big) > 0,$$
$$\det(S_2) = \frac{1}{64}\Big(16\sqrt{2}\rho^2 - 9\sqrt{2} - 6\Big) > 0,$$
$$\det(S_3) = \frac{1}{8}\Big(2\sqrt{2}\rho^2 - \sqrt{2} - 1\Big) > 0,$$
$$\det(S_4) = \frac{1}{32}\Big(8\sqrt{2}\rho^2 + 4\sqrt{2}\rho - 4\sqrt{2} - 3\Big) > 0,$$
$$\det(S_5) = \frac{1}{32}\Big(8\sqrt{2}\rho^2 - 4\sqrt{2} - 3\Big) > 0,$$
$$\det(S_6) = \frac{1}{64}\Big(16\sqrt{2}\rho^2 - 9\sqrt{2} - 9\Big) > 0,$$
$$\det(S_7) = \frac{1}{64}\Big(16\sqrt{2}\rho^2 + 8\sqrt{2}\rho - 8\sqrt{2} - 9\Big) > 0.$$

□

Lemma 3.6. *The diagram below is superhyperbolic for any $\rho > 1$.*



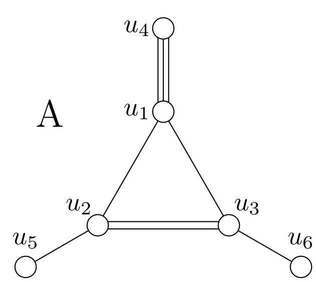
Proof. The determinant of the diagram is equal to

$$\frac{1}{32}\big(6\sqrt{2}\rho^2 + 4\sqrt{2}\rho - 5\sqrt{2} - 3\big) > 0.$$

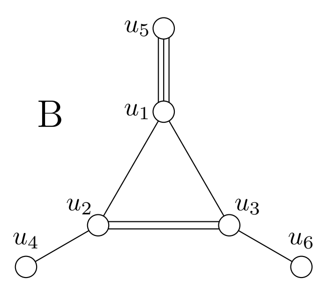
□

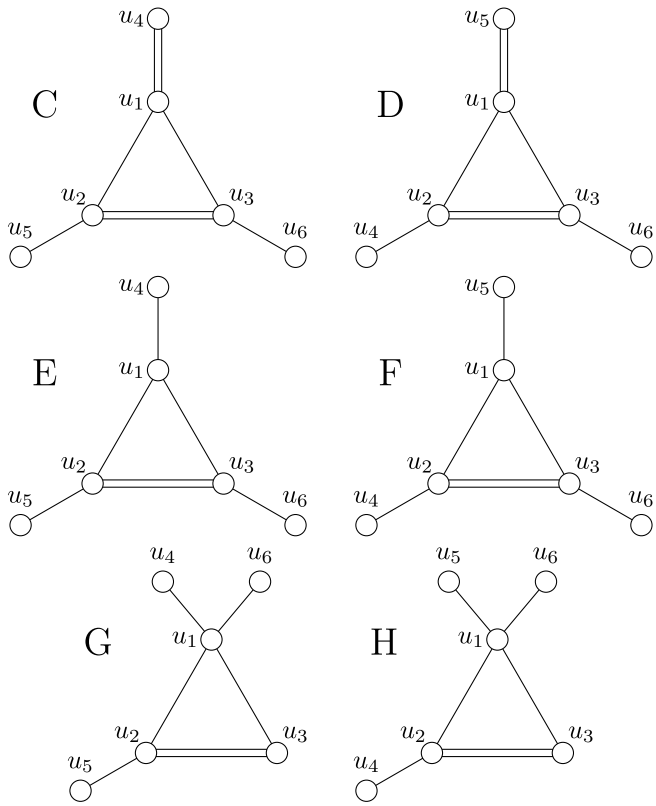
The subdiagram $\langle L_1, u_4, u_5, u_6\rangle$ must have one of the following forms.

A



B





Let us remind that we suppose that $[u_8, L_1] = [u_9, L_1] = 0$, $[u_4, L_1] \geq [u_7, L_1]$, and $[u_5, L_1] \geq [u_6, L_1] = 1$.

3.3.1 Case A

$$[u_5, u_4] = [u_5, u_7] = [u_4, u_8] = 0,$$
$$[u_4, u_6] = [u_4, u_9] = [u_6, u_7] = 0,$$
$$[u_5, u_6] = [u_5, u_9] = [u_6, u_8] = 0,$$

or some subdiagram that must be elliptic is parabolic or hyperbolic. This implies $[u_7, u_8] \neq 0$, $[u_8, u_9] \neq 0$, and $[u_9, u_7] \neq 0$. Then the subdiagram $\langle u_7, u_8, u_9 \rangle$ is not elliptic.

3.3.2 Case B

As above.

3.3.3 Case C

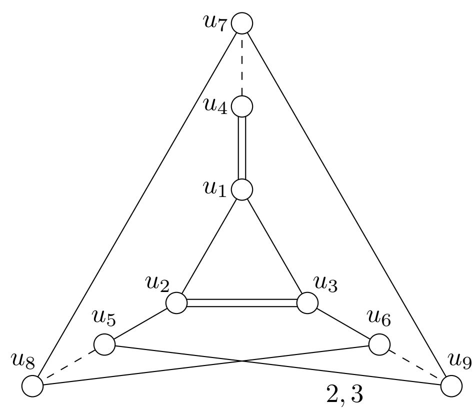
By the same argument we get

$$[u_5, u_4] = [u_5, u_7] = [u_4, u_8] = 0,$$
$$[u_4, u_6] = [u_4, u_9] = [u_6, u_7] = 0,$$
$$[u_5, u_6] = 0.$$

This yields that without loss of generality

$$[u_7, u_8] = [u_7, u_9] = 1, \quad [u_8, u_9] = 0,$$
$$[u_6, u_8] = 1, \quad [u_5, u_9] \in \{0, 1\},$$
$$[u_7, u_1] = [u_7, u_2] = [u_7, u_3] = 0.$$

Then the diagram is as shown below.



The subdiagram $\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9\rangle$ is superhyperbolic according to Lemma 3.6.

3.3.4 Case D

$$[u_7, u_1] = [u_7, u_3] = 0.$$

If $[u_7, u_2] = 1$ then

$$[u_4, u_5] = [u_4, u_8] = [u_7, u_5] = [u_7, u_8] = 0.$$

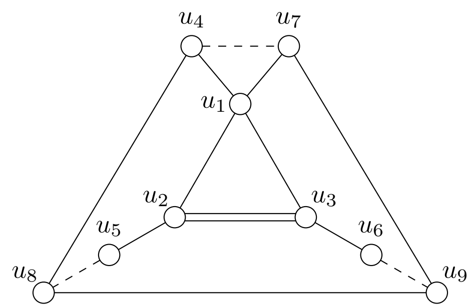
Thus, $[u_7, u_2] = 0$ which is the previous case.

3.3.5 Case E

$$[u_5, u_4] = [u_5, u_7] = [u_6, u_4] = [u_6, u_7] = 0.$$

If $[u_7, u_1] = 0$ then $[u_8, u_7] = 1$ or the diagram $\langle u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\rangle$ is superhyperbolic according to Corollary 3.4. In the same way $[u_9, u_7] = 1$. Without loss of generality $[u_8, u_6] = 1$. Then the subdiagram $\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9 \rangle$ is superhyperbolic according to Lemma 3.6.

If $[u_7, u_1] = 1$ then the diagram must have the following form.



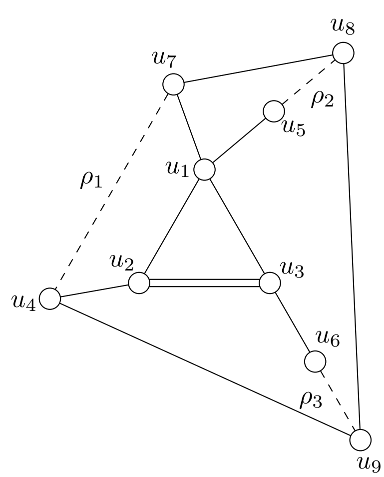
The subdiagram $\langle u_1, u_4, u_5, u_6, u_7, u_8, u_9 \rangle$ is superhyperbolic which follows from Corollary 3.4.

3.3.6 Case F

The case $[u_7, L_1] = 0$ is considered in the previous paragraph so $[u_7, L_1] \neq 0$. Then $[u_7, u_2] = [u_7, u_3] = 0$ and $[u_7, u_1] = 1$. The equality

$$[u_4, u_5] = [u_4, u_8] = [u_7, u_5] = 0$$

implies $[u_7, u_8] \neq 0$. The diagram have the following form.



But this diagram is superhyperbolic since

$$\det(\langle u_1, u_2, u_3, u_5, u_7, u_8, u_9 \rangle) = \frac{1}{32}\Big(\big(8\sqrt{2}+4\big)\rho_2^2 - 4\rho_2 - \big(5+4\sqrt{2}\big)\Big) > 0$$

for all $\rho_2 > 1$.

3.3.7 Case G

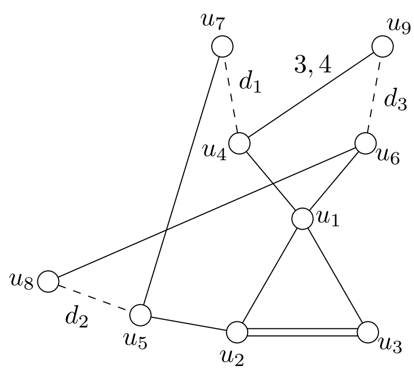
Suppose that $[u_7, L_1] = 0$ and consider the subdiagram $\langle u_1, u_2, u_3, u_4, u_7, u_8, u_9 \rangle$. Lemma 3.5 implies that either

$$[u_4, u_5] = [u_4, u_8] = [u_7, u_8] = 0 \text{ and } [u_7, u_5] = 1$$

or

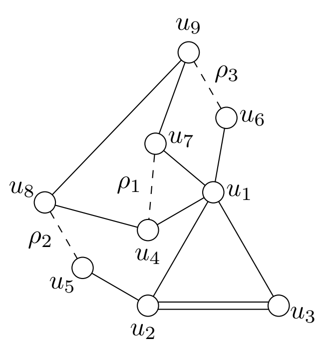
$$[u_4, u_6] = [u_4, u_9] = [u_7, u_9] = 0 \text{ and } [u_7, u_6] \geq 1.$$

In both cases the diagram must have the following form.



But Corollary implies that the subdiagram $\langle u_1, u_2, u_3, u_4, u_6, u_7, u_8, u_9 \rangle$ is superhyperbolic.

Now suppose that $[u_7, L_1] \neq 0$. Then $[u_7, u_2] = 0$. We also may suppose that $[u_7, u_1] = 1$ since $[u_7, u_3] = 1$ is already considered in Case F. The diagram must have the following form.

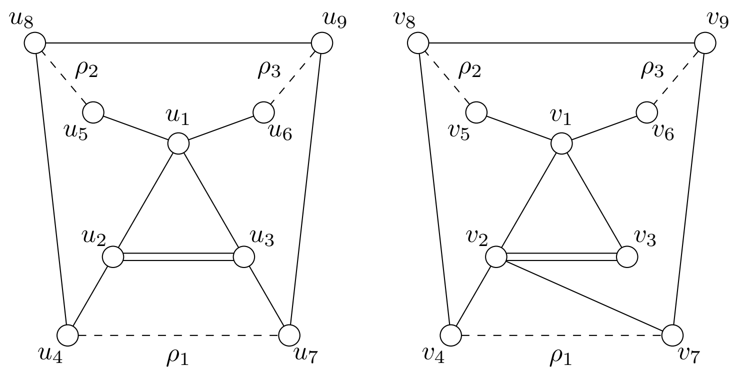


It is easy to calculate that

$$\det(\langle u_1, u_2, u_3, u_6, u_7, u_8, u_9 \rangle) = \frac{1}{32}\Big((4 + 8\sqrt{2})\rho_3^2 - 4\rho_3 - 4\sqrt{2} - 5\Big) > 0.$$

3.3.8 Case H

Suppose that $[u_7, L_1] \neq 0$. The opposite is considered in previous paragraph. The diagram must have one of the forms below.



Corollary 3.4 implies that the subdiagrams $\langle u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\rangle$ and $\langle v_2, v_4, v_5, v_6, v_7, v_8, v_9\rangle$ are superhyperbolic.

3.4 Products of simplices classification

The previous subsection proves the following lemma.

Lemma 3.7. *Let S be a Coxeter diagram whose partially ordered set of elliptic diagrams $\mathcal{F}(S)$ is isomorphic to the partially ordered set of faces of the triangle and 3-dimensional cube product. Then S is superhyperbolic.*

The whole section provides a proof of the following theorem.

Theorem A. *All compact hyperbolic Coxeter products of simplices are listed in [Lan50] (simplices), [Poi82] (2-dimensional cubes), [JT18] (cubes), [Kap74] (simplicial prisms), [Ess96] (products of two simplices), [Tum07] (products of three simplices).*

4 Coxeter polytopes with Lannér subdiagram of only order 2

If P is a compact hyperbolic Coxeter polytope that is a d-dimensional cube or a right-angled polytope then an order of its every Lannér subdiagram equals 2. It is known that d-dimensional cubes do not exist in $\mathbb{H}^{\geq 6}$ and right-angled polytopes do not exist in $\mathbb{H}^{\geq 5}$.

Theorem 4.1 (Mantel)**.** *Let $G = (V, E)$ be a triangle-free graph. Then*

$$|E| \leq \begin{cases} \frac{|V|^2}{4}, & |V| \textit{ is even;} \\ \frac{|V|^2-1}{4}, & |V| \textit{ is odd.} \end{cases}$$

The equality holds only for complete bipartite graphs with (almost) equal components.

Theorem 4.2 ([Nik81, Theorem 3.2.1])**.** *Let $\theta_0, \ldots, \theta_{k-1}$ be non-negative reals and let P be d-dimensional convex polytope for $d \geq 2k-1$. Then*

$$\frac{1}{\alpha_k^P} \sum_{\substack{Q<P \\ \dim Q = k}} \sum_{i=0}^{k-1} \theta_i \alpha_i^Q < \sum_{i=0}^{k-1} \theta_i A_d^{(i,k)},$$

where α_k^R is a number of k-dimensional faces of a polytope R, $Q < P$ means that Q is a face of P, and

$$A_d^{(i,k)} = \binom{d-i}{k-i} \frac{\binom{\lceil d/2 \rceil}{i} + \binom{\lfloor d/2 \rfloor}{i}}{\binom{\lceil d/2 \rceil}{k} + \binom{\lfloor d/2 \rfloor}{k}}.$$

Corollary 4.3. *Let $d \geq 3$. The mean edge number of 2-dimensional faces of a simple convex d-dimensional polytope is at least*

$$A_d^{(1,2)} = \begin{cases} \frac{4(d-1)}{d-2}, & d \text{ is even;} \\ \frac{4d}{d-1}, & d \text{ is odd.} \end{cases}$$

Lemma 4.4. *Let $d \geq 3$ and let P be simple convex d-dimensional polytope whose every 2-dimensional face is not a triangle. There is a vertex of P which is contained in at least*

$$\begin{cases} \binom{d}{2} - \frac{5d}{2}, & d \text{ is even;} \\ \binom{d}{2} - \frac{5(d-1)}{2}, & d \text{ is odd} \end{cases}$$

2-dimensional quadrilaterals.

Proof. Let us denote the number of the 2-dimensional quadrilaterals of P by α, the number of 2-dimensional faces of P by β, the number of the 2-dimensional quadrilaterals of P that contain v by α'_v, and the number of the 2-dimensional faces of P that contain v by β'_v. Let c and c' be reals such that $\alpha = c\beta$ and $4c - c' A_d^{(1,2)} = 0$. Then

$$\sum_{\substack{v<P \\ \dim v = 0}} \alpha'_v - c'\beta'_v \geq 4\alpha - c' A_d^{(1,2)} \beta \geq (4c - c' A_d^{(1,2)})\beta = 0.$$

This implies that the inequality $\alpha'_v \geq c'\beta'_v$ holds for at least one vertex. Let us denote this vertex by v.

The fact that the polytope does not have triangles as 2-dimensional faces implies

$$4c + 5(1-c) < A_d^{(1,2)}$$

and consequently

$$c > 5 - A_d^{(1,2)}.$$

Then

$$\alpha'_v \geq c'\beta'_v = \frac{4c}{A_d^{(1,2)}}\beta'_v > \left(\frac{20}{A_d^{(1,2)}} - 4\right)\beta'_v = \begin{cases} (1-\frac{5}{d-1})\beta'_v, & d \text{ is even;} \\ (1-\frac{5}{d})\beta'_v, & d \text{ is odd.} \end{cases}$$

The polytope P is simple so $\beta'_v = \binom{d}{2}$ and

$$\alpha'_v > \begin{cases} \binom{d}{2} - \frac{5d}{2}, & d \text{ is even;} \\ \binom{d}{2} - \frac{5(d-1)}{2}, & d \text{ is odd.} \end{cases}$$

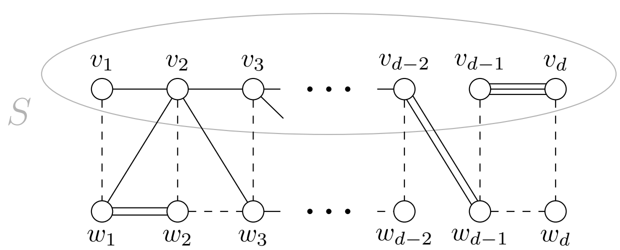
□

Theorem B. *Let P be a compact hyperbolic Coxeter polytope whose every Lannér subdiagram has order 2. Then $\dim P \leq 22$.*

Proof. Let S be a Coxeter diagram of the polytope P. Consider any 23-dimensional face Q of the polytope P. Denote the subdiagram which correspond to Q by T. Partially ordered set $\mathcal{F}(T)$ is isomorphic to partially ordered set of faces $\mathcal{F}(Q)$ since the diagram S contains no Lannér diagrams of order ≥ 3. The further proof is contained in the following lemma. □

Lemma 4.5. *Let T be a Coxeter diagram whose every Lannér subdiagram has order 2 and Q be a polytope with $\dim P = 23$. If $\mathcal{F}(T) = \mathcal{F}(Q)$ then the diagram T is superhyperbolic.*

Proof. Let $S = \langle v_1, \dots, v_d \rangle$ be an elliptic subdiagram that corresponds to the vertex from Lemma 4.4. For any v_k there is a unique w_k such that the diagram $\langle v_1, \dots, v_{k-1}, w_k, v_{k+1}, \dots, v_d \rangle$ is elliptic. The vertices v_k and w_k are connected with dotted line since the subdiagram $\langle S, w_k \rangle$ is hyperbolic.

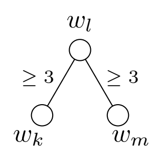


Denote by vv the number of edges that connect v_k with v_l for some k and l, by vw the number of elliptic edges that connect v_k with w_l for some k and l, by wwe the number of elliptic edges that connect w_k and w_l for some k and l, and by wwh the number of dotted edges that connect w_k and w_l for some k and l. The diagrams $\langle v_k, w_k \rangle$ are hyperbolic so they must be pairwise connected which implies

$$vv + vw + wwe + wwh \geq \binom{d}{2}. \tag{2}$$

The classification of elliptic diagrams implies $vv \leq d-1$ (there are no cycles in elliptic diagrams) and $vw \leq 3d$ (degree of an elliptic diagram vertex is at most 3). Let d be odd. Theorem 4.1 implies $wwh \leq \frac{d^2-1}{4}$. Lemma 4.4 implies $wwh < \frac{5(d-1)}{2}$.

For $d = 23$ we have the following estimations: $v \leq 22$, $vw \leq 69$, $wwe \leq 132$, and $wwh \leq 54$. If $vv = 22$ then every w_k is connected with S by at most one edge. This implies $vw \leq 23$ which contradicts inequality (2) so $vv \leq 21$. Inequality (2) implies $109 \leq wwe$. According to Theorem 4.1 there are at least 21 vertices from $\{w_1, \ldots, w_d\}$ which have an elliptic edge to some vertex from the same set. Every such vertex is connected with S by at most 2 elliptic edge which implies $vw \leq 48$ and $130 \leq wwe$. After applying the same argument once again we get $vv = 21$, $vw = 46$, $wwe = 132$, and $wwh = 54$. In this case Theorem 4.1 provides the structure of elliptic edges in the subdiagram $\langle w_1, \ldots, w_d \rangle$. For some w_k, w_l, and w_m the subdiagram $\langle w_k, w_l, w_m \rangle$ have the following form.



Then w_l is connected to S by at most one elliptic edge which yields $wwe < 132$. $\square$

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