Diophantine properties of fixed points and derivative of iterations of Minkowski question mark function.

Nikita A. Shulga

Abstract

We consider irrational fixed points of the Minkowski question mark function $?(x)$, that is irrational solutions of the equation $?(x) = x$. It is easy to see that there exist at least two such points. Although it is not known if there are other fixed points, we prove that the smallest and the greatest fixed points have irrationality measure exponent equal to 2. We give more precise results about the approximation properties of these fixed points. Moreover, we introduce a condition from which it follows that there are only two irrational fixed points. Also, we consider derivative of the function $f_n(x) \equiv \underbrace{?(?(...?}_{\text{n times}}(x)))$. It is easy to see that $f_n'(x) = 0$ almost everywhere. But apart from trivial cases(rational numbers for example) it is difficult to find explicit examples of numbers for which $f_n'(x) = 0$. In this paper we present a set of continued fractions, such that for every element x_0 of this set and for any $n \in \mathbb{N}$ one has $f_n'(x_0) = 0$.

This paper is prepared to the Moebius Contest 2020 and is based on the joint work with Dmitry Gayfulin [19] as well as on my yet unpublished paper about derivative of the iterations of Minkowski question mark function.

1 Introduction

For $x \in [0, 1]$ we consider its continued fraction expansion

$$x = [a_1, a_2, \ldots, a_n, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}, \quad a_j \in \mathbb{Z}_+$$

which is unique and infinite when $x \notin \mathbb{Q}$ and finite for rational x. Each rational x has just two representations

$$x = [a_1, a_2, \ldots, a_{n-1}, a_n] \quad \text{and} \quad x = [a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1], \quad \text{where} \quad a_n \geqslant 2.$$

By

$$\frac{p_k}{q_k} := [a_1, \ldots, a_k]$$

we denote the kth convergent fraction to x. By B_n we denote the nth level of the Stern-Brocot tree, that is

$$B_n := \{x = [a_1, \ldots, a_k] : a_1 + \ldots + a_k = n + 1\}.$$

In [12] Minkowski introduced a special function $?(x)$ which may be defined as the limit distribution function of sets B_n. This function was rediscovered and studied by many authors (see [11],[10],[1],[5],[16]). For irrational $x = [a_1, a_2, \ldots, a_n, \ldots]$ the formula

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\ldots+a_k-1}} \tag{1}$$

introduced by Denjoy [2, 3] and Salem [18] may be considered as one of the equivalent definitions of the function $?(x)$. If x is rational, then the infinite series in (1) is replaced by a finite sum. Note that $?([0; a_1, \ldots, a_t + 1]) = ?([0; a_1, \ldots, a_t, 1])$ and hence $?(x)$ is well-defined for rational numbers too. It is known that $?(x)$ is a continuous strictly increasing function, its derivative $?'(x)$ exists almost everywhere in $[0, 1]$ in the sense of Lebesgue measure, and $?'(x) = 0$ for $x \in \mathbb{Q}$.

The Minkowski question mark function has three trivial fixed points: $0, \frac{1}{2}$, and 1. As

$$?\Big(\frac{1}{5}\Big) = \frac{1}{16} < \frac{1}{5} < \frac{3}{7} < \frac{7}{16} = ?\Big(\frac{3}{7}\Big)$$

we see that there exists $x_1 \in (\frac{1}{5}, \frac{3}{7})$ such that $?(x_1) = x_1$. As $?(1 - x) = 1 - ?(x)$ for all $x \in [0, 1]$, one also can easily see that $?(1 - x_1) = 1 - x_1$.

A folklore conjecture states that

Conjecture 1. *The Minkowski question mark function $?(x)$ has exactly five fixed points. There is only one irrational fixed point of $?(x)$ in the interval $(0, \frac{1}{2})$.*

This conjecture has not yet been proved (for certain discussion see the survey preprint by Moshchevitin [13]). Our computations show that if there is more then one fixed point in the interval $(0, \frac{1}{2})$, then the first 5400 partial quotients in the continued fraction expansion of these numbers coincide. Although we do not know if there are exactly two irrational fixed points of $?(x)$, we are able to say something about Diophantine properties of some of them. In the present paper we give explicit lower bounds for the irrationality measure of the smallest and the greatest fixed points from $(0, \frac{1}{2})$, that is lower bounds of the form

$$\Big|x - \frac{p}{q}\Big| > \frac{1}{q^2 \cdot I(q)} = \frac{1}{q^{2+\delta(q)}}, \quad \delta(q) \geqslant 0$$

satisfied by all $p, q \in \mathbb{Z}, q \geqslant q_0$, where the dependence $I(q)$ on q is explicit and q_0 is given. Usually the infimum $\inf_{q \in \mathbb{Z}_+}(2 + \delta(q))$ is called irrationality measure (or exponent) of x.

As $?(1 - x) = 1 - ?(x)$ for all $x \in [0, 1]$, the set of fixed point of the Minkowski question mark function is symmetric with respect to the point $\frac{1}{2}$. Therefore, one can study the fixed points on the interval $[0, \frac{1}{2}]$ only.

Derivative of the Minkowski question mark function was studied in [16], [4], [5], [8], [9] and others. By considering n-th iteration of the $?(x)$ function, i.e. the function

$$f_n(x) := \underbrace{?(?(...?}_{\text{n times}}(x))),$$

we see that for $x_0 \in [0, 1]$

$$f'_n(x_0) = ?'(x_0) \cdot ?'(x)|_{x=?(x_0)} \cdot ?'(x)|_{x=?(?(x_0))} \cdot \ldots \cdot ?'(x)|_{x=\underbrace{?(?(...?}_{\text{n-1 times}}(x_0)))}. \tag{2}$$

As $?(x)$ is a continuous strictly increasing map $[0, 1] \to [0, 1]$ and $?'(x) = 0$ almost everywhere in $[0, 1]$, we see that $f'_n(x) = 0$ almost everywhere in $[0, 1]$.

Even though $f'_n(x) = 0$ for almost every $x \in [0, 1]$, it is hard to find explicit nontrivial examples of x_0 such that $f'_n(x_0) = 0$. By trivial examples we mean all rational numbers (since $?'(x_0) = 0$ and $?(x_0) \in \mathbb{Q}$ for any $x_0 \in \mathbb{Q}$), as well as some quadratic irrationals, satisfying following theorem:

Theorem A([4], Theorem 2). Let for real irrational $x_0 = [a_1, \dots, a_n, \dots]$ one has

$$\liminf_{t\to\infty} \frac{a_1 + ... + a_t}{t} > \kappa_2, \quad \text{where } \kappa_2 = 4,401^+$$

Then $?'(x_0)$ exists and $?(x_0) = 0$.

The exact form of κ_2 can be found in [4].

2 Main results

Our first result establishes some properties of the continued fraction expansion of certain fixed points of $?(x)$.

Theorem 1. *Let $x = [a_1, \dots, a_n, \dots]$ be the smallest or the greatest fixed point of the Minkowski question mark function on the interval $(0, \frac{1}{2})$. Then $a_1 = 2$ and*

$$a_{n+1} \leqslant \sum_{i=1}^{n} a_i \tag{3}$$

for all $n \in \mathbb{N}$.

We give a proof of Theorem 1 in Section 4. The following theorem is a stronger version of Theorem 1. It uses some new geometrical considerations.

Theorem 2. *Denote $\kappa_1 = 2\log_2(\frac{\sqrt{5}+1}{2}) - 1 \approx 0.38848383\dots$. Let x be fixed point from Theorem 1, then*

$$a_{n+1} < \kappa_1 \sum_{i=1}^{n} a_i + 2\log_2\left(\sum_{i=1}^{n} a_i\right) \tag{4}$$

for all $n \geqslant 1$.

Formula (4) gives an explicit irrationality measure lower estimate for the fixed points under considerations.

Theorem 3. *Let x be fixed point from Theorem 1, then there exists $q_0 \in \mathbb{Z}_+$ such that*

$$\left| x - \frac{p}{q} \right| > \frac{1}{\left(\frac{2\kappa_1}{\log 2}\log q + O(\log\log q)\right) q^2}$$

for all $q > q_0 \in \mathbb{N}, p \in \mathbb{N}$.

We give a proof of Theorem 3 in Section 6. The $O(\log\log q)$ term is calculated explicitly there.

The following statement reduces the problem of fixed points of $?(x)$ to the properties of values of $?(x)$ at rational points only.

Theorem 4. *Conjecture 1 follows from the inequality*

$$\left| ?\left(\frac{p}{q}\right) - \frac{p}{q} \right| > \frac{1}{2q^2} \tag{5}$$

for all $p, q \in \mathbb{Z}_+$ with $q \geqslant q_0$ for some $q_0 \in \mathbb{Z}_+$.

We prove Theorem 4 in Section 6.

Lastly, we apply one fact, which helped us to prove 1 to one problem about derivative and iterations of $?(x)$.

Theorem 5. *Let*

$$M = \Big\{[\overline{A}_1, \Gamma_1, \overline{A}_2, \Gamma_2, \ldots] : \forall k \ \Gamma_k = \sigma_{\overline{A_k}} + s_k, \ \ \textit{where } s_k > (\kappa_2 - 1)S_{\overline{A}_{k+1}} + \sigma_{\overline{A}_k} + 2\Big\}$$

Then for any $x_0 \in M$ *and for any* $n \in \mathbb{N}$ *we have* $f_n'(x_0) = 0$.

3 Preliminaries

By F_k we denote kth Fibonacci number, that is

$$F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}.$$

Lemma 3.1. *Let* $?([a_1, a_2, \ldots, a_{n-1}]) = [b_1, b_2, \ldots, b_k]$ *and* $\sum\limits_{i=1}^{n-1} a_i = s + 1 > 2$, *then* $\sum\limits_{i=1}^{k} b_i > s + 1$.

Proof. Reducing (1) to a common denominator we get

$$?([a_1, a_2, \ldots, a_{n-1}]) = \frac{2^{a_2+\ldots+a_{n-1}} - 2^{a_3+\ldots+a_{n-1}} + \ldots + (-1)^{n-1} \cdot 2^{a_{n-1}} + (-1)^n}{2^{a_1+a_2+\ldots+a_{n-1}-1}}. \quad (6)$$

Here the denominator is equal to 2^s, and the numerator is an odd number. Let us consider the level B_s of the Stern-Brocot tree, which contains the number $[a_1, a_2, \ldots, a_{n-1}]$. The greatest denominator on this level is equal to F_{s+2}. We know that for all $s > 2$ one has $F_{s+2} < 2^s$. This means that the image of the number $[a_1, a_2, \ldots, a_{n-1}]$, given by the formula (6), belongs to level B_{s+k} for some $k \geqslant 1$, since the denominator of the image is greater than the greatest denominator on the level B_s. □

Letting $S([a_1, \ldots, a_n]) = a_1 + \ldots + a_n$, Lemma 3.1 is actually proving the inequality $S(?(x)) > S(x)$ for every $x \in \mathbb{Q} \cap (0, 1)$, $x \neq \frac{1}{2}$.

Corollary 3.1. *The Minkowski question mark function has exactly 3 rational fixed points:* 0, $\frac{1}{2}$ *and* 1.

Proof. We see that $F_{s+2} = 2^s$ only for $s = 0, 1$, that is for numbers from the $0th$ and the $1st$ levels of the Stern-Brocot tree, which only contain the numbers $0, \frac{1}{2}$ and 1. For every other rational number, the sum of its partial quotients increases under the map $?(x)$. So the number is not mapped onto itself. □

The following lemma about the values of Minkowski function at rational points is related to a famous statement known as "Folding lemma"(see [17]).

Lemma 3.2. *Let* s *be an arbitrary nonnegative integer and*

$$?([a_1, a_2, \ldots, a_{n-1}]) = [b_1, b_2, \ldots, b_k], \ \ b_k \neq 1.$$

Consider the number

$$\theta = [a_1, a_2, \ldots, a_{n-1}, a_n], \ \ \textit{where } a_n = \sum_{i=1}^{n-1} a_i + s, \ \ s \geqslant 0.$$

Then

1. *If* $n \equiv k (mod\ 2)$*, then* $?(\theta) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]$.

2. *If* $n \equiv k + 1 (mod\ 2)$ *then* $?(\theta) = [b_1, b_2, \ldots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \ldots, b_1]$.

Proof. We know that $b_k \neq 1$. Let us choose one of the representations $\frac{p_l}{q_l} = [b_1, b_2, \ldots, b_k]$ or $\frac{p_l}{q_l} = [b_1, b_2, \ldots, b_k - 1, 1]$ so that the length l of the continued fraction expansion is of the same parity as $n + 1$, that is $l \equiv n + 1 \pmod 2$, and $l = k$ or $l = k + 1$. From (6) we know that $q_l = 2^{a_1 + \ldots + a_{n-1} - 1}$. Without loss of generality suppose that $n \equiv k + 1 \pmod 2$, then

$$?(\theta) = ?([a_1, a_2, \ldots, a_{n-1}]) + \frac{(-1)^{n+1}}{2^{2 \cdot \sum_{i=1}^{n-1} a_i - 1 + s}} = \frac{p_l}{q_l} + \frac{(-1)^{n+1}}{2^{s+1} \cdot q_l^2} = \frac{p_l}{q_l} + \frac{(-1)^l}{2^{s+1} \cdot q_l^2} = \frac{p_l q_l 2^{s+1} - (p_l q_{l-1} - q_l p_{l-1})}{2^{s+1} q_l^2} =$$

$$= \frac{p_l(2^{s+1} - \frac{q_{l-1}}{q_l}) + p_{l-1}}{q_l(2^{s+1} - \frac{q_{l-1}}{q_l}) + q_{l-1}} = [b_1, b_2, \ldots, b_k, 2^{s+1} - \frac{q_{l-1}}{q_l}] = [b_1, b_2, \ldots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \ldots, b_1].$$

In the last equality we use $1 - [b_k, \ldots, b_1] = [1, b_k - 1, b_{k-1}, \ldots, b_1]$. □

Remark 3.1. *Niederreiter [14] proved that if* m *is a power of* 2*, then there exists an odd integer* a *with* $1 \leqslant a \leqslant m$ *such that all partial quotients in the continued fraction expansion of* $\frac{a}{m}$ *are bounded by* 3*. In fact, he took iterations of the Minkowski question mark function of the continued fractions of a special form, where each partial quotient is equal to the sum of all previous ones or to the sum of all previous ones plus* 1*.*

Lemma 3.3. *Let* $a_1, \ldots, a_{n-1}$ *be the partial quotients of a fixed point* x*, then, depending on the parity of* n*, the next partial quotient* a_n *satisfies one of the following systems*

1. *If* n *is even, then* a_n *satisfies* $\begin{cases} [a_1, \ldots, a_{n-1}, a_n] < ?([a_1, \ldots, a_{n-1}, a_n + 1]), \\ [a_1, \ldots, a_{n-1}, a_n + 1] > ?([a_1, \ldots, a_{n-1}, a_n]). \end{cases}$

2. *If* n *is odd, then* a_n *satisfies* $\begin{cases} [a_1, \ldots, a_{n-1}, a_n] > ?([a_1, \ldots, a_{n-1}, a_n + 1]), \\ [a_1, \ldots, a_{n-1}, a_n + 1] < ?([a_1, \ldots, a_{n-1}, a_n]). \end{cases}$

Proof. Let $[a_1, \ldots, a_n, \ldots]$ be the fixed point. Let us show 1).
Since n is even, from the continued fraction theory we know

$$[a_1, \ldots, a_n] < [a_1, \ldots, a_n, \ldots] < [a_1, \ldots, a_n + 1].$$

That is $[a_1, \ldots, a_n]$ and $[a_1, \ldots, a_n + 1]$ lie on opposite sides with respect to x, and hence their images lie on different sides too, and we have

$$\begin{cases} [a_1, \ldots, a_{n-1}, a_n] < ?([a_1, \ldots, a_{n-1}, a_n + 1]), \\ [a_1, \ldots, a_{n-1}, a_n + 1] > ?([a_1, \ldots, a_{n-1}, a_n]). \end{cases}$$

Case 2) can be treated similarly, since for odd n one has

$$[a_1, \ldots, a_n + 1] < [a_1, \ldots, a_n, \ldots] < [a_1, \ldots, a_n].$$

□

The following lemma localizes fixed points.

Lemma 3.4. *All fixed points of ?(x) inside the interval* $(0, \frac{1}{2})$ *belong to the interval* $(\frac{2}{5}, \frac{3}{7})$.

Proof. First of all, we will show that there are no fixed points in $(0, \frac{1}{3})$. Decompose $(0, \frac{1}{3}) \setminus \mathbb{Q} = \bigcup_{n=3}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \setminus \mathbb{Q}$. Assume that for some n_0 there exists $x_0 \in (\frac{1}{n_0+1}, \frac{1}{n_0})$ such that $?(x_0) = x_0$. Then

$$\frac{1}{n_0+1} < x_0 = ?(x_0) < ?\left(\frac{1}{n_0}\right) = \frac{1}{2^{n_0-1}}.$$

So $n_0 + 1 > 2^{n_0-1}$, and this is not true for $\forall n_0 \geqslant 3$, hence there are no fixed points in $(0, \frac{1}{3})$.
This means that the first partial quotient is equal to 2 (1 is excluded, since we are on the interval $(0, \frac{1}{2})$).

Now we show that there are no fixed points in $(\frac{4}{9}, \frac{1}{2})$. Consider the decomposition $(\frac{4}{9}, \frac{1}{2}) \setminus \mathbb{Q} = \bigcup_{n=4}^{\infty} (\frac{n}{2n+1}, \frac{n+1}{2n+3}) \setminus \mathbb{Q}$. Assume that for some n_0 there exists $x_0 \in (\frac{n_0}{2n_0+1}, \frac{n_0+1}{2n_0+3})$ such that $?(x_0) = x_0$.

$$\frac{n_0+1}{2n_0+3} > x_0 = ?(x_0) > ?\left(\frac{n_0}{2n_0+1}\right) = \frac{1}{2} - \frac{1}{2^{1+n_0}}.$$

We get $2^{n_0} < 2n_0 + 3$, which holds for $n_0 = 1, 2, 3$ only, hence there are no fixed points in $(\frac{4}{9}, \frac{1}{2})$.

Since $?\circ?$ is increasing and $?(?(\frac{2}{5})) = ?(\frac{3}{8}) = \frac{5}{16} < \frac{1}{3}$, there are no fixed points in $(\frac{1}{3}, \frac{2}{5})$, while $\frac{4}{9} < ?(?(\frac{3}{7})) = ?(\frac{7}{16}) = \frac{29}{64}$ shows that there are no fixed points in $(\frac{3}{7}, \frac{4}{9})$. □

Lemma 3.4 means that the continued fraction expansion of every fixed point in $(0, \frac{1}{2})$ is of the form $[2, 2, \ldots]$.
The next statement is an obvious property of continuous functions. We formulate it without a proof.

Lemma 3.5. *Let* $f(x)$ *be a continuous function. Consider an interval* $[a, b]$ *such that there are no fixed points inside* (a, b). *Then* $f(x) - x$ *does not change sign on* (a, b).

For the last theorem we need a bit little more preparation.
Let $\overline{A}_i = (a_1^i, a_2^i, \ldots, a_k^i)$ be a list of natural numbers of arbitrary length $k \geqslant 0$. By $[\overline{A}_i]$ we denote a continued fraction of the list $\overline{A}_i$:

$$[\overline{A}_i] = [a_1^i, a_2^i, \ldots, a_k^i] = \frac{1}{a_1^i + \frac{1}{a_2^i + \ldots + \frac{1}{a_k^i}}} \tag{7}$$

Let us define a function $d(\overline{A}_i) = d(a_1^i, a_2^i, \ldots, a_k^i) \equiv k$ - length of the list.
When considering a set of lists $\overline{A}_1, \ldots \overline{A}_n, \ldots$, we will use notation $d_n \equiv d(\overline{A}_n)$.
If $d_n > 0$, then by $\overleftarrow{A}_n, (A_n)^-$ and $(A_n)_-$ we denote finite lists $(a_{d_n}^n, \ldots, a_1^n), (a_1^n, \ldots, a_{d_n-1}^n)$ and $(a_2^n, \ldots, a_{d_n}^n)$ respectively.
By $\langle A_i \rangle$ we denote the denominator of the continued fraction $[\overline{A}_i]$.
Using this notation, let us rewrite the following classical formula(see [6]):

$$\langle X \rangle \langle Y \rangle \leqslant \langle X, Y \rangle = \langle X \rangle \langle Y \rangle + \langle X^- \rangle \langle Y_- \rangle = \langle X \rangle \langle Y \rangle \Big(1 + [\overleftarrow{X}][Y]\Big) \leqslant 2 \langle X \rangle \langle Y \rangle.$$

Using this inequality twice, we get a result, which will be crucial for our proof:

$$a \langle X \rangle \langle Y \rangle \leqslant \langle X, a, Y \rangle \leqslant (a+2) \langle X \rangle \langle Y \rangle \tag{8}$$

Also, for the list $\overline{A}_i = (a_1^i, a_2^i, \ldots, a_{d_i}^i)$ we denote the sum of all elements of that list by $S_{\overline{A}_i}$

$$S_{\overline{A}_i} = a_1^i + a_2^i + \ldots + a_{d_i}^i$$

Now let us consider a set of lists $\overline{A}_i = (a_1^i, a_2^i, \ldots, a_{d_i}^i),\ i \in \mathbb{N}$

When considering continued fraction of the form $x_0 = [\overline{A}_1, \Gamma_1, \overline{A}_2, \Gamma_2, \ldots]$ we would use the following notation for the sum of all partial quotients up to $\overline{A}_k$.

$$\sigma_{\overline{A_k}} = \sum_{i=1}^{k} S_{\overline{A_i}} + \sum_{i=1}^{k-1} \Gamma_i$$

Next lemma is a corollary from the rules of comparison of continued fractions.

Lemma 3.6. *Let* $\mathbb{X} < \mathbb{Y}$ *and*

$$\mathbb{X} = [a_1, \ldots, a_n, X, ...],$$
$$\mathbb{Y} = [a_1, \ldots, a_n, Y, ...].$$

Then every fraction $\frac{p}{q}$, *for which* $\mathbb{X} < \frac{p}{q} < \mathbb{Y}$ *has a continued fraction of the form*

$$\frac{p}{q} = [a_1, \ldots, a_n, Z, \ldots],$$

where $min(X, Y) \leqslant Z \leqslant max(X, Y)$

By applying this result to the Minkowski question mark function we obtain

Corollary 3.2. *Let*

$$?([a_1, a_2, \ldots, a_{n-1}]) = [b_1, b_2, \ldots, b_k],\ \ b_k \neq 1.$$

and $a_n = \sum_{i=1}^{n-1} a_i + s$, *where* $s \in \mathbb{N}$.

Consider the number $\gamma = [a_1, \ldots, a_n, c_1, \ldots, c_s]$. *Then there exist a list* $(b_{k+2}, \ldots, b_p)$ *for which*

1. *If* $n \equiv k(mod\ 2)$, *then* $?(\gamma) = [b_1, \ldots, b_{k-1}, b_k - 1, 1, Z, b_{k+2}, \ldots, b_p]$.
2. *If* $n \equiv k + 1(mod\ 2)$ *then* $?(\gamma) = [b_1, \ldots, b_k, Z, b_{k+2}, \ldots, b_p]$.

where

$$2^{s+1} - 1 \leqslant Z \leqslant 2^{s+2} - 1.$$

Proof. Let us prove this lemma only in the case where n is even and k is odd. By Lemma 3.2 we have

$$?([a_1, \ldots, a_n]) = [b_1, \ldots, b_k, 2^{s+1} - 1, 1, b_k - 1, \ldots, b_1]$$
$$?([a_1, \ldots, a_n + 1]) = [b_1, \ldots, b_k, 2^{s+2} - 1, 1, b_k - 1, \ldots, b_1]$$

Then for the number $\gamma = [a_1, \ldots, a_n, c_1, \ldots, c_s]$ we have

$$[a_1, \ldots, a_n] < \gamma < [a_1, \ldots, a_n + 1]$$

and so

$$?([a_1, \ldots, a_n]) < ?(\gamma) < ?([a_1, \ldots, a_n + 1]).$$

Then by Lemma 3.6 we get

$$?(\gamma) = [b_1, \ldots, b_k, Z, b_{k+2}, \ldots, b_p],\ \text{where } 2^{s+1} - 1 \leqslant Z \leqslant 2^{s+2} - 1.$$

□

4 Proof of Theorem 1

Let us prove this theorem for the leftmost fixed point of $?(x)$ in the interval $(0, \frac{1}{2})$, which we denote by $x = [a_1, \dots, a_n, \dots]$. Our proof goes by contradiction. Assume that there exists $n \geqslant 3$ such that

$$a_n \geqslant \sum_{i=1}^{n-1} a_i. \tag{9}$$

Consider $[a_1, \dots, a_{n-1}]$ and let $?([a_1, \dots, a_{n-1}]) = [b_1, \dots, b_k],\ b_k \geqslant 2$. Now we distinguish cases 1) - 4) with several subcases. In each of them we will deduce a contradiction. We present a very detailed exposition of the case 1) below. Cases 2) - 4) are quite similar. We establish them with less details.
1) n odd, k odd. Then by Lemma 3.2, $?([a_1, a_2, \dots, a_n]) = [b_1, b_2, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, b_1]$. By Lemma 3.3 a_n should satisfy

$$\begin{cases} [a_1, \dots, a_{n-1}, a_n] > ?([a_1, \dots, a_{n-1}, a_n + 1]) = [b_1, b_2, \dots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \dots, b_1], \\ [a_1, \dots, a_{n-1}, a_n + 1] < ?([a_1, \dots, a_{n-1}, a_n]) = [b_1, b_2, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, b_1]. \end{cases} \tag{10}$$

Now we consider the possible three subcases 1.1), 1.2) and 1.3).
1.1) $k \leqslant n - 1$. Then by Lemma 3.1 we have $\sum\limits_{i=1}^{k} b_i > \sum\limits_{i=1}^{n-1} a_i$, hence there exists $i \in \{1, \dots, k\}$ such that $a_i \neq b_i$. By considering a partial quotient with the smallest index $i \leqslant k$ for which $a_i \neq b_i$, we get that the system (10) is incompatible by the rules of comparison of continued fractions.
1.2) $k > n$. If there is $i \in \{1, \dots, n-1\}$ such that $a_i \neq b_i$, then similarly to case 1.1) the system (10) is incompatible. Hence we assume that $a_i = b_i$ for all $i \in \{1, \dots, n-1\}$. Let us consider the four possible variants for b_n.
1.2.1) $b_n \leqslant a_n - 1$. The first inequality of (10) can be rewritten as

$$[a_1, \dots, a_{n-1}, a_n] > [a_1, a_2, \dots, a_{n-1}, b_n, \dots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \dots, b_1]. \tag{11}$$

But n is odd and $b_n \leqslant a_n - 1$, so (11) cannot be true.
1.2.2) $b_n \geqslant a_n + 2$. The second inequality of (10) can be rewritten as

$$[a_1, \dots, a_{n-1}, a_n + 1] < [a_1, a_2, \dots, a_{n-1}, b_n, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, b_1]. \tag{12}$$

But n is odd and $b_n \geqslant a_n + 2$, so (12) cannot be true.
1.2.3) $b_n = a_n + 1$. The second inequality in (10) can be rewritten in this case in the form

$$[a_1, \dots, a_n + 1] < [a_1, \dots, a_n + 1, b_{n+1}, \dots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \dots, b_1].$$

This inequality fails since the value of the continued fraction is always less than an odd convergent.
1.2.4) $b_n = a_n$. Now the equality $?([a_1, \dots, a_{n-1}]) = [b_1, \dots, b_k]$ gives

$$?([a_1, \dots, a_{n-1}]) = [a_1, \dots, a_{n-1}, a_n, \dots, b_k] > [a_1, \dots, a_{n-1}].$$

The last inequality is due to the fact that n is odd. But $[a_1, \dots, a_{n-1}] < x$ is also a convergent for our fixed point x, so we have

$$[a_1, \dots, a_{n-1}] < ?([a_1, \dots, a_{n-1}]) < [a_1, \dots, a_n, \dots] = x.$$

This contradicts Lemma 3.5 because by Lemma 3.4 we have $\frac{2}{5} < x < \frac{3}{7}$ and $?(\frac{2}{5}) = \frac{3}{8} < \frac{2}{5}$, hence $?(y) < y$ should hold for every $y \in (0, x)$, and in particular for $y = [a_1, \ldots, a_{n-1}]$, which is not possible.
1.3) $k = n$. Similarly to the case 1.2) we get $a_i = b_i$ for all $i = 1, \ldots, n-1$. Let us consider the following four possible subcases.
1.3.1) $b_n \leqslant a_n$. In this case we deduce a contradiction similarly to the case 1.2.1).
1.3.2) $b_n \geqslant a_n + 3$. In this case we deduce a contradiction similarly to the case 1.2.2).
1.3.3) $b_n = a_n + 1$. Equality $?([a_1, \ldots, a_{n-1}]) = [b_1, \ldots, b_k]$ under the assumptions of this case gives

$$?([a_1, \ldots, a_{n-1}]) = [a_1, \ldots, a_{n-1}, a_n + 1] > [a_1, \ldots, a_{n-1}].$$

But this contradicts Lemma 3.5 as in the case 1.2.4).
1.3.4) $b_n = a_n + 2$. The system (10) gives now in particular

$$[a_1, \ldots, a_n + 1] < [a_1, \ldots, a_n + 1, 1, 2^{s+1} - 1, a_n, \ldots, a_1].$$

This inequality fails since odd convergents are always greater than the value of the continued fraction.

2) n even, k even. Then by Lemma 3.2 $?([a_1, a_2, \ldots, a_n]) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]$. By Lemma 3.3 a_n should satisfy

$$\begin{cases} [a_1, \ldots, a_{n-1}, a_n] < ?([a_1, \ldots, a_{n-1}, a_n + 1]) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+2} - 1, b_k, \ldots, b_1], \\ [a_1, \ldots, a_{n-1}, a_n + 1] > ?([a_1, \ldots, a_{n-1}, a_n]) = [b_1, b_2, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1]. \end{cases} \tag{13}$$

As before, there are three possible cases.
2.1) $k \leqslant n - 1$. In this case we deduce a contradiction similarly to the case 1.1).
2.2) $k > n$. In the same way as in 1.2), we come to four options:
2.2.1) $b_n \leqslant a_n - 1$. In this case we deduce a contradiction similarly to the case 1.2.1).
2.2.2) $b_n \geqslant a_n + 2$. In this case we deduce a contradiction similarly to the case 1.2.2).
2.2.3) $b_n = a_n + 1$. The system (13) gives now

$$[a_1, \ldots, a_{n-1}, a_n + 1] > ?([a_1, \ldots, a_{n-1}, a_n]) = [a_1, \ldots, a_n + 1, b_{n+1}, \ldots, b_{k-1}, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1].$$

This inequality fails since even convergents are always smaller than the value of the continued fraction.
2.2.4) $b_n = a_n$. Let us consider the even convergent $[a_1, \ldots, a_n]$ of our fixed point, or rather its image. Taking into account the assumptions and using Lemma 3.2, we obtain

$$?([a_1, \ldots, a_n]) = [a_1, \ldots, a_n, b_{n+1}, \ldots, b_k - 1, 1, 2^{s+1} - 1, b_k, \ldots, b_1] > [a_1, \ldots, a_n].$$

We get a contradiction in the same way as in 1.2.4), since we found a number $[a_1, \ldots, a_n]$ less than the *the smallest* fixed point, whose image is greater than this number.
2.3) $k = n$. Similarly to the case 1.2) we come to the assumption that $a_i = b_i$ for all $i \in \{1, \ldots, n-1\}$ and so we consider the following four possible subcases:
2.3.1) $b_n \leqslant a_n$. In this case we deduce a contradiction similarly to the case 1.2.1).
2.3.2) $b_n \geqslant a_n + 3$. In this case we deduce a contradiction similarly to the case 1.2.2).
2.3.3) $b_n = a_n + 1$. We have

$$?([a_1, \ldots, a_n]) = [a_1, \ldots, a_n, 1, 2^{s+1} - 1, a_n, \ldots, a_1] > [a_1, \ldots, a_n].$$

Now we get a contradiction as in 1.2.4).
2.3.4) $b_n = a_n + 2$. The system (13) gives now in particular

$$[a_1, \ldots, a_n + 1] > [a_1, \ldots, a_n + 1, 1, 2^{s+1} - 1, a_n, \ldots, a_1].$$

This inequality fails since even convergents are always smaller than the value of the continued fraction.

Cases

$$3)\ n \text{ even},\ k \text{ odd} \qquad \text{and} \qquad 4)\ n \text{ odd},\ k \text{ even}$$

follow by a similar argument. For example, in case 3) by Lemma 3.2,

$$?([a_1, a_2, \dots, a_n]) = [b_1, \dots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1].$$

By Lemma 3.3, a_n satisfies

$$\begin{cases} [a_1, \dots, a_{n-1}, a_n] < [b_1, \dots, b_k, 2^{s+2} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1], \\ [a_1, \dots, a_{n-1}, a_n + 1] > [b_1, \dots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1], \end{cases}$$

and we need to consider only two subcases (subcase $n = k$ is impossible since the parity of n and k is different)

$$3.1)\ k \leqslant n - 1 \qquad \text{and} \qquad 3.2)\ k > n. \tag{14}$$

In the case 3.2), analogously to the case 1.2), we come to four options:

$$3.2.1)\ b_n = a_n + 1, \quad 3.2.2)\ b_n = a_n, \quad 3.2.3)\ b_n \leqslant a_n - 1, \quad 3.2.4)\ b_n \geqslant a_n + 2. \tag{15}$$

In every subcase we get a contradiction.
In the case 4) we have by Lemma 3.2

$$?([a_1, a_2, \dots, a_n]) = [b_1, \dots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1],$$

and by Lemma 3.3, a_n satisfies

$$\begin{cases} [a_1, \dots, a_{n-1}, a_n] > [b_1, \dots, b_k, 2^{s+2} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1], \\ [a_1, \dots, a_{n-1}, a_n + 1] < [b_1, \dots, b_k, 2^{s+1} - 1, 1, b_k - 1, b_{k-1}, \dots, b_1]. \end{cases}$$

Now we consider the same subcases as in (14), and the second subcase splits into the same subcases as in (15).
We exhausted all possibilities, getting a contradiction in each of them, hence the assumption (9) was false. □

Remark 4.1. *To prove Theorem 1 for the greatest fixed point y we should consider the interval $[y, \frac{1}{2}]$ and the number $\frac{3}{7}$ which belongs to it (by Lemma 3.4) to deduce a contradiction using Lemma 3.5.*

Remark 4.2. *One can see that a slight generalization of Theorem 1 can be proven not only for the smallest and the greatest fixed points, but for any fixed point x which is isolated and unstable at least at one side.*

For example, x is isolated and unstable at the left side if and only if

$$\exists \varepsilon > 0 : \forall y \in (x - \varepsilon, x) \ \text{ one has } \ |y - x| < |?(y) - x|.$$

For the isolated and unstable fixed points instead of Theorem 1 one can show that the inequality (3) is valid for all large enough n.

5 Proof of Theorem 2

Consider an arbitrary continued fraction $[a_1, \ldots, a_n, \ldots]$. Denote $S_n = a_1 + \ldots + a_n$. We need the following lemma from ([7], Theorem 4).

Lemma 5.1. *Denote $\varphi = \frac{\sqrt{5}+1}{2}$. For any $n \in \mathbb{N}$ one has*

$$q_n \leqslant F_{S_n+1} \leqslant \varphi^{S_n}. \tag{16}$$

Now we are ready to prove Theorem 2. We will only consider the case of the smallest fixed point in the interval $(0, \frac{1}{2})$. The case of the greatest fixed point is treated in the same way.

Proof. First of all, one can easily see that Theorem 2 holds for $n < 36$, as we know the first 36 partial quotients of x. The corresponding sequence is OEIS A058914 ([15]). Suppose that n is even. Then by Lemma 3.5 we have

$$?\left(\frac{p_n}{q_n}\right) < \frac{p_n}{q_n} < x = ?(x) < ?\left(\frac{p_{n+1}}{q_{n+1}}\right).$$

Hence

$$\frac{1}{(a_{n+1}+1)q_n^2} < x - \frac{p_n}{q_n} < ?(x) - ?\left(\frac{p_n}{q_n}\right) < ?\left(\frac{p_{n+1}}{q_{n+1}}\right) - ?\left(\frac{p_n}{q_n}\right) = \frac{1}{2^{S_n+a_{n+1}-1}}. \tag{17}$$

We obtain the inequality

$$\frac{(a_{n+1}+1)q_n^2}{2^{S_n+a_{n+1}-1}} > 1. \tag{18}$$

Suppose that $a_{n+1} \geqslant \kappa_1 S_n + 2\log_2 S_n$. We apply the upper estimate from Lemma 5.1 and use the fact that $\frac{x}{2^x}$ is strictly decreasing function for $x \geqslant 4$ to obtain

$$1 < \frac{(a_{n+1}+1)q_n^2}{2^{S_n+a_{n+1}-1}} < \frac{(\kappa_1 S_n + 2\log_2 S_n + 1)\varphi^{2S_n}}{2^{S_n(\kappa_1+1)+2\log_2 S_n - 1}} < \frac{2(\kappa_1 S_n + 2\log_2 S_n + 1)}{S_n^2}. \tag{19}$$

One can easily see that

$$\frac{2(\kappa_1 S_n + 2\log_2 S_n + 1)}{S_n^2} < 1$$

for $S_n \geqslant 4$. We obtain a contradiction.

The case when n is odd is slightly more complicated. Now we have $?(\frac{p_{n+1}}{q_{n+1}}) < \frac{p_{n+1}}{q_{n+1}} < x = ?(x)$. Using the same argument we obtain that

$$\frac{(a_{n+2}+1)q_{n+1}^2}{2^{S_n+a_{n+1}+a_{n+2}-1}} > 1.$$

As $q_{n+1} < (a_{n+1}+1)q_n$ and $\frac{a_{n+2}+1}{2^{a_{n+2}}} \leqslant 1$,

$$\frac{(a_{n+1}+1)^2 q_n^2}{2^{S_n+a_{n+1}-1}} > 1. \tag{20}$$

Suppose that $a_{n+1} \geqslant \kappa_1 S_n + 2\log_2 S_n$. Similarly to the previous case, we apply Lemma 5.1 and use the fact that $\frac{x^2}{2^x}$ is strictly decreasing function for $x \geqslant 7$. We have

$$1 < \frac{(a_{n+1}+1)^2 q_n^2}{2^{S_n+a_{n+1}-1}} < \frac{(\kappa_1 S_n + 2\log_2 S_n + 1)^2\varphi^{2S_n}}{2^{S_n(\kappa_1+1)+2\log_2 S_n - 1}} < \frac{2(\kappa_1 S_n + 2\log_2 S_n + 1)^2}{S_n^2}. \tag{21}$$

From (21) one can easily see that

$$\kappa_1 + \frac{2\log_2 S_n}{S_n} + \frac{1}{S_n} > \frac{1}{\sqrt{2}}. \tag{22}$$

But (22) is not true for $S_n \geqslant 36$ and we obtain a contradiction. □

Remark 5.1. *One can prove an even stronger statement, namely*

$$a_{n+1} + \ldots + a_{n+k} < \kappa_1 S_n + 2k \log_2 S_n.$$

Combining (20) *with* $n = m + k - 1$ *we obtain*

$$\frac{(a_{m+k}+1)^2 q_{m+k-1}^2}{2^{S_{m+k-1}}} > 1. \tag{23}$$

Let us estimate the left-hand side of (23):

$$1 < \frac{(a_{m+k}+1)^2 q_{m+k-1}^2}{2^{S_{m+k-1}}} < \frac{2(a_{m+k}+1)^2 \cdot \ldots \cdot (a_{m+1}+1)^2 q_m^2}{2^{S_m + a_{m+1} + \ldots + a_{m+k}}}.$$

By assuming that $a_{m+1} + \ldots + a_{m+k} \geqslant \kappa_1 S_m + 2k \log_2 S_m$ *and applying a similar argument*[1] *as in the proof of Theorem 2, we will get*

$$\frac{\kappa_1}{k} + 2\frac{\log_2 S_m}{S_m} + \frac{1}{S_m} > \frac{1}{2^{1/(2k)}},$$

and this inequality fails for every $k \geqslant 1$ *whenever* $S_n \geqslant 36$.

Remark 5.2. *Theorem 2 provides a (non-optimal) upper estimate on partial quotients of* x.

However, their mean behavior is much simpler. Denote

$$\lambda_i = \frac{i + \sqrt{i^2+4}}{2}, \quad \kappa_2 = \frac{5\log\lambda_4 - 4\log\lambda_5}{0.5\log 2 + \log\lambda_4 - \log\lambda_5} \approx 4.40104874\ldots. \tag{24}$$

In [5] Dushistova, Moshchevitin and Kan proved

Lemma 5.2 ([5], Theorem 3). *For an irrational number* x *there exists a constant* C *such that for all natural* n *one has*

$$S_n \geqslant \kappa_2 n - C. \tag{25}$$

Then $?'(x)$ *exists and equals* 0.

In fact, they showed that if (25) holds, then

$$\frac{q_n^2}{2^{S_n}} \to 0. \tag{26}$$

The fact that $?(x-\delta) < x - \delta$ for any positive δ implies that $x - \frac{p_n}{q_n} <?(x) - ?\left(\frac{p_n}{q_n}\right)$ for any even n. Hence the inequality (18) holds and we obtain a contradiction with (26). This implies that for any C the inequality $S_n < \kappa_2 n + C$ holds for all n large enough. Some calculations show that one can take $C = 0$ for all $n \geqslant 1$. Now we have an obvious consequence of Theorem 2 and Lemma 5.2.

Corollary 5.1. *Let* x *be fixed point from Theorem 1, then*

$$a_{n+1} < \kappa_1 \kappa_2 n + 2\log_2(\kappa_2 n) \tag{27}$$

for all $n \geqslant 2$.

[1] We use the fact that the product $\prod_{i=1}^{k} a_i,\ a_k \in \mathbb{Z}_+$ with the fixed sum of elements $S_k = \sum_{i=1}^{k} a_i$ attains its maximum when $\left|a_i - \frac{S_k}{k}\right| < 1$ for all $i \in \{1, \ldots, k\}$.

6 Proof of Theorem 3

Proof. From (18) and (20) we deduce that for any $n \in \mathbb{N}$ one has

$$\frac{(a_{n+1}+1)^2 q_n^2}{2^{S_n+a_{n+1}-1}} > 1. \tag{28}$$

As $\frac{(a_{n+1}+1)^2}{2^{a_{n+1}-1}} \leqslant \frac{9}{2}$, we have

$$\frac{2}{9} 2^{S_n} < q_n^2$$

or

$$S_n < \frac{2}{\log 2} \log q_n + \log_2 \frac{9}{2}.$$

As $a_{n+1} < \kappa_1 S_n + 2\log_2 S_n$, we have

$$a_{n+1} < \kappa_1 \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{9}{2} \right) + \frac{2}{\log 2} \log \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{9}{2} \right).$$

Consider an arbitrary convergent continued fraction to x. As

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{(a_{n+1}+1) q_n^2},$$

we see that

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{\left(\kappa_1 \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{9}{2} \right) + \frac{2}{\log 2} \log \left(\frac{2}{\log 2} \log q_n + \log_2 \frac{9}{2} \right) + 1 \right) q_n^2}.$$

□

Proof of Theorem 4

Proof. Suppose that there are at least two fixed points from the interval $(0, \frac{1}{2})$, namely x_1 and x_1'. Consider an arbitrary rational number $\frac{p}{q}$ between them. Note that the sequence of iterations $z_n = \underbrace{?(?(\ldots ?}_{n \text{ iterations}}(\frac{p}{q})\ldots))$ is monotonic and converges to some fixed point in the interval $[x_1, x_1']$. Suppose that the sequence z_n is decreasing and denote $\lim_{n\to\infty} z_n = x_1''$, which may coincide with x_1. Denote the n-th convergent fraction to x_1'' by $\frac{p_n}{q_n}$. There exists N such that for all $n > N$ one has $x_1'' < \frac{p_{2n-1}}{q_{2n-1}} < \frac{p}{q}$. As $?(\frac{p}{q}) < \frac{p}{q}$, by Lemma 3.5 we have $x_1'' < ?(\frac{p_{2n-1}}{q_{2n-1}}) < \frac{p_{2n-1}}{q_{2n-1}}$ for all $n > N$. Thus,

$$0 < \frac{p_{2n-1}}{q_{2n-1}} - ?\left(\frac{p_{2n-1}}{q_{2n-1}}\right) < \frac{p_{2n-1}}{q_{2n-1}} - x_1'' < \frac{1}{a_{2n} q_{2n-1}^2}. \tag{29}$$

Note that if the sequence z_n is increasing, then using the same argument we obtain that $x_1'' > \frac{p_{2n}}{q_{2n}} > \frac{p}{q}$ for all n big enough. The inequality (29) is replaced by

$$0 < ?\left(\frac{p_{2n}}{q_{2n}}\right) - \frac{p_{2n}}{q_{2n}} < x_1'' - \frac{p_{2n}}{q_{2n}} < \frac{1}{a_{2n+1} q_{2n}^2}. \tag{30}$$

We will finish the proof for decreasing z_n only, because the second case is treated in exactly the same way.

It follows from (29) that if $a_{2n} \geqslant 2$ for infinitely many $n > N$, we have a contradiction with (5). Hence there exists M such that $a_{2n} = 1$ for $n > M$. We know that

$$\frac{1}{q_{2n-1}^2} > \frac{p_{2n-1}}{q_{2n-1}} - x_1'' > ?\left(\frac{p_{2n-1}}{q_{2n-1}}\right) - ?(x_1'').$$

The right-hand side of the previous inequality may be estimated as follows:

$$?\left(\frac{p_{2n-1}}{q_{2n-1}}\right) - ?(x_1'') > \frac{1}{2^{S_{2n-1}+a_{2n}-1}} - \frac{1}{2^{S_{2n-1}+a_{2n}+a_{2n+1}-1}} > \frac{1}{2^{S_{2n-1}+a_{2n}}}.$$

Hence, as $a_{2n} = 1$ for $n > M$, we have

$$\frac{q_{2n-1}^2}{2^{S_{2n-1}+1}} < 1. \tag{31}$$

If there exist infinitely many s such that $?\left(\frac{p_{2s}}{q_{2s}}\right) > \frac{p_{2s}}{q_{2s}}$, then the following holds for $s > N$:

$$\begin{cases} 0 < \frac{p_{2s-1}}{q_{2s-1}} - ?\left(\frac{p_{2s-1}}{q_{2s-1}}\right) < \frac{p_{2s-1}}{q_{2s-1}} - x_1'' \\ 0 < ?\left(\frac{p_{2s}}{q_{2s}}\right) - \frac{p_{2s}}{q_{2s}} < x_1'' - \frac{p_{2s}}{q_{2s}}. \end{cases}$$

A classical theorem states that for any $s \in \mathbb{N}$ at least one of the inequalities $\frac{p_{2s-1}}{q_{2s-1}} - x_1'' < \frac{1}{2q_{2s-1}^2}$ or $x_1'' - \frac{p_{2s}}{q_{2s}} < \frac{1}{2q_{2s}^2}$ holds. Thus we obtain a contradiction with (5).

Now suppose that for all n big enough we have $?\left(\frac{p_{2n}}{q_{2n}}\right) < \frac{p_{2n}}{q_{2n}}$. Then, by (18) and $a_{2n} = 1$ we have

$$\frac{(a_{2n+1}+1)q_{2n}^2}{2^{S_{2n-1}+a_{2n+1}}} > 1. \tag{32}$$

From (31) and (32) we obtain that

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2 \frac{a_{2n+1}+1}{2^{a_{2n+1}-1}} > 1. \tag{33}$$

Note that

$$\frac{q_{2n}}{q_{2n-1}} = 1 + [a_{2n-1}, a_{2n-2}, a_{2n-2}, \ldots, a_1] < 1 + [1, a_{2n-2}, 1, \ldots, a_1]. \tag{34}$$

As we already mentioned, $a_{2n} = 1$ for all $n > M$. Hence one can estimate the right-hand side of (34) as follows

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2 < (\varphi + \varepsilon_n)^2 < 1.62^2 = 2.6244 \quad \text{for } n \text{ large enough.} \tag{35}$$

Here ε_n is some function, which exponentially tends to 0 as n tends to infinity. Now we can see that $a_{2n+1} \leqslant 4$, because if $a_{2n+1} \geqslant 5$, from (34) we have

$$\left(\frac{q_{2n}}{q_{2n-1}}\right)^2 \frac{a_{2n+1}+1}{2^{a_{2n+1}-1}} < 2.6244\frac{6}{16} < 1 \tag{36}$$

and we obtain a contradiction. Hence there exists K such that for any $n > K$ we have $a_n \leqslant 4$. By ([4], Theorem 3), we have[2] $?'(x_1'') = +\infty$. As

$$\frac{?\left(\frac{p_{2n-1}}{q_{2n-1}}\right) - ?(x_1'')}{\frac{p_{2n-1}}{q_{2n-1}} - x_1''} < 1$$

for all $n > N$, we obtain a contradiction. That finishes the proof. □

[2]Theorem 3 of that paper states that $?(x) = +\infty$ for numbers $x = [a_1, a_2, \ldots]$ with $a_i \leqslant 4$ for all i, but combining this with Denjoy's formula (1) for $?(x)$ it is easily seen that $?(x) = +\infty$ if $\exists N, \forall i > N, a_i \leqslant 4$.

Remark 6.1. *The inequality* (5) *has been verified for all* $\frac{p}{q} \in [0, \frac{1}{2}]$ *with* $q < 30000$ *and, separately, for 5400 first convergents to the irrational fixed point (or the set of points if Conjecture 1 is false) of* $?(x)$ *from the interval* $(0, \frac{1}{2})$*. The only counterexamples, apart from the three trivial rational fixed points* $0, \frac{1}{2}$ *and* 1*, were* $\frac{3}{7}$ *and* $\frac{8}{19}$*, both being the convergents to the fixed point. So the numerical calculations allow us to suggest that the inequality* (5) *holds for* $q > 19$*.*

Proof of Theorem 5

Proof. The idea is straightforward - we have to check two conditions:
1) If $x_0 \in M$, then $?'(x_0) = 0$.
2) If $x_0 \in M$, then $?(x_0) \in M$.
First, we have to define a form of the continued fraction of $?(x_0)$.

Let us define $\overline{B_1}$ as a sequence of partial quotients of image of Minkowski question mark of $[\overline{A}_1]$(which we can assume to be non-empty), that is

$$?([\overline{A_1}]) = [\overline{B_1}].$$

If $\overline{A}_2$ is non-empty, we apply Corollary 3.2 setting

$$(a_1, \ldots, a_{n-1}) = \overline{A_1},$$

$$(b_1, \ldots, b_k) = \overline{B_1},$$

$$a_n = \Gamma_1 = S_{\overline{A_1}} + s_1,$$

$$(c_1, \ldots, c_s) = \overline{A_2}.$$

Then for the number $\gamma = [\overline{A}_1, \Gamma_1, \overline{A}_2]$ we have $?(\gamma) = [\overline{B}_1, Z_1, \overline{B}_2]$, where

$$2^{s_1+1} - 1 \leqslant Z_1 \leqslant 2^{s_1+2} - 1$$

and by $\overline{B}_2$ we denoted sequence $(b_{k+2}, \ldots, b_p)$ from Corollary 3.2.
To iteratively define $\overline{B}_k$ with $k \geqslant 3$ and Z_r with $r \geqslant 2$ notice that by conditions of this theorem for every $k \in \mathbb{N}$, partial quotient Γ_k is greater than the sum of all previous ones by s_k and given that $\overline{A}_i$ is non-empty we can apply Corollary 3.2 with

$$(a_1, \ldots, a_{n-1}) = (\overline{A}_1, \Gamma_1, \overline{A}_2, \ldots, \Gamma_{i-2}, \overline{A}_{i-1}),$$

$$(b_1, \ldots, b_k) = (\overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}),$$

$$a_n = \Gamma_{i-1},$$

$$(c_1, \ldots, c_s) = \overline{A}_i.$$

We have $?([\overline{A}_1, \Gamma_1, \overline{A}_2, \ldots, \Gamma_{i-2}, \overline{A}_{i-1}]) = [\overline{B}_1, Z_1, \overline{B}_2, ..., Z_{i-2}, \overline{B}_{i-1}]$. Consider the number

$$\gamma = [\overline{A}_1, \Gamma_1, \overline{A}_2, \ldots, \Gamma_{i-2}, \overline{A}_{i-1}, \Gamma_{i-1}, \overline{A}_i].$$

Then by Corollary 3.2 we get

$$?(\gamma) = [\overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}, Z_{i-1}, \overline{B}_i],$$

where $2^{s_{i-1}+1} - 1 \leqslant Z_{i-1} \leqslant 2^{s_{i-1}+2} - 1$, and by $\overline{B}_i$ we denoted sequence $(b_{k+2}, \dots, b_p)$ from Corollary 3.2.

Now, if, for example $\overline{A}_k$ is empty, then by Lemma 3.2

$$?([\overline{A}_1, \dots, \overline{A}_{k-1}, \Gamma_{k-1}]) = [\overline{B}_1, \dots, \overline{B}_{k-1}, Z_{k-1}, \overline{B}_k], \text{where, depending on parities,}$$
$$\overline{B}_k = (1, b^{k-1}_{d(B_{k-1})} - 1, (\overleftarrow{B}_{k-1})_-, \dots, \overleftarrow{B}_1) \text{ or } (b^{k-1}_{d(B_{k-1})-1} + 1, (\overleftarrow{B}_{k-1})_{--}, \dots, \overleftarrow{B}_1) \tag{37}$$

As we can see, even if $\overline{A}_k$ is an empty list, $\overline{B}_k$ is defined and is non-empty, so if the next list $\overline{A}_{k+1}$ is non-empty, we can continue the procedure for non-empty lists(by Corollary 3.2). If it is empty, then we continue by Lemma 3.2.

By continuing this procedure, we conclude that

$$?([\overline{A}_1, \Gamma_1, \overline{A}_2, \dots, \Gamma_{i-2}, \overline{A}_{i-1}, \Gamma_{i-1}, \overline{A}_i, \dots]) = [\overline{B}_1, Z_1, \overline{B}_2, \dots, \overline{B}_{i-1}, Z_{i-1}, \overline{B}_i, \dots]$$

with $2^{s_i+1} - 1 \leqslant Z_i \leqslant 2^{s_i+2} - 1$ for every $i \in \mathbb{N}$.

Our first goal is to show that $?'(x_0) = 0$. For local convenience let us for a moment represent partial quotients of a continued fraction $[\overline{A}_1, \Gamma_1, \overline{A}_2, \dots]$ as a continuous list of partial quotients, by that I mean the notation

$$x_0 = [\overline{A}_1, \Gamma_1, \overline{A}_2, \dots] = [m_1, \dots, m_t, \dots].$$

By Theorem A, to secure $?'(x_0) = 0$ we need to show that

$$\liminf_{t\to\infty} \frac{m_1 + \dots + m_t}{t} > \kappa_2.$$

In the worst case scenario, $S_{\overline{A}_i} = d(\overline{A}_i)$ and so

$$\liminf_{t\to\infty} \frac{m_1 + \dots + m_t}{t} \geqslant \liminf_{t\to\infty} \frac{\sum\limits_{i=1}^{t} d(\overline{A}_i) + \sum\limits_{i=1}^{t-1} \Gamma_i}{\sum\limits_{i=1}^{t} d(\overline{A}_i) + (t-1)}. \tag{38}$$

Using the implication $\{e \leqslant \frac{a}{b} \leqslant \frac{c}{d}\} \Rightarrow \{\frac{a+c}{b+d} \geqslant \frac{a}{b} \geqslant e\}$ we can split the fraction on the right of (38) into following sum:

$$\frac{d(\overline{A}_1) + d(\overline{A}_2) + \Gamma_1}{d(\overline{A}_1) + d(\overline{A}_2) + 1} + \frac{d(\overline{A}_3) + \Gamma_2}{d(\overline{A}_3) + 1} + \dots + \frac{d(\overline{A}_t) + \Gamma_{t-1}}{d(\overline{A}_t) + 1}. \tag{39}$$

If we take

$$s_1 > (\kappa_2 - 1)[d(\overline{A}_1) + d(\overline{A}_2)] + \kappa_2 - S_{\overline{A}_1}$$

and for all $k \geqslant 2$

$$s_k > (\kappa_2 - 1) d(\overline{A}_{k+1}) + \kappa_2 - \sigma_{\overline{A}_k}, \tag{40}$$

we can conclude that every fraction in (39) is greater than κ_2 and so is the fraction on the right of (38), which numerator and denominator is equal to the sum of all numerators and denominator of the fractions in (39) consequently.

For the set M we have that

$$s_k > (\kappa_2 - 1) S_{\overline{A}_{k+1}} + \sigma_{\overline{A}_k} + 2,$$

which indeed satisfies (40). So we have shown that $?'(x_0) = 0$.

Now our goal is to prove that $?(x_0) \in M$.
We have to make sure that from

$$s_k > (\kappa_2 - 1)S_{\overline{A}_{k+1}} + \sigma_{\overline{A}_k} + 2$$

if follows that

$$g_k \equiv Z_k - \sigma_{\overline{B}_k} > (\kappa_2 - 1)S_{\overline{B}_{k+1}} + \sigma_{\overline{B}_k} + 2 \tag{41}$$

We know that $Z_k > 2^{s_k}$. Also, if $\frac{p_j}{q_j} = [a_1, \ldots, a_j]$, then

$$j \leqslant a_1 + \ldots + a_j \leqslant q_j = \langle a_1, \ldots, a_j \rangle,$$

so showing

$$2^{s_k} > (\kappa_2 - 1)\langle \overline{B}_{k+1} \rangle + 2\sigma_{\overline{B}_k} + 2 \tag{42}$$

would be enough to prove (41).
Now we need to get an upper estimates on $\sigma_{\overline{B}_k}$ and $\langle \overline{B}_{k+1} \rangle$.
Trivially, as

$$?([\overline{A}_1, \Gamma_1, \overline{A}_2, \ldots, \Gamma_{k-1}, \overline{A}_k]) = ?([\overline{B}_1, Z_1, \overline{B}_2, \ldots, Z_{k-1}, \overline{B}_k])$$

we conclude that

$$\sigma_{\overline{B}_k} < 2^{\sigma_{\overline{A}_k} - 1}. \tag{43}$$

Let us estimate the value of $\langle B_k \rangle$ with $k \in \mathbb{N}$:
First, we know that $\langle B_1 \rangle = 2^{S_{\overline{A}_1} - 1}$. Next, for estimating $\langle B_i \rangle$ with $i \geqslant 2$ we apply (8) with

$$X = (\overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}), \ a = Z_{i-1}, \ Y = \overline{B}_i$$

We get we following inequality:

$$Z_{i-1}\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1} \rangle \langle \overline{B}_i \rangle \leqslant \langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}, Z_{i-1}, \overline{B}_i \rangle \leqslant (Z_{i-1} + 2)\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1} \rangle \langle \overline{B}_i \rangle$$

from which we deduce

$$\frac{\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}, Z_{i-1}, \overline{B}_i \rangle}{(Z_{i-1} + 2)\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1} \rangle} \leqslant \langle \overline{B}_i \rangle \leqslant \frac{\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}, Z_{i-1}, \overline{B}_i \rangle}{Z_{i-1}\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1} \rangle} \tag{44}$$

Remember that

$$\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1} \rangle = 2^{\sigma_{\overline{A}_{i-1}} - 1},$$
$$\langle \overline{B}_1, Z_1, \overline{B}_2, ..., \overline{B}_{i-1}, Z_{i-1}, \overline{B}_i \rangle = 2^{\sigma_{\overline{A}_i} - 1},$$
$$2^{s_{i-1}+1} - 1 \leqslant Z_{i-1} \leqslant 2^{s_{i-1}+2} - 1,$$
$$\Gamma_{i-1} = \sigma_{\overline{A}_{i-1}} + s_{i-1}.$$

Now inequality (44) can be rewritten as

$$2^{\sigma_{\overline{A}_{i-1}} + S_{\overline{A}_i} - 2} \leqslant \frac{2^{\Gamma_{i-1} + \overline{A}_i}}{2^{s_{i-1}+2} - 1} \leqslant \langle \overline{B}_i \rangle \leqslant \frac{2^{\Gamma_{i-1} + \overline{A}_i}}{2^{s_{i-1}+1} - 1} \leqslant 2^{\sigma_{\overline{A}_{i-1}} + S_{\overline{A}_i}}. \tag{45}$$

Note that both upper and lower estimates do not depend on s_{i-1}.
Using estimates (43) and (45) in (42), as well as condition on s_k, we come to the following inequality:

$$2^{(\kappa_2 - 1)S_{\overline{A}_{k+1}} + \sigma_{\overline{A}_k}} > (\kappa_2 - 1)2^{\sigma_{\overline{A}_k} + S_{\overline{A}_{k+1}}} + 2^{\sigma_{\overline{A}_k}} + 2$$

or, dividing both sides by $2^{\sigma_{\overline{A}_k}}$,

$$2^{(\kappa_2-1)S_{\overline{A}_{k+1}}} > (\kappa_2-1)2^{S_{\overline{A}_{k+1}}} + 1 + \frac{2}{2^{\sigma_{\overline{A}_k}}}$$

which is true for any $S_{\overline{A}_{k+1}} \geqslant 1$ (i.e. any non-empty list $\overline{A}_{k+1}$).
If $\overline{A}_{k+1}$ is an empty list, then recall from (37) that

$$S_{\overline{B}_{k+1}} = \sigma_{B_k},$$

so using $Z_k > 2^{s_k}$, (41) can be rewritten as

$$2^{s_k} > (\kappa_2+1)\sigma_{\overline{B}_k} + 2,$$

and using estimates

$$s_k > (\kappa_2-1)S_{\overline{A}_{k+1}} + \sigma_{\overline{A}_k} + 2 = \sigma_{\overline{A}_k} + 2,$$

$$\sigma_{\overline{B}_k} < 2^{\sigma_{\overline{A}_k}-1},$$

we come to a

$$2^{\sigma_{\overline{A}_k}+3} > (\kappa_2+1)2^{\sigma_{\overline{A}_k}} + 4,$$

which is always true. □

Remark 6.2. *Condition*

$$s_k > (\kappa_2-1)S_{\overline{A}_{k+1}} + \sigma_{\overline{A}_k} + 2$$

is not optimal. We could have strengthen it by using more accurate estimates, for example the estimate of $\sigma_{\overline{B}_k}$. *We can do better if we consider this sum as*

$$\sigma_{\overline{B}_k} = \sum_{i=1}^{k-1} Z_i + \sum_{i=1}^{k} S_{\overline{B}_i}$$

and then bound every term separately. For the sake of readability we did not do it.

References

[1] G. Alkauskas, *Integral transforms of the Minkowski question mark function*, PhD thesis, Nottingham 2008.

[2] A. Denjoy, *Sur une fonction réelle de Minkowski*, C. R. Acad. Sci. Paris 194 (1932), 44-46.

[3] A. Denjoy, *Sur une fonction de Minkowski*, J. Math. Pures Appl. 17 (1938), 105-151.

[4] A.A. Dushistova, N.G. Moshchevitin, *On the derivative of the Minkowski question mark function ?(x)* (Russian), Fundam. Prikl. Mat. 16 (2010), 33-44; translation in J. Math. Sci. (N.Y.) 182 (2012), 463-471.

[5] A.A. Dushistova, I.D. Kan, N.G. Moshchevitin, *Differentiability of the Minkowski question mark function*, J. Math. Anal. Appl. 401 (2) (2013), 774-794.

[6] R. L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics* Addison-Wesley, 1994

[7] I. D. Kan, *Methods for estimating continuants*, J. Math. Sci. (NY), 182 (4) (2012), 508-517.

[8] I. D. Kan, *Differentiability of the Minkowski function* $?(x)$*. II* Izvestiya: Mathematics(2019), 83 (5):957

[9] I. D. Kan, *Differentiability of the Minkowski* $?(x)$*-function. III* Sbornik: Mathematics(2019), 210 (8):1148

[10] M. Kesseböhmer, B. O. Stratmann, *Fractal analysis for sets of non-differentiability of Minkowski's question mark function*, J. Number Theory 128 (2008), 2663-2686.

[11] J. R. Kinney, *Note on a singular function of Minkowski*, Proc. Amer. Math. Soc. 11 (5) (1960), 788-794.

[12] H. Minkowski, *Zur Geometrie der Zahlen*, Verhandlungen des III Internationalen Mathematiker-Kongresous, Heidelberg 1904, 164-173.

[13] N.G. Moshchevitin, *On some open problems in Diophantine approximation*, preprint. arXiv:1202.4539.

[14] H. Niederreiter, *Dyadic fractions with small partial quotients*, Monatsh. Math. 101, 309-315 (1986).

[15] OEIS: The on-line encyclopedia of integer sequences, https://oeis.org/A058914.

[16] J. Paradís, P. Viader, L. Bibiloni, *A new light on Minkowski's* $?(x)$ *function*, J. Number Theory 73 (2) (1998), 212-227.

[17] A. J. van der Poorten and J. Shallit, *Folded continued fractions*, J. Number Theory, 40:237-250, 1992.

[18] R. Salem, *On some singular monotonic functions which are strictly increasing*, Trans. Amer. Math. Soc. 53 (3) (1943), 427-439.

[19] D. Gayfulin, N. Shulga, *Diophantine properties of fixed points of Minkowski question mark function*, Acta Arithmetica 195 (2020), 367-382MSC