Faithful braid group actions via twists along ADE-configurations of spherical objects of non-positive Calabi-Yau dimension

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Abstract. We prove that the actions of the Artin groups on enhanced triangulated categories, generated by spherical twist functors along ADE-configurations of ω-spherical objects, are faithful for any $\omega \leq 0$.

1. Introduction

Spherical twists along spherical objects are a prominent type of autoequivalences of triangulated categories. The notion of a spherical twist functor was first introduced in [19] by P. Seidel and R. Thomas in connection with the Kontsevich's homological mirror symmetry program. Their original motivation was to look at the autoequivalences of the derived category of coherent sheaves on a variety that would arise as counterparts of generalised Dehn twists via mirror symmetry. In a general setting, a spherical twist is a functor constructed in a particular way from a spherical object, i.e. an object of a triangulated category whose Ext algebra is the same as the cohomology of a sphere.

The theory of Seidel and Thomas received a lot of attention and developed rapidly in numerous works of other mathematicians. For instance, in [1] the notion of a spherical twist along a spherical functor was introduced, generalising spherical twists along spherical objects. In [18], it was established that any triangulated autoequivalence is in fact a twist along some spherical functor. The theory of spherical twists constructed from spherical sequences was developed in [6]. Groups that can be generated by two spherical twists constructed from spherical sequences were described in [20]. Since the appearance of [19] in 2001, spherical twists have proved to be a useful tool in algebraic geometry and beyond. Among their applications are categorifications of Braid groups [15], Bridgeland stability conditions manifolds [10] and derived Picard groups [21].

Let Γ be a simply-laced Dynkin diagram. Seidel and Thomas showed that spherical twists along a so-called Γ-configuration of ω-spherical objects satisfy braid relations of type Γ modulo natural isomorphisms, hence induce an action of an Artin group (generalised braid group) B_Γ on the triangulated category in question. In the same paper they showed that for $\omega \geq 2$ and $\Gamma = A_n$ this action is faithful. By [6], their result can be also extended to the case $\omega = 0$. The paper [19] also provides an example when the action is not faithful for $\omega = 1$. Later several more results on the faithfulness have appeared. In [2], C. Brav and H. Thomas proved that the braid group action is faithful for $\omega = 2$ and all Γ. Recently Y. Qiu and J. Woolf generalised their result to $\omega \geq 2$ for the derived category of a Ginzburg algebra [14], which by the intrinsic formality result of A. Hochenegger and A. Krug [8] may be extended to any algebraic triangulated category, provided that $\omega \geq 4$.

Nevertheless, the existing results did not provide us with a complete picture. Moreover, none of the methods that had been already suggested seemed to be applicable to proving faithfulness for all Γ and all $\omega \neq 1$ simultaneously. For instance, the proof of Seidel and

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Thomas substantially relies on the existence of a faithful representation of the braid group B_{A_n} into the mapping class group of a surface, which does not exist for type E (see [22]). Brav and Thomas [2], on the other hand, suggest a purely algebraic proof, making use of the Garside structure on the braid groups, but relying on the assumption that $\omega = 2$. The approach of Qiu and Woolf is not suitable for obtaining the faithfulness for $\omega = 0$, required for a particular application to representation theory we had in mind when beginning to work on the problem. Finally, none of these results covered the case $\omega < 0$. Originally, Seidel and Thomas did not define $\omega-$spherical objects with negative ω, but they are also worth considering (for instance, see [9], [3], [4], [5]), despite being somewhat more exotic than those of non-negative CY dimension.

In this paper we present a new method that enables us to prove that the braid group action is faithful for any enhanced triangulated category, all $\Gamma = A_n, D_n, E_6, E_7, E_8$ and all $\omega \leq 0$. The current work is based on a joint preprint of the author and Yury Volkov [13], where the case $\omega \geq 2$ is also covered, using the same general strategy as presented here. However, it turns out that the proof for $\omega \leq 0$ requires significantly more work. It is also worth mentioning that among non-positive dimensions, the case $\omega = 0$, which we originally aimed for, happens to be the hardest to tackle.

The structure of the paper is as follows. Section 2 introduces basic notions crucial in the paper, i.e. we define spherical objects, spherical twist functors, configurations of spherical objects, etc. Section 3 contains the main result of the paper as well as the the key lemma on which its proof is based on and some useful technical observations. At the end of this section it is also explained how the key lemma implies the main result. The rest of the paper is mainly devoted to proving the key lemma. In Section 4 we outline the proof of the key lemma and then provide the details in the next three sections. In Section 5 we introduce two-term objects and study some of their properties that we use later. Section 6 explains the first step in the proof of the key lemma, the factorisation. Section 7 contains the second step, to which we refer as "braiding". This finishes the proof of the key lemma and hence the main result of the paper. The statement obtained in Section 7 deals with words in the generalised braid monoids and might be of interest independently, not only in connection with braid group actions on triangulated categories. Finally, in the last section one application of the main result of this paper is presented. Namely, we show how it may be used to make the first yet crucial step towards the description of the derived Picard groups of representation-finite self-injective algebras.

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2. Preliminaries

Throughout this paper $\mathfrak{D}$ is a triangulated category linear over a field k and with a fixed enhancement. For example, it can be an algebraic triangulated category in the sense of Keller (see [12]), i.e. the stable category of some Frobenius category. In this case we are equipped with functorial cones of natural transformations of exact functors, and for every object X of $\mathfrak{D}$ there is the derived Hom-complex functor $\mathrm{RHom}(X, -)\colon \mathfrak{D} \to D(\mathrm{k})$ and its right adjoint $- \otimes X\colon D(\mathrm{k}) \to \mathfrak{D}$, where $D(\mathrm{k})$ denotes the unbounded derived category of k-vector spaces.

Let $\mathrm{Hom}^k(A, B)$ denote $\mathrm{Hom}_{\mathfrak{D}}(A, B[k])$ and $\mathrm{Hom}^*(A, B) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k(A, B)$. For elements $g \in \mathrm{Hom}^k(A, B)$ and $f \in \mathrm{Hom}^l(B, C)$, we will write fg for the element $f[k]g \in \mathrm{Hom}^{k+l}(A, C)$.

Definition 2.1. (Seidel, Thomas [19]) Let $\omega \in \mathbb{Z}$. An object $P \in \mathfrak{D}$ is called *ω-spherical* if

(i) $\dim_{\mathrm{k}} \mathrm{Hom}^*(X, P) < \infty$ for any object $X \in \mathfrak{D}$.
(ii) $\mathrm{Hom}^*(P, P) \cong k[t]/(t^2)$ as graded k−algebras, where $deg(t) = \omega$.
(iii)

$$\mathrm{Hom}^*(P, X) \times \mathrm{Hom}^*(X, P) \xrightarrow{\circ} \mathrm{Hom}^*(P, P)/\langle Id_P \rangle \cong \mathrm{k}$$

is a perfect pairing for any $X \in \mathfrak{D}$ (defined by the composition).

Fix some undirected graph Γ. We will always assume that Γ is an ADE Dynkin diagram, but part of our arguments can be transferred to more general cases. We represent the set Γ_0 of vertices of Γ as a disjoint union of two sets V^0 and V^1 in such a way that each edge of Γ has one endpoint in V^0 and the other in V^1.

By $N(A)$ for $A \subseteq \Gamma_0$ we denote the set all neighbors of all vertices in A (i.e. $N(A)$ is formed by $j \in \Gamma_0$ such that $s_j s_k \neq s_k s_j$ for some $k \in A$).

Following Brav and Thomas ([2]), we now define a Γ-configuration of spherical objects.

Definition 2.2. A collection of ω-spherical objects $\{P_i\}_{i\in\Gamma}$ enumerated by vertices of Γ is a $\Gamma-$*configuration* if for any $i \neq j$

1) $\mathrm{Hom}^*(P_i, P_j)$ is one-dimensional if $i \in N(j)$;
2) $\mathrm{Hom}^*(P_i, P_j) = 0$ if $i \notin N(j)$.

Throughout the paper we will assume that $\omega \leq 0$, although some of the statements will remain true for any $\omega \neq 1$. Fix some integers ω_0 and ω_1 such that $\omega_0 + \omega_1 = \omega$. We will assume that $\omega_0, \omega_1 \leq 0$ as well. For example, one can simply take $\omega_0 = \omega_1 = \frac{\omega}{2}$ if ω is even and $\omega_0 = \frac{\omega+1}{2}$, $\omega_1 = \frac{\omega-1}{2}$ if ω is odd. It follows from the definition of an ω-spherical object that after shifting the objects P_i in a Γ-configuration, one may assume that $\mathrm{Hom}^*(P_i, P_j)$ is concentrated in degree ω_u, where u is such that i belongs to V^u. The index u in the notations ω_u and V^u will be always taken modulo 2.

Definition 2.3. (Seidel, Thomas [19]) Let P be an ω-spherical object. The *spherical twist* functor t_P along P is defined by

$$t_P(X) = cone(P \otimes \mathrm{RHom}(P, X) \xrightarrow{counit} X)$$

Remark. t_P is indeed a functor since we have functorial cones of natural transformations of exact functors.

Definition 2.4. The Artin group (or the generalised braid group) B_Γ of type Γ is generated by $s_i, i \in \Gamma_0$ subject to the braid relations $s_i s_j s_i = s_j s_i s_j$ for i, j adjacent in Γ and $s_i s_j = s_j s_i$ for i, j not adjacent in Γ. The braid monoid B_Γ^+ is a monoid given by the same generators and relations.

We are now going to recall some crucial facts about spherical twists and spherical objects (see [19]) that we will actively use throughout the paper.

1) If P is spherical, then t_P is an autoequivalence of $\mathfrak{D}$ with a quasi inverse t'_P defined by
$$t'_P(X) = cone(X \xrightarrow{unit} P \otimes \mathrm{RHom}(P, X)^*)[-1],$$
where * is the usual duality on the category $D(\mathrm{k})$.
2) If P is ω-spherical, then $t_P(P) = P[1-\omega]$.
3) For $\{P_i\}_{i\in\Gamma_0}$ forming a Γ-configuration, t_{P_i} satisfy braid relations of type Γ up to a natural isomorphism. In particular, for spherical P, Q not adjacent in their Γ-configuration (equivalently $\mathrm{Hom}^*(P, Q) = 0$), $t_P(Q) = Q$. In other words, there is a group homomorphism
$$F \colon B_\Gamma \to \mathrm{Aut}(\mathfrak{D})$$
where $\mathrm{Aut}(\mathfrak{D})$ is a group of autoequivalences of $\mathfrak{D}$ modulo natural isomorphisms. For $\alpha \in B_\Gamma$, we denote $F(\alpha)$ by t_α.

3. Main result

We have just defined an action of the braid group B_Γ of type Γ on the category $\mathfrak{D}$. The main result of this paper says that this action is faithful for every $\omega \neq 1$. The rest of this paper is mainly devoted to proving this claim:

Theorem 1. *If Γ is a simply-laced Dynkin diagram, $\omega \leq 0$ and $\{P_i\}_{i\in\Gamma_0}$ is a $\Gamma-$configuration of $\omega-$spherical objects in an enhanced triangulated category $\mathfrak{D}$, then the action of B_Γ on $\mathfrak{D}$ generated by the spherical twists t_{P_i} is faithful.*

Since the case $|\Gamma_0| = 1$ is clear, we will assume that Γ has at least two vertices. We will use the following auxiliary notation in our proof. For $i \in V^u$ and $j \in N(i)$, we fix some generator of $\mathrm{Hom}^{\omega_u}(P_i, P_j)$ and denote it by $\gamma_{i,j}$. We also introduce the morphisms $\rho_{i,j} \colon P_j \to t_iP_j$ and $\xi_{i,j} \colon t_iP_j \to P_i[1-\omega_u]$ coming from the following distinguished triangle

$$P_i[-\omega_u] \xrightarrow{\gamma_{i,j}[-\omega_u]} P_j \xrightarrow{\rho_{i,j}} t_iP_j \xrightarrow{\xi_{i,j}} P_i[1-\omega_u].$$

Let $\Lambda = \bigoplus_{i=1}^{n} P_i$. For brevity, we will write t_i for t_{P_i}. Now we define *the minimal nonzero degree* of an object in $\mathfrak{D}$:

Definition 3.1. Let $T \in \mathfrak{D}$. We say that $min(T)$ is *the minimal nonzero degree* of T if $\mathrm{Hom}^{k+\omega}(\Lambda, T) = 0$ for $k \leq min(T) - 1$ and $\mathrm{Hom}^{min(T)+\omega}(\Lambda, T) \neq 0$.

Definition 3.2. Let T be an object of $\mathfrak{D}$. We say that P_j with $j \in \Gamma_0$ is *a direct summand of T_r* if there exists a nonzero $f \in \mathrm{Hom}^{r+\omega}(P_j, T)$ such that $f\gamma_{k,j} = 0$ for any $k \in N(j)$. A morphism f satisfying this condition will be referred to as *long*. In other words, a nonzero morphism $f \colon P_j \to T$ is called long if the induced morphism $\mathrm{Hom}^*(P_k, f) \colon \mathrm{Hom}^*(P_k, P_j) \to \mathrm{Hom}^*(P_k, T)$ is zero for any $k \neq j$. We also say that P_j is a direct summand of $T_{[a,b]}$ if P_j is a direct summand of T_c for some $c \in [a, b]$.

Remark. Let $T \in \mathfrak{D}$ and $i \in V^u$. If $\mathrm{Hom}^{r+\omega}(P_i, T) \neq 0$ for some $r \leq min(T) - \omega_{u+1} - 1$, then P_i is a direct summand of T_r. Indeed, any composition of the form $P_j \to P_i[\omega_{u+1}] \to T[r + \omega + \omega_{u+1}]$ is zero by the definition of minimal nonzero degree. Moreover, if $\mathrm{Hom}^{min(T)+\omega}(P_j, T) = 0$ for every $j \in N(i)$ and $\mathrm{Hom}^{min(T)+\omega-\omega_{u+1}}(P_i, T) \neq 0$, then P_i is a direct summand of $T_{min(T)-\omega_{u+1}}$, because in this case any composition of the form $P_j \to P_i[\omega_{u+1}] \to T[min(T) + \omega]$ is zero too.

We denote $t_\alpha(\Lambda)$ by T_α. Let $min(\alpha)$ denote the minimal nonzero degrees of T_α. Following [2], we will deduce the faithfulness of the braid group action from the injectivity of the induced monoid homomorphism $B^+_\Gamma \to \mathrm{Aut}(\mathfrak{D})$. In turn, to prove the injectivity of the aforementioned monoid homomorphism, we require a tool that would allow us to find a leftmost factor of the reduced expression of $\alpha \in B^+_\Gamma$ using only information about T_α. This tool is presented in the following key lemma.

Lemma 1. *Let $\alpha \in B^+_\Gamma$, $\alpha \neq 1$. For $u \in \{0, 1\}$ let $I_{u,\alpha} = [min(\alpha), min(\alpha) - \omega_{u+1}]$. Then for any $u \in \{0, 1\}$ and any $j \in V^u$ such that the corresponding object P_j is a direct summand of $(T_\alpha)_{I_{u,\alpha}}$, the word α can be written as*

$$\alpha = s_j\alpha'$$

for some $\alpha' \in B^+_\Gamma$ with $l(\alpha) = l(\alpha') + 1$.

Before proving Lemma 1 we are going to show how it easily implies our main result. However, first we are going to prove some rather technical facts regarding the behaviour of minimal nonzero degrees and direct summands under the application of spherical twists. These statements will be required in our considerations throughout the paper.

Lemma 2. *Let $\omega \leq 0$, T be an object of $\mathfrak{D}$ and m be its minimal nonzero degree.*

1) *If $\mathrm{Hom}^r(P_i, t_i^{-1}T) \neq 0$, then $m \leq r - 1$.*
2) *For $k \in V^u$, $r \leq m - \omega_{u+1}$ and $i \neq k$, P_k is a direct summand of $(t_i^{-1}T)_r$ if and only if P_k is a direct summand of T_r.*
3) *The minimal nonzero degree of t_iT belongs to $[m - 1 + \omega, m]$ and P_k can be a direct summand of T_r with $r < m$ only if $k = i$.*

Proof. 1) As spherical twists are autoequivalences,

$$0 \neq \mathrm{Hom}^r(P_i, t_i^{-1}T) \cong \mathrm{Hom}^r(t_iP_i, T) = \mathrm{Hom}^r(P_i[1-\omega], T) \cong \mathrm{Hom}^{r-1+\omega}(P_i, T).$$

Hence the minimal nonzero degree m of T is indeed not greater than $r - 1$.

2) Suppose that $k \notin N(i)$. Since t_i is an autoequivalenvce and $t_iP_k = P_k$ in this case, we have an isomorphism $t_i : \mathrm{Hom}^{r+\omega}(P_k, t_i^{-1}T) \cong \mathrm{Hom}^{r+\omega}(P_k, T)$. It is sufficient to show that $f : P_k \to t_i^{-1}T[r+\omega]$ is long if and only if t_if is long. Pick some $l \in N(k)$. If $l \notin N(i)$, then $t_iP_l = P_l$ and we have $\mathrm{Im}\,\mathrm{Hom}^*(P_l, t_if) = t_i \,\mathrm{Im}\,\mathrm{Hom}^*(P_l, f)$. Since t_i is an autoequivalence, $\mathrm{Hom}^*(P_l, t_if) = 0$ if and only if $\mathrm{Hom}^*(P_l, f) = 0$.

If $l \in N(i)$, then $i \in V^u$ and there is a triangle

$$P_l \xrightarrow{\rho_{i,l}} t_iP_l \xrightarrow{\xi_{i,l}} P_i[1-\omega_u].$$

Since $\mathrm{Hom}^*(P_i, P_k) = 0$, we have $\gamma_{l,k} = g\rho_{i,l}$ for some $g \in \mathrm{Hom}^{\omega_{u+1}}(t_iP_l, P_k)$. Thus, if $(t_if)\gamma_{l,k} \neq 0$, then $f(t_i^{-1}g) \neq 0$, and $\mathrm{Hom}^*(P_l, f) \neq 0$ as well. On the other hand, if $f\gamma_{l,k} \neq 0$, then the composition

$$P_l \xrightarrow{\rho_{i,l}} t_iP_l \xrightarrow{t_i(f\gamma_{l,k})} T[r+\omega+\omega_{u+1}]$$

is nonzero, because $r+\omega-1 \leq m+\omega_u-1 < m$, and hence

$$\mathrm{Hom}^{r+\omega+\omega_{u+1}}(P_i[1-\omega_u], T) \cong \mathrm{Hom}^{r+2\omega-1}(P_i, T) = 0.$$

Then we have $\mathrm{Hom}^*(P_l, t_if) \neq 0$. Thus, f is long if and only if t_if is long.

Suppose now that $k \in N(i)$. Consider the triangle

(1) $$P_k \xrightarrow{\rho_{i,k}} t_iP_k \xrightarrow{\xi_{i,k}} P_i[1-\omega_{u+1}]$$

For any nonzero $f\colon P_k \to t_i^{-1}T[r+\omega]$, one has $t_if\rho_{i,k} \neq 0$, because $r+\omega_{u+1}-1 \leq m-1$, the minimal nonzero degree of T is m, and hence $\mathrm{Hom}^{r+\omega}(P_i[1-\omega_{u+1}], T) = 0$. Since $(t_if)\rho_{i,k}\gamma_{i,k}[-\omega_{u+1}] = 0$, it remains to show that $(t_if)\rho_{i,k}\gamma_{l,k}[-\omega_{u+1}] = 0$ for $l \in N(k) \setminus \{i\}$ if f is long. Indeed, suppose it is nonzero and apply t_i^{-1}. Since $l \notin N(i)$, we get a nonzero morphism $P_l[-\omega_{u+1}] \to P_k \xrightarrow{f} t_i^{-1}T[r+\omega]$, which contradicts f being a long morphism.

Pick now a long morphism $f'\colon P_k \to T[r+\omega]$. Then $f'\gamma_{i,k}[-\omega_{u+1}] = 0$ by the definition of a long morphism, and hence $f' = (t_if)\rho_{i,k}$ for some $f\colon P_k \to t_i^{-1}T[r+\omega]$. We have $0 = f\gamma_{i,k}\colon P_i \to t_i^{-1}T[r+\omega+\omega_{u+1}]$, because otherwise the minimal nonzero degree of T would be not greater than $r+\omega+\omega_{u+1}-1 \leq m+\omega-1 < m$ by the first assertion of the current lemma. Pick some $l \in N(k) \setminus \{i\}$. Since $\mathrm{Hom}^*(P_l, P_i) = 0$, the morphism $t_i\gamma_{l,k} : P_l \to t_iP_k[\omega_{u+1}]$ factors through $\rho_{i,k}[\omega_{u+1}]$, and hence equals $\rho_{i,k}\gamma_{l,k}$ modulo a nonzero scalar factor. Since $(t_if)\rho_{i,k}$ is long, one has $(t_if)\rho_{i,k}\gamma_{l,k} = 0$. Applying t_i^{-1}, we see that $f\gamma_{l,k} = 0$ as well. Thus, there exists a long morphism of P_k to $T[r+\omega]$ if and only if there exists a long morphism of P_k to $t_i^{-1}T[r+\omega]$.

3) First we show that the minimal nonzero degree of t_iT is not greater than m. There exists some $k \in \Gamma_0$ such that P_k is a direct summand of T_m. If $k \neq i$ and the minimal nonzero degree of t_iT is greater than m, then P_k is a direct summand of $(t_iT)_m$ by the second assertion of this lemma and we get a contradiction. If $k = i$, then $\mathrm{Hom}^{m+\omega}(P_i, T) \neq 0$ and the minimal nonzero degree of t_iT is not greater than $m+\omega-1 < m$ by the first assertion of the current lemma. Thus, the minimal nonzero degree of t_iT is not greater than m.

Now suppose P_k is a direct summand of $(t_iT)_r$ for $r < m$, $k \neq i$. If $k \notin N(i)$, then $0 \neq \mathrm{Hom}^{r+\omega}(P_k, t_iT) \cong \mathrm{Hom}^{r+\omega}(P_k, T)$, which contradicts the minimal nonzero degree of T being m. If $k \in N(i)$, then there exists a long morphism $P_k \xrightarrow{f} t_iT[r+\omega]$, in particular, $f\gamma_{i,k} = 0$. Consider the triangle

$$P_i[\omega_u - 1] \xrightarrow{t_i^{-1}\gamma_{i,k}[-\omega_{u+1}]} t_i^{-1}P_k \xrightarrow{t_i^{-1}\rho_{i,k}} P_k.$$

Since $t_i^{-1}(f\gamma_{i,k}[-\omega_{u+1}]) = 0$, the morphism $t_i^{-1}f$ factors through P_k and the minimal nonzero degree of T is not greater than $r < m$ which is impossible.

It now follows from what we have already established that if the minimal nonzero degree of t_iT equals $d < m$, then $\mathrm{Hom}^{d+\omega}(P_i, t_iT) \neq 0$. Applying t_i^{-1}, we get $\mathrm{Hom}^{d+1}(P_i, T) \neq 0$, and hence $d \geq m-1+\omega$.

□

We are now ready to deduce Theorem 1 from Lemma 1.

Proof of Theorem 1. According to [2, Proposition 2.3], a group homomorphism $B_\Gamma \to G$ is injective if and only if the induced monoid homomorphism $B_\Gamma^+ \hookrightarrow B \to G$ is injective. Hence, in our case it is sufficient to show that $B_\Gamma^+ \to \mathrm{Aut}(\mathfrak{D})$ is injective. Assume that it is not. Choose two words α, β with the smallest sum of lengths $l(\alpha)+l(\beta)$ among all pairs of words with coinciding images in $\mathrm{Aut}(\mathfrak{D})$ and $\alpha \neq \beta$. In particular, $t_\alpha(\Lambda) = T_\alpha \cong T_\beta = t_\beta(\Lambda)$ in $\mathfrak{D}$. Thus, for any $u \in \{0,1\}$, $I_{u,\alpha} = I_{u,\beta}$ and P_i $(i \in V^u)$ is a direct summand of $(T_\alpha)_{I_{u,\alpha}}$ if and only if it is a direct summand of $(T_\beta)_{I_{u,\beta}}$. First assume that one of α and β is 1, say α. Then $min(\alpha) = 0$. But since $\beta \neq 1$, we have $l(\beta) \geq 1$ and hence $min(\beta) < 0$ by Lemma 2. Thus, we may assume that $\alpha \neq 1$, $\beta \neq 1$.

Now Lemma 1 implies that there exists $i \in \Gamma_0$ such that

$$\alpha = s_i\alpha' \text{ and } \beta = s_i\beta'.$$

Obviously, the images of α' and β' also coincide in $\mathrm{Aut}(\mathfrak{D})$, and since $l(\alpha')+l(\beta') = l(\alpha) + l(\beta) - 2 < l(\alpha)+l(\beta)$ we get $\alpha' = \beta'$. But then $\alpha = s_i\alpha' = s_i\beta' = \beta$, which contradicts the assumption that $\alpha \neq \beta$. □

4. Outline of the proof of Lemma 1

In this section we give a general plan of our proof of Lemma 1.

We argue by contradiction. Take $\alpha \in B_\Gamma^+$ not satisfying the required condition and of minimal length. It is clear that $l(\alpha) > 0$, and hence α can be presented as $\alpha = s_i\beta$ for some $i \in \Gamma_0$ and $\beta \in B_\Gamma^+$ with $l(\beta) < l(\alpha)$. In particular, the statement of Lemma 1 holds for β and all its right factors. Without loss of generality we may assume that $i \in V^0$. Since Lemma 1 fails for α, there exists some $j \in V^u$ $(u = 0,1)$ such that P_j is a direct summand of $(T_\alpha)_{[m_\alpha, m_\alpha - \omega_{u+1}]}$ while α is not left-divisible by s_j. It is clear that $j \neq i$. Let $m \in [m_\alpha, m_\alpha - \omega_{u+1}]$ be the minimal degree such that P_j is a direct summand of $(T_\alpha)_m$. We may assume without loss of generality that if $r < m$ and P_k with $k \in V^u$ is a direct summand of $(T_\alpha)_r$, then α is divisible by s_k on the left.

Lemma 3. *1) $j \in N(i) \subset V^1$.*

2) If P_k for some $k \in V^1$ is a direct summand of $(T_\beta)_r$ with $r \leq m$, then $r = m$ and $k \in N(i)$.

3) If P_k is a direct summand of $(T_\beta)_m$ for some $k \in V^1 \setminus \{j\}$, then s_l does not divide α on the left for any $l \neq i$.

4) $\mathrm{Hom}^r(P_i, T_\beta) = 0$ for $r \leq m + \omega_0$.

Proof. 1) Suppose that $j \notin N(i)$. It follows from Lemma 2 that P_j is a direct summand of $(T_\beta)_m$ and $m \in [m_\beta, m_\beta - \omega_{u+1}]$. Since the statement of Lemma 1 holds for β, we have $\alpha = s_i\beta = s_is_j\beta' = s_js_i\beta'$ for some $\beta' \in B_\Gamma^+$ with $l(\beta') = l(\alpha)-2$. Thus, j does not give a contradiction to the statement of Lemma 1 which in turn contradicts the choice of j. Thus, $j \in N(i) \subset V^1$ and, in particular, we have $m \in [m_\alpha, m_\alpha - \omega_0]$.

2) Pick some $k \in V^1$ such that P_k is a direct summand of $(T_\beta)_r$ with $r \leq m$. Then P_k is a direct summand of $(T_\alpha)_r$ by Lemma 2. If $r < m$, then s_k divides α on the left by the definition of m. If $r = m$ but $k \notin N(i)$, then s_k divides α on the left by the argument from the proof of the first issue. In any case, we have $\alpha = s_k\alpha'$ for some $\alpha' \in B_\Gamma^+$ with $l(\alpha') = l(\alpha) - 1$. Note that Lemma 2 implies again that P_j is a direct summand of $(T_{\alpha'})_m$ and $m \in [m_{\alpha'}, m_{\alpha'} - \omega_0]$. Since the assertion of Lemma 1 is valid for α', we conclude that s_j divides α' on the left. Since s_j and s_k commute, s_j divides α on the left as well which contradicts the choice of j.

3) Suppose that s_l divides α on the left for some $l \neq i$. Then s_l commutes with at least one of the elements s_j and s_k. The argument we have just used in the proof of the second item shows that either s_j or s_k divides α on the left. Since s_k and s_j commute, the same argument shows that s_j divides α on the left in any case. This contradicts the choice of j.

4) Suppose that $\mathrm{Hom}^r(P_i, T_\beta) \neq 0$ for some $r \leq m + \omega_0$. By Lemma 2 one has $m_\alpha \leq r - 1 \leq m + \omega_0 - 1$. But in this case $m \notin [m_\alpha, m_\alpha - \omega_0]$ which yields a contradiction. □

Now set $s_A = \prod_{k\in A} s_k$, $t_A = \prod_{k\in A} t_k$ for any $A \subseteq V^u$ and $u \in \{0,1\}$. Define σ_u for $u \in \mathbb{Z}$ by $\sigma_0 = m$ and $\sigma_{u+1} = \sigma_u + 1 - \omega_{u+1}$. In other words, $\sigma_{2u} = m + u(2-\omega)$, $\sigma_{2u+1} = m + u(2-\omega) + 1 - \omega_{u+1}$. Let

$$\Delta_{-1} = \varnothing,\ \Delta_0 = \{i\},\ \Delta_1 = \{j \in V^1 \colon P_j \text{ is a direct summand of } (T_\beta)_m\}.$$

If $|\Delta_1| \geq 2$, then α cannot be written as $\alpha = s_k\alpha'$ with $l(\alpha') = l(\alpha) - 1$ and $k \in \Gamma_0 \setminus \{i\}$ by Lemma 3. Hence, we need to show that either $|\Delta_1| = 1$ and α is left-divisible by s_j where j is the unique element of Δ_1, or $|\Delta_1| \geq 2$ and α is left-divisible by s_k with some $k \neq i$. With this end in view, we will employ the following scheme:

Step I: **Factorisation.** First we are going to construct a presentation of α of a particular form.

We start with the presentation $\alpha = s_{\Delta_0}s_{\Delta_1}\beta_1 = s_i s_{\Delta_1}\beta_1$ obtained earlier. Set $\chi_0(i) = 1$. We continue the process inductively and obtain a presentation of the form $\alpha = s_{\Delta_0}s_{\Delta_1}\dots s_{\Delta_q}s_y\widetilde{\beta}$ satisfying the following conditions: for $1 \leq u \leq q$

(1) $\alpha = s_{\Delta_0}\dots s_{\Delta_u}\beta_u$ for some $\beta_u \in B_\Gamma^+$ with $l(\beta_u) = l(\alpha) - \sum_{v=0}^{u}|\Delta_v|$
(2) $\Delta_{u-2} \subseteq \Delta_u \subseteq N(\Delta_{u-1})$.
(3) P_l is not a direct summand of $(T_{\beta_u})_{[\sigma_{u-3}+1,\sigma_{u-1}]}$ for any $l \in V^u$.
(4) The minimal nonzero degree of T_{β_u} is not smaller than $\sigma_{u-2}+1$.
(5) For any $l \in \Delta_u$, P_l is a direct summand of $(t_lT_{\beta_u})_{[\sigma_{u-3}+1,\sigma_{u-1}]}$.
(6) $\chi_u(k) = \sum_{t\in N(k)} \chi_{u-1}(t) - \chi_{u-2}(k) > 0$ for any $k \in \Delta_u$, where we set for convenience $\chi_v(t) = 0$ if $v < 0$ or $t \notin \Delta_v$.

Moreover, $y \in \Delta_{q-1}$, s_y divides β_q on the left and

$$\chi_{q+1}(y) = \sum_{t\in N(y)\cap\Delta_q} \chi_q(t) - \chi_{q-1}(y) = 0.$$

Note that the sets Δ_0 and Δ_1 defined earlier satisfy the required conditions. Indeed, one has $\Delta_1 \subseteq N(i) = N(\Delta_0)$ by Lemma 3. The minimal nonzero degree of T_{β_1} is not smaller than $m_\alpha \geq m + \omega_0 = \sigma_{-1} + 1$ by Lemma 2. It follows from the same lemma and the fact that P_j is a direct summand of $(T_\beta)_m$ that P_j is a direct summand of $\left(t^{-1}_{\Delta_1\setminus\{j\}}T_\beta\right)_m = (t_jT_{\beta_1})_m$ for any $j \in \Delta_1$. Moreover, for any $r \leq m = \sigma_0$, P_k with $k \in V^1 \setminus \Delta_1$ is not a direct summand of $(T_\beta)_r$ by Lemma 3, and hence is not a direct summand of $(T_{\beta_1})_r$ by Lemma 2, and P_k with $k \in \Delta_1$ cannot be a direct summand of $(T_{\beta_1})_r$, because

$$\mathrm{Hom}^{r+\omega}(P_k, T_{\beta_1}) \cong \mathrm{Hom}^{r+2\omega-1}(t_{\Delta_1}P_k[\omega-1], T_\beta) = \mathrm{Hom}^{r+2\omega-1}(P_k, T_\beta) = 0$$

due to the inequality $r + \omega - 1 \leq m + \omega - 1 \leq m_\beta + \omega_1 - 1 < m_\beta$. Finally, we clearly have $\chi_1(j) = 1 > 0$ for any $j \in \Delta_1$.

Thus, it is enough to show that if we have sets $\Delta_0, \dots, \Delta_p$ such that the properties above are satisfied for any $1 \leq u \leq p$, then we either can construct Δ_{p+1} in such a way that the properties above are also satisfied for $u = p+1$ or find $y \in \Delta_{p-1}$ such that s_y divides β_p on the left and $\sum_{t\in N(y)\cap\Delta_p}\chi_p(t) = \chi_{p-1}(y)$. We introduce all the necessary technical tools and discuss this step in detail in Section 6.

Step II: **Braiding.** Once a presentation for α of the form $s_{\Delta_0}s_{\Delta_1}\dots s_{\Delta_p}s_y\widetilde{\beta}$ is obtained, it remains to show that either $\Delta_1 = \{j\}$ for some j such that s_j divides $s_{\Delta_0}s_{\Delta_1}\dots s_{\Delta_p}s_y$ on the left or $|\Delta_1| \geq 2$ and at least one of s_k with $k \in \Gamma_0 \setminus \{i\}$ divides ${}_{\Delta_0}s_{\Delta_1}\dots s_{\Delta_p}s_y$ on the left. Note that if $\Delta_1 = \{j\}$, then $\chi_2(i) = 0$, and hence our presentation is of the form $\alpha = s_i s_j s_i \widetilde{\beta} = s_j s_i s_j \widetilde{\beta}$. Thus, at this point it is enough to consider the case $|\Delta_1| \geq 2$ and show that some $k \in \Gamma_0 \setminus \{i\}$ can be pulled to the very left of the subword $s_{\Delta_0} s_{\Delta_1} \dots s_{\Delta_p} s_y$, applying a sequence of braid and commutator relations. This step is discussed in Section 7.

5. Two-term objects

In this section we introduce the notion of the a two-term object in $\mathfrak{D}$ and prove some facts about them required to fulfill the factorisation process announced above. The proofs in this section are not very difficult, but are mostly rather technically sophisticated. They can be found in the extended version of the current paper (see [13]).

Definition 5.1. An object X of a triangulated category $\mathfrak{D}$ with a fixed Γ-configuration of ω-spherical objects $\{P_j\}_{j\in\Gamma_0}$ is called *two-term* if there exists a triangle

$$X[-1] \xrightarrow{\beta_X} \bigoplus_{j\in V^u} P_j^{x_j}[-\omega_u] \xrightarrow{\varphi_X} \bigoplus_{k\in V^{u+1}} P_k^{x_k} \xrightarrow{\alpha_X} X$$

in $\mathfrak{D}$ for some $u \in \{0,1\}$ and some $x_j \geq 0$ ($j \in \Gamma_0$). A two-term object X is called *right-proper* if $f\varphi_X \neq 0$ for any split epimorphism $f\colon \bigoplus_{k\in V^{u+1}} P_k^{x_k} \to P_l$ with $l \in V^{u+1}$ and is called *left-proper* if $\varphi_X g[-\omega_u] \neq 0$ for any split monomorphism $g\colon P_l \to \bigoplus_{j\in V^u} P_j^{x_j}$ with $l \in V^u$. For a two-term object X we also define $\operatorname{lsupp}(X) = \{j \in V^u \mid x_j \neq 0\}$ and $\operatorname{rsupp}(X) = \{k \in V^{u+1} \mid x_k \neq 0\}$.

For example, P_l is a left-proper two-term object with $\alpha_{P_l} = id_{P_l}$ and $\beta_{P_l} = \varphi_{P_l} = 0$ for any $l \in \Gamma_0$. Moreover, $\operatorname{lsupp}(P_l) = \varnothing$ and $\operatorname{rsupp}(P_l) = \{l\}$. It is not difficult to show that any two-term object X can be represented in the form $X = X' \oplus \bigoplus_{k\in V^{u+1}} P_k^{r_k}$, where X' is a right-proper two-term object.

Remark. The notion of a two-term object in some sense generalises the notion of a two-term partial tilting complex to the setting of triangulated categories with a Γ-configuration of spherical objects.

Lemma 4. *Let X be a two-term object as in the definition above.*

1) *The following three conditions are equivalent:*
   - *X is right-proper;*
   - *if $l \in V^{u+1}$ and $g\colon P_l \to \bigoplus_{k\in V^{u+1}} P_k^{x_k}$ is not a split monomorphism, then $\alpha_X g = 0$;*
   - $\dim_{\Bbbk} \operatorname{Hom}^*(X, P_l) = \sum_{k\in N(l)} x_k$ *for any $l \in V^{u+1}$.*
2) *The following three conditions are equivalent:*
   - *X is left-proper;*
   - *if $l \in V^u$ and $f : \bigoplus_{j\in V^u} P_j^{x_j} \to P_l$ is not a split epimorphism, then $f[-\omega_u]\beta_X = 0$;*
   - $\dim_{\Bbbk} \operatorname{Hom}^*(X, P_l) = \sum_{k\in N(l)} x_k$ *for any $l \in V^u$.*

The first crucial fact about the class of two-term objects is that it is stable under certain autoequivalences of $\mathfrak{D}$. For $\Delta \subseteq V^u$ we denote $t_\Delta^+ = t_\Delta[\omega_{u+1} - 1]$ and $t_\Delta^- = t_\Delta^{-1}[1 - \omega_{u+1}]$.

Lemma 5. *Let X be as above.*

1) *If X is right-proper and $\operatorname{rsupp}(X) \subseteq \Delta \subseteq V^{u+1}$, then $t_\Delta^+ X$ is a left-proper two-term object with the defining triangle of the form*

$$t_\Delta^+ X[-1] \xrightarrow{\beta_{t_\Delta^+ X}} \bigoplus_{k\in V^{u+1}} P_k^{x'_k}[-\omega_{u+1}] \xrightarrow{\varphi_{t_\Delta^+ X}} \bigoplus_{j\in V^u} P_j^{x_j} \xrightarrow{\alpha_{t_\Delta^+ X}} t_\Delta^+ X,$$

   where $x'_k = \sum_{j\in N(k)} x_j - x_k$ for $k \in \Delta$ and $x'_k = 0$ for $k \in V^{u+1} \setminus \Delta$.

2) If X is left-proper and $\operatorname{lsupp}(X) \subseteq \Delta \subseteq V^u$*, then $t^-_\Delta X$ is a right-proper two-term object with the defining triangle of the form*

$$t^-_\Delta X[-1] \xrightarrow{\beta_{t^-_\Delta X}} \bigoplus_{k\in V^{u+1}} P_k^{x_k}[-\omega_{u+1}] \xrightarrow{\varphi_{t^-_\Delta X}} \bigoplus_{j\in V^u} P_j^{x'_j} \xrightarrow{\alpha_{t^-_\Delta X}} t^-_\Delta X,$$

where $x'_j = \sum\limits_{k\in N(j)} x_k - x_j$ *for* $j\in\Delta$ *and* $x'_j = 0$ *for* $j \in V^u\setminus\Delta$.

Next we need to study the behaviour of some relations between two-term objects with respect to autoequivalences $t^\pm_\Delta$.

Definition 5.2. Let

$$X{=}cone\left(\bigoplus_{j\in V^u} P_j^{x_j}[-\omega_u]\xrightarrow{\varphi_X}\bigoplus_{k\in V^{u+1}} P_k^{x_k}\right) \text{ and } Y{=}cone\left(\bigoplus_{j\in V^u} P_j^{y_j}[-\omega_u]\xrightarrow{\varphi_Y}\bigoplus_{k\in V^{u+1}} P_k^{y_k}\right)$$

be two-term objects. We will call X a *two-term subobject* of Y if there exist split monomorphisms $\iota_u\colon \bigoplus\limits_{j\in V^u} P_j^{x_j} \to \bigoplus\limits_{j\in V^u} P_j^{y_j}$ and $\iota_{u+1}\colon \bigoplus\limits_{k\in V^{u+1}} P_k^{x_k} \to \bigoplus\limits_{k\in V^{u+1}} P_k^{y_k}$ such that $\iota_{u+1}\varphi_X = \varphi_Y\iota_u[-\omega_u]$. The two-term subobject X of Y is called *trivial* if either $X=0$ or both of the maps ι_u, ι_{u+1} are isomorphisms. Otherwise X is called a *nontrivial two-term subobject* of Y.

We will say that a morphism $f\colon X\to Y[\omega]$ is a *right socle morphism* if it can be presented in the form $f=\alpha_Y[\omega]f'$ for some $f' : X \to \bigoplus\limits_{k\in V^{u+1}} P_k^{y_k}[\omega]$ such that for any split epimorphism $g : \bigoplus\limits_{k\in V^{u+1}} P_k^{y_k} \to P_l$ with $l\in V^{u+1}$ the morphism $g[\omega]f'\alpha_X\colon \bigoplus\limits_{k\in V^{u+1}} P_k^{x_k} \to P_l[\omega]$ is not a split epimorphism anymore.

Remark. The second condition in the definition of a right socle morphism is satisfied automatically if X is right-proper or $\omega\neq 0$. Moreover, if $\omega\neq 0$ and X is left-proper, then any morphism of the form $X\xrightarrow{f} Y[\omega]$ is automatically right socle. This follows from the fact that $\operatorname{Hom}_{\mathfrak{D}}(P_k, P_j[1+\omega_{u+1}])=0$ for any $k\in V^{u+1}$, $j\in V^u$ and $g[1-\omega_u]\beta_X[1]=0$ for any $g\colon \bigoplus\limits_{j\in V^u} P_j^{x_j} \to \bigoplus\limits_{j\in V^u} P_j^{y_j}[\omega]$. In fact, the definition of a right socle morphism is introduced solely to cover the case $\omega=0$ which nevertheless is of special interest for us in view of an application to the derived Picard groups of algebras.

It is worth mentioning that the assertions about right socle morphisms we provide below are trivial for $\omega\neq 0$.

Lemma 6. *Let X and Y be as above. Suppose also that both X and Y are right-proper. If X is a nontrivial two-term subobject of Y, then $t^+_\Delta X$ is a non-trivial two-term subobject of $t^+_\Delta Y$ for any* $\operatorname{rsupp}(Y)\subseteq\Delta\subseteq V^{u+1}$.

Lemma 7. *If X is right-proper and $l\in V^{u+1}$, then any right socle morphism from P_l to $X[\omega]$ is zero.*

Lemma 8. *Let X and Y be as above. Then any right socle morphism $f : X\to Y[\omega]$ factors through some right socle morphism $X'\to Y[\omega]$, where X' is a two-term object such that* $\operatorname{rsupp}(X')\subset\operatorname{rsupp}(Y)$.

Lemma 9. *Let X and Y be as above. Suppose that both X and Y are right-proper and* $\operatorname{rsupp}(X)\subset\operatorname{rsupp}(Y)$*. Then for any right socle $f : X\to Y[\omega]$ and any* $\operatorname{rsupp}(Y)\subseteq\Delta\subseteq V^{u+1}$*, the morphism $t^+_\Delta f : t^+_\Delta X \to t^+_\Delta Y[\omega]$ is right socle too.*

6. FACTORISATION

We return to the context of Lemma 1. Recall that $\sigma_{2u} = m+u(2-\omega)$, $\sigma_{2u+1} = m+u(2-\omega)+1-\omega_1$. Now suppose that we have sets $\Delta_0,\dots,\Delta_p$ for some $p\geq 1$ such that $\Delta_0=\{i\}$ for some $i\in V^0$. Let us recall that the numbers $\chi_u(k)$ ($0\leq u\leq p$, $k\in\Delta_u$) are defined inductively by $\chi_0(i):=1$ and $\chi_u(k) := \sum\limits_{t\in N(k)}\chi_{u-1}(t)-\chi_{u-2}(k)$ for $u\geq 1$, where we set for convenience $\chi_v(t)=0$ if $v<0$ or $t\notin\Delta_v$. Suppose also that the following conditions hold for $1\leq u\leq p$:

(1) $\alpha = s_{\Delta_0} \dots s_{\Delta_u} \beta_u$ for some $\beta_u \in B_\Gamma^+$ with $l(\beta_u) = l(\alpha) - \sum\limits_{v=0}^{u} |\Delta_v|$
(2) $\Delta_{u-2} \subseteq \Delta_u \subseteq N(\Delta_{u-1})$, where we set $\Delta_{-1} = \varnothing$ for convenience.
(3) P_l is not a direct summand of $(T_{\beta_u})_{[\sigma_{u-3}+1, \sigma_{u-1}]}$ for any $l \in V^u$.
(4) The minimal nonzero degree of T_{β_u} is not smaller than $\sigma_{u-2} + 1$.
(5) For any $l \in \Delta_u$, P_l is a direct summand of $(t_l T_{\beta_u})_{[\sigma_{u-3}+1, \sigma_{u-1}]}$.
(6) $\chi_u(k) > 0$ for any $k \in \Delta_u$.

We want to continue the process and construct Δ_{p+1} in such a way that the conditions (1) and (2) are fulfilled for $u = p+1$ and, whenever the condition (6) is valid for $u = p+1$, then so are the conditions (3)–(5). To this purpose, define

$$C_u := t^-_{\Delta_u} \dots t^-_{\Delta_1} P_i = (t^{-1}_{\Delta_u} \dots t^{-1}_{\Delta_1} P_i)[\sigma_u - m + \omega_u - \omega_0].$$

The first crucial fact that we will need is that C_u is a two-term object of a certain form.

Lemma 10. *For any $1 \le u \le p$ there exists a triangle of the form*

$$C_u[-1] \xrightarrow{\beta_u} \bigoplus_{j \in \Delta_{u-1}} P_j^{\chi_{u-1}(j)}[-\omega_{u-1}] \xrightarrow{\varphi_u} \bigoplus_{k \in \Delta_u} P_k^{\chi_u(k)} \xrightarrow{\alpha_u} C_u.$$

Proof. Since P_i is a left-proper two-term object, the required triangle can be obtained by iterated application of Lemma 5. □

The next proposition elaborates on the properties of the objects C_u that will allow us to find direct summands of $(T_{\beta_u})_{[\sigma_{u-2}+1, \sigma_u]}$.

Proposition 1. *For any $1 \le u \le p$, the two-term object C_u satisfies the following two conditions:*

1) $\mathrm{Hom}^r(C_u, T_{\beta_u}) = 0$ *for any* $r \le \omega_u + \sigma_u$.
2) For any nontrivial two-term subobject C' of C_u, there exists a nonzero morphism $f : C' \to T_{\beta_u}[r]$ with $\omega_u + \sigma_{u-2} + 1 \le r \le \omega_u + \sigma_u$ such that $f[\omega]g = 0$ for any right socle morphism $g : C'' \to C'[\omega]$.

Proof. We proceed by induction on u with the base case $u = 0$. Since $C_0 = P_i$, P_i does not have nontrivial two-term subobjects and $\mathrm{Hom}^r(P_i, T_\beta) = 0$ for any $r \le m + \omega_0$ by Lemma 3, there is nothing left to prove in the base case. Now we prove 1) and 2) for u, assuming they are true for $u-1$. The first property of C_u is clear, because

$$\mathrm{Hom}^r(C_u, T_{\beta_u}) = \mathrm{Hom}^r(t^-_{\Delta_u} C_{u-1}, t^{-1}_{\Delta_u} T_{\beta_{u-1}}) \cong \mathrm{Hom}^{r+\omega_{u+1}-1}(C_{u-1}, T_{\beta_{u-1}}).$$

Now we turn to the second property. Suppose that C' is a nontrivial two-term subobject of C_u. If C' is right-proper, then $t^+_{\Delta_u} C'$ is a nontrivial two-term subobject of $C_{u-1} = t^+_{\Delta_u} C_u$ by Lemma 6. By the induction hypothesis, there is a nonzero morphism $f \colon t^+_{\Delta_u} C' \to T_{\beta_{u-1}}[r]$ with $\omega_{u-1} + \sigma_{u-3} + 1 \le r \le \omega_{u-1} + \sigma_{u-1}$ such that $f[\omega]g = 0$ for any right socle morphism $g \colon C'' \to t^+_{\Delta_u} C'[\omega]$. Let us prove that the morphism $t^-_{\Delta_u} f \colon C' \to T_{\beta_u}[r + 1 - \omega_{u-1}]$ satisfies the required properties.

Suppose for a contradiction that there exists a right socle morphism $g \colon C'' \to C'[\omega]$ such that $(t^-_{\Delta_u} f)[\omega]g \neq 0$. Then applying Lemmas 8 and 7 we can find such a g with right-proper C'' satisfying the condition $\mathrm{rsupp}(C'') \subseteq \mathrm{rsupp}(C')$. Then $t^+_{\Delta_u} g : t^+_{\Delta_u} C'' \to t^+_{\Delta_u} C'[\omega]$ is right socle by Lemma 9 and satisfies the condition $f[\omega]\big(t^+_{\Delta_u} g\big) = t^+_{\Delta_u}\left((t^-_{\Delta_u} f)[\omega]g\right) \neq 0$ which is impossible. This shows that $t^{-1}_{\Delta_u} f$ is indeed the required morphism.

It remains to consider the case when C' is not right-proper. In this case we may assume that $C' = P_l$ for some $l \in \Delta_u$. Let a be the minimal integer such that P_l is a direct summand of $(t_l T_{\beta_u})_a$. Note that $a \in [\sigma_{u-3} + 1, \sigma_{u-1}]$ by the properties (3)–(5) of Δ_u. Picking some long morphism $f \colon P_l \to t_l T_{\beta_u}[a + \omega]$ and applying $t_l^{-1}[1 - \omega]$, we get a nonzero morphism $t_l^{-1} f[1 - \omega] \colon P_l \to T_{\beta_u}[a + 1]$. Note that

$$a + 1 \in [\sigma_{u-3} + 2, \sigma_{u-1} + 1] = [\omega_u + \sigma_{u-2} + 1, \omega_u + \sigma_u].$$

Thus, it remains to show that $(t_l^{-1}f)[1]g=0$ for any right socle morphism $g\colon C''\to P_l[\omega]$. Due to Lemma 8, we may assume that C'' has the form

$$C''=cone\left(\bigoplus_{j\in V^{u+1}}P_j^{x_j}[-\omega_{u+1}]\xrightarrow{\varphi_{C''}}P_l^x\right)$$

for some integers x_j,x. Note now that if $C''=P_l$, then $t_lg:P_l[1-\omega]\to P_l[1]$ is a right socle morphism, and hence factors though $P_j[1-\omega_{u+1}]$ for $j\in N(l)$. Then $f[1](t_lg)=0$ by the definition of a long morphism. Hence, we may assume that C'' is right-proper. Then t_lg is a morphism from $cone\left(P_l^y[1-\omega]\xrightarrow{\varphi_{t_l^+C''}[1-\omega_{u+1}]}\bigoplus_{j\in V^{u+1}}P_j^{x_j}[1-\omega_{u+1}]\right)$ to $P_l[1]$ by Lemma 5. Note that the composition

$$\bigoplus_{j\in V^{u+1}}P_j^{x_j}[-\omega_{u+1}]\xrightarrow{\alpha_{t_l^+C''}[-\omega_{u+1}]}t_lC''[-1]\xrightarrow{t_lg[-1]}P_l\xrightarrow{f}t_lT_{\beta_u}[a+\omega]$$

is zero by the definition of a long morphism. Hence $f(t_lg)[-1]$ factors through some morphism $\theta:P_l^y[1-\omega]\to t_lT_{\beta_u}[a+\omega]$. Since $a+\omega-1<a$, P_l is not a direct summand of $(t_lT_{\beta_u})_{a+\omega-1}$. If $f[1](t_lg)\neq 0$, then $\theta\neq 0$ and there is some $j\in N(l)$ such that $P_j\xrightarrow{\theta[\omega+\omega_u-1]\iota[\omega_u]\gamma_{j,l}}t_lT_{\beta_u}[a+2\omega+\omega_u-1]$ is nonzero, where $\iota:P_l\to P_l^y$ is some split monomorphism. Note that by property (4) and the choice of a, the minimal nonzero degree of $t_lT_{\beta_u}$ is not smaller than $min(a,\sigma_{u-2}+1)$ by Lemma 2. On the other hand, if $f[1](t_lg)\neq 0$, then the minimal nonzero degree of $t_lT_{\beta_u}$ does not exceed $a+\omega+\omega_u-1<a$. Then we have

$$\sigma_{u-2}+1\le a+\omega+\omega_u-1\le\sigma_{u-1}+\omega+\omega_u-1=\sigma_{u-2}+\omega,$$

i.e. $\omega>0$, a contradiction. This shows that $f[1](t_lg)=0$, and hence $(t_l^{-1}f)[1]g=0$, as required. □

Let us now construct $\Delta_{p+1}\subseteq V_{p+1}$. We will do this in the following way. Set $\Delta_{p+1}^0=\varnothing$. Suppose that we have defined the set Δ_{p+1}^c. Choose a pair l,a with $l\in V_{p+1}\setminus\Delta_{p+1}^c$, $a\in\mathbb{Z}$ such that P_l is a direct summand of $\left(t^{-1}_{\Delta^c_{p+1}}T_{\beta_p}\right)_a$, and with a the minimal possible among all such pairs. We define $\Delta_{p+1}^{c+1}=\Delta_{p+1}^c\cup\{l\}$ and continue the process if $a\in[\sigma_{p-2}+1,\sigma_p]$. If either such an integer a does not exist or $a>\sigma_p$, then we terminate the process, defining $\Delta_{p+1}:=\Delta_{p+1}^c$. Now we are ready to prove the factorisation theorem.

Theorem 2. *The set Δ_{p+1} constructed as described above satisfies the following conditions:*

1. *$\alpha=s_{\Delta_0}\dots s_{\Delta_{p+1}}\beta_{p+1}$ for some $\beta_{p+1}\in B_\Gamma^+$ with $l(\beta_{p+1})=l(\alpha)-\sum\limits_{v=0}^{p+1}|\Delta_v|$*
2. *$\Delta_{p-2}\subseteq\Delta_p\subseteq N(\Delta_{p-1})$.*

Moreover, if $\chi_{p+1}(l)>0$ for any $l\in\Delta_{p+1}$, then the following conditions are satisfied as well:

3. *P_l is not a direct summand of $(T_{\beta_{p+1}})_{[\sigma_{p-2}+1,\sigma_p]}$ for any $l\in V_{p+1}$.*
4. *The minimal nonzero degree of $T_{\beta_{p+1}}$ is not smaller than $\sigma_{p-1}+1$.*
5. *For any $l\in\Delta_{p+1}$, P_l is a direct summand of $(t_lT_{\beta_{p+1}})_{[\sigma_{p-2}+1,\sigma_p]}$.*

Proof. The first condition can be obtained applying Lemma 1 to the words β_p, $s^{-1}_{\Delta^1_{p+1}}\beta_p,\dots,s^{-1}_{\Delta_{p+1}}\beta_p$, all of which are of strictly smaller length than α. Indeed, if $\Delta^{c+1}_{p+1}=\Delta^c_{p+1}\cup\{l\}$, then P_l is a direct summand of $\left(t^{-1}_{\Delta^c_{p+1}}T_{\beta_p}\right)_a$ for $a\le\sigma_p$ and it is sufficient to prove that the minimal nonzero degree of $t^{-1}_{\Delta^c_{p+1}}T_{\beta_p}$ is not smaller than $a+\omega_p\le\sigma_{p-1}+1$. If this is not the case, there exists some $b\le min(a-1,\sigma_{p-1})$ and $k\in\Gamma_0$ such that P_k is a direct summand of $(t^{-1}_{\Delta^c_{p+1}}T_{\beta_p})_b$. Observe that k cannot belong to V_p by the conditions (3),

(4) with $u = p$ and Lemma 2 and cannot belong to $V^{u+1} \setminus \Delta^c_{p+1}$ by the choice of a. On the other hand, for $k \in \Delta^c_{p+1}$, one has

$$\mathrm{Hom}^{b+\omega}(P_k, t^{-1}_{\Delta^c_{p+1}} T_{\beta_p}) \cong \mathrm{Hom}^{b+\omega}(t_{\Delta^c_{p+1}} P_k, T_{\beta_p}) \cong \mathrm{Hom}^{b+2\omega-1}(P_k, T_{\beta_p}) = 0,$$

because the minimal nonzero degree of T_{β_p} is not smaller than $\sigma_{p-2} + 1$ and $b + \omega - 1 \le \sigma_{p-1} + \omega - 1 \le \sigma_{p-2}$.

Let us now prove the second condition. Suppose first that there exists some $l \in \Delta_{p+1} \setminus N(\Delta_p)$. Then it is clear that $l \notin N(\Delta_u)$ for all $0 \le u \le p$, in particular, $l \ne i$. Hence we have $\alpha = s_{\Delta_0} \dots s_{\Delta_p} s_l s_l^{-1} \beta_p = s_l \gamma$ for some $\gamma \in B^+_\Gamma$ with $l(\gamma) = l(\alpha) - 1$. Then the assertion of Lemma 1 is valid for α, a contradiction.

Suppose now that $l \notin \Delta_{p+1}$ for some $l \in \Delta_{p-1}$. By construction, this means that P_l is not a direct summand of $\left(t^{-1}_{\Delta_{p+1}} T_{\beta_p}\right)_{[\sigma_{p-2}+1, \sigma_p]}$. Let π denote the split epimorphism $\bigoplus_{j\in\Delta_{p-1}} P_j^{\chi_{p-1}(j)} \twoheadrightarrow P_l^{\chi_{p-1}(l)}$. The octahedral axiom applied to the composition $\pi[1-\omega_{p+1}] \circ \beta_{C_p}[1]$ gives us the following diagram:

$$\begin{array}{ccccc}
\bigoplus_{k\in\Delta_p} P_k^{\chi_p(k)} & \longrightarrow & C' & \longrightarrow & \bigoplus_{j\in\Delta_{p-1}\setminus\{l\}} P_j^{\chi_{p-1}(j)}[1-\omega_{p+1}] \\
\| & & \downarrow & & \downarrow \\
\bigoplus_{k\in\Delta_p} P_k^{\chi_p(k)} & \longrightarrow & C_p & \longrightarrow & \bigoplus_{j\in\Delta_{p-1}} P_j^{\chi_{p-1}(j)}[1-\omega_{p+1}] \\
 & & \downarrow & & \downarrow{\scriptstyle \pi[1-\omega_{p+1}]} \\
 & & P_l^{\chi_{p-1}(l)}[1-\omega_{p+1}] & = & P_l^{\chi_{p-1}(l)}[1-\omega_{p+1}]
\end{array}$$

Let ψ denote the morphism from $P_l^{\chi_{p-1}(l)}[-\omega_{p+1}]$ to C' arising from this diagram. Since $\chi_{p-1}(l) > 0$ by our assumptions, C' is a nontrivial two-term subobject of C_p and Proposition 1 can be applied. For some $\omega_p + \sigma_{p-2} + 1 \le r \le \omega_p + \sigma_p$, we have a nonzero morphism $f : C' \to T_{\beta_p}[r]$ such that $f[\omega]g = 0$ for any right socle morphism $g \colon C'' \to C'[\omega]$. Since $\mathrm{Hom}^r(C_p, T_{\beta_p}) = 0$ by Proposition 1, the morphism $f\psi'$ is nonzero for some component $\psi' : P_l[-\omega_{p+1}] \to C'$ of the map ψ. We are going to prove that $t^{-1}_{\Delta_{p+1}}(f\psi')[\omega_{p+1}] \colon P_l \to t^{-1}_{\Delta_{p+1}} T_{\beta_p}[r + \omega_{p+1}]$ is long. Since $r - \omega_p \in [\sigma_{p-2}+1, \sigma_p]$, this contradicts the assumption that P_l is not a direct summand of $\left(t^{-1}_{\Delta_{p+1}} T_{\beta_p}\right)_{[\sigma_{p-2}+1,\sigma_p]}$.
Let us pick some $r \in N(l)$. Since $P_r[1-\omega_p]$ is a right-proper two-term object with $\alpha_{P_r[1-\omega_p]} = \varphi_{P_r[1-\omega_p]} = 0$ and $\beta_{P_r[1-\omega_p]} = id_{P_r[\omega_p]}$, we have

$$t^+_{\Delta_{p+1}} P_r[1-\omega_p] \cong cone\left(\bigoplus_{j\in N(r)\cap\Delta_{p+1}} P_j[-\omega_{p+1}] \to P_r\right)$$

by Lemma 5. Note that the morphism $\psi'[\omega](t^+_{\Delta_{p+1}}\gamma_{r,l}[1-\omega_p])$ is right socle. Indeed, as the diagram above shows, ψ' factors through $\bigoplus_{k\in\Delta_p} P_k^{\chi_p(k)}$ and the map

$$P_r \xrightarrow{\alpha_{t^+_{\Delta_{p+1}} P_r[1-\omega_p]}} t^+_{\Delta_{p+1}} P_r[1-\omega_p] \xrightarrow{t^+_{\Delta_{p+1}}\gamma_{r,l}[1-\omega_p]} P_l[\omega_p] \to \bigoplus_{k\in\Delta_p} P_k^{\chi_p(k)}[\omega]$$

is not a split monomorphism, because it factors through $P_l[\omega_p]$. Then $(f\psi')[\omega](t^+_{\Delta_{p+1}}\gamma_{r,l}[1-\omega_p]) = 0$, and hence $t^{-1}_{\Delta_{p+1}}(f\psi')[\omega]\gamma_{r,l} = 0$. Thus, $t^{-1}_{\Delta_{p+1}}(f\psi')[\omega_{p+1}]$ is a long morphism.

Suppose now that $\chi_{p+1}(l) > 0$ for all $l \in \Delta_{p+1}$. We define $C_{p+1} := t^-_{\Delta_{p+1}} C_p$. Then we have $\mathrm{Hom}^r(C_{p+1}, T_{\beta_{p+1}}) = 0$ for $r \le \omega_{p+1} + \sigma_{p+1}$ (see the proof of the first part of Proposition

1). Due to Lemma 10, we have

$$C_{p+1} = cone\left(\bigoplus_{j\in\Delta_p} P_j^{\chi_p(j)}[-\omega_p] \xrightarrow{\varphi_{p+1}} \bigoplus_{k\in\Delta_{p+1}} P_k^{\chi_{p+1}(k)}\right)$$

for some morphism φ_{p+1}. We prove the third property by contradiction. If P_l with $l\in\Delta_{p+1}$ is a direct summand of $(T_{\beta_{p+1}})_a$ for some $a\le\sigma_p$, then there exists a morphism from $\bigoplus_{k\in\Delta_{p+1}} P_k^{\chi_{p+1}(k)}$ to $T_{\beta_{p+1}}[a+\omega]$ annihilated by φ_{p+1}. This gives a nonzero morphism from C_{p+1} to $T_{\beta_{p+1}}[a+\omega]$, which is impossible since $a+\omega\le\omega+\sigma_p<\omega_{p+1}+\sigma_{p+1}$. Observe that the minimal nonzero degree of $T_{\beta_{p+1}}$ is not smaller than $\sigma_{p-2}+1$ and P_l with $l\in V_{p+1}\setminus\Delta_{p+1}$ cannot be a direct summand of $(T_{\beta_{p+1}})_{[\sigma_{p-2}+1,\sigma_p]}$ by the construction of Δ_{p+1}. This finishes the proof of the third property. The forth property follows from the third one, the property (3) with $u=p$ and Lemma 2.

It remains to prove the fifth property. Let $l\in\Delta_{p+1}$. There is some c such that $l\notin\Delta_{p+1}^c$ and $l\in\Delta_{p+1}^{c+1}$. Then P_l is a direct summand of $\left(t^{-1}_{\Delta^c_{p+1}}T_{\beta_p}\right)_a$ for some $a\in[\sigma_{p-2}+1,\sigma_p]$. Choose the minimal such a number a. It follows from the construction of Δ_{p+1}, the property (3) with $u=p$ and Lemma 2 that the minimal nonzero degree of $t^{-1}_{\Delta^c_{p+1}}T_{\beta_p}$ is not smaller than $min(a,\sigma_{p-1}+1)$ (see the beginning of this proof). Then P_l is a direct summand of $\left(t^{-1}_{\Delta_{p+1}\setminus\Delta^{c+1}_{p+1}}t^{-1}_{\Delta^c_{p+1}}T_{\beta_p}\right)_a=(t_lT_{\beta_{p+1}})_a$ by Lemma 2 and we are done. □

If $\chi_{p+1}(l)=0$ for some $l\in\Delta_{p+1}$, we can write $\alpha=s_{\Delta_0}\cdots s_{\Delta_p}s_l\widetilde{\beta}$ and get the required presentation for α, as we have announced earlier. In this case we terminate the process of factorisation (step I) and go to braiding (step II), which is described in the next section. Otherwise condition (6) is also satisfied for $u=p+1$. Then write $\alpha=s_{\Delta_0}s_{\Delta_1}\cdots s_{\Delta_p}s_{\Delta_{p+1}}\beta_{p+1}$ and continue the process, applying the arguments above to $u=p+2$ instead of $u=p+1$.

7. Mesh braiding in B^+_Γ

The goal of this section is to show that any word of the form $s_{\Delta_0}\cdots s_{\Delta_p}s_l$ as obtained in the previous section is left-divisible by some s_j in B^+_Γ with $j\ne i$. As we explained in Section 5, this finishes the proof of Lemma 1.

Let $\mathbb{Z}\Gamma$ be a directed graph whose vertices are pairs (n,j) for every $n\in\mathbb{Z}$, $j\in V^n$. The arrows of $\mathbb{Z}\Gamma$ are of the form $((n,j)\to(n+1,i))$ for every edge (i,j) of Γ. The graph $\mathbb{Z}\Gamma$ is essentially the stable translation quiver associated to Γ, but with vertices enumerated differently. It is convenient to draw $\mathbb{Z}\Gamma$ in such a way that vertices with the same first coordinate form one "vertical slice". Set $p_1(a)=n$, $p_2(a)=j$ for $a=(n,j)\in(\mathbb{Z}\Gamma)_0$. Let Θ_n denote the set $p_1^{-1}(n)=\{(n,j)\in(\mathbb{Z}\Gamma)_0\mid j\in V^n\}\subset(\mathbb{Z}\Gamma)_0$.

Now any word γ representing an element of B^+_Γ can be depicted as some set of vertices Λ_γ of $\mathbb{Z}\Gamma$ in the following way. First let us choose a presentation $\gamma=s_{\Sigma_0}\dots s_{\Sigma_p}$, where $\Sigma_0,\dots,\Sigma_p$ are such that $\Sigma_k\subseteq V^k$ and the convention $s_\varnothing=1$ is used. Now we can define $\Lambda_\gamma=\{(k,j)\mid k\ge0,j\in\Sigma_k\}\subset(\mathbb{Z}\Gamma)_0$. On the other hand, to any finite set $\Lambda\subset(\mathbb{Z}\Gamma)_0$ one can assign a word γ_Λ in B^+_Γ. More precisely, set $\gamma_\Lambda=\prod\limits_{k=-\infty}^{+\infty}s_{\Sigma_k}$, where $\Sigma_k=\Lambda\cap\Theta_k$. It is easy to see that $\gamma_{\Lambda_\gamma}=\gamma$.

Example. Let $\Gamma=D_4$ with vertices enumerated in such a way that the vertex of degree three is labeled by 2. Let $\gamma=s_2(s_1s_3s_4)s_2(s_1s_3s_4)s_2s_4$. Then we have the following set Λ_γ:

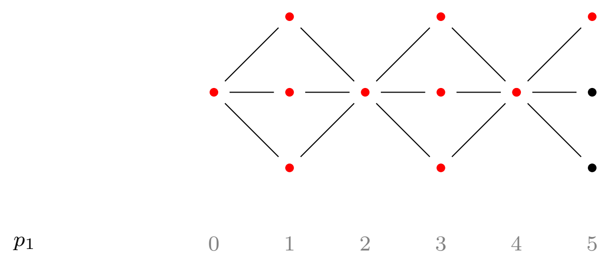

FIGURE 1

On the other hand, there are infinitely many sets Λ such that $\gamma_\Lambda = \gamma$. For instance, the following set of vertices of $(\mathbb{Z}\Gamma)_0$ also corresponds to the word γ:

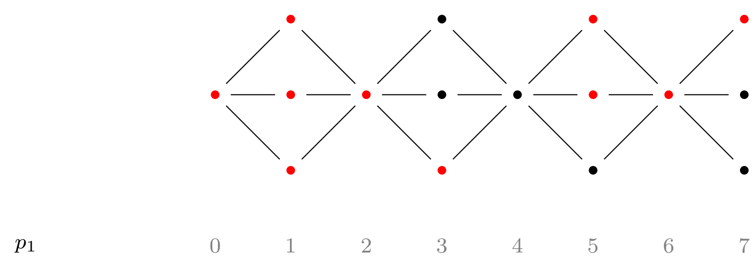

FIGURE 2

Now fix some finite $\Lambda \subset (\mathbb{Z}\Gamma)_0$.

Definition 7.1. Let $a, b \in \Lambda$. We say that there is a *generalised mesh* starting at a and ending at b if $p_2(a) = p_2(b) = t$, $p_1(a) < p_1(b)$ and $(k,t) \notin \Lambda$ for every k with $p_1(a) < k < p_1(b)$. In this case we set

$$mesh_\Lambda(a,b) = \{c \in \Lambda \mid p_2(c) \in N(t), p_1(a) < p_1(c) < p_1(b)\}.$$

We will sometimes refer to $mesh_\Lambda(a,b)$ as a generalised mesh as well.

For every $b \in \Lambda$, let $\tau_\Lambda(b)$ denote the element of Λ such that there is a generalised mesh starting at $\tau_\Lambda(b)$ and ending at b. If there is no such an element in Λ, we say that $\tau_\Lambda(b)$ is an "imaginary" vertex $(-\infty, p_2(b))$. In this case we also say that there is an infinite mesh starting at $\tau_\Lambda(b)$ and ending at b and set $mesh_\Lambda(\tau_\Lambda(b), b) = \{c \in \Lambda \mid p_2(c) \in N(p_2(b)), p_1(c) < p_1(b)\}$.

Now suppose there is a non-negative integer assigned to each element of Λ and each of the imaginary vertices at minus infinity, i.e. consider Λ together with a function $\theta\colon \Lambda \sqcup \big(\{-\infty\} \times \Gamma_0\big) \to \mathbb{Z}$.

Definition 7.2. We say that (Λ, θ) satisfies *mesh relations* if

$$\theta(b) + \theta\tau_\Lambda(b) = \sum_{c \in mesh_\Lambda(\tau_\Lambda(b), b)} \theta(c)$$

for every $b \in \Lambda$

The following statement follows immediately from the definitions:

Lemma 11. *Let γ be a word of the form $s_{\Delta_0} \dots s_{\Delta_p} s_l$ as described in the previous section. For every $(n,j) \in \Lambda_\gamma$ set $\theta(n,j) = \chi_n(j)$, $\theta(-\infty, i) = -1$, $\theta(-\infty, j) = 0$ for every $j \neq i$. Then (Λ_γ, θ) satisfies mesh relations.*

Example: The word $s_2(s_1s_3s_4)s_2(s_1s_3s_4)s_2s_4$ in $B^+_{D_4}$ as in the previous example is depicted below together with the function $\theta = \chi$.

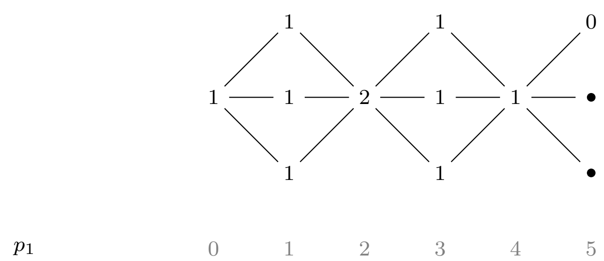

FIGURE 3

It is natural to ask which finite subsets of $(\mathbb{Z}\Gamma)_0$ correspond to words that are equal in the braid monoid B_Γ^+.

First let $a = (n, j) \in \Lambda$ be such that $b = (n + 2, j) \notin \Lambda$ (respectively $b = (n - 2, j) \notin \Lambda$) and $(n+1, k) \notin \Lambda$ (respectively $(n-1, k) \notin \Lambda$) for every $k \in N(j)$. Set $\overline{\Lambda} = \big(\Lambda \setminus \{a\}\big) \cup \{b\}$. In addition, let $\overline{\theta}(b) = \theta(a)$ and $\overline{\theta}|_{\overline{\Lambda}\setminus\{b\}} = \theta|_{\Lambda\setminus\{a\}}$. We refer to this procedure as commutation.

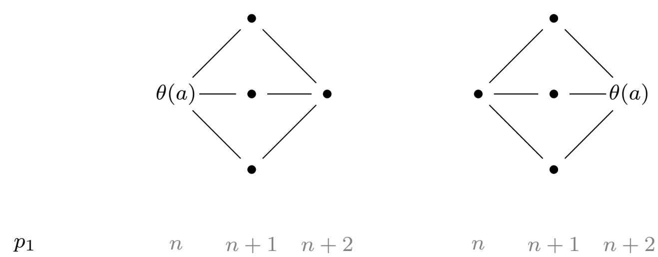

FIGURE 4

The following statement is clear:

Lemma 12. *Commutation does not change the corresponding word in B_Γ^+, i.e. $\gamma_\Lambda = \gamma_{\overline{\Lambda}}$. In addition, if (Λ, θ) satisfies mesh relations, then so does $(\overline{\Lambda}, \overline{\theta})$.*

Now braid relations in B_Γ^+ can be also depicted in terms of vertices of $\mathbb{Z}\Gamma$. We say that $a, b, c \in (\mathbb{Z}\Gamma)_0$ form *a braid* if $a = (n, j)$, $b = (n + 1, k)$, $c = (n + 2, j)$ for some $n \in \mathbb{Z}$ and some $j, k \in \Gamma_0$ such that $k \in N(j)$. Now let $a, b, c \in \Lambda$ as above form a braid and suppose in addition that $(n + 1, t) \notin \Lambda$ for any $t \in N(j) \setminus \{k\}$, $(n + 2, l) \notin \Lambda$ for any $l \in N(k) \setminus \{j\}$ and $d = (n + 3, k) \notin \Lambda$. Set $\overline{\Lambda} = (\Lambda \setminus \{a\}) \cup \{d\}$, $\overline{\theta}|_{\overline{\Lambda}\setminus\{b,c,d\}} = \theta|_{\Lambda\setminus\{a,b,c\}}$, $\overline{\theta}(b) = \theta(c), \overline{\theta}(c) = \theta(b), \overline{\theta}(d) = \theta(a)$. We refer to this procedure as braiding.

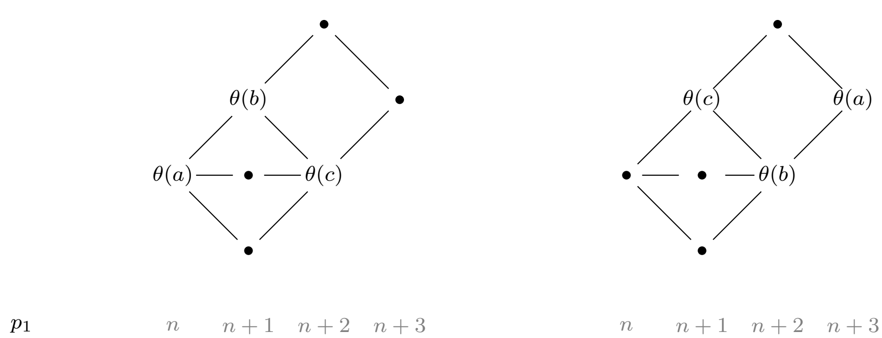

FIGURE 5

Lemma 13. *Braiding does not change the corresponding word in B_Γ^+, i.e. $\gamma_\Lambda = \gamma_{\overline{\Lambda}}$. Moreover, if (Λ, θ) satisfies mesh relations, then so does $(\overline{\Lambda}, \overline{\theta})$.*

Proof. First we establish that $\gamma_\Lambda = \gamma_{\overline{\Lambda}}$. It is sufficient to show that $s_{\Sigma_n} s_{\Sigma_{n+1}} s_{\Sigma_{n+2}} s_{\Sigma_{n+3}} = s_{\overline{\Sigma}_n} s_{\overline{\Sigma}_{n+1}} s_{\overline{\Sigma}_{n+2}} s_{\overline{\Sigma}_{n+3}}$ where $\Sigma_p = p_2\left(\Theta_p \cap \Lambda\right)$ and $\overline{\Sigma}_p = p_2\left(\Theta_p \cap \overline{\Lambda}\right)$. Observe that $\overline{\Sigma}_n = \Sigma_n \setminus \{j\}$, $\overline{\Sigma}_{n+1} = \Sigma_{n+1}$, $\overline{\Sigma}_{n+2} = \Sigma_{n+2}$ and $\overline{\Sigma}_{n+3} = \Sigma_{n+3} \cup \{k\}$. Hence

$$s_{\Sigma_n} s_{\Sigma_{n+1}} s_{\Sigma_{n+2}} s_{\Sigma_{n+3}} = s_{\overline{\Sigma}_n} s_j s_{\Sigma_{n+1}\setminus\{k\}} s_k s_j s_{\Sigma_{n+2}\setminus\{j\}} s_{\Sigma_{n+3}} =$$
$$s_{\overline{\Sigma}_n} s_{\Sigma_{n+1}\setminus\{k\}} s_j s_k s_j s_{\Sigma_{n+2}\setminus\{j\}} s_{\Sigma_{n+3}} = s_{\overline{\Sigma}_n} s_{\Sigma_{n+1}\setminus\{k\}} s_k s_j s_k s_{\Sigma_{n+2}\setminus\{j\}} s_{\Sigma_{n+3}} =$$
$$s_{\overline{\Sigma}_n} s_{\overline{\Sigma}_{n+1}} s_j s_{\Sigma_{n+2}\setminus\{j\}} s_k s_{\Sigma_{n+3}} = s_{\overline{\Sigma}_n} s_{\overline{\Sigma}_{n+1}} s_{\overline{\Sigma}_{n+2}} s_{\overline{\Sigma}_{n+3}}$$

Now we will show that braiding respects mesh relations. Since $\overline{\theta}$ differs from θ only on a, b, c and d, it is sufficient to check the relations only for generalised meshes these four vertices take part in. There are two types of such generalised meshes: those that have one of a, b, c and d as starting and/or ending vertices and those that do not.

1) Note that $\tau_{\overline{\Lambda}}(b) = \tau_\Lambda(b)$ and $mesh_{\overline{\Lambda}}(\tau_{\overline{\Lambda}}(b), b) = mesh_\Lambda(\tau_\Lambda(b), b) \setminus \{a\}$. We have

$$\theta\tau_\Lambda(b) + \theta(b) = \sum_{x \in mesh_\Lambda(\tau_\Lambda(b),b)} \theta(x) = \sum_{x \in mesh_{\overline{\Lambda}}(\tau_{\overline{\Lambda}}(b),b)} \theta(x) + \theta(a)$$

and $\theta(a) + \theta(c) = \theta(b)$, because a, b, c form a generlized mesh in Λ starting at a and ending at c. Then

$$\overline{\theta}\tau_{\overline{\Lambda}}(b) + \overline{\theta}(b) = \theta\tau_\Lambda(b) + \theta(c) = \sum_{x \in mesh_{\overline{\Lambda}}(\tau_{\overline{\Lambda}}(b),b)} \theta(x).$$

The remaining cases are either analogous to the one just discussed or obvious.

2) Now consider some generalised mesh $mesh_{\overline{\Lambda}}(x, y)$ in $\overline{\Lambda}$ such that $x, y \notin \{a, b, c, d\}$. If $b \in mesh_{\overline{\Lambda}}(x, y)$, then $d \in mesh_{\overline{\Lambda}}(x, y)$ and vise versa. The corresponding mesh relation remains valid after braiding, because $d \notin \Lambda$ and $\overline{\theta}(b) + \overline{\theta}(d) = \theta(c) + \theta(a) = \theta(b)$. If $c \in mesh_{\overline{\Lambda}}(x, y)$, then $a \in mesh_\Lambda(x, y)$ and vise versa. The corresponding mesh relation remains valid after braiding, because $\overline{\theta}(c) = \theta(b) = \theta(c) + \theta(a)$.

□

The main result of this section is the following theorem that in view of the facts we have already proved implies Lemma 1.

Theorem 3. *Let (Λ, θ) satisfying mesh relations be such that θ is non-negative on Λ, vanishing on precisely one vertex of Λ, $\theta(-\infty, i) = -1$ for some $i \in \Gamma_0$ and $\theta(-\infty, k) = 0$ for all $k \in \Gamma_0 \setminus \{i\}$. Then $\gamma = \gamma_\Lambda$ is left-divisible by some s_j in B^+_Γ with $j \neq i$.*

To prove this theorem we will need the following lemma.

Lemma 14. *Let (Λ, θ) be as in Theorem 3 and $a \in \Lambda$ such that $\theta(a) = 0$, $\tau_\Lambda(a) \neq -\infty$. Then there exists a sequence of commutations and braidings turning (Λ, θ) into some (Λ', θ') such that*

$$|\{x \in \Lambda' \mid p_1(x) \leq p_1(b)\}| < |\{x \in \Lambda \mid p_1(x) \leq p_1(a)\}|,$$

where b is the unique vertex of Λ' on which θ' vanishes.

Let us first show how Lemma 14 implies Theorem 3.

Proof of Theorem 3. It is sufficient to show that applying a sequence of commutations and braidings we can transform Λ into some Λ' such that $\Theta_n \cap \Lambda' \neq \{i\}$, where n is the smallest integer such that $\Theta_n \cap \Lambda' \neq \varnothing$. Note that commutation and braiding do not change the multiset $\{\theta(x)\}_{x\in\Lambda}$ of values of θ.

Applying Lemma 14 several times, one can obtain a pair (Λ'', θ'') satisfying mesh relations such that $\gamma_{\Lambda''} = \gamma_\Lambda$ and θ'' vanishes on (and only on) a vertex with the smallest first coordinate among all vertices of Λ''. Let $\theta''(x) = 0$. Then $\gamma_{\Lambda''} = \gamma_\Lambda$ is divisible by $s_{p_2(x)}$ on the left. It remains to ascertain that $p_2(x) \neq i$. Indeed, since x is a vertex with the smallest $p_1(x)$ among all vertices in Λ'', there is an infinite generalised mesh ending at x and starting at $(-\infty, j)$ for some $j \in \Gamma_0$. Since (Λ'', θ'') satisfies infinite mesh relations, $0 = \theta''(-\infty, j) + \theta''(x) = \theta''(-\infty, j)$. We immediately see that $j \neq i$, since $\theta''(-\infty, i) = -1$. □

Now we prove Lemma 14.

Proof of Lemma 14. Let $a_0 = a = (n, j)$. Since there is only one vertex of Λ on which θ vanishes and $\tau_\Lambda(a_0) \in \Lambda$, $\theta\tau_\Lambda(a_0) + \theta(a_0) = \theta\tau_\Lambda(a_0) > 0$. Hence $mesh_\Lambda(\tau_\Lambda(a_0), a_0) \neq \varnothing$. Let $a_1 = (n_1, t) \in mesh_\Lambda(\tau_\Lambda(a_0), a_0)$ be any vertex with the smallest first coordinate among vertices of $mesh_\Lambda(\tau_\Lambda(a_0), a_0)$. Then, possibly applying several commutations, one can assume that $\tau_\Lambda(a_0) = (n_1 - 1, j)$. If $|mesh_\Lambda(\tau_\Lambda(a_0), a_0)| = 1$, then, possibly applying several commutations, one can assume that $\tau_\Lambda(a_0), a_1, a_0$ form a braid. Then we can apply braiding to the triple $(\tau_\Lambda(a_0), a_1, a_0)$ after some commutations as described below and we are done. Suppose now that $q_0 = |mesh_\Lambda(\tau_\Lambda(a_0), a_0)| > 1$.

Note that $q_1 = |mesh_\Lambda(\tau_\Lambda(a_1), a_1)| > 0$, since $\tau_\Lambda(a_0) \in mesh_\Lambda(\tau_\Lambda(a_1), a_1)$. Suppose that $q_1 > 1$. Let $a_2 = (n_2, p) \in mesh_\Lambda(\tau_\Lambda(a_1), a_1)$ be any vertex with the smallest first coordinate among vertices of $mesh_\Lambda(\tau_\Lambda(a_1), a_1) \setminus \{\tau_\Lambda(a_0)\}$. Then, possibly applying several commutations, one can assume that $\tau_\Lambda(a_1) = (n_2 - 1, t)$.

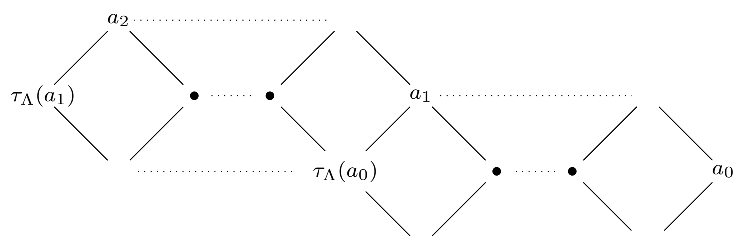

Figure 6

Continue in the same fashion to obtain a sequence of elements $a_0, \ldots, a_k$ of Λ such that $\tau_\Lambda(a_m) \in mesh_\Lambda(\tau_\Lambda(a_{m+1}), a_{m+1})$, $a_{m+1} \in mesh_\Lambda(\tau_\Lambda(a_m), a_m) \setminus \{\tau_\Lambda(a_{m-1})\}$ for every $m = 0, \ldots, k-1$ and $\tau_\Lambda(a_k) \neq -\infty$. Let us set $q_m := |mesh_\Lambda(\tau_\Lambda(a_k), a_k)|$ for $m = 0, \ldots, k$. We claim that if the sequence $a_0, \ldots, a_k$ is maximal with respect to inclusion among sequences satisfying this property, then $q_k = 1$. Indeed, if $q_k > 1$, then there is a vertex a_{k+1} with the smallest first coordinate among vertices of $mesh_\Lambda(\tau_\Lambda(a_k), a_k) \setminus \{\tau_\Lambda(a_{k-1})\}$. Because the sequence we consider is maximal, $\tau_\Lambda(a_{k+1}) = (-\infty, p_2(a_{k+1}))$. Since $\theta(-\infty, p_2(a_{k+1}))$ is either 0 or -1, we have $\theta(a_{k+1}) \geq \theta\tau_\Lambda(a_k)$. Then $\theta(a_k) \geq \theta\tau_\Lambda(a_{k-1})$, since

$$\theta(a_k) + \theta\tau_\Lambda(a_k) = \theta(a_{k+1}) + \theta\tau_\Lambda(a_{k-1}) + \sum_{y \in mesh_\Lambda(\tau_\Lambda(a_k), a_k) \setminus \{a_{k+1}, \tau_\Lambda(a_{k-1})\}} \theta(y)$$

Continuing in the same way we get $\theta(a_1) \geq \theta\tau_\Lambda(a_0)$. On the other hand, $\theta\tau_\Lambda(a_0) = \theta\tau_\Lambda(a_0) + \theta(a_0) > \theta(a_1)$, because $q_0 > 1$, a contradiction.

Take a sequence $a_0, \ldots, a_k$ maximal with respect to inclusion and satisfying the properties described above with the smallest corresponding sequence of integers $(q_0, \ldots, q_k)$ in the lexicographic order. Denote such a $(q_0, \ldots, q_k)$ by $seq(\Lambda)$. If $seq(\Lambda) = (1)$, there is nothing to prove, as we remarked earlier. Now suppose that the statement is known for all (Λ', θ') with $seq(\Lambda') < seq(\Lambda)$. We will show that there exists a sequence of commutations and braidings that turns (Λ, θ) into $(\widetilde{\Lambda}, \widetilde{\theta})$ such that $seq(\widetilde{\Lambda}) < seq(\Lambda)$ in the lexicographic order. Observe that commutations do not change $seq(\Lambda)$.

Since $q_k = 1$, $mesh_\Lambda(\tau_\Lambda(a_k), a_k) = \{\tau_\Lambda(a_{k-1})\}$ and, possibly applying several commutations, one can assume that $\tau_\Lambda(a_k), \tau_\Lambda(a_{k-1}), a_k$ form a braid. Let $\tau_\Lambda(a_k) = (n, j)$, $\tau_\Lambda(a_{k-1}) = (n+1, l)$, $a_k = (n+2, j)$. We already have $(n+1, t) \notin \Lambda$ for every $t \in N(j) \setminus \{l\}$.

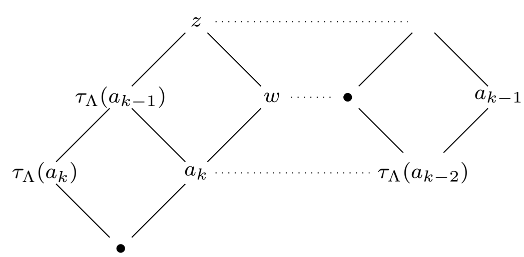

FIGURE 7. Λ

To perform a braiding on $\tau_\Lambda(a_k), \tau_\Lambda(a_{k-1}), a_k$, we also need to have $(n+3, j) \notin \Lambda$ and $(n+2, v) \notin \Lambda$ for every $v \in N(l) \setminus \{j\}$. To this end we first apply a sequence of commutations shifting all vertices x of Λ with $p_1(x) \geq n+2$, $x \neq a_k$, to the right. More precisely, the resulting set $\overline{\Lambda}$ is $\Lambda_1 \sqcup \Lambda_2$, where $\Lambda_1 = \{x \in \Lambda \mid p_1(x) \leq n+1\} \cup \{a_k\}$, $\Lambda_2 = \{(q+2, r) \mid (q, r) \in \Lambda \setminus \{a_k\}, q \geq n+2\}$ and $\overline{\theta}|_{\Lambda_1} = \theta|_{\Lambda_1}, \overline{\theta}(q+2, r) = \theta(q, r)$ for $(q+2, r) \in \Lambda_2$. It is clear that such a transformation can be obtained consequently applying commutations to all vertices of $\Lambda \setminus \Lambda_1$ starting with those with the largest first coordinate.

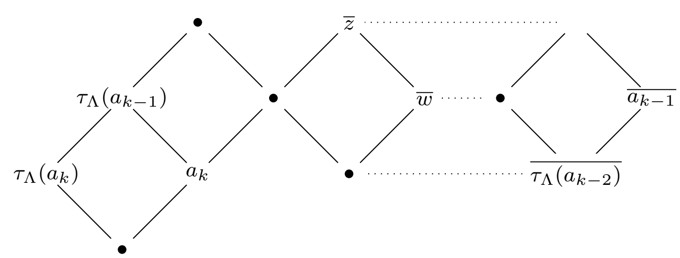

FIGURE 8. $\overline{\Lambda}$

Now let $(\widetilde{\Lambda}, \widetilde{\theta})$ be a pair obtained by braiding $(\tau_\Lambda(a_k), \tau_\Lambda(a_{k-1}), a_k)$ in $\overline{\Lambda}$. Clearly, $q_0, \ldots, q_{k-2}$ remain unchanged. However, the mesh ending at a_{k-1} now starts at $\tau_{\widetilde{\Lambda}}(a_{k-1}) = (n+3, l)$, and hence obviously does not contain the vertex $a_k = (n+2, j)$. We see that $|mesh_{\widetilde{\Lambda}}(\tau_{\widetilde{\Lambda}}(a_{k-1}), a_{k-1})| = q_{k-1} - 1$. Since any sequence that starts with $q_0, \ldots, q_{k-1} - 1$ is smaller in the lexicographic order than the sequence $(q_1, \ldots, q_{k-1}, q_k)$, we have $seq(\widetilde{\Lambda}) < seq(\Lambda)$.

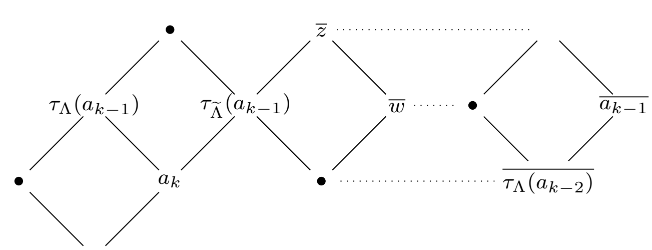

FIGURE 9. $\widetilde{\Lambda}$

$\square$

Example. Consider Λ_γ as in the previous example.

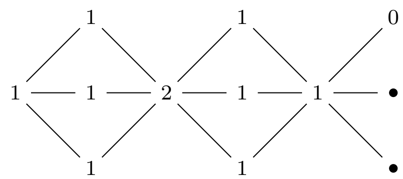

FIGURE 10. The first application of Lemma 16 requires just one braiding.

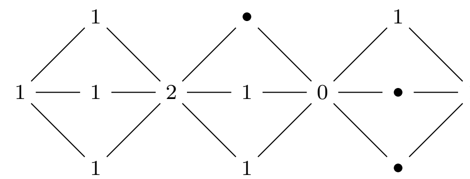

FIGURE 11. Now the sequence of meshes starting at 0 is $(2, 1)$.

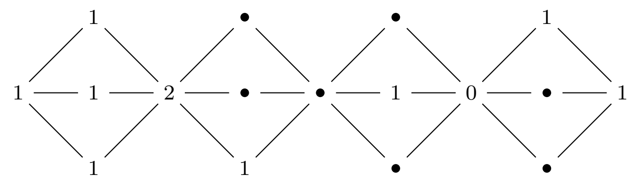

FIGURE 12. The second application of Lemma 14 begins with a sequence of commutations.

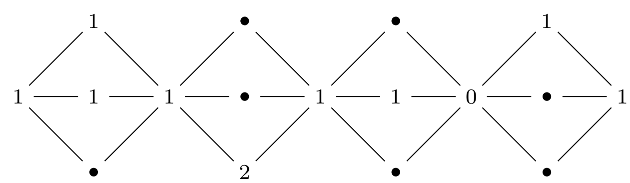

FIGURE 13. Now three vertices can be braided and the sequence of meshes starting at 0 is (1) again.

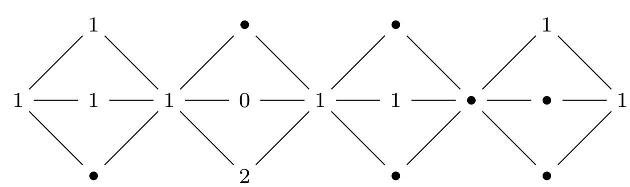

FIGURE 14. Braiding the remaining three vertices in the sequence of meshes finishes the second application of Lemma 14.

Remark. One can prove the assertion converse to Theorem 3. Namely, if (Λ, θ) satisfying mesh relations is such that θ is non-negative on Λ, $\theta(-\infty, i) = -1$ for some $i \in \Gamma_0$, $\theta(-\infty, k) = 0$ for all $k \in \Gamma_0 \setminus \{i\}$ and γ_Λ is left-divisible by some s_j in B_Γ^+ with $j \neq i$, then θ vanishes on some vertex of Λ. Indeed, in this case (Λ, θ) can be transformed using commutations and braidings into $(\widetilde{\Lambda}, \widetilde{\theta})$ such that there is a vertex $x = (n, j) \in \widetilde{\Lambda}$ with $\tau_{\widetilde{\Lambda}}(x) = (-\infty, j)$ and $mesh_{\widetilde{\Lambda}}\big(\tau_{\widetilde{\Lambda}}(x), x\big) = \varnothing$. Then the mesh relation corresponding to x implies $\widetilde{\theta}(x) = 0$ and the assertion follows from the fact that commutations and braidings do not change the multiset of values of θ.

8. An application to derived Picard groups

For this section we assume that k is algebraically closed. All modules in this section are assumed to be left unless explicitly stated otherwise. Let $\mathfrak{D}_\Gamma = D^b(\text{mod} - \Lambda_\Gamma)$, the bounded derived category of finitely generated Λ_Γ-modules, where Γ is a simply laced Dynkin diagram and Λ_Γ is the trivial extension algebra of the Γ diagram with alternating orientation. In other words, $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$, where Q_Γ is one of the following quivers:

Figure 15. $\Gamma = A_n$

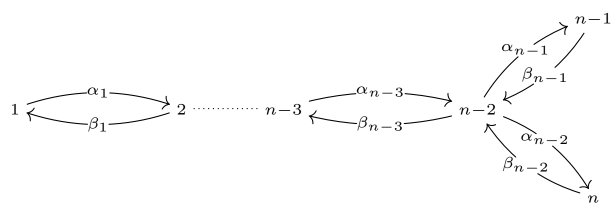

Figure 16. $\Gamma = D_n$

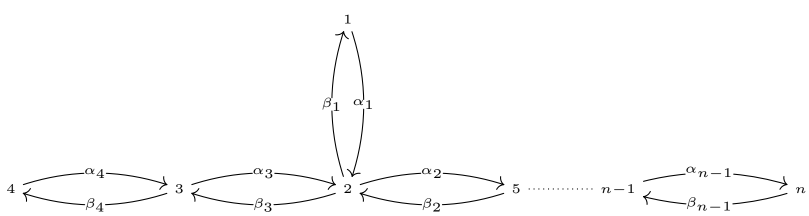

Figure 17. $\Gamma = E_n$ $(n = 6, 7, 8)$

The ideal I_Γ of kQ_Γ is generated by all paths of length greater or equal to 2, except for the paths of length 2 starting and ending at the same vertex, and the differences of any two paths of length two starting at the same vertex.

The k-algebra Λ_Γ is finite-dimensional and symmetric. Let e_i be the idempotent associated with the vertex i of the quiver Q_Γ. Denote by $P_i = \Lambda_\Gamma e_i$ the corresponding indecomposable projective module. Note that P_i is a 0-spherical object of $\mathfrak{D}_\Gamma$. Indeed, the first condition is satisfied automatically. Since P_i is projective, $\text{Ext}^m_{\Lambda_\Gamma}(P_i, -)$ vanishes for every $m \neq 0$, and hence the second condition simply means that $\text{End}_{\Lambda_\Gamma}(P_i) \cong \text{k}[t]/(t^2)$. Finally, since Λ_Γ is symmetric, the last condition is satisfied automatically as well due to an isomorphism of functors $\text{Hom}(P_i, -) \cong \text{Hom}(-, P_i)^*$. To sum up, it is now clear that $\{P_i\}_{i \in (Q_\Gamma)_0}$ is a Γ-configuration of 0-spherical objects of $\mathfrak{D}_\Gamma$.

Remark. In this less general setting we could have defined spherical twists in the following way:

Definition 8.1. (Grant, [7]) The spherical twist functor along P_i is

$$t_{P_i} \colon \mathfrak{D}_\Gamma \to \mathfrak{D}_\Gamma$$

$$t_{P_i}(-) = cone(\Lambda_\Gamma e_i \otimes_{\text{k}} e_i \Lambda_\Gamma \xrightarrow{m} \Lambda_\Gamma) \otimes^L_{\Lambda_\Gamma} -$$

where m is the multiplication map $m(ae_i \otimes e_i b) = ab$ of Λ_Γ-Λ_Γ bimodules.

It is easy to check that on objects it coincides with our original definition, which is

$$t_{P_i}(X) = cone(P_i \otimes_{\mathrm{k}} \mathrm{Hom}(P_i, X) \xrightarrow{ev} X)$$

with ev the obvious evaluation map. Indeed, since $cone(\Lambda_\Gamma e_i \otimes_{\mathrm{k}} e_i \Lambda_\Gamma \xrightarrow{m} \Lambda_\Gamma)$ is a two-term complex of right-projective bimodules,

$$cone(\Lambda_\Gamma e_i \otimes_{\mathrm{k}} e_i \Lambda_\Gamma \xrightarrow{m} \Lambda_\Gamma) \otimes^L_{\Lambda_\Gamma} X \cong cone\big((\Lambda_\Gamma e_i \otimes_{\mathrm{k}} e_i \Lambda_\Gamma \otimes_{\Lambda_\Gamma} X) \to (\Lambda_\Gamma \otimes_{\Lambda_\Gamma} X)\big)$$
$$\cong cone\big(P_i \otimes_{\mathrm{k}} (e_i \Lambda_\Gamma \otimes_{\Lambda_\Gamma} X) \to X\big) \cong cone\big(P_i \otimes_{\mathrm{k}} \mathrm{Hom}(P_i, X) \xrightarrow{ev} X\big).$$

Definition 8.2. (R. Rouquier, A. Zimmermann, [16]) Let A be a finite-dimensional algebra. The *derived Picard group* $TrPic(A)$ of A is the group of isomorphism classes of objects of the derived category of $A \otimes A^{\mathrm{op}}$-modules, invertible under $\otimes^L_A$. Equivalently, $TrPic(A)$ is the group of standard autoequivalences of $D^b(\mathrm{mod}-A)$ modulo natural isomorphisms.

It is known (see, for example, [21]) that in the case $\Gamma = A_n$ the subgroup of $TrPic(\Lambda_\Gamma)$ generated by the spherical twist functors t_{P_i} is isomorphic to the braid group B_{A_n} on $(n+1)$ strands. The next corollary of Theorem 1 generalises this result.

Corollary 1. *The subgroup of the derived Picard group $TrPic(\Lambda_\Gamma)$ of Λ_Γ generated by the spherical twist functors t_{P_i} is isomorphic to the generalised braid group B_Γ of type Γ.*

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