THE HARTOGS EXTENSION PHENOMENON IN SPHERICAL VARIETIES

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Abstract. We consider the phenomenon of removal of compact singularities for holomorphic functions in complex non-compact spherical varieties and its relation to $H^1_c(X,\mathcal{O})$ and a cone in the lattice of characters of a maximal algebraic torus T of the reductive group G.

1. Introduction

The classical Hartogs extension theorem states that for every domain $D \subset \mathbb{C}^n$, $(n > 1)$ and a compact set $K \subset D$ such that $D \setminus K$ is connected, any holomorphic function f on $D \setminus K$ extends holomorphically to D. A natural question arises if this is true for complex analytic spaces.

Definition 1. *We say that a connected complex space X admits the Hartogs phenomenon if for any domain $D \subset X$ and a compact set $K \subset D$ such that $D \setminus K$ is connected, the restriction homomorphism*

$$H^0(D,\mathcal{O}) \to H^0(D \setminus K,\mathcal{O})$$

is an isomorphism.

In this or a similar formulation this phenomenon has been extensively studied in many situations, including Stein manifolds and spaces, $(n-1)$-complete normal complex spaces and so on [1, 2, 3, 4, 5, 9, 16, 23, 26, 29].

Our goal is to study the Hartogs phenomenon in spherical G-varieties in terms of Luna-Vust theory. An example of such algebraic variety are toric varieties, for which the Luna-Vust theory is simple [10, 24]. For some classes of toric varieties the Hartogs phenomenon was first studied by M. A. Marciniak [20, 21] and for arbitrary normal toric variety was studied in [13].

Let G/H be a homogeneous space. Denote by $\mathcal{V}$ the cone of G-invariant geometric valuations of the rational functions field $\mathbb{C}(G/H)$, by $\mathcal{D}^B$ we denote the set of B-stable divisors of the homogeneous space G/H, and by $\mathfrak{X}_+(T)$ the semigroup of dominant weights. Let X_Σ be a noncompact spherical variety with the homogeneous space G/H and with the colored fan Σ such that $E := \mathcal{V} \setminus |\Sigma|$ is connected. Let $P := \mathfrak{X}_+(T) \cap \overline{E}^\vee$, here $\overline{E}^\vee$ is the dual cone of the set $\overline{E}$, and $P^B := \left\{\lambda \in \mathfrak{X}_+(T) \mid \langle \chi(v_L), \lambda\rangle \geq 0 \forall L \in \mathcal{D}^B\right\}$. The main result of this paper is the following

Main Theorem. *Let X_Σ be a noncompact spherical variety with the homogeneous space G/H and with the colored fan Σ such that $E := \mathcal{V} \setminus |\Sigma|$ is connected. The following conditions are equivalent*

1. *X_Σ admits the Hartogs phenomenon,*
2. *$H^1_c(X_\Sigma,\mathcal{O}) = 0$,*

(3) $P \cap P^B = 0$.

The equivalence of 1 and 2 is proved by using the long exact sequence of sheaf cohomology groups that relates the groups $H^p_K(X, \mathcal{O})$ and $H^p(X, \mathcal{O})$ for a compact set $K \subset X$, the excision lemma, and the isomorphism $\varinjlim_{V_n} H^1_{V_n}(X, \mathcal{O}) \cong H^1_c(X, \mathcal{O})$, where the limit is taken over a compact exhaustion. For the equivalence of 2 and 3 we use the description of global sections of the structure sheaf of the formal completion with respect to infinite divisor for some spherical compactification $X_{\Sigma'}$ of X_Σ. Since any infinite divisor is a compact algebraic subset, the GAGA principle allows us treat this problem with methods of algebraic geometry.

In particular, in the case of toric varieties we have $\mathcal{V} = \mathbb{Q}^n$, $\mathfrak{X}_+(T) = \mathfrak{X}(T)$, $P = \mathfrak{X}(T) \cap \overline{E}^\vee$, $P^B = \mathfrak{X}(T)$. Then a toric variety X_Σ admits the Hartogs phenomenon if and only if $\overline{E}^\vee = 0$.

2. Reductive groups

In this section we briefly review the necessary elements of the representation theory for algebraic reductive groups. See, for example, [18, 25, 31].

Let G be a connected linear algebraic group over $\mathbb{C}$ and $\mathfrak{g}$ be the Lie algebra of G.

Definition 2. A finite-dimensional G-module M is called rational if the representation map $R\colon G \to GL(M)$ is a homomorphism of algebraic groups. Generally, a rational G-module is a union of finite-dimensional rational submodules.

Definition 3.

(1) A connected algebraic group G over $\mathbb{C}$ is called reductive if any rational G-module is semisimple.
(2) A Borel subgroup B of an algebraic group G is a maximal Zariski closed and connected solvable algebraic subgroup.

Example 1. Examples of reductive groups: $GL_n, SL_n, SO_n, (\mathbb{C}^*)^n$. In GL_n (or SL_n) the subgroup of invertible upper triangular matrices is a Borel subgroup.

Example 2. Let X be a algebraic G-variety, G be a reductive group and $\mathcal{F}$ be an algebraic G-sheaf on X (i.e. a quasi-coherent sheaf (in Zariski topology on X) with G-linearization [33, Def. C.2]). Then by Theorem [33, Th. C.3] the cohomology groups $H^i(X, \mathcal{F})$ are rational G-modules.

Let $T \subset B$ be a maximal torus, $\mathfrak{t} \subset \mathfrak{b}$ is the corresponding Lie algebras, and $\mathfrak{X}(T)$ be a character group of torus T. We have a root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

with respect to the adjoint representation $Ad\colon T \to GL(\mathfrak{g}), Ad_x y = xyx^{-1}$, where $R \subset \mathfrak{X}(T)$ denotes the root system of G with respect to T, and

$$\mathfrak{g}_\alpha := \{y \in \mathfrak{g} \mid xyx^{-1} = \alpha(x)y\}$$

denotes the root subspace corresponding to a weight α. Notice that

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha,$$

where $R_+ \subset R$ is the set of positive roots.

Let $\mathfrak{X}^*(T)$ be a 1-parametric group of torus T. We have a non-degenerate $\mathbb{Z}$-pairing $\langle \cdot, \cdot \rangle$ between $\mathbb{Z}$-modules $\mathfrak{X}(T)$ and $\mathfrak{X}^*(T)$ which is defined from $\chi(l(t)) = t^{\langle \chi, l \rangle}$. The $\mathbb{Z}$-pairing can be extended to $\mathbb{Q}$-pairing between $\mathbb{Z}$-vector spaces $\mathfrak{X}(T) \otimes \mathbb{Q}$ and $\mathfrak{X}^*(T) \otimes \mathbb{Q}$.

The Weyl group $W := N_G(T)/T$ acts on $\mathfrak{X}(T)$ as follows: let w be a point of $N_G(T)$ representing the class $\widehat{w} \in W$, then $\widehat{w}.\chi(t) := \chi(w^{-1}tw)$. In the space $\mathfrak{X}(T) \otimes \mathbb{Q}$ there exists W-invariant inner product $(\cdot, \cdot)$. For a root $\alpha \in R$ we define a co-root $\alpha^\vee \in \mathfrak{X}^*(T)$ as $\langle x, \alpha^\vee \rangle := \frac{2(\alpha, x)}{(\alpha, \alpha)}$.

Definition 4.

1. The choice of the Borel subgroup defines the dominant Weyl chamber
$$C_+ := \{x \in \mathfrak{X}(T) \otimes \mathbb{Q} \mid \langle x, \alpha^\vee \rangle \geq 0 \forall \alpha \in R_+\}$$
2. Let $\mathfrak{X}_+(T) := \mathfrak{X}(T) \cap C_+$ denote the semigroup of dominant weights.
3. Let M be a G-module. For a character $\lambda \in \mathfrak{X}(T)$ define the space of B-eigenvectors with weight λ of the following
$$M_\lambda^{(B)} := \{v \in M \mid b.v = \lambda(b)v\}$$

There is a one-to-one correspondence between the set $\mathfrak{X}_+(T)$ of dominant weights and the set $\widehat{G}$ of isomorphism classes of simple G-modules. Namely, every $\lambda \in \mathfrak{X}_+(T)$ corresponds to simple G-module $(\mathbb{C}[G]_\lambda^{(B)})^*$, and every $V \in \widehat{G}$ corresponds to $\lambda \in \mathfrak{X}(T)$ such that B acts on the line V^U precisely by the character $\lambda \in \mathfrak{X}(T)$, where U is the unipotent radical of B.

So, any rational G-module M is isomorphic to a direct sum of simple G-modules $(\mathbb{C}[G]_\lambda^{(B)})^*$ (possibly with multiplicities). The multiplicity of $(\mathbb{C}[G]_\lambda^{(B)})^*$ in M is equal to $\dim(M_\lambda^{(B)})$ and we have an isomorphism of G-modules:

$$\bigoplus_{\lambda \in \mathfrak{X}_+(T)} M_\lambda^{(B)} \otimes (\mathbb{C}[G]_\lambda^{(B)})^* \cong M. \tag{1}$$

3. The Luna-Vust theory for spherical varieties

In this section we briefly review the necessary elements of the Luna-Vust theory for spherical varieties. See, for example, [14, 27, 33].

Let G be a connected reductive algebraic group over $\mathbb{C}$ and B be a Borel subgroup.

Definition 5.

1. A homogeneous space G/H is called spherical if it contains a Zariski open orbit of the action of the Borel subgroup B.
2. Let G/H be a spherical homogeneous space. A normal algebraic G-variety X with G-equivariant open embedding $G/H \hookrightarrow X$ is called a spherical variety.
3. A B-chart of the spherical variety X is a B-stable affine open subset of X.

Example 3. If G is the algebraic torus $T = (\mathbb{C}^*)^n$ and $H = \{e\}$, then $G = B = T$, $G/H = T$ and X is a normal toric variety.

A spherical homogeneous space G/H corresponds the following data:

- The weight lattice $\Lambda := \mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ which is a sublattice in $\mathfrak{X}(T)$.
- The set $\mathcal{V}$ of all G-invariant geometric valuations of the rational functions field $\mathbb{C}(G/H)$. This is a convex polyhedral cone in $\mathbb{Q}$-vector space $Hom(\Lambda, \mathbb{Q})$), and is called the valuation cone of G/H.
- The set $\mathcal{D}^B$ of all B-stable but not G-stable simple divisors of G/H. The set is called colors.
- The map χ which is for every valuation $v\colon \mathbb{C}(G/H) \to \mathbb{Q}$ sets $\chi(v) \in Hom(\Lambda, \mathbb{Q})$.

Every G-orbit $Y \subset X$ corresponds to the colored data $(\mathcal{V}_Y, \mathcal{D}^B_Y)$, where $\mathcal{V}_Y \subset \mathcal{V}$, $\mathcal{D}^B_Y \subset \mathcal{D}^B$ are the subsets corresponding to all prime divisors on X containing Y. Since any spherical variety has finitely many G-orbits, we have finitely many colored data.

Given the colored data $(\mathcal{V}_Y, \mathcal{D}^B_Y)$ we obtain B-chart

$$X_Y := Spec\Big(\bigcap_{D \in \mathcal{V}_Y \cup \mathcal{D}^B_Y} \mathcal{O}_{v_D} \cap \mathbb{C}(G/H)_B\Big),$$

here $\mathbb{C}(G/H)_B$ is the subalgebra of rational functions with B-stable divisor of poles on X, and v_D is the valuation associated with the divisor D.

The spherical variety has an open cover $X = \bigcup\limits_Y G.X_Y$ by simple spherical varieties $G.X_Y := \{g.x \mid g \in G, x \in X_Y\}$.

Instead of colored data, it is more convenient to consider a pair $(\mathcal{C}_Y, \mathcal{D}^B_Y)$, where $\mathcal{C}_Y := cone\langle \chi(v) \mid v \in \mathcal{V}_Y \cup \mathcal{D}^B_Y \rangle \subset Hom(\Lambda, \mathbb{Q})$. This leads to the concept of a colored cone.

Definition 6. A colored cone in $Hom(\Lambda, \mathbb{Q})$ is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{R} \subset \mathcal{D}^B$, $\mathcal{C}$ is a strictly convex cone generated by $\chi(\mathcal{R})$ and finitely many vectors from $\mathcal{V}$, such that $0 \notin \chi(\mathcal{R})$ and $int(\mathcal{C}) \cap \mathcal{V} \neq \emptyset$.

Therefore, given a G-orbit $Y \subset X$ we construct a colored cone $(\mathcal{C}_Y, \mathcal{D}^B_Y)$ in $Hom(\Lambda, \mathbb{Q})$.

Let Y_1, Y_2 are G-orbits in X and $Y_1 \subset \overline{Y_2}$, then $(\mathcal{C}_{Y_2}, \mathcal{D}^B_{Y_2})$ is a face of $(\mathcal{C}_{Y_1}, \mathcal{D}^B_{Y_1})$ in the following sense:

Definition 7. A colored face of $(\mathcal{C}, \mathcal{R})$ is a colored cone $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R} \cap \chi^{-1}(\mathcal{C}')$.

Let Y be the smallest G-orbit in X such that $Y_1 \subset \overline{Y}, Y_2 \subset \overline{Y}$. Then colored cones $(\mathcal{C}_{Y_1}, \mathcal{D}^B_{Y_1})$ and $(\mathcal{C}_{Y_2}, \mathcal{D}^B_{Y_2})$ intersect by the common colored face $(\mathcal{C}_Y, \mathcal{D}^B_Y)$. The collection $\{(\mathcal{C}_Y, \mathcal{D}^B_Y)\}$ of colored cones for all G-orbits $Y \subset X$ forms a fan in the following sense:

Definition 8. A colored fan is a finite set $\Sigma = \{(\mathcal{C}_i, \mathcal{R}_i)\}$ of colored cones which is closed under passing to a colored face and such that different cones intersect by faces inside $\mathcal{V}$.

The support of $\Sigma = \{(\mathcal{C}_i, \mathcal{R}_i)\}$ is $|\Sigma| := \bigcup\limits_i \mathcal{C}_i \cap \mathcal{V}$.

Spherical varieties with a given spherical homogeneous space G/H are classified by colored fans.

Theorem 1. (1) *G-orbits are in a bijection with colored cones.*
(2) *Spherical varieties are in a bijection with colored fans.*

(3) *If* Y_1, Y_2 *are two* G*-orbits, then* $Y_1 \subset \overline{Y_2}$ *if and only if* $(\mathcal{C}_{Y_2}, \mathcal{D}^B_{Y_2})$ *is a face of* $(\mathcal{C}_{Y_1}, \mathcal{D}^B_{Y_1})$.
(4) *Affine spherical varieties are in a bijection with colored cones of the form* $(\mathcal{C}, \mathcal{D}^B)$

Criterion of completeness: a shperical variety X_Σ with a colored fan Σ is complete (compact) if and only if $|\Sigma| = \mathcal{V}$.

Any noncompact spherical G-variety admits a G-equivariant embedding in a compact spherical variety by Sumihiro's theorem [32]:

Theorem 2. *Suppose a connected linear algebraic group* G *acts algebraically on an irreducible normal algebraic variety* X *over* $\mathbb{C}$*. Then* X *can be embedded as a* G*-invariant open subset of a complete irreducible normal algebraic variety* X' *on which* G *acts algebraically.*

For compact spherical G-varieties we have the vanishing cohomology theorem for some analytic sheaves [7, 33]:

Theorem 3. *Let* X *be a compact spherical* G*-variety and* D *is a nef-divisor. Then* $H^i(X, \mathcal{O}(D)) = 0, \forall i > 0$.

In particular, this is true for the sheaf of holomorphic functions on X.

4. The proof of the main theorem

4.1. **The equivalence of 1 and 2.** We prove this statement for a more general class of complex spaces. In this section we always deal with reducible, connected, paracompact complex spaces.

Let $(X, \mathcal{O}_X)$ be a noncompact complex space such that there exists a compactification X' of X with the following properties:

(1) X' is a reducible, connected, paracompact complex space
(2) $Z := X' \setminus X$ is a connected closed set in X'
(3) $H^1(X', \mathcal{O}_{X'}) = 0$

Example 4. Any spherical variety X_Σ (considered as complex space) with a colored fan Σ such that $\mathcal{V} \setminus |\Sigma|$ is connected and satisfies properties 1-3 by Theorems 2, 3.

Proposition 1. $(X, \mathcal{O}_X)$ *admits the Hartogs phenomenon if and only if* $H^1_c(X, \mathcal{O}_X) = 0$

Proof. Let X' be a compactification of X with properties 1-3 above. Since the set Z has a system of connected neighbourhoods $\{U_n\}_{n=1}^\infty$ with $U_n \Subset U_{n-1}$ ([8, Prop. 12.9]), the complex space X has an exhaustion by compact sets $\{V_n\}$ such that $X \setminus V_n$ is connected.

Now, let $H^1_c(X, \mathcal{O}_X) = 0$ and $K \subset X$ be a compact set. For the pair (X, K) we have an exact sequence [5, p. 11]:

(2)
$$0 \longrightarrow H^0_K(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X \setminus K, \mathcal{O}_X) \longrightarrow H^1_K(X, \mathcal{O}_X) \longrightarrow \cdots$$

By the excision property [5, c. 11] we have the canonical isomorphism for any n:

$$H^1_{V_n}(X', \mathcal{O}_{X'}) \cong H^1_{V_n}(X, \mathcal{O}_X).$$

For the pair (X', V_n) we obtain a short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(X' \setminus V_n, \mathcal{O}_{X'}) \longrightarrow H^1_{V_n}(X, \mathcal{O}_X) \longrightarrow 0\ . \tag{3}$$

Using homological algebra machinery, we get the following commutative diagram

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
0 \to \mathbb{C} & \longrightarrow & H^0(X' \setminus V_{n-1}, \mathcal{O}_{X'}) & \longrightarrow & H^1_{V_{n-1}}(X, \mathcal{O}_X) \to 0 \\
{\scriptstyle id_{n-1}}\downarrow & & {\scriptstyle p_{n-1}}\downarrow & & {\scriptstyle q_{n-1}}\downarrow \\
0 \to \mathbb{C} & \longrightarrow & H^0(X' \setminus V_n, \mathcal{O}_{X'}) & \longrightarrow & H^1_{V_n}(X, \mathcal{O}_X) \to 0 \\
{\scriptstyle id_n}\downarrow & & {\scriptstyle p_n}\downarrow & & {\scriptstyle q_n}\downarrow \\
0 \to \mathbb{C} & \longrightarrow & H^0(X' \setminus V_{n+1}, \mathcal{O}_{X'}) & \longrightarrow & H^1_{V_{n+1}}(X, \mathcal{O}_X) \to 0 \\
\downarrow & & \downarrow & & \downarrow
\end{array}\ . \tag{4}$$

Here id_n is the identity homomorphism, p_n is the restriction homomorphism induced by the embedding $X' \setminus V_{n+1} \subset X' \setminus V_n$, and q_n is the homomorphism induced by the embedding $V_n \subset V_{n+1}$.

Let X be a Hausdorff topological space and $\mathcal{F}$ be a sheaf of Abelian group on X. Then there is a canonical isomorphism [5, p. 11]:

$$\varinjlim_{V_n} H^p_{V_n}(X, \mathcal{F}) \cong H^p_c(X, \mathcal{F}).$$

Columns of the diagramm (4) form direct systems of Abelian groups. Since the funtor $\varinjlim_{V_n}$ is exact and $\varinjlim_{V_n} H^1_{V_n}(X, \mathcal{O}_X) \cong H^1_c(X, \mathcal{O}_X) = 0$, taking direct limits of the diagram, we obtain

$$\varinjlim_{V_n} H^0(X_{\Sigma'} \setminus V_n, \mathcal{O}) \cong \mathbb{C}.$$

By the uniqueness theorem for holomorphic functions we obtain

$$H^0(X' \setminus V_n, \mathcal{O}_{X'}) = \mathbb{C},$$

and

$$H^1_{V_n}(X, \mathcal{O}_X) = 0$$

for any n.

Let K be a compact set in a domain D such that $D \setminus K$ is connected. There is $n \in \mathbb{N}$ such that $K \subset V_n$. This implies $H^0(X' \setminus K, \mathcal{O}_{X'}) = \mathbb{C}$. By the excision property [5, p. 11] and by (2) for the pair (X', K) we obtain $H^1_K(D, \mathcal{O}_X) = 0$. Also by (2) but for the pair (D, K) we see that the restriction homomorphism

$$H^0(D, \mathcal{O}_X) \to H^0(D \setminus K, \mathcal{O}_X)$$

is an isomorphism.

Conversely, suppose that the Hartogs phenomenon holds in $(X, \mathcal{O}_X)$, i.e. for any domain $D \subset X$ and for any compact set $K \subset D$ such that $D \setminus K$ is connected, the restriction map $H^0(D, \mathcal{O}) \to H^0(D \setminus K, \mathcal{O})$ is an isomorphism.

We have a commutative diagram where all morphisms are restriction maps:

$$\begin{array}{ccc} H^0(X',\mathcal{O}_{X'}) & \longrightarrow & H^0(X'\setminus K,\mathcal{O}_{X'}) \\ \downarrow & & \downarrow \\ H^0(D,\mathcal{O}_X) & \longrightarrow & H^0(D\setminus K,\mathcal{O}_X) \end{array} . \tag{5}$$

By assumption, the lower arrow of the diagram is an isomorphism. But $H^0(X',\mathcal{O}_{X'}) = \mathbb{C}$ and $H^1(X',\mathcal{O}_{X'}) = 0$, then by (2) for pair (X', K) and by the excision property we obtain

$$H^1_K(X,\mathcal{O}_X) = 0.$$

Now we take an exhaustion of X by compact sets $\{V_n\}$ such that each $X \setminus V_n$ is connected. Replacing K by V_n in above, we obtain

$$H^1_c(X,\mathcal{O}_X) \cong \varinjlim_{V_n} H^1_{V_n}(X,\mathcal{O}_X) = 0.$$

□

It can be seen from the proof, that for complex spaces satisfying properties 1-3 the Hartogs phenomenon does not depend on a domain D that contains the compact set K, thus we have:

Corollary 1. *A complex space $(X,\mathcal{O}_X)$ with properties 1-3 admits the Hartogs phenomenon if and only if for any compact set $K \subset X$ such that $X\setminus K$ is connected, the restriction homomorphism*

$$H^0(X,\mathcal{O}_X) \to H^0(X\setminus K,\mathcal{O}_X)$$

is an isomorphism

For complex manifolds the '⇐'-statement in Proposition 1 may be proved differently [17, Th. 2.3.2]. The condition $H^1_c(X,\mathcal{O}) = 0$ is equivalent to solvability of a $\bar\partial$-problem in the class C^∞_c, then the holomorphic extension of $f \in H^0(D\setminus K,\mathcal{O})$ is the function $F := (1-\chi)f - v$, where $\chi \in C^\infty_c(D)$ and is identically 1 in a neighborhood of K, and the function $v \in C^\infty_c(D)$ is a solution of the equation $\bar\partial v = -f\bar\partial\chi$. This technique may be used for some normal complex spaces. For example, in [9] the problem of holomorphic continuation in a complex space X is reduced to a problem of holomorphic continuation in a complex manifold M, which is obtained by resolution of singularities of X and $H^1_c(M,\mathcal{O}) = 0$.

4.2. **The equivalence 2 and 3.**

Proof. Let $X_{\Sigma'}$ be a spherical compactification of X_Σ and let $Z := X_{\Sigma'} \setminus X_\Sigma$ be the support of a G-stable divisor $D = \sum_{i=1}^{k} D_i$ with the ideal sheaf $\mathcal{O}(-D)$.

Let $X_\Omega \subset X_{\Sigma'}$ be a spherical subvariety with the colored fan

$$\Omega := \{(\mathcal{C},\mathcal{R}) \in \Sigma' \mid int(\mathcal{C}) \subset |\Sigma'| \setminus int(|\Sigma|)\},$$

where $int(\mathcal{C})$ is a relative interior of the cone $\mathcal{C}$. Denote by $\Omega(1)$ the set of primitive generators of 1-dimensional cones in Ω. Clearly, $\chi(v_{D_i}) \in \Omega(1)$ for all $i = 1,\cdots,k$.

By Theorem 3 and by the following exact sequence [6, Section 2, §10]:

$$0 \longrightarrow H^0_c(U,\mathcal{A}) \longrightarrow H^0_c(X,\mathcal{A}) \longrightarrow H^0_c(F,\mathcal{A}) \longrightarrow H^1_c(U,\mathcal{A}) \longrightarrow \cdots,$$

where $\mathcal{F}$ is a sheaf of Abelian group on a topological space X, $U \subset X$ is an open set, $Z := X \setminus U$, we obtain

$$H^1_c(X_\Sigma, \mathcal{O}) \cong H^0(Z, \mathcal{O})/\mathbb{C}.$$

Recall that we have $H^0(Z, \mathcal{O}) = \varinjlim_{U \supset Z} H^0(U, \mathcal{O})$, where the inductive limit is taken over all neighborhoods of Z.

Consider the ringed space $(Z, \widehat{\mathcal{O}}|_Z)$ which is a formal completion of X_Ω with respect to Z. Here for any open set $U \subset X_\Omega$ we have [5, Ch. VI],[15, Ch. II. §4]:

$$H^0(U, \widehat{\mathcal{O}}) := \varprojlim_m H^0(U, \mathcal{O}/\mathcal{O}(-mD)).$$

Since we have an injection $\mathcal{O} \hookrightarrow \widehat{\mathcal{O}}$ induced by the residue maps $\mathcal{O} \to \mathcal{O}/\mathcal{O}(-mD)$ [15, Prop. 1.8], the restricton of the functor on Z is exact and $supp(\widehat{\mathcal{O}}) = Z$, we obtain

$$H^0(Z, \mathcal{O}) \hookrightarrow H^0(X_\Omega, \widehat{\mathcal{O}}) = \varprojlim_m H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD)).$$

Since Z is locally defined by polynomials, $(Z, (\mathcal{O}/\mathcal{O}(-mD))|_Z)$ is an analytification of the corresponding complete scheme of a finite type over $\mathbb{C}$. By the GAGA principle we may assume that sheaves $\mathcal{O}, \mathcal{O}(-mD), \forall m$ are algebraic.

Since G is a reductive group and sheaves $\mathcal{O}/\mathcal{O}(-mD)$ are with G-linearization [18], the G-modules $H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))$ are rational (see example 2).

For given m we have the isomorphism

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD)) \cong \bigoplus_{\lambda \in \mathfrak{X}_+(T)} H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \otimes (\mathbb{C}[G]^{(B)}_\lambda)^*,$$

where $H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda$ is the space of global B-eigensection with weight λ.

Denote

$$P := \big\{\lambda \in \mathfrak{X}_+(T) \mid \langle v, \lambda\rangle \geq 0 \forall v \in \Omega(1)\big\}$$

and

$$P^B = \big\{\lambda \in \mathfrak{X}_+(T) \mid \langle \chi(v_L), \lambda\rangle \geq 0 \forall L \in \mathcal{D}^B\big\}.$$

Note that $P = \overline{E}^\vee \cap \mathfrak{X}_+(T)$, where $E = \mathcal{V} \setminus |\Sigma|$.

Lemma 1.

(1) *For given m, if $\lambda \in \mathfrak{X}_+(T) \setminus P$, then*

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda = 0$$

(2) *For given m, if $\lambda \in \mathfrak{X}_+(T) \setminus P^B$, then*

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda = 0$$

(3) *For given m, if $\lambda \in \bigcap_{i=1}^{k} \{\langle \chi(v_{D_i}), \lambda\rangle \geq m\}$, then*

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda = 0$$

(4) *For any m we have $H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^B = \mathbb{C}$*

Proof.

(1) Let $\lambda \in \mathfrak{X}_+(T) \setminus P$, and assume $H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \neq 0$. There exists a nonzero section $f \in H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda$ with the weight λ and there exists $v \in \Omega(1)$ such that $\langle v, \lambda\rangle < 0$. The point v corresponds to a B-stable divisor L in X_Ω, i.e. $v = \chi(v_L)$, hence $v_L(f) = \langle v, \lambda\rangle < 0$.

Now, let X_L be a B-chart in X_Ω such that $X_L \cap L \neq \emptyset$. We have the restriction homomorphism

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \to (\mathbb{C}[X_L]/I^m_{Z\cap X_L})^{(B)}_\lambda,$$

where $I_{Z\cap X_L}$ is the ideal of $Z \cap X_L$. Then $f|_{X_L} \in (\mathbb{C}[X_L]/I^m_{Z\cap X_L})^{(B)}_\lambda$, i.e. $\langle v, \lambda\rangle \geq 0$. This is a contradiction; the proof of the statement is complete.

(2) Let $\lambda \in \mathfrak{X}_+(T) \setminus P^B$, and assume $H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \neq 0$. There exists a nonzero section $f \in H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda$ with the weight λ and there exists a B-stable but not G-stable divisor L in G/H such that $v_L(f) = \langle \chi(v_L), \lambda\rangle < 0$. Further on we proceed as in case 1 above.

(3) Let $\lambda \in \bigcap\limits_{i=1}^{k} \{\langle \chi(v_{D_i}), \lambda\rangle \geq m\}$ and consider a section $f \in H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda$. Let X_{D_i} be a B-chart in X_Ω such that $X_{D_i} \cap D_i \neq \emptyset$. We have the restriction homomorphism

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \to (\mathbb{C}[X_{D_i}]/I^m_{D_i\cap X_{D_i}})^{(B)}_\lambda = 0,$$

where $I_{D_i\cap X_{D_i}}$ is the ideal of $D_i \cap X_{D_i}$.

Therefore $f|_{X_{D_i}} = 0$. Since X_{D_i} is a dense open set in $G.X_{D_i}$ we have $f|_{G.X_{D_i}} = 0$. But $Z \cap \bigcup\limits_i G.X_{D_i}$ is a dense set in Z, hence $f = 0$.

(4) Consider a section $f \in H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^B$. Let X_{D_i} be a B-chart in X_Ω such that $X_{D_i} \cap D_i \neq \emptyset$. We have the restriction homomorphism

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^B \to (\mathbb{C}[X_{D_i}]/I^m_{D_i\cap X_{D_i}})^B = \mathbb{C}.$$

Therefore $f|_{X_{D_i}} = c_i = \mathit{const}$ for all i. Since X_{D_i} is a dense open set in $G.X_{D_i}$, we get $f|_{G.X_{D_i}} = c_i = \mathit{const}$. But Z is a connected set and $Z \cap \bigcup\limits_i G.X_{D_i}$ is dense in Z, hence $c_i = c_j$ for all $i \neq j$ and $f = \mathit{const}$.

□

Denote $C_m := \{\lambda \in \mathfrak{X}_+(T) \mid \langle \chi(v_{D_i}), \lambda\rangle \geq m\}$. By Lemma 1 we obtain

$$H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD)) \cong \bigoplus_{\lambda \in P\cap P^B \setminus C_m} H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \otimes (\mathbb{C}[G]^{(B)}_\lambda)^*$$

Note that $C_{m+1} \subset C_m$ and $P \cap P^B \setminus C_m$ is a finite set, for all m. We have the homomorphism

$$\bigoplus_{\lambda \in P\cap P^B \setminus C_{m+1}} H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-(m+1)D))^{(B)}_\lambda \otimes (\mathbb{C}[G]^{(B)}_\lambda)^* \to$$

$$\to \bigoplus_{\lambda \in P\cap P^B \setminus C_m} H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^{(B)}_\lambda \otimes (\mathbb{C}[G]^{(B)}_\lambda)^*$$

which is defined as follows:

$$\sum_{\lambda\in A} f_\lambda \otimes v_\lambda \to \sum_{\lambda \in A'} r_m(f_\lambda) \otimes v_\lambda,$$

where

$$A \subset P \cap P^B \setminus C_{m+1}, A' = A \setminus C_m,$$

and

$$r_m \colon H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-(m+1)D)) \to H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))$$

is the canonical homomorphism.

Taking the inverse limit over m we obtain:

$$H^0(X_\Omega, \widehat{\mathcal{O}}) \cong \widehat{\bigoplus_{\lambda \in P \cap P^B}} H^0(X_\Omega, \widehat{\mathcal{O}})^{(B)}_\lambda \otimes (\mathbb{C}[G]^{(B)}_\lambda)^*$$

Here on the right side we have the algebra of formal series of the form $\sum\limits_{\lambda \in A} f_\lambda \otimes v_\lambda$, where $A \subset P \cap P^B$.

Assume that $P \cap P^B = 0$, then $H^0(X_\Omega, \widehat{\mathcal{O}}) \cong H^0(X_\Omega, \widehat{\mathcal{O}})^B \otimes (\mathbb{C}[G]^B)^*$. But $(\mathbb{C}[G/B])^* = \mathbb{C}$ and by Lemma 1 we have

$$H^0(X_\Omega, \widehat{\mathcal{O}})^B = \varprojlim_m H^0(X_\Omega, \mathcal{O}/\mathcal{O}(-mD))^B = \mathbb{C}$$

Then $H^0(X_\Omega, \widehat{\mathcal{O}}) = \mathbb{C}$, and $H^0(Z, \mathcal{O}) = \mathbb{C}$.

Now, let $P \cap P^B \neq 0$. Note that there exists an injection

$$H^0(X_\Omega, \mathcal{O}) \hookrightarrow H^0(X_\Omega, \widehat{\mathcal{O}})$$

and the G-module $H^0(X_\Omega, \mathcal{O})$ is rational also (recall that we consider the sheaf of regular functions). Therefore we have the decomposition:

$$H^0(X_\Omega, \mathcal{O}) \cong \bigoplus_{\lambda \in P \cap P^B} H^0(X_\Omega, \mathcal{O})^{(B)}_\lambda \otimes (\mathbb{C}[G]^{(B)}_\lambda)^*$$

Consider a nonzero weight $\lambda_0 \in P \cap P^B$ and denote by Bo the B-stable affine open set in G/H. Since

$$H^0(X_\Omega, \mathcal{O})^{(B)}_{\lambda_0} = H^0(G/H, \mathcal{O})^{(B)}_{\lambda_0} = \mathbb{C}(Bo)^{(B)}_{\lambda_0} \neq \mathbb{C},$$

there exists a nonconstant global regular function $f_{\lambda_0} \in H^0(X_\Omega, \mathcal{O})^{(B)}_{\lambda_0}$. Since the function f_{λ_0} can be considered as a holomorphic function in a neighbourhood of Z, $H^0(Z, \mathcal{O}) \neq \mathbb{C}$. □

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