Surfaces Containing Two Parabolas Through Each Point

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Abstract

We prove (modulo some technical assumptions) that each surface in $\mathbb{R}^3$ containing two arcs of parabolas with axes parallel to Oz through each point has a parametrization $\left(\frac{P(u,v)}{R(u,v)}, \frac{Q(u,v)}{R(u,v)}, \frac{Z(u,v)}{R^2(u,v)}\right)$ for some $P, Q, R, Z \in \mathbb{R}[u,v]$ such that P, Q, R have degree at most 1 in u and v, and Z has degree at most 2 in u and v. The proof is based on the observation that one can consider a parabola with vertical axis as an isotropic circle; this allows us to use methods of the recent work by R. Krasauskas and M. Skopenkov in which all surfaces containing two Euclidean circles through each point are classified. Such approach also allows us to find a similar parametrization for the surfaces in $\mathbb{R}^3$ containing two arbitrary isotropic circles through each point (modulo the same technical assumptions). Finally, we get some results concerning the top view (the projection along the Oz axis) of the surfaces in question.

1 Introduction and background

Various problems about classification of surfaces in $\mathbb{R}^3$ containing two special curves through each point have rich history and natural applications in architecture. Probably the most classical examples are a one-sheeted hyperboloid and Shukhov's architectural structures based on it (although the fact that one-sheeted hyperboloid is a ruled surface was discovered by Wren back in 1669). Another example is the problem of finding all surfaces containing several circles through each point. Although the problem has been studied since XIXth century, it was completely solved only in the recent paper [12] (a brief but very informative history of the question can be also found there).

The aim of this paper is finding all surfaces in $\mathbb{R}^3$ containing two parabolas with vertical axes through each point (Theorem 1). The new idea is to consider parabolas with vertical axes as circles in isotropic geometry and apply isotropic stereographic projection (see [5], [10]). This approach allows us to solve the problem using quite standard methods similar to [12]. In passing, we find all surfaces containing two arbitrary isotropic circles through each point (Theorem 2).

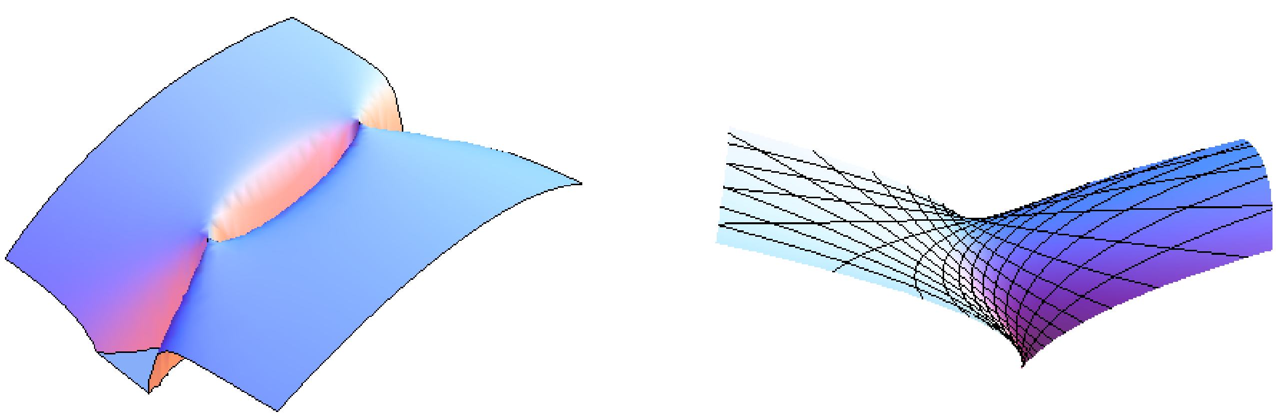

Figure 1: examples of surfaces containing two parabolas through each point.

An interesting feature of isotropic geometry is the concept of "top view". Projecting the isotropic circles through each point of the surface to a horizontal plane, we obtain a configuration of lines and (Euclidean)

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Table 1: a summary of known results on surfaces containing two special curves through each point in various dimensions and geometries.

Problem	Assumptions	Readily-calculable solution	Author(s)	Reference
CCE3	—	yes	Skopenkov-Krasauskas	[12, Theorem 1.1]
CCE4	two circles through each point are noncospheric, finitely many circles through each point	no		[12, Corollary 5.1]
	infinitely many circles through each point	yes	Kollár	[4, Theorems 3, 8 and Propositions 11, 12] (stated also in [12, Remark 5.3])
	the surface is algebraic	no	Lubbes	[7, Theorem 1]
CCE9, CCI9	—	yes (there are no such surfaces, i. e., any surface in question is contained in $\mathbb{R}^8$)	Schicho	remark after [12, Theorem 4.1]
CCI3	many	yes	see Theorem 2 below	
CLE3	—	yes (such surfaces are quadrics only)	Nilov-Skopenkov	[9, Theorem 1.1]
CLE4, CLE5	the surface is algebraic	no	Lubbes	[7, Corollary 2]
CLE6		yes (there are no such surfaces, i. e., any surface in question is contained in $\mathbb{R}^5$)		
CLS3	the surface is algebraic and contains *exactly* one circle and one line through each point	no		[6, Corollary 3b]

circles in the plane. For the case of surfaces containing two parabolas through each point we give a complete classification of the resulting line configurations (Corollary 1) and for the case of surfaces containing two isotropic circles through each point we give a partial classification (Corollary 2). It seems to be not difficult to obtain a complete classification for the latter case using the methods of §§4.6 – 4.8 but this is beyond the scope of the paper (see also Conjecture 4).

Our proofs use Schicho's theorem (Theorem 4) which parametrizes all surfaces containing two conics through each point. Although Schicho's theorem is powerful, it requires many technical assumptions, which leads to quite long statements of our main theorems. We give a detailed plan of the proof in §3.

The problem of finding all surfaces containing two circles through each point is interesting also in higher dimensions and other geometries (for instance, classification of surfaces in $\mathbb{R}^3$ was actually deduced from the classification of surfaces in $\mathbb{R}^4$ in [12]). A closely related problem is finding all surfaces containing a line and a circle through each point. All known results on these problems are summarized in Table 1.

How to read Table 1. Each row depicts a problem of finding all surfaces containing two special curves through each point in space of given dimension with given geometry. The columns mean the following:

- *Problem.* A label in this column abbreviates a particular problem as follows:
  - the first two letters refer to curves passing through each point of the surface: "CC" means "two circles" and "CL" means "a line and a circle";
  - the third letter refers to geometry: "E" means "Euclidean", "I" means "isotropic", "S" means "spherical"; similar problems in pseudoeuclidean and hyperbolic geometries are unsolved; note that for the "CC" problems the spherical case is equivalent to the Euclidean one, but this is not the case for the "CL" ones;
  - the number refers to the dimension of the ambient space; we assume that the surface is not contained in a subspace of lower dimension.

  For example, the label "CCE3" denotes a problem of finding all surfaces in $\mathbb{R}^3$ which are not contained in $\mathbb{R}^2$ and which contain two Euclidean circles through each point.
- *Assumptions.* These are the assumptions which the surface must satisfy. The analyticity of the surface and the analytical dependence of the curves through each point on the point are the default ones.
- *Readily-calculable solution.* We say that a problem has a *readily-calculable solution* if there is an explicit *parametrization* of the set of surfaces in question, i. e. an explicit surjection from a p-dimensional subset in $\mathbb{R}^p$ (or a finite union of such subsets with different p) to the set of surfaces in question. Such parametrization is the best possible answer to the problem. See [2, Remark 1.2] for a detailed discussion.

The problems in Table 2 have not yet been solved in any sense but seem to be accessible by known methods.

Table 2: Remarks on some unsolved problems.

Problem	Remark
CCE8	The problem might possibly be solved via octonions (just as CCE4 was approached by quaternions, see [12]).
CCI5	Using the method of the paper the problem can be reduced to the classification of Pythagorean 6-tuples from [12, Theorem 1.3].

The 3-dimensional pseudoeuclidean case may be interesting due to relation to *circle patterns* (see [10] for the isotropic case).

Organization of the paper. In §2 we state the main results and in §3 we state a number of auxiliary theorems. Theorems 1 and 2 are proved in §§4.3-4.4 respectively. Corollary 1 is proved in §4.5. Corollary 2 is much more complicated; it is proved in §4.8 using a small theory developed in §§4.6-4.8. Finally, in §5 we state several open problems.

2 Statements

Consider 3-dimensional Euclidean space $\mathbb{R}^3$ with the Cartesian coordinates (x, y, z). Each line parallel to Oz is called *vertical*.

An *isotropic circle* in $\mathbb{R}^3$ is either a parabola with vertical axis or an ellipse whose orthogonal projection to the plane Oxy is a circle. The latter projection is called *the top view* of the isotropic circle. An *isotropic sphere* in $\mathbb{R}^3$ is either a paraboloid of revolution with vertical axis or a circular cylinder with vertical axis. A *pencil of lines* in a plane is either a set of all lines passing through some fixed point or a set of all lines parallel to some fixed line. A *cyclic* is a subset of the plane given by the equation

$$a(x^2 + y^2)^2 + (x^2 + y^2)(bx + cy) + Q(x, y) = 0, \tag{1}$$

where $a, b, c \in \mathbb{R}$ and $Q \in \mathbb{R}[x, y]$ has degree at most 2 (and a, b, c, Q do not vanish simultaneously).

An *analytic surface* in $\mathbb{R}^3$ is the image of an injective real analytic map of a planar domain into $\mathbb{R}^3$ with nondegenerate differential at each point. An isotropic circular arc *analytically dependent* on a point is a real analytic map of an analytic surface into the real analytic variety of all isotropic circular arcs in $\mathbb{R}^3$.

Denote by $\mathbb{R}_{ij}$ (respectively, $\mathbb{C}_{ij}$) the set of all polynomials $F \in \mathbb{R}[u, v]$ (respectively, $\mathbb{C}[u, v]$) such that $\deg_u F \le i$ and $\deg_v F \le j$.

Theorem 1. *Assume that through each point of an analytic surface in $\mathbb{R}^3$ one can draw two transversal parabolic arcs with vertical axes fully contained in the surface (and analytically depending on the point). Assume that these two arcs lie neither in the same isotropic sphere nor in the same plane. Assume that through each point in some dense subset of the surface one can draw only finitely many (not nested) arcs of isotropic circles and line segments contained in the surface. Then the surface (possibly besides a one-dimensional subset) has a parametrization*

$$\Phi(u, v) = \left(\frac{P(u, v)}{R(u, v)}, \frac{Q(u, v)}{R(u, v)}, \frac{Z(u, v)}{R^2(u, v)} \right) \tag{2}$$

for some $P, Q, R \in \mathbb{R}_{11}$ and $Z \in \mathbb{R}_{22}$, where $R \ne 0$, such that the parabolic arcs are the curves $u = \text{const}$ and $v = \text{const}$.

Corollary 1. *For each surface satisfying the assumptions of Theorem 1 the top views of the two parabolas through each point are either tangent to one conic or lie in a union of two pencils of lines (see Fig. 2).*

Figure 2: to Corollary 1: the top views of parabolas through each point.

Remark 1. Surfaces described in Theorem 1 admit the following geometric construction. Fix a set U of lines in the plane Oxy which are tangent to one conic. Take three parabolas p_1, p_2, p_3 with the top views being lines lying in U. For any other line $l \in U$ take the plane π such that $\pi \perp Oxy$ and $\pi \cap Oxy = l$. Draw a parabola with vertical axis through the points $p_1 \cap \pi, p_2 \cap \pi, p_3 \cap \pi$. Then one can see by direct computation that the surface formed by all such parabolas for all $l \in U$ contains two parabolas through each point.

A similar construction works if U is a union of two pencils of lines. In this case the top views of p_1, p_2, p_3 must lie in one pencil and l must be taken from the other one.

Theorem 2. *Assume that through each point of an analytic surface in $\mathbb{R}^3$ one can draw two transversal arcs of isotropic circles fully contained in the surface (and analytically depending on the point). Assume that these two arcs lie neither in the same isotropic sphere nor in the same plane. Assume that through each point in some dense subset of the surface one can draw only finitely many (not nested) arcs of isotropic circles and* *line segments contained in the surface. Then the surface (possibly besides a one-dimensional subset) has a parametrization*

$$\Phi(u,v) = \left(\frac{P_0P_1 - P_2P_3}{P_0^2+P_3^2}, \frac{P_1P_3+P_0P_2}{P_0^2+P_3^2}, \frac{Z}{P_0^2+P_3^2}\right) \tag{3}$$

for some $P_0, P_1, P_2, P_3 \in \mathbb{R}_{11} \subset \mathbb{R}[u,v]$ and $Z \in \mathbb{R}_{22} \subset \mathbb{R}[u,v]$, where $P_0^2+P_3^2 \neq 0$, such that the arcs of the isotropic circles are the curves $u = \mathrm{const}$ and $v = \mathrm{const}$.

Corollary 2. *Consider a surface satisfying the assumptions of Theorem 2. Then if the top views of the isotropic circles $u = \mathrm{const}$ are all tangent to some regular smooth curve, then this curve is contained in a cyclic. In this case, if the top views of the isotropic circles $v = \mathrm{const}$ are all tangent to some regular smooth curve, then this curve is contained in the same cyclic (see Fig. 3 top row).*

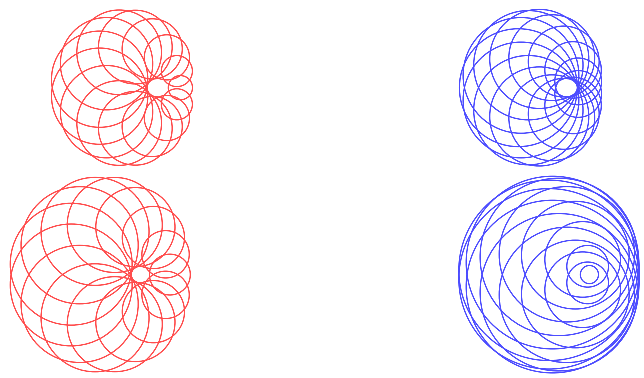

Figure 3: to Corollary 2: top views of isotropic circles through each point.

Remark 2. Actually it is possible that one family of top views has a (real) envelope and the other one has not (see Fig. 3 bottom row). A more general assertion covering this case is stated in Conjecture 4.

3 Outline of the proofs

We begin with some notation. Let $\mathbb{RP}^4$ be the 4-dimensional real projective space with the homogeneous coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$. Consider the following subsets:

- the cylinder $S^2 \times \mathbb{R}$ given by $x_1^2 + x_2^2 + x_4^2 = x_5^2$ and $x_5 \neq 0$;
- the affine subspace H_5 given by $x_4 = 0, x_5 = 1$;
- the line $l \subset S^2 \times \mathbb{R}$ given by $x_1 = x_2 = 0, x_4 = x_5$.

A *cylindrical section* is a nonempty section of $S^2 \times \mathbb{R}$ by a 2-dimensional projective plane that contains no rulings of $S^2 \times \mathbb{R}$.

The proofs of both main theorems are similar and consist of three steps.

1. Reducing the problem of finding all surfaces in $\mathbb{R}^3$ containing two isotropic circles through each point (or two parabolas through each point) to the problem of finding all surfaces in $S^2 \times \mathbb{R}$ containing two cylindrical sections through each point (in case of Theorem 1 these sections must intersect the line l).
2. Solving the resulting problems using parametrization of surfaces containing two conics through each point.
3. Extracting the solutions of the initial problems from the solutions obtained in the previous step.

For the first and the third steps we perform the *inverse isotropic stereographic projection* of a surface in $\mathbb{R}^3$ to obtain a surface in $S^2 \times \mathbb{R}$. The *isotropic stereographic projection* $\pi\colon (S^2 \times \mathbb{R}) \setminus l \to H_5$ is defined by the formula

$$\pi\colon (x_1 : x_2 : x_3 : x_4 : x_5) \mapsto \left(\frac{x_1}{x_5 - x_4} : \frac{x_2}{x_5 - x_4} : \frac{x_3}{x_5 - x_4} : 0 : 1\right).$$

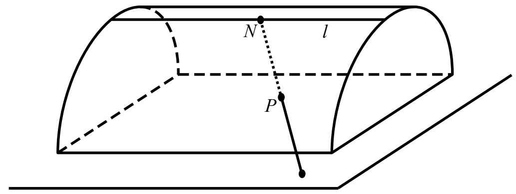

Figure 4: the isotropic stereographic projection (in 3-dimensional space).

The inverse map $\pi^{-1}\colon H_5 \to (S^2 \times \mathbb{R}) \setminus l$ is given by

$$\pi^{-1}\colon (x_1 : x_2 : x_3 : 0 : 1) \mapsto (2x_1 : 2x_2 : 2x_3 : x_1^2 + x_2^2 - 1 : x_1^2 + x_2^2 + 1).$$

The point $N = (0 : 0 : 0 : 1 : 1)$ is called *the projection center.* Thinking geometrically, the map π maps a point $P \in (S^2 \times \mathbb{R}) \setminus l$ to the intersection of the line NP and H_5 (see Fig. 4). In particular, the image of any section of $(S^2 \times \mathbb{R}) \setminus l$ by a projective subspace containing N is an affine subspace of H_5.

The key property of the isotropic stereographic projection is stated in the following theorem, which is proved in [5].

Theorem 3 ([5, Theorem 3]). *Identify H_5 with $\mathbb{R}^3$ by the map $(x_1 : x_2 : x_3 : 0 : 1) \mapsto (x_1, x_2, x_3)$. Then the maps π and π^{-1} define a 1-1 correspondence between isotropic spheres (respectively, isotropic circles) in H_5 and nonempty sections of $(S^2 \times \mathbb{R}) \setminus l$ by a 3-dimensional (respectively, 2-dimensional) projective subspace that does not pass through the projection center N and contains no rulings of $S^2 \times \mathbb{R}$.*

For the second step we need several auxiliary theorems. Our main instrument is the following theorem by J. Schicho.

Theorem 4 ([11, Theorem 21], [12, Theorem 4.1]). *Assume that through each point of an analytic surface Φ in a domain in $\mathbb{CP}^n$ one can draw two transversal conic sections intersecting each other only at this point (and analytically depending on the point) such that their intersections with the domain are contained in the surface. Assume that through each point in some dense subset of the surface one can draw only finitely many conic sections such that their intersections with the domain are contained in the surface. Then the surface is algebraic and has a parametrization (possibly besides a one-dimensional subset):*

$$\Phi(u, v) = (X_1(u, v) : \cdots : X_{n+1}(u, v)),$$

for some $X_1, \ldots, X_{n+1} \in \mathbb{C}_{22}$ such that the conic sections are the curves $u = \mathrm{const}$ and $v = \mathrm{const}$.

Using this theorem we find a convenient parametrization of surfaces in $S^2 \times \mathbb{R}$ containing two cylindrical sections through each point.

Theorem 5. *Assume that through each point P of an analytic surface Ψ in $S^2 \times \mathbb{R}$ one can draw two transversal arcs of cylindrical sections fully contained in the surface (and analytically depending on the point). Assume that the conics containing these two arcs intersects only at P. Assume that through each point in some dense subset of the surface one can draw only finitely many (not nested) arcs of cylindrical sections fully contained in the surface. Then the surface (possibly besides a one-dimensional subset) has a parametrization*

$$\Psi(u, v) = (X_1(u, v) : X_2(u, v) : X_3(u, v) : X_4(u, v) : X_5(u, v)), \tag{4}$$

for some $X_1, \ldots, X_5 \in \mathbb{R}_{22}$ satisfying the equation

$$X_1^2 + X_2^2 + X_4^2 = X_5^2, \tag{5}$$

and such that the cylindrical sections are the curves $u = \mathrm{const}$ and $v = \mathrm{const}$.

In fact, the proof of this theorem is quite technical and goes along the lines of the proof of [12, Theorem 1.2], so it is presented in the appendix.

The following theorem gives an algebraic interpretation for the condition that both isotropic circles through each point of the initial surface in $\mathbb{R}^3$ are parabolas. It is used in the proof of Theorem 1 only.

Theorem 6. *Suppose that a surface Ψ has parametrization* (4)*, where $X_1, \dots, X_5 \in \mathbb{R}_{22}$ satisfy* (5)*. Assume that all the curves $u = \mathrm{const}$ and $v = \mathrm{const}$ on the surface Ψ intersect the line l. Then the surface Ψ has another parametrization (possibly besides a one-dimensional subset as well)*

$$\Psi(u, v) = (Y_1(u, v) : \dots : Y_5(u, v)),$$

for some $Y_1, \dots, Y_5 \in \mathbb{R}_{22}$ satisfying (5) *and such that Y_1, Y_2, and $Y_5 - Y_4$ have a common divisor of degree at least 1 in u and v.*

The required parametrization (2) in Theorem 1 is deduced from the following theorem.

Theorem 7. *Let $X_1, X_2, X_4, X_5 \in \mathbb{R}_{22}$ be four polynomials satisfying* (5)*. Assume that X_1, X_2, and $X_5 - X_4$ have a common divisor of degree at least 1 in u and v. Then there exist polynomials $P, Q, R, T \in \mathbb{R}_{22}$ such that*

$$\begin{aligned} X_1 &= 2PRT, \\ X_2 &= 2QRT, \\ X_4 &= (P^2 + Q^2 - R^2)T, \\ X_5 &= (P^2 + Q^2 + R^2)T. \end{aligned} \tag{6}$$

Finally, the required parametrization (3) in Theorem 2 is deduced from the following theorem, which is proved in [1].

Theorem 8 (Parametrization of Pythagorean 4-tuples [1, Theorem 2.2]). *Let $X_1, X_2, X_4, X_5 \in \mathbb{R}[u, v]$ be four polynomials satisfying* (5)*. Then there exist polynomials $P_0, P_1, P_2, P_3, T \in \mathbb{R}[u, v]$ such that*

$$\begin{aligned} X_1 &= 2(P_0P_1 - P_2P_3)T, \\ X_2 &= 2(P_1P_3 + P_0P_2)T, \\ X_4 &= (P_1^2 + P_2^2 - P_0^2 - P_3^2)T, \\ X_5 &= (P_0^2 + P_1^2 + P_2^2 + P_3^2)T. \end{aligned} \tag{7}$$

Remark 3. Actually from [1, Theorem 2.2] it follows that under the assumptions of Theorem 8 there exist parametrization (7) with $X_5 = \pm(P_0^2 + P_1^2 + P_2^2 + P_3^2)T$. However one can exclude the case of minus sign in the latter formula by the change of variables $(P_0, P_1, P_2, P_3, T) \mapsto (-P_1, P_0, P_3, -P_2, -T)$.

For the proof of Corollary 2 we need the following auxiliary results. Consider a surface of form (3). Identify the plane Oxy with $\mathbb{C}$ by the map $(x, y) \mapsto x + yi$. Then the projection of a point $\Phi(u, v)$ to the plane Oxy is identified with the complex number

$$\frac{P_0P_1 - P_2P_3}{P_0^2 + P_3^2} + \frac{P_1P_3 + P_0P_2}{P_0^2 + P_3^2} \cdot i = \frac{P_1(u, v) + P_2(u, v)i}{P_0(u, v) - P_3(u, v)i} =: \frac{a_{11}uv + a_{10}u + a_{01}v + a_{00}}{b_{11}uv + b_{10}u + b_{01}v + b_{00}} \tag{8}$$

for some $a_{ij}, b_{ij} \in \mathbb{C}$. Therefore studying the top view of isotropic circles on Φ is closely related to classification of complex *bilinear*-fractional maps of the form

$$F(u, v) = \frac{a_{11}uv + a_{10}u + a_{01}v + a_{00}}{b_{11}uv + b_{10}u + b_{01}v + b_{00}}. \tag{9}$$

This is the missing *complex* analogue of the *quaternionic* classification from [12, §3], cf. [13].

Two rational functions $F(u, v)$ and $G(u, v)$ of form (9) are *equivalent*, if there are invertible complex linear-fractional maps $f(z), f_u(u), f_v(v)$ such that

$$F(u, v) = f(G(f_u(u), f_v(v))).$$

Theorem 9 (classification of complex bilinear-fractional maps). *Each rational function $F(u, v)$ of form* (9) *is equivalent to one of the following 5 polynomials: $uv, u + v, u, v, 0$.*

The latter theorem allows us to obtain the following description of the top view as a "Minkowski sum or product" of circles or lines. A *generalized circle* in the plane is either a circle or a line. For any $A \subset \mathbb{C}$ and $z \in \mathbb{C}$ denote $z \cdot A = \{za \mid a \in A\}$ and $z + A = \{z + a \mid a \in A\}$. Note that if A is a circle (respectively, a line) in the complex plane, then both sets $z \cdot A$ and $z + A$ are circles (respectively, lines) as well.

Theorem 10. *Let Φ be a surface satisfying the assumptions of Theorem 2. Then there exist generalized circles $\omega_1, \omega_2 \subset \mathbb{C}$ and a linear-fractional map $f(z)$ such that*

(i) *for any $u_0 \in \mathbb{R}$ there exists $z_0 \in \omega_1$ such that the top view of the isotropic circle $u = u_0$ on Φ is either $\mathrm{Cl}\, f(z_0 \cdot \omega_2)$ or $\mathrm{Cl}\, f(z_0 + \omega_2)$;*

(ii) *for any $v_0 \in \mathbb{R}$ there exists $w_0 \in \omega_2$ such that the top view of the isotropic circle $v = v_0$ on Φ is either $\mathrm{Cl}\, f(w_0 \cdot \omega_1)$ or $\mathrm{Cl}\, f(w_0 + \omega_1)$.*

Here "Cl" stands for the closure of a set.

The last result required for the proof of Corollary 2 is the following one.

Theorem 11. (i) *Let ω_1 and ω_2 be two generalized circles in the complex plane. Suppose that the generalized circles from the family $\Pi_1 = \{w \cdot \omega_1 \mid w \in \omega_2\}$ are tangent to some regular smooth curve; then this curve is contained in a cyclic. If, in addition, the generalized circles from the family $\Pi_2 = \{z \cdot \omega_2 \mid z \in \omega_1\}$ are tangent to some regular smooth curve as well, then the latter curve is contained in the same cyclic.*

(ii) *The same assertions hold for the families $\Sigma_1 = \{w + \omega_1 \mid w \in \omega_2\}$ and $\Sigma_2 = \{z + \omega_2 \mid z \in \omega_1\}$.*

4 Proofs

4.1 Proof of Theorem 6.

Lemma 1. *Let Ψ be a surface satisfying the assumptions of Theorem 6. Then Ψ has a parametrization (possibly besides a one-dimensional subset)*

$$\Psi(u, v) = (Y_1(u, v) : \cdots : Y_5(u, v)),$$

for some $Y_1, \ldots, Y_5 \in \mathbb{R}_{22}$ satisfying (5) and such that

(i) *there are infinitely many $u_0 \in \mathbb{R}$ such that the polynomials $Y_1(u_0, v)$, $Y_2(u_0, v)$, $Y_5(u_0, v) - Y_4(u_0, v) \in \mathbb{R}[v]$ have a common root v_1;*

(ii) *there are infinitely many $v_0 \in \mathbb{R}$ such that the polynomials $Y_1(u, v_0)$, $Y_2(u, v_0)$, $Y_5(u, v_0) - Y_4(u, v_0) \in \mathbb{R}[u]$ have a common root u_1.*

Proof of Lemma 1. In parametrization (4), the pair (u, v) runs through some open subset of $\mathbb{R}^2$. Take any (u_0, v_0) from this subset and consider the conic $u = u_0$ in Ψ (recall that by a conic we mean an *irreducible* degree 2 curve; in particular, $\max_{i=1}^5 \deg_v X_i = 2$). This conic intersects the line l. By the definition of l it follows that two cases are possible:

Case 1: there exist $v_1, a \in \mathbb{R}$, such that

$$(X_1(u_0, v_1) : \cdots : X_5(u_0, v_1)) = (0 : 0 : a : 1 : 1).$$

Case 2: for some $a \in \mathbb{R}$ we have

$$(v^2 X_1(u_0, 1/v) : \cdots : v^2 X_5(u_0, 1/v))|_{v=0} = (0 : 0 : a : 1 : 1).$$

If for infinitely many u_0 we have the first case, then condition (i) of Lemma 1 holds. Otherwise for infinitely many u_0 we have the second case. Then let us change the parametrization taking $X_i'(u, v) = v^2 X_i(u, 1/v)$ for $i = 1, \ldots, 5$. For the resulting parametrization $(X_1'(u, v) : \cdots : X_5'(u, v))$ condition (i) holds. Repeating this procedure with u we obtain a parametrization $(Y_1(u, v) : \cdots : Y_5(u, v))$ which satisfies both conditions (i) and (ii). $\square$

Lemma 2. *Let F_1, F_2, and F_3 be three polynomials in $\mathbb{R}[u, v]$. Suppose that for infinitely many $u_0 \in \mathbb{R}$ the polynomials $F_1(u_0, v)$, $F_2(u_0, v)$, and $F_3(u_0, v)$ have a common root. Then F_1, F_2, and F_3 have a common divisor of degree at least one in v.*

Proof of Lemma 2. Let D be one of the greatest common divisors of F_1, F_2, and F_3. First let us prove that $\deg D \geq 1$. Without loss of generality it can be assumed that $F_1 \neq 0$. For the case $F_2 = F_3 = 0$ there is nothing to prove. Otherwise without loss of generality it can be assumed that $F_2 \neq 0$. Then the curves $F_1(u,v) = 0$ and $F_2(u,v) = 0$ have infinitely many common points. By Bezout's theorem it follows that F_1 and F_2 are not coprime. Let A be one of the greatest common divisors of F_1 and F_2, so that $F_1 = AF_1'$, $F_2 = AF_2'$ for some $F_1', F_2' \in \mathbb{R}[u,v]$ and $\deg A \geq 1$. If $F_3 = 0$, then A is a common divisor of F_1, F_2, and F_3; in particular, $A \mid D$ and we are done. Otherwise the curves $F_1 = 0$, $F_2 = 0$, and $F_3 = 0$ have infinitely many common points, whereas the curves $F_1' = 0$ and $F_2' = 0$ have finitely many common points. Hence the curves $A = 0$ and $F_3 = 0$ have infinitely many common points. By Bezout's theorem it follows that A and F_3 are not coprime. Thus F_1, F_2, and F_3 are not coprime as well.

Further, let us prove that actually $\deg_v D \geq 1$. Assume the converse. Then $D \in \mathbb{R}[u]$. Suppose $F_i = DF_i''$ for some $F_i'' \in \mathbb{R}[u,v]$ (here $i = 1,2,3$). Then there exist infinitely many u_0 such that $D(u_0) \neq 0$ but the polynomials $F_1(u_0,v)$, $F_2(u_0,v)$, and $F_3(u_0,v)$ have a common root. For any such u_0 the polynomials $F_1''(u_0,v)$, $F_2''(u_0,v)$, and $F_3''(u_0,v)$ have a common root. From the previous paragraph it follows that F_1'', F_2'', and F_3'' are not coprime. This contradicts the assumption that D is the greatest common divisor of F_1, F_2, and F_3, which completes the proof. $\square$

Proof of Theorem 6 follows from Lemmas 1 and 2. $\square$

4.2 Proof of Theorem 7.

First let us prove the following simple lemma.

Lemma 3. *Let F_1 and F_2 be two coprime polynomials in $\mathbb{R}[u,v]$ (in particular, $F_1^2 + F_2^2 \neq 0$). Suppose that a real polynomial I divides $F_1^2 + F_2^2$ and $I \neq \mathrm{const}$; then $\deg_u I \geq 2$ or $\deg_v I \geq 2$.*

Proof of Lemma 3. Assume the converse. Without loss of generality it can be assumed that $\deg_v I = 1$. Then there exist infinitely many $u_0 \in \mathbb{R}$ such that $I(u_0,v)$ has a real root v_0. For any such pair (u_0,v_0) we have $F_1^2(u_0,v_0) + F_2^2(u_0,v_0) = 0$, i. e., $F_1(u_0,v_0) = F_2(u_0,v_0) = 0$. In particular, the curves $F_1 = 0$ and $F_2 = 0$ have infinitely many common points. From Bezout's theorem it follows that F_1 and F_2 are not coprime. This contradiction concludes the proof. $\square$

Proof of Theorem 7. If X_1 and X_2 vanish simultaneously, then by (5) we have $X_4 = \pm X_5$ and the required parametrization is given by either $P = Q = 0, R = 1, T = X_5$ (in the case when $X_4 = -X_5$) or $Q = R = 0, P = {}$ $= 1, T = X_4$ (in the case when $X_4 = X_5$). Assume further that $X_1^2 + X_2^2 \neq 0$.

Let D be one of the greatest common divisors of X_1, X_2, and $X_5 - X_4$. Denote $X_1 = DP$, $X_2 = DQ$, $X_5 - X_4 = DR$. By (5) we have

$$D(P^2 + Q^2) = R(X_5 + X_4). \tag{10}$$

Let us show that $R \mid D$; then taking $T = D/2R$ we obtain the required parametrization. By (10) it suffices to show that R and $P^2 + Q^2$ are coprime. Since $X_1^2 + X_2^2 \neq 0$, we have $P^2 + Q^2 \neq 0$. Hence P and Q have some greatest common divisor D' so that $P = D'P'$, $Q = D'Q'$, where P' and Q' are coprime. Then R and D' are coprime because otherwise D is not a greatest common divisor of X_1, X_2, and $X_5 - X_4$.

It remains to show that R and $(P')^2 + (Q')^2$ are coprime. By the assumptions of Theorem 7 we have $\deg_v D \geq 1$ and $\deg_u D \geq 1$. However $DR = X_5 - X_4 \in \mathbb{R}_{22}$, hence $R \in \mathbb{R}_{11}$. Let I be an arbitrary irreducible (in $\mathbb{R}[u,v]$) nonconstant divisor of $(P')^2 + (Q')^2$. By Lemma 3 it follows that $\deg_u I \geq 2$ or $\deg_v I \geq 2$. Since $R \in \mathbb{R}_{11}$, it follows that $I \nmid R$. Thus R and $(P')^2 + (Q')^2$ are coprime and $R \mid D$. $\square$

4.3 Proof of Theorem 1.

Identify space $\mathbb{R}^3$ with H_5 by the map $(x,y,z) \mapsto (x : y : z : 0 : 1)$. Consider the image Ψ of the given surface Φ under the inverse isotropic stereographic projection π^{-1}.

Let us show that Ψ satisfies the assumptions of Theorem 5. By Theorem 3 it follows that each isotropic circle in H_5 maps to a cylindrical section (in particular, by the assumptions of Theorem 1 it follows that through each point of some dense subset of Ψ one can draw only finitely many not nested arcs of cylindrical sections). Moreover, each parabola with vertical axis is mapped to a cylindrical section that intersects the line l (with the point on l excluded). Indeed, the projective closure of each parabola with a vertical axis passes through the point $(0 : 0 : 1 : 0 : 0)$ while the line joining this point with the projection center N is exactly l.

It remains to show that the two conics containing the two arcs of cylindrical sections through a point P of Ψ intersect only at P. Indeed, otherwise both conics are contained in one 3-dimensional subspace. If this subspace does not contain N, then by Theorem 3 it follows that the π-images of the conics are contained in one isotropic sphere, which is forbidden. If this subspace contains N, then the π-images of the conics are contained in one plane, which is forbidden as well.

By Theorems 5, 6, and 7 the surface Ψ has parametrization (4), where $X_1, \dots, X_5$ satisfy (6). We have $R \neq 0$, otherwise $X_1 = X_2 = 0$, $X_4 = X_5$, and the surface is contained in a line, which is impossible.

Now, using the stereographic projection π, we obtain the parametrization of the initial surface Φ:

$$\Phi(u,v) = \left(\frac{P}{R}, \frac{Q}{R}, \frac{Z}{R^2 T} \right),$$

for which the parabolas through each point are the curves $u = \text{const}$ and $v = \text{const}$.

Let us show that actually $T = \text{const}$. Assume the converse. Then without loss of generality we may assume that $\deg_u T \geq 1$. Then (in the notation of Theorem 7) from $X_5 = (P^2 + Q^2 + R^2)T$ and the condition $X_5 \in \mathbb{R}_{22}$ we obtain $P, Q, R \in \mathbb{R}_{01}$. This means that all points of Φ with $v = \text{const}$ are contained in a vertical line, i. e., the curve $v = \text{const}$ is not a parabola with vertical axis. This contradiction implies that $T = \text{const}$. Since $X_5 \in \mathbb{R}_{22}$ and $X_5 = (P^2 + Q^2 + R^2)T$, we obtain $P, Q, R \in \mathbb{R}_{11}$. Absorbing the constant T into $Z(u,v)$ we arrive at parametrization (2). $\square$

4.4 Proof of Theorem 2

As in the proof of Theorem 1, identify space $\mathbb{R}^3$ with H_5 by the map $(x,y,z) \mapsto (x:y:z:0:1)$ and consider the image Ψ of the given surface Φ under inverse isotropic stereographic projection π^{-1}.

Let us show that Ψ satisfies the assumptions of Theorem 5. By Theorem 3 it follows that each isotropic circle in H_5 is mapped to a cylindrical section under this projection (in particular, by the assumptions of Theorem 2 it follows that through each point of some dense subset of Ψ one can draw only finitely many not nested arcs of cylindrical sections). Literally as in the 3rd paragraph of the proof of Theorem 1 it is shown that the two conics containing the two cylindrical sections through a point P of Ψ intersect only at P.

By Theorems 5 and 8 the surface Ψ has parametrization (4), where $X_1, \dots, X_5$ now satisfy (7). We have $P_0^2 + P_3^2 \neq 0$ and $P_1^2 + P_2^2 \neq 0$, otherwise $X_1 = X_2 = 0$, $X_4 = \pm X_5$, and the surface is contained in a line, which is impossible.

Using the stereographic projection π, we obtain the parametrization of the initial surface Φ:

$$\Phi(u,v) = \left(\frac{P_0P_1 - P_2P_3}{P_0^2 + P_3^2}, \frac{P_1P_3 + P_0P_2}{P_0^2 + P_3^2}, \frac{Z}{(P_0^2 + P_3^2)T} \right),$$

for which the isotropic circles through each point are the curves $u = \text{const}$ and $v = \text{const}$.

Let us show that $T = \text{const}$. Assume the converse. Then without loss of generality we may assume that $\deg_u T \geq 1$. Then (in the notation of Theorem 8) from $X_5 = (P_0^2 + P_1^2 + P_2^2 + P_3^2)T$ and the condition $X_5 \in \mathbb{R}_{22}$ we obtain $P_i \in \mathbb{R}_{01}$ for $i = 1,2,3,4$. This means that all points of Φ with $v = \text{const}$ are contained in a vertical line, i. e., the curve $v = \text{const}$ is not an isotropic circle. This contradiction implies that $T = \text{const}$. Since $X_5 \in \mathbb{R}_{22}$ and $X_5 = (P_0^2 + P_1^2 + P_2^2 + P_3^2)T$, we obtain $P_i \in \mathbb{R}_{11}$ for $i = 1,2,3,4$. Absorbing the constant T into $Z(u,v)$ we arrive at parametrization (3). $\square$

4.5 Proof of Corollary 1

Let us start from parametrization (2) given by Theorem 1. Identify our space $\mathbb{R}^3$ with $\{(x:y:z:t) \mid t \neq 0\} \subset \mathbb{RP}^3$ by the map $(x,y,z) \mapsto (x:y:z:1)$. Write

$$P(u,v) = p_{11}uv + p_{10}u + p_{01}v + p_{00},\ Q(u,v) = q_{11}uv + q_{10}u + q_{01}v + q_{00},\ R(u,v) = r_{11}uv + r_{10}u + r_{01}v + r_{00}.$$

The top views of the parabolas through a point $\Phi(u_0,v_0)$ of the surface Φ are the two lines parametrized by

$$(P(u,v_0) : Q(u,v_0) : 0 : R(u,v_0)) \text{ and } (P(u_0,v) : Q(u_0,v) : 0 : R(u_0,v)).$$

These lines are the images of the rulings $u/w = u_0, v/w = v_0$ of the quadric $\Sigma = \{(s:u:v:w) \mid sw = uv\}$ under the map

$$(s:u:v:w) \mapsto (p_{11}s + p_{10}u + p_{01}v + p_{00}w : q_{11}s + q_{10}u + q_{01}v + q_{00}w : r_{11}s + r_{10}u + r_{01}v + r_{00}w : 0).$$

Now the corollary follows from the following well-known assertion: *under a projective map $\mathbb{RP}^3 \to \mathbb{RP}^2$ the rulings of a ruled quadric are mapped to either the tangents to a conic, or two pencils of lines, or one line, or one point.* $\square$

4.6 Proof of Theorem 9.

Let $\mathrm{Mat}_2(\mathbb{C})$ be the set of all (2×2)-matrices with entries in $\mathbb{C}$. Take $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}), A = \left(\begin{smallmatrix} a_{11} & a_{10} \\ a_{01} & a_{00} \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{C}), B = \left(\begin{smallmatrix} b_{11} & b_{10} \\ b_{01} & b_{00} \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{C}) \setminus \{0\}$. By definition, put

$$f_M(z) := \frac{az+b}{cz+d},$$
$$F_B^A(u,v) := \frac{a_{11}uv + a_{10}u + a_{01}v + a_{00}}{b_{11}uv + b_{10}u + b_{01}v + b_{00}}.$$

Lemma 4. *For any $M, N \in \mathrm{GL}_2(\mathbb{C})$ we have $f_N \circ f_M = f_{NM}$. For any $A \in \mathrm{Mat}_2(\mathbb{C}), B \in \mathrm{Mat}_2(\mathbb{C}) \setminus \{0\}$ and $C, D \in \mathrm{GL}_2(\mathbb{C})$ we have $F_B^A(f_C(u), f_D(v)) = F_{C^T BD}^{C^T AD}(u,v)$.*

Lemma 5. *Let $P(u,v) = c_{11}uv + c_{10}u + c_{01}v + c_{00} \in \mathbb{R}_{11}$; then $P = QR$ for some $Q \in \mathbb{R}_{10}, R \in \mathbb{R}_{01}$ if and only if $c_{00}c_{11} - c_{10}c_{01} = 0$.*

Proofs of Lemmas 4 and 5 can be obtained by a direct computation.
Proof of Theorem 9. Suppose that $F(u,v) = F_B^A(u,v)$ for some $A \in \mathrm{Mat}_2(\mathbb{C}), B \in \mathrm{Mat}_2(\mathbb{C}) \setminus \{0\}$.

Case 1: $\det A = \det B = 0$. Then by Lemma 5 we have

$$F_B^A(u,v) = \frac{a_{11}uv + a_{10}u + a_{01}v + a_{00}}{b_{11}uv + b_{10}u + b_{01}v + b_{00}} = \frac{au+b}{cu+d} \cdot \frac{a'v+b'}{c'v+d'} =: f_u(u) \cdot f_v(v)$$

for some $a, b, c, d, a', b', c', d' \in \mathbb{C}$. If $f_u(u), f_v(v) \neq \mathrm{const}$, then this shows that $F_B^A(u,v)$ is equivalent to uv. Otherwise $F_B^A(u,v)$ is clearly equivalent to either u, or v, or 0.

Case 2: $\det B \neq 0$. Let us show that there exist matrices $C, D, M \in \mathrm{GL}_2(\mathbb{C})$ such that $f_M(F_B^A(f_C(u), f_D(v)))$ equals one of the polynomials $uv, u+v, u, v, 0$. From Lemma 4 it follows that $f_M(F_B^A(f_C(u), f_D(v))) = f_M(F_{C^T BD}^{C^T AD}(u,v))$. Suppose that $X \in \mathrm{GL}_2(\mathbb{C})$ is such that $J = XAB^{-1}X^{-1}$ is the Jordan normal form of the matrix AB^{-1}. Consider the following 3 subcases, depending on the Jordan cells of the matrix J.

Subcase 2.1: $J = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right), \lambda \neq \mu$. Then take $C = X^T, D = B^{-1}X^{-1}, M = \left(\begin{smallmatrix} -1 & \mu \\ 1 & -\lambda \end{smallmatrix}\right)$. We have

$$f_M(F_{C^T BD}^{C^T AD}(u,v)) = f_M(F_{\mathrm{Id}}^J(u,v)) = f_M\left(\frac{\lambda uv + \mu}{uv+1}\right) = uv.$$

Subcase 2.2: $J = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ *(that is, $F_B^A(u,v) = \mathrm{const}$)*. Then take $C = X^T, D = B^{-1}X^{-1}, M = \left(\begin{smallmatrix} 1 & -\lambda \\ 0 & 1 \end{smallmatrix}\right)$. We have

$$f_M(F_{C^T BD}^{C^T AD}(u,v)) = f_M(F_{\mathrm{Id}}^J(u,v)) = f_M\left(\frac{\lambda uv + \lambda}{uv+1}\right) = 0.$$

Subcase 2.3: $J = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. Then take $C = X^T I, D = B^{-1}X^{-1}, M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -\lambda \end{smallmatrix}\right)$, where $I = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We have

$$f_M(F_{C^T BD}^{C^T AD}(u,v)) = f_M(F_I^{IJ}(u,v)) = f_M\left(\frac{\lambda(u+v)+1}{u+v}\right) = u+v.$$

Case 3: $\det A \neq 0, \det B = 0$. Note that $F_B^A(u,v)$ is equivalent to $F_A^B(u,v)$ because $F_A^B(u,v) = f_I(F_B^A(u,v))$. Thus this case reduces to the previous one. □

4.7 Proof of Theorem 10

Let us start from parametrization (3) given by Theorem 2. The projection of a point $\Phi(u,v)$ onto the plane Oxy is given by formula (8). The top views of the isotropic circles through a point $\Phi(u_0, v_0)$ are the two generalized circles $\mathrm{Cl}\,\{F(u_0, v) \mid v \in \mathbb{R}\}$ and $\mathrm{Cl}\,\{F(u, v_0) \mid u \in \mathbb{R}\}$, where $F(u,v)$ is given by (9).

By Theorem 9 there exist complex linear-fractional maps f, f_z, f_w such that $G(z,w) := f(F(f_z(z), f_w(w)))$ equals one of the polynomials $zw, z+w, z, w, 0$. Consider the following three cases.

Case 1: $G(z,w) = zw$. Denote $\omega_1 = \mathrm{Cl}\, f_z^{-1}(\mathbb{R})$, $\omega_2 = \mathrm{Cl}\, f_w^{-1}(\mathbb{R})$, and also $z_0 = f_z^{-1}(u_0), w_0 = f_w^{-1}(v_0)$. Then

$$\mathrm{Cl}\,\{F(u_0,v) \mid v \in \mathbb{R}\} = \mathrm{Cl}\,\{f^{-1}(f(F(f_z(z_0), f_w(w)))) \mid w \in \omega_2\} = \mathrm{Cl}\,\{f^{-1}(G(z_0,w)) \mid w \in \omega_2\} = \mathrm{Cl}\, f^{-1}(z_0 \cdot \omega_2).$$

Analogously, $\mathrm{Cl}\,\{F(u, v_0) \mid u \in \mathbb{R}\} = \mathrm{Cl}\, f^{-1}(w_0 \cdot \omega_1)$.

Case 2: $G(z,w)=z+w$ is completely analogous to the previous case, only the product is replaced by the sum.

Case 3: $G(z,w)$ *equals either* z, *or* w, *or* 0. Then $G(z,w)$, hence $F(u,v)$, does not depend on one of the variables. Thus the top view of one of the isotropic circles $u=\text{const}$ or $v=\text{const}$ is a single point, which is impossible. $\square$

4.8 Proofs of Theorem 11 and Corollary 2

Recall that any generalized circle in a complex plane can be given by an equation

$$\alpha z\bar{z}+\beta z+\bar{\beta}\bar{z}+\gamma=0, \tag{11}$$

where $\alpha,\gamma\in\mathbb{R},\beta\in\mathbb{C}$. Suppose that $P(z,\bar{z})$ and $Q(z,\bar{z})$ are two nonproportional equations of form (11); then the family of generalized circles $\{\lambda P(z,\bar{z})+\mu Q(z,\bar{z})=0\mid\lambda,\mu\in\mathbb{R}\}$ is *a pencil of circles.* Equivalently, a pencil of circles can be defined as the image of either a pencil of lines or the set of circles $\{|z|=R\mid R\in\mathbb{R}\}$ under a linear-fractional map. In particular, a pencil of circles never has an envelope.

Proof of Theorem 11. (i) (A. A. Zaslavsky, private communication). Let us show that the envelope of the family Π_1 is contained in a cyclic. Suppose that $\omega_2=\{\frac{av+b}{cv+d}\mid v\in\mathbb{R}\setminus\{-\frac{d}{c}\}\}\cup\{\frac{a}{c}\}$ and that ω_1 is given by (11). Then the generalized circle $\frac{av+b}{cv+d}\cdot\omega_1\in\Pi_1$ has equation

$$\alpha z\bar{z}(cv+d)(\bar{c}v+\bar{d})+\beta z(\bar{a}v+\bar{b})(cv+d)+\bar{\beta}\bar{z}(av+b)(\bar{c}v+\bar{d})+\gamma(av+b)(\bar{a}v+\bar{b})=0. \tag{12}$$

It can be rewritten as

$$A(z,\bar{z})v^2+B(z,\bar{z})v+C(z,\bar{z})=0, \tag{13}$$

where

$$\begin{aligned}
A(z,\bar{z})&=\alpha c\bar{c}z\bar{z}+\beta\bar{a}cz+\bar{\beta}a\bar{c}\bar{z}+\gamma a\bar{a},\\
B(z,\bar{z})&=\alpha(c\bar{d}+\bar{c}d)z\bar{z}+\beta(\bar{b}c+\bar{a}d)z+\bar{\beta}(b\bar{c}+a\bar{d})\bar{z}+\gamma(\bar{a}b+a\bar{b}),\\
C(z,\bar{z})&=\alpha d\bar{d}z\bar{z}+\beta\bar{b}dz+\bar{\beta}b\bar{d}\bar{z}+\gamma b\bar{b}.
\end{aligned}$$

In other words, Π_1 is a *quadratic* family of generalized circles. To find the equation of the envelope, differentiate (13) with respect to v and get

$$2A(z,\bar{z})v+B(z,\bar{z})=0. \tag{14}$$

The system of equations (13) and (14) gives the envelope. Eliminating v from the system, we get

$$B^2(z,\bar{z})-4A(z,\bar{z})C(z,\bar{z})=0. \tag{15}$$

It is easy to see that the substitution of $z=x+yi,\bar{z}=x-yi$ into (15) gives an equation in x and y of form (1), i. e., a cyclic.

Let us check that equation (15) does not hold identically. Indeed, otherwise we have $(2Av+B)^2=4A(Av^2+Bv+C)$, i. e., for each $v\in\mathbb{R}$ the generalized circle given by (13) lies in the pencil of circles given by $2A\lambda+B\mu=0$ for $\lambda,\mu\in\mathbb{R}$. Hence the family Π_1 has no envelope, which contradicts to the assumptions of the theorem.

Now suppose that ω_1 and ω_2 are parametrized by some parameters $s_1\in\mathbb{R}$ and $s_2\in\mathbb{R}$ respectively. Then the family Π_1 can be considered as a family of curves parametrized by s_2. Moreover, each curve of this family is itself parametrized by s_1. Changing the order of parameters, we obtain the family Π_2. If both families have an envelope, then these envelopes coincide due to a general fact, see [3].

(ii) In this case the envelope of the family Σ_1 is either a pair of concentric circles, or a circle, or a pair of parallel lines (the proof is trivial). All these sets are cyclics. $\square$

Proof of Corollary 2 follows from Theorems 10 and 11 and the fact that cyclics are mapped to cyclics under complex linear-fractional transformations. $\square$

5 Open problems

We believe that the technical assumptions in the main theorems can be dropped, or more precisely, the following conjectures hold.

Conjecture 1. *Assume that through each point of an analytic surface in* $\mathbb{R}^3$ *one can draw two transversal parabolic arcs with vertical axes fully contained in the surface (and analytically depending on the point). Then the surface (possibly besides a one-dimensional subset) has a parametrization* (2).

Conjecture 2. *Assume that through each point of an analytic surface in* $\mathbb{R}^3$ *one can draw two transversal arcs of isotropic circles fully contained in the surface (and analytically depending on the point). Then the surface (possibly besides a one-dimensional subset) has a parametrization* (3).

The following conjecture gives a classification of surfaces containing a line and a parabola through each point.

Conjecture 3. *Assume that through each point of an analytic surface in* $\mathbb{R}^3$ *one can draw a line segment and a parabolic arc with vertical axis fully contained in the surface, intersecting transversally, and depending analytically on the point. Then the surface (possibly besides a one-dimensional subset) has a parametrization*

$$\Phi(u,v) = \left(\frac{P}{UV}, \frac{Q}{UV}, \frac{Z}{U^2V}\right)$$

for some $P, Q \in \mathbb{R}_{11}, U \in \mathbb{R}_{10}, V \in \mathbb{R}_{01}, Z \in \mathbb{R}_{21}$, *where* $U, V \neq 0$, *such that the line segments are the curves* $u = \mathrm{const}$ *and the parabolic arcs are the curves* $v = \mathrm{const}$.

The following conjecture gives a complete description of the top views for the surfaces containing two isotropic circles through each point similar to Corollary 1.

Conjecture 4. *For each surface satisfying the assumptions of Conjecture 2 the top views of the two isotropic circles through each point are either tangent to one cyclic (probably without real points) at two points (which may coincide or be complex conjugate) or lie in a union of two pencils of circles.*

Actually for a general position cyclic there are four families of circles which are tangent to this cyclic at points (which may coincide or be complex conjugate). This observation motivates the following brave conjecture.

Conjecture 5. *Is it true that if a surface contains two isotropic circles through each point, then it also contains a third one? And that all three form a hexagonal web (see [8] for a definition and Fig. 5 for an example)?*

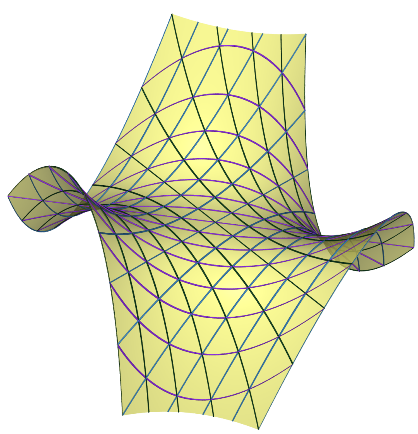

Figure 5: the surface $z = xy(x + y)$ contains three parabolas through each point which form a hexagonal web.

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Appendix. Proof of Theorem 5

We sometimes write just an "arc" instead of an "arc of cylindrical section" throughout this proof. Hereafter Φ is a surface satisfying the assumptions of Theorem 5.

An *analytic surface* in $\mathbb{CP}^n$ is the image of an injective complex analytic map from a domain in $\mathbb{C}^2$ into $\mathbb{CP}^n$ with nondegenerate differential at each point. An *algebraic subset* in $\mathbb{CP}^n$ or $\mathbb{R}^3$ is a solution set of some system of algebraic equations. An algebraic subset of dimension 2 is called an *algebraic surface*.

Lemma 6 ([12, Lemma 4.4]). *If an open subset of an analytic surface in $\mathbb{CP}^n$ or $\mathbb{R}^3$ is contained in an algebraic surface, then the whole analytic surface is contained in the algebraic one.*

Lemma 7. *The surface Φ (possibly besides a one-dimensional subset) has parametrization* (4)*, where $X_1, \dots, X_5 \in \mathbb{C}_{22}$ satisfy equation* (5)*, and (u, v) runs through some (not open) subset of $\mathbb{C}^2$.*

Proof of Lemma 7. Extend $\Phi \subset S^2 \times \mathbb{R}$ analytically to a complex analytic surface $\hat{\Phi}$ in a sufficiently small neighborhood of Φ in $\mathbb{CP}^4$ modulo the boundary (so that the boundaries of Φ and $\hat{\Phi}$ are contained in the boundary of the neighbourhood). Extend the two real analytic families of arcs of cylindrical sections in Φ to complex analytic families of (complex) conics in $\mathbb{CP}^4$. By the analyticity $\hat{\Phi}$ satisfies the assumptions of Theorem 4 (for $n = 4$). By Theorem 4 the surface $\hat{\Phi}$ (possibly besides a one-dimensional subset) has parametrization (4) with $X_1, \dots, X_5 \in \mathbb{C}_{22}$ such that the arcs of cylindrical sections have the form $u = \text{const}$ and $v = \text{const}$. Since the surface $\hat{\Phi}$ is contained in the complex quadric extending the cylinder $S^2 \times \mathbb{R}$, it follows that the polynomials satisfy equation (5). $\square$

Let us reparametrize the surface to make the domain of the map Φ and the coefficients of the polynomials $X_1, \dots, X_5$ real.

Lemma 8. *The surface Φ (possibly besides a one-dimensional subset) has parametrization* (4)*, where $X_1, \dots, X_5 \in \mathbb{C}_{22}$ satisfy equation* (5)*, and (u, v) runs through certain open subset of $\mathbb{R}^2$.*

Proof of Lemma 8. Start with the parametrization given by Lemma 7. Draw two arcs of cylindrical sections of the form $u = \text{const}$ and $v = \text{const}$ through a point of the surface Φ. Through another pair of points of the first arc, draw two more arcs of cylindrical sections of the form $v = \text{const}$. Perform a complex fractional-linear transformation of the parameter v so that the second, the third, and the fourth arcs obtain the form $v = 0, \pm 1$, respectively (we consider the part of the surface where the denominator of the transformation does not vanish). Perform an analogous transformation of the parameter u so that $u = 0, \pm 1$ become arcs of cylindrical sections intersecting the arc $v = 0$. After performing the transformations and clearing denominators we get parametrization (4), where still $X_1, \dots, X_5 \in \mathbb{C}_{22}$ and (u, v) runs through a subset $\Psi \subset \mathbb{C}^2$. By the inverse function theorem, Ψ is a 2-dimensional real analytic surface, because Φ is the image of an injective real analytic function with nonvanishing differential.

Let us prove that actually $\Psi \subset \mathbb{R}^2$. Take $(\hat{u}, \hat{v}) \in \Psi$ sufficiently close to $(0, 0)$. Then $\Phi(\hat{u}, \hat{v})$ is a point of the surface Φ sufficiently close to $\Phi(0, 0)$. Draw the two arcs of cylindrical sections $u = \hat{u}$ and $v = \hat{v}$ through the point $\Phi(\hat{u}, \hat{v})$. By continuity it follows that the arc $v = \hat{v}$ intersects the arc $u = 0$ in Φ. The intersection point can only be $\Phi(0, \hat{v})$. In particular, we get $(0, \hat{v}) \in \Psi$. In a quadratically parametrized conic, the cross-ratio of any four points equals the cross-ratio of their parameters. Since three (real) points $v = 0, \pm 1$ of the arc $u = 0$ have real v-parameters it follows that all (but one) points of the arc have real v-parameters. In particular, $\hat{v} \in \mathbb{R}$. Analogously, $\hat{u} \in \mathbb{R}$. We have proved that all $(\hat{u}, \hat{v}) \in \Psi$ sufficiently close to the origin are real. Since Ψ is real analytic, by Lemma 6 Ψ is an open subset of $\mathbb{R}^2$. $\square$

Lemma 9 ([12, Lemma 4.15]). *Let $X_1, \dots, X_n \in \mathbb{C}[u, v]$. Assume that for all the points (u, v) from some open subset of $\mathbb{R}^2$ the point $X_1(u, v) : \dots : X_n(u, v)$ is real. Then $X_1 = X_1'Y, \dots, X_n = X_n'Y$ for some real $X_1', \dots, X_n' \in \mathbb{R}[u, v]$ and complex $Y \in \mathbb{C}[u, v]$.*

Proof of Theorem 5 follows from Lemmas 7—9. $\square$

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