BASIC COHOMOLOGY OF CANONICAL FOLIATIONS ON COMPLEX MOMENT-ANGLE MANIFOLDS

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Abstract. Battaglia and Zaffran [4] computed the basic Betti numbers for the canonical holomorphic foliation on a moment-angle manifold corresponding to a shellable fan. They conjectured that the basic cohomology ring in the case of any complete simplicial fan has a description similar to the cohomology ring of a complete simplicial toric variety due to Danilov and Jurkiewicz [8]. In this work we prove the conjecture.

1. Introduction

The *moment-angle complex* $\mathcal{Z}_\mathcal{K}$ corresponding to a simplicial complex $\mathcal{K}$ is a topological space build up as a union of products of polydiscs and tori with respect to combinatorial data given by $\mathcal{K}$; see [6] where it was first defined in this fashion. The spaces $\mathcal{Z}_\mathcal{K}$ carry natural torus action. The research around these spaces has flourished during the last two decades with the works of Buchstaber and Panov [6, 7], Bahri, Bendersky, Cohen and Gitler [1], Barreto and Verjovsky [2, 3] and many others. Panov, Ustinovskiy [14] and Tambour [16] showed that an even-dimensional moment-angle manifold $\mathcal{Z}_\Sigma$ corresponding to a complete simplicial fan Σ admits complex structures invariant under the torus action. Then Ishida showed in [10] that $\mathcal{Z}_\mathcal{K}$ admits an invariant complex structure only if $\mathcal{K}$ is the underlying complex of a complete simplicial fan (in other words, $\mathcal{K}$ is a star-shaped sphere triangulation).

Another class of complex manifolds with holomorphic torus action was constructed by Bosio in [5] and became known as *LVMB-manifolds*. Ishida showed in [10, Theorem 9.4] that complex moment-angle manifolds are biholomorphic to LVMB-manifolds of certain type. Both complex moment-angle manifolds and LVMB-manifolds are examples of *complex manifolds with maximal torus action*, completely classified by Ishida in terms of simplicial fans [10].

Battaglia and Zaffran [4] considered a certain holomorphic foliation on a complex LVMB-manifold, which later was shown by Ishida [11] to be a particular case of the *canonical* foliation on any complex manifold with a holomorphic torus action. We review the construction of the canonical foliation in Section 2. Battaglia and Zaffran computed the basic Betti numbers for their foliation in the case when the associated complete fan is shellable. Their method consisted in applying the Mayer–Vietoris sequence. They conjectured that the basic cohomology ring has a description similar to the cohomology ring of a complete simplicial toric variety due to Danilov and Jurkiewicz [8]. The conjecture was justified by the fact that in the case of a complete regular fan the foliation becomes a locally trivial bundle over the associated toric variety with fibre a holomorphic torus (see Remark 2.5).

In this paper we prove this conjecture for all complex moment-angle manifolds with invariant complex structure (see Theorem 3.4). Our approach is very different from that of Battaglia–Zaffran and Ishida: it is based on the Eilenberg–Moore spectral sequence and formality of the Cartan model for the torus action on $\mathcal{Z}_\mathcal{K}$ (see Lemma 3.2).

We note that our approach is applicable in the more general case of moment-angle manifolds $\mathcal{Z}_\mathcal{K}$ admitting a smooth structure only (rather than a complex one). Nevertheless, it is important for us to emphasise the holomorphic nature of the foliation under consideration. The reason is that we hope that our methods can be applied for calculation of basic Dolbeault cohomology for the foliation and Dolbeault cohomology of complex moment-angle manifolds. Recently Ishida has computed these rings for the case when the foliation is transverse Kähler, which is the case if and only if the fan Σ is polytopal (see [12]). In the general case, the description of the Dolbeault cohomology rings is an open problem.

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2. Preliminaries

2.1. **The moment-angle complex.** An abstract *simplicial complex* on the set $[m] = \{1, 2, \dots, m\}$ is a collection $\mathcal{K}$ of subsets $I \subseteq [m]$ such that if $I \in \mathcal{K}$ then each $J \subseteq I$ also belongs to $\mathcal{K}$.

The *moment-angle complex* $\mathcal{Z}_\mathcal{K}$ corresponding to $\mathcal{K}$ is a topological space constructed as follows. Consider the unit m-dimensional polydisc:

$$\mathbb{D}^m = \{(z_1, ..., z_m) \in \mathbb{C}^m : |z_i|^2 \leqslant 1 \text{ for } i = 1, ..., m\}.$$

Then

$$\mathcal{Z}_\mathcal{K} := \bigcup_{I\in\mathcal{K}} \Big(\prod_{i\in I} \mathbb{D} \times \prod_{i\notin I} \mathbb{S}\Big) \subseteq \mathbb{D}^m,$$

where $\mathbb{S}$ is the boundary of the unit disk $\mathbb{D}$.

The moment-angle complex is equipped with the natural action of the torus

$$T^m = \{(t_1, \dots, t_m) \in \mathbb{C}^m \colon |t_i| = 1\}.$$

When $\mathcal{K}$ is simplicial subdivision of a sphere, $\mathcal{Z}_\mathcal{K}$ is a topological manifold [7, Theorem 4.1.4], called the *moment-angle manifold*.

We define an open submanifold $U(\mathcal{K}) \subseteq \mathbb{C}^m$ in a similar way:

$$U_\mathcal{K} = \bigcup_{I\in\mathcal{K}} \Big(\prod_{i\in I} \mathbb{C} \times \prod_{i\notin I} \mathbb{C}^\times\Big),$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. The manifold $U_\mathcal{K}$ has a coordinate-wise action of the algebraic torus $(\mathbb{C}^\times)^m$, in which T^m is a maximal compact subgroup.

Given a commutative ring R with unit, the *face ring* (or the *Stanley–Reisner ring*) of $\mathcal{K}$ is

$$R[\mathcal{K}] := R[v_1, ..., v_m]/I_\mathcal{K},$$

where $R[v_1, ..., v_m]$ is the polynomial algebra, $\deg v_i = 2$, and $I_\mathcal{K}$ is the *Stanley–Reisner ideal*, generated by those monomials $v_I = \prod_{i \in I} v_i$ for which I is not a simplex of $\mathcal{K}$.

2.2. **Complex structure on moment-angle manifolds.** When $\mathcal{K}$ is the underlying complex of a complete simplicial fan, the moment-angle manifold $\mathcal{Z}_\mathcal{K}$ has a structure of a complex manifold with a holomorphic foliation by the orbits of a Lie subgroup $H_A \subset T^m$. We review this construction below.

Let $N_\mathbb{R} \cong \mathbb{R}^n$ be a real vector space. A *simplicial fan* Σ in $N_\mathbb{R}$ is uniquely determined by generators $\boldsymbol{a}_1, ..., \boldsymbol{a}_m$ of its one-dimensional cones and a simplicial complex $\mathcal{K} = \mathcal{K}_\Sigma$, which is the collection of $I \subseteq [m]$ such that $\{\boldsymbol{a}_i \colon i \in I\}$ spans a cone σ_I of Σ. The generators of each cone σ_I must be linearly independent, and the relative interiors of different cones σ_I and σ_J must be non-intersecting. Under this conditions we say that the data $\{\mathcal{K}; \boldsymbol{a}_1, ..., \boldsymbol{a}_m\}$ *define a fan* Σ.

Construction 2.1. Let $\{\boldsymbol{a}_1, ..., \boldsymbol{a}_m\}$ be a set of vectors generating $N_\mathbb{R}$ and assume that $m-n = 2l$ is even. Consider the map

$$A \colon \mathbb{R}^m \to N_\mathbb{R}, \quad \boldsymbol{e}_i \mapsto \boldsymbol{a}_i, \quad i = 1, \dots, m.$$

Choose a linear map $\varPsi \colon \mathbb{C}^l \to \mathbb{C}^m$ satisfying the following conditions:

(a) the composite $\operatorname{Re} \circ \varPsi \colon \mathbb{C}^l \to \mathbb{R}^m$ is a monomorphism, where $\operatorname{Re} \colon \mathbb{C}^m \to \mathbb{R}^m$ denotes the natural projection;

(b) the composite $A \circ \operatorname{Re} \circ \varPsi \colon \mathbb{C}^l \to N_\mathbb{R}$ is a zero map.

We denote $\mathfrak{c}_\varPsi := \varPsi(\mathbb{C}^l) \subseteq \mathbb{C}^m$ and $\mathfrak{h}_A := \operatorname{Re} \mathfrak{c}_\varPsi = \operatorname{Ker} A \subseteq \mathbb{R}^m$. Now define a Lie group

$$C_\varPsi = \{(e^{\langle \psi_1, \boldsymbol{w} \rangle}, ..., e^{\langle \psi_m, \boldsymbol{w} \rangle}) \in (\mathbb{C}^\times)^m\} \tag{1}$$

where $\boldsymbol{w} \in \mathbb{C}^l$ and $\psi_k \in (\mathbb{C}^l)^*$ is given by the k-th coordinate projection: $\psi_k = \operatorname{pr}_k \circ \varPsi$. As $C_\varPsi$ is a complex subgroup of $(\mathbb{C}^\times)^m$, it acts on $U(\mathcal{K})$ holomorphically.

The following result provides complex structures on moment-angle manifolds.

Theorem 2.2 ([14, Theorem 3.3]). *Assume that data $\{\mathcal{K}; \boldsymbol{a}_1, ..., \boldsymbol{a}_m\}$ define a complete simplicial fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$ and $m - n = 2l$. Let $C_\varPsi$ be a group given by* (1)*. Then the holomorphic action of $C_\varPsi$ on $U(\mathcal{K})$ is free and proper, and the complex manifold $U(\mathcal{K})/C_\varPsi$ is T^m-equivariantly diffeomorphic to the moment-angle manifold $\mathcal{Z}_\mathcal{K}$.*

Furthermore, it was proved in [10, Theorem 9.4] that any complex structure on $\mathcal{Z}_\mathcal{K}$ invariant under the action of T^m is obtained from the construction above. So, from now on we assume that $m - n = 2l$ is even and that Σ is a *complete* simplicial fan. Also, we denote by $\mathcal{Z}_\Sigma$ the moment-angle manifold $\mathcal{Z}_\mathcal{K}$ for $\mathcal{K} = \mathcal{K}_\Sigma$ with a complex structure defined above.

Now we define a holomorphic foliation on $\mathcal{Z}_\Sigma$. Consider the following Lie subgroup of the torus T^m:

$$H_A = \exp(\operatorname{Ker} A) = \{(e^{2\pi i y_1}, ..., e^{2\pi i y_m}) \in T^m \colon (y_1, ..., y_m) \in \operatorname{Ker} A\} \tag{2}$$

with Lie algebra $\mathfrak{h}_A = \operatorname{Ker} A$. Conditions (a) and (b) from Construction 2.1 imply that the intersection of the subgroups C_Ψ and H_A in $(\mathbb{C}^\times)^m$ is trivial. The action of H_A on $U(\mathcal{K})/C_\Psi$ is almost free (i. e., all stabiliser subgroups are finite), see [7, Proposition 5.4.6]. Therefore, we obtain a *holomorphic foliation* of $\mathcal{Z}_\Sigma$ by the orbits of the H_A-action.

The foliation defined above is a particular case of the following construction described by Ishida [11, §3]. Let M be a complex manifold with an effective holomorphic action of a compact torus T^m. The complexification of this action gives a holomorphic action of $(\mathbb{C}^\times)^m$ on M, which is not necessarily effective. Let $H \subset (\mathbb{C}^\times)^m$ be the subgroup of global stabilisers. Denote the Lie algebra of T^m by $\mathfrak{g}$, and let $\mathfrak{g}^\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ be its complexification. Let J be the operator of the complex structure on M, and X_v the fundamental vector field corresponding to $v \in \mathfrak{g}$. Then

$$\mathfrak{h} = \{v + iu \in \mathfrak{g}^\mathbb{C} \colon X_v + JX_u = 0\} \subseteq \mathfrak{g}^\mathbb{C}$$

is the Lie algebra of the global stabilisers subgroup H. Let $\mathfrak{h}_\mathbb{R}$ be the image of $\mathfrak{h}$ under the projection $\operatorname{Re}\colon \mathfrak{g}^\mathbb{C} \to \mathfrak{g}$; then we have $\mathfrak{h}_\mathbb{R} = \mathfrak{g} \cap J\mathfrak{g}$. Then the action of $\exp(\mathfrak{h}_\mathbb{R}) \subset T^m$ is locally free and defines a holomorphic foliation on M.

In our case, M is $\mathcal{Z}_\Sigma$, H is C_Ψ, $\mathfrak{h}$ is $\mathfrak{c}_\Psi$, and $\mathfrak{h}_\mathbb{R}$ is $\mathfrak{h}_A$.

Remark 2.3. A fan Σ is *rational* when its generator vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ can be chosen so that they lie in a full-dimensional lattice $N \subset N_\mathbb{R}$. A rational fan defines a *toric variety* V_Σ. When Σ is a complete simplicial rational fan, the holomorphic foliation of $\mathcal{Z}_\Sigma$ by the orbits of H_A becomes a holomorphic Seifert *fibration* over the toric orbifold V_Σ with fibres compact complex tori.

2.3. **Basic cohomology.** Let K be a connected Lie group with Lie algebra $\mathfrak{k}$. Consider the symmetric (polynomial) algebra $S(\mathfrak{k}^*)$ on $\mathfrak{k}^*$ with generators of degree 2, and the exterior algebra $\Lambda(\mathfrak{k}^*)$ with generators of degree 1. The *Weil algebra* of K is the differential graded algebra (DGA for short)

$$\mathcal{W} := \bigl(\Lambda(\mathfrak{k}^*) \otimes S(\mathfrak{k}^*), d\bigr)$$

with the standard acyclic (Koszul) differential d. A $\mathcal{W}^*$*-algebra* is a $\mathcal{W}$-algebra together with an action of operators i_ξ (concatenation) and L_ξ (Lie derivative) for $\xi \in \mathfrak{k}$, see [9, Definition 3.4.1] for the details. Note that $\mathcal{W}$ is naturally a $\mathcal{W}^*$-algebra.

Let (B, d_B) be a DGA over $\mathbb{R}$ which is also a $\mathcal{W}^*$*-algebra.* Define the subalgebra of basic elements

$$B_{\mathrm{bas}} = \{\omega \in B \colon \iota_\xi \omega = L_\xi \omega = 0 \quad \text{for any } \xi \in \mathfrak{k}\},$$

and define the *basic cohomology* of B as

$$H_{\mathrm{bas}}(B) = H(B_{\mathrm{bas}}, d_B).$$

The are two classical models for basic cohomology of B. The *Cartan model* is defined as

$$\mathcal{C}_K(B) = \bigl((B \otimes S(\mathfrak{k}^*))^K, d_C\bigr).$$

We may think of an element $\omega \in C_K(B)$ as an K-equivariant polynomial map from $\mathfrak{k}$ to B. The differential d_K is given by

$$d_K(\omega)(\xi) = d_B(\omega(\xi)) - \iota_\xi(\omega(\xi)).$$

The *Weil model* is

$$\mathcal{W}_K(B) = \big((B \otimes \mathcal{W})_{\text{bas}}, d_{\mathcal{W}}\big),$$

where $d_{\mathcal{W}} = d_B \otimes d$.

Theorem 2.4 ([9, Section 5.1])**.** *The natural inclusions $B_{\text{bas}} \hookrightarrow \mathcal{C}_K(B)$ and $B_{\text{bas}} \hookrightarrow \mathcal{W}_K(B)$ are quasi-isomorphisms of DGAs.*

When the Lie group K is compact and B is the DGA $\Omega(M)$ of differential forms on a compact smooth manifold M, the algebra $H_{\text{bas}}(B)$ is isomorphic to the *equivariant cohomology* $H^*_K(M) := H^*(EK \times_K M)$, see [9, Theorem 2.5.1].

Now we apply this general setup to calculating basic cohomology of $\mathcal{Z}_\Sigma$ with respect to the foliation given by the action of the Lie group H_A, which is non-compact in general, see (2). For this purpose, we take $K = H_A$ and B a subalgebra of $\Omega(\mathcal{Z}_\Sigma)$. As examples of B we consider $\Omega(\mathcal{Z}_\Sigma)$ itself and its subalgebras $\Omega(\mathcal{Z}_\Sigma)^{T^m}$ and $(\Omega(\mathcal{Z}_\Sigma))^{H_A}$ of T^m-invariant and H_A-invariant forms, respectively. Since H_A is a connected abelian group, it acts trivially on its Lie algebra $\mathfrak{h}_A$, so we have

$$\mathcal{C}_{H_A}(\Omega(\mathcal{Z}_\Sigma)) = \Omega(\mathcal{Z}_\Sigma)^{H_A} \otimes S(\mathfrak{h}^*_A).$$

By Theorem 2.4, we have an isomorphism

$$H^*_{\text{bas}}(\mathcal{Z}_\Sigma) := H_{\text{bas}}(\Omega(\mathcal{Z}_\Sigma)) \cong H\big(\mathcal{C}_{H_A}(\Omega(\mathcal{Z}_\Sigma))\big).$$

The cohomology of the Cartan algebra $\big(\mathcal{C}_{H_A}(\Omega(\mathcal{Z}_\Sigma)), d_{\mathcal{C}}\big)$ is described in the next section.

Remark 2.5. When Σ is a *regular* fan, H_A is a compact torus and the foliation under consideration becomes a fibration over the toric manifold V_Σ. In this case, the basic cohomology $H_{\text{bas}}(\Omega(\mathcal{Z}_\Sigma))$ is $H^*(V_\Sigma)$, which is given by the Danilov–Jurkiewicz Theorem [8] (see also [7, Theorem 5.3.1]).

3. Basic cohomology of $\mathcal{Z}_\Sigma$

We first reduce the computation of $H^*_{\text{bas}}(\mathcal{Z}_\Sigma)$ to cohomology of a special DGA:

Lemma 3.1. *Consider the algebra*

$$\mathcal{N} := \mathcal{C}_{H_A}\big(\Omega(\mathcal{Z}_\Sigma)^{T^m}\big) = \Omega(\mathcal{Z}_\Sigma)^{T^m} \otimes S^*(\mathfrak{h}^*_A).$$

Then we have an isomorphism

$$H^*_{\text{bas}}(\mathcal{Z}_\Sigma) \cong H(\mathcal{N}).$$

Proof. By a result of Ishida [12, Lemma 6.1], the basic cohomologyy of $\mathcal{Z}_\Sigma$ can be defined using the subalgebra $\Omega(\mathcal{Z}_\Sigma)^{T^m} \subset \Omega(\mathcal{Z}_\Sigma)$, that is, there is an isomorphism $H^*_{\text{bas}}(\mathcal{Z}_\Sigma) \cong H_{\text{bas}}\big(\Omega(\mathcal{Z}_\Sigma)^{T^m}\big)$. This isomorphism together with Theorem 2.4 imply an isomorphism $H^*_{\text{bas}}(\mathcal{Z}_\Sigma) \cong H(\mathcal{N})$. □

Recall that a DGA B is called *formal* if it is weak equivalent to its cohomology algebra: $(B, d_B) \simeq (H^*(B, d_B), 0)$. (A *weak equivalence* is the equivalence generated by quasi-isomorphisms; it may not be realised by a single quasi-isomorphism of DGA, but rather by a zigzag of quasi-isomorphisms.)

The cohomology of the Cartan model

$$\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma)) = \big(\Omega(\mathcal{Z}_\Sigma)^{T^m} \otimes S(\mathfrak{g}^*),\, d_C\big)$$

is the equivariant cohomology $H^*_{T^m}(\mathcal{Z}_\Sigma)$, which is a module over $S(\mathfrak{g}^*) = H^*_{T^m}(\mathrm{pt}) = H^*(BT^m)$.

Lemma 3.2. *The algebra $\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma))$ is formal. Furthermore, there is a zigzag of quasi-isomorphisms of DGAs between $\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma))$ and $H_{T^m}(\mathcal{Z}_\Sigma)$ which respect the $S(\mathfrak{g}^*)$-module structure.*

Proof. In this proof, $\mathcal{W} = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ is the Weil algebra of the torus T^m. Let $E = EU(m)$ be the space of orthonormal m-frames in $\mathbb{C}^\infty$. By $\Omega(E)$ we understand the inverse limit of the algebras of differential forms on smooth manifolds of m-frames in $\mathbb{C}^N$. We consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \Omega(\mathcal{Z}_\Sigma)^{T^m} \otimes \mathcal{W} & \xhookrightarrow{\iota} & \Omega(\mathcal{Z}_\Sigma)^{T^m} \otimes \Omega(E) & & \\
 & & \uparrow & & \uparrow & & \\
\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma)) & \xleftarrow[\varphi]{\simeq} & (\Omega(\mathcal{Z}_\Sigma)^{T^m} \otimes \mathcal{W})_{\mathrm{bas}} & \xhookrightarrow[\iota_{\mathrm{bas}}]{\simeq} & (\Omega(\mathcal{Z}_\Sigma)^{T^m} \otimes \Omega(E))_{\mathrm{bas}} & \xrightarrow[\psi]{\cong} & \Omega(\mathcal{Z}_\Sigma \times_{T^m} E) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S(\mathfrak{g}^*) & = & \mathcal{W}_{\mathrm{bas}} & \xhookrightarrow{\simeq} & \Omega(E)_{\mathrm{bas}} & \xrightarrow{\cong} & \Omega(BT^m)
\end{array}$$

Here ι and ι_{bas} are the quasi-isomorphisms induced by the inclusion $\mathcal{W} \hookrightarrow \Omega(E)$ of a free acyclic $\mathcal{W}^*$-algebra (see [9, Proposition 2.5.4 and §4.4]). The quasi-isomorphism φ is given by Cartan's Theorem (see [9, Theorem 4.2.1]). The isomorphism ψ follows from the fact that T^m acts freely on E.

The middle line of the diagram above gives a zigzag of quasi-isomorphisms between $\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma))$ and $\Omega(\mathcal{Z}_\Sigma \times_{T^m} E)$ which respect the $S(\mathfrak{g}^*)$-module structure.

Now, the Borel construction $\mathcal{Z}_\Sigma \times_{T^m} E$ is homotopy equivalent to the polyhedral product $(\mathbb{C}P^\infty)^{\mathcal{K}}$, which is a rationally formal space by [13, Theorem 4.8] or [7, Theorem 8.1.6, Corollary 8.1.7]. Rational formality implies a zigzag of quasi-isomorphisms between $\Omega(\mathcal{Z}_\Sigma \times_{T^m} E)$ and $H^*_{T^m}(\mathcal{Z}_\Sigma) = H^*(\mathcal{Z}_\Sigma \times_{T^m} E)$, as the de Rham forms Ω is a commutative cochain model. This zigzag can be chosen to respect the $H^*(BT^m)$-module structure (see [7, 8.1.11–8.1.12]). □

We have the following extended functoriality property of Tor in the category of DGAs, which is a standard corollary of the Eilenberg-Moore spectral sequence:

Lemma 3.3 ([15, Corollary 1.3])**.** *Let A and B be DGAs, let L, L' be a pair of A-modules and let M, M' be a pair of B-modules given together with morphisms*

$$f\colon A \to B, \quad g\colon L \to M, \quad g'\colon L' \to M'$$

where g and g' are f-linear. If f, g and g' are quasi-isomorphisms, then

$$\mathrm{Tor}_f(g, g')\colon \ \mathrm{Tor}_A(L, L') \to \mathrm{Tor}_B(M, M')$$

is an isomorphism.

Now we are ready to prove the main result:

Theorem 3.4. *There is an isomorphism of algebras:*

$$H^*_{\mathrm{bas}}(\mathcal{Z}_\Sigma) \cong \mathbb{R}[v_1, ..., v_m]/(I_{\mathcal{K}} + J),$$

where $I_{\mathcal{K}}$ is the Stanley–Reisner ideal of simplicial complex $\mathcal{K}$, and J is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle \boldsymbol{u}, \boldsymbol{a}_i \rangle v_i, \quad \boldsymbol{u} \in N^*_{\mathbb{R}} = (\mathfrak{g}/\mathfrak{h}_A)^*.$$

Proof. Denote $\mathfrak{g}' := \mathfrak{g}/\mathfrak{h}_A$. We have a splitting $\mathfrak{g} \cong \mathfrak{g}' \oplus \mathfrak{h}_A$. Hence, $S(\mathfrak{g}^*) \cong S(\mathfrak{g}'^*) \otimes S(\mathfrak{h}_A^*)$, and $S(\mathfrak{g}^*)$ is an $S(\mathfrak{g}'^*)$-module via the linear monomorphism $A^*\colon \mathfrak{g}'^* \to \mathfrak{g}^*$. We also obtain a DGA isomorphism

$$\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma)) \cong S(\mathfrak{g}'^*) \otimes \mathcal{N}, \tag{3}$$

where the right hand side is understood as the Cartan model of $\mathcal{N}$ with respect to the Lie algebra $\mathfrak{g}'$. Recall that $H^*_{T^m}(\mathcal{Z}_\Sigma; \mathbb{R}) \cong \mathbb{R}[\mathcal{K}]$ (see [6, Corollary 3.3.1]). Since the fan Σ is complete, the Stanley–Reisner ring $\mathbb{R}[\mathcal{K}]$ is Cohen–Macaulay, that is, it is a finitely-generated free module over its polynomial subalgebra. Furthermore, the composite $\mathfrak{g}'^* \hookrightarrow \mathfrak{g}^* \to \mathfrak{g}_I^*$ is onto for any $I \in \mathcal{K}$, where $\mathfrak{g}_I$ is the coordinate subspace generated by all $\boldsymbol{e}_i$ with $i \in I$. Therefore, the criterion [7, Lemma 3.3.1] applies to show that $\mathbb{R}[\mathcal{K}]$ is a free module over $S(\mathfrak{g}'^*)$.

Consider the following pushout diagram of DGAs:

$$\begin{array}{ccc}
\mathcal{N} \cong \mathbb{R} \otimes_{S^*(\mathfrak{g}'^*)} \mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma)) & \xleftarrow{\tilde{f}^*} & \mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma)) \cong S(\mathfrak{g}'^*) \otimes \mathcal{N} \\
\uparrow{\scriptstyle \tilde{\pi}^*} & & \uparrow{\scriptstyle \pi^*} \\
\mathbb{R} & \xleftarrow[f^*]{} & S(\mathfrak{g}'^*)
\end{array}$$

where the morphisms are given by

$$f^*\colon p \mapsto p(0), \quad \pi^*\colon p \mapsto p \otimes 1, \quad \tilde{f}^*\colon \omega \mapsto 1 \otimes \omega, \quad \tilde{\pi}^*\colon c \mapsto c \otimes 1.$$

We have a sequence of algebra isomorphisms:

$$\begin{aligned}
\mathrm{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, S^*(\mathfrak{g}'^*) \otimes \mathcal{N}\big) &\cong \mathrm{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, \mathcal{C}_{T^m}(\Omega(\mathcal{Z}_\Sigma))\big) \cong \mathrm{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, H^*_{T^m}(\mathcal{Z}_\Sigma)\big) \\
&\cong \mathrm{Tor}^0_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, H^*_{T^m}(\mathcal{Z}_\Sigma)\big) \cong \mathbb{R} \otimes_{S(\mathfrak{g}'^*)} H^*_{T^m}(\mathcal{Z}_\Sigma) \cong H^*_{T^m}(\mathcal{Z}_\Sigma)/S^+(\mathfrak{g}'^*) \\
&\cong \mathbb{R}[v_1, ..., v_m]/(I_{\mathcal{K}} + J),
\end{aligned} \tag{4}$$

Here the first isomorphism follows from (3). The second isomorphism follows from Lemma 3.2 and Lemma 3.3. In the third isomorphism, the higher Tor vanish because $H^*_{T^m}(\mathcal{Z}_\Sigma)$ is a free module over $S^*(\mathfrak{g}'^*)$. The fourth and fifth isomorphisms are clear. For the last isomorphism, recall that $A\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}_A = \mathfrak{g}'$ is given by $\boldsymbol{e}_i \mapsto \boldsymbol{a}_i$, so that $A^*(\boldsymbol{u}) = \sum_{i=1}^m \langle \boldsymbol{u}, \boldsymbol{a}_i \rangle x_i$ for any $\boldsymbol{u} \in \mathfrak{g}'^*$.

On the other hand, we have a sequence of isomorphisms

$$\begin{aligned} (5)\quad \operatorname{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, S(\mathfrak{g}'^*)\otimes\mathcal{N}\big) &\cong \operatorname{Tor}^0_{S(\mathfrak{g}'^*)}\big(\mathbb{R}, S(\mathfrak{g}'^*)\otimes\mathcal{N}\big) \cong H(\mathbb{R})\otimes_{S(\mathfrak{g}'^*)} H\big(S(\mathfrak{g}'^*)\otimes\mathcal{N}\big)\\ &\cong H\big(\mathbb{R}\otimes_{S(\mathfrak{g}'^*)}(S(\mathfrak{g}'^*)\otimes\mathcal{N})\big)\cong H(\mathcal{N})\cong H^*_{\mathrm{bas}}(\mathcal{Z}_\Sigma). \end{aligned}$$

Here the vanishing of the higher Tor in the first isomorphism follows from (4). The second isomorphism is by definition of Tor^0. The third isomorphism follows from the Künneth Theorem, since $H\big(S(\mathfrak{g}'^*)\otimes\mathcal{N}\big) = H^*_{T^m}(\mathcal{Z}_\Sigma)$ is a free module over $S^*(\mathfrak{g}'^*)$. The fourth isomorphism is clear. The last isomorphism is Lemma 3.1.

The theorem follows from (4) and (5). □

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