# Homogeneous coordinate ring for semi-infinite Veronese curve of degree 2

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**Abstract**

In this work we describe a linear basis in the homogeneous coordinate ring for the jet scheme of Veronese curve of degree 2 and prove that this ring is embedded to the polynomial algebra. For the case of our basic field being $\mathbb{C}$, we prove that this coordinate ring is a direct sum of cocyclic $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ representations.

## 1 Introduction

Let $H, E, F$ be a standard basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with relations $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$. Denote the $\lambda+1$ dimensional irreducible representation of $\mathfrak{sl}_2$ by $V_\lambda$. Denote the $\lambda$-th symmetric power and the symmetric algebra of a vector space $V$ by $S^\lambda(V)$ and $S(V)$ respectively. Recall that $V_\lambda \cong S^\lambda V_1$.

Fix the basis $(v_\lambda^0, \ldots, v_\lambda^\lambda)$ of $V_\lambda$. Consider the action of $SL_2$ on $\mathbb{P}(V_\lambda)$. Note that $\mathrm{stab}([v_\lambda^0])$ is a Borel subgroup $B \subset SL_2$ of upper triangular matrices with determinant 1. Thus,

$$SL_2 \cdot [v_\lambda^0] = SL_2/(\mathrm{stab}([v_\lambda^0])) = SL_2/B = \mathbb{P}^1.$$

Let us consider $\exp(cF) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in SL_2$ for $c \in \mathbb{C}$. Assuming that $V_\lambda = S^\lambda V_1$, we get:

$$\exp(cF)v_\lambda^0 = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)(v_1^0)^\lambda = (v_1^0 + cv_1^1)^\lambda = \sum_{i=0}^{\lambda} \binom{\lambda}{i} c^i (v_1^0)^{\lambda-i}(v_1^1)^\lambda = \sum_{i=0}^{\lambda} \binom{\lambda}{i} c^i v_\lambda^i.$$

After the renormalisation of basis vectors and a projectivization, we obtain the map

$$\mathbb{CP}^1 \hookrightarrow \mathbb{P}(V_\lambda) = \mathbb{CP}^\lambda$$
$$(a:b) \longmapsto (a^\lambda : a^{\lambda-1}b : \ldots : b^\lambda).$$

The image of this map is called Veronese curve (or rational normal curve) of degree $\lambda$.

In what follows we consider the case $\lambda = 2$.

Let us fix a field $\Bbbk$ of zero characteristic. Then the Veronese curve of degree 2 is defined as the image of the map: $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, $(a : b) \longmapsto (a^2 : ab : b^2)$. It is easy to see that the coordinate ring of this curve is $\Bbbk[x, y, z]/\langle xz - y^2\rangle$ and a map $\Bbbk[x, y, z]/\langle xz - y^2\rangle \longrightarrow \Bbbk[a, b]$, defined by $x \longmapsto a^2, y \longmapsto ab, z \longmapsto b^2$ is injective. Also one can check that $\mathbb{C}[x, y, z]/\langle xz - y^2\rangle \simeq \bigoplus_{\lambda\geq 0} V_{2\lambda}$, Further detail can be found in [F] or [G].

We describe a semi-infinite analog of this. We have $\mathbb{C}[[t]]/\langle t^{N+1}\rangle$ instead of $\mathbb{C}$ and $\mathfrak{sl}_2 \otimes \mathbb{C}[[t]]/\langle t^{N+1}\rangle$ instead of $\mathfrak{sl}_2$.

Suppose $N \in \mathbb{Z}_{\geq 0} \cup \infty$. Consider a map

$$\Bbbk^2 \otimes \Bbbk[[t]]/\langle t^{N+1}\rangle \hookrightarrow \Bbbk^3 \otimes \Bbbk[[t]]/\langle t^{N+1}\rangle$$
$$(a(t), b(t)) \mapsto (a^2(t), a(t)b(t), b^2(t))$$

Here $a(t) = \sum_i a_i t^i, b(t) = \sum_i b_i t^i \in \mathbb{C}[[t]]/\langle t^{N+1}\rangle$ and a point $(a(t), b(t))$ means a point $(a_0, a_1, \ldots, b_0, \ldots)$.

This map induces a map

$$\mathbb{P}(\Bbbk^2 \otimes \Bbbk[[t]]/\langle t^{N+1}\rangle) \hookrightarrow \mathbb{P}(\Bbbk^3 \otimes \Bbbk[[t]]/\langle t^{N+1}\rangle)$$

We call the image of the last map by the jet scheme of Veronese curve of degree 2. In particular, for $N = \infty$ this gives the semi-infinite Veronese curve.

Below we describe some properties of the homogeneous coordinate ring of this curve.

# 2 Preliminaries

## 2.1 Jet Schemes

Let $I = \langle g_1, \ldots, g_s\rangle$ be an ideal in the polynomial ring $\Bbbk[u_1, ..., u_n]$.

By $J_N(I)$ we will denote the $N$-th jet ideal of $I$ which is an ideal in the polynomial ring $\Bbbk[u_1^{(0)}, u_1^{(1)}, \ldots, u_1^{(N)}, u_2^{(0)}, \ldots, u_2^{(N)}, \ldots, u_n^{(0)}, \ldots, u_n^{(N)}]$, generated by the coefficients of the power series

$$g_i(\sum_{j\geq 0} u_1^j t^j, \ldots, \sum_{j\geq 0} u_n^j t^j) \in \Bbbk[u_1, \ldots, u_n][[t]]/\langle t^{N+1}\rangle,$$

where $i = 1, \ldots, s$.

The ring $\mathbb{k}[u_1^{(0)},\ldots,u_1^{(N)},u_2^{(0)},\ldots,u_2^{(N)},\ldots,u_n^{(0)},\ldots,u_n^{(N)}]/J_N(I)$ is called $N$-th jet scheme.

We also define $J_\infty(I)$ as an ideal in the polynomial ring

$$\mathbb{k}[u_1^{(0)},u_1^{(1)},\ldots,u_2^{(0)},u_2^{(1)},\ldots,u_n^{(0)},u_n^{(1)},\ldots],$$

generated by the coefficients of the power series

$$g_i(\sum_{j\geq 0} u_1^j t^j,\ldots,\sum_{j\geq 0} u_n^j t^j)\in \mathbb{k}[u_1,\ldots,u_n][[t]],$$

where $i=1,\ldots,s$.

The ring $\mathbb{k}[u_1^{(0)},u_1^{(1)},\ldots,u_2^{(0)},u_2^{(1)},\ldots,u_n^{(0)},u_n^{(1)},\ldots]/J_\infty(I)$ will be named arc scheme.

In what follows for variables $F_0,F_1,\ldots$, belonging to some ring $A$ we will use the notation $F(t)=\sum_{i\geq 0}F_i t^i\in A[[t]]$.

By $\langle F(t)\rangle\subset A$ we denote the ideal, generated by the coefficients of $F(t)$: $\langle F(t)\rangle=\langle F_0,F_1,F_2,\ldots\rangle\subset A$.

For a ring homomorphism $\phi: A\longrightarrow B$ the following notation will be used: $\phi(F(t))=\sum_{n\geq 0}\phi(F_i)t^i\in B[[t]]$.

**Example.** *For $I=\langle xz-y^2\rangle\subset\mathbb{k}[x,y,z]$, we obtain the following 1-jet scheme:*

$$R^1=\mathbb{k}[x_0,x_1,y_0,y_1,z_0,z_1]/\langle x_0z_0-y_0^2,x_0z_1+x_1z_0-2y_0y_1\rangle,$$

*and the following arc scheme:*

$$R^\infty=\mathbb{k}[x_0,x_1,\ldots,y_0,y_1,\ldots,z_0,z_1,\ldots]/\langle x_0z_0-y_0^2,\ x_0z_1+x_1z_0-2y_0y_1,\ldots\rangle = \mathbb{k}[x_i,y_i,z_i]/\langle x(t)z(t)-y^2(t)\rangle. \tag{1}$$

Below the ring $A[F_0,F_1,F_2,\ldots]$ will be simply denoted by $A[F_i]$.

## 2.2 q-binomials

In what follows we need the following notation:

$$(x)_\infty=\prod_{i=0}^{\infty}(1-xq^i);$$

$$(x)_n=\frac{(x)_\infty}{(xq^n)_\infty}.$$

Note that for $x = q$ and $n \in \mathbb{N}$ we obtain:

$$(q)_n = \begin{cases} (1-q)(1-q^2)\dots(1-q^n), & \text{for } n \geq 1; \\ 1, & \text{for } n = 0. \end{cases}$$

We will use the following
partitions generating function

**Lemma 1.**

$$\sum_{k\geq 0} q^k p_n(k) = \frac{1}{(q)_n},$$

*where $p_n(k)$ is a number of partitions of $k$ into at most $n$ summands.*

*Proof.* Indeed, for a partition $\{0 \leq \lambda_1 \leq \ldots \leq \lambda_n,\ \lambda_1 + \ldots + \lambda_n = k\}$, we consider

$$\begin{aligned} f_1 &= \lambda_n - \lambda_{n-1}; \\ f_2 &= \lambda_{n-1} - \lambda_{n-2}; \\ &\vdots \\ f_{n-1} &= \lambda_2 - \lambda_1; \\ f_n &= \lambda_1. \end{aligned}$$

Then $f_i \geq 0$, $\sum_i f_i i = k$. That means that $\{f_1, \ , f_n\}$ corresponds to a partition of $k$ with $f_i$ summands equal to $i$. Then our generating function is equal to

$$\sum_{k\geq 0} q^k p_n(k) = \sum_{k\geq 0} q^k |\{f_1, \ldots, f_n : \sum_i f_i i = k\}| = \sum_{f_1,\ldots,f_n \geq 0} q^{1\cdot f_1 + \ldots + n\cdot f_n} =$$

$$(1 + q + q^2 + \ldots)(1 + q^2 + q^4 + \ldots)\ldots(1 + q^n + q^{2n} + \ldots) =$$

$$\prod_{i=1}^{n} \frac{1}{1-q^i} = \frac{1}{(q)_n}.$$

□

Also we state

**Lemma 2.** *Let $p_n(k, M)$ denote the number of partitions of $k$ into at most $n$ summands, each $\leq M$. Then*

$$\sum_{k\geq 0} p_n(k, M) q^k = \frac{(q)_{n+M}}{(q)_n (q)_M}.$$

For the proof of this Lemma we refer reader to [A].

Following [SW], we introduce q-Supernomial coefficients.

**Definition.** The $q-$binomial coefficient is defined as

$$\binom{L}{a}_q = \begin{cases} \dfrac{(q^{L-a+1})_a}{(q)_a}, & \text{for } a \in \mathbb{Z}_+, L \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition.** Let $N$ be a positive integer.

Define $N$-dimensional matrix $T$ as $T_{i,j} = \delta_{i,j}(2 - \delta_{j,N}) - \delta_{|i-j|,1}$. Then $T^{-1}_{i,j} = \min(i,j)$.

Suppose $(L_1, \ldots, L_N) = \mathbf{L} \in \mathbb{Z}^N$ and $(l_1, \ldots, l_N) = T^{-1}\mathbf{L}$. Then for $a + \frac{1}{2}l_N \in \mathbb{Z}_+$

$$\binom{\mathbf{L}}{a}_q = \sum_{j_1+\ldots+j_N = a+\frac{l_N}{2}} q^{\sum_{k=2}^N j_{k-1}(L_k+\ldots+L_N-j_k)} \binom{L_N}{j_N}_q \binom{L_{N-1}+j_N}{j_{N-1}}_q \cdots \binom{L_1+j_2}{j_1}_q. \tag{2}$$

In [SW], $T(\mathbf{L}, a) = q^{\frac{1}{4}\mathbf{L}T^{-1}\mathbf{L} - \frac{a^2}{N}} \binom{\mathbf{L}}{a}_{1/q}$ is called as $q-$supernomial coefficient. We will use the same notation for $\binom{\mathbf{L}}{a}_q$.

## 2.3 Gröbner bases

For given ordering $\succ$ on the set of monomials in some polynomial algebra $A$, we will use the notation $LM(P), P \in A$ for the leading monomial of $P$ with respect to $\succ$.

Following [CLO], we will use the following notation:

**Definition.** Fix a monomial order on $A$.

1. For given ideal $I \subset A$ we denote $LM(I) = \{LM(P) : P \in I\}$.
2. We denote by $\langle LM(I) \rangle$ the ideal, generated by the elements of $LM(I)$.
3. We say that a finite subset $\{g_1, \ldots, g_s\}$ of an ideal $I$ is its Gröbner basis if
$$\langle LM(g_1), \ldots, LM(g_s) \rangle = \langle LM(I) \rangle.$$

It follows from the definition that for a Gröbner basis $\{g_1, \ldots, g_s\}$ of an ideal $I$ we have $\langle g_1, \ldots, g_s \rangle = I$.

We will use the following lemmas:

**Lemma 3.** *Fix a monomial order in A. Let* $\langle g_1, \ldots, g_s \rangle \subset A$ *be an ideal, and let the monomials* $LM(g_1), \ldots, LM(g_s)$ *be pairwise relatively prime. Then* $\{g_1, \ldots, g_s\}$ *is a Gröbner basis.*

**Lemma 4.** *Fix a monomial order in A. Suppose* $\{g_1, \ldots, g_s\}$ *is a Gröbner basis of an ideal* $I \subset A$. *Then the set of monomials, that are not divisible by any of* $LM(g_i)$, *forms the linear basis of* $A/I$ *(as a vector space).*

Lemma 3 is proved in [CLO] by consecutive implementation of Theorem 3 and Proposition 4 in §2.9.

Lemma 4 is proved in [CLO] in Proposition 1 of §2.6.

# 3 Basis of $R$

## 3.1 Basis of $R^\infty$

In this subsection we describe the linear basis of $R^\infty$ and find its character.

Consider the arc scheme $R = R^\infty = \Bbbk[x_i, y_i, z_i]/\langle x(t)z(t) - y^2(t)\rangle$, defined in (1).

We introduce three different gradings $\deg, \mathrm{sum}, \Delta$ on the algebra $R$ by their values on variables:

$$\begin{array}{lll} \deg(x_i) = 1 & \mathrm{sum}(x_i) = i & \Delta(x_i) = 2 \\ \deg(y_i) = 1 & \mathrm{sum}(y_i) = i & \Delta(y_i) = 0 \\ \deg(z_i) = 1 & \mathrm{sum}(z_i) = i & \Delta(z_i) = -2 \end{array}$$

So, for arbitrary monomial we have

$$\deg(x_{i_1} \ldots x_{i_{n_1}} y_{l_1} \ldots y_{l_{n_2}} z_{j_1} \ldots z_{j_{n_3}}) = n_1 + n_2 + n_3;$$
$$sum(x_{i_1} \ldots x_{i_{n_1}} y_{l_1} \ldots y_{l_{n_2}} z_{j_1} \ldots z_{j_{n_3}}) = \sum_s i_s + \sum_s j_s + \sum_s l_s;$$
$$\Delta(x_{i_1} \ldots x_{i_{n_1}} y_{l_1} \ldots y_{l_{n_2}} z_{j_1} \ldots z_{j_{n_3}}) = 2n_1 - 2n_3.$$

Note that all the coefficients of $x(t)z(t) - y^2(t)$ are homogeneous with respect to $\deg, \mathrm{sum}$ and $\Delta$ and that is why these gradings are well-defined on $R = \Bbbk[x_i, y_i, z_i]/\langle x(t)z(t) - y^2(t)\rangle$.

Let us describe a linear basis of $R$.

To do this, we need to describe a basis of the algebra $R' = \Bbbk[x_i, z_i]/\langle x(t)z(t)\rangle$. Note that the gradings $\deg_x$ and $\deg_z$, defined by $\deg_x(x_i) = \deg_z(z_i) = 1$, $\deg_x(z_i) = \deg_z(x_i) = 0$, are well-defined on $R'$, because all the coefficients of $x(t)z(t)$ are homogeneous by $\deg_x$ and $\deg_z$.

Consider $R'_{n_1,n_2;k} = \{r' \in R' : \deg_x(r') = n_1, \deg_z(r') = n_2, \mathrm{sum}(r') = k\}$.

Define $\mathrm{ch}(R'_{n_1,n_2})(q) = \sum_{k \geq 0} q^k \cdot \dim(R'_{n_1,n_2;k})$.

**Lemma 5.** $\mathrm{ch}(R'_{n_1,n_2})(q) = q^{n_1 n_2}\dfrac{1}{(q)_{n_1}(q)_{n_2}}$.

*Proof.* Consider the series of $n_1+n_2$ variables:

$$x(t_1)x(t_2)\dots x(t_{n_1})z(s_1)\dots z(s_{n_2}) = \sum_{\substack{i_1,\dots,i_{n_1}\\ j_1,\dots,j_{n_2}}} (x_{i_1}\dots x_{i_{n_1}} z_{j_1}\dots z_{j_{n_2}}) t_1^{i_1}\dots t_{n_1}^{i_{n_1}} s_1^{j_1}\dots s_{n_1}^{j_{n_2}}.$$

Let us introduce the map

$$d : (R'_{n_1,n_2;k})^* \longrightarrow \Bbbk[t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}];$$
$$d : \xi \longmapsto \xi(x(t_1)x(t_2)\dots x(t_{n_1})z(s_1)\dots z(s_{n_2})) =$$
$$\sum_{\substack{i_1,\dots,i_{n_1}\\ j_1,\dots,j_{n_2}}} \xi(x_{i_1}\dots x_{i_{n_1}} z_{j_1}\dots z_{j_{n_2}}) t_1^{i_1}\dots t_{n_1}^{i_{n_1}} s_1^{j_1}\dots s_{n_1}^{j_{n_2}}.$$

Note that $d$ is injective. Indeed, suppose $d(\xi)=0$ for some $\xi$. It means that $\xi(x_{i_1}\dots x_{i_{n_1}} z_{j_1}\dots z_{j_{n_2}})=0$ for any monomial, which means that simply $\xi\equiv 0$.

Now we will describe $\mathrm{Im}(d)$.

Since $d$ is defined on $(R'_{n_1,n_2;k})^*$, $\mathrm{Im}(d)$ lies in the space of homogeneous polynomials of degree $k$. Let us denote the space of homogeneous polynomials of degree $i$ by $\Bbbk[t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}]_i$. So, $\mathrm{Im}(d)\subset \Bbbk[t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}]_k$.

Recall that $x(t)z(t)=0$ in $R'$. Thus, for any $\xi$ the following equation holds

$$d(\xi)(t_1,\dots,t_{n_1},s_1,\dots,s_{n_2})|_{t_i=s_j} = 0.$$

It means that $\prod_{i,j}(t_i-s_j) | d(\xi)$, and, therefore, $\mathrm{Im}(d)\subset \prod_{i,j}(t_i-s_j)\cdot \Bbbk[t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}]_{k-n_1n_2}$.

Finally, let us note that $d(\xi)(t_1,\dots,t_{n_1},s_1,\dots,s_{n_2})$ does not change under the interchange of variables $t_1,\dots,t_{n_1}$ or $s_1,\dots,s_{n_2}$. So,

$$\mathrm{Im}(d)\subset \prod_{i,j}(t_i-s_j)\Bbbk[t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}]_{k-n_1n_2}^{S_{n_1}\times S_{n_2}}.$$

Now let us prove that any $P\in \prod_{i,j}(t_i-s_j)\cdot \Bbbk[t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}]_{k-n_1n_2}^{S_{n_1}\times S_{n_2}}$ lies in $\mathrm{Im}(d)$.

Let

$$P(t_1,\dots,t_{n_1},s_1,\dots,s_{n_2}) = \sum_{\substack{i_1\le\dots\le i_{n_1}\\ j_1\le\dots\le j_{n_2}}} a_{i_1,\dots,i_{n_1},j_1,\dots,j_{n_2}} t_1^{i_1}\dots t_{n_1}^{i_{n_1}} s_1^{j_1}\dots s_{n_1}^{j_{n_2}}.$$

Then $\xi \in (\Bbbk[x_i, z_i]_{n_1,n_2;k})^*$, simply defined by $\xi(x_{i_1} \dots x_{i_{n_1}} z_{j_1} \dots z_{j_{n_2}}) = a_{i_1,\dots,i_{n_1},j_1,\dots,j_{n_2}}$, satisfies $\xi(\prod_{1\le i\le n_1} x(t_i) \prod_{1\le j\le n_2} z(s_j)) = P$.

Note that $P$ lies in $\prod_{i,j}(t_i - s_j) \cdot \Bbbk[t_1, \dots, t_{n_1}, s_1, \dots, s_{n_2}]^{S_{n_1}\times S_{n_2}}_{k-n_1n_2}$, and it implies that for constructed $\xi$, we have $\xi(x(t)z(t)) = 0$. That means that $\xi$ is well defined on $(\Bbbk[x_i, z_i]/\langle x(t)z(t)\rangle)_{n_1,n_2;k} = R'_{n_1,n_2;k}$.

So, we proved that $d$ is injective, and

$$\mathrm{Im}(d) = \prod_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} (t_i - s_j) \cdot \Bbbk[t_1, \dots, t_{n_1}, s_1, \dots, s_{n_2}]^{S_{n_1}\times S_{n_2}}_{k-n_1n_2}.$$

Hence,

$$\dim(R'_{n_1,n_2;k}) = \dim((R'_{n_1,n_2;k})^*) = \dim(\Bbbk[t_1, \dots, t_{n_1}, s_1, \dots, s_{n_2}]^{S_{n_1}\times S_{n_2}}_{k-n_1n_2}) = \\ |\{0 \le i_1 \le \dots \le i_{n_1}, 0 \le j_1, \le \dots \le j_{n_2} : (i_1+\dots+i_{n_1})+(j_1+\dots+j_{n_2}) = k-n_1n_2\}| \tag{3}$$

Using Lemma 1, we have

$$\mathrm{ch}(R'_{n_1,n_2})(q) = \sum_{k\ge 0} q^k \cdot \dim(R'^k_{n_1,n_2}) = q^{n_1n_2}\frac{1}{(q)_{n_1}(q)_{n_2}},$$

and that was the statement to be proven. □

The same can be said in the following way. Suppose $R'_{n,k,\delta} = \{r' \in R' : \dim(r') = n, \mathrm{sum}(r') = k, \Delta(r') = \delta\}$ and $\mathrm{ch}(R') = \sum_{n,k,\delta} u^n q^k s^\delta \dim(R'_{n,k,\delta})$. Then:

$$\mathrm{ch}(R') = \sum_{n,k,\delta} u^n q^k s^\delta \dim(R'_{n,k,\delta}) = \\ \sum_{n\ge 0} u^n (\sum_{n_1+n_2=n} q^{n_1n_2}\frac{1}{(q)_{n_1}(q)_{n_2}} s^{2(n_1-n_2)}). \tag{4}$$

Now, when we know the dimension of $R'_{n_1,n_2;k}$, we can describe its basis.

**Lemma 6.** *The set of monomials*

$$B' = \{x_{i_1} \dots x_{i_{n_1}} z_{j_1} \dots z_{j_{n_2}} : \forall\ u\ i_u \ge n_2; i_1 \ge \dots \ge i_{n_1}, j_1 \ge \dots \ge j_{n_2}\}$$

*forms a linear basis of $R'$.*

*Proof.* First we check that $|B' \cap R'_{n_1,n_2;k}| = \dim(R'_{n_1,n_2;k})$. Indeed, using (3):

$$|B' \cap R'_{n_1,n_2;k}| =$$
$$|\{i_1 \leq \ldots \leq i_{n_1}, j_1, \leq \ldots \leq j_{n_2} : \forall\ u\ i_u \geq n_2; \sum_u i_u + \sum_u j_u = k\}| =$$
$$|\{i_1 \leq \ldots \leq i_{n_1}, j_1, \leq \ldots \leq j_{n_2} | \forall\, u\, i_u \geq n_2; \sum_u (i_u - n_2) + \sum_u j_u = k - n_1 n_2\}| =$$
$$|\{i'_1 \leq \ldots \leq i'_{n_1}, j_1, \leq \ldots \leq j_{n_2} : \sum_u i'_u + \sum_u j_u = k - n_1 n_2\}| = \dim(R'_{n_1,n_2;k}).$$

Hence, it is enough to prove that $B'$ is a spanning set in $R'$.

Let us prove that any monomial $x_{i_1} \ldots x_{i_{n_1}} z_{j_1} \ldots z_{j_{n_2}}$ belongs to $\mathrm{span}(B')$. The case $n_1 = 0$ is trivial. We proceed by induction on $n_1$ for $n_1 \geq 1$.

**Induction step.** For $n_1 \geq 2$ we write $x_{i_1} \ldots x_{i_{n_1}} z_{j_1} \ldots z_{j_{n_2}}$ as

$$x_{i_1}(x_{i_2} \ldots x_{i_{n_1}} z_{j_1} \ldots z_{j_{n_2}}).$$

By the induction hypothesis we have

$$x_{i_1} \ldots x_{i_{n_1}} z_{j_1} \ldots z_{j_{n_2}} = x_{i_1}\Big( \sum_{\substack{i'=(i'_2,\ldots,i'_{n_1}) \\ j'=(j'_1,\ldots,j'_{n_2})}} \alpha_{i',j'} x_{i'_2} \ldots x_{i'_{n_1}} z_{j'_1} \ldots z_{j'_{n_2}}\Big)$$

where $\forall\ u\ i'_u \geq n_2$.

Then we write each $x_{i_1} x_{i'_2} \ldots x_{i'_{n_1}} z_{j'_1} \ldots z_{j'_{n_2}}$ as $x_{i'_2}(x_{i_1} \ldots x_{i'_{n_1}} z_{j'_1} \ldots z_{j'_{n_2}})$ and apply the induction hypothesis to each of monomials $x_{i_1} \ldots x_{i'_{n_1}} z_{j'_1} \ldots z_{j'_{n_2}}$. As result we obtain the decomposition of $x_{i_1} \ldots x_{i_{n_1}} z_{j_1} \ldots z_{j_{n_2}}$ to the linear combination of the elements of $B'$.

**Induction base.** To represent $x_{i_1} z_{j_1} \ldots z_{j_{n_2}}$ as an element of $\mathrm{span}(B')$ we recall that in $R'$ we have $\sum_{i+j=m} x_i z_j = 0$. So, we will use that

$$x_u z_v \in \mathrm{span}(\{x_i z_j\ :\ i + j = u + v,\ \{i, j\} \neq \{u, v\}\}). \tag{5}$$

Suppose there exist monomials that do not lie in $\mathrm{span}(B')$. We introduce a partial order on the set of such monomials:

$$x_{i_1} z_{j_1} \ldots z_{j_{n_2}} \succ x_{i'_1} z_{j'_1} \ldots z_{j'_{n_2}} (j_1 \geq \ldots \geq j_{n_2},\ j'_1 \geq \ldots \geq j'_{n'_2}),$$
$$\text{if } (j_{n_2}, \ldots, j_1) \succ (j'_{n_2}, \ldots, j'_1) \text{ in a ususal lexicgraphic order.}$$

Let $M = x_{i_1} z_{j_1} \ldots z_{j_{n_2}}$ $(j_1 \geq \ldots \geq j_{n_2})$ be any of minimal monomials, which do not lie in $\mathrm{span}(B')$. $i_1 < n_2$, because otherwise $M \in B'$.

Below we assume that for any $i < 0$ we have $x_i = z_i = 0 \in \mathrm{span}(B')$.

At the first step we apply (5) to $x_{i_1}z_{j_1}$. Note that all the monomials $x_{i_1+u_1}z_{j_1-u_1}\dots z_{j_{n_2}}$ (for positive $u_1$) are less than $M$, and, therefore, lie in $\mathrm{span}(B')$. So, in the vector space $R'/\mathrm{span}(B')$ we have

$$M \in \mathrm{span}(\{x_{i_1-u_1}z_{j_1+u_1}\dots z_{j_{n_2}} : u_1 \geq 1\}).$$

At the second step we apply (5) in each of the monomials $x_{i_1-u_1}z_{j_1+u_1}\dots z_{j_{n_2}}$ to $x_{i_1-u_1}z_{j_2}$. As in the previous step, we note that every monomial

$$x_{i_1-u_1+u_2}z_{j_1+u_1}z_{j_2-u_2}\dots z_{j_{n_2}}$$

is less than $M$, and, therefore, lies in $\mathrm{span}(B')$. So, in the vector space $R'/\mathrm{span}(B')$ we have:

$$M \in \mathrm{span}(\{x_{i_1-u_1-u_2}z_{j_1+u_1}z_{j_2+u_2}\dots z_{j_{n_2}} : u_1, u_2 \geq 1\}).$$

Making $n_2$ such steps (applying (5) to $x_{i_1-u_1-\dots-u_s}z_{j_{s+1}}$ on the step $s+1$), we get:

$$M \in \mathrm{span}(\{x_{i_1-u_1-\dots-u_{n_2}}z_{j_1+u_1}\dots z_{j_{n_2}+u_{n_2}} : u_1\dots u_{n_2} \geq 1\}).$$

But then $i_1 - u_1 - \dots - u_{n_2} \leq i_1 - n_2 < 0$, so we have $M \in \mathrm{span}(0) \subset R'/\mathrm{span}(B')$ and we are done. □

Now we describe a linear basis in $R$.

**Theorem 7.** *The set of monomials*

$$B = \{x_{i_1}\dots x_{i_{n_1}}y_{l_1}\dots y_{l_{n_2}}z_{j_1},\dots,z_{j_{n_3}} : \forall\ u\ i_u \geq n_3;\\ i_1 \geq \dots \geq i_{n_1}, l_1 \geq \dots \geq l_{n_2}, j_1 \geq \dots \geq j_{n_3}\}$$

*forms a basis in $R$.*

*Proof.* The fact that $\mathrm{span}(B) = R$ can be proven similarly to the fact that $\mathrm{span}(B') = R'$ in the proof of Lemma 6. The same algorithm of presenting a monomial as an element of $\mathrm{span}(B)$ works, we just need to apply the relation

$$x_u z_v \in \mathrm{span}(\{x_i z_j - \sum_{a+b=u+v} y_a y_b : i+j = u+v,\ \{i,j\} \neq \{u,v\}\})$$

instead of (5).

Let us prove that elements of $B$ are linearly independent. Denote by $w_i$ the $i$-th coefficient of the series $x(t)z(t) - y^2(t)$, and by $w'_i$ the $i$-th coefficient of $x(t)z(t)$. So, $x(t)z(t) - y^2(t) = \sum_i w_i t^i$, $x(t)z(t) = \sum_i w'_i t^i$ and $w_i = w'_i - \sum_{0\leq j\leq i} y_j y_{j-i}$.

Suppose, the elements of $B$ are not linearly independent. This means that there exist $b_1, \ldots, b_l \in B$, $\alpha_1, \ldots, \alpha_l \in \mathbb{k}$, $P_1, \ldots, P_m \in \mathbb{k}[x_i, y_i, z_i]$ and $u_1, \ldots, u_m \in \mathbb{N}$ such that in $\mathbb{k}[x_i, y_i, z_i]$ holds:

$$\sum_{0 \le i \le l} \alpha_i b_i = \sum_{0 \le i \le m} P_i w_{u_i}. \tag{6}$$

There is a natural homomorphism $\phi : \mathbb{k}[x_i, y_i, z_i] \longrightarrow (\mathbb{k}(y_i))[x_i, z_i]$. Applying this homomorphism to both parts of (6), we have that for some $b_i' \in B'$ $\alpha_i' \in \mathbb{k}(y_i)$, $P_i' \in \mathbb{k}(y_i)[x_i, z_i]$ the following holds in $\mathbb{k}(y_i)[x_i, z_i]$:

$$\sum_{0 \le i \le l} \alpha_i' b_i' = \sum_{0 \le i \le m} P_i' w_{u_i} = \sum_{0 \le i \le m} P_i' (w_i' - \sum_{0 \le j \le i} y_j y_{j-i}). \tag{7}$$

Here $B'$ and $w_i'$ are considered as the sets of elements of $\mathbb{k}(y_i)[x_i, z_i]$, and all the polynomials in $\mathbb{k}(y_i)[x_i, z_i]$ are considered as polynomials in $x_i, z_i$ with coefficients in the field $\mathbb{k}(y_i)$.

By $HD(P), P \in \mathbb{k}(y_i)[x_i, z_i]$ we denote the homogeneous component of $P$ of the highest degree with respect to the grading $\deg = \deg_x + \deg_z$. Taking $HD$ from both sides of (7), we obtain:

$$\sum_{0 \le i \le l} HD(\alpha_i') b_i' = \sum_{0 \le i \le l} HD(\alpha_i') HD(b_i') =$$
$$\sum_{0 \le i \le m} HD(P_i') HD(w_i' - \sum_{0 \le j \le i} y_j y_{j-i}) = \sum_{0 \le i \le m} HD(P_i') w_i'.$$

But this means that the elements of $B'$ are not linearly independent in $\mathbb{k}(y_i)[x_i, z_i]/(w_1, \ldots)$. That contradicts Lemma 6 (Lemma 6 was proved for any field, including $\mathbb{k}(y_i)$). □

Define

$$\mathrm{ch}(R) = \sum_{n,k,\delta} u^n q^k s^\delta \dim(\{r \in R : \dim(r) = n, \mathrm{sum}(r) = k, \Delta(r) = \delta\}).$$

**Corollary 7.1.** *We have*

$$\mathrm{ch}(R) = \sum_{n \ge 0} \frac{u^n}{(q)_n} \Big( \sum_{n_1+n_2+n_3=n} q^{n_1 n_3} \frac{(q)_n}{(q)_{n_1} (q)_{n_2} (q)_{n_3}} s^{2(n_1-n_3)} \Big) =$$
$$\sum_{n \ge 0} \frac{u^n}{(q)_n} \Big( \sum_{-n \le a \le n} \binom{(0,n)}{a}_q s^{2a} \Big).$$

*Proof.* Note that the gradings $\deg, \mathrm{sum}, \Delta$ are well defined on the rings $R, R'$ and $\Bbbk[y_i] \subset R$. The descriptions of bases of $R$ and $R'$ in Lemma 6 and Theorem 7 implies that the following holds for vector spaces

$$R \simeq R' \otimes \Bbbk[y_i],$$

and the isomorphism respects the gradings $\deg, \mathrm{sum}, \Delta$. In other terms, that means the equality of characters:

$$\mathrm{ch}(R) = \mathrm{ch}(R')\mathrm{ch}(\Bbbk[y_i]) =$$
$$\Big(\sum_{n\geq 0} u^n \big( \sum_{n_1+n_3=n} q^{n_1 n_3} \frac{1}{(q)_{n_1}(q)_{n_3}} s^{2(n_1-n_3)} \big)\Big)\Big(\sum_{n\geq 0} u^n \frac{1}{(q)_n}\Big) =$$
$$\sum_{n\geq 0} u^n \big( \sum_{n_1+n_2+n_3=n} q^{n_1 n_3} \frac{1}{(q)_{n_1}(q)_{n_2}(q)_{n_3}} s^{2(n_1-n_3)} \big) =$$
$$\sum_{n\geq 0} \frac{u^n}{(q)_n} \big( \sum_{n_1+n_2+n_3=n} q^{n_1 n_3} \frac{(q)_n}{(q)_{n_1}(q)_{n_2}(q)_{n_3}} s^{2(n_1-n_3)} \big).$$

This can be expressed in terms of q-supernomial coefficients, defined in (2):

$$\binom{(0,n)}{a}_q = \sum_{j_1+j_2=a+n} q^{j_1(n-j_2)} \binom{n}{j_2}_q \binom{j_2}{j_1}_q .$$

In the last sum all the summands with $j_2 < j_1$ are equal to 0, so we can assume that $j_2 \geq j_1$ and introduce new variables $n_1 = j_1$, $n_2 = j_2 - j_1$, $n_3 = n - j_2$, with conditions $n_1 + n_2 + n_3 = n$, $n_1 - n_3 = j_1 + j_2 - n = a$. So we have

$$\binom{(0,n)}{a}_q = \sum_{\substack{n_1+n_2+n_3=n \\ 2n_1-2n_3=2a}} q^{n_1 n_3} \binom{n}{n_1+n_2}_q \binom{n_1+n_2}{n_1}_q =$$
$$\sum_{\substack{n_1+n_2+n_3=n \\ 2n_1-2n_3=2a}} q^{n_1 n_3} \frac{(q)_n}{(q)_{n_1}(q)_{n_2}(q)_{n_3}}.$$

Thus,

$$\mathrm{ch}(R) = \sum_{n\geq 0} \frac{u^n}{(q)_n} \big( \sum_{-n\leq a\leq n} \binom{(0,n)}{a}_q s^{2a} \big).$$

□

## 3.2 Basis of $R^N$

We define $R^N, N \in \mathbb{Z}_{\geq 0}$ as $N$-th jet scheme of $\langle xz - y^2 \rangle \subset \Bbbk[x,y,z]$.

In this subsection we describe the linear basis of $R^N$ for $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

We prove that the set of monomials that are not divisible by any of $z_0x_1, z_1x_2, \ldots, z_{N-1}x_N$, $y_0^2, y_1^2, \ldots, y_N^2$ forms a linear basis of $R^N$. More formally, the following Theorem holds.

**Theorem 8.** *The set of monomials*

$$B_N = \{x_{i_1} \ldots x_{i_{n_1}} y_{q_1} \ldots y_{q_{n_2}} z_{j_1}, \ldots, z_{j_{n_3}} : \forall\ u_1, u_2\ i_{u_1} - 1 \neq j_{u_2}; \forall\ u_1 \neq u_2\ q_{u_1} \neq q_{u_2}; \\ i_1 \geq \ldots \geq i_{n_1}, q_1 \geq \ldots \geq q_{n_2}, j_1 \geq \ldots \geq j_{n_3}\}$$

*forms a linear basis of $R^N$.*

*Proof.* Denote by $w_i$ the $i$-th coefficient of the series $x(t)z(t) - y^2(t) \in \Bbbk[x_0, \ldots, x_N, y_0, \ldots, y_N, z_0, \ldots, z_N][[t]]/\langle t^{N+1} \rangle$. So, $J_N = J_N(\langle xz - y^2 \rangle) = \langle w_0, \ldots, w_N \rangle$.

Let us introduce the reverse lexicographic order on $\Bbbk[x_0, \ldots, x_N, y_0, \ldots, y_N, z_0, \ldots, z_N]$ with respect to the following order on the variables:

$$x_0 \succ y_0 \succ z_0 \succ x_1 \succ y_1 \succ \ldots z_{i-1} \succ x_i \succ y_i \succ z_i \succ x_{i+1} \succ \ldots \succ z_N.$$

Reverse lexicographic monomial order on the ring $\Bbbk[F_1, \ldots, F_n]$ with respect to the order on the variables

$$F_0 \succ F_1 \succ \ldots \succ F_n$$

means that $F^\alpha = F_0^{\alpha_1} \ldots F_n^{\alpha_n} \succ F^\beta = F_0^{\beta_1} \ldots F_n^{\beta_n}$ if and only if the rightmost nonzero entry of $\alpha - \beta$ is negative.

Note that

$$LM(w_{2i}) = LM(x_0z_{2i} + \ldots + x_{2i}z_0 - y_0y_{2i} - \ldots - y_{2i}y_0) = y_i^2;$$
$$LM(w_{2i+1}) = LM(x_0z_{2i+1} + \ldots + x_{2i+1}z_0 - y_0y_{2i+1} - \ldots - y_{2i+1}y_0) = z_ix_{i+1}.$$

So, $LM(w_0), \ldots, LM(w_N)$ are pairwise relatively prime, and, by Lemma 3 we can say that $\{w_0, \ldots, w_N\}$ is a Groöbner basis of $J_N$ with respect to this order.

Using Lemma 4 we immediately obtain the statement of Theorem 8. □

Let us calculate the character of $R^N$, defined by:

$$\mathrm{ch}(R^N) = \sum_{n,k,\delta} u^n q^k s^\delta \dim(R^N_{n,k,\delta}) =$$
$$\sum_{n,k,\delta} u^n q^k s^\delta \dim(\{r \in R^N : \dim(r) = n, \mathrm{sum}(r) = k, \Delta(r) = \delta\}).$$

First let us define $\mathrm{ch}(\Bbbk[x_0,\dots,z_N]) = \sum_{n,k,\delta} u^n q^k s^\delta \dim(\Bbbk[x_0,\dots,z_N]_{n,k,\delta}) = \sum_{n,k,\delta} u^n q^k s^\delta \dim(\{P \in \Bbbk[x_0,\dots,z_N] : \dim(P) = n, \mathrm{sum}(P) = k, \Delta(P) = \delta\})$ and using Lemma 2 we have

$$\mathrm{ch}(\Bbbk[x_0,\dots,z_N]) =$$

$$\sum_{n,k,\delta} u^n q^k s^\delta \cdot |\{i_1 \le \dots \le i_{n_1} \le N, l_1 \le \dots \le l_{n_2} \le N, j_1 \le \dots \le j_{n_3} \le N :$$

$$n_1+n_2+n_3 = n, \sum_s i_s + \sum_s l_s + \sum_s j_s = k, 2n_1 - 2n_3 = \delta\}| =$$

$$\sum_{n\ge 0} u^n \Big( \sum_{n_1+n_2+n_3=n} \frac{(q)_{n_1+N}}{(q)_N (q)_{n_1}} \cdot \frac{(q)_{n_2+N}}{(q)_N (q)_{n_2}} \cdot \frac{(q)_{n_2+N}}{(q)_N (q)_{n_2}} \cdot s^{2n_1-2n_3} \Big).$$

Note that $\dim(R^N_{n,k,\delta})$ is equal to the number of monomials in $\Bbbk[x_0,\dots,z_N]_{n,k,\delta}$ that are not divisible by any of $y_i^2$, $z_i x_{i+1}$. Using the inclusion–exclusion principle, we get:

$$\dim(R^N_{n,k,\delta}) = \sum_{a,b\ge 0} \dim(\Bbbk[x_0,\dots,z_N]_{n-2a,k-b,\delta}) \cdot (-1)^a p_a(b),$$

where $p_a(b)$ is a number of partitions of $b$ into at most $a$ summands.

In terms of characters this means:

$$\mathrm{ch}(R^N) = \sum_{a,b\ge 0} \mathrm{ch}(\Bbbk[x_0,\dots,z_N]) u^{2a} q^b p_a(b) =$$

$$\mathrm{ch}(\Bbbk[x_0,\dots,z_N]) \sum_{a\ge 0} (-u^2)^a \sum_{b\ge 0} p_a(b) q^b =$$

$$\Big( \sum_{n_1,n_2,n_3} u^{n_1+n_2+n_3} \frac{(q)_{n_1+N}(q)_{n_2+N}(q)_{n_3+N}}{(q)_N^3 (q)_{n_1}(q)_{n_2}(q)_{n_3}} \cdot s^{2n_1-2n_3} \Big) \Big( \sum_{a\ge 0} (-u^2)^a \frac{1}{(q)_a} \Big)$$

# 4 Embedding to the Polynomial Algebra

Below we denote a localization of a commutative ring $A$ by a multiplicative set $\{a^i : i \ge 0\}$, $a \in A$ by $S_a^{-1}A$.

**Remark.** *Note that for any series $\sum_{i\ge 0} F_i t^i = F(t) \in \Bbbk[F_0, F_1, \dots][[t]]$, all the coefficients of $F^{-1}(t)$ can be expressed in terms of $\{F_0^{-1}, F_0, F_1, F_2, \dots\}$.*

*In other words, $F^{-1}(t) \in (S_{F_0}^{-1}(\Bbbk[F_0, F_1, \dots]))[[t]]$.*

*The same holds for the finite case. If $F(t) = \sum_{i=0}^N F_i t^i \in \Bbbk[F_0, \dots, F_N][[t]]/\langle t^{N+1}\rangle$, then $F^{-1}(t) \in (S_{F_0}^{-1}(\Bbbk[F_0, F_1, \dots]))[[t]]/\langle t^{N+1}\rangle$.*

In what follows instead of defining a map $\phi : A[F_i] \longrightarrow B$ on each of variables $F_i$, we will define $\phi(F(t)) \in B[[t]]\langle t^{N+1} \rangle$.

**Theorem 9.** *$f : R^N \longrightarrow \Bbbk[a_0, \dots, b_N]$, defined by the rules*

$$\begin{aligned} f &: x(t) \longmapsto a^2(t); \\ f &: y(t) \longmapsto a(t)b(t); \\ f &: z(t) \longmapsto b^2(t) \end{aligned}$$

*is injective.*

**Remark.** *Theorem 9 means that $R$ is a homogeneous coordinate ring for the jet scheme of Veronese curve of degree 2.*

*Also it implies that $R$ is a domain and, therefore, the the jet scheme of Veronese curve of degree 2 is an irreducible variety.*

Let us prove it.

*Proof.* Let us consider the following diagram:

$$\begin{array}{ccc} R^N & \xrightarrow{\quad f \quad} & \Bbbk[a_i, b_i] \\ \big\downarrow l & & \\ S_{x_0}^{-1} R^N & & \\ \big\updownarrow \phi_1 & & \big\downarrow \psi \\ S_{x_0}^{-1}(\Bbbk[x_i, Y_i, Z_i]/\langle Z(t) - Y^2(t) \rangle) & & \\ \big\updownarrow \phi_2 & & \\ S_{x_0}^{-1}\Bbbk[x_i, Y_i] & \xhookrightarrow{\quad \tau \quad} & \Bbbk(c_i, d_i) \end{array}$$

Where the maps are defined as follows:
$l : R^N \longrightarrow S_{x_0}^{-1} R^N$ is a usual localization map.
$\phi_1 : S_{x_0}^{-1} R^N \longrightarrow S_{x_0}^{-1}(\Bbbk[x_i, Y_i, Z_i]/\langle Z(t) - Y^2(t) \rangle)$ is defined by:

$$\begin{aligned} \phi_1 &: x(t) \longmapsto x(t); \\ \phi_1 &: y(t) \longmapsto Y(t)x(t); \\ \phi_1 &: z(t) \longmapsto Z(t)x(t); \\ \phi_1 &: x_0^{-1} \longmapsto x_0^{-1}. \end{aligned}$$

$\phi_2 : S_{x_0}^{-1}(\Bbbk[x_i, Y_i, Z_i]/\langle Z(t) - Y^2(t)\rangle) \longrightarrow S_{x_0}^{-1}\Bbbk[x_i, Y_i]$ is defined by:

$$\begin{aligned}\phi_2 &: x(t) \longmapsto x(t);\\ \phi_2 &: Y(t) \longmapsto Y(t);\\ \phi_2 &: Z(t) \longmapsto Y^2(t);\\ \phi_2 &: x_0^{-1} \longmapsto x_0^{-1}.\end{aligned}$$

$\tau : S_{x_0}^{-1}\Bbbk[x_i, Y_i] \longrightarrow \Bbbk(c_i, d_i)$ is defined by:

$$\begin{aligned}\tau &: x(t) \longmapsto c^2(t);\\ \tau &: Y(t) \longmapsto d^2(t);\\ \tau &: x_0^{-1} \longmapsto c_0^{-2}.\end{aligned}$$

$\psi : \Bbbk[a_i, b_i] \longrightarrow \Bbbk(c_i, d_i)$ is defined by:

$$\begin{aligned}\psi &: a(t) \longmapsto c(t);\\ \psi &: b(t) \longmapsto c(t)d^2(t).\end{aligned}$$

Note that all the maps are well-defined on the corresponding quotient-rings.

First of all, we check that this diagram is commutative. It is easy to check that:

$$\begin{aligned}\tau\phi_2\phi_1 l(x(t)) &= c^2(t) = \psi f(x(t));\\ \tau\phi_2\phi_1 l(y(t)) &= c^2(t)d^2(t) = \psi f(y(t));\\ \tau\phi_2\phi_1 l(z(t)) &= c^2(t)d^4(t) = \psi f(z(t)).\end{aligned}$$

Now we will prove that $l, i$ are injective and $\phi_1, \phi_2$ are bijective. That will automatically imply that $f$ is injective.

To prove that $l$ is injective we use the following

**Lemma 10.** *Suppose $A$ is a commutative ring, $a \in A$, $l : A \longrightarrow S_a^{-1}A$ is a localization map.*

*Then* $\ker(l) = \{v \in A : \ \exists\, n \in \mathbb{N} : va^n = 0\}$.

Let us prove this Lemma afterwards and use it now.

Using Lemma 10 it is enough to prove that $x_0$ is not a zero divisor in $R$ to state that $\ker(l) = 0$. But we know by Lemma 8 that multiplication by $x_0$ preserves the basis $B_N$ of $R^N$. Hence, for any $r \in R^N$ that is a nontrivial combination of elements of $B$, $x_0 r$ is also a nontrivial combination of elements $B_N$.

Now let us define $\phi_1^{-1} : S_{x_0}^{-1}(\Bbbk[x_i, Y_i, Z_i]/\langle Z(t) - Y^2(t)\rangle) \longrightarrow S_{x_0}^{-1}R^N$:

$$\begin{aligned}\phi_1^{-1} &: x(t) \longmapsto x(t);\\ \phi_1^{-1} &: Y(t) \longmapsto y(t)x^{-1}(t);\\ \phi_1^{-1} &: Z(t) \longmapsto z(t)x^{-1}(t);\\ \phi_1^{-1} &: x_0^{-1} \longmapsto x_0^{-1},\end{aligned}$$

and $\phi_2^{-1} : S_{x_0}^{-1}\Bbbk[x_i, Y_i] \longrightarrow S_{x_0}^{-1}(\Bbbk[x_i, Y_i, Z_i]/\langle Z(t) - Y^2(t)\rangle)$:

$$\begin{aligned}\phi_2^{-1} &: x(t) \longmapsto x(t);\\ \phi_2^{-1} &: Y(t) \longmapsto Y(t);\\ \phi_2^{-1} &: x_0^{-1} \longmapsto x_0^{-1}.\end{aligned}$$

One can easily check that $\phi_1^{-1}$ and $\phi_2^{-1}$ are inverse to $\phi_1$ and $\phi_2$ correspondingly.

Finally, let us check that $\tau$ is injective. Suppose $0 \neq r \in \ker(\tau)$. Then there exists $u$ such that $r = (x_0)^{-u}r'$, $r' \in \Bbbk[x_i, Y_i]$. Note that $\tau(r') = (c_0)^{2u}\tau(r) = 0$. So, there exists $r' \in \Bbbk[x_i, Y_i]$ such that $\tau(r') = 0$.

Let us introduce the lexicographic orders on $\Bbbk[x_i, Y_i]$ and $\Bbbk[c_i, d_i]$ defined by the following orderings on variables:

| | |
|---|---|
| $\succ_1$ on $k[x_i,Y_i]$ : | $\succ_2$ on $k[c_i,d_i]$ : |
| $x_i \succ_1 x_j$, for $i > j$; | $c_i \succ_2 c_j$, for $i > j$; |
| $x_i \succ_1 Y_j$, for any $i, j$; | $c_i \succ_2 d_j$, for any $i, j$; |
| $Y_i \succ_1 Y_j$, for $i > j$. | $d_i \succ_2 d_j$, for $i > j$. |

Let $x_{i_1} \dots x_{i_{n_1}} Y_{j_1} \dots Y_{j_{n_2}}$ be an arbitrary monomial in $\Bbbk[x_i,Y_i]$. Then:

$$\begin{aligned}&LM(\tau(x_{i_1} \dots x_{i_{n_1}} Y_{j_1} \dots Y_{j_{n_2}})) =\\ &\quad LM((c_0c_{i_1} + c_1c_{i_1-1} + \dots + c_{i_1}c_0) \dots (c_0c_{i_{n_1}} + \dots + c_{i_{n_1}}c_0)\cdot\\ &\quad (d_0d_{j_1} + \dots + d_{j_1}d_0) \dots (d_0d_{j_{n_2}} + \dots + d_{j_{n_2}}d_0)) =\\ &\quad c_{i_1} \dots c_{i_{n_1}} c_0^{n_1} d_{j_1} \dots d_{j_{n_2}} d_0^{n_2}.\end{aligned} \tag{8}$$

This calculation implies that for any monomials $M_1, M_2 \in \Bbbk[x_i,Y_i]$ the following holds:

$$M_1 \succ_1 M_2 \Longleftrightarrow LM(\tau(M_1)) \succ_2 LM(\tau(M_2)).$$

Thus,

$$LM(\tau(r')) = LM(\tau(LM(r'))) \neq 0, \tag{9}$$

because $LM(r') \neq 0$ and (8) shows that $LM(\tau(\cdot))$ maps a nonzero monomial to nonzero.

Formula (9) contradicts with the fact that $\tau(r') = 0$, and, therefore, finishes the proof. □

Here is the proof of Lemma 10.

*Proof.* Note that $S_a^{-1}A \cong A[x]/(ax-1)$. We will assume that $l$ maps $A$ to $A[x]/(ax-1)$.

First we prove that $\{v \in A : \exists\, n \in \mathbb{N} : va^n = 0\} \subset \ker(l)$. Indeed, if $va^n = 0$, then $l(t) = l(t)(l(a)x)^n = l(va^n)x = 0$.

Now we prove that $\ker(l) \subset \{v \in A : \exists\, n \in \mathbb{N} : va^n = 0\}$. Suppose $l(v) = 0$. That implies that for some $(a_nx^n + \ldots + a_0) \in A[x]$ we have:

$$v = (ax-1)(a_nx^n+\ldots+a_0) = x^{n+1}(aa_n)+x^n(aa_{n-1}-a_n)+\ldots+x(aa_0-a_1)-a_0.$$

So, we obtain the following equations:

$$\begin{cases} a_0 &= -v \\ a_1 &= aa_0 \\ & \vdots \\ a_n &= aa_{n-1} \\ 0 &= aa_n \end{cases} \implies a^{n+1}(-v) = 0.$$

□

## 5 Representations of $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t]$

Below we assume that $\Bbbk = \mathbb{C}$.

Let us recall (see [F]) the $\mathfrak{sl}_2$-module structure on the finite-dimensional coordinate ring of Veronese curve $\mathbb{C}[x,y,z]/(xz-y^2) \subset \mathbb{C}[a,b]$.

First we identify $\mathbb{C}[x,y,z]$ with $S(V_2)$ and $\mathbb{C}[a,b]$ with $S(V_1)$. The representation $\rho_{finite}$ of $\mathfrak{sl}_2$ on $(x,y,z) = V_2$ is defined by:

$$\begin{array}{lll} H(x) = 2x; & E(x) = 0; & F(x) = 2y; \\ H(y) = 0; & E(y) = x; & F(y) = z; \\ H(z) = -2z; & E(z) = 2y; & F(z) = 0. \end{array}$$

Note that $H(xz-y^2) = E(xz-y^2) = F(xz-y^2) = 0$, and, therefore, $\langle xz-y^2\rangle \subset \mathbb{C}[x,y,z]$ is a subrepresentation.

The action of $\mathfrak{sl}_2$ on $(a, b) = V_1$ is defined by:

$$\begin{aligned} H(a) &= a; & E(a) &= 0; & F(a) &= b; \\ H(b) &= -b; & E(b) &= a; & F(b) &= 0. \end{aligned}$$

Then the embedding $\phi : \mathbb{C}[x, y, z]/\langle xz - y^2\rangle \hookrightarrow \mathbb{C}[a, b] \simeq \bigoplus_{\lambda\geq 0} V_\lambda$ is a morphism of representations and $\mathbb{C}[x, y, z]/\langle xz - y^2\rangle \simeq \mathrm{Im}(\phi) \simeq \bigoplus_{n\geq 0} V_{2n}$.

Similarly, the homogeneous coordinate ring for the jet scheme of Veronese curve of degree 2 has a structure of a representation of $\mathfrak{sl}_2$ current algebra.

**Definition.** The current algebra $\mathfrak{g}[t]$ of a Lie algebra $\mathfrak{g}$ is the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ where the bracket is given by $[g_1 \otimes P_1, g_2 \otimes P_2] = [g_1, g_2] \otimes P_1P_2$ for all $g_1, g_2 \in \mathfrak{g}, P_1, P_2 \in \mathbb{C}[t]$.

In what follows we denote $g \otimes t^i \in \mathfrak{sl}_2[t]$ by $g_i$.

Below we assume that $N \in \mathbb{Z}_{\geq 0} \cup \infty$.

We identify $\mathbb{C}[x_0, \ldots, z_N]$ with $S(V_2 \otimes (\mathbb{C}[[t]]/\langle t^{N+1}\rangle))$, which is $\mathfrak{sl}_2[t]$-representation $\rho$, that satisfies the rule $\rho(g_i)(r \otimes t^j) = \rho_{finite}(g)(r) \otimes t^{j+i}$.

Let us consider a representation $\rho^*$ of $\mathfrak{sl}_2[t]$ on the space $\mathbb{C}[x_0, \ldots, z_N]^* = \bigoplus_{n\geq 0} \mathbb{C}[x_0, \ldots, z_N]^*_n$ (here $\mathbb{C}[x_0, \ldots, z_N]_n$ means the space of polynomials of degree $n$).

Using the fact that $V_2^* \simeq V_2$ we obtain that $\rho^*$ can be defined by the rule

$$\rho(g_i)(r \otimes t^j) = \begin{cases} \rho_{finite}(g)(r) \otimes t^{j-i}, & \text{if } j - i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of action on series $x(t), y(t), z(t) \in \mathbb{C}[x_0, \ldots, z_N][[t]]/\langle t^{N+1}\rangle)$ it has the form:

$$\begin{aligned} H_i(x(t)) &= 2t^i x(t); & E_i(x(t)) &= 0; & F_i(x(t)) &= 2t^i y(t); \\ H_i(y(t)) &= 0; & E_i(y(t)) &= t^i x(t); & F_i(y(t)) &= t^i z(t); \\ H_i(z(t)) &= -2t^i z(t); & E_i(z(t)) &= 2t^i y(t); & F_i(z(t)) &= 0. \end{aligned}$$

Now one can easily check that any $g \in \{H_i, E_i, F_i\}$ satisfies $g(x(t)z(t) - y^2(t)) = 0$ and, therefore, $\mathbb{C}[x_0, \ldots, z_N]/\langle x(t)z(t) - y^2(t)\rangle$ is a $\mathfrak{sl}_2[t]$-module.

**Definition.** The representation of the Lie algebra $\mathfrak{g}$ on the vector space $V$ is called cocyclic if there exists an element (cogenerator) $v \in V$ such that for any element $u \in V$ there exists an element $q \in U(\mathfrak{g})$ such that $qu = v$ .

**Theorem 11.** *Consider the decomposition* $R^N = \bigoplus_{n\geq 0} R^N_n = \bigoplus_{n\geq 0}\{r \in R : \deg(r) = n\}$. *Then* $R^N_n$ *is a cocyclic representation of* $\mathfrak{sl}_2[t]$.

*Proof.* Let us construct an action of $\mathfrak{sl}_2[t]$ on $\mathbb{C}[a_0, \ldots, b_N]$ in such a way that $f : R^N \hookrightarrow \mathbb{C}[a_0, \ldots, b_N] = L$ is a morphism of representations.

One can check that the following representation is the one we are looking for:

$$H_i(a(t)) = t^i a(t); \qquad E_i(a(t)) = 0; \qquad F_i(a(t)) = t^i b(t);$$
$$H_i(b(t)) = -t^i b(t); \qquad E_i(b(t)) = t^i a(t); \qquad F_i(b(t)) = 0.$$

Now we prove that $L_n$ (which is a space of polynomials in $L$ of degree $n$) is a cocyclic with a cogenerator $a_0^n$ and this implies that $R_n$ is cocyclic with a cogenerator $x_0^n$.

Let us define $L_n^k = \{l \in L : \deg(l) = n, \operatorname{sum}(l) = k\}$. For $l \in L_n^k$ we prove that there exists $g \in \mathfrak{sl}_2[t]$ such that $\operatorname{sum}(g(l)) < k$. It is enough because $L_n^0 \simeq S^n(V_1) \simeq V_n$ and it is an irreducible representation of $\mathfrak{sl}_2 \otimes t^0$.

Let $b_m$ be the "$b$ with the maximal index" in $l$ and $l = \operatorname{sum}_u b_m^{i_u} \cdot s_u$. Then

$$E_m(l) = E_m(\sum_u b_m^{i_u} \cdot s_u) = \sum_{u \text{ s.t. } i_u > 1} b_m^{i_u - 1} a_0 \cdot s_u \neq 0$$

and

$$\operatorname{sum}(E_m(l)) = \operatorname{sum}(l) - m < \operatorname{sum}(l),$$

that finishes the proof unless $l$ does not depend on $b_i$ and $l \in \mathbb{C}[a_i]$. In this case we consider $a_m$ as "$a$ with the maximal index" in $l$ and take $F_m(l)$ instead of $E_m(l)$. $\square$

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