KÄHLER SUBMANIFOLDS IN IWASAWA MANIFOLDS

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Abstract. Iwasawa manifold is a compact complex homogeneous manifold isomorphic to a quotient G/Λ, where G is the group of complex unipotent 3×3 matrices and $\Lambda \subset G$ is a cocompact lattice. We prove that any compact complex curve in an Iwasawa manifold is contained in a holomorphic subtorus. We also show that any Kähler surface in an Iwasawa manifold is either an abelian surface or a non-projective isotrivial elliptic surface of Kodaira dimension one. In the Appendix we show that any subtorus in Iwasawa manifold is of maximal Picard number.

Contents

1. Introduction

Recall that a *nilmanifold* is a compact manifold which admits a transitive action of nilpotent Lie group. The classical theorem of Mal'čev [Mal] states that any nilmanifold is diffeomorphic to a quotient of a connected simply connected nilpotent Lie group G by a cocompact lattice $\Lambda \subset G$. Moreover, the group G can be obtained as the pro-unipotent completion of Λ (which is also known as *Mal'čev completion*).

A *complex nilmanifold* is a pair (N, J), where $N = G/\Lambda$ is a nilmanifold and J is a G-invariant integrable complex structure on N.

The topology of nilmanifolds is well understood: any nilmanifold is diffeomorphic to an iterated tower of principal toric bundles. However, the complex geometry of nilmanifolds is much more varied and rich: for example, one is not always able to choose these toric bundles to be holomorphic. For further discussion on this and related problems see e.g. [Rol]

One of the reasons why nilmanifolds provoke such an interest in complex geometry is that they deliver a number of examples of non-Kähler complex manifolds. Indeed, a complex nilmanifold never admits Kähler metric unless it is a torus ([BG]). Moreover, a nilmanifold which is not a torus is not homotopically equivalent to any Kähler manifold: its de Rham algebra is never formal ([Has]). In [BR] Bigalke and Rollenske constructed complex nilmanifolds with arbitrary long non-degenerate Frölicher spectral sequence, which means that complex nilmanifolds can be in some sense arbitrary far from being Kähler.

At this point it is rather natural to try to describe Kähler submanifolds in complex nilmanifolds.

We solve this problem for Iwasawa manifolds which are a special but important class of 3-dimensional complex nilmanifolds.

2. Iwasawa manifolds

Starting from now, let G be the group of complex unipotent 3×3-matrices, i.e. matrices of the kind $\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$ with $z_i \in \mathbb{C}$.

Definition 1. An *Iwasawa manifold* is a compact complex manifold which admits a transitive holomorphic action of G with discrete stabilizer.

As follows from the Mal'čev theorem [Mal], any Iwasawa manifold is isomorphic to G/Λ for a cocompact lattice $\Lambda \subset G$. Complex structure on it is induced by the standard complex structure on G. Hence any Iwasawa manifold is a homogeneous 3-dimensional complex manifold. Any two Iwasawa manifolds are diffeomorphic, but for a different choice of Λ one in general gets pairwise non-biholomorphic complex manifolds.

The group G can be viewed as a central extension of a two-dimensional commutative complex Lie group by one-dimensional:

$$1 \to \mathbb{C} \to G \to \mathbb{C}^2 \to 1.$$

The corresponding cocycle for its Lie algebra $\mathfrak{g}$ is cohomological to the standard (complex) symplectic form on $\mathbb{C}^2$.

The algebra $\mathfrak{g}$ is three-dimensional. The first differential $d\colon \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ in the Chevalley complex of $\mathfrak{g}$ is given by

$$de_1 = de_2 = 0$$

$$de_3 = e_1 \wedge e_2$$

for a certain basis (e_1, e_2, e_3) in $\mathfrak{g}^*$.

Consider an Iwasawa manifold I isomorphic to G/Λ. Let Λ^{ab} be the abelianization of the lattice Λ and $\Lambda_Z := \Lambda \bigcap Z(G)$ the intersection of Λ with the group center $Z(G)$. The above-mentioned Mal'čev theorem ([Mal]) also implies that both Λ^{ab} and Λ_Z are lattices in $G^{ab} = \mathbb{C}^2$ and $Z(G) = \mathbb{C}$ respectively. Thus $E = \mathbb{C}/\Lambda_Z$ is an elliptic curve and $T = \mathbb{C}^2/\Lambda^{ab}$ is a compact complex torus. The projection $p\colon G \to G/Z(G)$ descends to holomorphic map $\pi\colon I \to T$. The curve E acts on its fibers freely and transitively, hence π is a holomorphic principal E-bundle. Further on we are going to call it the *Iwasawa bundle*.

The cohomology of Iwasawa manifold are computed in [FG].

Our aim is to study curves in Iwasawa manifold. We will show that any curve in Iwasawa manifold is either elliptic or is contained in some abelian surface $A \subset I$. Moreover, both this elliptic curve and abelian surface carry complex multiplication (see Section 6). From this we will deduce the classification theorem for Kähler surfaces in Iwasawa manifold.

3. Principal elliptic bundles.

3.1. **Holomorphic principal elliptic bundles.** For the duration of this section fix a compact complex manifold B and an elliptic curve $E = \mathbb{C}/\Gamma$.

The aim of this section is to recall some techniques of working with holomorphic principal E-bundles and to apply them to the case of Iwasawa bundle.

Let $\mathcal{E}_B$ denote the sheaf of holomorphic functions on B with values in E. The isomorphism classes of holomorphic principal bundles are in one-to-one correspondence with the elements of $H^1(B, \mathcal{E}_B)$.

Let $\mathcal{P}\colon M \to B$ be a principal holomorphic E-bundle. Denote by $\mathcal{V} = \operatorname{Ker} D\mathcal{P}$ the subbundle in TM tangent to the fibers of $\mathcal{P}$ (the *vertical subbundle*). Choose any $\mathcal{E}_B$-invariant Ehresmann connection $\mathcal{H}$, that is a $\mathcal{E}_B$-equivariant subbundle in TM, for which $TM = \mathcal{V} \oplus \mathcal{H}$. It induces an affine connection on $\mathcal{P}$ with values in the Lie algebra $\mathfrak{e}$ of

E. By the standard results of the theory of principal G-bundles (see e.g.[S]) the curvature form of this affine connection descends to a closed $\mathfrak{e}$-valued 2-form on B (here we also use that the group E is abelian) . Its cohomology class $c_1(\mathcal{P})$ (the first Chern class) is a topological invariant of $\mathcal{P}$. In particular, if $c_1(\mathcal{P}) \neq 0$, the bundle admits no holomorphic sections.

As far as E is a complex commutative Lie group, $\mathfrak{e}$ as a real vector space can be canonically identified with $\mathbb{C}$. Hence it is natural to think about $c_1(\mathcal{P})$ as an element of $H^2(B, \mathbb{C})$.

Since E is a commutative group, $H^1(B, \mathcal{E}_B)$ carries a structure of commutative group itself. With respect to this structure

$$c_1 \colon H^1(B, \mathcal{E}_B) \to H^2(B, \mathbb{C})$$

is a homomorphism. What is important is that this structure can be described explicitly in the following way:

Proposition 3.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two holomorphic principal E-bundles over B with total spaces M_1 and M_2. Let $[\mathcal{P}_i] \in H^1(B, \mathcal{E}_B)$ be the corresponding cocycles. Define $\mathcal{P}_1 \times_E \mathcal{P}_2$ as the quotient of $M_1 \times_B M_2$ over the diagonal action of E. Then*

$$[\mathcal{P}_1 \times_E \mathcal{P}_2] = [\mathcal{P}_1] + [\mathcal{P}_2]$$

Proof: Follows from a simple computation and Eckmann-Hilton argument [EH]. □

Proposition 3.2. *Assume that an elliptic bundle $\mathcal{P} \colon M \to B$ admits a multisection, i.e. there exists a submanifold $Z \subset M$, such that the restriction $\mathcal{P}|_Z \colon Z \to B$ is a finite covering. Then $c_1(\mathcal{P})$ is a torsion class.*

Proof: Let $\mathcal{P}|_Z \colon Z \to B$ be of degree k and $S^n\mathcal{P}$ the fiberwise n-th symmetric power of the bundle $\mathcal{P}$. For any $b \in B$ the intersection of Z with the fiber $E_b = \mathcal{P}^{-1}(b)$ is a zero-dimensional subscheme Z_b on E_b of degree k. So the multisection Z defines a section $b \mapsto Z_b$ of the fiberwise Hilbert scheme of points $\mathrm{Hilb}^k(\mathcal{P})$. It can be contracted on the fiberwise k-th symmetric power $S^k\mathcal{P}$, which projects on the total space of the bundle $\mathcal{P}^k$. The section of $S^k\mathcal{P}$ produces a section of $\mathcal{P}^k$.

Hence, the k-th power of $\mathcal{P}$ is trivial.

□

There is also another point of view on $c_1(\mathcal{P})$, which was studied by T. Höfer in [Hof].

The short exact sequence

$$0 \to \Gamma \to \mathbb{C} \to E \to 0$$

gives rise to exact sequence of sheaves

$$0 \to \underline{\Gamma} \to \mathcal{O}_B \to \mathcal{E}_B \to 0. \tag{1}$$

Here $\underline{\Gamma}$ is the subsheaf of the constant sheaf $\underline{\mathbb{C}}$ consisting of the sections which take values in Γ. Therefore one gets the cohomology long exact sequence:

$$\dots H^1(B,\underline{\Gamma}) \to H^1(B,\mathcal{O}_B) \to H^1(B,\mathcal{E}_B) \xrightarrow{c^{\mathbf{z}}} H^2(B,\underline{\Gamma}) \to H^2(B,\mathcal{O}_B) \to \dots$$

Proposition 3.3 ([Hof])**.** *Consider the first Chern class c_1 as a homomorphism from $H^1(B,\mathcal{E}_B)$ to $H^2(B,\mathbb{C})$. Then c_1 coincides with the composition of $c^{\mathbf{z}}$ and the natural map $H^2(B,\underline{\Gamma}) \to H^2(B,\mathbb{C})$ induced by the embedding $\Gamma \hookrightarrow \mathbb{C}$.*

Furthermore, if B is Kähler then the map

$$H^2(B,\underline{\Gamma}) \to H^2(B,\mathcal{O}_B)$$

equals to the composition of natural morphism $H^2(B,\underline{\Gamma}) \to H^2(B,\mathbb{C})$ and the projection in the Hodge decomposition

$$H^2(B,\mathbb{C}) \to H^{2,0}(B,\mathbb{C}) = H^2(B,\mathcal{O}_B).$$

This shows that the image of c_1 has vanishing $(0,2)$-part.

The following theorem was proved by Blanchard [Bl]:

Theorem 3.4 (Blanchard)**.** *Let $\mathcal{P} = (p\colon M \to B)$ be a holomorphic principal elliptic bundle. Assume that B is Kähler and $H^2(B,\mathbb{Z})$ is torsion-free. Then M is Kähler if and only if $c_1(\mathcal{P}) = 0$.*

We sketch the proof for the case when B is a curve.

Proposition 3.5. *Let B be a compact complex curve. Consider a holomorphic principal elliptic bundle $\mathcal{P} = (p\colon M \to B)$. Then M is Kähler if and only if $c_1(\mathcal{P}) = 0$.*

Sketch of proof. Assume that $c_1(\mathcal{P}) \neq 0$ and M is Kähler.

It is easy to see that the pull-back of $\mathcal{P}$ to its own total space admits a section, and hence $p^*(c_1(\mathcal{P})) = 0$. Either $c_1(\mathcal{P})$ or $-c_1(\mathcal{P})$ can be represented by a Kähler form ω_B. Let ω_M be a Kähler form on M. Thus $p^*\omega_B = d\eta$ for some 1-form η on M. Now let us compute the volume of M:

$$\mathrm{Vol}(M) = \int_B \Big(\int_{p^{-1}(b)} \omega_M\Big)\omega_B = \int_M p^*\omega_B \wedge \omega_M = \int_M d(\eta \wedge \omega_M) = 0.$$

Clearly, this is impossible. This proves the "only if" part.

To prove the sufficiency one can use the fact that the vanishing of $c_1(\mathcal{P})$ is equivalent to existence of a flat E-connection on $\mathcal{P}$. On a small open set $U \subset B$ one is able to choose a splitting $p^{-1}(U) = U \times E$ and find there a closed positive $(1,1)$ form ξ_0 which is Poincaré-dual to the class of the fiber $[E_b]$. Averaging by the action of E one can make this form to be constant among the fibers. Then, translating by the flat connection on $\mathcal{P}$, glue it into a global form ξ.

If ω_B is a Kähler form on the base, a Kähler form on M is given by

$$\omega_M = p^*\omega_B + \xi.$$

□

Remark: Indeed c_1 is the only topological invariant of a principal elliptic bundle. To see this, replace the sheaves $\mathcal{O}_B$ and $\mathcal{E}_B$ in the exact sequence (1) with the sheaves of smooth functions $C^\infty(B, \mathbb{C})$ and $C^\infty(B, E)$. Consider the corresponding cohomology long exact sequence. Since $C^\infty(B, \mathbb{C})$ is acyclic, $c^{\mathbf{z}}$ defines a bijection between the isomorphism classes of smooth principal E-bundles and the elements of $H^2(B, \underline{\Gamma})$. As a consequence we get the following fact: if M is the total space of a principal holomorphic E-bundle over B and both B and M are Kähler, then M is diffeomorphic to $B \times E$.

The first Chern class and other invariants for the Iwasawa bundle and other examples of holomorphic principal toric bundles were computed in [Hof](§3, §10) .

3.2. Iwasawa bundle.

Proposition 3.6. *Let $\pi\colon I \to T$ be an Iwasawa bundle. Then $c_1(\pi)$ can be represented by a holomorphically symplectic form. In particular $c_1(\pi) \in H^{2,0}(T)$.*

Proof: Recall that G is the group consisting of matrices of the kind $\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}$ with $z_i \in \mathbb{C}$. This group acts on I transitively. Let $\mathfrak{g}$ be its Lie algebra. Again denote by Z the center of G and by G^{ab} its abelianization. Their Lie algebras will be denoted as $\mathfrak{z}$ and $\mathfrak{g}^{ab}$ respectively.

The exact sequence of Lie algebras

$$0 \to \mathfrak{z} \to \mathfrak{g} \xrightarrow{p} \mathfrak{g}^{ab} \to 0$$

corresponds to a certain cocycle $s \in H^2(\mathfrak{g}^{ab}, \mathfrak{z}) = \mathrm{Hom}(\Lambda^2\mathbb{C}^2, \mathbb{C})$. Direct computation of s shows that it is proportional to the standard holomorphic symplectic form on $\mathbb{C}^2$.

Choose any map of $\mathbb{C}$-vector spaces $h\colon \mathfrak{g}^{ab} \to \mathfrak{g}$, which is right inverse to the projection p and $\mathfrak{g} = \mathfrak{z} \oplus \mathrm{Im}\, h$. It induces an Ehresmann G-connection $\mathcal{H}$ on π and hence affine G-connection ∇.

Let σ be the curvature form of ∇, so $c_1(\pi) = [\sigma] \in H^2(T, \mathbb{C})$. If we restrict σ on the tangent space to the origin in G^{ab}, then

$$\sigma(X, Y) = [h(X), h(Y)] - h([X, Y]) = s(X, Y)$$

Since both ∇ and σ are G^{ab}-invariant, σ coincides with the form on G^{ab} obtained by left translations from s. This means that σ is holomorphicaly symplectic (this is a local property and it is preserved by descending to T).

□

Corollary 3.7. *Let $\pi\colon I \to T$ be an Iwasawa bundle and $i\colon C \hookrightarrow T$ an embedding of a curve. Since any $(2,0)-$form vanishes on any curve in T, one has*

$$c_1(i^*\pi) = 0.$$

4. Curves in Iwasawa manifold.

For this section let $I = G/\Lambda$ be an Iwasawa manifold, $E = Z(G)/\Lambda_Z$ an elliptic curve, and $\pi\colon I \to T$ the Iwasawa bundle which is a holomorphic principal E-bundle. Our aim is to describe all closed complex curves in I.

For the purposes of this section we will need the techinque of Douady spaces.

4.1. **Douady spaces.** Recall that for any complex manifold X there exists its *Douady space*: a complex analytic space $\mathcal{D}(X)$ which classifies its closed complex subvarieties ([Dou]).

The Douady spaces can be viewed as the analytical counterpart for the Hilbert schemes. For a more detailed introduction into Douady spaces see ([CDGP], ch. VIII).

For a subvariety $S \subset X$ we denote the corresponding point in $\mathcal{D}(X)$ by $[S]$.

In general the Douady spaces are very singular and even non-reduced. However, the Zariski tangent space in a point $[Z] \in \mathcal{D}(X)$ can be described as the space of global sections of normal bundle:

$$T_{[Z]}\mathcal{D}(X) = H^0(Z, \mathcal{N}_{Z/X}).$$

(see [CDGP], ch. VIII).

One can also consider the *marked Douady space* $\mathcal{D}^+(X)$ which is the space of pairs (S, s), where $S \subset X$ is a closed subvariety and $s \in S$ is a point. It is equipped with two forgetfull projections

$$\begin{tikzcd} & \mathcal{D}^+(X) \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathcal{D}(X) & & X \end{tikzcd}$$

For a point $[S] \in \mathcal{D}(X)$ the fiber $\Phi^{-1}([S])$ is isomorphic to S.

Let C be a curve of genus g in complex manifold X. Denote by $\mathcal{D}(C, X)$ the reduction of the connected component of $\mathcal{D}(X)$ which contains C. We call it *the space of deformations of C in X*.

We also consider *the space of isotrivial deformations of C in X*, that is the space $\mathcal{M}(C, X)$ of all curves $[C'] \in \mathcal{D}(C, X)$, such that $C \simeq C'$. It is a complex analytical space since it can be viewed as the reduction of the fiber of the characteristic rational morphism $\mu \colon \mathcal{M}(C, X) \dashrightarrow \mathfrak{M}_g$ to the moduli space[1] of curves of genus g. The *space of marked isotrivial deformations* $\mathcal{M}^+(C, X) = \Phi^{-1}(\mathcal{M}(C, X))$ is the total space of an isotrivial family of curves over $\mathcal{M}(C, X)$.

4.2. **Weakly Singular Kähler spaces and Fujiki class $\mathcal{C}$.** Now we will discuss the definitions and the basic properties of two important classes of complex spaces, which are Fujiki class $\mathcal{C}$ manifolds and weakly singular Kähler spaces. We will also deduce certain properties of the spaces of deformations of curves in Iwasawa manifolds.

Definition 2. A compact (possibly singular) complex manifold Z is said to be *of Fujiki class $\mathcal{C}$* if it is bimeromorphic to a Kähler manifold.

Proposition 4.1. *A compact subvariety in a manifold of Fujiki class $\mathcal{C}$ is again from Fujiki class $\mathcal{C}$. Moreover, if X belongs to class $\mathcal{C}$ and $f \colon X \to Y$ is a proper map, then $f(X)$ is of Fujiki class $\mathcal{C}$ as well.*

Proof: See [Fuj78]. □

Proposition 4.2. *Iwasawa manifold is not from Fujiki class $\mathcal{C}$.*

Proof: The classical paper of Deligne, Griffiths, Morgan and Sullivan dedicated to formality of compact Kähler manifolds also contains proof of formality of compact manifolds of Fujiki class $\mathcal{C}$ ([DGMS], Theorem 5.22) However, Iwasawa manifold is a complex nilmanifold and hence it is not formal. In fact there is a non-vanishing Massey product in its cohomology ([Has]). □

[1] Starting from this point we assume that C is smooth. In order to study singular curves in Iwasawa manifolds one can replace $\mathfrak{M}_g$ with the moduli space of marked curves $\mathfrak{M}_{g,n}$. All the rest considerations stay without any changes.

The Douady spaces of Fujiki class $\mathcal{C}$ manifolds are described by the following theorem from [Fuj82]:

Theorem 4.3 (Fujiki). *Let X be a compact smooth manifold of Fujiki class $\mathcal{C}$ and $D_0 \subset \mathcal{D}(X)$ be a connected component of Douady space. Then D_0 is compact and $D_{0,red}$ is of Fujiki class $\mathcal{C}$ (notice that in general D_0 itself is not necessarily a reduced complex analytical space).*

One of the most powerful tools of proving that a certain manifold is of Fujiki class $\mathcal{C}$ is the theorem of Demailly and Păun.

Definition 3. A closed $(1,1)$-current T on complex manifold X is a *Kähler current* if there exist a positive Hermitian form h on X with $\omega = \mathrm{Im}(h)$, such that $T - \delta\omega > 0$ for certain $\delta > 0$.

Theorem 4.4 (Demailly, Păun). *A smooth compact complex manifold is of Fujiki class $\mathcal{C}$ if and only if it admits a Kähler current.*

The proof of this theorem is contained in [DP].

Since Iwasawa manifold is not of Fujiki class $\mathcal{C}$, we cannot claim that its Douady space is of Fujiki class $\mathcal{C}$, neither that it is compact. Nevertheless $\mathcal{D}(I)$ is still in some sense close to Kähler manifolds.

Let X be a reduced complex space and $i\colon X^0 \hookrightarrow X$ the embedding of the smooth locus of X. Let $C^\infty_+(X)$ be the sheaf of continuous $\mathbb{C}$-valued function on X which are smooth outside the singular locus. Clearly, this is a sheaf of $\mathcal{O}_X$-modules.

Consider also the p-th exterior power of the Zariski cotangent sheaf Ω^p_X (which is also known as the space of Kähler differential forms). There exists a canonical isomorphism between $i^*\Omega^1_X$ and the sheaf of holomorphic 1-forms on X^0. Define *the sheaf of (p,0)-forms on X* as

$$\mathcal{A}^{p,0}(X) := \Omega^p_X \otimes_{\mathcal{O}_X} C^\infty_+(X).$$

Put also $\mathcal{A}^{0,q}(X) := \overline{\mathcal{A}^{q,0}}$ and

$$\mathcal{A}^{p,q}(X) := \mathcal{A}^{p,0}(X) \otimes_{C^\infty_+(X)} \mathcal{A}^{0,q}.$$

The injective morphism $i^*C^\infty_+(X) \hookrightarrow C^\infty(X^0)$ gives us an embedding of $i^*\mathcal{A}^{p,q}(X)$ to the sheaf of smooth (p,q)-forms on X^0. If $X = X^0$, it is an isomorphism.

If $f\colon X \to Y$ is a map of complex analytic spaces and Y is smooth, there is a natural morphism $f^*\colon \mathcal{A}^{p,q}(Y) \to \mathcal{A}^{p,q}(X)$.

We say that a $(1,1)$-form ω on complex space X is *strictly positive*, if for any Zariski tangent vector v one has

$$-\sqrt{-1}\omega(v,\overline{v}) > 0.$$

Definition 4. A reduced complex analytical space X is *weakly singular Kähler* if it admits a closed $(1,1)$-form $\omega \in \mathcal{A}^{1,1}(X)$, such that $\omega|_{X^0}$ is Kähler.

Clearly, any closed subspace of weakly singular Kähler space is again weakly singular Kähler. A smooth manifold is weakly singular Kähler if and only if it is Kähler.

Remark: Our definition of a (p,q)-form on a reduced complex space is close to the definition given by Demailly in [Dem], though it is strictly weaker. There also exists the notion of *singular Kähler space*, in which, roughly speaking, the existence of a Kähler form in the sense of [Dem] is required (see [Gr] and [HP] for the precise definition). This notion also seems stronger rather than the notion of weakly singular Kähler space.

Proposition 4.5. *Assume that X is a weakly singular Kähler space, Y is a smooth compact complex manifold and $f\colon X \to Y$ is a surjective proper holomorphic map. Then Y is of Fujiki class $\mathcal{C}$.*

Proof: Let ω_X be the Kähler form on the smooth locus of X .Choose a positive Hermitian form (i.e. the imaginary part of a positively-defined Hermitian metric) ω_Y on Y, such that $\omega_X > f^*\omega_Y$.

Consider ω_X as a current on X and let $T := f_*\omega_X$ be its push-forward on Y. Clearly, T is closed and of $(1,1)$-type. Since for any positive $(n-1,n-1)$ form η one has

$$\langle T, \eta\rangle_y = \int_{f^{-1}(y)} f^*\eta \wedge \omega_X > (\eta \wedge \omega_Y)_y,$$

we get that $T - \omega_Y > 0$ and the current T is Kähler. Thus the Demailly - Păun theorem is appliable. $\square$

Corollary 4.6. *There is no surjective morphism from a compact weakly singular Kähler space onto an Iwasawa manifold.*

Proof: Follows from Propositions 4.2 and 4.5. $\square$

Now we turn to the Douady space of an Iwasawa manifold.

Proposition 4.7. *Let I be an Iwasawa manifold and $C \subset I$ be a closed complex curve. Then the reduced space $\mathcal{D}(C, I)$ of deformations of C in I is weakly singular Kähler.*

In [Ver] and [KV] it was proved that the space of lines in twistor space of a holomorphically symplectic surface is Kähler. The proof of this Proposition is pretty similar to that one, so we will only sketch it:

Sketch of proof: Iwasawa manifolds are balanced: *balanced* n-dimensional complex manifolds are those which admit a strictly positive closed $(n-1, n-1)$ form. For an Iwasawa manifold it can be constructed as follows: let ω_T be a Kähler form on the base of the Iwasawa bundle and let ξ be any $(1,1)$-form which is tangent to the fibers of the Iwasawa bundle and becomes positive after restriction on any of them. Then put

$$\zeta := \pi^*\omega_T \wedge \xi + \pi^*\omega_T^2.$$

An easy computation shows that ζ is a closed strictly positive $(2,2)$-form. It induces a $(1,1)$ form Ω on the Douady space $\mathcal{D}(C, I)$. Its value in a point $[C]$ is defined as

$$\Omega(s_1, \overline{s_2}) = \int_C \zeta(s_1, \overline{s_2}, -, -)$$

(here s_1 and s_2 are two sections of the normal bundle of C). The smoothness, positivity and closedness of ζ imply that Ω is a smooth Kähler form on the smooth part of $\mathcal{D}(C, I)$. $\square$

The projection $\pi\colon I \to T$ defines a map of Douady spaces

$$\pi_*\colon \mathcal{D}(I) \to \mathcal{D}(T).$$

The groups G and $\mathbb{C}^2$ act by translations on Iwasawa manifold and the torus T and hence on their Douady spaces $\mathcal{D}(I)$ and $\mathcal{D}(T)$ respectively. Slightly abusing the language we denote both of this actions by τ.

Any element $g \in G$ acts on $I = G/\Lambda$ by multiplication from the left:

$$\tau_g\colon g_1\Lambda \mapsto gg_1\Lambda.$$

This action is holomorphic, so it preserves the space of isotrivial deformations $\mathcal{M}(C, I)$ for any curve $C \subset I$.

If $p\colon G \to \mathbb{C}^2$ is the abelianization map, for any $g \in G$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}(I) \arrow[r, "\tau_g"] \arrow[d, "\pi_*"'] & \mathcal{D}(I) \arrow[d, "\pi_*"'] \\ \mathcal{D}(T) \arrow[r, "\tau_{p(g)}"] & \mathcal{D}(T) \end{tikzcd}$$

4.3. **Projections of curves.** Let us fix a smooth complex curve $C \subset I$ of genus g. A priori there are three possibilities:

(1) $\pi(C)$ is a point, i.e. C is a fiber of the Iwasawa bundle;
(2) $\pi(C) \subset T$ is an elliptic curve;
(3) $\pi(C) = B \subset T$ is a curve of genus $g' > 1$.

Clearly, if $g = 1$, the case 3 is not possible. From now we suppose that $g \geqslant 2$. We are going to show that the third case still cannot occur.

Lemma 4.8. *Let $C \subset I$ be a smooth compact curve in an Iwasawa manifold I. Then $\pi(I)$ is an elliptic curve.*

Proof: The strategy of the proof is the following. Assuming the opposite, we construct a certain compact subspace K^+ in the space $\mathcal{M}^+(C, I)$ of marked isotrivial deformations of C in I. We prove that K^+ projects onto I surjectively and that K^+ is weakly singular Kähler. Combined with the Corollary 4.6 this leads us to the contradiction.

So assume that $C \subset I$ is a curve and $B = \pi(C)$ is of genus $g > 1$. The torus T acts on its own Douady space $\mathcal{D}(T)$ by translations. Let T_B be the orbit of $[B]$ under this action.

Consider the space $K := \mathcal{M}(C, I) \bigcap \pi_*^{-1}(T_B)$. This is the space of all isotrivial deformations C' of C in I, such that $\pi(C') = B'$ is a translation of B.

Proposition 4.9. *The projection $\pi_* \colon K \to T_B$ is a locally trivial fibration.*

Proof: Consider any curve $B' \in T_B$. Denote its pre-image $\pi_*^{-1}(B')$ by $K_{B'}$.

The curve B' differs from B by a translation on an element $v \in T$. One is always able to find such $g \in G$, that its image $p(g)$ in $G^{ab} = \mathbb{C}^2$ acts on T as v (see the commutative diagram in the end of the Subsection 4.2). This defines an automorphism $\tau_g \colon \mathcal{M}(C, I) \to \mathcal{M}(C, I)$ with $\tau_g(C) \in K_{B'}$. Therefore π_* is surjective.

Since τ_g is holomorphic and invertible it defines an isomorphism

$$K_B \simeq K_{B'}.$$

It is a standard fact that a holomorphic map of complex spaces with constant fibers is a locally trivial fibration. □

Proposition 4.10. *The space K is compact.*

Proof: As proved in the previous proposition, K is the total space of a certain locally trivial fibration over T_B. The subgroup of T which preserves B is finite, so the orbit T_B is isomorphic to the quotient of T over a finite subgroup, and hence compact.

Now what is the fiber of the bundle π_*?

For any curve $B' \in T$ let $S_{B'} = \pi^{-1}(B')$ denote the total space of the restriction $\pi_{B'}$ of the Iwasawa bundle π onto B'. The Corollary 3.7 claims that $c_1(\pi_{B'}) = 0$, so due to the Blachard theorem (Theorem 3.4) $S_{B'}$ is Kähler. The fiber $K_{B'}$ is a closed subspace in a connected component of the Douady space $\mathcal{D}(S_{B'})$. Applying the Fujiki theorem (Theorem 4.3) one gets that each fiber $K_{B'}$ is compact as well. □

Recall that the marked space of isotrivial deformations $\mathcal{M}^+(C, I)$ is equipped with the natural projections $\Phi\colon \mathcal{M}^+(C, I) \to \mathcal{M}(C, I)$ and $\Psi\colon \mathcal{M}^+(C, I) \to I$. Let us consider the space $K^+ := \Phi^{-1}(K)$.

It is clear that $\Psi|_{K^+}$ is surjective. Indeed choose any point $y \in C$ and any other point $y' \in I$. There exists an element g in the group of unipotent matrices G, such that $\tau_g(y) = y'$. As far as $\pi(\tau_g(C)) = \tau_{p(g)}(\pi(C))$, we get that $\tau_g(C, y) \in K^+$ and $\Psi(\tau_g(C, y)) = y'$.

Proposition 4.11. *The space K^+ is weakly singular Kähler.*

Proof: The space K^+ is the total space of an isotrivial family over K with fibers isomorphic to C. All of such families are classified by $H^1(K, \mathrm{Aut}(C))$. Since the group $\mathrm{Aut}(C)$ is finite ([Hur]), this family becomes trivial after a finite covering. Of course, if K is a weakly singular Kähler space, then $K \times C$ is also weakly singular Kähler. But $K \times C$ finitely covers K^+. It is obvious that then K^+ is also weakly singular Kähler. □

So $\Psi\colon K^+ \to I$ is a surjective map from a compact weakly singular Kähler manifold to an Iwasawa manifold. As discussed above, it is not possible. This finishes the proof of Lemma 4.8

□

Remark: There is also another point of view on this lemma. Its result can be interpreted as non-existence of multisections of the restriction of Iwasawa bundle onto any curve of general type in T, since any such multisection would be a curve of type 3 . Such a restriction has vanishing first Chern class, so it admits a flat E-connection. The question about existence of a multisection is equivalent to the question about finiteness of the monodromy of such connection.

Since $c_1(\pi)$ is represented by a holomorphically symplectic form, this lemma might have a sibling from symplectic geometry. Consider a (e.g. holomorphically) symplectic manifold M and a (non-holomorphic) linear $\mathbb{C}$-bundle L on it. Assume that L is polarizing, i.e. its first Chern class is cohomological to the symplectic form. For any Lagrangian submanifod $Z \subset M$ the restriction $L|_Z$ has vanishing c_1 hence it is flat. One might ask what is the monodromy of this flat bundle. It is expectable that, as an analogy with the Iwasawa case, for some natural examples its monodromy will be infinite, though the general answer is not known to the author.

4.4. **Curves in Iwasawa manifold.**

Proposition 4.12. *Let E' be an elliptic curve in T. Then the restriction of π on E' is trivial.*

Proof: Without loss of generality we might assume that E' passes through the origin.

The exact sequence of complex Lie groups

$$1 \to \mathbb{C} \to G \to \mathbb{C}^2 \to 1$$

splits over any line $L \subset \mathbb{C}^2$. This splitting defines a trivialization of π over any 1-dimensional complex subgroup in T. $\square$

Theorem 4.13. *Any curve $C \subset I$ is contained in some holomorphic subtorus.*

Proof: If C is in the fiber of π, then it is isomorphic to E as a complex manifold. Otherwise $\pi(C) = E'$ is necessarily an elliptic curve in T and $C \subset \pi^{-1}(E') = E' \times E$.

$\square$

In the Appendix we prove that such a torus always has maximal Picard number. Indeed both E and E' necessarily carry (the same) complex multiplication.

Remark: Our considerations can be slightly generalized. Let H be any central extension of a commutative complex Lie group $\mathbb{C}^k$ by one-dimensional center. For any cocompact lattice $\Lambda \subset H$ we get a nilmanifold $N = H/\Lambda$ which is fibered over a k-dimensional complex torus $T = \mathbb{C}^k/\Lambda^{ab}$. Then, after some modifications, the arguments from the Lemma 4.8 show that the projection of any curve in N to T is necessarily contained in a proper subtorus $T' \subsetneq T$. Using induction on k one can deduce that any curve in such a nilmanifold is contained in a certain holomorphic subtorus.

5. Surfaces in Iwasawa manifold

Now let $S \subset I$ be a complex surface. Since π does not admit a multisection, $\pi(S) \neq T$. This means that $\pi(S)$ is a curve. One easily gets the following classification theorem:

Theorem 5.1. *Let $S \subset I$ be a complex surface. Then there are two possibilities :*

(1) *S is an abelian surface isomorphic to a product of two elliptic curves.*

(2) *S is a Kähler non-projective isotrivial elliptic surface of Kodaira dimension 1. Thus S is diffeomorphic to $C \times E$, where E is an elliptic curve and C is a curve of general type.*

Proof: The first case takes place when $\pi(S) = C$ is of genus one. Then S is the total space of π_C i.e. the total space of trivial elliptic bundle over C with fibre isomorphic to E.

The second case takes place when genus of C is greater or equal then 2. Then S is a total space of principal E-bundle π_C over C which is the restriction of π to C. The bundle π_C has vanishing Chern class (Corollary 3.7), hence S is Kähler (Proposition 3.5). Consequently, S is diffeomorphic to $C \times E$ (see the Remark on the page 6) . However, π_C admits no holomorphic multisections (a multisection would provide a counter-example to Lemma 4.8). We claim that S cannot be projective.

Indeed, suppose that S is embbeded into projective space. Then one could find an irreducible hyperplane section of S transversal to the fibers of π_C. This would deliver a finite multisection of π_C.

□

Notice that a posteriori all complex surfaces in Iwasawa manifolds are Kähler.

Corollary 5.2. *Let $f\colon X \to I$ be a holomorphic map from a smooth projective variety to an Iwasawa manifold. Then f factors through an abelian variety.*

Remark:

In [FGV] Fino, Grantcharov and Verbitsky proved that any holomorphic morphism from a complex nilmanifold to a projective variety factors through an abelian variety. Our results might be considered as a particular case of the (conjectural) dual statement. It is not known to the author if it is true or not in the general case.

6. Appendix: Iwasawa manifolds and complex multiplication.

One might be interested in the following question: which tori T and elliptic curves E can be obtained as the base and the fiber of an Iwasawa bundle.

Let $\mathfrak{g}$ be the Lie algebra of G. Denote its center by $\mathfrak{z}$ and its abelinization by $\mathfrak{g}^{ab}$. Since the group G is nilpotent, the exponential map $\exp\colon \mathfrak{g} \to G$ is affine and the cocompact lattices in G are in one to one correspondence with lattices of full rank in $\mathfrak{g}$, which are closed under the Lie bracket ([Mal], [Rol]).

So let Λ' be a cocompact lattice in G and $\Lambda = \log(\Lambda')$ its formal logarithm which is a Lie subring in $\mathfrak{g}$. Consider the lattices $\Gamma := \log(\Lambda) \bigcap \mathfrak{z}$ and $\Delta := \log(\Lambda)/\Gamma$. The above-mentioned Mal'čev theorem implies these are lattices of the full rank in $\mathfrak{z} = \mathbb{C}$ and $\mathfrak{g}^{ab} = \mathbb{C}^2$ respectively.

For any $\mathbb{C}$-linear section of the natural projection $h\colon \mathfrak{g}^{ab} \to \mathfrak{g}$ the cocycle map

$$\Lambda^2\mathfrak{g}^{ab} \to \mathfrak{z},\ v_1 \wedge v_2 \mapsto [h(v_1), h(v_2)]$$

defines a $\mathbb{C}$-linear non-degenerate symplectic form $q\colon \Lambda^2\mathbb{C}^2 \to \mathbb{C}$, such that

$$q(\Lambda^2\Delta) \subset \Gamma$$

(see Proposition 3.6).

Conversely, if for a fixed non-degenerate form $q \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^2\mathbb{C}^2, \mathbb{C})$ and for some lattices $\Delta \subset \mathbb{C}^2$ and $\Gamma \subset \mathbb{C}$ the condition

$$q(\Lambda^2\Delta) \subset \Gamma$$

is satisfied, then there exists a lattice Λ in $\mathfrak{g}$, such that $\Lambda \bigcap \mathfrak{z} = \Gamma$ and $\Lambda/(\Lambda \bigcap \mathfrak{z}) = \Delta$. It is given by

$$\Lambda := \left\{ \begin{pmatrix} 1 & \delta_1 & \frac{\gamma}{2} \\ 0 & 1 & \delta_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \delta_1, \delta_2 \in \Delta, \gamma \in \Gamma \right\}.$$

Notice that for the corresponding Iwasawa manifold $I = G/\exp(\Lambda)$ the base of the Iwasawa bundle $T = G^{ab}/\exp(\Lambda)^{ab}$ is isomorphic to $\mathbb{C}^2/\Delta$ and its fiber $E = Z(G)/\exp(\Lambda \bigcap \mathfrak{z})$ is isogenous to the elliptic curve $\mathbb{C}/\Gamma$.

It turns out that the condition $q(\Lambda^2\Delta) \subset \Gamma$ is rather strong. It implies that the curve E always carries a complex multiplication and the torus T is isogenous to a product of two elliptic curves with the same complex multiplication.

Recall that for a general complex torus of complex dimension g its endomorphism ring is isomorphic to $\mathbb{Z}$. A complex torus A of dimension g is said to carry *complex multiplication* (or to be *of CM-type*) if its rational endomorphism algebra has dimension $\dim_{\mathbb{Q}} \mathrm{End}(A) \otimes \mathbb{Q} = 2g^2$.

For elliptic curves this notion is especially well-known:

Proposition 6.1. *Let E be an elliptic curve. Then the following conditions are equivalent:*

(1) *There exists a holomorphic automorphism $\tau\colon E \to E$, which fixes the origin and is not equal to ± 1;*
(2) $\mathrm{End}(E) \otimes \mathbb{Q} = K$ *for some imaginary quadratic number field K;*

(3) *$E = \mathbb{C}/\sigma(O)$, where O is an order in an imaginary quadratic number field K and $\sigma\colon K \hookrightarrow \mathbb{C}$ is an embedding;*
(4) *E is an elliptic curve with complex multiplication.*

Proof: See in ([Sil], ch.V) □

The classical sources on the theory of complex multiplication on elliptic curves and abelian varieties are [Mum] and [Sil].

In a more general situation a close statement about complex multiplication on tori in nilmanifolds was proved by J. Winkelmann in [W, ch. 9]

Theorem 6.2 (Winkelmann)**.** *Let G be a nilpotent complex Lie group which is irreducible(i.e. is not a product of two other complex nilpotent groups) and not abelian.Consider a nilmanifold $N = G/\Lambda$.*

Then N decomposes into a sequence of holomorphic principal toric bundles

$$N = N_0 \xrightarrow{T_0} N_1 \xrightarrow{T_1} N_2 \xrightarrow{T_2} \dots \xrightarrow{T_{s-2}} N_{s-2} \xrightarrow{T_{s-1}} N_{s-1} = T_s.$$

Moreover for any $j < s$ the torus T_j is isogenous to a product of simple tori with complex multiplication. The same holds for T_s if and only if it is algebraic.

Winkelmann's proof is rather sophisticated and tells nothing about the geometry of the basic torus when it is not algebraic. We are going to show that in the Iwasawa case it always has the maximal Picard number, which not only yields its algebraicity, but gives the result of the Theorem 6.2 for Iwasawa manifolds.

Lemma 6.3. *The base T of an Iwasawa bundle π has maximal Picard number.*

Proof: Consider the space $H^{2,0+0,2}_{\mathbb{Q}} := H^2(T,\mathbb{Q}) \bigcap (H^{2,0}(T) \oplus H^{0,2}(T))$. It is enough to prove that $\dim_{\mathbb{Q}} H^{2,0+0,2}_{\mathbb{Q}} = 2$.

Assume that $E = \mathbb{C}/\Gamma$ is the structure group of π and consider $\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q}$ viewed as a $\mathbb{Q}$-vector subspace of $\mathbb{C}$.

Recall that the image of the cocycle $q\colon \Lambda^2\mathfrak{g}^{ab} \to \mathfrak{z}$ generates the whole center $\mathfrak{z}$ of the Lie algebra $\mathfrak{g}$. The cohomology class of the induced left-invariant form on T is equal to $c_1(\pi)$ and generates $H^{2,0}(T)$. Since q is defined over $\mathbb{Q}$ we get that

$$\dim_{\mathbb{Q}} H^2(T, \Gamma_{\mathbb{Q}}) \bigcap H^{2,0}(T) = \dim_{\mathbb{Q}} \log(\Gamma) \bigcap \mathfrak{z} = 2.$$

If γ_1 and γ_2 are two linearly independent classes in $H^2(T,\Gamma_{\mathbb{Q}}) \bigcap H^{2,0}(T)$, then $\gamma_1 + \overline{\gamma_1}$ and $\gamma_2 + \overline{\gamma_2}$ are two linearly independent rational classes in $H^{2,0+0,2}_{\mathbb{Q}}$. □

Next we use the following lemma:

Lemma 6.4. *Let A be a complex torus of dimension $g \geqslant 2$ and $\rho(A)$ its Picard number. Then the following conditions are equivalent:*

1. *A has the maximal Picard number, i.e. $\rho(A) = h^{1,1}(A) = g^2$;*
2. *$H^{1,1}(A)$ is defined over $\mathbb{Q}$;*
3. *$H^{2,0}(A) \oplus H^{0,2}(A)$ is defined over $\mathbb{Q}$.*
4. *$\dim_{\mathbb{Q}} \mathrm{End}(A) \otimes \mathbb{Q} = 2g^2$;*
5. *A is isogenous to the g-th power of E, where E is an elliptic curve with complex multiplication.*
6. *A is isomorphic to $E_1 \times \ldots \times E_s$, where all E_s are mutually isogenous elliptic curves with complex multiplication.*

Proof: See e.g. [Beau] □

Hence if $T = \mathbb{C}^2/\Delta$ is the base of Iwasawa bundle π, then it is isomorphic to the product of two elliptic curves E' and E'' with the same complex multiplication. Particularly, this means that the $\mathbb{Q}$-vector space $\Delta \otimes \mathbb{Q} \subset \mathbb{C}^2$ can be decomposed into a direct sum $\sigma(K) \oplus \sigma(K)$, where K is an imaginary quadratic number field and σ is one of its two conjugated embeddings into $\mathbb{C}$.

Let $E = \mathbb{C}/\Gamma$ be the structure group of the Iwasawa bundle over T. As it was discussed in the beginning of this section, there is a holomorphically symplectic form $q\colon \Lambda^2\mathbb{C} \to \mathbb{C}$ such that

$$q(\Lambda^2\Delta \otimes \mathbb{Q}) = \Gamma \otimes \mathbb{Q}.$$

Clearly, this implies that $\Gamma \subset \sigma(K)$ and E carries the same complex multiplication as E' and E''.

Corollary 6.5. *Let $\pi\colon I \to T$ be an Iwasawa bundle with the structure group E. Then T is isogenous to a product of two elliptic curves $E' \times E''$. All the three curves E, E' and E'' carry the same complex multiplication.*

Proposition 6.6. *Let A be a complex torus of maximal Picard number and $B \subset A$ be an elliptic curve. Then B has complex multiplication.*

Proof: For the first step notice that B carries complex multiplication if and only if it is isogenous to a certain curve B' with complex multiplication. Due to the property 6 from Lemma 6.4 the torus A is isomorphic to $E_1 \times \ldots \times E_s$, where E_j are isogenous elliptic curves with the same complex multiplication.

Let $pr_j\colon A \to E_j$ be the natural projections. At least one of this projections defines an isogeny $pr_j\colon B \to E_j$. Hence B carries the same complex multiplication as E_j. □

Proposition 6.7. *Let $A \subset I$ be a holomorphic torus. Then A carries complex multiplication.*

Proof: Suppose that $\dim_{\mathbb{C}} A = 1$, i.e. A is an elliptic curve. As follows from Lemma 4.7 either A is a fiber of π and hence $A \simeq E$, or $\pi(A) = B$ is an isogenous to A curve in T. Due to Proposition 6.6 and Lemma 6.3 B is an elliptic curve with complex multiplication.

Now assume that $\dim_{\mathbb{C}} A = 2$. Then $\pi(A) = B$ is an elliptic curve. Moreover, as it is proved in Proposition 4.11, $A \simeq B \times E$. Thus, Proposition 6.6 and Lemma 6.4 imply that the Picard number of A equals 4. $\square$

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