

Unification in Linear Modal Logic on Non-transitive Time with the Universal Modality

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1. Introduction

Modal logics play an important role in the design of systems that provide components of reasoning about the knowledge and time. Temporal logics actively develops the fields of mathematical logic, philosophy, computer science and artificial intelligence. The first study of temporal logics as modal systems was proposed by A. Prior [1], for the next half-century this area has become a complex technical discipline [2].

The idea of non-transitive time, in the aspect of knowledge, proceeds from the observation that the transfer of knowledge from the past to the future may not always be successfully performed: the available information in the past may not be available in the present. A detailed consideration of different points of view on non-transitive time and its expression by means of logical systems is considered in [3].

At the stage of its formation, the unification problem consisted in answering the question: is it possible to transform two terms into syntactically equivalent ones by changing variables to other terms. In the field of nonstandard logics this problem is equivalent to (and more often considered in the form of) possibility of a formula to become a theorem after replacing variables, preserving the values of the coefficient-parameters [4].

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V.V. Rybakov solved this problem for modal $\mathcal{S4}$, \mathcal{Grz} and intuitionistic logics, [5], in [6] he proposed an approach to the definition of all non-unifiable formulas for the extensions of $\mathcal{S4}$ and $(K4 + [\Box\perp \equiv \perp])$. Using this technique, the criteria of non-unifiability in linear transitive temporal logics of knowledge with multi-modal relations were found: over the \mathbb{N} (\mathcal{LTK} , [7]) and over the \mathbb{Z} with alternative relations (\mathcal{LFPK} , [8]).

To study the unification S. Ghilardi proposed a new approach based on the projective formulas [9], which allowed to algorithmize the construction of a finite complete sets of unifiers for the series of logics, [10, 11]. Based on this approach, W. Dzik and P. Wojtylak established a projective unification in the extensions of the logic $\mathcal{S4.3}$ [12]. In [13–16] it was found that a solution of the admissibility problem follows from the existence of computable complete sets of unifiers, which significantly increased the importance of the approach to unification through the projective formulas. In [17] V.V. Rybakov found the modification of linear temporal logic \mathcal{LTL} with the operator *Until*, for which the projective unification was established. From the projectivity of unification follows the existence of the most general unifier (mgu), but not vice versa. For example, in [18] the existence of mgu for each unified formula in \mathcal{LTL} with the operators *Next* and *Until* is proved and counterexample is constructed: an unified, but not a projective formula. In [19] the projective unification is proved for \mathcal{LFPK} , $\mathcal{LFPK}_{U_-}^{U_+}$, $\mathcal{LFPK}_{U_-,P}^{U_+,N}$.

The unification problem is reducible to the admissibility problem: the formula φ is unifiable in the logic \mathcal{L} if the inference rule φ/\perp is not admissible in \mathcal{L} . In some cases, when logic has a finitary type of unification, the admissibility problem is also reducible to the problem of unification [20, 21].

The approach based on the construction of a ground unifier (i.e., obtained by the substitution of constants) demonstrates wide applica-

bility: both in a way of proving the unifiability of an arbitrary formula, and in constructing projective unifiers [17, 19, 23]. The idea of constructing a projective unifier using a ground one, however, is not universal and all-applicable: in [10] it was shown that not for every formula in \mathcal{Int} a ground unifier gives a construction of a projective unifier, in [12] it was proved that for $\mathcal{S4.3}$ the projective unifier can't be simply described on the basis of the ground one. Despite this, the use of ground unifiers in solving unification problems is stay appropriate even when logic has nullary (worst) type of unification and mgu for some formulas do not exist: the construction of the ground unifier remains possible.

Simultaneously with intensive studies of unification in transitive logics, analogous questions remain extremely poorly studied for non-transitive cases, where they appear to carry much greater complexity, and many methods and even definitions turns out to be inapplicable or require considerable modification. However, it would be unfair ignore the existence of works for logics with non-standard relations. For example, E. Jerabek proved the nullary type of unification in minimal normal logic \mathcal{K} [22], and W. Dzik — the best — unitary type for $\mathcal{S5}$ and its extensions [23]. F. Wolter and M. Zakharyashev [24] proved the unsolvability of unification over the \mathcal{K} with additional universal modality.

In this paper, we investigate linear modal logic based on non-transitive time with a universal modality. It is proved that unifiability of any formula in this logic can be effectively detemined and a ground unifier can be found, if one exists. The projective unification is established, which guarantees its unitary type [9] and (almost) structurally completeness [25] in this logic.

2. Definitions and semantics

We give some definitions and formulations, and also semantic construction of linear bimodal logic of non-transitive time with universal modality (in our notation \mathcal{ULITL}).

The alphabet of the language $L^{\mathcal{ULITL}}$ includes a countable set of propositional variables $P = \{p_1, \dots, p_n, \dots\}$, brackets $(,)$, standard Boolean operations and two modal operators: non-transitive \diamond and universal \Box_U modalities.

Kripke frame (or scale) F is a pair $\langle W, R \rangle$, where W is a nonempty set of elements, and R is a binary relation on W . If for $a, b \in W$ is fulfilled aRb , then say a «sees» b . A frame F is said to be reflexive and transitive if its binary relation R is such kind. Let $\{p_1, \dots, p_n\}$ be the set of propositional variables. *Valuation* V on the frame F is a mapping associating with each variable p_i subset $V(p_i) \subseteq W$. *Kripke model* M (or shortly *model*) is a triple $\langle W, R, V \rangle$, where $\langle W, R \rangle$ is a frame, and V is a valuation of propositional variables from the set $Dom(V) = \{p_1, \dots, p_n\}$ called *sign domain* V . Let the model $M = \langle F, V \rangle$ be given. Then $\forall w \in F$:

- a. $\langle F, w \rangle \Vdash_V p \Leftrightarrow w \in V(p)$;
- b. $\langle F, w \rangle \Vdash_V \varphi \vee \psi \Leftrightarrow [(\langle F, w \rangle \Vdash_V \varphi) \vee (\langle F, w \rangle \Vdash_V \psi)]$;
- c. $\langle F, w \rangle \Vdash_V \varphi \wedge \psi \Leftrightarrow [(\langle F, w \rangle \Vdash_V \varphi) \wedge (\langle F, w \rangle \Vdash_V \psi)]$;
- d. $\langle F, w \rangle \Vdash_V \neg\varphi \Leftrightarrow [\neg(\langle F, w \rangle \Vdash_V \varphi)]$;
- e. $\langle F, w \rangle \Vdash_V \diamond\varphi \Leftrightarrow [\exists v \in F : (wRv) \Rightarrow (\langle F, v \rangle \Vdash_V \varphi)]$;
- f. $\langle F, w \rangle \Vdash_V \Box\varphi \Leftrightarrow [\forall v \in F : (wRv) \Rightarrow (\langle F, v \rangle \Vdash_V \varphi)]$.

For the logic \mathcal{L} a frame F is called a \mathcal{L} -*frame* or a frame that is adequate to the logic \mathcal{L} , if for any formula $\alpha \in \mathcal{L}$ for any valuation V we have $F \Vdash_V \mathcal{L}$. The logic \mathcal{L} defined by the frame F will be written as $\mathcal{L}(F)$.

In this paper we consider the Kripke frame $F = \langle \mathbb{N}, Next_{inf} \rangle$, where \mathbb{N} is the set of integers, and $Next_{inf}$ is the binary relation «next natural

number»: $\forall a, b \in \mathbb{N} : aNext_{inf}b \Leftrightarrow b = a + 1$. The model on the infinite frame $F = \langle \mathbb{N}, Next_{inf} \rangle$ will be denoted as $M = \langle F, V \rangle$.

In accordance with the definition, the frame F is linear non-transitive with irreflexive points, therefore the truth values of the modality \Box on any such model M coincides with \Diamond .

In addition to the non-transitive modality \Diamond , the language of logic \mathcal{ULITL} contains the model operator \Box_U , the truth values of formulas containing \Box_U on $M = \langle F, V \rangle$ is given as follows:

$$\forall x \in F, \langle F, x \rangle \Vdash_V \Box_U \varphi \Leftrightarrow [\forall y \in F, \langle F, y \rangle \Vdash_V \varphi].$$

The modal operator \Diamond_U is expressed in terms of the pairwise \Box_U by the ordinary way: $\Diamond_U \varphi := \neg \Box_U \neg \varphi$.

In other words, $\Box_U \varphi$ means that the formula φ always and everywhere valid. In this case, \Box_U is called a universal modality, and logic \mathcal{ULITL} containing \Box_U is called the linear bimodal logic based on non-transitive time with universal modality.

Definition 1. *Logic \mathcal{ULITL} is the set of all formulas of the language $L^{\mathcal{ULITL}}$ valid on the frame F :*

$$\mathcal{ULITL} := \{A \in Fma(L^{\mathcal{ULITL}}) \mid F \in \mathcal{ULITL}(F \Vdash_V A)\}.$$

3. Unification

The length $l(\alpha)$ of the formula α is defined as follows: $l(p) = 0$, where p is a proposition variable; $l(\alpha \circ \beta) = l(\alpha) + l(\beta) + 1$, where $\circ \in \{\vee, \wedge\}$; $l(\bigcirc \alpha) = l(\alpha) + 1$, where $\bigcirc \in \{\neg, \Diamond, \Box_U\}$.

Before proceeding to the main results, we prove an auxiliary, almost obvious, fact.

Proposition 1. *For all $c_1, \dots, c_r \in \{\top, \perp\}$ and any formula $\delta(p_1, \dots, p_r)$ there is $c \in \{\top, \perp\}$, s.t. $\forall x \in F, \langle F, x \rangle \Vdash \delta(c_1, \dots, c_r) \equiv c$.*

Proof. We carry out the proof by induction on the length of the formula δ . Let $\delta = p$, then as a result of the substitution we get $\delta = \top$, so $V(\top) = F$, or $\delta = \perp$, which means $V(\perp) = \emptyset$.

If $\delta = c_1 \vee c_2$, where $c_1, c_2 \in \{\top, \perp\}$, then $\delta = \max(c_1, c_2)$, if $\delta = c_1 \wedge c_2$, then $\delta = \min(c_1, c_2)$ and, by the inductive hypothesis, $V(\delta) = F$ or $V(\delta) = \emptyset$.

If $\delta = \neg c_1$, where $c_1 \in \{\top, \perp\}$, then $\delta = \top$, if $c_1 = \perp$, or $\delta = \perp$, if $c_1 = \top$ and, again accordingly to the inductive hypothesis, $V(\delta) = F$ or $V(\delta) = \emptyset$.

Let $\delta = \bigcirc c_1$, where $\bigcirc = \{\diamond, \square_U\}$ and $c_1 \in \{\top, \perp\}$. If $c_1 = \perp$ then, because of $V(\perp) = \emptyset$, we get $V(\bigcirc\perp) = \emptyset$. If $c_1 = \top$ then, because of $V(\top) = F$, we also get $V(\bigcirc\top) = F$. \square

Definition 2. A formula $\alpha(p_1, \dots, p_s)$ is said to be unifiable in a logic \mathcal{L} iff exists a substitution $\sigma : p_i \mapsto \sigma_i$ for each p_i , s.t. $\alpha(\sigma_1, \dots, \sigma_s) \in \mathcal{L}$. In this case, this substitution σ is called a unifier of the formula α .

A ground unifier is a unifier obtained by the substitution constants $\{\top, \perp\}$ in place of the variables of the formula.

Earlier, in [26] we proved the criterion of non-unifiability for arbitrary \mathcal{L} with expressible universal modality:

Theorem 1. A formula A is non-unifiable in $\mathcal{L} \Leftrightarrow \square_U A \rightarrow \left[\bigvee_{p \in \text{Var}(A)} \diamond_U p \wedge \diamond_U \neg p \right] \in \mathcal{L}$.

Definition 3. A unifier σ of the formula $\alpha(p_1, \dots, p_s)$ is called more general than another σ^1 in \mathcal{L} , if there exists a substitution σ^2 , s.t. for any variable p_i : $\sigma^1(p_i) \equiv \sigma^2(\sigma(p_i)) \in \mathcal{L}$.

A unifier σ of the formula $\alpha(p_1, \dots, p_s)$ is called a most general unifier (shortly mgu), if for any other σ^i unifier σ is more general than σ^i .

A most general unifier can be interpreted as the best solution to the unification problem. Logic has a *unitary* type of unification, if for

any unified formula there is a mgu; *finitary* (*infinitary*) type if there is a finite (respectively infinite) number of the best solutions (in this case, all of them are called *a maximal unifiers*). The worst type of unification is *nullary* type: some of the unifiable formulas do not have maximal unifiers, [22].

Definition 4. A formula $\alpha(p_1, \dots, p_s)$ is said to be *projective in logic \mathcal{ULITL}* , if there is a unifier τ (which called *projective unifier*) for a formula α , s.t. $\Box_U \alpha \rightarrow [p_i \equiv \tau(p_i)] \in \mathcal{ULITL}$ for any variable p_i of the formula α .

For an arbitrary formula in the logic \mathcal{ULITL} it is possible to establish its unifiability using only ground unifiers:

Theorem 2. *Unifiability of an arbitrary formula $\varphi(p_1, \dots, p_s)$ in \mathcal{ULITL} can be effectively established using the substitution $\sigma(\varphi)$ of the following form: $\forall p_i \in \text{Var}(\varphi) \sigma(p_i) \in \{\top, \perp\}$.*

Proof. Lets show that to check unifiability of any given formula φ it is enough to establish only the existence of the ground unifier $gu := \{\top, \perp\}$, obtained by substituting variables for constants.

Let a formula $\varphi(p_1, \dots, p_s)$ be unified in \mathcal{ULITL} and the set $\delta_1(q_1, \dots, q_r), \dots, \delta_s(q_1, \dots, q_r)$ is its unifier. Then

$$\delta(\varphi) := \varphi(\delta_1(q_1, \dots, q_r), \dots, \delta_s(q_1, \dots, q_r)) \in \mathcal{ULITL}.$$

We replace the variables q_1, \dots, q_r by the constants $c_i \in \{\top, \perp\} (i \in [1, r])$ in an arbitrary way. Because of we are dealing with a valid formula in logic, as a result of substitution we again obtain valid formula:

$$\varphi(\delta_1(c_1, \dots, c_r), \dots, \delta_s(c_1, \dots, c_r)) \in \mathcal{ULITL}.$$

Let us denote $gu(p_i) := \delta_i(c_1, \dots, c_r)$, then

$$\varphi(gu(p_1), \dots, gu(p_s)) \in \mathcal{ULITL},$$

where each of $gu(p_i) \in \{\top, \perp\}$ is a constant. Therefore, $gu(\varphi)$ is a ground unifier, which existence for an arbitrary formula can be verified in \mathcal{ULITL} as follows.

Due to the fact that $gu(p_1), \dots, gu(p_s)$ is nothing more than a set of constants for which φ is true, for an arbitrary (not necessarily unified) formula $\psi(p_1, \dots, p_s)$ it suffices to go through no more than 2^s substitution options of $\{\top, \perp\}$ instead of variables. If among them there is such that $\psi(gu(p_1), \dots, gu(p_s)) \equiv_{\mathcal{ULITL}} \top$, it will mean that formula ψ is unifiable in \mathcal{ULITL} and $gu(\psi) \in \mathcal{ULITL}$ is its ground unifier. Otherwise, if for all 2^s substitution options $gu(p_1), \dots, gu(p_s)$, $\psi(gu(p_1), \dots, gu(p_s)) \notin \mathcal{ULITL}$, then such formula ψ does not have a ground unifier, which means that it is non-unifiable in \mathcal{ULITL} . \square

Now we are ready to prove the main result of the article.

Theorem 3. *Any unifiable in \mathcal{ULITL} formula is projective.*

Proof. Assuming that $\varphi(p_1, \dots, p_s)$ is unifiable in \mathcal{ULITL} formula. For any variable $p_i \in Var(\varphi)$ we set the following substitution $\sigma(p_i)$:

$$\sigma(p_i) := (\Box_U \varphi \wedge p_i) \vee (\neg \Box_U \varphi \wedge gu(p_i)),$$

where $gu(p_1), \dots, gu(p_s)$ is a ground unifier of the formula $\varphi(p_1, \dots, p_s)$, obtained by the algorithm from the previous theorem.

Take any infinite model $M := \langle F, V \rangle$ with the arbitrary notation V . If σ is a unifier for φ , then $\sigma(\varphi) \in \mathcal{ULITL}$ and $\forall x \in F \langle M, x \rangle \Vdash_V \sigma(\varphi)$. Let us prove that the substitution σ is a unifier for φ in the logic \mathcal{ULITL} .

1. If $\forall x \in F : \langle M, x \rangle \Vdash_V \varphi$, then $\langle M, x \rangle \Vdash_V \Box_U \varphi$ and hence the second disjunctive term is refuted at x . If $\langle M, x \rangle \Vdash_V p_i$, then $\langle M, x \rangle \Vdash_V \Box_U \varphi \wedge p_i$, hence $\langle M, x \rangle \Vdash_V \sigma(p_i)$. If $\langle M, x \rangle \Vdash_V \neg p_i$, then $\langle M, x \rangle \not\Vdash_V \Box_U \varphi \wedge p_i$ and therefore $\langle M, x \rangle \Vdash_V \neg \sigma(p_i)$. Consequently, the truth value of $\varphi(p_1, \dots, p_s)$ at the point x w.r.t. V coincides with the value of $\varphi(\sigma(p_1), \dots, \sigma(p_s))$ at the same point w.r.t. V and hence in this case $\langle M, x \rangle \Vdash_V \sigma(\varphi)$.

2. If $\exists x \in F : \langle M, x \rangle \Vdash_V \neg\varphi$, then $\langle M, x \rangle \not\Vdash_V \Box_U\varphi$, which is possible for the second disjunctive term, but the first one is immediately disproved at x . Then the truth values of all $\sigma(p_i)$ at x coincide with $gu(p_i)$, and because $\langle M, x \rangle \Vdash_V gu(\varphi)$ (by virtue of the selection of the ground unifier $gu(\varphi) \in \mathcal{ULITL}$), again $\langle M, x \rangle \Vdash_V \sigma(\varphi)$. Hence, $\sigma(\varphi) \in \mathcal{ULITL}$ for the unifiable in \mathcal{ULITL} formula φ .

Lets prove that $\sigma(\varphi)$ is a projective unifier. If we substitute $\sigma(p_i)$ into the definition of the projective formula, we obtain the following:
 $\forall p_i \in Var(\varphi)$

$$\Box_U\varphi \rightarrow (p_i \leftrightarrow [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))]) \in \mathcal{ULITL},$$

if σ is a projective unifier for φ . Assume the converse: let σ be a not projective substitution. Then $\exists x$

$$\langle M, x \rangle \Vdash_V \Box_U\varphi, \tag{1}$$

but

$$\langle M, x \rangle \not\Vdash_V p_i \leftrightarrow [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))]. \tag{2}$$

In this case

$$\langle M, x \rangle \not\Vdash_V p_i \rightarrow [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))], \tag{3}$$

or

$$\langle M, x \rangle \not\Vdash_V [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))] \rightarrow p_i. \tag{4}$$

If (3), then $\langle M, x \rangle \Vdash_V p_i$, but in this instance $\langle M, x \rangle \Vdash_V \Box_U\varphi \wedge p_i$, by virtue of (1) and p_i at x , and therefore $\langle M, x \rangle \Vdash_V p_i \rightarrow [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))]$.

If (4), consequently $\langle M, x \rangle \Vdash_V [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))]$, but it is possible only with $\langle M, x \rangle \Vdash_V p_i$, because $\langle M, x \rangle \Vdash_V \Box_U\varphi$ following from (1), hence in the disjunction of $\sigma(p_i)$ only first term can be fulfilled. Therefore the conclusion (4) is true and $\langle M, x \rangle \Vdash_V [(\Box_U\varphi \wedge p_i) \vee (\neg\Box_U\varphi \wedge gu(p_i))] \rightarrow p_i$. Hence, σ is a projective unifier for φ in logic \mathcal{ULITL} , and therefore φ is a projective formula.

□

By the theorem above, for any φ unifiable in \mathcal{ULITL} the substitution σ is a projective unifier, and hence the most general one [9]. Besides, the existence of mgu for every unified formula implies the finiteness of all complete sets of unifiers in the logic, and all of them can be obtained from the given projective substitution σ , and the logic \mathcal{ULITL} has a unitary type of unification [9].

A remarkable consequence of the projective unification in the logic \mathcal{ULITL} is also its almost structural completeness [25]: each admissible rule in \mathcal{ULITL} is derivable.

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