MORAVA-ORIENTABLE ORIENTABLE THEORIES

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Abstract. We prove that there are many generalized orientable cohomology theories in algebraic geometry for which there exist Chern class-like operations from Morava K-theories $K(n)^*$. We call these theories p^n-typical. These operations generate all operations to those p^n-typical theories whose ring of coefficients is free as a $\mathbb{Z}_{(p)}$-module. Examples of such theories are mn-th Morava K-theories $K(nm)^*$ for all $m \in \mathbb{N}$ and $CH^* \otimes \mathbb{Z}_{(p)}$. There is also the universal p^n-typical orientable theory which we call $BP\{n\}^*$.

Chern classes from Morava K-theory $K(n)^*$ to itself allow us to introduce the gamma filtration on $K(n)^*$ which satisfies properties analogous to the gamma-filtration on K_0.

As an application of constructed operations we give a new estimate on the torsion in Chow groups of codimension less than 2^n+1 of quadrics in I^{n+2} following an idea of N. Semenov.

Introduction

Complex orientable cohomology theories appeared in algebraic topology as rigid examples of the objects in the stable homotopy category. The definition of orientability is strongly connected to the notion of Chern classes: all invarivants of vector bundles in such theories are expressed in terms of Chern classes. Moreover all relations between these invariants can be expressed in terms of relations coming from vector bundles.

In algebraic geometry the notion of an orientable cohomology theory did not seem to appear until the fundamental work of F.Morel and V.Voevodsky on the motivic homotopy theory. The setting developed by them allowed geometers to 'borrow' many notions and constructions from topology in a clear conceptual (but technically uneasy) way. In particular, V.Voevodsky has performed a motivic construction of the Thom spectrum to introduce the theory of algebraic cobordisms $MGL^{*,*}$ which was later proved to be the universal orientable theory.

The notion of orientability of more general cohomology theories than those which are representable in the stable motivic homotopy category was studied by I.Panin and A.Smirnov. They observed that orientability has three equivalent avatars: it can be specified either by Thom classes of line bundles, Chern classes of line bundles or pushforward maps for proper morphisms. Perhaps, it shows crucial importance of Chern classes in defining orientable theories.

While the study of oriented motivic spectra is not easy, any motivic-representable cohomology theory has a *pure* part, sometimes referred to as a *small* theory as opposite to the whole *big* theory. Small theories are presheaves of graded rings which often allow a more geometric description in comparison with big theories. For example, the pure part of motivic cohomology is Chow groups, and the pure part of algebraic K-theory is the Grothendieck group of vector bundles $K_0[\beta,\beta^{-1}]$.

The pure part of algebraic cobordisms $MGL^{*,*}$ has been developed in a seminal paper by M.Levine and F.Morel. It is usually denoted as Ω^* and is reffered to as Levine-Morel algebraic cobordisms. (However, the comparison of Ω^* and $MGL^{*,*}$ turned out to be difficult and was proved only years later). They also gave a definition of an orientable theory which is more restrictive than that of Panin-Smirnov, for which they proved that Ω^* is the universal oriented theory. This allowed to introduce the whole bunch of orientable theories which were investigated before in algebraic topology and are in some vague sense *freely generated by Chern classes*. More precisely, for any formal group law F_R over any ring R there exist an oriented theory $\Omega^* \otimes_{\Omega^*(k)} R$ with the ring of coefficients R and the corresponding formal group law F_R. These theories are called *free theories*.

In particular, in this way one may introduce algebraic Morava K-theories $K(n)^*$ defined by specific formal group laws over $\mathbb{Z}_{(p)}$. In the definition we use in the paper for each prime p and each number n there is a whole bunch of Morava K-theories $K(n)^*$ which are not multiplicatively isomorphic. Nevertheless we prove that these theories are additively isomorphic, however, non-canonically.

Theorem. *Let $K(n)^*$, $\bar{K}(n)^*$ be two n-th Morava K-theories over $\mathbb{Z}_{(p)}$, i.e. localisations of algebraic cobordisms defined by FGLs F_1, F_2 with logarithms of the form $\sum_{i=0}^\infty a_i x^{p^{ni}}$, $a_0=1$, $a_1 \in \frac{1}{p}\mathbb{Z}_{(p)}^\times$.*

Then there exist an isomorphism of presheaves of abelian groups $K(n)^* \xrightarrow{\sim} \bar{K}(n)^*$.

The goal of this paper is to show that Morava K-theories behave in similar ways to the K-theory. Consider Chern classes as (not neccesarily additive) operations from K_0, then one can prove that for any orientable theory A^* all operations from K_0 to it are freely generated by Chern classes as $A^*(pt)$-algebra. In this paper we define so called p^n-typical orientable theories and construct operations $c_i : K(n)^* \to A^*$ to them. If $A^*(pt)$ is a free $\mathbb{Z}_{(p)}$-module, then these operations generate all operations to A^*.

Definition. A formal group law over a torsion-free $\mathbb{Z}_{(p)}$-algebra A is called p^n-typical if its logarithm is of the form $\sum_{i=0}^{\infty} a_i x^{p^{ni}}$, $a_i \in A \otimes \mathbb{Q}$.

Orientable theory A^* is called p^n-typical if the corresponding FGL is p^n-typical.

There is a universal p^n-typical theory which we denote by $BP\{n\}^*$. It is a factor of BP^* by an ideal in $BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_k]$ generated by v_i with $i \nmid n$.

Theorem. *For any Morava K-theory $K(n)^*$ and any p^n-typical theory A^* there exist a series of non-additive operations $c_i : K(n)^* \to A^* \otimes \mathbb{Z}_{(p)}$ for $i \geq 1$ (which we call Chern classes) satisfying the following conditions.*

i) Operation c_i is supported on $K(n)^{i \mod (p^n-1)}$, i.e. $c_i(x) = 0$ for $x \in K(n)^j$, where $j \neq i \mod p^n - 1$.

ii) Denote by $c_{tot} = \sum_{i\geq 1} c_i$ the total Chern class. Then the Cartan's formula holds:

$$c_{tot}(x + y) = F^{K(n)}(c_{tot}(x), c_{tot}(y)).$$

iii) If A is a free $\mathbb{Z}_{(p)}$-module, then any operation $\tilde{K}(n)^* \to CH^* \otimes \mathbb{Z}_{(p)}$ can be written uniquely as a series in $\{c_j\}_{j\geq 1}$ over $\mathbb{Z}_{(p)}$:

$$[\tilde{K}(n)^*, A^* \otimes \mathbb{Z}_{(p)}] = A[[c_1, \ldots, c_i, \ldots]].$$

It should be mentioned that operations $c_1, \ldots, c_{p^n}$ from a particular $K(n)^*$ to $CH^* \otimes \mathbb{Z}_{(p)}$ were introduced by V. Petrov and N. Semenov ([6]). Their methods were different and allowed these operations to be constructed only to $CH^* \otimes \mathbb{Z}_{(p)}$ modulo torsion.

Now let us explain which tools we use to construct Chern classes from Morava K-theories. At the moment there seems to be no easy way to construct operations between big theories like $MGL^{*,*}$, motivic cohomology $H^{*,*}_{\mathcal{M}}$ and other representable motivic theories. The method we use does not allow to construct operations between big theories either. Note also that there is a mystery both shared by topology and algebraic geometry of a geometric or intrinsic description of Morava K-theories. The only way constructing values of Morava K-theories on an arbitrary variety at the moment is the localisation of cobordisms which is rarely amenable for computations. Thus, no geometric interepretation of constructed Chern classes seems possible for the time being. Nevertheless, operations can be constucted by purely algebraic methods with the use of the recent work of A. Vishik.

Vishik has developed a definition of orientable theories of rational type, for which it is possible to construct the value of a theory on a particular variety by induction on its dimension. It turned out that these theories are precisely free theories appearing above. The inductive description allows to classify completely all operations from a free theory to any orientable theory. More precisely, operation can be uniquely reconstructed by its restriction to the values on products of projective spaces which commutes with pullbacks along several morphisms between them. These morphisms are the Segre embeddings, the diagonal maps, the point inclusions, the projections and the permutations. Using the projective bundle theorem (over a point) one can make a 1-to-1 correspondence between operations and solutions of a certain system of linear equations which depends *solely* on formal group laws of theories in consideration.

To construct Chern classes from Morava K-theories to a p^n-typical theory A^* we first consider the case $A^* = CH^* \otimes \mathbb{Z}_{(p)}$. Using the Cartan's formula we look for a formula relating these conjectural operations to the 'Chern character' $ch : K(n)^* \to CH^* \otimes \mathbb{Q}$. For example, one may define usual Chern classes using the formulas $ch_1 = c_1$, $ch_2 = \frac{1}{2}(c_1^2 - c_2)$, $ch_3 = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3)$ and so on, provided that operations c_i defined by these formulas in terms of ch act integrally on products of projective spaces. Thus, by Vishik's theorem they define operations to CH^*.

However, we do not give explicit formulas relating ch_i and $c_1, \ldots, c_i$. Instead of ch_i we work with ϕ_i – an integral generator of a free rank one $\mathbb{Z}_{(p)}$-module of additive operations to $CH^i \otimes \mathbb{Z}_{(p)}$. We express c_i as a polynomial in $\phi_1, \ldots, \phi_i$ and prove that it acts integrally on products of projective spaces.

To deal with the general case we introduce Chern filtration $F^\bullet$ on the space $[A^*, B^*]$ of all operations from a free theory A^* to any orientable theory B^*. There is a natural injective map $gr^k_F[A^*, B^*] \to [A^*, CH^* \otimes B^*]$ which allows to reduce the general case to the case of Chow groups. The condition that the target theory has to be p^n-typical is needed for a nice classification of additive operations.

Using constructed Chern classes we are able to give an application of Morava K-theories to torsion in Chow groups of quadrics. Recall that A. Grothendieck has introduced the gamma filtration on K_0 with the use of Chern classes $c_i : K_0 \to K_0$. The definition is the following:

$$\gamma^m K_0(X) := < c_{i_1}(\alpha_1)\cdots c_{i_k}(\alpha_k) | \sum_j i_j \geq m, \alpha_j \in K_0(X) > .$$

Having constructed Chern classes $c_i^{K(n)} : K(n)^* \to K(n)^*$ we define the gamma filtration on $K(n)^*$ *verbatim* as for K_0.

Theorem. *The gamma filtration on $K(n)^*$ satisfies the following properties:*

i) the gamma filtration is strictly compatible with additive isomorphisms between different n-th Morava K-theories;
ii) $c_i^{CH}|\gamma^{i+1}K(n)^* = 0$;
iii) the operation c_i^{CH} is additive when restricted to $\gamma^i K(n)^*$, and it induces an isomorphism of $\mathbb{Q}$-vector spaces;
iv) c_i^{CH} when resrticted to $\gamma^i K(n)^*$ is surjective for $1 \leq i \leq p^n$;
v) $gr^i_\gamma K(n)^* = gr^i_\gamma K(n)^{i \mod p^n - 1}$.

It seems interesting to note that historically a kind of 'loop' is made. The gamma filtration was introduced as a tool for the proof Riemann-Roch theorem, and the key ingredient of the Vishik's classification theorem is the general Riemann-Roch theorem which thus allows us to introduce the gamma filtration on $K(n)^*$. Of course, the end of this 'loop' does not coincide with the beginning which perhaps makes our considerations not so uninteresting.

The gamma filtration on $K(n)^*$ may be used to estimate Chow groups in codimensions less or equal to p^n. For $p = 2$ we provide such an estimate for quadrics of special type.

Recall that algebraic Morava K-theories (with $p = 2$) first appeared in the Voevodsky's plan of the proof of the Milnor conjecture. The reason was an observation of Voevodsky that Pfister quadrics in I^{n+2} are 'seen as split quadrics' by Morava K-theories $K(m)$ with $m \leq n$ and $p = 2$. Later N. Semenov generalized this result to all quadrics in I^{n+2}.

This allows to calculate the gamma filtration on the n-th Morava K-theory of such quadrics with the use of the general Riemann-Roch theorem. Using the surjectivity of several first Chern classes from $K(n)^*$ to $CH^* \otimes \mathbb{Z}_{(2)}$ we are able then to prove the following result.

Theorem. *Let $q \in I^{n+2}$ be an anisotropic quadratic form, Q a corresponding smooth quadric of dimension $2d$. Let $j \in [1, 2^n - 1]$ be s.t. $d \equiv j \mod (2^n - 1)$.*

Then $CH^i(Q)$ is torsion-free for all i: $1 \leq i \leq 2^n$ except for $i = 2^n$ when $j = 1$ and for $i = j$ in the other case.

Parts of this paper can be found in the preprint [7] and were reported at the MFO conference "Algebraic Cobordism and Projective Homogenous Varieties" (2016).

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1. Preliminaries

In this section we recall the notions of a generalized orientable cohomology theory, a theory of rational type and state the main tool needed for our paper, Vishik's Classification of Operations Theorem.

Fix a field k with char $k = 0$.

1.1. **Orientable theories.**

Definition 1.1 (Vishik, [9, 2.1]; cf. Panin-Smirnov, [5, 2.0.1], Levine-Morel, [4, 1.1.2])**.** A generalized oriented cohomology theory (g.o.c.t.) is a presheaf A^* of commutative rings on a category of smooth varieties over k supplied with the data of push-forward maps for proper morphisms.

The structure of push-forwards has to satisfy the following axioms: projection formula, projective bundle theorem, $\mathbb{A}^1$-homotopy invariance and the localisation axiom (sometimes called excision).

Notation. Star in the superscript of a g.o.c.t. does not mean neccessarily that the theory is graded. However, if it is, then we will freely replace the star by a number or a variable, e.g. CH^2 or CH^i. In non-graded cases we do not drop the superscript in order to distinguish the theory itself with its ring of coefficients, i.e. A^* is a presheaf of rings, and A is usually the corresponding value on a point, $A = A^*(Spec(k))$. Nevertheless, we always write K_0 to denote the Grothendieck group of vector bundles as a g.o.c.t. with the ring of coefficients $\mathbb{Z}$.

We would like to present an exact form of the localisation axiom here as it varies among definitions in the literature. Let X be a smooth variety over k, let $j : U \to X$ be an open embedding and let $i : Z \to X$ be its closed complement. Define $A^*(Z)$ as a direct limit of $A^*(V)$ over the system of projective morhisms $f : V \to Z$ from smooth varieties V. Push-forward maps $(i \circ f)_*$ induce the map $i_* : A^*(Z) \to A^*(V)$.

The localisation axiom says that the following sequence of abelian groups is exact:

$$A^*(Z) \xrightarrow{i_*} A^*(X) \xrightarrow{j^*} A^*(U) \to 0 \tag{LOC}$$

For any g.o.c.t. one can define Chern classes of vector bundles in a usual way ([5]) as follows. Let X be a smooth variety, L_X be a line bundle over X, denote by $s : X \to L_X$ the zero section of L_X. Then $c_1^A(L_X) := s^* s_*(1_X)$ is the first Chern class. Higher Chern classes are defined using the projective bundle theorem and are uniquely determined by the Cartan's formula and the property that $c_i^A(V) = 0$ for $i > rk(V)$.

One may associate with each g.o.c.t. a formal group law (FGL) over its ring of coefficients via the following formula: $c_1^A(L \otimes L') = F_A(c_1^A(L), c_1^A(L'))$.

Example. The universal g.o.c.t. is the algebraic cobordisms of Levive-Morel Ω^* ([4, Ch. II]). Other examples of g.o.c.t. are the Grothendieck group of vector bundles K_0 and Chow groups CH^*.

The corresponding formal groups are the universal FGL over the Lazard ring $\mathbb{L}$, the multiplicative FGL $F_m(x, y) = x + y + xy$, the additive FGL $F_a(x, y) = x + y$, respectively.

Note that the (unique) morphism of oriented theories $\Omega^* \to A^*$ preserves Chern classes, and since $c_i^{\Omega}(V_X) = 0$ when $i > \dim X$ for dimension reasons the same is true for any g.o.c.t. The same argument shows that Chern classes are nilpotent in any g.o.c.t.

One can construct a g.o.c.t. with any FGL in the following way.

Definition 1.2 (Levine-Morel, [4, Rem. 2.4.14])**.** Let R be a ring, let $\mathbb{L} \to R$ be a ring morphism classifying a formal group law F_R over R.

Then $\Omega^* \otimes_{\mathbb{L}} R$ is a g.o.c.t. which is called a **free theory**. Its ring of coefficients is R, and its associated FGL is F_R.

Remark 1.3. Note that *any* formal group law yields a g.o.c.t, which is mainly due to the kind of localisation axiom in the definition. The tensor product is exact on the right and it suffices for the property (LOC) to stay true after the change of coefficients.

This shows the difference with orientable theories in topology or in the stable motivic homotopy category where additional conditions on the formal group law are imposed in order for it to be realized.

Theorem 1.4 (Levine-Morel, [4, Th. 1.2.2 and 1.2.3])**.** *Chow groups CH^* and K-theory of vector bundles K_0 are free theories, i.e. natural morphisms*

$$\Omega^* \otimes_{\mathbb{L}, F_a} \mathbb{Z} \to CH^*, \quad \Omega^* \otimes_{\mathbb{L}, F_m} \mathbb{Z} \to K_0$$

are isomorphisms.

Theories of rational type were introduced by A.Vishik in [9] and are those g.o.c.t. which satisfy an additional axiom (CONST) and a really strong 'inductive' property (but rather technical to state it here precisely). One crucial feature of this property is that values on varieties can be described by induction on dimension.

Definition 1.5. The **axiom (CONST)** for a g.o.c.t. A^* says that for any smooth irreducible variety X value of the theory in its generic point $A^*(k(X)) := \lim_{U\subset X} A^*(U)$ is canonically isomorphic to the value on the base field point $A := A^*(Spec(k))$.

This allows to split A^* as presheaf of abelian groups into two summands: $A^* = \tilde{A}^* \oplus A$, where A is a constant presheaf and $\tilde{A}^*$ is an ideal subpresheaf of elements which are trivial in generic points.

The following theorem allows us to skip the definition of a theory of rational type, which we will not use explicitly.

Theorem 1.6 (Vishik, [9, Prop. 4.9])**.** *Theories of rational type are precisely free theories.*

1.2. **Operations between theories of rational type.** Though the notion of a theory of rational type does not yield new examples of cohomology theories, their intrinsic inductive description allows to study operations and poly-operations between them in a very efficient way.

Definition 1.7. Let A^*, B^* be presheaves of abelian groups (or rings, or graded rings) on the category of smooth varieties over a field.

An **operation** $\phi : A^* \to B^*$ is a morphism of presheaves of sets. The set of operations is denoted by $[A^*, B^*]$. If B^* is a presheaf of rings, the set $[A^*, B^*]$ has the natural ring structure given by the multiplication on the target theory.

An **additive operation** $\phi : A^* \to B^*$ is a morphism of presheaves of abelian groups. The set of additive operations is denoted by $[A^*, B^*]^{add}$.

We will need to consider not only operations, but poly-operations as well. There are two types of them: external and internal ones. It is not hard to see that there is a 1-to-1 correspondence between these two notions ([10, p.8]). As we will be concerned only with external operations, we omit the adjective in the following definition.

Definition 1.8. Let A^*, B^* be presheaves of abelian groups (or rings, or graded rings) on the category of smooth varieties over a field k.

An **r-ary poly-operation** from A^* to B^* is a moprhism of presheaves of sets on the r-product category of smooth varieties over a field k from $(A^*)^{\times r}$ to $B^* \circ \prod^r$.

Explicitly, for varieties $X_1, \ldots, X_r$ poly-operation yields a map of sets

$$A^*(X_1) \times A^*(X_2) \times \ldots \times A^*(X_r) \to B^*(X_1 \times X_2 \times \ldots \times X_r).$$

The set of r-ary poly-operations is denoted by $[(A^*)^{\times r}, B^* \circ \prod^r]$.

Classification of Operations Theorem (COT) (Vishik, [9, Th. 5.1], [10, Th. 5.2])**.**
Let A^* be a theory of rational type and let B^* be a g.o.c.t.

Then the set of r-ary poly-operations from A^* to B^* is in 1-to-1 correspondence with the following data: maps of sets $\times_{i=1}^r A^*((\mathbb{P}^\infty)^{\times l_i}) \to B^*(\times_{i=1}^r (\mathbb{P}^\infty)^{\times l_i})$ for $l_i \geq 0$ (restrictions of a poly-operation), which commute with the pull-backs for:

1. *the permutation action of a product of symmetric groups $\times_{i=1}^r \Sigma_{l_i}$;*
2. *the partial diagonals for each i;*
3. *the partial Segre embeddings for each i;*
4. *the partial point embeddings for each i;*
5. *the partial projections for each i.*

Remark 1.9. If the target theory is graded, then the theorem allows one to compute poly-operations to each of the components of the target.

To see this note that grading on B^* yields (additive) projectors $p_n : B^* \to B^n$ and an operation to a component B^n is just an operation which is zero when composed with p_m, $m \neq n$. As follows from the theorem, this property may be checked on products of projective spaces.

Remark 1.10. Analogous result was proved in topology by T.Kashiwabara, [3, Th. 4.2].

However, Kashiwabara's theorem demands topological g.o.c. theories to satisfy several additional conditions. We did not check these conditions and thus cannot claim that our results hold in topological context as well.

The following straight-forward Proposition allows us to simplify statements about spaces of operations in future considerations. Namely, we will claim several times that some operations from A^* to B^* generate all operations meaning the space $[\tilde{A}^*, B^*]$.

Proposition 1.11. *Let A^* be a g.o.c.t. satisfying the (CONST) axiom, let B^* be any g.o.c.t. Then there are natural isomorphisms between spaces of operations:*

$$[A^*, B^*] = \prod_A [\tilde{A}^*, B^*], \qquad [A^*, B^*]^{add} = Hom(A, B) \oplus [\tilde{A}^*, B^*].$$

Proof. As $A^* = A \oplus \tilde{A}^*$ by the (CONST) property, there is a natural map $\prod_A [\tilde{A}^*, B^*] \to [A^*, B^*]$ which sends a set of operations $(\phi_a)_{a \in A}$ to an operation $(a, x) \to \phi_a(x)$. The inverse map is just the restriction of an operation to subspaces $a \oplus \tilde{A}^*$ for all $a \in A$.

The case of additive operations is done likewise. □

1.3. **Continuity of operations.** Recall that by the projective bundle theorem for any g.o.c.t. A^* the value on the product of infinite projective spaces is a formal power series ring $A^*((\mathbb{P}^\infty)^{\times l}) = A[[c_1^A(\mathcal{O}(1)_1), \ldots, c_1^A(\mathcal{O}(1)_l)]]$, where $\mathcal{O}(1)_i$ is the pull-back of the canonical line bundle from the i-th factor. Throughout the article we will use the notation $z_i^A := c_1^A(\mathcal{O}(1)_i)$, or just z_i if theory A^* is clear from the context.

The restriction of any operation to products of projective spaces satisfies the property of continuity which we now explain.

Let $G : A^* \to B^*$ be any operation from a theory of rational type A^* to any g.o.c.t. B^*.

By the COT G is determined by maps of sets $G_{\{l\}} : A[[z_1^A, \ldots, z_l^A]] \to B[[z_1^B, \ldots, z_l^B]]$ for all $l \geq 0$. As $G_{\{l\}}$'s have to commute with pull-backs along partial projections the following diagram is commutative for any $l \geq 0$:

$$\begin{array}{ccc} A[[z_1^A, \ldots, z_l^A]] & \xrightarrow{G_{\{l\}}} & B[[z_1^B, \ldots, z_l^B]] \\ \downarrow & & \downarrow \\ A[[z_1^A, \ldots, z_{l+1}^A]] & \xrightarrow{G_{\{l+1\}}} & B[[z_1^B, \ldots, z_{l+1}^B]] \end{array}$$

This allows to use only one transform, the inductive limit of maps $G_{\{l\}}$

$$G : A[[z_1^A, \ldots, z_l^A, \ldots]] \to B[[z_1^B, \ldots, z_l^B, \ldots]]$$

which uniquely determines $G_{\{l\}}$ for any l.

Denote by F_A^k an ideal in $A[[z_1^A, \ldots, z_l^A, \ldots]]$ of series of degree $\geq k$ (F_B^k is defined analogously). Without loss of generality (as we may subtract $G(0)$), we may assume that $G(0) = 0$.

Proposition 1.12 (Vishik, [10, Prop. 5.3])**.**

Let $P, P' \in A[[z_1^A, \ldots, z_l^A, \ldots]]$ be s.t. $P \equiv P' \mod F_A^k$. Then

$$G(P) \equiv G(P') \mod F_B^k.$$

This Proposition allows to calculate approximation of $G(P)$ approximating P. In particular, operation G is determined by its restriction to the products of finite-dimensional projective spaces, or equivalently by the maps

$$G_{r,n} : A^*((\mathbb{P}^n)^{\times l}) = A[[z_1^A, \ldots, z_l^A]]/F_A^{n+1} \to B^*((\mathbb{P}^n)^{\times l}) = B[[z_1, \ldots, z_l]]/F_B^{n+1}.$$

In other words, maps $G_{\{l\}}$ or map G are determined by their restriction to the polynomial rings $A[z_1^A, \ldots, z_l^A] \subset A[[z_1^A, \ldots, z_l^A]]$.

Analogous statements are true for poly-operations as well.

1.4. **General Riemann-Roch theorem.** Let A^* be a theory of rational type, let B^* be a g.o.c.t., $\phi : A^* \to B^*$ is an operation.

For a smooth variety X let $i : Z \to X$ be its closed smooth subvariety. Denote by G_Z^c the composition

$$A^*(Z) \xrightarrow{z_1^A \cdots z_c^A} A^*(Z \times (\mathbb{P}^\infty)^{\times c}) \xrightarrow{\phi} B^*(\mathbb{P}^\infty)^{\times c}) \to B^*(Z)[[z_1^B, \ldots, z_c^B]].$$

The following result is a general form of Riemann-Roch-type theorems for non-additive operations.

Theorem 1.13 (Vishik, [9, Prop. 5.19])**.** *Let $\alpha \in A^*(Z)$, denote by $\mu_1, \ldots, \mu_c$ the B-roots of the normal bundle $N_{Z/X}$.*

Let L_i be line bundles over Z for $1 \leq i \leq k$, and denote $x_i = c_1^A(L_i)$, $y_i = c_1^B(L_i)$ their first Chern classes. Then

$$\phi\left(i_*(\alpha \prod_{i=1}^k x_i)\right) = i_* Res_{t=0} \frac{G_Z^{c+k}(\prod_{i=1}^c z_i^A \prod_{j=c+1}^{c+k} z_j^A \cdot \alpha)|_{z_i^B = t +_B \mu_i, z_{c+j}^B = y_j}}{t \cdot \prod_{i=1}^c (t + \mu_i)} dt.$$

1.5. **Classification of additive operations.** The data of an additive operation specified in the COT can be rewritten as a set of formal power series satisfying a system of linear equations. We specify this system here.

Let A^* be a theory of rational type, let B^* be a g.o.c.t. and let ϕ be an additive operation from A^* to B^*. For $l \geq 0$ define maps $G_l \in Hom_{Ab}(A, B[[z_1^B, \cdots, z_l^B]])$ to be the values of the operation on 'basis' monomials of z-degree l:

$$G_l(\alpha)(z_1^B, \ldots, z_l^B) := \phi(\alpha z_1^A \cdots z_l^A).$$

It is clear that maps of abelian groups $G_{\{l\}} : A^*((\mathbb{P}^\infty)^{\times l}) \to B^*(((\mathbb{P}^\infty)^{\times l}))$ are defined by maps G_l.

Algebraic Classification of Additive Operations Theorem (CAOT) (Vishik, [9, Th. 6.2])**.** *Let A^* be a theory of rational type and let B^* be any g.o.c.t. Denote by $F_A(x,y) = \sum_{i,j} a_{ij}x^iy^j$, $F_B = \sum_{i,j} b_{ij}x^iy^j$ the corresponding formal group laws.*

Then the abelian group of additive operations $[A^*, B^*]^{add}$ is in 1-to-1 correspondence with the set of maps $G_l \in Hom_{Ab}(A, B[[z_1^B, \cdots, z_l^B]])$ for $l \geq 0$ which satisfy the following properties:

i) for any $\alpha \in A$ series $G_l(\alpha)$ is divisible by $z_1^B \cdots z_l^B$;
ii) for any $\alpha \in A$ series $G_l(\alpha)$ is symmetric;
iii) for any $\alpha \in A$ the following system of equations is satisfied

$$G_l(\alpha)(z_1^B, z_2^B, \ldots, z_{l-1}^B, F_B(x,y)) = \sum_{i,j} G_{l+i+j-1}(\alpha a_{i,j})(z_1^B, z_2^B, \ldots, z_{l-1}^B, x^{\times i}, y^{\times j}) \tag{1}$$

Here $x^{\times i}$ and $y^{\times j}$ denote i-tuple $(x, x, \ldots, x)$ and j-tuple $(y, y, \ldots, y)$ respectively.

Note that i) is an instance of continuity discussed in Section 1.3.

Remark 1.14. If theories A^* and B^* are graded, one can use Remark 1.9 to specify the data of additive operations between graded components.

Additive operations from A^n to B^m are in 1-to-1 correspondence with maps $G_l \in Hom_{Ab}(A^{n-l}, B[[z_1, \ldots, z_l]]_{(m)})$ satisfying properties i)-iii) of the CAOT.

1.6. **Derivatives and products of poly-operations.** There are two straight-forward ways to produce some poly-operations from operations, or in other words to increase the *arity* of operations.

First, if ϕ_1, ϕ_2 are r_1-ary and r_2-ary poly-operations, respectively, then we define an $(r_1 + r_2)$-ary poly-operation $\phi_1 \odot \phi_2$ as their external product:

$$(\phi_1 \odot \phi_2)(x_1, x_2, \ldots, x_{r_1}, y_1, y_2, \ldots, y_{r_2}) = \phi_1(x_1, x_2, \ldots, x_{r_1})\phi_2(y_1, y_2, \ldots, y_{r_2}).$$

This construction defines a morphism of algebras

$$[(A^*)^{\times r_1}, B^* \circ \prod^{r_1}] \otimes_B [(A^*)^{\times r_2}, B^* \circ \prod^{r_2}] \xrightarrow{\odot} [(A^*)^{\times (r_1+r_2)}, B^* \circ \prod^{r_1+r_2}].$$

One may interprete the statement that the latter map is an isomorphism for particular A^* and B^* as some kind of Kunneth-type property. When this property is satisfied for all r_1, r_2 (e.g. Th. 3.44, Prop. 3.47), we will write $[(A^*)^{\times r}, B^* \circ \prod^r] = [A^*, B^*]^{\odot r}$.

Second, if ϕ is an r-ary poly-operation, then we define an $(r+1)$-ary poly-operation $\partial_i^1\phi$ as its derivative with respect to the i-th component ([10, Def.3.1]). Denote by $Z_{<i} = (z_1, \ldots, z_{i-1})$, $Z_{>i} = (z_{i+1}, \ldots, z_r)$, then

$$\partial_i\phi(Z_{<i}, x, y, Z_{>i}) := \phi(Z_{<i}, x+y, Z_{>i}) - \phi(Z_{<i}, x, Z_{>i}) - \phi(Z_{<i}, y, Z_{>i}).$$

It is clear that ϕ is poly-additive if and only if $\partial_i\phi = 0$ for $1 \leq i \leq r$.

If $r = 1$, i.e. ϕ is an operation, we will omit the subscript and write $\partial\phi$ to mean its derivative. Iterating the procedure one can easily define $\partial^s_{(r_1,\ldots,r_s)} = \partial_{r_s} \circ \partial_{r_{s-1}} \circ \cdots \circ \partial_{r_1}$. However, it is easy to see that all s-derivatives of an operation are symmetric and thus derivatives do not depend on the order of derivation. We will write $\partial^s\phi$ to denote any of them.

By definition of the derivative of ϕ one can express values of ϕ on the sum of *two* elements as the sum of values of ϕ and $\partial^1\phi$. It is useful for computations to have analogous formulas for the values on the sum of any number of elements.

Proposition 1.15 (Discrete Taylor Expansion, Vishik, [10, Prop. 3.2])**.** *Let $f : A \to B$ be a map between abelian groups. Denote by $\partial^i f : A^{\times i} \to B$ its derivatives.*

For any set $\{a_i\}_{i\in I}$ of elements in A the following equality holds:

$$f(\sum_{i\in I} a_i) = \sum_{J\subset I} \partial^{|J|-1} f(a_j|j\in J).$$

2. Motivation: operations from K_0 to orientable theories

In this section we mention several results on operations from K_0 to other orientable theories. As we use these only as motivation for what will follow about algebraic Morava K-theories we leave them without proofs.

2.1. Chern classes as free generators of operations from K_0.

The following Theorem was communicated to the author by A.Vishik. The result is analogous to the usual calculation of generalized cohomology of a product of infinite Grassmannians, though it does not formally follow from it.

Theorem 2.1 (Vishik, for the proof see [7, Th. 2.1]). *Let A^* be a g.o.c.t. Then the ring of r-ary poly-operations from a presheaf $\tilde{K}_0$ to A^* is freely generated over A by external products of Chern classes.*

Using notations from section 1.6, we write $[(\tilde{K}_0)^{\times r}, A^*\circ\prod^r] = A[[c_1^A,\ldots,c_i^A,\ldots]]^{\odot r}$.

Remark 2.2. Note that there is no issue of convergence of a series of Chern classes for any particular element of K_0. Due to a discussion after Def. 1.1, Chern classes c_i^A are nilpotent and thus a formal power series reduces to a polynomial for each particular variety.

2.2. Additive operations from K_0 to an oriented theory A^*.

One can we deduce the description of all additive operations $[\tilde{K}_0, A^*]^{add}$ in terms of series in Chern classes. The result provides some intution about what one could expect of additive operations from higher Morava K-theories to Chow groups $[K(n)^*, CH^*\otimes \mathbb{Z}_{(p)}]^{add}$ (compare Cor. 2.5 and Cor. 3.21).

Denote by $P_n\in\mathbb{Q}[c_1,\ldots,c_n]$ a polynomial, which is the n-th graded component of $\log(1+c_1+c_2+\ldots+c_n)$ multiplied by n (variable c_i has degree i here). For example, $P_1=c_1, P_2=2c_2+(c_1)^2, P_3=3c_3+3c_1c_2+(c_1)^3$.

Proposition 2.3 ([7, Prop. 2.3]).
(1) $P_n\in\mathbb{Z}[c_1,\ldots,c_n]$;
(2) P_n's produce linearly independent (over A) additive operations from K_0 to A^*;
(3) all additive operations from $\tilde{K}_0$ to A^* are infinite A-linear combinations of P_n's.

Remark 2.4. This proposition is a generalization of a result of A.Vishik classifiying additive operations from K_0 to itself. Vishik has shown that any additive operation is a unique infinite linear combination of operations $\Upsilon_k := \sum_{i=1}^k (-1)^{i-1}\binom{k}{i}\psi_i$, where ψ_i are Adams operations. The proofs of Vishik's result and of ours are quite similar and one can check that Υ_k equals to a polynomial P_k on Chern classes $c_i^{K_0}$.

Corollary 2.5. *The natural map $[K_0, CH^i\otimes\mathbb{Z}_{(p)}]^{add}/p\to[K_0, CH^i/p]^{add}$ is an isomorphism of 1-dimensional vector spaces.*

In particular, take ϕ_i,ϕ_{ip} to be two generators of additive integral operations from K_0 to $CH^i\otimes\mathbb{Z}_{(p)}$, $CH^{ip}\otimes\mathbb{Z}_{(p)}$ respectively. Then $(\phi_i)^p = a\phi_{ip} \mod p$ for some $a\in\mathbb{F}_p$.

2.3. The gamma-filtration on K_0.

As for any g.o.c.t. one can define Chern classes for K_0, i.e. operations $c_i^{K_0}: K_0\to K_0$. These operations are strictly connected with γ-operations γ_i and can be expressed in terms of so called λ-operations ([1, Exp. 0]).

One then defines the following γ-filtration on K_0:

$$\gamma^i K_0(X) := < c_{i_1}^{K_0}(\alpha_1)\cdots c_{i_k}^{K_0}(\alpha_k) | \sum_j i_j\geq m, \alpha_j\in K_0(X) > . \tag{2}$$

We summarise properties of the gamma filtration in the following proposition.

Proposition 2.6.
(1) $\gamma^i K_0\subset\tau^i K_0$, where τ^i is the topological (codimension) filtration on K_0;
(2) Chern class $c_i^{CH}: K_0\to CH^i$ induces an additive morphism $c_i: gr^i_\gamma K_0\to CH^i$;
(3) $c_i\otimes id_{\mathbb{Q}}$ is an isomorphism between $gr^i_\gamma K_0\otimes\mathbb{Q}$ and $CH^i\otimes\mathbb{Q}$;
(4) c_1 is an isomorphism, c_2 is surjective when restricted to the graded components of the gamma filtration.

3. Operations from Morava K-theories

Fix a prime p. From now on all theories are defined over $\mathbb{Z}_{(p)}$-algebras and the adjective *integral* will always mean *defined over* $\mathbb{Z}_{(p)}$.

3.1. p^n-typical formal group laws.

Definition 3.1. A series $\gamma \in A[[x]]$ is called p^n-gradable (w.r.to x) if it is of the form $\sum_{j\geq 0} a_j x^{1+j(p^n-1)}$.

A series $\eta \in A[[x_1, \ldots, x_n]]$ is called p^n-gradable if it is p^n-gradable w.r.to any variable x_i.

The reason for this name is that a p^n-gradable series over A can be made a homogenous series of degree 1 over the ring $A[v_n]$ with $\deg v_n = 1 - p^n$, and $\deg x = 1$.

The following is straight-forward.

Proposition 3.2. *(1) Whenever composition of p^n-gradable series is defined, it is p^n-gradable.*
(2) If a series γ is p^n-gradable and invertible w.r.to composition, then its inverse is also p^n-gradable.
(3) If a series if p^{kn}-gradable, then it is also p^n-gradable.

Recall that there exist a notion of a p-typical FGL, and over a torsion-free ring an FGL is p-typical if its logarithm has the form $\sum_{i=1}^{\infty} l_i x^{p^i}$ ([2, Th. 4]).

There exist the universal p-typical FGL F_{BP} over the ring BP which is non-canonically isomorphic to $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. Its logarithm is $\sum_i l_i x^{p^i}$ where $pl_n = \sum_{i=0}^{n} l_i v_{n-i}^{p^i}$ (that is we take v_i to be Araki generators).

For these generators we have $[p]\cdot_{BP} \equiv \sum^{F_{BP}} v_i x^{p^i} \mod p$.

Definition 3.3. Formal group law F over a $\mathbb{Z}_{(p)}$-algebra is called p^n-typical, if it is p-typical and $[p]\cdot_F x$ is a p^n-gradable series.

An orientable theory is called p^n-typical, if the corresponding FGL is p^n-typical.

Proposition 3.4. *There exist a graded ring $BP\{n\}$ classifying p^n-typical FGL's.*

The ring $BP\{n\}$ can be naturally identified with a factor ring $\mathbb{Z}_{(p)}[v_n, v_{2n}, \ldots]$ of BP where Araki generators v_i are sent to zero for $i \nmid n$.

In particular, $BP\{1\}$ is BP, and $BP\{kn\}$ is a natural factor of $BP\{n\}$ for any $k, n \in \mathbb{N}$.

Proof. It is clear that the ring $BP\{n\}$ exists and can be identifed with a factor of the ring BP classifying p-typical FGLs with relations given by those coefficients of $[p]\cdot_{F_{BP}} x$ being equal to zero so that it is a p^n-gradable series. Denote by $F_{BP\{n\}}$ the universal p^n-typical FGL.

Let v_i be Araki's generators of BP. Our goal now is to show that the map $\phi : BP \to BP\{n\}$ sends v_i to zero for $i \nmid n$.

Suppose that $i_0 = \min\{j : \phi(v_j) \neq 0, j \nmid n\}$ is finite. Recall that $[p]\cdot_{BP} x = \sum_i^{BP} v_i x^{p^i}$, and therefore $[p]\cdot_{BP\{n\}} x = \sum_i^{F} \phi(v_i) x^{p^i}$. By our assumption

$$[p]\cdot_{BP\{n\}} x = \sum_j^{BP\{n\}} \phi(v_{jn}) x^{p^{jn}} +_{BP\{n\}} \phi(v_{i_0}) x^{p^{i_0}} +_{BP\{n\}} \ldots. \tag{3}$$

The first summand of the RHS of (3) is p^n-gradable up to summands of degree greater than $x^{p^{i_0}}$. Indeed, $\log_F$ is p^n-gradable up to x^{p^i}, and therefore F is also. It is clear then that the sum over F of $x^{p^{kn}}$ is p^n-gradable up to certain degree of x strictly bigger than x^{p^i}.

The rightmost summands of (3) have degree in x strictly bigger than p^{i_0}, and therefore in $[p]\cdot_{BP\{n\}} x$ there appears the monomial $\phi(v_{i_0})x^{p^{i_0}}$, so the series is not p^n-gradable and we get a contradiction with the finiteness of i_0.

It is clear for degree reasons that the FGL of $BP/(v_i, i \nmid n)$ is p^n-typical, and thus it is the ring classifying p^n-typical FGLs. ☐

Corollary 3.5. *A FGL F over a ring R is p^n-typical iff it is p-typical and $F = \sum a_{ij} x^i y^j$ is p^n-gradable (i.e. $a_{ij} = 0$ whenever $i + j \neq 1 \mod (p^n - 1)$).*

Proof. For degree reasons $F_{BP\{n\}}$ has the form as in the Corollary, and therefore any p^n-typical FGL has such form.

The converse is straight-forward: if F is p^n-gradable, then $[k]\cdot_F x$ is p^n-gradable for any $k \in \mathbb{Z}$. ☐

Corollary 3.6. *If A is a torsion-free $\mathbb{Z}_{(p)}$-algebra, then a FGL F over A is p^n-typical iff its logarithm is of the form $\sum_{i=0}^{\infty} l_i x^{p^{ni}}$.*

Proof. It is known that p-typical FGLs over torsion-free $\mathbb{Z}_{(p)}$-algebras are precisely those which have the logarithm of the form $\sum_{i=0}^{\infty} l_i x^{p^i}$. Thus, F having logarithm of the form prescribed in Corollary is p-typical.

Note that $[p]\cdot_F x = \log^{-1}(\log(px))$. Thus, if log is p^n-gradable, then so is $[p]\cdot_F x$ and F is p^n-typical.

Conversely, it is enough to check that the logarithm of $BP\{n\}$ is p^n-gradable. To do this we need to recall the formulas linking coefficients of the logarithm of BP in terms of Araki generators v_i:

$$pl_m = \sum_{i=0}^{m} l_i v_{m-i}^{p^i},$$

where $v_0 = p$.

Denote by $\bar{l_j} \in \mathbb{Q}[v_n, v_{2n}, \ldots]$ the coefficients of the logarithm of the universal p^n-typical FGL, i.e. $\log_{BP\{n\}} = \sum_{j=0}^{\infty} \bar{l}_j x^{p^j}$. Let $i_0 := \min\{j : l_j \neq 0, n \nmid j\}$, and assume that it is finite. Then $pl_{i_0} = \sum_{i=0}^{m-1} l_i v_{m-i}^{p^i} + l_{i_0} p^{p^{i_0}}$ in BP, and in the sum in the RHS everything maps to zero in $BP\{n\}$. Indeed, if $n \nmid i$ then l_i is zero, and if $n|i$, then $n \nmid i_0 - i$ and v_{i_0-i} maps to zero.

Thus, the equation reduces to $p\bar{l}_{i_0} = p^{p^{i_0}} \bar{l}_{i_0}$, and we arrive to a contradiction: $l_{i_0} = 0$. □

Definition 3.7. Denote by $BP\{n\}^*$ a theory of rational type with the ring of coefficients $BP\{n\}$ and the corresponding formal group law is the universal p^n-typical FGL.

Clearly, from the universality of algebraic cobordisms the following is straight-forward.

Proposition 3.8. *Theory $BP\{n\}^*$ is the universal p^n-typical orientable theory, i.e. for any g.o.c.t. A^* s.t. the correspoding FGL is p^n-typical there exist a unique morphism of theories $BP\{n\}^* \to A^*$.*

3.2. **Definitions and grading.**

Definition 3.9. Let $n \geq 1$. A theory of rational type with a p^n-typical FGL F over $\mathbb{Z}_{(p)}$ is called **n-th Morava K-theory** $K(n)^*$ if the height of $F \mod p$ has height n.

Later on we will need the following.

Proposition 3.10. *Let F be a FGL of an n-th Morava K-theory.*

Then its logarithm has the form

$$\log_{K(n)}(x) = x + \frac{a_1}{p} x^{p^n} + \frac{a_2}{p^2} x^{p^{2n}} + \ldots$$

where $a_i \in \mathbb{Z}_{(p)}^{\times}$.

Moreover, $a_k \equiv (a_1)^k \mod p$.

Proof. The condition that $F \mod p$ has height n means that it is classified by a map $\psi : BP\{n\} \to \mathbb{Z}_{(p)}$ sending the Araki generator v_n to an invertible element of $\mathbb{Z}_{(p)}$.

The logarithm of BP-theory has the form $\sum_{i=0}^{\infty} l_i x^{p^i}$ where $pl_m = \sum_{i=0}^{m} l_i v_{m-i}^{p^i}$. Denote by $\bar{l}_m \in \mathbb{Q}$ the image of $l_m \in BP \otimes \mathbb{Q}$ under the map classifying the FGL F.

As we have shown in Prop. 3.6, for the logarithm of any p^n-typical FGL l_m of BP is sent to zero when $m \nmid n$. Let us show now by induction that for n-th Morava K-theory $p^k \bar{l}_{kn} \in \mathbb{Z}_{(p)}^{\times}$, and $p^k \bar{l}_{kn} \equiv (pl_n)^k \mod p$.

Base of induction ($k = 1$) is clear, as $p\bar{l}_n = \psi(v_n) =: a_1$.

We have

$$p^k(1 - p^{kn-1})\bar{l}_{kn} = \sum_{i=0}^{k-1} p^{k-1} \bar{l}_{in} \psi(v_{(k-i)n})^{p^{in}},$$

and by induction we see that the RHS is integral and reducing modulo p we get

$$p^k \bar{l}_{kn} \equiv p^{k-1} \bar{l}_{(k-1)n} \psi(v_n)^{p^{(k-1)n}} \equiv (a_1)^{k-1} a_1 = (a_1)^k \mod p.$$

□

One can easily check that the first terms of the formal group law of any Morava K-theory $K(n)^*$ look like this:

$$F_{K(n)}(x, y) = x + y - a_1 \frac{1}{p} \sum_{i=1}^{p^n-1} \binom{p^n}{i} x^i y^{p^n-i} + \text{higher degree terms},$$

where $a_1 \in \mathbb{Z}_{(p)}^{\times}$.

Remark 3.11. The Artin-Hasse exponential establishes an isomorphism between formal group laws $F_m = x + y + xy$ and a p-typical FGL over $\mathbb{Z}_{(p)}$ of height 1.

Therefore $K_0 \otimes \mathbb{Z}_{(p)}$ is isomorphic to a first Morava K-theory as a presheaf of rings.

It is not true that any two n-th Morava K-theories as defined above are multiplicatively isomoprhic. We will show, however, in Theorem 5.3 that any two n-th Morava K-theories are isomoprhic as presheaves of abelian groups.

Proposition 3.12. *(1) Morava K-theories $K(n)^*$ are $\mathbb{Z}/(p^n-1)$-graded.*

We denote its graded components as $K(n)^1$, $K(n)^2$, …, $K(n)^{p^n-1}$, and freely use the following expressions $K(n)^i$, $K(n)^{i \mod p^n-1}$, $K(n)^{i+r(p^n-1)}$ to denote the component $K(n)^j$ where $j \equiv i \mod p^n-1$, $1 \le j \le p^n-1$.

(2) The grading on $K(n)^*$ is respected by Adams operations.

(3) The grading is compatible with push-forwards, i.e. for a proper morphism $f: X \to Y$ of codimension c push-forward maps increase grading by c, $f_*: K(n)^i(X) \to K(n)^{i+c}(Y)$.

In particular, $c_1^{K(n)}(L) \in K(n)^1(X)$ for any line bundle L over a smooth variety X.

Proof. Let $\sum_{i=0}^\infty a_i \frac{x^{p^{ni}}}{p^i}$ be the logarithm of $K(n)$ (Prop. 3.10). Consider the FGL over the graded ring $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ (with $\deg v_n = 1-p^n$) defined by its logarithm $\sum_{i=0}^\infty \frac{1}{p^i} v_n^{\frac{p^{in}-1}{p^n-1}} x^{p^{in}}$. Denote by $\hat{K}(n)^*$ the corresponding theory of rational type.

Since the logarithm is homogenous of degree 1 (with x having degree 1), the map $\mathbb{L} \to \mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ classifying the FGL preserves grading and the theory $\hat{K}(n)^*$ inherits grading from algebraic cobordisms.

It is easy to see from the COT that Adams operations respect grading on algebraic cobordisms and commute with morphisms to orientable theories. Thus, properties (2), (3) are straight-forward for $\hat{K}(n)^*$.

The FGL $F_{K(n)}$ can be defined by setting $v_n = 1$ in $F_{\hat{K}(n)}$ and theory $K(n)^*$ is the localisation of $\hat{K}(n)^*$ along $v_n = 1$, so that components $\hat{K}(n)^{i+r(p^n-1)}$ glue together for all $r \in \mathbb{Z}$. The claim now follows. ☐

3.3. Chern classes: statement of the main theorem.

Theorem 3.13. *For any Morava K-theory $K(n)^*$ and any p^n-typical theory A^* there exist a series of non-additive operations $c_i : K(n)^* \to A^*$ for $i \ge 1$ (which we call Chern classes) satisfying the following conditions.*

i) Operation c_i is supported on $K(n)^{i \mod (p^n-1)}$, i.e. $c_i(x) = 0$ for $x \in K(n)^j$, where $j \ne i \mod p^n-1$.

ii) Denote by $c_{tot} = \sum_{i\ge1} c_i$ the total Chern class. Then the Cartan's formula holds:

$$c_{tot}(x+y) = F_{K(n)}(c_{tot}(x), c_{tot}(y)).$$

iii) If A is a free $\mathbb{Z}_{(p)}$-module, then all operations from $K(n)^*$ to A^* are uniquely expressible as series in Chern classes:

$$[\tilde{K}(n)^*, A^* \otimes \mathbb{Z}_{(p)}] = A[[c_1, \ldots, c_i, \ldots]].$$

Remark 3.14. V.Petrov and N.Semenov introduced operations $c_1, c_2, \ldots, c_{p^n}$ from a specific Morava K-theory $K(n)^*$ to Chow groups in [6].

They did not use the COT, which was not available at that time, and the operations were constructed to $CH^* \otimes \mathbb{Z}_{(p)}$ modulo torsion.

Remark 3.15. The Cartan's formula in the Theorem 3.13 should be understood as a relation for each particular Chern class (in the same way as in the classical case).

Namely, $c_i(x+y) = (F_{K(n)}(c_1(x)+c_2(x)+\ldots+c_i(x), c_1(y)+c_2(y)+\ldots+c_i(y)))_i$, where $(F_{K(n)})_i$ means part of the degree i.

Remark 3.16. No uniqueness of Chern classes is claimed in the Theorem, though for operations to Chow groups it is not hard to explain how big is the difference of possible Chern classes. For operations from any n-th Morava K-theory to itself Chern classes are in some sense 'less unique', however, one can define the gamma filtration (Section 6) which does not depend on any choices.

It should be noted that one can substitute the formal group law of $K(n)^*$ in the Cartan's formula to any p^n-typical FGL over $\mathbb{Z}_{(p)}$ of height n. The proof remamins the same. There is no advantage of using one or another FGL for the Cartan's formula for Chern classes from $K(n)^*$, however, using the same FGL as $K(n)^*$ looks similar to the classical case of K_0. The usual Cartan's formula for Chern classes from K_0 can be written as $c_{tot}(x+y) = F_{K_0}(c_{tot}(x), c_{tot}(y))$ where $c_{tot} = \sum_{i\ge1} c_i$.

Remark 3.17. If A is not a free $\mathbb{Z}_{(p)}$-module, then it is not true that all operations from $K(n)^*$ to A^* are expressible in terms of Chern classes (for $n > 1$).

More precisely, we will show in Section 3.6 that for $n > 1$ there exist additive operations from $K(n)^*$ to CH^*/p which are not liftable to $CH^* \otimes \mathbb{Z}_{(p)}$ even as non-additive operations.

To prove the theorem first we consider the case $A^* = CH^* \otimes A$ where A is free as a $\mathbb{Z}_{(p)}$-module. Clearly, as $CH^* \otimes A = \oplus CH^* \otimes \mathbb{Z}_{(p)}$, it reduces to the case $A = \mathbb{Z}_{(p)}$. In this case we construct Chern classes by induction on degree. From the Cartan's formula it follows that derivative $\partial^1 c_i$ of operation c_i is equal to a polynomial in Chern classes of smaller degree. To define c_i one calculates an *anti-derivative* of $\partial^1 c_i$ as a $\mathbb{Q}$-polynomial in c_j, $j < i$. This is defined uniquely and any anti-derivative differs from it by an additive operation. Our goal is to find an additive operation ψ_i s.t. its sum with the anti-derivative above is an integral operation, to be denoted c_i. We reduce the problem of existence of ψ_i to a certain question about additive operations from $K(n)^*$ to CH^*/p (Lemma 3.23). This is done in Section 3.5.

In Section 3.6 we investigate additive operations from $K(n)^*$ to CH^*/p. Though we notice that there are many of them which are non-liftable, we find a sufficient condition for liftability as an additive operation which is sufficient for our purposes.

It is not hard to show using Theorem 2.1 and the COT that operations c_i provide rational generators of the ring of all operations to $CH^* \otimes \mathbb{Q}$. Thus, to prove iv) of Theorem 3.13 it is enough to show that a non-integral polynomial in Chern classes will yield a non-integral operation. To do this we make a careful study of derivatives of poly-operations defined by polynomials in Chern classes finally reducing the question to poly-additive poly-operations. The latter problem is quite easy. This is done in Section 3.7. In fact, we also show that external products of Chern classes provide free generators of poly-operations.

To prove the existence part of the theorem for a general p^n-typical theory it is enough to consider the case of universal p^n-typical theory. Using the continuity of operations we develop a tool which allows to truncate operations to theory B^* to operations to $CH^* \otimes B$. Thus we are able to reduce the problem to the already solved case of the Chow groups. The part iv) of the Theorem is in fact proved by the same tool.

In what follows n-th Morava K-theory is fixed, its FGL is denoted by $F_{K(n)}$ and its logarithms is $\log_{K(n)}(x) = x + \sum_{i=1}^{\infty} \frac{a_i}{p^i} x^{p^{ni}}$.

3.4. **Corollaries of the main theorem.** In this section we deduce several corollaries of Theorem 3.13 in the case when $A^* = CH^* \otimes \mathbb{Z}_{(p)}$. These corollaries are not applications of our result, but on the contrary they provide us a clue of how to construct Chern classes and prove the theorem. Lemma 3.19 is proved unconditionally and will be used in subsequent sections as a tool, while Cor. 3.18 and 3.21 will be proved by the end of Section 3.6.

Notation. Denote by $\nu_p(a)$ the p-valuation of a rational number $a \in \mathbb{Q}$. Let Q be a polynomial with rational coefficients: $Q = \sum a_{(i_1,\ldots,i_n)} t_1^{i_1} \cdots t_n^{i_n}$. Denote by $\nu_p(Q)$ the smallest p-valuation of coefficients of monomials of Q, i.e. $\nu_p(Q) = \min \nu_p(a_{(i_1,\ldots,i_n)})$. Thus, Q has coefficients in $\mathbb{Z}_{(p)}$ if and only if $\nu_p(Q) \geq 0$. If Q is p-integral, then it is divisible by p if and only if $\nu_p(Q) > 0$.

Let P be a polynomial in variable t over some ring A. Denote by $P[t^s]$ the coefficient of t^s in P (which is an element of A).

Define rational polynomials $P_i \in \mathbb{Q}[c_1, \ldots, c_{i-1}]$ for each $i \geq 1$ by the following formula: $(\log_{K(n)} c_{tot})_i = c_i - P_i$. Here $(\log_{K(n)} c_{tot})_i$ is the i-th graded component of the series $\log_{K(n)}(c_1 + c_2 + \ldots)$ where c_i has degree i. For example, $P_1 = \ldots = P_{p^n-1} = 0$, $P_{p^n} = -\frac{1}{p}(c_1)^{p^n}$.

Note that from the Cartan's formula it follows that $(\log_{K(n)} c_{tot})_i$ defines an additive operation from $K(n)^*$ to $CH^i \otimes A \otimes \mathbb{Q}$, where c_j's are operations from Theorem 3.13. There exist the minimal $\mu_i \in \mathbb{N}$ s.t. $p^{\mu_i}(\log_{K(n)} c_{tot})_i := \phi_i$ is integral.

By iv) of Theorem 3.13 the operation ϕ_i should be uniquely expressed as an integral polynomial in $c_1, \ldots, c_i$. Moreover, this polynomial should not be zero modulo p as ϕ_i is not zero modulo p by the choice of μ_i. Combined with formulas above this gives us an equation $\phi_i = p^{\mu_i} c_i - p^{\mu_i} P_i$. Thus, $\mu_i \geq 0$ (as P_i does not contain c_i) and p^{μ_i} is bigger or equal to the least common multiple of p-factors of denominators of P_i.

More precisely, $\mu_i = \max(0, -\nu_p(P_i))$. Indeed, if P_i is integral (i.e. $\nu_p(P_i) \geq 0$), then the expression $c_i - P_i$ is already integral and not zero modulo p (as P_i does not depend on c_i). Otherwise, if $\nu_p(P_i) < 0$, we have $\mu_i = -\nu_p(P_i)$ and the expression $\phi_i = p^{-\nu_p(P_i)} c_i - p^{-\nu_p(P_i)} P_i$ is integral and not zero modulo p.

Thus, we have proven the following Corollary, which can be used to define operations c_i inductively.

Corollary 3.18. *Let P_i, μ_i be as above.*

Then there exist an additive operation $\phi_i : K(n)^* \to CH^i \otimes A$ s.t. the following equality between operations to $CH^i \otimes A \otimes \mathbb{Q}$ holds

$$c_i = \frac{\phi_i}{p^{\mu_i}} - P_i(c_1, \ldots, c_{i-1}).$$

Actually we can calculate μ_i precisely, and this is done in the next Lemma. The computation together with the structure of polynomials P_i allows to prove some relations between additive operations from $K(n)^*$ to $CH^* \otimes \mathbb{Z}_{(p)}$ in Cor. 3.21.

The following Lemma is proved unconditionally.

Lemma 3.19. *Let $C := c_1t + c_2t^2 + \ldots = \sum_{i=1}^{\infty} c_it^i$ be a formal power series in variables c_i, i.e. $C \in \mathbb{Z}[c_1, c_2, \ldots][[t]]$. Define $P_s \in \mathbb{Q}[c_1, c_2, \ldots, c_{s-1}]$ to be equal to $c_s - \log_{K(n)}(C)[t^s]$ for $s \geq 1$.*

Fix $i > 0$.

1. Let j s.t. $j \neq i \mod (p^n - 1)$. Set variables c_s to be zero whenever $s \neq j \mod (p^n - 1)$. Then $P_i = 0$.

2. Let v be s.t. $i = p^{nk}v$ and $p^n \nmid v$. Then $\nu_p(P_i) \geq -k$.

2p. If moreover $k > 0$, then $\nu_p(P_i) = -k$.

Set variables c_j to be zero in P_i and P_v whenever $j \neq i \mod (p^n - 1)$. Denote these polynomials by $\tilde{P}_i, \tilde{P}_v$. Then over $\mathbb{F}_p[c_1, \ldots, c_i]$ polynomials $(p^k\tilde{P}_i \mod p)$ and $((c_v - \tilde{P}_v)^{p^{nk}} \mod p)$ are proportional.

Remark 3.20. Let us briefly explain how it will be used later.

In the course of an inductive construction of Chern classes the operation c_i will be expressed as a sum of P_i and some additive operation. Part 1 of the Lemma will be enough to prove that c_i is supported on $K(n)^{i \mod (p^n-1)}$ (which is Theorem 3.13, i)). Part 2p will be used to show that p^kP_i defines a non-trivial operation modulo p. Also Corollary 3.21 follows from it, which *indirectly* confirms and motivates results of Section 3.6.

Proof. 1. Denote by $C_{[j]} := \sum_{r=0}^{\infty} c_{j+r(p^n-1)}t^{j+r(p^n-1)}$, which equals to the formal power series C with variables c_s taken to be zero whenever $s \neq j \mod (p^n - 1)$.

We need to show that the coefficient of t^i in $\log_{K(n)} C_{[j]} = C_{[j]} + \frac{a_1}{p}C_{[j]}^{p^n} + \ldots$ equals to zero. Indeed, a product of p^{kn} monomials $c_{j+r(p^n-1)}t^{j+r(p^n-1)}$ has the power of t equal to $jp^{kn} \equiv j$ modulo $p^n - 1$.

2. Let us prove that p^kP_i is an integral polynomial. We need to show that the coefficient of t^i in $p^kC + a_1p^{k-1}C^{p^n} + \ldots + a_kC^{p^{nk}} + \frac{a_{k+1}}{p}C^{p^{n(k+1)}} + \ldots$ is a polynomial over $\mathbb{Z}_{(p)}$. First $(k+1)$ summands of this expression obviously produce an integral polynomial. Other summands are of the form $a_{k+r}p^k\frac{C^{p^{n(k+r)}}}{p^{k+r}}$ for $r > 0$. The derivative by t of this expression is equal to $a_{k+r}p^{kn+r(n-1)}C^{p^{n(k+r)}-1}C'$, which is an integral polynomial with coefficients divisible by $p^{kn+r(n-1)}$. On the other hand, the coefficient of t^i is multiplied by $i = p^{nk}v$ after differentiation. Comparing these two statements and using comparison $\nu_p(i) = nk + \nu_p(v) \leq nk + n - 1 \leq nk + r(n-1)$ we get that the coefficient of t^i is integral, and the claim is proved.

2p. Two cases can be dealt with separately.

First, assume $\nu_p(v) < n - 1$. Then we can actually prove the proportionality of P_i and $(c_v - P_v)^{p^{nk}}$.

We have $\nu_p(i) < nk + n - 1$ and by the argument in 2 summands $a_{k+r}p^k\frac{C^{p^{n(k+r)}}}{p^{k+r}}$ do not add anything non-divisible by p for $r > 0$. Thus, polynomial $-p^kP_i$ modulo p equals to $a_kC^{p^{nk}}[t^i]$. The latter is simply $a_kc_v^{p^{nk}}$.

Using the same argument one gets that $c_v - P_v$ modulo p equals to $C[t^v]$ and thus is c_v. The claim follows.

Second, $\nu_p(v) = n - 1$. Then again using the same argument polynomial $-p^kP_i$ equals $(a_kC^{p^{nk}} + \frac{a_{k+1}}{p}C^{p^{n(k+1)}})[t^i]$ modulo p and $-P_v$ equals $\frac{a_1}{p}C^{p^n}[t^v]$ modulo p. Our goal is to prove that $\frac{1}{p}C^{p^{n(k+1)}}[t^{p^{nk}v}]$ is equal to $(\frac{1}{p}C^{p^n}[t^v])^{p^{nk}}$ modulo p when specific variables in C are taken to be zero. As $a_{k+1} \equiv (a_1)^{k+1} \mod p$, $a_k = (a_1)^k$ by Prop. 3.10 we will have $p^kP_i = (a_1)^kc_v^{p^{nk}} + (a_1)^{k+1}\frac{1}{p}C^{p^n}[t^v])^{p^{nk}}$ and $(c_v - \tilde{P}_v)^{p^{nk}} \equiv c_v^{p^{nk}} + (a_1)^{p^{nk}}\frac{1}{p}C^{p^n}[t^v])^{p^{nk}}$. As $(a_1)^{p^{nk}} \equiv a_1 \mod p$ the claimed propotionality would follow.

Let l be s.t. $1 \leq l \leq p^n - 1$, $i \equiv l \mod (p^n - 1)$ or in other terms, $v = l + s(p^n - 1)$. From the equality $\nu_p(v) = n - 1$ one easily gets that there exists $u \in \mathbb{Z}$ non-divisible by p s.t. $s = l + up^{n-1}$, i.e. $v = l + (l + up^{n-1})(p^n - 1)$.

Define f to be $\sum_{r\geq 0} c_{l+r(p^n-1)}t^r$, thus, $C_{[l]} = t^lf(t^{p^n-1})$. In this notation we have two equalities:

$$-p^k\tilde{P}_i = (c_v^{p^{nk}} + \frac{1}{p}t^{lp^{n(k+1)}}f(t^{p^n-1})^{p^{n(k+1)}})[t^{p^{nk}v}] \mod p, \qquad -\tilde{P}_v = \frac{1}{p}t^{lp^n}f(t^{p^n-1})^{p^n}[t^v]. \tag{4}$$

We need to prove the equality $\frac{1}{p}t^{lp^{n(k+1)}}f(t^{p^n-1})^{p^{n(k+1)}}[t^{p^{nk}v}] = (\frac{1}{p}t^{lp^n}f(t^{p^n-1})^{p^n}[t^v])^{p^{nk}} \mod p$. Simplify it by changing variables $x = t^{p^n-1}$ and recalling that $v - lp^n = up^{n-1}(p^n - 1)$ and $p^{nk}v - lp^{n(k+1)} = up^{nk+n-1}(p^n - 1)$. Thus, we need to prove that $\frac{1}{p}f(x)^{p^{n(k+1)}}[x^{up^{nk+n-1}}]$ equals to $(\frac{1}{p}f(x)^{p^n}[x^{up^{n-1}}])^{p^{nk}}$. This is already a rather universal equality since it has to be true for any positive integer u non-divisible by p, any $k > 0$ and any series f (as it has coefficients which are independent variables).

To see this recall two simple facts about multinomial coefficients.

(1) For any $m > 0$ p-valuation $\nu_p\binom{p^m}{r_1,\ldots,r_k} = 1$ if and only if $r_j = a_j p^{m-1}$ for all j, $0 < a_j < p$.

In particular, k is at most p and allowing a_k to be equal to 0 we may write these coefficients as $\binom{p^m}{a_1p^{m-1},\ldots,a_pp^{m-1}}$.

(2) For any $a_j : 0 \leq a_j < p$ we have

$$\frac{1}{p}\binom{p^m}{a_1p^{m-1},\ldots,a_pp^{m-1}} \equiv \frac{1}{p}\binom{p}{a_1,\ldots,a_p} \mod p.$$

One writes $\frac{1}{p}f(x)^{p^m}[x^{up^{m-1}}]$ explicitly for any $m > 0$ and sees that it equals $(\frac{1}{p}f(x)^p[x^u])^{p^{m-1}}$ modulo p using the identities above.

$\square$

Assuming Theorem 3.13 we deduce the following result, which will be proven unconditionally later.

Corollary 3.21. *Let $i = p^{nk}v$ where $p^n \nmid v$, and let ϕ_i, ϕ_v be generators of integral additive operations from $K(n)^*$ to $CH^i \otimes \mathbb{Z}_{(p)}$ and $CH^v \otimes \mathbb{Z}_{(p)}$, respectively.*

Then $\phi_i \equiv a(\phi_v)^{p^{nk}} \mod p$ for some $a \in \mathbb{F}_p^\times$.

Proof. If $k = 0$ the claim is trivial, so assume $k > 0$.

It is clear that to prove the Corollary we may choose any generators ϕ_i, ϕ_v. In accordance with Cor. 3.18 and using Lemma 3.19 we choose them to satisfy the following equalities $\phi_i = p^kc_i - p^kP_i$, $\phi_v = c_v - P_v$.

As additive operations are supported only on the one component, which in this case is $K(n)^v$, to compare them we may restrict operations to it. Thus, we get $\phi_v = c_v - \tilde{P}_v$, $\phi_i = p^kc_i - p^k\tilde{P}_i$.

The comparison between $(\phi_v)^{p^{nk}}$ and ϕ_i modulo p now follows from part 2p of Lemma 3.19. $\square$

3.5. **Construction of Chern classes.** In this section we reduce the existence of operations $c_i : K(n)^* \to CH^i \otimes \mathbb{Z}_{(p)}$ satisfying conditions i), ii) of Theorem 3.13 to Lemma 3.23, which is proved in the next section.

For now let us provide another point of view on the Cartan's formula as stated in ii) of Theorem 3.13. We may look at each graded component of $c_{tot}(x+y)$ separately: in degree i the derivative of c_i is expressed as a polynomial in external products of Chern classes of smaller degree. For example, for $i = p^n$ we get $\partial^1 c_{p^n}(x,y) := c_{p^n}(x+y) - c_{p^n}(x) - c_{p^n}(y) = -\frac{a_1}{p}\sum_{j=1}^{p^n-1}\binom{p^n}{i}(c_1(x))^i(c_1(y))^{p^n-i}$. As the derivative of an operation defines it uniquely up to an additive operation, this gives a way of an inductive construction of Chern classes.

It is rather easy to integrate the derivative of the i-th Chern class as $\mathbb{Q}$-polynomial P_i in $c_1, \ldots, c_{i-1}$ (notation is consistent with Section 3.4). For many values of i this polynomial is not integral (Lemma 3.19), and to construct c_i one needs to find an additive operation s.t. the sum of P_i with it will be integral (cf. Cor. 3.18). For example, $-\frac{a_1}{p}(c_1)^{p^n}$ has the same derivative as the derivative of c_{p^n} predicted by the Cartan's formula. To prove the existence of an integral operation c_{p^n} we need to find an additive operation ψ_{p^n} from $K(n)^*$ to $CH^{p^n} \otimes \mathbb{Q}$ s.t. $-\frac{a_1}{p}(c_1)^{p^n} + \psi_{p^n}$ is an integral operation.

Proof of Theorem 3.13. We construct operations c_i, satisfying i) and ii) by induction on i s.t. they satisfy the following property:

iiibis) a generator of integral additive operations ϕ_i is expressible as an integral polynomial in $c_1, \ldots, c_i$.

We will show in Prop. 3.29 that for any $j \geq 1$ the space $[K(n)^*, CH^j \otimes \mathbb{Z}_{(p)}]^{add}$ of additive operations is a free module over $\mathbb{Z}_{(p)}$ of rank 1.

Base of induction. For $1 \leq i \leq p^n - 1$ choose c_i to be *any* generator of $[K(n)^*, CH^j \otimes \mathbb{Z}_{(p)}]^{add}$.

By Prop. 3.29 condition i) is satisfied. As there are no terms in $F_{K(n)}$ of degree bigger than 1 and less than $p^n - 1$ the Cartan's formula can be reformulated as additivity of operations c_i for $1 \leq i \leq p^n - 1$. Therefore ii) is satisfied. Property iiibis) is obviously true.

Induction step. Let $i > p^n - 1$. Assume that operations $c_1, c_2, \ldots c_{i-1}$, satisfying i) and ii), are defined. In particular, this means that we can calculate derivatives of polynomials in Chern classes c_j, $j < i$, as polynomials in these Chern classes c_j, $j < i$.

Define the rational polynomial $P_i \in \mathbb{Q}[c_1, \ldots, c_{i-1}]$ via the following formula: $(\log_{K(n)} c_{tot})_i = c_i - P_i$. Here $(\log_{K(n)} c_{tot})_i$ is the i-th graded component of the series $\log_{K(n)}(c_1 + c_2 + \ldots)$.

We claim that the derivative of c_i as predicted by Cartan's formula is equal to the derivative of P_i as a polynomial in Chern classes. Indeed, $(\log_{K(n)} c_{tot})_i$ is an additive operation, thus $\partial^1((\log_{K(n)} c_{tot})_i) = 0 = \partial^1(c_i - P_i)$. (*A priori* there could be relations between external products of Chern classes. However, the equality here is between derivatives of P_i and c_i as polynomials in external products of $c_1, \ldots, c_{i-1}$.)

Therefore if we define c_i as a sum of P_i and some additive operation, condition ii) will be satisfied.

Lemma 3.22. *[cf. Cor. 3.18] Let $\mu_i = \max(0, -\nu_p(P_i))$.*

Then there exist a generator ϕ_i of additive operations $[K(n)^*, CH^i \otimes \mathbb{Z}_{(p)}]^{add}$, s.t. operation $c_i : K(n)^* \to CH^i \otimes \mathbb{Q}$ defined by the formula $c_i = P_i(c_1, \cdots, c_{i-1}) + \frac{\phi_i}{p^{\mu_i}}$ acts integrally on products of projective spaces.

The existence of operation $c_i : K(n)^* \to CH^i \otimes \mathbb{Z}_{(p)}$ satisfying conditions i), ii), iiibis) follows from this Lemma. Indeed, the COT yields existence of integral operation c_i, s.t. $c_i \otimes id_{\mathbb{Q}}$ (which is loosely denoted by c_i in the Lemma) satisfies the formula in Lemma 3.22. The Cartan's formula will be true for integral c_i, since it is an equality between two operations which can be checked on products of projective spaces. As there is no torsion in values of our theories on products of projective spaces, the statement can be checked rationally where it is true by the formula defining c_i.

Let us show now that condition i) is satisfied. Choose $j \neq i \mod p^n - 1$. By Lemma 3.19 the polynomial P_i equals zero if we take c_s to be zero whenever $s \equiv j \neq i \mod (p^n - 1)$. By the induction assumption this is what happens to operation P_i when restricted to $K(n)^j$. By Prop. 3.29 the additive operation $\frac{\phi_i}{p^{\mu_i}}$ is supported on $K(n)^i$. Therefore by the formula defining c_i it is zero when restricted to $K(n)^j$.

Condition iiibis) is satisfied by the choice of μ_i as explained in Section 3.4.

We finish this section by reducing Lemma 3.22 to a result on additive operations $[K(n)^*, CH^*/p]^{add}$, Lemma 3.23, which is proved in the next section. In fact, much stronger version of it will be proved saying that the reduction modulo p of *many* integral operations, whenever it is additive, is proportional to the reduction of an integral additive operation.

Lemma 3.23. *Suppose that for some $\mu > 0$, $a \in \mathbb{Q}$ operation $p^\mu P_i + a\phi_i$ acts integrally on products of projective spaces and, thus, (by the COT) defines an integral operation π.*

Then π is proportional to ϕ_i modulo p. In particular, operation π is additive modulo p.

As a matter of fact let us show first that if π is integral, then it is additive modulo p. It is enough to show that its derivative is zero modulo p as an integral polynomial in Chern classes. Recall that the derivative of $p^\mu P_i$ equals to $p^\mu \partial^1 c_i$ by construction. Here $\partial^1 c_i$ is a formal notation meaning an integral polynomial in Chern classes $c_1, \ldots, c_{i-1}$ which is predicted by the Cartan's formula. Poly-operation $p^\mu \partial^1 c_i$ is equal to zero modulo p, since $\mu > 0$. The derivative of $a\phi_i$ is zero as well.

Lemma 3.22 follows from Lemma 3.23. Fix any generator ϕ_i of integral additive operations $[K(n)^*, CH^i \otimes \mathbb{Z}_{(p)}]$.

Let us prove the following statement by (finite) induction on r using Lemma 3.23.

Claim. For $0 \le r \le \mu_i$ there exist $a_r \in \mathbb{Z}_{(p)}^\times$ s.t. $p^{\mu_i - r} P_i + \frac{a_r}{p^r}\phi_i$ is an integral operation.

Base of induction ($r = 0$). Recall that μ_i is chosen so that $p^{\mu_i} P_i$ is integral, and so if $r = 0$, one may take $a_0 = 1$.

Induction step. Suppose we have shown that $p^{\mu_i - r} P_i + \frac{a_r}{p^r}\phi_i$ is an integral operation.

If $r = \mu_i$, then the induction is finished and we define $c_i := P_i + \frac{a_{\mu_i}}{p^{\mu_i}}\phi_i$, which is proved to be an integral operation. As $a_{\mu_i} \in \mathbb{Z}_{(p)}^\times$ the operation $\frac{a_{\mu_i}}{p^{\mu_i}}\phi_i$ is a generator of integral additive operations and Lemma 3.22 is proved.

If $\mu_i - r > 0$ then Lemma 3.23 is applicable to it as follows from a discussion above. Thus, operation $p^{\mu_i - r} P_i + \frac{a_r}{p^r}\phi_i$ equals to $b\phi_i$ modulo p for some $b \in \mathbb{Z}_{(p)}$. Operation $p^{\mu_i - r} P_i + (\frac{a_r}{p^r} - b)\phi_i$ is zero modulo p and hence operation $p^{\mu_i - (r+1)} P_i + \frac{a_r - bp^r}{p^{r+1}}\phi_i$ is an integral operation. Define $a_{r+1} = a_r - bp^r$.

If $r > 0$ then $a_r - bp^r \in \mathbb{Z}_{(p)}^\times$, since $a_r \in \mathbb{Z}_{(p)}^\times$, $b \in \mathbb{Z}_{(p)}$, and the induction step is proved.

However, for the induction step with $r = 0$ we need to show that $b \neq a_0 \mod p$.

Additional details on the induction step $r = 0 \to r = 1$.

Note that this applies only if $p^n | i$, as otherwise by Lemma 3.19 $\mu_i = 0$ and the induction stops at the base.

As we want a_1 not to be p-divisible, polynomial $p^{\mu_i} P_i$ should not be equal to zero modulo p.

To prove it use Lemma 3.19, 2p which says that polynomials $-p^{\mu_i} P_i \mod p$ and $(c_v - P_v)^{p^{nk}} \mod p$ are proportional. However, by induction assumption of the construction (not of this proof) property iiibis) is satisfied, i.e. $c_v - P_v = \phi_v$ for some generator ϕ_v of integral additive operations. Thus, it is not equal to zero modulo p. From the COT it follows that powers of $\phi_v \mod p$ are not zero as well, since the coefficient ring of the target theory has no zero divisors. Therefore $-p^{\mu_i} P_i \mod p$ also is not zero as an operation to CH^i/p.

The induction step is proved and reduction of Lemma 3.22 to Lemma 3.23 as well. □

Remark 3.24. One could avoid the use of *ugly* Lemma 3.19, 2p in the latter argument by proving all parts of the theorem together by induction. Namely, to show that $p^{\mu_i} P_i$ is not zero modulo p it would be enough to prove that there are no polynomial relations between constructed operations c_i modulo p. This is equivalent to part iii) of the Theorem.

As the technique which we develop in Section 3.7 to prove iii) differs from the discussion above we advocate the use of Lemma 3.19 as a clearer explanation.

3.6. **Additive operations to Chow groups modulo p.** In this section we prove Lemma 3.23 as Corrolary 3.33 of the study of additive operations $[K(n), CH^*/p]^{add}$. This Lemma is the main technical part of the proof of the theorem, it will be used not only for the construction of Chern classes to Chow groups but in the general case as well.

The proof is based on a discussion of the system of linear equations, which defines additive operations according to the CAOT. Roughly speaking it goes as follows. This system is finite-dimensional (for operations to a particular component of CH^*/p) and is upper-triangular when written in a naturally chosen basis. Over rings $\mathbb{Z}_{(p)}$, $\mathbb{Q}$ coefficients on the diagonal of this system are non-zero, and therefore the space of solutions is 1-dimensional (cf. Prop.3.29). However, over $\mathbb{F}_p$ many of coefficients on the diagonal are zero which leads to a higher dimension of the space of solutions and to the existence of additive operations to CH^*/p which are not liftable as additive operations to $CH^* \otimes \mathbb{Z}_{(p)}$. It turns out that for rather a natural set of additive operations (*gradable* operations, Definition 3.30) it is possible to use equations with zeros on the diagonal to express all variables in terms of one of them. In other words, one can transform this system into an upper-triangular one without zeros on the diagonal, but the choice of the new basis for this transformation is not so natural in the coordinates we work with. Anyway this proves that the space of gradable additive operations to CH^i/p is 1-dimensional (Cor.3.33) and it is easy to show that it is generated by a reduction of an integral additive operation. Lemma 3.23 then easily follows.

Notation (cf. Section 1.5). Let B be any $\mathbb{Z}_{(p)}$-algebra. If $\phi : \tilde{K}(n)^* \to CH^* \otimes B$ is an operation, denote by $G_l = G_l(t_1, \ldots, t_l) \in B[t_1, \ldots, t_l]$ the value of ϕ on $\prod_{i=1}^l c_1^{\tilde{K}(n)}(\mathcal{O}(1)_i) := z_1 \cdots z_l \in \tilde{K}(n)(\prod(\mathbb{P}^\infty)^{\times l})$ expressed as a symmetric polynomial in $t_i = c_1^{CH}(\mathcal{O}(1)_i)$. Here $\mathcal{O}(1)_i$ is the pullback of the canonical line bundle on the i-th component of the product of projective spaces.

Remark 3.25. From continuity of operations it follows that G_l is divisible by $t_1 \cdots t_l$.

Now we rewrite the CAOT for the case $A^* = K(n)^*$, $B^* = CH^m \otimes B$.

Theorem 3.26 (CAOT). *Additive operations $[\tilde{K}(n)^*, CH^m \otimes B]^{add}$ are in 1-to-1 correspondence with the set of symmetric polynomials $G_l \in B[t_1, \ldots, t_i] \cdot (t_1 \cdots t_i)$ (cf. Remark 3.25) of degree m for $l \geq 1$ which satisfy the following equations:*

$$G_l(t_1, t_2, \ldots, t_{l-1}, u + v) - \sum_{i,j \geq 0} a_{ij} G_{i+j+l-1}(t_1, \ldots, t_{l-1}, u^{\times i}, v^{\times j}) = 0, \qquad (A_l)$$

where a_{ij} are the coefficients of the formal group law of $K(n)$: $F_{K(n)}(x, y) = \sum_{i,j} a_{ij} x^i y^j$.

Remark 3.27. If we investigated operations from $K(n)^*$, and not from $\tilde{K}(n)^*$, then there would be also a 'polynomial' $G_0 \in A$ defining an operation. It is equal to zero in our considerations.

Let $\vec{r} = (r_1, \ldots, r_i)$ be a partition of m. We do not imply in our notations that numbers r_j are ordered somehow and $(1, 2, 3)$ and $(2, 1, 3)$ are both acceptable notations of the same partition. Denote by $\alpha^{(i)}_{(r_1,\ldots,r_i)}$ or $\alpha^{(i)}_{\vec{r}}$ a coefficient of any monomial $t_{j_1}^{r_1} \cdots t_{j_i}^{r_i}$ in the symmetric polynomial G_i where $\{j_1, \ldots, j_i\} = \{1, 2, \ldots, i\}$. We will call these coefficients *variables* as we consider them unknown in equations (A_l).

Proposition 3.28. *The system of equations (A_l) is a finite-dimensional linear homogenous system on variables $\alpha^{(i)}_{(r_1,\ldots,r_i)}$ with $i \leq m$.*

The matrix of this system is defined over $\mathbb{Z}_{(p)}$, it is upper-triangular when variables $\alpha^{(i)}_{(r_1,\ldots,r_i)}$ are ordered by the superscript (and no matter how we order variables with the same superscript).

The value of this matrix on the diagonal corresponding to a variable $\alpha^{(i)}_{(r_1,\ldots,r_i)}$ is non-zero if and only if $i \neq m$.

Moreover, there exist a value on the daigonal corresponding to a variable $\alpha^{(i)}_{(r_1,\ldots,r_i)}$ which is divisible by p if and only if the partition $(r_1,\ldots,r_i)$ is p-special, i.e. r_j is a p-th power for all $j : 1 \leq j \leq i$.

Proof. As G_l has to be of degree m and is divisible by $t_1\cdots t_l$, i.e. has degree at least l, then $G_l = 0$ for $l > m$. Thus, the system is finite-dimensional, and equations (A_l) are clearly linear and homogenous on coefficients of G_l and are defined over $\mathbb{Z}_{(p)}$.

Let $a, b, r_1, \ldots, r_{i-1} > 0$ s.t. $\vec{r} := (r_1,\ldots,r_{i-1}, a+b)$ is a partition of m. We need to show that the coefficient of the monomial $t_1^{r_1}\cdots t_{l-1}^{r_{l-1}}u^av^b$ in the equation A_l lies in $\mathbb{Z}_{(p)}\alpha^{(l)}_{\vec{r}} + \oplus_{k>l}\mathbb{Z}_{(p)}\alpha^{(k)}_{(s_1,\ldots,s_k)}$.

Variables $\alpha^{(i)}_{\vec{t}}$ with $i < l$ do not appear in the equation A_l. Thus, it is enough to show that variables $\alpha^{(l)}_{(s_1,\ldots,s_l)}$ do not appear as coefficients of monomial $t_1^{r_1}\cdots t_{l-1}^{r_{l-1}}u^av^b$ in equation A_l for $\vec{s} := (s_1,\ldots,s_l) \neq \vec{r}$.

In fact these variables can appear only in the expression $G_l(t_1,t_2,\ldots,t_{l-1},u+v) - G_l(t_1,t_2,\ldots,t_{l-1},u) - G_l(t_1,t_2,\ldots,t_{l-1},v)$ of equation A_l. That is we have to look at $\alpha^{(l)}_{(s_1,\ldots,s_l)}t_1^{s_1}\cdots t_{l-1}^{s_{l-1}}\left((u+v)^{s_l} - u^{s_l} - v^{s_l}\right)$, which obviously does not contain the monomial under investigation if partitions $\vec{s}$ and $\vec{r}$ are different.

Moreover, whenever $s_i \neq 1$ for all i (which is the case $l = m$), $(u+v)^{s_i} - u^{s_i} - v^{s_i}$ is not a zero polynomial and therefore there is a non-zero element on the diagonal of the matrix corresponding to the coefficient $\alpha^{(l)}_{(s_1,\ldots,s_l)}$. Clearly, all these elements are divisible by p if and only if s_i is a power of p for all i. □

Proposition 3.29. *Let B be a torsion-free $\mathbb{Z}_{(p)}$-algebra, then the B-module of additive operations $[\tilde{K}(n)^*, CH^i \otimes B]$ is free of rank 1, and it is generated by an operation from $[\tilde{K}(n), CH^i \otimes \mathbb{Z}_{(p)}]$.*

All operations from $\tilde{K}(n)^*$ to $CH^i \otimes B$ are supported on $\tilde{K}(n)^i$, i.e. if $\phi : K(n)^* \to CH^i \otimes B$ is an additive operation, then $\phi(x) = 0$ for $x \in K(n)^j$ whenever $j \neq i \mod p^n - 1$.

Proof. As was shown in Prop. 3.28 the matrix of the system defining additive operations to $CH^i \otimes B$ has rank 1 over $\mathbb{Q}$ and is upper-triangular. This proves the first claim.

It is enough to compute the support of a generator of operations from $\tilde{K}(n)^*$ to $CH^i \otimes \mathbb{Z}_{(p)}$. However, there is a non-zero additive operation $ch_i : K(n)^* \to CH^i \otimes \mathbb{Q}$ (the i-th component of the Chern character). There exist $N = N(i)$ s.t. $p^N ch_i$ acts integrally on products of projective spaces (indeed, an operation is defined by a finite number of polynomials with a finite number of deenominators), and thus it is enough to check that ch_i has support on $K(n)^i$ for products of projective spaces.

In the notations of this section let us show that

$$ch(z) \in \oplus_{r=0}^{\infty} CH^{1+r(p^n-1)}(\mathbb{P}^\infty) \otimes \mathbb{Q} \tag{5}$$

From the multiplicativity of the Chern character it would follow that $ch(z_1\cdots z_i) \in \oplus_{r=0}^{\infty} CH^{i+r(p^n-1)}((\mathbb{P}^\infty)^{\times i}) \otimes \mathbb{Q}$ for any $i \geq 1$.

Note that the element $ch(z)$ can be expressed as a series in $t := c_1^{CH}(\mathcal{O}(1)) \in CH^1(\mathbb{P}^\infty)$ which is inverse to the logarithm of the FGL $F_{K(n)}$. As $\log_{K(n)}(x)$ is p^n-gradable, the same is true for its inverse. The formula 5 now follows and the claim about support is proved. □

Definition 3.30. Operation $\phi : \tilde{K}(n)^* \to CH^i \otimes B$ is called gradable, if for any $l \geq 1$ symmetric polynomial $G_l(t_1,\ldots,t_l)$ is p^n-gradable in each variable t_j (i.e. admits only monomials where every variable has its power equal to 1 modulo $p^n - 1$).

We will show later that there are many gradable operations from any n-th Morava K-theories. For example, all additive operations to $CH^* \otimes B$ when B is torsion-free are gradable.

The goal of what now follows is to show that for a $\mathbb{F}_p$-algebra B all gradable additive operations to $CH^* \otimes B$ are actually generated by reductions of additive operations to $CH^* \otimes \mathbb{Z}_{(p)}$. More precisely, we will prove that the system of equation (A_l) over $\mathbb{F}_p$ is upper-triangular of corank 1 for gradable operations. This will allow us to prove Lemmas neccessary for the construction of Chern classes to $CH^* \otimes \mathbb{Z}_{(p)}$.

By the definition of a gradable operation variables $\alpha^{(i)}_{\vec{r}}$ corresponding to it have to be zero whevener there exist $r_i \in \vec{r}$ s.t. $r_i \neq 1 \mod (p^n - 1)$. We now investigate whenever variables $\alpha^{(l)}_{\vec{r}}$ corresponding to p-special partitions can be non-zero for gradable operations as precisely these variables are in charge of the possible non-liftable additive operations.

Proposition 3.31. *Number p^r is equal to 1 modulo $p^n - 1$ if and only if r is divisible by n.*

Therefore variables $\alpha^{(l)}_{\vec{r}}$ corresponding to p-special partition $\vec{r}$ are zero for gradable operation whenever there exists r_i s.t. p^n does not divide it.

Proof. From the equality $r = 1 + v(p^n - 1) = p^m$ we get that $p^n - 1 | p^m - 1$ and hence $n|m$ ($p^m \equiv p^{m \mod n} \mod (p^n-1)$ and thus $m \equiv 0 \mod n$). By definition of gradable operations the claim follows. $\square$

We will call partitions $(p^{ns_1}, \ldots, p^{ns_l})$ – p^n-special.

Proposition 3.32. *Let $\{\alpha^{(i)}_{(r_1,\ldots,r_i)}\}$ satisfy the system of equations (A_l) and let $\alpha^{(i)}_{\vec{r}} = 0$ whenever there exists $r_i \neq 1 \mod (p^n-1)$. In other words, $\{\alpha^{(i)}_{\vec{r}}\}$ correspond to a gradable additive operation.*

Let $\vec{r} := (r_1, \ldots, r_l)$ be a p^n-special partition s.t. there are at least p^n equal numbers among r_i, $1 \le i \le l$.

Then using equations A_{l-p^n+1} and A_l one may express $\alpha^{(l)}_{\vec{r}}$ of in terms of $\alpha^{(k)}_{\vec{s}}$ for $k > l$.

Proof. As there are at least p^n equal numbers in the partition $\vec{r}$, it follows that $l > p^n - 1$. Denote by $r_{l-p^n+1} = r_{l-p^n+2} = \ldots = r_l = p^{sn}$ for some $s \ge 0$.

Recall that modulo p the formal group law of n-th Morava K-theory looks like this $x+y-\frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}x^{p^{n-1}i}y^{p^{n-1}(p-i)}+$ higher degree terms, i.e. $a_{t,p^n-t} \neq 0 \mod p$ if and only if $t = ip^{n-1}$, $0 < i < p$.

Look at the coefficient of monomial $t_1^{r_1}\cdots t_{l-p^n}^{r_{l-p^n}} u^{p^{sn+n-1}} v^{p^{sn+n-1}(p-1)}$ in the equation A_{l-p^n+1}. Variable $\alpha^{(l)}_{(r_1,\ldots,r_l)}$ appears non-trivially as a coefficient of this monomial coming from term

$$a_{p^{n-1},(p-1)p^{n-1}} G_l(t_1, \ldots, t_{l-p^n}, u^{\times p^{n-1}}, v^{\times (p-1)p^{n-1}}).$$

Here we have used the assumption that there are at least p^n equal numbers in $(r_1, \ldots, r_l)$. Otherwise we would get some multiple of $\alpha^{(l)}_{(r_1,\ldots,r_l)}$, because $G_l(t_1, \ldots, t_l)$ is symmetric and monomials may 'glue' in $G_l(t_1, \ldots, t_{l-p^n}, u^{\times p^{n-1}}, v^{\times(p-1)p^{n-1}})$.

The first claim is that variables $\alpha^{(k)}_{\vec{m}}$ with $k < l$ do not appear in the coefficient of this monomial. In equation A_{l-p^n+1} these variables may appear only for $k = l - p^n + 1$, however they do not as monomials in question are additive (Prop. 3.28).

The second claim is that variables $\alpha^{(l)}_{(m_1,\ldots,m_l)}$ which appear in the coefficient of the monomial in question are only those which come from non-additive monomials, i.e. p does not divide m_i for some $1 \le i \le l$. These can be expressed in terms of variables $\alpha^{(k)}_{\vec{s}}$ with $k > l$ via equation A_l by Prop. 3.31 and so the proof will be finished.

Indeed, the only variables coming from additive gradable monomials are $\alpha^{(l)}_{(p^{nk_1},\ldots,p^{nk_l})}$. They may come in equation A_{l-p^n+1} from $a_{ip^{n-1},(p-i)p^{n-1}} G_l(t_1, \ldots, t_{l-p^n}, u^{\times ip^{n-1}}, v^{\times(p-i)p^{n-1}})$ for some $i : 1 \le i \le p-1$. To appear as a coefficient of monomial $t_1^{r_1}\cdots t_{l-p^n}^{r_{l-p^n}} u^{p^{sn+n-1}} v^{p^{sn+n-1}(p-1)}$ we need $p^{nk_i} = r_i$ for $i < l - p^n$, and moreover $\sum_{i=l-p^n+1}^{l} p^{nk_i} = p^n p^{ns} = p^{n(s+1)}$.

It is easy to see that the latter equation has a unique solution with $k_{l-p^n+1} = \ldots = k_l = s$. This corresponds to the coefficient we were interested in, so no other variable $\alpha^{(l)}_{(p^{nk_1},\ldots,p^{nk_l})}$ appears in the coefficient of the monomial in question. $\square$

Proposition 3.33. *Let B be an $\mathbb{F}_p$-algebra, then the B-module of gradable additive operations from $\tilde{K}(n)^*$ to $CH^m \otimes B$ is a free module of rank 1 generated by a reduction of an operation from $\tilde{K}(n)^*$ to $CH^m \otimes \mathbb{Z}_{(p)}$.*

Proof. Clearly, it is enough to prove that the system of equations (A_l) modulo p is upper-triangular and of corank 1 for gradable operations. To show this we express all variables $\alpha^{(l)}_{(r_1,\ldots,r_l)}$ as multiples of one of them.

We say that the variable α is expressible in terms of some variables β_s if there exist a consequence of the system of equations (A_l) which looks like this: $\alpha = \sum b_s\beta_s$, where $b_s \in \mathbb{F}_p$. From Prop. 3.31 and 3.32 we know that the variable $\alpha^{(l)}_{(r_1,\ldots,r_l)}$ is expressible in terms of variables with a bigger superscript whenever there exists $r_i \neq p^{ns_i}$ for any s_i or if there are at least p^n equal numbers among $(r_1, \ldots, r_l)$. Consider now other variables $\alpha^{(l)}_{(r_1,\ldots,r_l)}$ corresponding to p^n-special partitions which do not satisfy this condition. Denote by m_i the number of p^{in} among $(r_1, \ldots, r_l)$, where $i \ge 0$. We know that $0 \le m_i < p^n$ for any i, and we know that $j = m_0 + m_1p^n + m_2p^{2n} + \ldots + m_sp^{sn}$ because it is partition of m. Thus m_i are uniquely defined by m as they are the digits of m in p^n-ary digit system. This means that the 'exceptional' variable which can not be expressed in terms of variables with higher superscript by propositions 3.31, 3.32 is unique. (Note that we do not claim that it could not be expressible in this way by some other argument.)

If $m < p^n$, then the space of gradable additive operations is 1-dimensional as it has to be determined by the only coefficient $\alpha^{(m)}_{(1,1,\ldots,1)}$ which is the 'exceptional' one.

If $m \geq p^n$, then the variable $\alpha^{j}_{(1,1,\ldots,1)}$ equals 0 due to the Prop. 3.32. Any variable except for the exceptional one can be expressed in terms of the exceptional one and $\alpha^{j}_{(1,1,\ldots,1)}$. The Proposition is thus proved. $\square$

Proposition 3.34. *(1) The sum of gradable operations is gradable;*
(2) product of N gradable operations is gradable if $N \equiv 1 \mod p^n - 1$;
(3) an operation $\phi : K(n)^* \to CH^i \otimes \mathbb{Z}_{(p)}$ is gradable iff $\phi \otimes id_{\mathbb{Q}} : K(n) \to CH^i \otimes \mathbb{Q}$ is gradable.

Proposition 3.35. *All additive operations from $K(n)^*$ to $CH^* \otimes \mathbb{Z}_{(p)}$ are gradable.*

Proof. Only a finite number of (non-zero) polynomials needs to be specified in the COAT to define an operation to $CH^j \otimes \mathbb{Z}_{(p)}$ for each particular j. Therefore one can find N such that $p^N ch_j$ has these polynomials integral, and thus defines an integral operation. By Prop. 3.29 any additive operation is thus rationally proportinal to ch_j, and it is enough to prove that all components of the Chern character are gradable.

Since the Chern character is multiplicative, we have $G_l(t_1, t_2, \ldots, t_l) = \prod_i G_1(t_i)$. It is neccessary and sufficient for ch to be gradable that $\gamma(t) := G_1(t)$ admits only monomials with t of the power equal to 1 modulo $p^n - 1$.

Series γ defines a morphism from the FGL $F_{K(n)}$ to the additive FGL. Thus, by definition of the logarithm, γ is the composition inverse of $\log_{K(n)}$. One may consider the inverse to the homogenous series $x + v_n \frac{x^{p^n}}{p} + \ldots$ over $\mathbb{Z}_{(p)}[v_n]$, where $\deg v_n = 1 - p^n$ and $\deg x = 1$. Its inverse is homogenous as well. However, as $\log_{K(n)}$ can be obtained from this series by setting $v_n = 1$, γ can be obtained from its inverse by the same procedure. Therefore, it's 'gradable', the operation ch is gradable and the proposition is proved. $\square$

Remark 3.36. One can show that not all operations $K(n)^* \to CH^* \otimes \mathbb{Z}_{(p)}$ are gradable, e.g. ϕ_1^2 is not.

For $n > 1$ not all additive operations $K(n)^* \to CH^*/p$ are gradable as well, the easiest example being ϕ_1^p.

Proposition 3.37. *Let $i \geq p^n$. Assume that operations $c_1, \ldots, c_{i-1} : K(n)^* \to CH^* \otimes \mathbb{Z}_{(p)}$, satisfying i), ii) and iiibis) of Theorem 3.13 exist.*

Then these operations are gradable.

Denote by $P_j = -(\log_{K(n)} c_{tot})_j + c_j$, $j < i$, a rational polynomial in Chern classes $c_1, \ldots, c_{j-1}$.

Then $aP_j + b\phi_j$ defines a gradable operation for any $a, b \in \mathbb{Q}$.

Proof. Operation defined by the polynomial P_j is gradable as follows from the gradability of $c_1, \ldots, c_{j-1}$ and Prop. 3.34, (2). Indeed, any monomial of $\log_{K(n)} c_{tot}$ is a product of p^{kn} Chern classes for $k \geq 0$ and $p^{kn} \equiv 1 \mod p^n - 1$.

Base of induction. Operations $c_1, c_2, \ldots, c_{p^n-1}$ are additive and are proved to be gradable in Prop. 3.35.

Induction step. Recall from Section 3.5 that rationally c_j is equal to a sum of a multiple of P_j and a rational additive operation ψ_j. Operation P_j is gradable by induction and ψ_j is gradable as additive oepration. By Prop. 3.34 the induction step is proved.

Operation $aP_j + b\phi_j$ is a sum of gradable operations and, thus, gradable. $\square$

Proposition 3.38. *Lemma 3.23 and Corollary 3.21 are unconditionally true.*

Proof. To prove Lemma 3.23 note that due to Prop. 3.37 expression $p^\mu P_i + a\phi_i$ defines a gradable operation. It was shown in Section 3.5 that if it is integral then it is additive modulo p. Therefore it is propotional to a generator ϕ_i of additive operations to $CH^i \otimes \mathbb{Z}_{(p)}$ modulo p.

Cor. 3.21 follows as $(\phi_v)^{p^{kn}}$ is a gradable additive operation by Prop. 3.34 and thus it is proportional to $\phi_{vp^{kn}}$. $\square$

One can show after the proof of iii) of Theorem 3.13 (and in the same manner as the proof in Section 3.7) that the only liftable additive operations $K(n) \to CH^*/p$ are gradable operations and their p-primary powers (however p-primary powers do not lift in general as additive operations). From this it is easy to show that there are many additive operations to CH^*/p which are not liftable. Let us produce now a construction of such operations with a particular example which is proved to be non-liftable directly.

It is well-known (cf. [9, Th. 6.6]) that to any p-partition $(p^{s_1}, \ldots, p^{s_i})$ of j corresponds an additive operation $CH^i/p \to CH^j/p$ which sends the product of i first Chern classes $t_1 \cdots t_i$ to the symmetrization of the monomial $t_1^{p^{s_1}} \cdots t_i^{p^{s_i}}$ (as in the notations of Section 1.5).

It is clear that the composition of such an operation with a gradable operation is not always gradable. We expect however that all additive operations from $K(n)^*$ to CH^*/p are generated by the action of the Steenrod algebra on gradable operations though we do not prove it here.

For example, let $n > 1$, and consider an additive operation $Q : CH^2/p \to CH^{p+1}/p$ which sends t_1t_2 to $t_1^pt_2 + t_1t_2^p$. The composition $Q \circ c_2$ is a non-zero additive operation, which is supported on $K(n)^2$. Assume that $p \neq 2$. From iii) of Theorem 3.13 it follows that there is the only (up to a scalar) integral operation from $K(n)^2$ to $CH^{p+1} \otimes \mathbb{Z}_{(p)}$ which is $c_2^{\frac{p+1}{2}}$. One easily checks that this is not additive modulo p therefore not proportional to $Q \circ c_2$, and therefore $Q \circ c_2$ is not liftable.

3.7. **Chern classes freely generate all operations.** In this section we prove the statement iii) of Theorem 3.13.

In Sections 3.5, 3.6 we have constructed operations $c_i : K(n)^* \to CH^i \otimes \mathbb{Z}_{(p)}$, $i \geq 1$, satisfying properties i), ii) of Theorem 3.13 and iiibis) of Section 3.5. The latter says that generators of additive integral operations ϕ_i can be expressed as an integral polynomial in $c_1, \ldots, c_i$. We now prove that Chern classes c_i generate freely the ring of integral operations to Chow groups. In fact, we get a more general statement that external products of Chern classes freely generate rings of poly-operations (Prop. 3.48).

The proof is independent of the construction of Chern classes and uses only the notion of a gradable operation (Def. 3.34) and the fact that integral additive operations are gradable (Prop. 3.35).

The starting point for the proof of iii) is the natural inclusion $[\tilde{K}(n)^*, CH^* \otimes \mathbb{Z}_{(p)}] \subset [\tilde{K}(n)^*, CH^* \otimes \mathbb{Q}]$ provided by the COT. We will show that the latter space is freely generated by components of the Chern character reducing the problem to the Theorem 2.1, and then it is easy to show that it is freely generated by Chern classes (Prop. 3.44 below). Thus if we have an integral operation it can be uniquely expressed as a rational series in Chern classes. The problem is then to prove that this is in fact an integral series. To do this we study derivatives of this operation and, roughly speaking, reduce everything to the case of additive poly-operations (Lemma 3.47).

We start, however, with several general observations.

3.7.1. *Adams operations are universally central.* The following proposition was communicated to the author by A. Vishik.

Proposition 3.39 (Vishik, cf. [9, Th. 6.15])**.** *Let A^* be a theory of rational type. Then there exist multiplicative Adams operations $\psi_i^A : A^* \to A^*$ for $i \in \mathbb{Z}$, which are uniquely defined by the property $\psi_i^A(c_1^A(L)) = c_1^A(L^{\otimes i})$, where L is a line bundle over some smooth variety.*

Moreover, Adams operations do not depend on the orientatation.

Proof. The existence of Adams operations was proved in [9] and uniqueness follows from the COT.

Using the transformation of the first Chern class under a change of an orientation, one checks the last claim of the proposition. □

Proposition 3.40. *Let $\phi : A^* \to B^*$ be an operation between two theories of rational type.*

Then ϕ commutes with Adams operations, i.e. $\phi \circ \psi_k^A = \psi_k^B \circ \phi$.

Proof. According to the COT we may check the equality $\phi \circ \psi_k^A = \psi_k^B \circ \phi$ on products of projective spaces. However, the action of Adams operations ψ_k on products of projective spaces $(\mathbb{P}^\infty)^r$ is nothing more than the pull-back along the composition of Segre maps with diagonals (i.e. products of k-Veronese maps) and, thus, commutes with any operation. □

3.7.2. *Localisation of non-additive operations.*

Proposition 3.41. *Let S be a subset in $\mathbb{Z} \setminus \{0\}$, denote by $\mathbb{Z}_S := S^{-1}\mathbb{Z}$ the localisation of integers in S.*

Let A^* be a theory of rational type s.t. the map $A \to A \otimes \mathbb{Z}[S^{-1}]$ is injective, and let B^* be any g.o.c.t such that S is invertible in B.

Then the natural map $[\tilde{A}^* \otimes \mathbb{Z}_S, B^*] \to [\tilde{A}^*, B^*]$ is an isomorphism.

This Proposition is obviously true for additive operations, though to deal with non-additive ones we use the COT.

Proof. From the COT it follows that any operation from A^* to B^* factors through a theory of rational type $\Omega^* \otimes_{\mathbb{L}} B$. Thus, without loss of generality we may assume that B^* is of rational type as well.

Let ϕ be any operation from $\tilde{A}^*$ to B^*. We need to show that there exist unique $\bar{\phi} : \tilde{A}^* \otimes \mathbb{Z}_S \to B^*$, s.t. its composition with the natural map $\tilde{A}^* \to \tilde{A}^* \otimes \mathbb{Z}_S$ is equal to ϕ.

Denote by $A_{r,n} := \tilde{A}^*((\mathbb{P}^n)^{\times r})$ a factor ring of $A[[z_1, \ldots, z_r]]$ by the ideal of power series of degree $\geq (n+1)$. By the COT and continuity of operations $\bar{\phi}$ is determined by its restriction to maps of sets from $A_{r,n} \otimes \mathbb{Z}_S$ to $B_{r,n}$ for all r, n.

We claim that for any $P \in A_{r,n} \otimes \mathbb{Z}_S$ there exists $M \in \mathbb{Z}_S^\times$, s.t. $\psi^A_M(P) \in A_{r,n}$. Recall that operation ψ^A_M is multiplicative and $\psi^A_M(z) = Mz + Q(z)$ where $Q \in Az^2[[z]]$. Write $P = P_{<k} + P_k + P_{>k}$ where summands are of degree less than k, exactly k and bigger than k, respectively. Assume that summands of degree less than k have coefficients from A, i.e. $P_{<k} \in A_{r,n}$. Let d be the common denominator of coefficients of P_k. Apply ψ^A_d to P. Polynomial $\psi^A_d(P_{<k})$ still has coefficients in A. Polynomial $\psi^A_d(P_{>k})$ still has degree bigger than k. At the same time polynomial $\psi^A_d(P_k)$ has its coefficients in degree k multiplied by d^k and thus lying in A. Therefore $\psi^A_d(P)$ has its summands of degree less than $k+1$ lying in A. Iterating this procedure and using the fact that $\psi^A_m \circ \psi^A_n = \psi^A_{mn}$, the claim is proved.

By Prop. 3.40 Adams operations commute with any operation, $\bar{\phi}(\psi^A_M(P)) = \psi^B_M \circ \bar{\phi}(P)$. On the other hand ψ^B_M is an invertible operation when M is invertible in B. Therefore this equality allows to express $\bar{\phi}(P)$ in terms of ϕ uniquely, which proves the unicity of $\bar{\phi}$.

One can define $\bar{\phi}$ by the procedure above. It is enough to show that maps $\bar{\phi} : A_{r,n} \to B_{r,n}$ commute with the restriction maps of the list of morphisms in COT (Segre, diagonals, etc.). However, this follows quite formally as ϕ and Adams operations commute with pull-backs along all morphisms between projective spaces. □

Proposition 3.42. *Let K be a theory of rational type with the ring of coefficients being a subring in $\mathbb{Q}$.*

Then any r-ary poly-operation from $\tilde{K}$ to $CH^* \otimes \mathbb{Q}$ can be uniquely written as a series in external products of monomials in $\{ch_i\}_{i\geq 1}$, where $ch : K \to CH^* \otimes \mathbb{Q}$ is the unique stable multiplicative operation (aka Chern character).

In other words, $\left[\tilde{K}^{\times r}, CH^*_{\mathbb{Q}} \circ \prod^r\right] = \mathbb{Q}[[ch_1, \ldots, ch_i, \ldots]]^{\odot r}$.

Proof. By Prop. 3.41 the natural map $[\tilde{K} \otimes \mathbb{Q}, CH^* \otimes \mathbb{Q}] \to [\tilde{K}, CH^* \otimes \mathbb{Q}]$ is an isomorphism. As any two formal group laws over $\mathbb{Q}$-algebras are isomorphic, there exists the unique stable multiplicative isomoprhism $\tilde{K}_0 \otimes \mathbb{Q} \cong \tilde{K} \otimes \mathbb{Q}$. Obviously, it maps components of the Chern character $K \otimes \mathbb{Q} \to CH^* \otimes \mathbb{Q}$ to components of the Chern character $K_0 \otimes \mathbb{Q} \to CH^* \otimes \mathbb{Q}$. Thus, it is enough to prove the statement for $K = K_0$.

Note that there is a standard equality $(\log(1 + c_{tot}))_n = (n-1)!ch_n$ between operations from K_0 to $CH^n \otimes \mathbb{Q}$, where $\log(1+x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$ and $c_{tot} = c_1 + c_2 + c_3 + \ldots$ is the total Chern class. Using it one can uniquely express Chern classes as polynomials in the components of the Chern character ch_n and vice versa. The claim now follows from Theorem 2.1. □

Corollary 3.43. *The dimension of the space of operations $[K, CH^i \otimes \mathbb{Q}]$ equals to $p(i)$, where $p(i)$ is the number of partitions of i.*

The dimension of the space of r-ary poly-operations from K to $CH^i \otimes \mathbb{Q}$ does not depend on K (and can be interpreted as a number of some partition-type objects).

3.7.3. *Proof that Chern classes freely generate all operations.*

Proposition 3.44 ([7, Prop. 4.28])**.** *Chern classes are free generators of the $\mathbb{Q}$-algebra of poly-operations from $\tilde{K}(n)^*$ to $CH^* \otimes \mathbb{Q}$.*

Proof. From Prop. 3.42 one easily sees that it is enough to show that components of Chern character can be expressed as polynomials in Chern classes. Indeed, by iiibis) a generator of integral additive operations ϕ_i is equal to an integral polynomial in Chern classes. By Prop. 3.29 rationally ϕ_i is a multiple of ch_i and thus ch_i is a rational polynomial in $c_1, \ldots, c_i$. Therefore Chern classes generate all poly-operations to $CH^* \otimes \mathbb{Q}$.

Suppose that there is a rational relation of degree i (i.e. as a poly-operation to $CH^i \otimes \mathbb{Q}$) between external products of monomials of Chern classes. Then the dimension of the space of poly-operations from $K(n)^*$ to $CH^i \otimes \mathbb{Q}$ would be less than the dimension of poly-operations from K_0 to $CH^i \otimes \mathbb{Q}$ which contradicts Cor. 3.43. Indeed, in both cases the dimension can be calculated as the number of external products of monomials in variables $1, c_1, \ldots, c_i$ which have degree i. □

Recall from Section 1 that we denote by $\odot$ the external product of operations. Proposition 3.44 thus can be written as $[\tilde{K}(n)^{*\times r}, CH^* \otimes \mathbb{Q}] = \mathbb{Q}[[c_1 \ldots, c_n, \ldots]]^{\odot r}$.

We call ψ_i the i-th component of poly-operation $\psi_1 \odot \psi_2 \odot \cdots \odot \psi_r$.

Proposition 3.45. *Let A^* be a theory of rational type, let B^* be a g.o.c.t. and let $\{\phi_s\}_{s\in S}$ be a set of B-linearly independent additive operations from A^* to B^*.*

Then external products of operations $\{\phi_{s_1} \odot \phi_{s_2} \odot \cdots\}_{s_1,s_2,\ldots\in S}$ are B-linearly independent.

Proof. We prove the statement by induction on arity of poly-operations.

Base of induction is the assumption of the Proposition.

Induction step. Suppose we know the statement of the Proposition for i-ary poly-operations, where $i < r$. Consider a non-trivial linear combination $T := \sum \beta_{(s_1,s_2,\ldots,s_r)}\phi_{s_1} \odot \phi_{s_2} \odot \cdots \odot \phi_{s_r}$. Choose a monomial with a non-zero coefficient in this linear combination $\beta\phi_{s_1} \odot \phi_{s_2} \odot \cdots \odot \phi_{s_r}$, and collect all the terms which differ from it only in the first component. Denote it by $R \odot \phi_{s_2} \odot \cdots \odot \phi_{s_r} := (\sum_l \beta_{l,s_2,\ldots,s_r}\phi_l) \odot \phi_{s_2} \odot \cdots \odot \phi_{s_r}$.

By assumption of the Proposition and from the COT it follows that there exist x in $A^*((\mathbb{P}^\infty)^{\times k})$ for some $k \geq 0$, s.t. $R(x) \neq 0$. Restrict poly-operation T in the first component to this element. Thus, we get a natural transformation $T_x := T(x,-)$ from the functor $\times_{i=1}^{k-1} A^*$ to the functor $B^* \circ ((\mathbb{P}^\infty)^{\times k} \times \prod)$.

Note that $B^*(X \times (\mathbb{P}^\infty)^{\times k}) \cong B^*(X) \otimes_B B[[z_1,\ldots,z_k]]$ for any X by the projective bundle theorem. Choose any B-linear projection $p : B^*(\mathbb{P}^\infty)^{\times k} \cong B[[z_1,\ldots,z_k]] \to B$ s.t. $p(R(x)) \neq 0$ and compose it with T_x to get an $(k-1)$-ary poly-operation $(id \otimes p) \circ T_x$. It can be expressed as a sum of $p(R(x))\phi_{s_2} \odot \cdots \odot \phi_{s_r}$ and a linear combination of external products of ϕ_i which does not contain this summand. By induction assumption this is a non-trivial poly-operation, and therefore T is non-trivial as well. □

Corollary 3.46. *External products of additive operations $\{\phi_i^{p^s}\}_{\{p^n\nmid i\}}$ are linearly independent over $\mathbb{F}_p$ as poly-operations to CH^*/p.*

Note that by Cor. 3.21 (unconditionally proved in Cor. 3.33) operation $\phi_j^{p^n}$ is proportional to ϕ_{jp^n} modulo p for any $j \geq 1$, which explains a somewhat strange set of additive operations in the statement.

Proof. By Prop. 3.45 it is enough to show that these additive operations are linearly independent.

Suppose we have a linear combination $\sum_i \alpha_i\phi_i^{p^{r_i}}$, which is zero modulo p as an operation. Let $r = \min_i r_i$. If $r > 0$ then consider p^r-th root of this expression. If it were not a trivial operation, then from the COT it would follow that its p-primary powers are non-trivial (as there are no p-nilpotents in the coefficient ring of a target theory). Thus, we obtain a relation in which there is a unique summand $\alpha_i\phi_i$, $\alpha_i \neq 0$. It is a gradable operation by Prop.3.29.

Since i is not divisible by p^n, then by degree reasons all other summands look like $\alpha_j\phi_j^{p^{r_j}}$ where $n \nmid r_j$. It follows from the fact that ϕ_j are gradable that this operations are non-gradable and so is their sum. Therefore such relation cannot exist if $\alpha_i \neq 0$. □

To prove the next Lemma we will use derivatives of poly-operations (Section 1), and mainly derivatives of Chern classes (Section 3.5). Recall that by Cartan's formula $\partial^1 c_n$ is expressible as an integral polynomial in Chern classes $c_1,\ldots,c_{n-1}$.

Lemma 3.47. *Let $\phi \in \mathbb{Q}[[c_1,c_2,\ldots]]^{\odot N}$ be any N-ary poly-operation from $\tilde{K}(n)^*$ to $CH^* \otimes \mathbb{Z}_{(p)}$. Assume that all its first derivatives can be expressed as a $\mathbb{Z}_{(p)}$-series in Chern classes.*

Then $\phi \in \mathbb{Z}_{(p)}[[c_1,c_2,\ldots]]^{\odot N}$.

Proof. We will prove the Lemma by contradiction. First, it is enough to consider components of ϕ to each graded component $CH^j \otimes \mathbb{Z}_{(p)}$ separately. Thus, we may assume that ϕ is in fact a polynomial and not a series.

Second, if a counter-example to the statement of Lemma existed, then there would be a counter-example ϕ s.t. denominators of ϕ are at most p. Otherwise one can multiply ϕ by p to get a counter-example which denominators are smaller. So we assume that $p\phi \in \mathbb{Z}_{(p)}[c_1,\ldots,c_n]^{\odot N}$ for some $n \geq 1$.

Third, we may assume that all coefficients of ϕ are not p-integral, as we can subtract integral part without breaking a counter-example.

We will now continue to reduce a counter-example in order to finally get $\sum_{\mathbf{i}=(i_1,i_2,\ldots,i_N);(r_1,\ldots,r_N)} \frac{\beta_{\mathbf{i}}}{p}\phi_{i_1}^{p^{r_1}} \odot \phi_{i_2}^{p^{r_2}} \odot \cdots \odot \phi_{i_N}^{p^{r_N}}$, where $\beta_{\mathbf{i}} \in \mathbb{Z}_{(p)}^\times$ and $p^n \nmid i_l$ for all $1 \leq l \leq N$. This would contradict Cor. 3.46.

Denote by i the highest index of the Chern class c_i appearing in the first component of ϕ. One can write down ϕ as a sum $\sum_{\mathbf{s}=(s_i,s_{i-1},\ldots,s_1)} a_{\mathbf{s}}c_i^{s_i}c_{i-1}^{s_{i-1}}\cdots c_1^{s_1} \odot P_{\mathbf{s}}$, where $P_{\mathbf{s}} \in \mathbb{Z}_{(p)}[c_1,\ldots,c_n]^{\odot(N-1)}$ is an integral polynomial in external products of Chern classes, s.t. $P_{\mathbf{s}}$ is not zero modulo p.

Note that if $a_{\mathbf{s}}$ is not zero and thus not integral by our assumptions, then derivatives $\partial_j P_{\mathbf{s}}$ for any $j : 1 \leq j \leq N-1$ are divisible by p as polynomials in Chern classes. Indeed, the derivative in the $(j+1)$-th component of ϕ contains $a_{\mathbf{s}}c_i^{s_i}c_{i-1}^{s_{i-1}}\cdots c_1^{s_1} \odot \partial_j P_{\mathbf{s}}$ which is integral iff $a_{\mathbf{s}}\partial_j P_{\mathbf{s}}$ is integral.

Define a (lexico-graphical) order on i-tuples of non-negative numbers as follows: $(s_i, s_{i-1}, \dots, s_1) \succ (r_i, r_{i-1}, \dots, r_1)$ if and only if there exist $k : 1 \le k \le i$ s.t. $s_j = r_j$ for $i \ge j \ge k$ and $s_{k-1} > r_{k-1}$. For example, $(1, 0, \dots, 0) \succ (0, 2, \dots, 0)$.

Let $\mathbf{r} = (r_i, r_{i-1}, \dots, r_1)$ be the $\succ$-highest index of non-zero coefficients $a_{\mathbf{s}}$ of ϕ. Without loss of generality we may assume that $a_{\mathbf{r}} = \frac{1}{p}$ and by the choice of i we have $r_i \neq 0$. Denote $P := P_{\mathbf{r}}$.

We will show that ϕ having an integral derivative in the first component implies strong restrictions on the tuple $\mathbf{r}$.

Step 1. Tuple $\mathbf{r}$ has to be of the form $(p^m, 0, 0, \dots, 0)$ for some $m \ge 0$.

Let us recall how to differentiate monomials in Chern classes:

$$(\partial c_i^{r_i} \cdots c_1^{r_1})(x, y) = (c_i(x) + c_i(y) + \partial c_i(x, y))^{r_i} \cdots (c_1(x) + c_1(y))^{r_1} - \prod_{l=1}^{i} (c_l(x))^{r_l} - \prod_{l=1}^{i} (c_l(y))^{r_l}.$$

In particular, if there exist $r_j \neq 0$, $j < i$, then the expression on the right contains $c_i(x)^{r_i} \prod_{l=1}^{i-1} c_l(y)^{r_l}$ which does not cancel.

Thus, if there exist $r_j \neq 0$, $j < i$, then in the derivative $\partial_1 \phi$ we get a summand $\frac{a_1}{p} c_i^{r_i} \odot \prod_{l=1}^{i-1} c_l^{r_l} \odot P$, which can not be cancelled by derivatives of other summands in ϕ due to the highest $\succ$-order of tuple $\mathbf{r}$.

If r_i is not a power of p then there exist $k : 1 \le k \le r_i - 1$, s.t. $\binom{r_i}{k}$ is not divisible by p. Thus $\partial^1(\frac{1}{p} c_i^{r_i} \odot P)$ contains non-integral $\binom{r_i}{k} \frac{1}{p} c_i^{r_i - k} \odot c_i^k \odot P$ which can not be cancelled by derivatives of other monomials in ϕ, because of the highest order of $(r_i, 0, \dots, 0)$.

We have shown that the highest $\succ$-order term is proportional to $\frac{1}{p} c_i^{p^m} \odot P$.

Step 2. Number i has to be non-divisible by p^n.

Assume the contrary and denote by $v = \frac{i}{p^n}$. Look at the derivative of c_i as given by the Cartan's formula. It contains, in particular, the expression $-\frac{a_1}{p} \sum_{j=1}^{p^n - 1} \binom{p^n}{j} c_v^j \odot c_v^{p^n - j}$. Note that it has a summand which is not zero modulo p (e.g. $j = p^{n-1}$). Therefore $\partial_1(\frac{1}{p} c_i^{p^m} \odot P)$ contains non-integral summand $\binom{p^n}{p^{n-1}} \frac{a_1}{p^2} c_v^{p^{m+n-1}} \odot c_v^{(p-1)p^{m+n-1}} \odot P$, which we will call '*bad*'.

We claim that in order for 'bad' monomial to be cancelled in the derivative of ϕ there has to be monomial $-\frac{1}{p^2} \binom{p^n}{p^{n-1}} \binom{p^{m+n}}{p^{m+n-1}}^{-1} c_v^{p^{m+n}} \odot P$ in ϕ. This would contradict our assumption that demoninators are at most p in ϕ.

The only Chern classes c_j, $j \le i$ s.t. $\partial^1 c_j$ contains $b_{(e,g)} c_v^e \odot c_v^g$ for some $b_{(e,g)} \neq 0, e, g$ are c_i and c_v. To see this recall that formal group law $F_{K(n)}$ has only monomials $x^\alpha y^\beta$ with $\alpha + \beta \equiv 1 \mod (p^n - 1)$ (Prop. 3.12, (1)). Thus, by Cartan's formula if $\partial^1 c_j$ contains external product $b_{(e,g)} c_v^e \odot c_v^g$, then $e + g \equiv 1 \mod (p^n - 1)$, though $v(e + g)$ has to be less or equal to $i = vp^n$.

Therefore 'bad' monomial in $\partial^1 \phi$ may be cancelled only by derivatives of monomials $b_{(e,g)} c_i^e c_v^g \odot P$ for some $e \ge 0$ and $g > 0$. Fix maximal $e > 0$ s.t. $b_{(e,g)} \neq 0$, it is not integral by our assumptions. The derivative $\partial_1(b_{(e,g)} c_i^e c_v^g \odot P)$ contains non-integral monomial $b_{(e,g)} c_i^e \odot c_v^g \odot P$, which can be cancelled only by the derivative of $b_{(e',g')} c_i^{e'} c_v^{g'} \odot P$ for some $e' > e, g' \ge 0$ and $b_{(e',g')} \in \frac{1}{p}\mathbb{Z}_{(p)}$. By maximality of e it has to be cancelled by $\frac{1}{p} c_i^{p^m} \odot P$, but its derivative does not contain $b c_i^e \odot c_v^g \odot P$ for any $e > 0$, $g > 0$ and any $b \in \mathbb{Q}$.

Thus, $e = 0$ and 'bad' momonial has to be cancelled by the derivative of $b c_v^{p^{n+m}} \odot P$. As $\partial(c_v^{p^{n+m}})$ contains $c_v^{p^{m+n-1}} \odot c_v^{(p-1)p^{m+n-1}}$ with coefficient $\binom{p^{m+n}}{p^{m+n-1}}$, b has to be equal to $-\frac{a_1}{p^2} \binom{p^n}{p^{n-1}} \binom{p^{m+n}}{p^{m+n-1}}^{-1}$ which has p-valuation equal to -2. We do not allow so big denominators in our counter-example ϕ and the claim of Step 2 is proved.

Step 3. Reduce the non-integral monomial with the highest lexico-graphical order.

If i is non-divisible by p^n, then in the notaion of Lemmas 3.19 and 3.22 $\mu_i = 0$. Thus, a generator ϕ_i of additive operations to $CH^i \otimes \mathbb{Z}_{(p)}$ is expressible as integral polynomial $c_i - P_i \in \mathbb{Z}_{(p)}[c_1, \dots, c_i]$ with a summand c_i. Thus, the derivative of $\psi := \phi - \frac{\phi_i^{p^r}}{p} \odot P$ in the first component is still integral, because $\partial^1 \phi_i = 0$, and $\partial^1 (\phi_i)^{p^m} = \sum_{k=1}^{p^r - 1} \binom{p^r}{k} \phi_i^k \odot \phi_i^{p^r - k}$, which is divisible by p. Derivatives of ψ in other components are integral because derivatives of P are p-divisible as explaine in the beginning of the proof.

Note that the highest $\succ$-order of $\mathbf{r} = (r_i, \dots, r_1)$ of non-zero coefficients $a_{\mathbf{r}}$ in ψ is smaller than for ϕ.

Step 4. Reduce the counter-example to the form $\sum_s \frac{\alpha_s}{p} \phi_{i_s}^{p^{r_s}} \odot P_s$.

If ψ is not an integral polynomial we apply Steps 1-3 reducing the highest $\succ$-order of non-integral monomials 'in the first component'. Note that we have not used the assumption that the poly-operation is integral anywhere above. In the end we have $\phi - \sum_s \frac{\alpha_s}{p}\phi_{i_s}^{p^{r_s}} \odot P_s$, which is an integral polynomial in Chern classes and $\sum_s \frac{\alpha_s}{p}\phi_{i_s}^{p^{r_s}} \odot P_s$ has derivatives in all components which are integral as polynomials in Chern classes. Here $\alpha_s \in \mathbb{Z}_{(p)}^{\times}$ and i_s is not divisible by p^n.

Thus a counterexample to Lemma is reduced to the case of $\sum_{s;r_s} \frac{\alpha_s}{p}\phi_{i_s}^{p^{r_s}} \odot P_s$. Indeed, if it were integral, then ϕ would be integral as well.

Step 5. Apply Steps 1-3 to other components of the poly-operation.

Reducing counter-example of the form $\sum_s \frac{\alpha_s}{p}\phi_{i_s}^{p^{r_s}} \odot P_s$ by the procedure above (applied not to the first component) does not change expressions in the first component.

Thus, finally we get a counter-example of the form $\sum_{\mathbf{i}=(i_1,i_2,\ldots,i_N);(r_1,\ldots,r_N)} \frac{\beta_{\mathbf{i}}}{p}\phi_{i_1}^{p^{r_1}} \odot \phi_{i_2}^{p^{r_2}} \odot \cdots \odot \phi_{i_N}^{p^{r_N}}$, where $\beta_{\mathbf{i}} \in \mathbb{Z}_{(p)}^{\times}$ and $p^n \nmid i_l$ for all $1 \leq l \leq N$.

Step 6. If this poly-operation is integral, then multiplying it by p, we get a non-trivial relation in poly-operations modulo p: $\sum_{\mathbf{i}} \beta_{\mathbf{i}}\phi_{i_1}^{p^{r_1}} \odot \phi_{i_2}^{p^{r_2}} \odot \cdots \odot \phi_{i_N}^{p^{r_N}} = 0 \mod p$. Due to Corollary 3.46 we have arrived to a contradiction and the Lemma is proved. □

Proposition 3.48. *Any integral poly-operation ϕ from $K(n)^*$ to $CH^* \otimes \mathbb{Z}_{(p)}$ is uniquely expressible as an integral series in external products of Chern classes.*

Proof. It is enough to work with polynomials (not series) of Chern classes, i.e. to prove that a poly-operation to a component of CH^* is expressible as an integral polynomial in external products of Chern classes. Let ϕ be an integral r-ary poly-operation to $CH^n \otimes \mathbb{Z}_{(p)}$.

By Lemma 3.44 operation ϕ and all its derivatives are uniquely expressible as rational polynomials in external products of Chern classes. We prove by decreasing induction on s that all s-th derivatives are integral.

Base of induction ($s >> 1$). Poly-operation ϕ can be written as a finite $\mathbb{Q}$-linear combination of external products of polynomials in components of the Chern character ch_j.

Recall that if ψ is an additive operation, then the j-th derivative of ϕ^N looks like: $\sum \binom{N}{r_1,r_2,\ldots,r_j}\phi^{r_1} \odot \phi^{r_2} \odot \cdots \odot \phi^{r_j}$. In particular, for $j > N$ we have $\partial^j \phi^N = 0$.

As the Chern character is additive, the same applies to polynomials in it and external products in them.

One could prove the base step directly from the COT using continuity of poly-operations.

Induction step ($s \to (s-1)$). Suppose that all derivatives $\partial^s\phi$ are integral polynomials in products of Chern classes. Previous derivatives $\partial^{s-1}\phi$ are integral poly-operations for which Lemma 3.47 is applicable. Therefore they are integral polynomials in external products of Chern classes. □

This finishes the proof of Theorem 3.13 for the case of $A^* = CH^* \otimes \mathbb{Z}_{(p)}$, the general case will be proven by the end of the next section as we develop a technical tool to reduce operations to an arbitrary g.o.c.t. to operations to Chow groups.

4. Continuity and truncation of operations

In this section we study operations between theories of rational type which 'increase' topological filtration on products of projective spaces.

Let $\phi : A^* \to B^*$ be any operation from theory of rational type A^* to g.o.c.t. B^*. By the COT it is determined by its restrictions to the products of projective spaces $G_{\{l\}} : A[[z_1^A, \ldots, z_l^A]] \to B[[z_1^B, \ldots, z_l^B]]$.

Denote by $F^\bullet$ a filtration on values of a g.o.c.t. B^* on a product of projective spaces generated by the ideal $\tilde{B}^*$. In other words, $F^iB^*((\mathbb{P}^\infty)^{\times k}) \subset B[[z_1^B, \ldots, z_k^B]]$ are series of at least degree i in variables z_j^B (which have degree 1). One can easily see, that this filtration coincides with the topological (codimension) filtration, however, in the following definition we

Definition 4.1. Operation $\phi : \tilde{A}^* \to B^*$ is said to increase Chern filtration by c if G_l takes values in $F^cB^*(\mathbb{P})$ for all $l \geq 1$.

Remark 4.2. It seems plausible that for any g.o.c.t. B^* one can define Chern bi-filtration $F^{\bullet,*}$ satisfying the following property. If $i : Z \subset X$ is a smooth closed subvariety of codimension at least n in a smooth variety, c is a polynomial in Chern classes of vector bundles on Z of degree at least m and any $\alpha \in B^*(Z)$, then $i_*(\alpha c) \in F^{m,n}B^*(X)$.

Then using the general Riemann-Roch theorem one should be able to prove that any operation ϕ from a theory of rational type A^* respects Chern bi-filtration: $\phi F^{m,n}A^* \subset F^{m,n}B^*$. Moreover, if the operation ϕ

increases Chern filtration by c (in the sense of Definition 4.1) then $\phi F^{m,n}A^* \subset F^{m+c-n,n}B^*$ (note that this is a new condition only for $n<c$).

There are some technical issues in defining the Chern bi-filtration and as throughout the paper we will use it only on products of projective spaces we decided to leave aside the general definition of the Chern bi-filtration. However, we hope that this may be done and it could give some kind of interpretation of what does it mean for an operation to increase the Chern filtration.

Definition 4.3. Denote by $F^c[\tilde{A}^*, B^*]$ the subset of operations increasing Chern filtration by $c\geq 1$.

This defines a decreasing filtration on the set of operations.

Let $\mathbb{P}_n := (\mathbb{P}^\infty)^{\times n}$ be a product of infinite-dimensional projective spaces.

Let B^* be a g.o.c.t. Denote by π_n the morphism of B-algebras $B^*(\mathbb{P}_n)\to (CH^*\otimes B)(\mathbb{P}_n)$ which sends z_i^B to z_i^{CH}.

Note that in general maps π_n do not commute with pullbacks in theories B^* and $CH^*\otimes B$, however, a weaker statement holds.

Lemma 4.4. *Let $f:\mathbb{P}_n\to\mathbb{P}_m$ be one of the morphisms between products of projective spaces appearing in the list of the COT.*

Then for any $c\geq 1$ the following diagram is commutative:

$$\begin{array}{ccc} F^cB^*(\mathbb{P}_m) & \xrightarrow{\pi_n} & (CH^c\otimes B)(\mathbb{P}_m) \\ {\scriptstyle f^*_{B^*}}\downarrow & & \downarrow{\scriptstyle f^*_{CH^*\otimes B}} \\ F^cB^*(\mathbb{P}_n) & \xrightarrow{\pi_m} & (CH^c\otimes B)(\mathbb{P}_n). \end{array}$$

Proof. Consider the case when f is a partial Segre embedding acting on the last two components of $\mathbb{P}_n$, other cases being similar (and, perhaps, more obvious).

The pull-back along Segre map $f^*_{B^*}$ sends z_n^B to $F_B(z_n^B, z_{n+1}^B)$, $f^*_{CH^*\otimes B}$ sends z_n to z_n+z_{n+1}. As $F_B(z_n^B, z_{n+1}^B)\equiv z_n^B+z_{n+1}^B \mod (z_n^B z_{n+1}^B)$ the claim is checked by a straight-forward computation. ☐

Proposition 4.5. *Let A^* be a theory of rational type and let B^* be a g.o.c.t.*

Let $\phi: A^*\to B^*$ be an operation which increases Chern filtration by c.

Then we can associate an operation $tr_c\phi: A^*\to CH^c\otimes B$, s.t. for any $n\geq 0$ and any $\alpha\in A^*(\mathbb{P}_n)$

$$tr_c\phi(\alpha)=\pi(\phi(\alpha)). \tag{6}$$

This defines a functorial map between groups of operations $tr_c: F^c[\tilde{A}^*, B^*]\to[\tilde{A}^*, CH^c\otimes B]$.

Proof. According to the COT and Lemma 4.4 the equation 6 correctly defines an operation to $CH^c\otimes B$. ☐

Proposition 4.6. *Suppose that $[A^*, CH^i\otimes B]$ is a free B-module with generators $\phi_1^{(i)},\ldots,\phi_{k_i}^{(i)}$, and assume that the map $gr_i^F[A^*, B^*]\to[A^*, CH^i\otimes B]$ is an isomorphism for all i.*

Therefore the lifts of operations $\phi_j^{(i)}$, denote them by $\psi_j^{(i)}\in F^i[A^*, B^*]$, freely generate the B-module of all operations from $[A^*, B^*]$.

The same is true for additive operations in the place of all operations.

Proposition 4.7. *Let A^* be a theory of rational type, B^* a g.o.c.t.*

Assume that the truncation maps $gr_i^F[\tilde{A}^*, B^*]\to[\tilde{A}^*, CH^i\otimes B]$ are isomorphisms for all i. Assume that the ring $[\tilde{A}^*, CH^*\otimes B]$ is freely generated as B-algebra by operations $\bar{t}_i:\tilde{A}^*\to CH^{m_i}\otimes B$, $i\in I$, and assume that $[\tilde{A}^*, CH^i\otimes B]$ is a finite rank (free) B-module.

Denote by $t_i\in F^{m_i}[\tilde{A}^*, B^*]$ s.t. $tr_{m_i}t_i=\bar{t}_i$.

Then $[\tilde{A}^*, B^*]$ is freely generated as B-algebra by oeprations t_i.

Proof. First, let us prove that there are no relations between operations t_i. Suppose that $P\in B[[t_1,\ldots,t_i,\ldots]]$ defines a zero operation. Define degree of a monomial $\prod_k t_{i_k}^{r_k}$ to be $\sum_k m_{i_k}r_k$, i.e. this monomial defines an operation increasing Chern filtration (at least) by its degree. Let j be the minimal degree of monomial summands of P, note that this is finite whenever $P\neq 0$.

Thus, tr_jP defines an operation to $CH^j\otimes B$ which is defined a polynomial in operations $\bar{t}_i$. As there are no relations between $\bar{t}_i$ by the assumption, we deduce that j is infinite, or equivalently $P=0$.

Second, let ϕ be any operation from $\tilde{A}^*$ to B^*. Then approximating its truncations by operations Namely, let us prove by induction on j that there exist $P_j\in B[[t_1,\ldots,t_i,\ldots]]$ s.t. $\phi-P_j$ increases Chern filtration by j.

Base of induction is proved the same way as step so we skip it. Operation $tr_j(\phi - P_j)$ can be lifted as a polynomial Q_j in operations t_i by the assumption of the proposition. Therefore $\phi - (P_j - Q_j)$ increases Chern filtration by $j+1$. As the Chern filtration on the space of operations is complete, operation ϕ is equal to a series in operations t_i. □

4.1. **Truncation of operations from Morava K-theories.** Let A^* be a p^n-typical theory. In this section we construct Chern classes from $K(n)^*$ to A^* and finish the proof of Theorem 3.13.

It is enough to consider the case of $A^* = BP\{n\}^*$, the universal p^n-typical theory. Thus, for any A^* Chern classes are just compositions of $c_i : K(n)^* \to BP\{n\}^*$ with the unique morphism of theories $BP\{n\}^* \to A^*$.

Later we will show how to prove iii) and iv) of Theorem 3.13 for any p^n-gradable theory.

Proposition 4.8. *There exist additive operations $\phi_i : K(n)^{i \mod p^n-1} \to BP\{n\}^i$, s.t.*

(1) operation ϕ_i increases Chern filtration by i;
(2) operation $tr_i\phi_i : K(n)^{i \mod p^n-1} \to CH^i \otimes BP\{n\}$ is a generator of $BP\{n\}$-module of additive operations.

Proof. First, let us show that there exist operations $\psi_i : K(n)^* \to BP\{n\}^i \otimes \mathbb{Q}$ which increase Chern filtration by i and s.t. $tr_i\psi_i$ is a generator of additive operations to $CH^* \otimes BP\{n\}$. Let us call such generator an integral generator in this proof.

Clearly, we have an isomorphism $[K(n)^*, BP\{n\}^* \otimes \mathbb{Q}]^{add} = [K(n)^* \otimes \mathbb{Q}, BP\{n\}^* \otimes \mathbb{Q}]^{add}$. There exist Chern characters $K(n)^* \otimes \mathbb{Q} \to CH^* \otimes \mathbb{Q}$ and $BP\{n\}^* \otimes \mathbb{Q} \to CH^* \otimes BP\{n\} \otimes \mathbb{Q}$

The space $gr_i^F[K(n)^* \otimes \mathbb{Q}, BP\{n\}^* \otimes \mathbb{Q}]^{add}$ is a free $BP\{n\} \otimes \mathbb{Q}$-module of rank 1 by Prop. 3.29. Take any generator of this space χ_i. The truncation of this operation is clearly a generator of the $BP\{n\} \otimes \mathbb{Q}$-module of operations to $CH^i \otimes BP\{n\} \otimes \mathbb{Q}$. Recall that the module of 'integral' additive operations $[K(n)^*, CH^i \otimes BP\{n\}]^{add}$ is also a free $BP\{n\}$-module of rank 1. Therefore there exist $a_i \in \mathbb{Q}$, s.t. $a_i\chi_i$ is integral and $tr_i(a_i\chi_i)$ is a generator of integral additive operations.

Second, let us define ϕ_i as an infinite $BP\{n\} \otimes \mathbb{Q}$-linear combination of operations ψ_i. For each i we define ϕ_i by an induction procedure.

On the k-th induction step we construct an additive operation $\phi_i^k : K(n) \to BP\{n\}^* \otimes \mathbb{Q}$, s.t. ϕ_i^k increases Chern filtration by i, $tr_i\phi_i$ is an integral generator of additive operations and ϕ_i^k is integral on products of projective spaces modulo F^k.

Base of induction. For ϕ_i^0 one may take ψ_i.

Induction step.

Let $m := \min\{p^m\phi_i^k isintegral\}$, i.e. ϕ_i^k has denominators at most p^m in degree k. If $m = 0$ then one can define $\phi_i^{k+1} = \phi_i$.

Assume now $m > 0$. We will decrease m by a new induction procedure. Loosely speaking, operation $p^m\phi_i^k$ is 'integral modulo F^{k+1}' and its truncations (if properly defined) $tr_j(p^m\phi_i^k) \mod p$ are all zero for $j < k$. At the same time the k-th truncation modulo p is a gradable operation and therefore is proportional to the truncation of ψ_k. Thus, we will be able to find $b \in BP\{n\}$ s.t. $p^m\phi_i^k - b\psi_i$ has denominators at most p^{m-1} in the k-th part of filtration.

To be more precise, by the CAOT operation ϕ_i^k is determined by series $G_j \in BP\{n\} \otimes \mathbb{Q}[[z_1, \ldots, z_j]]$ for $j \geq 0$. The conditions on the operation reformulate in terms of this series as G_j has degree at least k in variables z, and $p^mG_j \in BP\{n\}[[z_1, \ldots, z_m]] \mod F^{k+1}$. Denote the latter elements by $H_j = p^mG_j \in BP\{n\}[[z_1, \ldots, z_m]]/F^{k+1}$, in particular, $H_j = 0$ for $j \geq k+1$ by continuity. Note that $H_j \mod p$ has degree at least k, since G_j were integral in degrees less than k. Therefore by the considerations in Section 4 $H_j \mod p$ determines a truncated operation $K(n)^* \to CH^k \otimes BP\{n\}/p$. This operation is gradable by Prop. 3.35, and by Prop. 3.33 it is proportional to the truncation of the operation ψ_k. Therefore there exist $b \in BP\{n\}$ s.t. $p^m\phi_i^k - b\psi_k$ has denominators at most p^{m-1} in the k-th part of filtration.

This reduces m and by the induction on m one defines ϕ_i^{k+1} as $\phi_i^k + x\psi_k$ for some $x \in BP\{n\} \otimes \mathbb{Q}$. Note that since ψ_k increases Chern filtration by k $\phi_i^{k+1} \equiv \phi_i^k \mod F^{k+1}$, and thus it is clear that the induction process on k converges to an operation ϕ_i.

It is clear from the definition that the composition of ϕ_i with a projection π_i to the i-th component of $BP\{n\}^*$ has the same i-th truncation, so we may take $\pi_i \circ \phi_i$ to be the operation of the Proposition. □

Corollary 4.9. *(1) The $BP\{n\}$-module of additive operations from $K(n)^*$ to $BP\{n\}^*$ is freely generated by ϕ_i, i.e.*

$$[K(n)^*, BP\{n\}^*]^{add} = \{\sum_{i=1}^{\infty} a_i\phi_i | a_i \in BP\{n\}\}.$$

(2) Any additive gradable operation to $BP\{n\}^*/p$ is uniquely represented as a linear combination of operation of reductions of operations ϕ_i:

$$[K(n)^*, BP\{n\}^*/p]^{add,grad} = \{\sum_{i=1}^{\infty} a_i(\phi_i \mod p) | a_i \in BP\{n\}/p\}.$$

Proof. The first part follows from Proposition 4.7.

Recall that all additive operations ϕ_i are gradable, and by Prop. 3.33 the truncation map

$$gr_i^F[\tilde{K}(n)^*, BP\{n\}^*/p]^{add,grad} \to [\tilde{K}(n)^*, CH^i \otimes BP\{n\}/p]^{add,grad}$$

is an isomorphism. Arguments analogous to the proof of Prop. 4.7 can be applied here to prove the second part. □

Proposition 4.10. *There exist operations $c_i : K(n)^* \to BP\{n\}^i$ s.t.*

(1) they satisfy a Cartan-type formula (cf. Remark 3.15):

$$c_{tot}(x+y) = F_{K(n)}(c_{tot}(x), c_{tot}(y));$$

(2) operation c_i increases Chern filtration by i;
(3) operation $tr_i c_i$ is equal to c_i^{CH} as operations to $CH^i \otimes BP\{n\}$.

Proof. We will construct operations c_i by induction, in the same way as was done for operations to Chow groups.

Base of induction. Take c_i to be equal the additive generator ϕ_i for $1 \le i \le p^n - 1$.

Induction step. Assume that operations $c_1, \ldots, c_{i-1}$ are constructed and satisfy properties of the proposition.

As in the case of Chern classes with values in Chow groups we define operations c_i as a sum of a polynomial in $c_1, \ldots, c_{i-1}$ and an additive operation increasing Chern filtration by i.

As usually denote by $P_i := c_i - (\log_{K(n)}(c_1 + c_2 + \ldots) \in \mathbb{Q}[c_1, \ldots, c_{i-1}]$. Let $\mu_i = \max(0, -\nu_p(P_i))$.

By the induction assumption operation defined by P_i increases Chern filtration by i. Moreover, it is clear that $tr_i P_i(c_1, \ldots, c_{i-1}) = P_i(tr_1 c_1, \ldots, tr_{i-1} c_{i-1}) = P_i(c_1^{CH}, \ldots, c_{i-1}^{CH})$. From Prop. 4.8 it clear that there exists an additive operation ψ_i s.t. $tr_i \psi_i$ is equal to the additive generator of operations to $CH^i \otimes \mathbb{Z}_{(p)}$ defined by the formula $p^{\mu_i}(c_i^{CH} - P_i(c_1^{CH}, \ldots, c_{i-1}^{CH}))$. Then $P_i + p^{-\mu_i}\psi_i$ is an operation to $BP\{n\}^* \otimes \mathbb{Q}$ increasing the Chern filtration by i and s.t. its i-th truncation is equal to c_i^{CH}.

Let us give a sketch of a proof analogous to the construction of operations ϕ_i in Prop. 4.8 that there exist $\tilde{\psi}_i = \psi_i + \sum_{j>i} b_i \phi_i$, $b_i \in \mathbb{Q}$, s.t. $P_i + p^{-\mu_i}\tilde{\psi}_i$ is an integral operation.

Suppose that we have found $b_{i+1}, \ldots, b_{i+k} \in \mathbb{Q}$ s.t. $P_i + p^{-\mu_i}\tilde{\psi}_i$ acts integrally on products of projective spaces up to the $(i+k)$-th part of filtration and the denominators in this degree are at most p^N. It is clear that $p^N(P_i + p^{-\mu_i}\tilde{\psi}_i)$ is integral up to the $(i+k)$-th part of filtration and thus one can look at its reduction modulo p, F^{i+k}. Since P_i is integral, this reduction will be a gradable additive map which can be approximated by ϕ_{i+k}. Reducing N one finds b_{i+k}.

Clearly, $P_i + p^{-\mu_i}\tilde{\psi}_i$ increases the Chern filtration by i, define c_i to be equal to it and the induction step is finished. The Cartan's formula is satisfied by the definition of polynomials P_i.

As P_i has degree i in Chern classes, then by induction it takes values in $BP\{n\}^i$, and so is c_i. □

The following proposition finishes the proof of Theorem 3.13.

Proposition 4.11. *Let A^* be a p^n-typical theory s.t. A is a free $\mathbb{Z}_{(p)}$-module.*

Then all operations from $K(n)^*$ to A^* are uniquely expressible as series in Chern classes:

$$[\tilde{K}(n)^*, A^*] = A[[c_1, \ldots, c_i, \ldots]].$$

Proof. As $CH^* \otimes A$ is isomorphic to a direct sum of $CH^* \otimes \mathbb{Z}_{(p)}$ as a presheaf of abelian groups, then it is clear, that the A-module of operations $[\tilde{K}(n)^*, CH^* \otimes A]$ is isomorphic to $A \otimes [\tilde{K}(n)^*, CH^* \otimes \mathbb{Z}_{(p)}]$.

From Prop. 3.48 it follows that the latter is freely generated by polynomials in Chern classes c_i^{CH} of degree i, which by the construction of Chern classes to A^* are equal to i-th truncations of polynomials in c_i of degree i. By Prop. 4.7 the claim now follows. □

5. On the uniqueness of Morava K-theories

Proposition 5.1. *Let $\phi_i : K(n)^* \to K(n)^*$ be a basis of additive endo-operations of Morava K-theory, s.t. ϕ_i increases Chern filtration by i.*

Then an additive endo-operation $\phi := \sum_i a_i\phi_i$ is invertible if and only if $a_i \in \mathbb{Z}_{(p)}^\times$ for $1 \le i \le p^n - 1$.

Proof. Denote by β_i, b^i_k the coefficients in the expansion of $\phi_i^{\circ 2}$ in the basis ϕ_i: $\phi_i^{\circ 2} = \beta_i\phi_i + \sum_{k>i} b^i_k\phi_k$.

The key observation is that

Lemma 5.2. *$\beta_i \in \mathbb{Z}_{(p)}^\times$ for $i < p^n$, and $\beta_i \in p\mathbb{Z}_{(p)}$ for $i \ge p^n$.*

Proof of Lemma. First, let us explain how one can express numbers β_i in terms of the series G_l defining an additive operation (for the notation see Section 1.5).

It is clear that $\phi_i^{\circ 2}$ increases the Chern filtration by i, and so it is represented by a linear combinations of ϕ_j with $j \ge i$. The series G_i of ϕ_i is equal to $\beta_i z_1 \cdots z_i$+higher degree terms, and for ϕ_j the series G_i is of degree at least j.

Now for $\phi_i^{\circ 2}$ we see that the series G_i starts with $\beta_i^2 z_1 \cdots z_i$, and thus it is clear that $\phi_i^{\circ 2} - \beta_i\phi_i$ increases the Chern filtration by $i + 1$, and can be represented by a linear combinations of ϕ_j with $j > i$.

Recall that truncations of generators of additive operations to $BP\{n\}^*$ are generators of additive operations to $CH^* \otimes \mathbb{Z}_{(p)}$, and the same is true for the composition $K(n)^* \to BP\{n\}^* \to K(n)^*$. Thus, our goal is to show that for a generator of additive operations $\tilde{K}(n)^* \to CH^i \otimes \mathbb{Z}_{(p)}$ the corresponding polynomial G_i is of the form $\beta_i z_1 \cdots z_i$ where β_i is invertible in $\mathbb{Z}_{(p)}$ iff $i < p^n$. This also proves that the statement of Lemma does not depend on the choice of generators ϕ_i.

For $i < p^n$ one checks that ch_i is a generator of integral operations, and for it $G_i = z_1 \cdots z_i$. For $i \ge p^n$ consider the mod-p reduction of a generator of integral operations. The polynomial $G_i = \alpha^{(i)}_{(1,1,\ldots,1)} z_1 \cdots z_i$ is equal to zero by Prop. 3.32. □

Let $\phi := \sum_{i=1}^\infty a_i\phi_i$ be an additive endo-operation of $K(n)^*$, and assume that ϕ_i is the projection on the i-th graded component for $i < p^n$ (it is clear that the statement of Proposition does not depend on the choice of generators).

First, assume that ϕ is invertible, and let ψ be its inverse. The series G_i for ϕ, ψ are equal to $a_i z_1 \cdots z_i + \ldots$, $b_i z_1 \cdots z_i + \ldots$ for $i < p^n$. As the composition $\psi \circ \phi$ is the identity, we see that $a_i b_i = 1$, i.e. $a_i \in \mathbb{Z}_{(p)}^\times$ has to be invertible for $i < p^n$.

Second, assume that $a_i \in \mathbb{Z}_{(p)}^\times$ for $i < p^n$ and let us construct the inverse operation by induction.

The induction hypothesis is that we have found ψ_k s.t. $\psi_k \circ \phi - Id$ increases the Chern filtration by k, i.e. it is equal to $\sum_{i \ge k} \alpha_i\phi_i$ for some $\alpha_i \in \mathbb{Z}_{(p)}$.

Base of induction ($k = p^n$).

Define the operation $\psi_{p^n} := \sum_{i=1}^{p^n-1} a_i^{-1}\phi_i$. Using the fact that $\phi_i \circ \phi_j = \delta_{ij}\phi_i$ for $i \ne j$ and $i, j < p^n$, a direct computation shows that $\psi_{p^n} \circ \phi - Id$ is a linear combination of operations ϕ_i for $i \ge p^n$. Thus, the b

Induction step ($p^n \le k \to k + 1$).

Let us define $\psi_{k+1} = (Id + x\phi_k) \circ \psi_k$ with x defined so that $\psi_{k+1} \circ \phi - Id$ increases Chern filtration by $k + 1$.

Note that $\psi_{k+1} \circ \phi = Id + a_k\phi_k + x\phi_k + xa_k\alpha_k\phi_k + \sum_{i>k} c_k\phi_k$, and therefore setting $x = -\frac{a_k}{1+\beta_k}$ the claim is obtained. Number $x \in \mathbb{Z}_{(p)}$ since $\beta_k \in p\mathbb{Z}_{(p)}$ for $k \ge p^n$.

This process 'converges' as $\psi_{k+1} - \psi_k = x\phi_k \circ \psi_k$ increases the Chern filtration at least by k. □

Theorem 5.3. *Let $K(n)^*$, $\bar{K}(n)^*$ be two n-th Morava K-theories over $\mathbb{Z}_{(p)}$ (i.e. localisations of algebraic cobordisms defined by p^n-gradable FGLs F_1, F_2 respectively of height n).*

Then there exist an isomorphism of presheaves of abelian groups $K(n)^* \to \bar{K}(n)^*$.

Proof. 1. There exist an additive operation ϕ_1 from $K(n)^*$ to $\bar{K}(n)^*$ s.t. the corresponding series $G_i = z_1 \cdots z_i$+higher degree terms for $1 \le i \le p^n - 1$.

By the construction of additive operations from $K(n)^*$ to $\bar{K}(n)^*$: it starts with the Chern character and then it is somehow adjusted. Denote the corresponding operations by ϕ_1. Note however that $\phi_1(z_1 \cdots z_i) = z_1 \cdots z_i + \ldots$ for $1 \le i \le p^n - 1$.

Denote by ψ_1 an additive operation from $\bar{K}(n)^*$ to $K(n)^*$ satisfying the same property.

2. Compositions $\psi_1 \circ \phi_1$, $\phi_1 \circ \psi_1$ are invertible additive operations.

For these compositions we have $G_i = z_1 \cdots z_i$+higher degree terms, which means that they satisfy the conditions of Prop. 5.1. □

Remark 5.4. Note that not all of Morava K-theories are multiplicatively isomorphic. The condition to be checked is whether $\log_1^{-1} \circ \log_2$ is an integral series. However, by a direct computation one checks that if v_n is sent to $a, b \in \mathbb{Z}_{(p)}^\times$ and $a \neq b \mod p$, then it is not integral.

Recall, however, that $K(n)^* \otimes \bar{\mathbb{F}}_p$ and $\bar{K}(n) \otimes \bar{\mathbb{F}}_p$ are multuplicatively isomorphic by a theorem of Lazard.

Remark 5.5. The theorem above, perhaps, suggest that there is a uniquely defined n-th Morava K-theory as a presheaf of abelian groups (this does not follow from the theorem as additive isomorphism constructed therea are neither canonical, nor unique).

At the same time a multiplicative structure on $K(n)^*$ is certainly not unique, and this breaks the hope that one could find some algebraic structure similar to FGLs which controls $K(n)^*$-orientability. However, it is possible that there is some weaker than multiplicative additional structure on Morava K-theories which is in some sense canonical and could account for the orientability.

6. The gamma filtration on Morava K-theories

In this section we define a functorial filtration on $K(n)^*(X)$ for all $n \geq 1$. The definition of this filtration is verbatim the definition of the gamma filtration for K_0 with the use of Chern classes.

We prove that thus introduced gamma filtration does not depend on the choice of Chern classes and it is strictly compatible with additive isomorphisms between n-th Morava K-theories. The filtration on $K(1)^*$ coincides with the gamma filtration on $K_0 \otimes \mathbb{Z}_{(p)}$ under an additive isomorphism.

We call this filtration on $K(n)^*$ *the* gamma filtration.

6.1. Definitions and properties.

Definition 6.1. Define the gamma filtration on $K(n)^*$ of a smooth variety by the following formula:

$$\gamma^m K(n)^*(X) := < c_{i_1}(\alpha_1) \cdots c_{i_k}(\alpha_k) | \sum_j i_j \geq m, \alpha_j \in K(n)^*(X) > .$$

It is clear from the definition that $\gamma^m K(n)^*$ is an ideal subpresheaf of $K(n)^*$.

Proposition 6.2. *The gamma filtration satisfies the following properties:*

i) $c_i^{CH}|\gamma^{i+1}K(n)^* = 0$*;*
ii) operations c_i^{CH} *is additive when restricted to* $\gamma^i K(n)^*$*, and it induces an isomorphisms of* $\mathbb{Q}$*-vector spaces;*
iii) c_i^{CH} *when resrticted to* $\gamma^i K(n)^*$ *is surjective for* $1 \leq i \leq p^n$*;*
iv) $gr_\gamma^i K(n)^* = gr_\gamma^i K(n)^{i \mod p^n - 1}$.

Remark 6.3. Unfortunately, we can not yet prove that the gamma filtration is weaker than the topological one, i.e. $\gamma^i \subset \tau^i$, however, we conjecture this to be true.

Proof. The property i) follows by continuity, since monomials of Chern classes of degree at least $i+1$ increase the Chern filtration at least by $i+1$ as poly-operations. Thus, the composition of them with c_i^{CH} increases the Chern filtration at least by $i+1$ and there are no such operations to $CH^i \otimes \mathbb{Z}_{(p)}$.

The operation c_i^{CH} is additive on $\gamma^i K(n)^*$ as follows from i) and the Cartan's formula. Recall that there is a surjective morphism $ch_i : K(n)^* \to CH^i \otimes \mathbb{Q}$, it is proportional to $c_i - P_i(c_1, \ldots, c_{i-1})$ as an additive operation, and thus it is proportional to c_i when restricted to γ^i. It is left to show that the map $\gamma^i K(n)^* \xrightarrow{ch_i} CH^i \otimes \mathbb{Q}$ is still surjective.

To do this recall that an additive operation $\psi_i : c_i - P_i : K(n)^* \to K(n)^* \otimes \mathbb{Q}$ increases the Chern filtration by i and $tr_i \psi_i \neq 0$. As the polynomial P_i has degree i in Chern classes, the image of ψ_i lies in $\gamma^i K(n)^*$. Operation ψ_i has G_i equal to $\alpha z_1 \cdots z_i + \ldots$, where $\alpha \neq 0$. Thus, $ch_i \circ \psi_i$ has G_i equial to $\alpha z_1 \cdots z_i$ and is not zero. As the vector space of additive operations to $CH^i \otimes \mathbb{Q}$ is 1-dimensional, $ch_i \circ \psi_i$ is proportional to ch_i, and thus it is clear that ch_i is surjective on γ^i.

The property iii) is proved analogously. Consider the morphism of theories $p_i : BP^i \otimes \mathbb{Z}_{(p)} \to CH^i \otimes \mathbb{Z}_{(p)}$ as a surjective operation, then let us show that it is proportional to $c_i^{CH} \circ c_i \circ \pi_i$ where π_i is the projection to $K(n)^i$. If the coefficient of the proportionality is invertible, then it is clear that c_i^{CH} is surjective on the image of c_i.

Note that both these operations are additive as $c_i^{CH} \circ c_i$ is additive by the considerations above. Thus, it is enough to show that corresponding polynomials $G_1, \ldots, G_i$ (notation from Section 1.5) for operations these operations coincide. It is clear for degree reasons that for p_i polynomial $G_j = 0$ for $j \neq i$, and $G_i(z_1 \cdots z_i) = t_1 \cdots t_i$.

The projection π_i has $G_j = 0$ for $j < i$ for the same degree reasons, and it is clear that if the operation $c_i^{CH} \circ c_i \circ \pi_i$ sends $z_1 \cdots z_i$ to to $\alpha_i t_1 \cdots t_i$, then $\alpha_i p_i = c_i^{CH} \circ \pi_i$.

By construction of Chern classes one may take c_i^{CH} to be ch_i for $i \leq p^n - 1$, and then $\alpha_i = 1$. Thus it is left to show that $\alpha_{p^n} \in \mathbb{Z}_{(p)}^\times$.

By definition $c_p^n = \frac{-(c_1)^{p^n} + \phi_{p^n}}{p}$ for some additive operation ϕ_{p^n}. By the construction one can take c_1 to be the projection on $K(n)^1$, and therefore $(c_1)^{p^n}$ sends $z_1 \cdots z_{p^n}$ to $(z_1 \cdots z_{p^n})^{p^n}$. The p^n-th truncation of ϕ_{p^n} is a generator of additive operations to $CH^{p^n} \otimes \mathbb{Z}_{(p)}$, and one easily checks that it is equal to pch_{p^n}, so it sends $z_1 \cdots z_{p^n}$ to $pz_1 \cdots z_{p^n}$. The same considerations apply to $c_{p^n}^{CH}$, and one sees that $c_{p^n}^{CH} \circ c_{p^n}$ sends $z_1 \cdots z_{p^n}$ to $t_1 \cdots t_{p^n}$.

To prove iv) recall that the Chern class $c_i : K(n)^* \to BP\{n\}^i$ takes values in the i-th graded part, and as the classifying morphism of theories from $BP\{n\}^*$ to $K(n)^*$ respects the grading, the i-th Chern class from $K(n)^*$ to itself takes values in $K(n)^i$.

The space of all polynomials in Chern classes is split into $p^n - 1$ summands by their degree modulo $p^n - 1$. It is clear that the filtration by the degree of polynomials jumps on each summand every $p^n - 1$ steps, and thus the same is true for the gamma filtration and the graded components of $K(n)^*$. This proves iv). □

6.2. **Uniqueness of the gamma filtration.**

Proposition 6.4. *For any $m \geq 0$*

$$\gamma^m K(n)^*(X) := < \phi(\alpha_1, \ldots, \alpha_k) | \phi \in F^m[(K(n)^*)^{\times k}, K(n)^*] >,$$

i.e. the m-th part of the gamma filtration is generated by the image of all poly-operations increasing the Chern filtration by m.

In particular, it does not depend on the choice of operations c_i satisfying properties of Theorem 3.13.

Proof. The statement is equivalent to the fact that poly-operations increasing the Chern filtration by m are precisely series in external products of Chern classes of degree at least m. However it is clear from the classification of poly-operations and the fact that c_i increase the Chern filtration by i. □

Proposition 6.5. *Let $K(n)^*$, $\bar{K}(n)^*$ be two n-th Morava K-theories, and let $\phi : K(n)^* \to \bar{K}(n)^*$ an additive isomorphism between them (which exists by Th. 5.3).*

Then ϕ is strictly compatible with the gamma filtration.

Proof. To show the strict compatibility it is enough to show the compatibility for ϕ, since the inverse also will be compatible with the gamma filtration.

Thus, let P be a series of degree i in Chern classes defining an r-ary poly-operation in $K(n)^*$, we need to show that $\phi \circ P = P' \circ \phi$, where P' has also degree i. To define P', consider an endo-operation of $\bar{K}(n)^*$ by the formula $\phi \circ P \circ \psi^{\times r}$. It is then clear that P' increases the Chern filtration by i, and thus by Prop. 6.4. □

Proposition 6.6. *Denote by $\theta : K_0 \otimes \mathbb{Z}_{(p)} \to K(1)$ an invertible multiplicative operation defined by the Artin-Hasse exponential which gives an isomorphism of K_0 and a first Morava K-theory. Denote by $F_\gamma^\bullet K_0$ the classical gamma-filtration on K_0.*

Then $\theta F_\gamma^i K_0 = F_\gamma^i K(n)$.

Proof. The proof is the same as of Prop. 6.5, one needs only to note that the usual Chern classes $c_i^{K_0} : K_0 \to K_0$ increase the Chern filtration by i. We leave it to the reader to prove as an exercise. □

6.3. **An application: Chow groups of quadrics in I^n.**

In this section we prove several bounds on torsion in Chow groups of small codimension of quadrics in I^n. The idea of such an application is due to N. Semenov.

As there is only 2-torsion in Chow groups of quadrics, $p = 2$ in this section and all Morava K-theories have $\mathbb{Z}_{(2)}$-coefficients.

The following result is a corollary of a result of N. Semenov.

Proposition 6.7 (Semenov, [8, Prop. 6.14])**.** *Let $q \in I^{n+2}$ be a quadratic form, and denote by Q a corresponding smooth quadric.*

Then the restriction morphism $K(m)^*(Q) \to K(m)^*(\bar{Q})$ is an isomorphism for $1 \leq m \leq n$.

Proof. By Semenov $K(m)^*$-motive of Q, denote it by $M(Q)$, is a Tate motive, and thus is isomorphic to the motive of $\bar{Q}$.

As $K(m)^*(Q) = Hom_{K(m)}(\mathbb{Z}(0), M(Q))$, the Proposition follows. □

Corollary 6.8. *The restriction morphisms induces canonical isomorphisms for all $i \geq 0$:*

$$gr_i^{\gamma} K(n)^*(Q) \to gr_i^{\gamma} K(n)^*(\bar{Q}).$$

Proof. It follows from the functoriality and the definition of the gamma filtration. □

It follows from Corollary 6.8 and Prop. 6.2 that one can estimate Chow groups $CH^i(Q) \otimes \mathbb{Z}_{(2)}$ for $i \leq 2^n$ by the graded factrors of the gamma filtration on $K(n)^*(\bar{Q})$.

6.3.1. *An estimate of the gamma filtration for a split quadric.* Denote by $\bar{Q}$ a split quadric of dimension $2d$, and let $d \geq 2^{n+1} - 1$. Recall that by the result of Pfister all quadrics in I^{n+2} have dimension at least $2^{n+2} - 2$.

Denote by $i : \mathbb{P}^d \to \bar{Q}$ an inclusion of the maximal linear subspace.

Proposition 6.9 (Neshitov)**.** *The linear projection map $p : \bar{Q} \setminus \mathbb{P}^d \to \mathbb{P}^d$ induces an isomorphism $p^* : K(n)^*(\mathbb{P}^d) \to K(n)^*(\bar{Q} \setminus \mathbb{P}^d)$.*

Thus, there is a short exact sequence of abelian groups :

$$0 \to \oplus_{i=0}^d \mathbb{Z}_{(2)} l_i \xrightarrow{i_*} K(n)^*(\bar{Q}) \xrightarrow{\pi^*} \mathbb{Z}_{(2)}[H]/(H^{d+1}) \to 0,$$

where the map π^* is a morphism of rings.

Note that p^* induces surjective maps of abelian groups $gr_i^{\gamma} K(n)^*(\bar{Q}) \to gr_i^{\gamma} K(n)^*(\mathbb{P}^d)$. A direct calculation shows that $gr_i^{\gamma} K(n)^*(\mathbb{P}^d)$ has no torsion for all i, i.e. it equals to $\mathbb{Z}_{(2)}$ for $1 \leq i \leq d$ and 0 for $i > d$. As we will be interested in $gr_i^{\gamma} K(n)^*$ for $i \leq 2^n < d$, the rational generator of it is given by H^i and the torsion equals to the torsion in $gr_i^{\gamma} K(n)^*(\bar{Q}) \cap Imi_*$.

Proposition 6.10. *Denote by $\mathfrak{m} = \oplus_{i=1}^d \mathbb{Z}_{(2)} h^i + \oplus_{i=0}^d \mathbb{Z}_{(2)} l_i$ the maximal ideal of $K(n)^*(\bar{Q})$.*

Then for $2 \leq i \leq 2^n + 1$ we have $l_j \in \mathfrak{m}^{i+1}$ for $j \geq i$, and $2l_j \in \mathfrak{m}^{i+1}$ for $j \geq 1$.

Proof. Multiplication in the graded pieces of topological filtration of free theories is the same as in Chow groups for torsion-free rings.

Thus, $hl_i \equiv l_{i+1} \mod (l_j | j \geq i + 2^n)$, $h^{d+1} \equiv 2l_1 \mod (l_j | j \geq 2^n)$. □

Proposition 6.11. *We have $c_{2^n}(h) \equiv h \mod (h^{2^n})$, therefore h lies in 2^n-th part of the gamma filtration.*

Proof. This follows from the calculation of $c_{2^n}^{CH}(z_1 \cdots z_{2^n}) = t_1 \cdots t_{2^n}$ done in the proof of Prop. 6.2 and the general Riemann-Roch theorem 1.13. □

Proposition 6.12. *Let $\bar{Q}$ be a split quadric of dimension $2d$ with $d \geq 2^{n+1} - 1$. Let $j \in [1, 2^n - 2]$ be s.t. $d \equiv j \mod (2^n - 1)$.*

Then $gr_i^{\gamma} K(n)(\bar{Q})$ is torsion-free for all i: $1 \leq i \leq 2^n$ except for $i = 2^n$ when $j = 1$ and for $i = j$ in the other case.

Proof. As was noted above all torsion in the graded piece of the gamma-factor comes from classes l_i.

For $i \geq 1$ we can obtain l_i as a product of $c_{2^n}(h)$ and l_0 modulo elements l_j, $j > i$. By induction on i it is clear that they all lie in the $2^n + 1$-th piece of the gamma-filtration.

However, there is no way to obtain l_0 by multiplication, and it might give some torsion in the gamma filtration in $K(n)^j$. However, if $j = 1$ then $c_{2^n}(l_0) = l_0$ modulo $(l_i | i \geq 1)$ and thus l_0 lies in the 2^n-th piece of the gamma filtration. □

Combining all the results of this section we obtain the following theorem.

Theorem 6.13. *Let $q \in I^{n+2}$ be an anisotropic quadratic form, Q a corresponding smooth quadric of dimension $2d$. Let $j \in [1, 2^n - 2]$ be s.t. $d \equiv j \mod (2^n - 1)$.*

Then $CH^i(Q)$ is torsion-free for all i: $1 \leq i \leq 2^n$ except for $i = p^n$ when $j = 1$ and for $i = j$ in the other case.

Remark 6.14. One can give the bound for the torsion in the 'exceptional' component of Chow groups, however, it seems to depend on the dimension of a quadric.

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