Two-dimensional Riemann problem for rigid representations on an elliptic curve

A. A. Matveeva

National Research University Higher School of Economics, Russian Federation

Abstract

We discuss a generalization of Riemann-Hilbert problem on elliptic curves. We consider rank two rigid monodromy representations on elliptic curve with three singular points. For a given elliptic curve and representation we construct explicitly a semistable vector bundle of degree zero obeying a logarithmic connection with required monodromy and singular points.

Introduction

Classical Riemann-Hilbert problem deals with Fuchsian systems on the Riemann sphere. It explores the existence of Fuchsian system of linear differential equations with given singular points and prescribed monodromy representation. A. Bolibrukh proved that in general setting the problem has negative solution [1]. There also exist certain sufficient conditions for positive solvability. The problem can be solved positively for irreducible monodromy representations, representations of dimension two and some other cases.

The natural way to generalize the Riemann-Hilbert problem to other than Riemann sphere surfaces appeals to geometric approach to the problem. One can consider a Fuchsian system on the sphere as a logarithmic connection in trivial vector bundle on the Riemann sphere. It appears that in this approach essential properties of trivial vector bundle are its semi-stability and equality of its degree to zero [2]. The trivial bundle appears here because on Riemann sphere in any dimension it is the only holomorphic semi-stable vector bundle of degree zero. To this reason the generalization that we consider is to be given an elliptic curve, set of marked points on it and representation of fundamental group of punctured curve to construct over that curve a semi-stable vector bundle of degree zero equipped with logarithmic connection with prescribed singular points and monodromy representation.

We restrict ourselves to the case of three singular points and two-dimensional irreducible representations. The specifics of that case is the rigidity of monodromy representation. Rigid monodromy representations are uniquely defined by local monodromy data, that fact drastically simplify the problem and allows to construct solution to Riemann-Hilbert problem explicitly, in general setting there are only existence theorems.

In order to simplify and shorten explicit calculations and expressions we assume the monodromy in consideration to be in $\mathrm{SL}(2, \mathbb{C})$.

1 2d RHP on Riemann sphere

It it known that in dimension two on Riemann sphere the Riemann-Hilbert problem has positive solution for any poles positions and monodromy representations. Rigid representations are distinguished by the possibility of explicit solution construction while in general setting there are only existence theorems.

1.1 Rigid representations

We call the representation rigid if it can be restored from the spectra of local monodromy.

The two-dimensional irreducible representations of the fundamental group of a sphere with three punctures are rigid. In what follows considering only irreducible monodromy representations we construct defined modulo overall conjugation triple of monodromy matrices G_1, G_2, G_3 corresponding to loops encircling the punctures at a_1, a_2, a_3 from spectra $(\lambda_1, \lambda_2), (\mu_1, \mu_2), (\nu_1, \nu_2)$. Reducible case can be treated separately, it is not very difficult but need special approach and study of a number of degenerate cases. The criterion of representation irreducibility in terms of eigenvalues is well-known, if for all i, j, k there holds $\lambda_i \mu_j \nu_k \neq 1$ then the representation is irreducible.

We construct the basis from the vectors e_1 and e_2, the non-collinear eigenvectors of G_1 and G_2 respectively. For irreducible representations such a pair obviously exists. One can always normalize one of basis vectors in a way that G_1 has the form

$$G_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$$

By construction G_2 in this basis is lower triangular:

$$G_2 = \begin{pmatrix} \mu_1 & 0 \\ k & \mu_2 \end{pmatrix}$$

Matrix G_3 can be obtained from the relation $G_1 G_2 G_3 = 1$ or $G_3 = (G_1 G_2)^{-1}$ that gives

$$G_3 = \frac{1}{\lambda_1 \lambda_2 \mu_1 \mu_2} \begin{pmatrix} \lambda_2 \mu_2 & -\mu_2 \\ -\lambda_2 k & \lambda_1 \mu_1 + k \end{pmatrix}$$

The only parameter k sets the representation and can be obtained from the relation on trace of G_3:

$$\frac{\lambda_1 \mu_1 + \lambda_2 \mu_2 + k}{\lambda_1 \lambda_2 \mu_1 \mu_2} = \nu_1 + \nu_2.$$

Therefore the spectra $(\lambda_1, \lambda_2), (\mu_1, \mu_2)$ and (ν_1, ν_2) together with relation $G_1 \cdot G_2 \cdot G_3 = 1$ uniquely define the triple

$$G_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} \mu_1 & 0 \\ (\nu_1 + \nu_2)\lambda_1 \lambda_2 \mu_1 \mu_2 - \lambda_1 \mu_1 - \lambda_2 \mu_2 & \mu_2 \end{pmatrix}, \quad G_3 = (G_1 \cdot G_2)^{-1}$$

modulo an overall conjugation and hence, monodromy representation is fixed and therefore rigid.

1.2 Explicit construction

A logarithmic connection ∇ on Riemann sphere has the form $\nabla = d - \omega(z)$, where $\omega(z)$ is a matrix differential one-form having only simple poles as singular points.

Below we enlist some properties of logarithmic connections required for our construction. Proofs can be found for example at [1]. Consider that $a_i \neq \infty$.

Statement 1.1. *A matrix one-form of logarithmic connection with three singular points $\{a_1, a_2, a_3\}$ on Riemann sphere is set by the triple of residue matrices (B_1, B_2, B_3) defined up to an overall conjugation and satisfying $B_1 + B_2 + B_3 = 0$:*

$$\omega(z) = \left(\frac{B_1}{z-a_1} + \frac{B_2}{z-a_2} + \frac{B_3}{z-a_3} \right) dz,$$

If the eigenvalues of B_i do not differ by a natural number the point a_i is called non-resonant.

Statement 1.2. *In non-resonant point, local monodromy of connection is conjugated to the exponent of the corresponding residue multiplied by* $2\pi\imath$

$$G_i \sim \exp\left(2\pi\imath \operatorname*{Res}_{z=a_i} \omega(z)\right) = e^{2\pi\imath B_i}.$$

Statement 1.3. *For logarithmic connection, the eigenvalues of the local monodromy* G_i *coincide with the eigenvalues of* $\exp\bigl(2\pi\imath \operatorname{Res}_{z=a_i} \omega(z)\bigr) = \exp(2\pi\imath B_i)$.

Statement 1.4 (Fuchs relation)**.** *The sum of eigenvalues of* $B_i = \operatorname{Res}_{z=a_i} \omega(z)$ *over all singular points of a logarithmic connection is equal to zero.*

The statements above together with results of section 1.1 shows the way to explicit construction of logarithmic connection with three singular points and irreducible monodromy.

Corollary 1. *Consider an irreducible representation* $\chi : \pi_1(\mathbb{CP}^1 \setminus \{a_1, a_2, a_3\}) \to \mathrm{GL}(2, \mathbb{C})$ *with eigenvalues of* $\chi(\gamma_{1,2,3})$ *being equal to* (λ_1, λ_2), (μ_1, μ_2), (ν_1, ν_2) *respectively and fix complex logarithms of these eigenvalues in a way that Fuchs relation*

$$\ln\lambda_1 + \ln\lambda_2 + \ln\mu_1 + \ln\mu_2 + \ln\nu_1 + \ln\nu_2 = 0$$

holds. If the triple of residues (B_1, B_2, B_3) *of logarithmic connection*

$$\omega(z) = \left(\frac{B_1}{z-a_1} + \frac{B_2}{z-a_2} + \frac{B_3}{z-a_3} \right) dz,$$

has eigenvalues $(\frac{1}{2\pi i}\ln\lambda_1, \frac{1}{2\pi i}\ln\lambda_2)$, $(\frac{1}{2\pi i}\ln\mu_1, \frac{1}{2\pi i}\ln\mu_2)$ *and* $(\frac{1}{2\pi i}\ln\nu_1, \frac{1}{2\pi i}\ln\nu_2)$ *respectively and* $B_1 + B_2 + B_3 = 0$*, then, this logarithmic connection has monodromy* χ*.*

As it is proved below such a triple does always exist.

Theorem 1. *For any set* $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$ *such that* $\lambda_1\lambda_2\mu_1\mu_2\nu_1\nu_2 = 1$ *there exists a triple of matrices* (B_1, B_2, B_3) *satisfying requirements of corollary 1.*

Proof. Consider B_1 to be an upper triangular matrix.

$$B_1 = \begin{pmatrix} \frac{1}{2\pi i}\ln\lambda_1 & 1 \\ 0 & \frac{1}{2\pi i}\ln\lambda_2 \end{pmatrix}$$

and B_2 to be a lower triangular matrix.

$$B_2 = \begin{pmatrix} \frac{1}{2\pi i}\ln\mu_1 & 0 \\ \kappa & \frac{1}{2\pi i}\ln\mu_2 \end{pmatrix}$$

As we know, B_3 is defined by matrices B_1 and B_2:

$$B_3 = -B_1 - B_2 = -\begin{pmatrix} \frac{1}{2\pi i}(\ln\lambda_1 + \ln\mu_1) & 1 \\ \kappa & \frac{1}{2\pi i}(\ln\lambda_2 + \ln\mu_2) \end{pmatrix}.$$

Now the only parameter κ can be computed from the relation on eigenvalues of B_3.

$$\det B_3 = -\frac{1}{4\pi^2}\ln\nu_1\ln\nu_2$$

$$\frac{1}{4\pi^2}(\ln\lambda_1 + \ln\mu_1)(\ln\lambda_2 + \ln\mu_2) + \kappa = -\frac{1}{4\pi^2}\ln\nu_1\ln\nu_2$$

$$\kappa = -\frac{1}{4\pi^2}\bigl((\ln\lambda_1 + \ln\mu_1)(\ln\lambda_2 + \ln\mu_2) + \ln\nu_1\ln\nu_2\bigr).$$

□

Summarizing all above we obtain following statement.

Corollary 2. *If two-dimensional representation of $\pi_1(\mathbb{C}P^1 \setminus \{a_1, a_2, a_3\}, z_0)$ is irreducible, the corresponding Riemann-Hilbert problem can be solved explicitly.*

2 Vector bundles on an elliptic curve

The essential tool for studying vector bundles on an elliptic curve is theta-functions. Below we give definition and some basic properties of $\theta-$functions basing on [3].

2.1 θ-function

Consider on the complex plane the function $\theta(z)$ defined by

$$\theta(z) = \theta_1(z|\tau) = \imath \sum_{m\in\mathbb{Z}} (-1)^m q^{(m-\frac{1}{2})^2} e^{(m-\frac{1}{2})2\pi\imath z},$$

where $q(\tau) = e^{\imath\pi\tau} = e^{\imath\pi x - \pi y}$ sets the mapping of the upper half-plane $H = \{\tau \in \mathbb{C}|\,\mathrm{Im}\,\tau > 0\}$ into the unit circle $D = \{q \in \mathbb{C}\,|\,|q| \leqslant 1\}$.

It is easy to check that $\theta(z)$ is entire and odd function so $\theta(z) = -\theta(-z)$ and $\theta(0) = 0$. We also need an information about branching of $\theta(z)$ and $\theta'(z)$. Directly from definition we derive

$$\begin{array}{l} \theta(z+1) = -\theta(z) \\ \theta(z+\tau) = -q^{-1}e^{-2\pi iz}\theta(z). \end{array} \tag{1}$$

That implies the relations on derivatives:

$$\begin{array}{l} \theta'(z+1) = -\theta'(z) \\ \theta'(z+\tau) = q^{-1}e^{-2\pi iz}\big(2\pi i\theta(z) - \theta'(z)\big). \end{array} \tag{2}$$

Therefore,

$$\begin{array}{l} \dfrac{\theta'(z+1)}{\theta(z+1)} = \dfrac{\theta'(z)}{\theta(z)} \\ \dfrac{\theta'(z+\tau)}{\theta(z+\tau)} = \dfrac{\theta'(z)}{\theta(z)} - 2\pi i. \end{array} \tag{3}$$

The relations above imply that the integral of logarithmic derivative of $\theta(z)$ over the perimeter of fundamental parallelogram equals to $2\pi\imath$. Since $\theta(z)$ has no poles inside the fundamental parallelogram, it has the only simple zero there and as we have already seen it is located at the point $z = 0$.

Since the zero of $\theta(z)$ is simple $\theta^\alpha(z)$ under analytic continuation along the loop around $z = 0$ changes similar to z^α. Denoting g_0^* operator of the monodromy around zero, it is

$$g_0^*\left(\theta^\alpha(z)\right) = \theta^\alpha(z) \cdot e^{2\pi\imath\alpha}. \tag{4}$$

2.2 Line bundles on an elliptic curve

Denote Λ_τ an elliptic curve, obtained by factorization of the complex plane by lattice $\{1, \tau\}$, $\mathrm{Im}\tau > 0$. On the curve Λ_τ vector bundle can be set by action of two shifts: by 1 and by τ on sections of the bundle. It suffices to consider sections over the fundamental parallelogram.

Consider a holomorphic line bundle E of degree zero over the elliptic curve Λ_τ and $\varphi(z)$ to be a meromorphic section of E. Since $\deg E = 0$ section $\varphi(z)$ has equal number of zeroes and poles in the fundamental parallelogram. It also has some monodromy corresponding to a- and b-cycles, or, which is the same corresponding to shifts by 1 and τ. This monodromies are not uniquely defined, after an appropriate gauge one can always set monodromy corresponding to 1-shift to be equal to 1 and monodromy corresponding to τ-shift equal to some constant ν.

In that setting parameter ν still is not uniquely defined. Multiplying the section by $e^{2\pi\imath z}$ preserves its zeroes, poles, invariance under shifting by one and changes ν to $\nu \cdot e^{2\pi\imath\tau}$. Hence ν is defined up to multiplication by an integer power of $e^{2\pi\imath\tau}$. To work with it is more convenient to take parameter λ connected with ν via the relation $\nu = e^{2\pi\imath\lambda}$. Parameter λ is defined on the complex plane up to shifts along the lattice $\{1, \tau\}$ i.e. parameter λ encoding line bundles on the curve Λ_τ takes values from the curve Λ_τ itself. It is a well known fact that the moduli space of line bundles of fixed degree on an elliptic curve is isomorphic to an elliptic curve itself.

Hence considering a section of E we can assume $\varphi(z)$ to be invariant when z shifts by one and multiplied by $e^{2\pi\imath\lambda}$ when z shifts by τ. Such an objects we can effectively investigate using θ-functions.

Consider $\varphi_\lambda(z) = \frac{\theta(z-\lambda)}{\theta(z)}$. From (1) it follows that

$$\begin{aligned} \varphi_\lambda(z+1) &= \varphi(z) \\ \varphi_\lambda(z+\tau) &= \varphi(z) \cdot e^{2\pi\imath\lambda} \end{aligned} \tag{5}$$

$\varphi_\lambda(z)$ has exactly one zero and one pole on the elliptic curve. Therefore $\varphi_\lambda(z)$ is a section of some line bundle E of degree zero. Denote this bundle as $\mathcal{O}_\lambda(0)$. Further, ${}^b\varphi_\lambda(z) = \varphi_\lambda(z-b)$ differs from $\varphi_\lambda(z)$ by multiplication on meromorphic function and thus for any point b on elliptic curve it is also a section of $\mathcal{O}_\lambda(0)$. Hence the modular parameter λ together with degree k completely define the line bundle $\mathcal{O}_\lambda(k)$, the ratio of two sections with equal λ and k is a meromorphic function on Λ_τ.

Now consider $\varphi(z)$ to be the product of k different sections of the type ${}^{b_i}\varphi_\lambda(z)$. It is a section of $\mathcal{O}_{k\lambda}(0)$. Denoting zeros and poles of this product by a_i we get

$$\varphi(z) = \theta^{k_1}(z-a_1)\cdots\theta^{k_n}(z-a_n),$$

where k_i are integers and

$$\sum_{i=1}^{n} k_i = 0.$$

From the relations (1) we obtain

$$\begin{aligned} \varphi(z+1) &= (-1)^{\sum k_i}\varphi(z) = \varphi(z) \\ \varphi(z+\tau) &= \varphi(z) \cdot e^{2\pi\imath\sum k_i a_i} \end{aligned} \tag{6}$$

Since $\varphi(z)$ is a section if the $\mathcal{O}_{k\lambda}(0)$ it implies

$$\sum_{i=1}^{n} k_i a_i = k\lambda.$$

It is easy to see that for any set of points a_i an expression

$$\varphi(z) = \theta^{\alpha_1}(z-a_1)\cdots\theta^{\alpha_n}(z-a_n) \tag{7}$$

with any complex α_i such that $\sum \alpha_i = 0$, gives a (multivalued) section of the bundle $\mathcal{O}_\lambda(0)$ where

$$\lambda = \sum_{i=1}^{n} \alpha_i a_i \tag{8}$$

2.3 Rank 2 vector bundles on elliptic curve and logarithmic connections

In this section we examine two-dimensional vector bundles of degree zero over Λ_τ. From the results of previous section it follows that $\mathcal{O}_\lambda(k) \oplus \mathcal{O}_\mu(-k)$ gives an example of such a bundle. From general theory [5] it is known that there also exists exceptional indecomposable two-dimensional vector bundles of degree zero. Roughly speaking one can differ these two classes of bundles by decomposable or indecomposable monodromy representation generated by a- and b-cycles. In our work we shall only consider decomposable bundles.

Definition 1. *Vector bundle E is semi-stable if for any subbundle $F \subset E$ there holds* $\deg F / \operatorname{rk} F \leqslant \deg E / \operatorname{rk} E$

Theorem 2. *If F is a line sub-bundle of $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_\mu(0)$ then $\deg F \leqslant 0$.*

Proof. Consider φ to be a meromorphic section of F. Then $\deg F = N_\varphi - P_\varphi$ where N_φ, P_φ are total numbers of zeroes and poles of φ in a fundamental parallelogram. Being a section of F, φ is also a section of $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_\mu(0)$ and hence $\varphi = \varphi_1 \oplus \varphi_2$ where φ_1, φ_2 are some sections of $\mathcal{O}_\lambda(0)$ and $\mathcal{O}_\mu(0)$ respectively. Therefore zeroes of φ are the common zeroes of φ_1 and φ_2 while poles of φ are both poles of φ_1 and poles of φ_2. Hence $N_\varphi \leqslant \min(N_{\varphi_1}, N_{\varphi_2})$ and $P_\varphi \geqslant \max(P_{\varphi_1}, P_{\varphi_2})$ implying $N_\varphi - P_\varphi \leqslant \min(N_{\varphi_i} - P_{\varphi_i}) = 0$ ☐

Corollary 3. *Vector bundle $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_\mu(0)$ is semistable.*

According to our formulation of generalized Riemann-Hilbert problem we need to construct on elliptic curve Λ_τ vector bundle $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}$ equipped with logarithmic connection with prescribed monodromy representation and singular points location.

Let us describe the explicit form of logarithmic connection in that bundle. Consider a canonical base (s_1, s_2) in meromorphic sections of E, taking s_1 to be a section of $\mathcal{O}_\lambda(0)$ and s_2 to be a section of $\mathcal{O}_{-\lambda}(0)$ respectively. Any meromorphic section of E in that base has the form

$$\varphi(z) = \begin{pmatrix} f_1(z)s_1(z) \\ f_2(z)s_2(z) \end{pmatrix} = \begin{pmatrix} \varphi_\lambda(z) \\ \varphi_{-\lambda}(z) \end{pmatrix}$$

where $f_{1,2}(z)$ are meromorphic functions on Λ_τ and $\varphi_{\pm\lambda}(z)$ are the sections of $\mathcal{O}_{\pm\lambda}(0)$ respectively. Section $\varphi(z)$ is horizontal for some meromorphic connection with matrix differential 1-form ω:

$$d\varphi(z) = \omega(z)\varphi(z)$$

From relations 5 it follows

$$\begin{aligned} &\varphi(z+1) = \varphi(z) \\ &\varphi(z+\tau) = \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda} \end{pmatrix} \varphi(z) \end{aligned}$$

and hence

$$\begin{aligned} &\omega(z+1) = \omega(z) \\ &\omega(z+\tau) = \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda} \end{pmatrix} \omega(z) \begin{pmatrix} e^{-2\pi\imath\lambda} & 0 \\ 0 & e^{2\pi\imath\lambda} \end{pmatrix} \end{aligned} \tag{9}$$

Theorem 3. *Consider $\{a_1, \ldots, a_n\} \in \Lambda_\tau$, $a_i \neq a_j$ and complex $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, \ldots, n$ such that*

$$\sum_{i=1}^{n} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta \end{pmatrix} = 0$$

Then matrix 1-form

$$\Omega(z) = \sum_{i=1}^{n} \frac{\begin{pmatrix} \alpha_i \theta'(z-a_i) & \beta_i \frac{\theta'(0)}{\theta(-2\lambda)} \theta(z-a_i-2\lambda) \\ \gamma_i \frac{\theta'(0)}{\theta(2\lambda)} \theta(z-a_i+2\lambda) & -\delta_i \theta'(z-a_i) \end{pmatrix}}{\theta(z-a_i)} dz, \tag{10}$$

defines a logarithmic connection on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ with residues

$$\operatorname*{Res}_{z=a_i} \Omega(z) = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

Proof. Relations 3 and 5 imply

$$\begin{aligned}\frac{\theta'(z-a_i+1)}{\theta(z-a_i+1)} &= \frac{\theta'(z-a_i)}{\theta(z-a_i)}\\ \frac{\theta'(z-a_i+\tau)}{\theta(z-a_i+\tau)} &= \frac{\theta'(z-a_i)}{\theta(z-a_i)} - 2\pi\imath.\end{aligned} \tag{11}$$

and

$$\begin{aligned}\frac{\theta(z-a_i\mp 2\lambda+1)}{\theta(z-a_i+1)} &= \frac{\theta(z-a_i\mp 2\lambda)}{\theta(z-a_i)}\\ \frac{\theta(z-a_i\mp 2\lambda+\tau)}{\theta(z-a_i+\tau)} &= \frac{\theta(z-a_i\mp 2\lambda)}{\theta(z-a_i)}e^{\pm 4\pi\imath\lambda}\end{aligned} \tag{12}$$

Therefore from $\sum\alpha_i = \sum\delta_i = 0$ it follows

$$\begin{aligned}&\Omega(z+1) = \Omega(z)\\ &\Omega(z+\tau) = \begin{pmatrix} e^{2\pi\imath\lambda} & 0\\ 0 & e^{-2\pi\imath\lambda}\end{pmatrix}\Omega(z)\begin{pmatrix} e^{-2\pi\imath\lambda} & 0\\ 0 & e^{2\pi\imath\lambda}\end{pmatrix}\end{aligned} \tag{13}$$

and $\Omega(z)$ is a 1-form of meromorphic connection on some vector bundle $F\simeq\mathcal{O}_\lambda(k)\oplus\mathcal{O}_{-\lambda}(l)$.

Since $\operatorname{tr}\Omega(z)=0$ degree of F equals to zero and therefore $F\simeq\mathcal{O}_\lambda(k)\oplus\mathcal{O}_{-\lambda}(-k)$ for some integer k. Consider a section Φ of bundle F written down as

$$\Phi(z) = \begin{pmatrix}\varphi_\lambda(z)\\ \varphi_{-\lambda(z)}\end{pmatrix}$$

where $\varphi_{\pm\lambda}(z)$ are some sections of $\mathcal{O}_{\pm\lambda}(\pm k)$. Any connection on F maps sections of F to sections of $F\otimes T^*\Lambda_\tau$. For our Ω that imply that in the first row of $\Omega\Phi$

$$\left(\sum_{i=1}^n \alpha_i\frac{\theta'(z-a_i)}{\theta(z-a_i)}\right)\varphi_\lambda(z) + \left(\sum_{i=1}^n\beta_i\frac{\theta'(0)}{\theta(-2\lambda)}\frac{\theta(z-a_i-2\lambda)}{\theta(z-a_i)}\right)\varphi_{-\lambda}(z) \tag{14}$$

should be a section of $\mathcal{O}_\lambda(k)$.

Since $A(z) = \left(\prod_{i=1}^n\theta^{\alpha_i}(z-a_i)\right)'/\left(\prod_{i=1}^n\theta^{\alpha_i}(z-a_i)\right)$ is a ratio of two sections of $\mathcal{O}_{\sum\alpha_i a_i}(0)$ it is a single-valued function on Λ_τ or a section of $\mathcal{O}_0(0)$. Therefore $A(z)\varphi_\lambda(z)$ is a section of $\mathcal{O}_\lambda(k)$. Hence, $B(z)\varphi_{-\lambda}(z) = \left(\sum_{i=1}^n\beta_i\frac{\theta'(0)}{\theta(-2\lambda)}\frac{\theta(z-a_i-2\lambda)}{\theta(z-a_i)}\right)\varphi_{-\lambda}(z)$ as a difference of two sections should also be a section of $\mathcal{O}_\lambda(k)$. But by explicit construction $B(z)$ is a section of $\mathcal{O}_{2\lambda}(0)$ and therefore $B(z)\varphi_{-\lambda}(z)$ is a section of $\mathcal{O}_\lambda(-k)$. Hence $k=0$ and $\Omega(z)$ defines a connection on $E\simeq\mathcal{O}_\lambda(0)\oplus\mathcal{O}_{-\lambda}(0)$.

Differential 1-form $\Omega(z)$ is holomorphic on $\Lambda_\tau\setminus\{a_1,\dots,a_n\}$ and since $\theta(z)$ is an entire function and $\theta'(0)$ and $\theta(\pm 2\lambda)$ do not equal to zero $\Omega(z)$ has a simple poles in $z=a_i$. Therefore the connection defined by Ω is logarithmic with prescribed polar locus.

Calculation of residues is evident. □

It is important to notice that unlike to logarithmic connections over the Riemann sphere logarithmic connection over an elliptic curve is not uniquely defined by its residues. Since in the bundle E can exist holomorphic matrix 1-forms one can add them to $\Omega(z)$ and obtain new connection with the same residues. Explore in greater details the construction of such a 1-form $\Upsilon(z)$. Denote

$$\Upsilon(z) = \begin{pmatrix}\Upsilon_1(z) & \Upsilon_2(z)\\ \Upsilon_3(z) & \Upsilon_4(z)\end{pmatrix}dz$$

and consider relations (9). We get

$$\begin{aligned}\Upsilon_{1,4}(z+1) &= \Upsilon_{1,4}(z)\\ \Upsilon_{1,4}(z+\tau) &= \Upsilon_{1,4}(z)\\ \Upsilon_{2,3}(z+1) &= \Upsilon_{2,3}(z)\\ \Upsilon_{2,3}(z+\tau) &= \Upsilon_{2,3}(z)e^{\mp 4\pi\imath\lambda}\end{aligned} \tag{15}$$

Since all $\Upsilon_i(z)$ are holomorphic and $\Upsilon_{1,4}(z)$ are double-periodic, $\Upsilon_{1,4}(z)$ are constant. Relations (15) imply

$$\begin{array}{c} \frac{\Upsilon_{2,3}'(z+1)}{\Upsilon_{2,3}(z+1)} = \frac{\Upsilon_{2,3}'(z)}{\Upsilon_{2,3}(z)} \\ \frac{\Upsilon_{2,3}'(z+\tau)}{\Upsilon_{2,3}(z+\tau)} = \frac{\Upsilon_{2,3}'(z)}{\Upsilon_{2,3}(z)} \end{array}$$

Therefore integral of logarithmic derivative of $\Upsilon_{2,3}(z)$ along the perimeter of fundamental parallelogram is zero and $\Upsilon_{2,3}(z)$ has equal number of zeroes and poles in the parallelogram. Since $\Upsilon_{2,3}(z)$ is holomorphic it has no poles and hence no zeroes. But [4] the only entire functions obeying relations (15) with no zeroes in complex plane are $f(z) = Ce^{2\pi k\imath z}$ with integer k inducing $2\lambda = k\tau$. Since λ is defined modulo $\{1, \tau\}$ it follows that λ equals either zero, or $\tau/2$. The first case corresponds to $\Upsilon_{2,3}(z) = 0$, the second to $\Upsilon_{2,3}(z) = C_{\mp}e^{\mp 2\pi\imath z}$.

Finally, all holomorphic matrix 1-forms $\Upsilon(z)$ on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ have the form

$$\Upsilon(z) = \left(\begin{array}{ll} C_1 & C_- e^{-2\pi\imath z} \\ C_+ e^{2\pi\imath z} & C_4 \end{array} \right) dz$$

with constant $C_1, C_{\mp}, C_4$ and $C_{\mp} = 0$ if $\lambda \neq \tau/2$. Logarithmic connections defined by 1-forms Ω and $\Omega(z) + \Upsilon(z)$ have coinciding residues, but in general different monodromy representations.

For shortness exclude in what follows from consideration the exceptional case $2\lambda = 0$ modulo $\{1, \tau\}$. In that case $\mathcal{O}_\lambda(0) \simeq \mathcal{O}_{-\lambda}(0)$, and the two-dimensional vector bundle $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ is analogous to trivial vector bundle in a sense that all four matrix entries of connection form are just $\left(\sum_{i=1}^{3} c_i \cdot \theta'(z - a_i)/\theta(z - a_i)\right) dz$

3 2d RHP on Elliptic curve

Two-dimensional Riemann problem on elliptic curve we consider consists in establishing on an elliptic curve a semi-stable vector bundle of degree zero with logarithmic connection having prescribed monodromy and singularities. For shortness of explicit expressions and calculations we restrict ourselves to the case of $\mathrm{SL}_(2, \mathbb{C})$-monodormy.

3.1 Monodromy data

Suppose we are given a logarithmic connection on an elliptic curve Λ_τ with singular points $a_1, \ldots, a_n$ and monodromy representation

$$\chi : \pi_1 \left(\Lambda_\tau \setminus \{a_1, \ldots, a_n\}, z_0\right) \to \mathrm{SL}(2, \mathbb{C}).$$

Namely we are given a set of matrix multipliers G_1, G_2, G_3 corresponding to the change of local horizontal sections basis under continuation along the loops encircling singular points and G_a, G_b corresponding to trivialisation deformation along $a-, b-$ cycles respectively.

$$\begin{array}{ll} \gamma_i & : Y(z) \mapsto Y(z)G_i \\ \gamma_{a,b} & : Y(z) \mapsto G_{a,b}Y(z) \end{array} \tag{16}$$

Let us find the conditions that monodromy satisfy and the most convenient way to encode it.

The fundamental group of an elliptic curve obey the relation $aba^{-1}b^{-1} = \mathrm{id}$, the loop, encircling the fundamental parallelogram along perimeter can be contracted inside it. Obviously, that for some natural ordering points a_i and choice of classes of basic loops γ_i encircling them, the sequential bypassing all the punctures is equivalent to bypassing the perimeter of the fundamental parallelogram and hence the relation in the fundamental group of punctured torus is $\gamma_1 \cdots \gamma_n = aba^{-1}b^{-1}$ or $\gamma_1 \cdots \gamma_n bab^{-1}a^{-1} = \mathrm{id}$. It corresponds to the condition

$$G_a G_b G_a^{-1} G_b^{-1} Y(z) G_n \cdots G_1 = Y(z)$$

on monodromy matrices. In general setting, and particularly for irreducible monodromy in the right hand-side that implies

$$\begin{cases} G_a G_b G_a^{-1} G_b^{-1} = 1 \\ G_1 \cdots G_n = 1 \end{cases}$$

As we have already seen for line bundles monodromy corresponding to the periods is not uniquely defined. Here situation is similar. Remind that we consider only decomposible bundles. Then by constant and holomorphic gauges acting on the left on $Y(z)$ one can easily get

$$G_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad G_b = \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda} \end{pmatrix}$$

preserving all other G_i.

Finally the input monodromy data for Riemann problem on elliptic curve Λ_τ in our approach is

$$\left\{ G_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, G_b = \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda} \end{pmatrix}, \{G_1, G_2, G_3 | G_i \in \mathrm{SL}(2,\mathbb{C}), G_1 G_2 G_3 = 1)\}\big/\!\sim \right\}$$

3.2 Explicit construction of solution

We shall construct required logarithmic connection ∇ on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ in the form $\nabla = d - \Omega(z)$, where $\Omega(z)$ is a matrix differential one-form described in Theorem 3.

Results of section 2.3 imply the explicit form of $\Omega(z)$.

Lemma 3.1. *A logarithmic connection with singular points $\{a_1, a_2, a_3\}$ on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ can be given by the set of residue matrices (B_1, B_2, B_3),*

$$B_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \qquad \sum_{i=1}^{3} B_i = 0$$

in the form:

$$\Omega(z) = \sum_{i=1}^{3} \frac{\begin{pmatrix} \alpha_i \theta'(z-a_i) & \beta_i \frac{\theta'(0)}{\theta(-2\lambda)} \theta(z-a_i-2\lambda) \\ \gamma_i \frac{\theta'(0)}{\theta(2\lambda)} \theta(z-a_i+2\lambda) & \delta_i \theta'(z-a_i) \end{pmatrix}}{\theta(z-a_i)} dz + \Upsilon(z), \tag{17}$$

up to an overall conjugation and holomorphic 1-form $\Upsilon(z)$ described in 2.3.

Following statements analogous to 1.2, 1.3 are essentially local and are valid for an elliptic curve as well.

Statement 3.1. *In non-resonant point, local monodromy of connection is conjugated to the exponent of the corresponding residue multiplied by $2\pi\imath$*

$$G_i \sim \exp\left(2\pi\imath \operatorname*{Res}_{z=a_i} \Omega(z)\right) = e^{2\pi\imath B_i}.$$

Statement 3.2. *For logarithmic connection, the eigenvalues of the local monodromy G_i coincide with the eigenvalues of* $\exp\big(2\pi\imath \operatorname{Res}_{z=a_i} \Omega(z)\big) = \exp(2\pi\imath B_i)$.

Being essentially non-local the Fuchs relation also holds true for a logarithmic connection on E as it deals with the sum of residues of determinant connection on determinant bundle, and the latter is trivial.

Statement 3.3 (Fuchs relation)**.** *The sum of eigenvalues of $B_i = \operatorname{Res}_{z=a_i} \Omega(z)$ over all singular points of a logarithmic connection is equal to zero.*

Similarly to the Riemann sphere case altogether that leads to

Lemma 3.2. *Consider an irreducible representation $\chi : \pi_1(\Lambda_\tau \backslash \{a_1, a_2, a_3\}) \to \mathrm{SL}(2, \mathbb{C})$ where*

$$G_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, G_b = \begin{pmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{pmatrix}.$$

and eigenvalues of $\chi(\gamma_{1,2,3})$ equals to $(\lambda_1, \lambda_1^{-1})$, (μ_1, μ_1^{-1}), (ν_1, ν_1^{-1}) respectively and fix complex logarithms of these eigenvalues in a way that Fuchs relation

$$\ln \lambda_1 + \ln \lambda_1^{-1} + \ln \mu_1 + \ln \mu_1^{-1} + \ln \nu_1 + \ln \nu_1^{-1} = 0$$

fulfills. If the triple of residues (B_1, B_2, B_3) of logarithmic connection $\Omega(z)$, described in lemma 3.1 has eigenvalues $(\frac{1}{2\pi i} \ln \lambda_1, \frac{1}{2\pi i} \ln \lambda_1^{-1})$, $(\frac{1}{2\pi i} \ln \mu_1, \frac{1}{2\pi i} \ln \mu_1^{-1})$, $(\frac{1}{2\pi i} \ln \nu_1, \frac{1}{2\pi i} \ln \nu_1^{-1})$ respectively and $B_1 + B_2 + B_3 = 0$, then, this logarithmic connection has monodromy χ.

As we already know from theorem 1 there always exists triple of residues B_1, B_2, B_3 satisfying $B_1 + B_2 + B_3 = 0$ with prescribed spectra. Therefore we obtain the main result

Theorem 4. *An irreducible representation $\chi : \pi_1(\Lambda_\tau \setminus \{a_1, a_2, a_3\}, z_0) \to \mathrm{SL}(2, \mathbb{C})$ with eigenvalues of $\chi(\gamma_{1,2,3})$ equal to $(\lambda_1, \lambda_1^{-1})$, (μ_1, μ_1^{-1}), (ν_1, ν_1^{-1}) respectively for any choice of complex logarithms branches such that*

$$\ln \lambda_1 + \ln \lambda_1^{-1} + \ln \mu_1 + \ln \mu_1^{-1} + \ln \nu_1 + \ln \nu_1^{-1} = 0$$

can be realized as a monodromy of a logarithmic connection on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ with one-form

$$\Omega(z) = \left[\frac{\begin{pmatrix} \frac{\ln \lambda_1}{2\pi i} \theta'(z - a_1) & \frac{\theta'(0)}{\theta(-2\lambda)} \theta(z - a_1 - 2\lambda) \\ 0 & \frac{\ln \lambda_1^{-1}}{2\pi i} \theta'(z - a_1) \end{pmatrix}}{\theta(z - a_1)} + \right.$$
$$+ \frac{\begin{pmatrix} \frac{\ln \mu_1}{2\pi i} \theta'(z - a_2) & 0 \\ k \frac{\theta'(0)}{\theta(2\lambda)} \theta(z - a_2 + 2\lambda) & \frac{\ln \mu_1^{-1}}{2\pi i} \theta'(z - a_2) \end{pmatrix}}{\theta(z - a_2)} +$$
$$\left. + \frac{\begin{pmatrix} -\frac{(\ln \lambda_1 + \ln \mu_1)}{2\pi i} \theta'(z - a_3) & -\frac{\theta'(0)}{\theta(-2\lambda)} \theta(z - a_3 - 2\lambda) \\ -k \frac{\theta'(0)}{\theta(2\lambda)} \theta(z - a_3 + 2\lambda) & -\frac{(\ln \lambda_1^{-1} + \ln \mu_1^{-1})}{2\pi i} \theta'(z - a_3) \end{pmatrix}}{\theta(z - a_3)} \right] dz + \Upsilon(z) \qquad (18)$$

where

$$k = -\frac{1}{4\pi^2} \left[\ln \nu_1 \ln \nu_1^{-1} + (\ln \lambda_1 + \ln \mu_1)(\ln \lambda_1^{-1} + \ln \mu_1^{-1}) \right],$$

and $\Upsilon(z)$ is holomorphic 1-form on E

References

[1] Bolibrukh, A. A. *Hilbert's twenty-first problem for linear Fuchsian systems.* (Russian) Trudy Mat. Inst. Steklov. 206 (1994), 160 pp.; translation in Proc. Steklov Inst. Math. 1995, no. 5 (206), viii + 145pp

[2] R. R. Gontsov, V. A. Poberezhny, *Various versions of the Riemann-Hilbert problem for linear differential equations*, Russian Mathematical Surveys (2008), 63(4):603

[3] V. V. Prasolov, Yu. P. Solovyov, *Elliptic functions and algebraic equations*, (Russian) Faktorial, Moscow, 1997

[4] A. A. Matveeva, V. A. Poberezhny, *One-dimensional Riemann problem on an elliptic curve*, preprint ITEP-TH-34/15

[5] M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math.Soc. (3) 7 (1957), 414–452