EXPLICIT CONSTRUCTION OF AN ISOMORPHISM BETWEEN QUIVER VARIETIES OF TYPE A AND TRANSVERSAL SLICES IN THE AFFINE GRASSMANIAN.

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Abstract. This paper will appear as an appendix to new version of the paper [MV1]. In the paper we write down the isomorphism between Nakajima quiver varieties $\mathfrak{M}_{\mathbf{0}}^0(v,d)$ of type A and transversal slices in the affine Grassmanian explicitely and explain it geometrically. After that we show that the isomorphism which is given by our formula coincides with the one constructed in [MV1].

Contents

1. Introduction

1.1. **Transversal slices in the affine Grassmanian.** Let $K := \mathbb{C}((z))$ and let $O := \mathbb{C}[[z]]$. Let $\mathcal{G}_{GL_m} := GL_m(K)/GL_m(O)$. For a cocharacter λ of GL_m we denote by z^λ the corresponding element in $T(K) \subset \mathcal{G}_{GL_m}$ where T is the diagonal torus in GL_m. Let $L^{\geq 0}G$ and $L^{<0}G$ be subgroups of non-negative and negative loops respectively in $GL_m(K)$. ($L^{\geq 0}G := GL_m[[z]]$, $L^{<0}G$ is the kernel of the natural evaluation homomorphism $GL_m[z^{-1}] \to GL_m$) Let $\mathcal{W}_\lambda^\mu := L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)}$ and let $\overline{\mathcal{W}}_\lambda^\mu := L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap \overline{L^{\geq 0}GL_m \cdot z^{-w_0(\mu)}}$.

1.2. **Affine quiver varieties $\mathfrak{M}_{\mathbf{0}}^0(v,d)$ of type A.** Here we follow [MV1, Section 2].

Let us consider the Dynkin graph of type A_{n-1} with vertices $I = \{1,\dots,n-1\}$ and the orientation Ω given by $0 \to 1 \to \cdots \to n-1$.

Let $\Omega \xrightarrow{\cong} \overline{\Omega}$, $\omega \mapsto \overline{\omega}$, be the reversal of orientation. Our quiver (I, H) has the set of arrows $H = \Omega \sqcup \overline{\Omega}$. For an arrow $h \in H$ we denote by $h' \in I$ its initial vertex and by $h'' \in I$ its terminal vertex.

Following Nakajima we position vector spaces V_i and D_i of dimensions $\dim V_i = v_i$ and $\dim D_i = d_i$, $i \in I$, at the vertices of our quiver i.e. we consider the I-graded vector spaces $V = \oplus_{i \in I} V_i$ and $D = \oplus_{i \in I} D_i$. Let $v = (v_1, \ldots, v_{n-1})$ and $d = (d_1, \ldots, d_{n-1})$ and define an affine space

$$M(v, w) \ = \ \bigoplus_{h \in H} \mathrm{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, D_i).$$

Following Lusztig and Maffei we will consider an element in $M(v, w)$ as a quadruple $(x, \overline{x}, p, q)$.

The group $G(V) = \prod_{i \in I} GL(V_i)$ acts on $M(v, w)$ so that for $g = (g_i)_{i \in I}$

$$g(x, \overline{x}, p, q) \ \stackrel{\text{def}}{=} \ (g_{i+1} x_i g_i^{-1}, g_i \overline{x}_i g_{i+1}^{-1}, g_i p_i, q_i g_i^{-1})_{i \in I}. \tag{1}$$

Let $\mathfrak{g}(V)$ be the Lie algebra of $G(V)$. For the corresponding moment map $\mu : M(v, d) \to \mathfrak{g}(V)$ the fiber $\Lambda^c(v, d) \stackrel{\text{def}}{=} \mu^{-1}(c)$ at $c = (c_1, \ldots, c_{n-1}) \in Z[\mathfrak{g}(V)]$ consists of all $(x, \overline{x}, p, q)$ such that

$$\begin{aligned} c_1 + \overline{x}_1 x_1 &= p_1 q_1, \\ c_i + \overline{x}_i x_i &= x_{i-1} \overline{x}_{i-1} + p_i q_i \quad 2 \leq i \leq n-2, \\ c_{n-1} &= x_{n-2} \overline{x}_{n-2} + p_{n-1} q_{n-1}. \end{aligned} \tag{2}$$

Let $\mathfrak{M}^0_{\mathbf{0}}(v, d)$ be a geometric quotient of $\Lambda^0(v, d)$ by the action of $G(V)$.

Let us fix some basic $\{e^i_j | i \in I, 1 \leq j \leq d_i\}$ of D such that $e^i_j \in D_i$. We have a natural order on our basis (we need it because we want to talk about dominant coweights of $GL(D)$). Let w_0 denote the longest element in the Weyl group for $GL(D)$ (corresponding to our basis).

1.3. **A variety $\mathfrak{M}^{reg}_{\mathbf{0}}(v, d)$ and cocharacters λ, μ.** (see [N2, Section 3] and [MV1, Subsection 5.1.1]) A quadruple $(x, \bar{x}, p, q) \in \Lambda^c(v, d)$ is called stable if for any I-graded subspace V' of V which contains $\mathrm{Im}(p)$ and preserved by x and $\bar{x}$ we have $V' = V$. A quadruple $(x, \bar{x}, p, q) \in \Lambda^c(v, d)$ is called costable if for any I-graded subspace V' of V contained in $\mathrm{Ker}(q)$ and preserved by x and $\bar{x}$ we have $V' = 0$. Denote by $\Lambda^0_{reg}(v, d)$ the set of stable and costable quadruples in $\Lambda^0(v, d)$. Let $\mathfrak{M}^{reg}_{\mathbf{0}}(v, d)$ [N2, Section 3] be the quotient of $\Lambda^0_{reg}(v, d)$ by the free action of $GL(V)$. One can see that $\mathfrak{M}^{reg}_{\mathbf{0}}(v, d)$ can be embedded in $\mathfrak{M}^0_{\mathbf{0}}(v, d)$ as an open dence subset.

For the dimension vectors (v, d) we define the following dominant cocharacters λ, μ of $GL(D)$: the cocharacter λ acts with eigenvalue t^i on the space of dimension d_i (thus $w_0(\lambda)$ is antidominant and acts with eigenvalue t^i on D_i) and the cocharacter μ acts with eigenvalue t^i on the subspace of dimension $v_{i+1}+v_{i-1}+d_i-2v_i$. Note that this definition is in accordance with [MV1, Subsection 5.1.1].

1.4. **Variety $\mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$ and its torus fixed points.** (see [BF1, Subsection 4.4])

Let $\mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$ denote the moduli space of principal GL_m-bundles on $\mathbb{P}^2$ of second Chern class a with a trivialization at the line at infinity l_∞.

Consider the action of $\mathbb{C}^*$ on $\mathbb{A}^2$ which sends (x,y) to $(t^{-1}x, ty)$. Note that $GL_m \times \mathbb{C}^*$ acts on $\mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$: the first factor acts by changing a trivialization at l_∞ and the second factor via its action on $\mathbb{A}^2$. Now for every cocharacter $\rho_\lambda : \mathbb{C}^* \to GL_m$ obtain the diagonal action of $\mathbb{C}^*$ on $\mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$. Let $\mathrm{Bun}^a_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ denote the fixed point set of this action. The point $(0,0) \in \mathbb{A}^2$ is fixed under the $\mathbb{C}^*$-action. So for every $E \in \mathrm{Bun}^a_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ the group $\mathbb{C}^*$ acts on the fiber $E_{(0,0)}$ of E at the point $(0,0) \in \mathbb{A}^2$. Let us denote by $\mathrm{Bun}^{\mu,a}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ the subvariety of $\mathrm{Bun}^a_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ formed by all $E \in \mathrm{Bun}^a_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ such that $\mathbb{C}^*$ acts on $E_{(0,0)}$ by the cocharacter ρ_μ. (Here μ and λ are the dominant cocharacters of GL_m and we use notations ρ_λ, ρ_μ when we talk about actions). Also let us denote by $\mathrm{Bun}^{\mu}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ the variety $\mathrm{Bun}^{\mu,\frac{(\mu,\mu)}{2}-\frac{(\lambda,\lambda)}{2}}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ (according to [BF1, Theorem 5.2(1)] for $a \neq \frac{(\mu,\mu)}{2}-\frac{(\lambda,\lambda)}{2}$ the variety $\mathrm{Bun}^{\mu,a}_{GL_m,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ is empty).

1.5. In [MV1] an isomorphism between quiver varieties $\mathfrak{M}^0_{\mathbf{0}}(v,d)$ (see [MV1]) of type A and slices $\overline{\mathcal{W}}^\mu_\lambda$ is constructed. Let us describe it. Firstly the isomorphism between quiver varieties of type A and certain transversal slices to nilpotent orbits (not Slodowy slices but similar) is constructed. (this isomorphism is denoted by ϕ in [MV1]) Also the isomorphism between slices to nilpotent orbits and slices $\overline{\mathcal{W}}^\mu_\lambda$ is constructed (it is denoted by ψ in [MV1]). Thus the composition $\psi \circ \phi$ gives us an isomorphism between quiver varieties of type A $\mathfrak{M}^0_{\mathbf{0}}(v,d)$ and slices $\overline{\mathcal{W}}^\mu_\lambda$.

We use a different approach to that isomorphism. We prove that $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$ are isomorphic to $\mathrm{Bun}^{-w_0(\mu)}_{GL_m,-w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m)$ where λ and μ are constructed from v,d as in 1.3. In paper [BF1] the isomorphism between $\mathcal{W}^\mu_\lambda$ and $\mathrm{Bun}^{-w_0(\mu)}_{GL_m,-w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m)$ is constructed. Thus independently from the paper [MV1] we get an isomorphism between $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$ and $\mathcal{W}^\mu_\lambda$. We calculate this isomorphism and see that it is given by a very explicit simple formula. (Theorem 2.1) After that we prove that this formula gives exactly the isomorphism $\psi \circ \phi$ (restricted on $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$) which was constructed in [MV1]. Thus from the continuity argument (see Subsection 5.2) it follows that the whole isomorphism $\psi \circ \phi$ is given by our explicit formula 3 (Theorem 2.2).

Remark. (We do not use this remark in the proof of theorems of the paper)

Actually in [BF1] authors constructed an isomorphism between $\overline{\mathcal{W}}^{\mu}_{\lambda}$ and $\mathcal{U}^{-w_0(\mu),\frac{(\mu,\mu)}{2}-\frac{(\lambda,\lambda)}{2}}_{GL(m),-w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m)$ (wich is a closure of $\mathrm{Bun}^{-w_0(\mu)}_{GL_m,-w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m)$ in the reduced Uhlenbeck space $\mathcal{U}^{\frac{(\mu,\mu)}{2}-\frac{(\lambda,\lambda)}{2}}_{GL_m}$ (see [BFG]). Also from the decomposition [BFG, (1)] for the Uhlenbeck space, the same decomposition for $\mathfrak{M}^0_{\mathbf{0}}(V,D)$ (affine Gieseker space) and the isomorphism between $\mathrm{Bun}^{-w_0(\tilde{\mu})}_{GL_m,-w_0(\tilde{\lambda})}(\mathbb{A}^2/\mathbb{G}_m)$ and $\mathfrak{M}^{reg}_{\mathbf{0}}(\tilde{v},\tilde{d})$ for different $\tilde{\mu},\tilde{\lambda},\tilde{v},\tilde{d}$ should follow the existence of the isomorphism between $\mathfrak{M}^0_{\mathbf{0}}(v,d)$ and $\mathcal{U}^{-w_0(\mu),\frac{(\mu,\mu)}{2}-\frac{(\lambda,\lambda)}{2}}_{GL(m),-w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m)$. Thus we can construct the isomorphism between $\mathfrak{M}^0_{\mathbf{0}}(v,d)$ and $\overline{\mathcal{W}}^{\mu}_{\lambda}$. Note that to prove that it is given by certain explicit formula it is enough to show that the restriction of this isomorphism on the dense open subvariety $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$ is given by that explicit formula. Thus for our purposes it is enough to deal with the explicit formula for the isomorphism between $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$ and $\mathcal{W}^{\mu}_{\lambda}$.

2. Main theorems

2.1. **Theorem.** The map

$$(x_i,\bar{x}_i,p_i,q_i)\mapsto z^{-w_0\lambda}(1+z^{-1}\sum_{n,l=0}^{\infty}z^{-n}q\bar{x}^n x^l p), \tag{3}$$

where $(x,\bar{x},p,q):=(\oplus x_i,\oplus\bar{x}_i,\oplus p_i,\oplus q_i)$ and λ,μ are as in Subsection 1.3 gives an isomorphism between $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$ and $\mathcal{W}^{\mu}_{\lambda}$.

Recall the isomorphisms $\phi:\mathfrak{M}^0_{\mathbf{0}}(v,d)\xrightarrow{\sim}T_\lambda\cap\overline{\mathcal{O}}_\mu$ (see [MV1, Section 8], [MV2, Subsection 3.3]) and $\psi: T_\lambda\cap\overline{\mathcal{O}}_\mu\xrightarrow{\sim}L^{<0}GL_m\cdot z^{-w_0(\lambda)}\cap\overline{L^{\geq 0}GL_m\cdot z^{-w_0(\mu)}}$ (see [MV1, Subsection 4.4.11]).

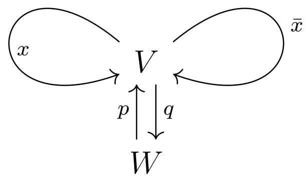
2.2. **Theorem.** The isomorphism $\psi\circ\phi$ is given by the formula (3).

The proof of the Theorem 2.1 will be given in the Section 4

The proof of the Theorem 2.2 will be given in the Section 5.

3. Geometric interpretation of the formula (3)

3.1. **ADHM description.** We set $\mathfrak{M}^{reg}_{\mathbf{0}}(V,D)=\{(x,\bar{x},p,q)\in\mu^{-1}(0)\mid \text{stable and costable }\}/GL_m$, where $(x,\bar{x},p,q)$ are Jordan quiver quadruples:



$\dim V = a$, $\dim D = m$

The ADHM description [N1, Theorem 2.1] identifies $\mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$ with $\mathfrak{M}^{reg}_{\mathbf{0}}(V, D)$.

The vector bundle $E_{(x,\bar{x},p,q)}$ corresponding to a quadruple $(x, \bar{x}, p, q)$ can be obtained as the middle cohomology of the following monad:

$$V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{a=\begin{bmatrix} z_0 x - z_1 \\ z_0\bar{x} - z_2 \\ z_0 q \end{bmatrix}} \begin{array}{c} V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ D \otimes \mathcal{O}_{\mathbb{P}^2} \end{array} \xrightarrow{b=\begin{bmatrix} -(z_0\bar{x} - z_2) & z_0 x - z_1 & z_0 p \end{bmatrix}} V \otimes \mathcal{O}_{\mathbb{P}^2}(1).$$

3.2. **Induced torus action on $\mathfrak{M}^{reg}_{\mathbf{0}}(V, D)$.** The $\mathbb{C}^*$-action on $\mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$ corresponding to a cocharacter λ defines an action on $\mathfrak{M}^{reg}_{\mathbf{0}}(V, D)$ via the ADHM isomorphism 3.1.

3.3. **Lemma.** This action can be described as follows:

$$(x, \bar{x}, p, q) \mapsto (t^{-1}x, t\bar{x}, p\rho_\lambda(t)^{-1}, \rho_\lambda(t)q).$$

Proof. Take $t \in \mathbb{C}^*$. Consider a vector bundle $tE_{(x,\bar{x},p,q)}$ that is obtained from $E_{(x,\bar{x},p,q)}$ by the action of t. It can be described as the middle cohomology of the following monad:

$$V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\begin{bmatrix} z_0 x - t z_1 \\ z_0\bar{x} - t^{-1} z_2 \\ z_0 q \end{bmatrix}} \begin{array}{c} V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ D \otimes \mathcal{O}_{\mathbb{P}^2} \end{array} \xrightarrow{\begin{bmatrix} -(z_0\bar{x} - t^{-1} z_2) & z_0 x - t z_1 & z_0 p \end{bmatrix}} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

We have to emphasize that the trivialization of $tE_{(x,\bar{x},p,q)}$ at infinity is $\rho_\lambda(t)$.

We have the following commutative diagram giving the isomorphism between the monad for $tE_{(x,\bar{x},p,q)}$ and the one for the quadruple $(t^{-1}x, t\bar{x}, p\rho_\lambda(t)^{-1}, \rho_\lambda(t)q)$:

$$\begin{array}{ccccccccc}
V\otimes\mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & V\otimes\mathcal{O}_{\mathbb{P}^2} & \oplus & V\otimes\mathcal{O}_{\mathbb{P}^2} & \oplus & D\otimes\mathcal{O}_{\mathbb{P}^2} & \longrightarrow & V\otimes\mathcal{O}_{\mathbb{P}^2}(1) \\
\downarrow Id & & \downarrow t^{-1} & & \downarrow t & & \downarrow \rho_\lambda(t) & & \downarrow Id \\
V\otimes\mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & V\otimes\mathcal{O}_{\mathbb{P}^2} & \oplus & V\otimes\mathcal{O}_{\mathbb{P}^2} & \oplus & D\otimes\mathcal{O}_{\mathbb{P}^2} & \longrightarrow & V\otimes\mathcal{O}_{\mathbb{P}^2}(1)
\end{array}$$

The morphism on cohomology induces the isomorphism between corresponding vector bundles with trivializations. □

3.4. An isomorphism between $\coprod_{v,\sum v_i=a}\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ and $\mathrm{Bun}^a_{GL_m,-w_0(\lambda)}$.

Let $\coprod_{v,\sum v_i=a}\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ be the disjoint union of quiver varieties of type A with vertices numbered by integers with a framing of dimension d and $\sum v_i = a$. We define a morphism [N2, Section 3]

$$\tilde{\Theta}_{v,d} : \mathfrak{M}_{\mathbf{0}}^{reg}(v,d) \to \mathfrak{M}_{\mathbf{0}}^{reg}(V,D) \simeq \mathrm{Bun}^a_{GL_m}(\mathbb{A}^2),$$

$$\tilde{\Theta}_{v,d}(x_i,\bar{x}_i,p_i,q_i) = (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i).$$

The maps $\tilde{\Theta}_{v,d}$ for different v induce the map $\tilde{\Theta}_d : \coprod_{v,\sum v_i=a}\mathfrak{M}_{\mathbf{0}}^{reg}(v,d) \to \mathrm{Bun}^a_{GL_m}(\mathbb{A}^2)$.

3.4.1. **Lemma.** $\tilde{\Theta}_d$ induces an isomorphism between $\coprod_{v,\sum v_i=a}\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ and $\mathrm{Bun}^a_{GL_m,-w_0(\lambda)}$ where λ is as in Subsection 1.3.

Proof. We describe the inverse map. Let $(x,\bar{x},p,q)$ be a fixed point under the $\mathbb{C}^*$-action on $\mathfrak{M}_{\mathbf{0}}^{reg}(V,D)$ corresponding to $-w_0(\lambda)$. Then using Lemma 3.3 we have that for every $t\in\mathbb{C}^*$ there exists $\rho_V(t)\in GL(V)$ such that

$$(t^{-1}x, t\bar{x}, p\rho_{-w_0(\lambda)}(t)^{-1}, \rho_{-w_0(\lambda)}(t)q) = (\rho_V(t)x\rho_V(t)^{-1}, \rho_V(t)\bar{x}\rho_V(t)^{-1}, \rho_V(t)p, q\rho_V(t)^{-1}). \tag{4}$$

Note that $\rho_V(t)$ is uniquely determined by t because of the freeness of $GL(V)$-action on stable and costable quadruples. In particular ρ_V defines a cocharacter of $GL(V)$. We decompose V into a direct sum $\oplus V_i$ (where V_i is the t^{-i}-eigenspace of ρ_V) and similarly decompose D into a direct sum $\oplus D_i$ with respect to $\rho_{-w_0(\lambda)}$ (because of our definition D_i is the t^{-i}-eigenspace of $-w_0(\lambda)$). It is easy to see that the condition (4) implies that $\forall i\subset\mathbb{Z}$, $x(V_i)\subset V_{i+1}$, $\bar{x}(V_i)\subset V_{i-1}$, $p(D_i)\subset V_i$, $q(V_i)\subset D_i$. So $(x,\bar{x},p,q)$ defines a point in a quiver variety of type A with vertices numbered by integers such that $\sum_{i=-\infty}^{+\infty} v_i = a$, and the framing is d. The inverse map is constructed.

$\square$

3.5. **An isomorphism between** $\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ **and** $\mathrm{Bun}_{GL_m,-w_0(\lambda)}^{-w_0(\mu),\sum v_i}$**.** *Lemma.* $\tilde{\Theta}_d$ induces the isomorphism [N2, Section 4] $\Theta : \mathfrak{M}_{\mathbf{0}}^{reg}(v,d) \simeq \mathrm{Bun}_{GL_m,-w_0(\lambda)}^{-w_0(\mu),\sum v_i}$ where λ,μ are as in Subsection 1.3.

Proof. It is enough to prove that $\mathbb{C}^*$ acts on the fibre of $E \in \Theta(\mathfrak{M}_{\mathbf{0}}^{reg}(v,d))$ at the origin by ρ_μ. Let us denote the cocharacter corresponding to the framing by ρ_d ($\rho_d = \rho_{-w_0(\lambda)}$) and let ρ_v be the cocharacter of $GL(\oplus V_i)$ that acts with eigenvalue t^{-i} on the space V_i.

Let $(x,\bar{x},p,q) := \Theta(x_i,\bar{x}_i,p_i,q_i)$ and $E_{(x,\bar{x},p,q)}$ be the corresponding vector bundle. The bundle $tE_{(x,\bar{x},p,q)}$ is the middle cohomology of the following monad:

$$V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\begin{bmatrix} z_0x - tz_1 \\ z_0\bar{x} - t^{-1}z_2 \\ z_0q \end{bmatrix}} \begin{matrix} V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ V \otimes \mathcal{O}_{\mathbb{P}^2} \\ \oplus \\ D \otimes \mathcal{O}_{\mathbb{P}^2} \end{matrix} \xrightarrow{\begin{bmatrix} -(z_0\bar{x} - t^{-1}z_2) & z_0x - tz_1 & z_0p \end{bmatrix}} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

The map $(\rho_v(t), t\rho_v(t) \oplus t^{-1}\rho_v(t) \oplus \rho_d(t), \rho_v(t))$ provides an isomorphism between the monads corresponding to $E_{(x,\bar{x},p,q)}$ and $tE_{(x,\bar{x},p,q)}$. In particular this map induces a $\mathbb{C}^*$-module structure on the fibre at the origin of the monad corresponding to $E_{(x,\bar{x},p,q)}$:

$$\overset{\rho_v \circlearrowleft}{V} \longrightarrow \overset{t^{-1}\rho_v \oplus t\rho_v \oplus \rho_d \circlearrowleft}{V \oplus V \oplus D} \longrightarrow \overset{\rho_v \circlearrowleft}{V}$$

Now we can calculate the action of $\mathbb{C}^*$ on cohomology of this complex. The cocharacter corresponding to this action is the difference of the cocharacters on $V \oplus V \oplus D$ and the double cocharacter on V. It means that the desired cocharacter acts with an eigenvalue t^{-i} on a subspace of dimension $v_{i-1} + v_{i+1} + d_i - 2v_i$. So this cocharacter is $-w_0(\mu)$. $\square$

3.6. **An isomorphism** $\mathrm{Bun}_{GL_m,-w_0(\lambda)}^{-w_0(\mu)}(\mathbb{A}^2/\mathbb{G}_m) \simeq (L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)})$**.**

Recall the construction of the isomorphism

$$\eta : \mathrm{Bun}^{-w_0(\mu)}_{GL_m,-w_0(\lambda)}(\mathbb{A}^2/\mathbb{G}_m) \simeq (L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)})$$

[BF1, Theorem 5.2].

Let us think about vector bundles on $\mathbb{P}^2$ with trivialization on the line at infinity (and fixed rank and second Chern class) as of bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with trivialization on

$\mathbb{P}^1 \times \infty \bigsqcup \infty \times \mathbb{P}^1$.

The morphism η is constructed as follows: a bundle $E \in \mathrm{Bun}^{\mu}_{GL_r,\lambda}(\mathbb{A}^2/\mathbb{G}_m)$ has to be trivial on $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$. (It is trivial on the line $\mathbb{P}^1 \times \infty$ hence on a neighbourhood of that line. It means that the number of horizontal jumping lines has to be finite. Using invariance of E under $\mathbb{C}^*$-action we see that the only jumping line must be $\mathbb{P}^1 \times 0$. Alternatively we can look at the monad corresponding to E and see that we can write down an explicit trivialization of E restricted on $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$ (we will do it)). But E is also trivialised on the line $\infty \times \mathbb{P}^1$. We can uniquely extend this trivialization to the whole variety $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$. Now we restrict E with the trivialization to the line $(1:1) \times \mathbb{P}^1$. We get a point in the affine Grassmannian $\mathcal{G}_{GL_m}$. Finally, we apply $z^{-w_0(\lambda)}$ to this point to obtain the desired point in the slice.

3.6.1. *An isomorphism between* $\mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$ *and* $(L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)})$. Composing the isomorphisms from subsections 3.5 and 3.6 we obtain an isomorphism

$$\eta \circ \Theta : \mathfrak{M}^{reg}_{\mathbf{0}}(v,d) \simeq (L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)}).$$

4. Proof of Theorem 2.1

The isomorphism $\eta \circ \Theta$ can be described as follows:

$$(x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-w_0\lambda}(1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p),$$

where $(x, \bar{x}, p, q) := (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i)$.

Proof. (Similar to [Hen, Proposition 4.8]) Take $(x_i, \bar{x}_i, p_i, q_i) \in \mathfrak{M}^{reg}_{\mathbf{0}}(v,d)$.

Let $E_{(x,\bar{x},p,q)} := \Theta(x_i, \bar{x}_i, p_i, q_i)$. The vector bundle $E_{(x,\bar{x},p,q)}$ can be described as the middle cohomology of the following monad [BF2, Subsection 2.4]:

$$V \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \xrightarrow[a]{} \begin{matrix} V \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1) \\ \oplus \\ V \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0) \\ \oplus \\ D \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \end{matrix} \xrightarrow[b]{} V \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$$

$$a = \begin{bmatrix} tx - y \\ h\bar{x} - z \\ thq \end{bmatrix}, b = \begin{bmatrix} -(h\bar{x} - z), & tx - y, & p \end{bmatrix},$$

where $((y:t),(z:h))$ are the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $(\infty, \infty) := ((1:0),(1:0))$. We want to describe the trivialization of $E_{(x,\bar{x},p,q)}$ restricted to $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$. For this it suffices to construct a map $D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)} \to \mathrm{Ker(b)} |_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}$ transversal to $\mathrm{Im(a)} |_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}$. It is easy to see that the map:

$$D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)} \xrightarrow[\tau_1 = \begin{bmatrix} (h\bar{x} - z)^{-1}p \\ 0 \\ \mathrm{Id} \end{bmatrix}]{} \begin{matrix} V \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}(0,-1) \\ \oplus \\ V \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)}(-1,0) \\ \oplus \\ D \otimes \mathcal{O}_{\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)} \end{matrix}$$

satisfies the requirement.

Note that τ_1 is well defined because $h\bar{x}$ is nilpotent ($\bar{x} = \oplus \bar{x}_i$, and $\bar{x}_i$ sends V_i to V_{i-1}, so that $\oplus \bar{x}_i$ acts nilpotently on $\oplus V_i$), hence $h\bar{x} - z$ is invertible when restricted to $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$ (since $z \neq 0$ on $\mathbb{P}^1 \times (\mathbb{P}^1 \setminus 0)$ and $h\bar{x}$ is nilpotent).

For the same reasons the map:

$$D \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1} \xrightarrow[\tau_2 = \begin{bmatrix} 0 \\ (y - tx)^{-1}p \\ \mathrm{Id} \end{bmatrix}]{} \begin{matrix} V \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1}(0,-1) \\ \oplus \\ V \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1}(-1,0) \\ \oplus \\ D \otimes \mathcal{O}_{(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1} \end{matrix}$$

induces the trivialization of $E_{(x,\bar{x},p,q)}$ restricted to $(\mathbb{P}^1 \setminus 0) \times \mathbb{P}^1$. Note that these two trivializations agree at the point (∞, ∞) and extend the trivialization of $E_{(x,\bar{x},p,q)}$ restricted to two infinite lines. Now we can construct $\eta(E_{(x,\bar{x},p,q)})$. To this end we have to calculate the transition function $(\tau_1^{-1} \circ \tau_2)_{|(1:1) \times (\mathbb{P}^1 \setminus \{0,\infty\})}$ it is the point in $\mathcal{G}_{GL_m}$ corresponding to $E_{|(1:1) \times (\mathbb{P}^1 \setminus \{0,\infty\})}$ and the trivialization induced by

$$\tau_1 : D \otimes \mathcal{O}_{(1:1) \times (\mathbb{P}^1 \setminus \{0,\infty\})} \to D \otimes \mathcal{O}_{(1:1) \times (\mathbb{P}^1 \setminus \{0,\infty\})}.$$

Let us compute $\tau_1^{-1} \circ \tau_2$ on the fibre at a point $(y_0 : t_0),(z_0 : h_0) := g$. On the fibre of τ_1, τ_2 at g we have the following morphisms:

$$D \xrightarrow[(\tau_1)_{|g}]{} V \oplus V \oplus D \xleftarrow[(\tau_2)_{|g}]{} D$$

They induce isomorphisms:

$$D \xrightarrow[(\tau_1)_{|g}]{\qquad\qquad} \mathrm{Ker(b)} \,/\, \mathrm{Im(a)}_{|g} \xleftarrow[(\tau_2)_{|g}]{\qquad\qquad} D$$

For a vector $w \in D$ we want to find $\tau_1^{-1} \circ \tau_2(w)$ i.e. a vector $\tilde{w} \in D$ such that $\tau_2(w) - \tau_1(\tilde{w}) \in$ Im(a). It means that there exists a vector $u \in V$ such that $\tau_2(w) - \tau_1(\tilde{w}) = a(u)$. It gives us the system of equations:

$$\begin{cases} (z - h\bar{x})^{-1} p(\tilde{w}) = tx(u) - yu \\ (y - tx)^{-1} p(w) = h\bar{x}(u) - zu \\ w - \tilde{w} = thq(u) \end{cases} \Rightarrow \tag{5}$$

$$\begin{cases} u = (h\bar{x} - z)^{-1}(y - tx)^{-1} p(w) \\ \tilde{w} = w - thq(u) \end{cases} \tag{6}$$

Hence $\tilde{w} = w - thq(h\bar{x} - z)^{-1}(y - tx)^{-1} p(w)$.

So

$$(\tau_1^{-1} \circ \tau_2)_{|(1:1) \times (\mathbb{P}^1 \setminus \{0_2, \infty_2\})} = (1 + q(\bar{x} - z)^{-1}(x - 1)^{-1} p).$$

Thus we obtained the following point in $\mathcal{G}_{GL_m}$: $1 + z^{-1} \sum\limits_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p$. It remains to multiply it by $z^{-w_0(\lambda)}$. We have calculated $\eta(E_{(x,\bar{x},p,q)})$. Thus

$$\eta \circ \Theta(x_i, \bar{x}_i, p_i, q_i) = z^{-w_0(\lambda)}(1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p).$$

□

Note that from (5) it also follows $w = \tilde{w} + thq(y - tx)^{-1}(h\bar{x} - z)^{-1} p(w)$. So

$$(1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p)^{-1} = (1 - z^{-1} \sum_{n,l=0}^{\infty} z^{-n} q x^l \bar{x}^n p). \tag{7}$$

5. Proof of Theorem 2.2

5.1. **Lemma.** The isomorphism $\psi \circ \phi$ restricted to $\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ induces an isomorphism between $\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ and $(L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)})$ and is given by the formula:

$$(x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-w_0\lambda}(1 + z^{-1}\sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p) \tag{8}$$

where $(x, \bar{x}, p, q) := (\oplus x_i, \oplus \bar{x}_i, \oplus p_i, \oplus q_i)$.

Proof. In Subsection 3 we proved that the map

$$\eta \circ \Theta : (x_i, \bar{x}_i, p_i, q_i) \mapsto z^{-w_0\lambda}(1 + z^{-1}\sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p)$$

is an isomorphism between $\mathfrak{M}_{\mathbf{0}}^{reg}(v,d)$ and $(L^{<0}GL_m \cdot z^{-w_0(\lambda)} \cap L^{\geq 0}GL_m \cdot z^{-w_0(\mu)})$. Let us think of $\mathcal{G}_{GL_m}$ as of the moduli space of lattices $L \subset D(K)$. Then the above isomorphism sends $(x_i, \bar{x}_i, p_i, q_i)$ to the lattice

$$L := z^{-w_0\lambda}(1 + z^{-1}\sum_{n,l=0}^{\infty} z^{-n} q \bar{x}^n x^l p)(L_0) = (1 + z^{-1}\sum_{n,l=0}^{\infty} z^{-l} q \bar{x}^n x^l p)(L_b),$$

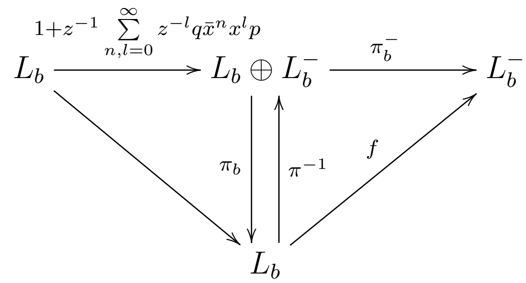
where L_0 is the standard lattice $D(O)$, b is a permutation of λ and L_b is the lattice corresponding to $z^{-w_0(\lambda)}$ (see also [MV1, Subsection 4.4.1]) According to [MV1, Subsection 4.4] this lattice L is uniquely determined by a $\mathbb{C}$-linear map $f : L_b/L_0 \to L_b^-$ (L_b^- is a subspace in $D(K)$ spanned by $\{z^{-l}e_j^i | i \in I, 1 \leq j \leq d_i, l > i\}$ see [MV1, Subsection 4.4.3]) and f is uniquely determined by $f_1 : L_b/L_0 \to U_b$ (where U_b is spanned by $\{z^{-i}e_j^i | i \in I, 1 \leq j \leq d_i\}$ it is the same as V_b in [MV1, Subsection 4.4.4]). Let us compute f_1. (Note that f_1 is nothing but $\psi^{-1}(L) - \phi(0)$ (see [MV1, Subsection 4.4.11] and [MV2, Subsection 3.3]) thus we only have to prove that $f_1 = \phi(x, \bar{x}, p, q) - \phi(0)$)

Recall the definition of f. Denote the projection of $L_b \oplus L_b^-$ to L_b^- along L_b by π_b^-. Note that π_b (see Subsection[MV1, Subsection 4.4]) induces the isomorphism $\pi : L \xrightarrow{\sim} L_b$.

$$f := \pi_b^- \circ \pi^{-1}.$$

We have the following commutative diagram:

(9)



Let $f = \sum\limits_{k=1}^{\infty} z^{-k} f_k$ as in [MV1, Subsection 4.4.4]. For a vector $z^{-h'}e_{j'}$, $e_{j'} \in D_{j'}$, $1 \leq h' \leq j'$,

$\pi^{-1}(z^{-h'}e_{j'}) = z^{-h'}e_{j'} + \sum\limits_{k=1}^{\infty} z^{-k} f_k(z^{-h'}e_{j'}) = z^{-h'}e_{j'} + \sum\limits_{j} z^{-j-1}w_j + \sum\limits_{k=2}^{\infty} z^{-k} f_k(z^{-h'}e_{j'})$ for some $w_j \in D_j$(we want to compute them).

Conjugating (7) by $z^{-w_0(\lambda)}$ we note that the map

$$1 - z^{-1} \sum_{n,l=0}^{\infty} z^{-l} q x^l \bar{x}^n p : L_b \oplus L_b^- \to L_b \oplus L_b^-$$

is inverse to the map

$$1 + z^{-1} \sum_{n,l=0}^{\infty} z^{-l} q \bar{x}^n x^l p : L_b \oplus L_b^- \to L_b \oplus L_b^-.$$

Now we see that the diagram (9) gives us the condition

$$\Xi_{h',j'} := (1 - z^{-1} \sum_{n,l} z^{-l} q x^l \bar{x}^n p)(z^{-h'}e_{j'} + \sum_j z^{-j-1} w_j + \sum_{k=2}^{\infty} z^{-k} f_k(z^{-h'}e_{j'})) \in L_b. \tag{10}$$

Note that we have two gradings on $D(K)$. One is by degree of z and the other comes from the decomposition $D = \oplus D_i$. A straightforward computation shows that $(-j-1, j)$-component of the vector $\Xi_{h',j'}$ is $z^{-j-1}(w_j - q x^{j-h'} \bar{x}^{j'-h'} p(e_{j'}))$ (to prove it we observe that sum of degrees of components of vectors $z^{-k} f_k(z^{-h'}e_{j'})$ is equal to $-k$ thus less than -1 for $k > 1$ while operator $z^{-l-1} q x^l \bar{x}^n p$ shifts a sum of degrees on $-n-1$, from that two observations our claim follows). Using (10) and the fact that vectors of L_b do not have any components of degree $(-j-1, j)$ we see that

$$w_j = q_j x_{j-1} \dots x_{h'} \bar{x}_{h'} \dots \bar{x}_{j'-1} p_{j'}(e_{j'}).$$

It follows directly from the definition of ϕ (see [MV2, Subsection 3.3]) that $\phi(0) + f_1 = \phi(x_i, \bar{x}_i, p_i, q_i)$. So

$$\psi \circ \phi(x_i, \bar{x}_i, p_i, q_i) = L = \eta \circ \Theta(x_i, \bar{x}_i, p_i, q_i).$$

□

5.2. **Proof of Theorem 2.2.**

Proof. According to Lemma 5.1, the morphism $\psi\circ\phi$ restricted to the dense open subvariety $\mathfrak{M}_0^{reg}(v,d)\subset\mathfrak{M}_0(v,d)$ is given by the formula

$$(x_i,\bar{x}_i,p_i,q_i)\mapsto z^{-w_0\lambda}(1+z^{-1}\sum_{n,l=0}^{\infty}z^{-n}q\bar{x}^n x^l p).$$

Now continuity of the map $\psi\circ\phi$ implies Theorem 2.2. □

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