Automorphisms of K3-surfaces

Renat Abugaliev

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Introduction

Before we start speaking about the question, it is necessary to give the definition of the main object of our research.

Definition 1. K3-surface is a projective surface X with $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ and $\Omega_X^2 \simeq \mathcal{O}_X$.

Let us discuss their automorphisms groups. All $K3$-surfaces are holomorphic symplectic varieties. Thus, if G act on X, we can consider natural representation on $\mathrm{H}^0(X, \Omega_X^2) = \mathbb{C}\,\omega$. We denote by G_s the kernel of this action, and by G_n the image of this map. Thus, we have the following exact sequence of groups:

$$1 \to G_\mathrm{s} \to G \to G_\mathrm{n} \to 1 \qquad (1)$$

In our paper we suppose that the exacts sequence (1) splits and that G_n has an element g, whose order is an odd prime number[1] . We consider the quotient variety of X by g. We denote it by Y. We prove that Y is a rational surface with the action of $G/\langle g\rangle$. We study the case where Y is a $G/\langle g\rangle$-*conic bundle* (a $G/\langle g\rangle$- equivariant fibration whose general fiber is $\mathbb{P}^1$).

Theorem 1. *Let X be a $K3$ surface and g be a non-symplectic automorphism of X. We denote the order of g as n. Suppose that n is not a power of 2. Moreover, the quotient variety of X by g is a conic bundle. Then there is an isotrivial elliptic fibration of X whose j-invariant equal to 0. Moreover, 3 divides n.*

We refer to the paper of Justin Sawon [Saw14] to the classification of isotrivial elliptic fibrations of $K3$-surfaces. In addition, we achive the following particular result about such fibrations.

Theorem 2. *Let $\pi : X \to \mathbb{P}^1$ be an isotrivial jacobian elliptic fibration of a $K3$ surface whose j-invariant is 0 and $G_\mathrm{s} = \mathbb{Z}/3\mathbb{Z}$ and G_n be a complex multiplication by $\frac{1+\sqrt{-3}}{2}$. Then all singular fibers of this fibration has type IV or IV^*. All possible configurations of the singular fibers contain in this list: 6 IV, IV+IV^*, 2 IV+2IV^*, 3 IV^*.*

[1] The case of antisymplectic involution was studied in detail by Kristina Frantzen in her paper [Fra11]

Background

In this sectin we recall the background knowledge and of the history of the question. Shafarevich and Piateckii-Shapiro [PS71] proved the famous Torelli theorem on K3-surfaces. It brings to us the following corollary.

Theorem 3 (Shafarevich and Piateckii-Shapiro, 74)**.** *Let X be a K3-surface, then the natural map* $\operatorname{Aut} X \to O(\mathrm{H}^2(X,\mathbb{Z}))$ *is injective.*

See e.g. [Huy15, chapter 15.2] for more details. To apply Theorem 1 let us remind something about the structure of $H^2(X,\mathbb{Z})$.

$$H^2(X,\mathbb{Z}) = E_8(-1)^2 \oplus H^3$$

. Particularly, the period lattice is a principally polarized lattice of rank 22 and signature $(3,19)$. The intersection of the period lattice with the Hodge part of the second cohomology is the Neron-Severi group denoted by $NS(X)$. The transcendental lattice of X is the ortogonal complement of $NS(X)$. We denote it as $T(X)$. The classifcation of all possible G_{n} is well-known.

Proposition 4. *G_{n} is a cyclic group whose order divides* 66*,* 44*,* 42*,* 36*,* 28 *or* 12

Proof. See e.g. [Huy15, chapter 15.1]. □

Clearly, the fixed locus of G_{n} is the disjoint union of k smooth curves and n isolated fixed points. In the paper [AST11] Michela Artebani, Alessandra Sarti and Shingo Taki find all possible n, k and topological structures of the fixed curves.

Theorem 5 (Artebani, Sarti and Taki)**.** *Let X be a K3-surface and σ be a non-symplectic automorphism whose order is a prime number. Then at most one of the fixed curves may have positive genus g. Moreover, the numbers g, n and k can be expressed in terms of the second cohomology lattice of X.*

Now let us remind some results about the symplectic automorphisms groups of $K3$ surfaces.

Vyacheslav Nikulin in his work "Finite groups of automorphisms of Kählerian K3 surfaces" [Nik79] gives the list of all abelian groups that can act symplectically on some $K3$. This result based on the action on the second cohomology lattice.

Theorem 6 (Nikulin, 79)**.** *There are exactly 14 non-trivial finite abelian groups G that can be realized as subgroups of* $\mathrm{Aut}(X)$ *of a complex K3-surface X. Moreover, the induced action on the abstract lattice* $\mathrm{H}^2(X,\mathbb{Z})$ *is unique up to orthogonal transformations. Apart from the cyclic groups* $\mathbb{Z}/\mathrm{n}\mathbb{Z}$*,* $2 \leq n \leq 8$ *the list comprises the following groups:* $\mathbb{Z}/2\mathbb{Z}^2$*,* $\mathbb{Z}/2\mathbb{Z}^3$*,* $\mathbb{Z}/2\mathbb{Z}^4$*,* $\mathbb{Z}/3\mathbb{Z}^2$*;* $\mathbb{Z}/4\mathbb{Z}^4$*,* $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ *and* $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$*.*

Nine years later Mukai Shigeru extended the previous result to the case of non-commutative G. In his paper "Finite groups of automorphisms of K3 surfaces and the Mathieu group" [Muk88], by application of the Niemeier lattices, he classified all finite symplectic groups of automorphisms of K3-surfaces.

Theorem 7 (Mukai, 88)**.** *For a finite group G the following conditions are equivalent:*

(i) *There exists a complex (projective) K3 surface X such that G is isomorphic to a subgroup of* $\mathrm{Aut}(X)$.

(ii) *There exists an monomorphism $G \hookrightarrow M_{23}$ into the Mathieu group M_{23} such that the induced action of G on $\Omega := 1, ..., 24$ has at least five orbits.*

There are 11 maximal subgroups of finite symplectic automrphisms groups acting faithfully of complex K3-surface. The orders of these groups are 48, 72, 120, 168, 192, 288, 360, 384, 960. Some orders appear twice.
The goal of our work is to study the case in which the both parts of G are not trivial, and the last map in the exact sequence (1) is a trivial extension,

$$G = G_{\mathrm{n}} \times G_{\mathrm{s}}.$$

The main idea of our work is to take the quotient of X by an ellement of G_{n} and study how G_{s} can act here. These quotients can be the so-called Enriques surfaces, which were discovered by the Italian mathematician Federigo Enriques. However, we do not need to focus on this case, as all the groups of the automorphisms of these surfaces are classified in [BP83]. Otherwise, X/G_{n} is a rational (possibly singular) surface.

Quotient variety by the non-symplectic part and G-MMP

In this section we consider the quotient of our K3-surface by g, where $g \in G_{\mathrm{n}}$ and order of g is odd. We denote it as Y. We prove that Y is a rational surface and describe the classification of rational surfaces with a group action. Notice that this variety can be singular. Let P be an isolated fixed point G_{n}. The group G_{n} act non-trivialy on the symplectic form. Thus, $G_{\mathrm{n}}|_{T_P} \bigcap SL(T_P) = id$. Hence, the singularities are not canonical. Theory of singularities of surfaces provides us with following fact.

Proposition 8. *Let S be a surface and $P \in S$ be a quotient singularity by a cyclic group G. Supose that $G \bigcap SL(T_P) = id$. Let $\pi \bar{S} \to S$ be the minimal desingularisation of S. Then the class of divisors $E = -\pi^* K_S + K_{\bar{S}}$ contains an effective divisor.*

Proof. See e.g. [Rei] $\square$

Lemma 9. *Let X be a K3 surface and g be a automorphism of X. Suppose that the action of g is free. Then order of g equal to 2.*

Proof. We denote the quotient variety X be g. Since the action of g is free, the natural map from X to Y is étale. Hence,

$$\chi(X, \mathcal{O}_X) = \operatorname{ord} g \cdot \chi(Y, \mathcal{O}_Y)$$

. Because X is a K3, $\chi(X, \mathcal{O}_X) = 2$. Thus, g is an involution. $\square$

Proposition 10. *Let X be a K3-surface and G_{n} be the group of its non-symplectic automorphisms. Suppose that the order of G_{n} is not equal to 2. Then $Y := X/G_{\mathrm{n}}$ is a rational (possibly singular) surface.*

Proof. By the Hurwitz formula for the finite maps we have:

$$K_X = \pi^*(K_Y + B)$$

,

where B is the ramification divisor.

Applying the push forward to the first equality, we get:

$$\pi_*\pi^*(K_Y + B) = \pi_* K_x = 0$$

$$-K_Y = \frac{B}{\operatorname{ord} g}$$

Let $p : \bar{Y} \to Y$ be the minimal resolution of singularities of Y. Then

$$-K_{\bar{Y}} = -p^* K_Y + E$$

By proposition 8 the divisor E equal to 0 if only and if there is no isolated point with non-trivial stabilizer. The support of the divisor B is the set non-isolated fixed points with non-trivial stabilizer. By lemma 9 the action of g is not free, hence the divisor $E + B$ is non-zero effective divisor. As we noticed above

$$-K_{\bar{Y}} = p^* B + E$$

. Thus, $\mathrm{H}^0(\bar{Y}, \mathcal{O}_{\bar{Y}}) \neq 0$. In particular, Y has the negative Kodaira dimension. Since X is a K3, we have $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Hence, $\mathrm{H}^1(\bar{Y}, \mathcal{O}_{\bar{Y}}) = 0$. Thus by Castelnuovo criterion, $\bar{Y}$ is a smooth rational surface. □

Since G_{s} commutes with G_{n} and G_{n} is a cyclic group, each element $g \in G_{\mathrm{n}}$ commutes with all group G. Hence, G/g acts on Y. To classify types of this actions we apply G-minimal model program.

Proposition 11. *All G-minimal rational surfaces has one of the following type:*

- *Y_d is a del Pezzo surface of degree d whose invariant Picard lattice is generated by the canonical bundle.*
- *$\phi : Y \to \mathbb{P}^1$ is a G-minimal conic bundle. The invariant Picard lattice of X has rank 2.*
- *Projective plane. And G is subgroup of $PGL_3(\mathbb{C})$*

Proof. See [DI09]. □

In this paper we consider only the case of a conic bundle.

Elliptic fibrations

In the present section we prove theorem 1
Proof of Theorem 1. The quotient variety of X by g is a conic bundle. Hence, we have the following diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

The map from X to $\mathbb{P}^1$ is a fibration of a $K3$ surface. Hence, it is an elliptic fibration. Let E be a smooth fiber of this elliptic fibration. The curve E/g is rational. Since the order of g is not 2 or 4, some power of g is a complex multiplication of order 3 or 6. Hence, $E = \mathbb{C}\,/\,\mathbb{Z}[\frac{1-\sqrt{-3}}{2}]$. This elliptic curve has j-invariant 0. □

1 Isotrivial elliptic fibrations whith the complex multiplication of order 3

In the last section we study the automorphism group of isotrivial elliptic $K3$ surfaces with complex multiplication of order 3. We classify all such fibrations and describe their automorphisms. First, we prove that the fibration has a section using the involution h from the lemma 7. Since $h \in G_{\mathrm{n}}$ and G_{n} is cyclic group, g and h commute. Hence, their product is an element of order 6. We use it to construct a section of the fibrationπ. Now we can consider only Jacobian fibrations. It allows us to use the Weierstrass equation. We study the automorphisms of isotrivial elliptic fibrations whose j-invariant equals to 0 in the following sections.

Proposition 12. *Let X be a K3-surface and $\pi : X \to \mathbb{P}^1$ be an isotrivial elliptic fibration whose j-invariant equals to 0. Then X can be determined as zeros of cubic equation*

$$y^2 = x^3 + f(t)z^3$$

in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6))$ and $f(t)$ is degree 12 polynomial on the base.

Proof. Consider the Weierstrass equation of X:

$$y^2 = x^3 + g(t)xz^2 + f(t)z^3$$

in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6))$, where $f(t)$ has degree 12 and $g(t)$ has degree 8. See e.g. [Huy15, chapter 11.2]. What we need is to show that g vanishes. The j-invariant of any fiber equals 0. Let us calculate it using $g(t)$ and $f(t)$.

$$0 = j(t) = 1728\frac{4g(t)^3}{4g(t)^3 + 27f(t)^2}$$

Hence, $g = 0$. □

Lemma 13. *Let $\pi : X \to \mathbb{P}^1$ be an isotrivial elliptic fibration whose j-invariant equal to 0. Then it has a complex multiplication of order 6 if and only if π admits a section.*

Proof. Indeed, the set of fixed points of the complex multiplication of order 6 is a section. Conversely, if the fibration is jacobian, we can write its Weierstrass equation (proposition 12). The required complex multiplication derminated by formula

$$(x, y) \mapsto (\frac{1-\sqrt{-3}}{2}x, -y)$$

□

A fiber is singular at a point b if only and if $f(b) = 0$, and depends only on the order of zero at b. We denote the order of zero as m. All fibers with $m \leqslant 5$ are listed in the table below (see [Saw14]). In the last column of this table we describe the group structure of smooth loci of these singular fibers. See e.g. [Huy15, chapter 11.2].

m	Kodaira type	Dynkin diagram	Euler number	Group structure of singular loci
1	II	$\tilde{A}_0$	2	$\mathbb{G}_\mathrm{a}$
2	IV	$\tilde{A}_2$	4	$\mathbb{G}_\mathrm{a} \times \mathbb{Z}/3\mathbb{Z}$
3	I_0^*	$\tilde{D}_4$	6	$\mathbb{G}_\mathrm{a} \times \mathbb{Z}/2\mathbb{Z}^2$
4	IV^*	$\tilde{E}_6$	8	$\mathbb{G}_\mathrm{a} \times \mathbb{Z}/3\mathbb{Z}$
5	II^*	$\tilde{E}_8$	10	$\mathbb{G}_\mathrm{a}$

Remark. Let F be a singular fiber fixed by G_s.The group $G_\mathrm{s} \times G_\mathrm{n}$ acts faithfully on it. Indeed, if kernel of action is non-trivial, it have singular loci of fixed points, but it is impossible.

Proof of Theorem 2. Firstly, note that $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ can act faithfully only on a fiber of type IV and IV^* and symplectic part has 1 and 2 fixed points on each fiber respectively. We prove that G_s act by a translation. Otherwise, the group G_s acts non-trivially on the base of this fibration. Hence, it has two fixed fibers. The group G_s fixes 2 point on base. Hence, the number of fixed point of G_s cannot be greater then 4. However, it is well-know that number of fixed points of a symplectic automorphism of order 3 is equal to 6 (see [Huy15][chapter 11.1]). Thus, we have a contradiction. Since G_s acts trivialy on the base, it act on each fiber. As we noticed in the begining of this proof, it implies that every singular fiber has a type IV or IV^*. Remark that the sum of the Euler characteristics of all singular fiber is equal to 24. □

Example. Consider a surfaces X with three singulars of type IV^*. We can describe this surface in a more geometric way. Let E is elliptic curve with $j(E) = 0$ and ω its complex multiplication of order 3. Consider the quotient variety of $E \times E$ by the group acting diagonally by ω on this abelian surface. Fibration to the E/ω has three singular fiber. Every singular fiber has three singularities of type $A3$. Thus, we obtain X as the desingulariation of this surface.

Suppose that G_s acts trivially on the base of fibration. We acts by the translations. By the table above, the group G_s can be only $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}^2$. The case of $\mathbb{Z}/3\mathbb{Z}$ is described above. The complex multiplication by $\frac{1-\sqrt{-3}}{2}$ does not fix non-zero points of 2-torsion. Hence, in our case G_n and G_s do not commute. However, if we neglect condition of the commutativity, by the table, we get that all singular fibers have type I_0^*. The group of automorphisms of this surface is a non-trivial exterior of $\mathbb{Z}/6\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}^2$.

Example. Let E and ω be as in the previous example and B be arbitrary elliptic curve. Let X be the Kummer surface of $E \times B$ and π is the projection to $B/\{\pm 1\}$. Then X is a K3 with $G_{\mathrm{n}} = \mathbb{Z}/6\mathbb{Z}$ and $G_{\mathrm{s}} = \mathbb{Z}/2\mathbb{Z}^2$, which do not commute.

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