Multidimensional Permanents and an Upper Bound on the Number of Transversals in Latin Squares

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Abstract

The permanent of a multidimensional matrix is the sum of products of entries over all diagonals. A nonnegative matrix whose every 1-dimensional plane sums to 1 is called polystochastic.

A latin square of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in each row and each column. A transversal of such a square is a set of n entries such that no two entries share the same row, column, or symbol. Let $T(n)$ be the maximum number of transversals over all latin squares of order n.

Here we prove that over the set of multidimensional polystochastic matrices of order n the permanent has a local extremum at the uniform matrix for whose every entry is equal to $1/n$. Also, we obtain an asymptotic value of the maximal permanent for a certain set of nonnegative multidimensional matrices. In particular, we get that the maximal permanent of polystochastic matrices is asymptotically equal to the permanent of the uniform matrix, whence as a corollary we have an upper bound on the number of transversals in latin squares

$$T(n) \leq n^n e^{-2n+o(n)}.$$

Keywords: permanent, multidimensional matrix, polystochastic matrix, latin square, transversal.
MSC 05A16; 05B15

Introduction

Let A be a matrix of order n, $A = (a_{i,j})_{i,j=1}^n$. The *permanent* of a matrix A is defined as

$$\mathrm{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma i},$$

where S_n is the symmetric group on a set of n symbols.

A matrix A is called *nonnegative* if $a_{i,j} \geq 0$ for all $i, j \in \{1, \dots, n\}$. Here we consider only nonnegative matrices. A nonnegative matrix A is said to be *doubly stochastic* if $\sum\limits_{j=1}^{n} a_{i,j} = 1$ for all $i \in \{1, \dots, n\}$ and $\sum\limits_{i=1}^{n} a_{i,j} = 1$ for all $j \in \{1, \dots, n\}$. By J_n denote the matrix of order n all of whose entries are $1/n$.

It is not hard to prove [6], [8] that J_n is a local minimum for the permanent among all doubly stochastic matrices. In Section 3 we prove that the uniform matrix is a local extremum for the permanent among all multidimensional polystochastic matrices.

In 1926, van der Waerden conjectured that the uniform matrix J_n has strictly the smallest permanent over all doubly stochastic matrices. The conjecture was proved in 1980 by Egorychev [3] and Falikman [4]. Also, in 1963 Minc conjectured that the permanent of n-ordered (0,1)-matrices is not greater than $\prod\limits_{i=1}^{n} r_i!^{1/r_i}$, where r_i is the number of 1's in the ith row. This conjecture was proved by Bregman [2], Schrijver [10], and Radhakrishnan [9]. For additional information about the permanent of matrices see [8].

Here we extend the set of polystochastic matrices and obtain an asymptotic upper bound on their permanent which depends on their dimension and sum of entries.

The permanent may be generalized to higher-dimensional matrices by different ways. We consider only one of them. Then the number of tilings of some regular graph (the number of partitions into copies of some subgraph) is equal to the permanent of a certain multidimensional nonnegative matrix. In particular, the number of 1-perfect codes is expressed as a multidimensional permanent [1]. Moreover, the number of transversals in a latin square coincides with the permanent of a certain 3-dimensional matrix. Another generalization of the permanent makes it possible to estimate the number of latin hypercubes [5].

A *latin square of order* n is an $n \times n$ array of n symbols, in which each symbol occurs exactly once in each row and each column. A *transversal* is a set of n entries, one selected from each row and each column of a latin square of order n such that no two entries contain the same symbol. Define $T(n)$ to be the maximum number of transversals over all latin squares of order n.

In [7], McKay, McLeod, and Wanless proved that $b_1^n \leq T(n) \leq b_2^n \sqrt{n} n!$ for $n \geq 5$, where $b_1 \approx 1.719$ and $b_2 \approx 0.614$. As $n \to \infty$, the right-hand side of the inequality may be written as $T(n) \leq n^n e^{-cn+o(n)}$, where $c \approx 1.487$. In Section 5 we obtain the asymptotic upper bound on the number of transversals in latin squares

$$T(n) \leq n^n e^{-2n+o(n)}$$

as a corollary from the bound on the permanent of tristochastic matrices.

By using the symbols of a latin square to index its rows and columns, each latin square can be interpreted as the Cayley table of a quasigroup. In [11], Wanless proposes:

Conjecture 1. *Let L_n be the Cayley table of the cyclic group of order n and let n be odd. Denote by z_n*

the number of transversals in L_n. Then

$$\lim_{n\to\infty} \frac{1}{n} \ln(z_n/n!) = -1.$$

Let us remark that Conjecture 1 implies that our upper bound on $T(n)$ is achieved.

For additional information about transversals in latin squares see [11].

1 Definitions

Let $n, d \in \mathbb{N}$, and $I = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in \{1, \ldots, n\}\}$. A *$d$-dimensional matrix A of order n* is an array $(a_\alpha)_{\alpha \in I}$, $a_\alpha \in \mathbb{R}$.

Let $k \in \{0, \ldots, d\}$. A *k-dimensional plane* in A is the set of entries obtained by fixing $d-k$ indices and letting the other k indices vary from 1 to n. A $(d-1)$-dimensional plane is said to be a *hyperplane*. By $L^k(A)$ denote the set of k-dimensional planes in a matrix A.

Let α belong to I. Let $(A|\alpha)$ denote the d-dimensional matrix of order $n-1$ obtained from the matrix A by deleting the entries a_β such that $\alpha_i = \beta_i$ for some $i \in \{1, \ldots, d\}$.

A nonnegative matrix A is said to be *polystochastic* if the sum of its entries in each 1-dimensional plane is equal to 1. In the sequel, 3-dimensional polystochastic matrices are called *tristochastic*.

Denote by w the function that maps a matrix (or a part of a matrix) to the sum of all its entries. The function w is said to be the *norm* of a matrix.

For a d-dimensional matrix A of order n, denote by $D(A)$ the set of its diagonals

$$D(A) = \left\{(a_{\alpha^1}, \ldots, a_{\alpha^n}) \mid a_{\alpha^i} \in A \ \forall i \neq j \ \rho(\alpha^i, \alpha^j) = d\right\},$$

where ρ is the Hamming distance (the number of positions at which the corresponding entries are different). Then the *permanent* of a matrix A is

$$\mathrm{per} A = \sum_{p \in D} \prod_{a_\alpha \in p} a_\alpha.$$

The correspondence between the latin squares of order n and 3-dimensional (0,1)-matrices is given by the next rule: an element of a latin square with coordinates (i, j) equals k iff $a_{i,j,k}$ equals 1. Note that the permanent of the matrix coincides with the number of transversals in the latin square.

For all even n there exists a tristochastic matrix of order n whose permanent vanishes. Indeed, consider the matrix A such that $a_{i,j,k} = 1$ if $i + j \equiv k \mod n$ and $a_{i,j,k} = 0$ otherwise. It can easily be checked that the matrix A is tristochastic. Assume that the permanent of A is nonzero, that is, there exists a diagonal in A composed of 1. Summing $i+j \equiv k \mod n$ over all entries of the diagonal, we find

$$0 \equiv n(n+1) = \sum_{i=1}^{n} i + \sum_{j=1}^{n} j = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \not\equiv 0 \mod n;$$

a contradiction.

Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ 0 \leq \gamma \leq n^{d-2}$. Consider the following set of d-dimensional matrices of order n:

$$M^d_{n,\gamma} = \left\{ (a_\alpha)_{\alpha \in I} \mid a_\alpha \geq 0, \sum_{\alpha \in I} a_\alpha = \gamma n, \forall l \in L^1(A) \sum_{a_\alpha \in l} a_\alpha \leq 1 \right\}.$$

By definition, put

$$P^d_n(\gamma) = \max_{A \in M^d_{n,\gamma}} \mathrm{per} A$$

and

$$\varphi^d_n(\gamma) = \frac{\ln P^d_n(\gamma)}{n} - \ln \gamma + d - 1, \text{ i.e., } P^d_n(\gamma) = \gamma^n e^{-(d-1)n + \varphi^d_n(\gamma) n}.$$

Since $M^d_{n,\gamma}$ is a compact set, $P^d_n(\gamma)$ and $\varphi^d_n(\gamma)$ are well defined. Note that $M^d_{n,n^{d-2}}$ is the set of d-dimensional polystochastic matrices of order n.

Our main aim is to prove that

$$P^d_n(\gamma) = \gamma^n e^{-(d-1)n + o(n)}$$

for $d \geq 3$ and $\gamma = n^{d-3+\delta}$, where $\delta \in (0, 1]$ is arbitrary.

2 Properties of $P^d_n(\gamma)$ and $\varphi^d_n(\gamma)$

Property 1. *$P^d_n(\gamma)$ and $\varphi^d_n(\gamma)$ are continuous because the permanent is continuous and the set $M^d_{n,\gamma}$ changes continuously as γ varies.*

Property 2. *$P^d_n(\gamma)$ and $\varphi^d_n(\gamma)$ have a left derivative for all $\gamma \in (0, n^{d-2}]$.*

The points of differentiability of $\varphi^d_n(\gamma)$ and $P^d_n(\gamma)$ are the same. If there exist several maximizing matrices for fixed $\gamma = \gamma_0$, then $P^d_n(\gamma)$ may be not differentiable at γ_0. For each d-dimensional matrix from $M^d_{n,\gamma}$ we can construct a matrix of the same order and dimension, but with lesser γ and permanent, by decreasing one entry (there are some examples of these matrices in Property 5). The entries of maximizing matrices are continuous piecewise linear functions of γ. This implies that $P^d_n(\gamma)$ has the left derivative equal to a linear combination of the permanents of submatrices.

Below, all derivatives are considered as left derivatives.

Property 3. *Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ 0 \leq \gamma \leq n^{d-2}$. Then $0 \leq \varphi_n^d(\gamma) \leq d-1$.*

Let $A \in M_{n,\gamma}^d$. We project the matrix A to one of its hyperplanes until we get the 2-dimensional matrix $\tilde{A}$, $\tilde{a}_{i,j} = \sum\limits_{\alpha=(i,j,*\ldots*)} a_\alpha$. At each step of the projection the permanent does not decrease. This is an easy generalization of the observation that the permanent of a nonnegative 2-dimensional matrix is not greater than the product of its row sums. Therefore $\mathrm{per}A \leq \mathrm{per}\tilde{A}$. By r_i we denote the sum of entries in the ith row of the matrix $\tilde{A}$. Then

$$\mathrm{per}A \leq \mathrm{per}\tilde{A} \leq \prod_{i=1}^{n} r_i \leq \gamma^n.$$

On the other hand, if $J_{n,\gamma}^d$ is the d-dimensional matrix of order n all of whose entries are equal to γ/n^{d-1}, then by Stirling's formula we have

$$P_n^d(\gamma) \geq \mathrm{per}J_{n,\gamma}^d = \frac{(n!)^{d-1}\gamma^n}{n^{(d-1)n}} > \gamma^n e^{-(d-1)n}.$$

Property 4. *Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ 0 < \gamma < n^{d-2}$, and let $J_{n,\gamma}^d$ be the d-dimensional matrix of order n all of whose entries are γ/n^{d-1}. Then $\mathrm{per}J_{n,\gamma}^d < P_n^d(\gamma)$.*

Proof. Consider ε such that $0 < \varepsilon \leq \frac{n^{d-2}-\gamma}{n^{d-2}}$ and the matrix B with entries

$b_{1,\ldots,1} = b_{2,\ldots,2} = \frac{\gamma}{n^{d-1}} + \varepsilon$,

$b_{1,\ldots,1,2} = b_{2,\ldots,2,1} = \frac{\gamma}{n^{d-1}} - \varepsilon$,

and the other entries equal $\frac{\gamma}{n^{d-1}}$.

It is easy to show that $B \in M_{n,\gamma}^d$. Using the Laplace expansion for the permanent along some hyperplane, we see by direct calculation that $\mathrm{per}J_{n,\gamma}^d < \mathrm{per}B \leq P_n^d(\gamma)$. □

Property 5. *Let n and d be fixed. Then $\varphi_n^d(\gamma)$ is a nonincreasing function.*

Proof. Using the definition of $\varphi_n^d(\gamma)$, we have

$$\frac{d\varphi_n^d(\gamma)}{d\gamma} = \frac{1}{nP_n^d(\gamma)}\frac{dP_n^d(\gamma)}{d\gamma} - \frac{1}{\gamma}.$$

Let $A \in M_{n,\gamma}^d$ be a d-dimensional matrix of order n such that $\mathrm{per}A = P_n^d(\gamma)$, and let ε be nonnegative. Note that the difference between the permanent of the matrix A and the permanent of an arbitrary matrix from $M_{n,\gamma-\varepsilon}^d$ is not less than the difference between $P_n^d(\gamma)$ and $P_n^d(\gamma-\varepsilon)$. Consequently,

$$\frac{dP_n^d(\gamma)}{d\gamma} \leq \frac{\mathrm{per}A - \mathrm{per}A^\varepsilon}{\varepsilon}$$

for all sufficiently small ε and matrices $A^\varepsilon \in M^d_{n,\gamma-\varepsilon}$ sufficiently close to A.

Let us construct appropriate matrices A^ε. Let Γ be a hyperplane of the matrix A with the norm at least γ. Such a hyperplane exists because the mean norm of hyperplanes of the matrix A equals γ. Find a positive element a_β in Γ such that

$$\mathrm{per}(A|\beta) = \min\left\{\mathrm{per}(A|\alpha) \mid a_\alpha \in \Gamma, a_\alpha > 0\right\}.$$

Using the Laplace expansion for the permanent of A along the hyperplane Γ, we obtain

$$\mathrm{per}A = \sum_{a_\alpha \in \Gamma} a_\alpha \mathrm{per}(A|\alpha) \geq \mathrm{per}(A|\beta) \sum_{a_\alpha \in \Gamma} a_\alpha \geq \gamma \mathrm{per}(A|\beta).$$

Hence, $\frac{\mathrm{per}(A|\beta)}{\mathrm{per}A} \leq \frac{1}{\gamma}$.

For sufficiently small $\varepsilon > 0$ consider the set of matrices $A^\varepsilon \in M^d_{n,\gamma-\varepsilon}$ whose entries are equal to the entries of the matrix A except $a^\varepsilon_\beta = a_\beta - \varepsilon n$. Using the definition of the permanent, we get

$$\frac{\mathrm{per}A - \mathrm{per}A^\varepsilon}{\varepsilon} = n\mathrm{per}(A|\beta).$$

This yields

$$\frac{d\varphi^d_n(\gamma)}{d\gamma} = \frac{1}{nP^d_n(\gamma)} \frac{dP^d_n(\gamma)}{d\gamma} - \frac{1}{\gamma} \leq \frac{n\mathrm{per}(A|\beta)}{n\mathrm{per}A} - \frac{1}{\gamma} \leq 0,$$

and $\varphi^d_n(\gamma)$ is a nonincreasing function. □

3 A local extremum of a multidimensional permanent

Now we generalize the theorem about a local extremum of the permanent over the set of doubly stochastic matrices [8]. Let us first prove the following lemma.

Lemma 1. *Let A be a d-dimensional polystochastic matrix of order n. Then the norm of $(A|\alpha)$ equals*

$$\sum_{j=0}^{d-1} (-1)^j \binom{d}{j} n^{d-j-1} + (-1)^d a_\alpha.$$

Proof. We prove the lemma using the inclusion-exclusion principle. Note that the norm of $(A|\alpha)$ is equal to

$$\sum_{j=0\dots d, l^j \in L^j(A): a_\alpha \in l^j} (-1)^{d-j} w(l^j).$$

There are $\binom{d}{j}$ j-dimensional planes containing a_α. Using the condition for the norm of 1-dimensional planes, we get that the norm of these j-dimensional planes is equal to n^{j-1} if $j \neq 0$ and a_α if $j = 0$. Thus

$$w(A|\alpha) = \sum_{j=0}^{d-1} (-1)^j \binom{d}{j} n^{d-j-1} + (-1)^d a_\alpha.$$

□

Let J_n^d be the d-dimensional matrix of order n all of whose entries are equal to $1/n$.

Theorem 1. *The matrix J_n^d is a local extremum for the permanent among all d-dimensional polystochastic matrices of order n. In addition, if d is even, then J_n^d is a local minimum, and if d is odd, then J_n^d is a local maximum.*

Proof. Over the set of d-dimensional polystochastic matrices the permanent has a second derivative at the point J_n^d. Consequently, it suffices to prove that J_n^d is a local extremum in any direction. Consider a d-dimensional polystochastic matrix A of order n that is approximate to J_n^d. Let $a_\alpha = \min_{\beta \in I} a_\beta$, $a_\alpha > 0$ and $\theta_0 = 1 - n a_\alpha > 0$. Then

$$B = \frac{1}{\theta_0}(A - (1 - \theta_0) J_n^d)$$

is a polystochastic matrix containing a null entry, and $A = \theta_0 B + (1 - \theta_0) J_n^d$. By definition, put

$$Lin(A) = \left\{ S | S = \theta B + (1 - \theta) J_n^d, 0 \leq \theta \leq 1 \right\}.$$

$Lin(A)$ is a linear subset of the set of polystochastic matrices, and the matrices A and J_n^d belong to $Lin(A)$.

Consider the function

$$f(\theta) = \mathrm{per}(\theta B + (1 - \theta) J_n^d), \ \theta \in [0, 1].$$

Notice that $\mathrm{per}A = f(\theta_0)$. The function $f(\theta)$ is infinitely differentiable, and in a neighborhood of zero $f(\theta)$ can be written as

$$\mathrm{per}A = f(\theta_0) = f(0) + \theta_0 f'(0) + \frac{\theta_0^2}{2} f''(0) + O(\theta_0^3).$$

Using the definition of the permanent, we have

$$f'(\theta) = \sum_{\alpha \in I} (b_\alpha - 1/n) \mathrm{per}(\theta B + (1 - \theta) J_n^d | \alpha).$$

Consequently,

$$f'(0) = \sum_{\alpha \in I} (b_\alpha - 1/n) \mathrm{per}(J_n^d|\alpha) = \frac{(n-1)!^{d-1}}{n^{n-1}} \sum_{\alpha \in I} (b_\alpha - 1/n) = 0,$$

because the norms of B and J_n^d are the same and $\mathrm{per}(J_n^d|\alpha) = \frac{(n-1)!^{d-1}}{n^{n-1}}$ is independent of α.

Find the second derivative of $f(\theta)$:

$$f''(\theta) = \sum_{\alpha \in I} (b_\alpha - 1/n) \sum_{\beta \in I: \alpha_i \neq \beta_i \forall i} (b_\beta - 1/n) \mathrm{per}(\theta B + (1-\theta) J_n^d | \alpha, \beta).$$

Therefore,

$$f''(0) = \sum_{\alpha \in I} (b_\alpha - 1/n) \sum_{\beta \in I: \alpha_i \neq \beta_i \forall i} (b_\beta - 1/n) \mathrm{per}(J_n^d | \alpha, \beta).$$

The permanent of $(J_n^d|\alpha, \beta)$ does not depend on α and β and equals $\frac{(n-2)!^{d-1}}{n^{n-2}}$.

Note that $\sum\limits_{\beta \in I: \alpha_i \neq \beta_i \forall i} (b_\beta - 1/n)$ is equal to the difference between the norms of $(B|\alpha)$ and $(J_n^d|\alpha)$. Applying Lemma 1 to $(B|\alpha)$ and $(J_n^d|\alpha)$, we get

$$f''(0) = \frac{(n-2)!^{d-1}}{n^{n-2}} (-1)^d \sum_{\alpha \in I} (b_\alpha - 1/n)^2.$$

Recall that there exists a null entry in the matrix B. Consequently, $\sum\limits_{\alpha \in I} (b_\alpha - 1/n)^2 > 0$. Finally, $f''(0) > 0$ if d is even, and $f''(0) < 0$ if d is odd. This completes the proof. □

4 A differential inequality for $P_n^d(\gamma)$

In this section we obtain a differential inequality for $P_n^d(\gamma)$. The inequality will be used to prove the upper bound on $P_n^d(\gamma)$. First we demonstrate the following two lemmas.

Lemma 2. *Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ 1 \leq \gamma \leq n^{d-2}$, and let $A \in M_{n,\gamma}^d$ be a matrix whose permanent is nonzero. Then there exists an element $a_\alpha > 0$ such that*

$$w(A|\alpha) \leq \gamma \left(n - d + \binom{d}{2} \frac{n^{d-3}}{\gamma} \right).$$

Proof. Since the permanent of A is nonzero, there exists a diagonal $\{a_{\alpha^i}\}_{i=1}^n$ with all $a_{\alpha^i} > 0$.

Denote by $S_i = \gamma n - w(A|\alpha^i) = w(A) - w(A|\alpha^i)$ the norm of the 'shell' of $(A|\alpha^i)$. Consider $\sum\limits_{i=1}^n S_i$. Since the elements $\{a_{\alpha^i}\}_{i=1}^n$ form a diagonal, every element of A occurs d times in this sum except the elements in $(d-2)$-dimensional planes that contain some a_{α^i}. The norm of every $(d-2)$-dimensional plane is not greater than n^{d-3}. Then

$$\sum_{i=1}^{n} S_i \geq dw(A) - \binom{d}{2} n^{d-2} = d\gamma n - \binom{d}{2} n^{d-2}.$$

Thus the mean norm of the 'shells' equals

$$\frac{\sum_{i=1}^{n} S_i}{n} \geq d\gamma - \binom{d}{2} n^{d-3}.$$

Therefore there exists $j \in \{1, \dots, n\}$ such that $a_{\alpha^j} > 0$ and $S_j \geq d\gamma - \binom{d}{2} n^{d-3}$. Then

$$w(A|\alpha^j) = w(A) - S_j \leq \gamma \left(n - d + \binom{d}{2} \frac{n^{d-3}}{\gamma} \right).$$

□

The following statement binds the permanent of a matrix to the permanent of a matrix with smaller value of order and γ.

Lemma 3. *Let $n, d \in \mathbb{N}$, $d \geq 3$, $\gamma \in \mathbb{R}$, $dn^{d-3} \leq \gamma \leq n^{d-2}$, and let $A \in M^d_{n,\gamma}$ be a matrix with nonzero permanent. Then for all sufficiently small $\varepsilon > 0$ there exist a matrix $A^\varepsilon \in M^d_{n,\gamma-\varepsilon}$ and a matrix $B \in M^d_{n-1,\gamma_B}$, where $\gamma - \frac{dn^{d-2} - \gamma}{n-1} \leq \gamma_B \leq \gamma \left(1 - \frac{d-1-\binom{d}{2} n^{d-3}/\gamma}{n-1} \right)$, such that*

$$\mathrm{per} A = \mathrm{per} A^\varepsilon + \varepsilon n \mathrm{per} B.$$

Proof. By Lemma 2, the matrix A contains an element $a_\alpha > 0$ such that $w(A|\alpha) \leq \gamma(n - d + \binom{d}{2} \frac{n^{d-3}}{\gamma})$.

Suppose that $B = (A|\alpha)$, $B \in M^d_{n-1,\gamma_B}$. Decreasing the element a_α by εn, we get the matrix A^ε. For sufficiently small $\varepsilon > 0$, the matrices A^ε belong to $M^d_{n,\gamma-\varepsilon}$. By direct calculation and the definition of the permanent, we obtain

$$\mathrm{per} A = \mathrm{per} A^\varepsilon + \varepsilon n \mathrm{per} B.$$

Using Lemma 2, we have

$$\gamma_B = \frac{w(B)}{n-1} \leq \gamma \frac{n - d + \binom{d}{2} n^{d-3}/\gamma}{n-1} = \gamma \left(1 - \frac{d - 1 - \binom{d}{2} n^{d-3}/\gamma}{n-1} \right).$$

On the other hand, $w(B) \geq \gamma n - dn^{d-2}$, because the norm of each of d hyperplanes bordering B is not greater than n^{d-2}. Thus

$$\gamma_B \geq \frac{\gamma n - dn^{d-2}}{n-1} \geq \gamma - \frac{dn^{d-2} - \gamma}{n-1},$$

and the proof is complete. □

Finally, let us prove a differential inequality for $P_n^d(\gamma)$.

Statement 1. *Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ dn^{d-3} \leq \gamma \leq n^{d-2}$. Then*

$$\frac{dP_n^d(\gamma)}{d\gamma} \leq nP_{n-1}^d(\tilde{\gamma})$$

for some $\tilde{\gamma}$ from $\left[\gamma - \frac{dn^{d-2}-\gamma}{n-1}, \gamma\left(1 - \frac{d-1-\binom{d}{2}n^{d-3}/\gamma}{n-1}\right)\right]$.

Proof. Let $A \in M_{n,\gamma}^d$ have the maximal permanent over $M_{n,\gamma}^d$. Then $\mathrm{per}A > 0$. By Lemma 3, there exist a set of matrices $A^\varepsilon \in M_{n,\gamma-\varepsilon}^d$ and a matrix $B \in M_{n-1,\gamma_B}^d$ such that

$$\gamma - \frac{dn^{d-2} - \gamma}{n-1} \leq \gamma_B \leq \gamma\left(1 - \frac{d - 1 - \binom{d}{2}n^{d-3}/\gamma}{n-1}\right)$$

and

$$P_n^d(\gamma) = \mathrm{per}A = \mathrm{per}A^\varepsilon + \varepsilon n \mathrm{per}B.$$

Maximizing the right-hand side of the equality, we get

$$P_n^d(\gamma) \leq \max_{A^\varepsilon \in M_{n,\gamma-\varepsilon}^d} \mathrm{per}A^\varepsilon + \varepsilon n \max_{B \in M_{n-1,\gamma_B}^d} \mathrm{per}B = P_n^d(\gamma - \varepsilon) + \varepsilon n P_{n-1}^d(\gamma_B).$$

Suppose that $\tilde{\gamma} = \gamma_B$. Dividing the inequality by ε and letting $\varepsilon \to 0$, we obtain

$$\frac{dP_n^d(\gamma)}{d\gamma} \leq nP_{n-1}^d(\tilde{\gamma}).$$

□

Corollary 1. *Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ dn^{d-3} \leq \gamma \leq n^{d-2}$. Consider sufficiently large n and assume that $n^{d-3} = o(\gamma)$. Then*

$$\frac{dP_n^d(\gamma)}{d\gamma} \leq nP_{n-1}^d(\tilde{\gamma})$$

for some $\tilde{\gamma}$ from $\left[\gamma - dn^{d-3}, \gamma\left(1 - \frac{d-1}{n-1} + o(1/n)\right)\right]$.

5 An asymptotic upper bound on a multidimensional permanent

In this section we prove that for all $\delta \in (0,1]$ and $\gamma = n^{d-3+\delta}$

$$P_n^d(\gamma) = \gamma^n e^{-(d-1)n+o(n)} \text{ as } n \to \infty,$$

whence an asymptotic upper bound on the number of transversals in latin squares follows

$$T(n) \leq P_n^3(n) = n^n e^{-2n+o(n)}.$$

Let $\delta, \varepsilon \in (0,1]$. Consider the function

$$F_n^d(\gamma) = \gamma^n e^{-n\left(d-2+\frac{\ln(\gamma/n^{d-3})}{\delta \ln n}\right)(1-\varepsilon)}. \tag{1}$$

Now we define two functions which will be used as boundaries of some intervals. Suppose that $0 < \sigma \leq \delta$ and $C \geq 1$ are some constants. Put $g_2(n,C) = C \ln n$ for $d=3$, and $g_2(n,C) = Cn^{d-3+\sigma}$ for $d > 3$. Let $g_1(n) = g_2(n,1) - dn^{d-3}$. Note that

$$\frac{n^{d-3}}{g_1(n)} \to 0 \text{ as } n \to \infty. \tag{2}$$

The main idea of the proof is to show that $P_n^d(\gamma)$ is majorized by $F_n^d(\gamma)$ for all sufficiently large n and $\gamma \in \Lambda(n) = \left[g_1(n), n^{d-3+\delta}\right]$. For this purpose, we prove that if n is sufficiently large and if the function $F_{n-1}^d(\gamma)$ majorizes $P_{n-1}^d(\gamma)$ for all γ from some interval $\Delta(n-1)$, then $F_n^d(\gamma)$ majorizes $P_n^d(\gamma)$ for all γ from some interval $\Theta(n) \supset \Delta(n)$.

Statement 2. *Let $n, d \in \mathbb{N}$, $d \geq 3$, $\delta, \varepsilon \in (0,1]$, n be sufficiently large, and let the function $F_n^d(\gamma)$ be defined by (1).*

Assume that $F_{n-1}^d(\gamma) \geq P_{n-1}^d(\gamma)$ for all $\gamma \in \Delta(n-1) = [g_1(n-1), g_2(n-1,C)]$, and $F_n^d(\gamma) \geq P_n^d(\gamma)$ for all $\gamma \in \Delta_0(n) = [g_1(n), g_2(n,1)]$. Then

$$F_n^d(\gamma) \geq P_n^d(\gamma)$$

for all $\gamma \in \Theta(n) = \left[g_1(n), g_2(n,C)\frac{n-1}{n-2}\right]$.

Proof. Taking into account the condition (2) for $g_1(n)$, the definition of $\Theta(n)$, and Corollary 1, we have

$\frac{dP^d_n(\gamma)}{d\gamma} \leq nP^d_{n-1}(\tilde{\gamma})$ for all $\gamma \in \Theta(n)$ and for some

$$\tilde{\gamma} \in \left[\gamma - dn^{d-3}, \gamma\left(1 - \frac{d-1}{n-1} + o(1/n)\right)\right].$$

Note also that there exist $N_0 \in \mathbb{N}$ and $\mu \in \mathbb{R},\ \mu > 0$ such that for $n > N_0$ and for all $\gamma \in \Theta(n)$ the following inequalities hold:

$$1 - \frac{d-1}{n-1} + o(1/n) \geq 1 - \frac{\mu}{n}$$

and

$$\left(1 - \frac{d-1}{n-1} + o(1/n)\right)^{n-1} \leq e^{-(d-1)+(d-2)\varepsilon}.$$

Let us prove that

$$\frac{dF^d_n(\gamma)}{d\gamma} \geq nF^d_{n-1}(\tilde{\gamma}) \tag{3}$$

for all $\gamma \in \Theta(n)$.

Indeed, note that $F^d_{n-1}(\gamma)$ is an increasing function for all sufficiently large n. Using these inequalities and the definition of $F^d_{n-1}\left(\gamma\left(1 - \frac{d-1}{n-1} + o(1/n)\right)\right)$, we have

$$F^d_{n-1}(\tilde{\gamma}) \leq F^d_{n-1}\left(\gamma\left(1 - \frac{d-1}{n-1} + o(1/n)\right)\right) \leq \gamma^{n-1}e^{-(d-1)+(d-2)\varepsilon-(n-1)\left(d-2+\frac{\ln(\chi(1-\mu/n))}{\delta\ln(n-1)}\right)(1-\varepsilon)},$$

where $\chi = \gamma/n^{d-3}$.

Consequently, to obtain the inequality (3) it is sufficient to prove that

$$\frac{dF^d_n(\gamma)}{d\gamma} \geq n\gamma^{n-1}e^{-(d-1)+(d-2)\varepsilon-(n-1)\left(d-2+\frac{\ln(\chi(1-\mu/n))}{\delta\ln(n-1)}\right)(1-\varepsilon)} \tag{4}$$

for some $\mu > 0$ and for all $\gamma \in \Theta(n)$.

Let us prove the inequality (4) now. It can be rewritten as:

$$n\gamma^{n-1}e^{-n\left(d-2+\frac{\ln(\gamma/n^{d-3})}{\delta\ln n}\right)(1-\varepsilon)}\left(1 - \frac{1-\varepsilon}{\delta\ln n}\right) \geq n\gamma^{n-1}e^{-(d-1)+(d-2)\varepsilon-(n-1)\left(d-2+\frac{\ln(\chi(1-\mu/n))}{\delta\ln(n-1)}\right)(1-\varepsilon)}. \tag{5}$$

Reducing (5) by the factor $n\gamma^{n-1}e^{-(d-2)n(1-\varepsilon)}$, we rewrite (5) as follows:

$$1 - \frac{1-\varepsilon}{\delta\ln n} \geq e^{-1+(1-\varepsilon)\kappa_n(\gamma)},$$

where

$$\kappa_n(\gamma) = n\left(\frac{\ln\chi}{\delta\ln n} - \left(1-\frac{1}{n}\right)\frac{\ln(\chi(1-\mu/n))}{\delta\ln(n-1)}\right).$$

Using the Taylor series of the natural logarithm and ignoring sufficiently small summands, we obtain that

$$\kappa_n(\gamma) \leq 1+\varepsilon$$

for all large enough n and $\gamma \in \Theta(n)$.

Since ε is nonzero, there exists $N \in \mathbb{N}$ such that for all $n > N$ and $\gamma \in \Theta(n)$

$$1 - \frac{1-\varepsilon}{\delta\ln n} \geq e^{-\varepsilon^2} \geq e^{-1+(1-\varepsilon)\kappa_n(\gamma)},$$

whence the inequality (3) follows.

Since for all $\gamma \in \Theta(n)\setminus\Delta_0(n)$ we have that $\gamma \leq \frac{n-1}{n-2}g_2(n,C)$ and $\tilde{\gamma} \leq \gamma\left(1-\frac{d-1}{n-1}+o(1/n)\right)$, it can be checked that $\tilde{\gamma} \leq g_2(n-1,C)$ for sufficiently large n. Also, by the definitions of g_1 and g_2, we obtain that $g_1(n-1) < g_2(n,1) - dn^{d-3} \leq \tilde{\gamma}$. Consequently, $\tilde{\gamma}$ belongs to $\Delta(n-1)$ for all $\gamma \in \Theta(n)\setminus\Delta_0(n)$.

Recall that $F^d_{n-1}(\gamma) \geq P^d_{n-1}(\gamma)$ for all $\gamma \in \Delta(n-1)$. It follows that

$$\frac{dF^d_n(\gamma)}{d\gamma} \geq nF^d_{n-1}(\tilde{\gamma}) \geq nP^d_{n-1}(\tilde{\gamma}) \geq \frac{dP^d_n(\gamma)}{d\gamma}$$

for all $\gamma \in \Theta(n)\setminus\Delta_0(n)$.

Since $\frac{dF^d_n(\gamma)}{d\gamma} \geq \frac{dP^d_n(\gamma)}{d\gamma}$ for all $\gamma \in \Theta(n)\setminus\Delta_0(n)$, and $F^d_n(\gamma) \geq P^d_n(\gamma)$ for all $\gamma \in \Delta_0(n)$, we obtain

$$F^d_n(\gamma) \geq P^d_n(\gamma)$$

for all $\gamma \in \Theta(n)$. □

Statement 3. *Let $n, d \in \mathbb{N},\ d \geq 3,\ \gamma \in \mathbb{R},\ \delta, \varepsilon \in (0,1]$, and let the function $F^d_n(\gamma)$ be defined by (1). Let us fix sufficiently large k.*

Assume that $F^d_n(\gamma) \geq P^d_n(\gamma)$ for all $\gamma \in \Delta_0(n) = [g_1(n), g_2(n,1)]$ and for all $n \geq k$. Then there exists m such that for all $n \geq k+m$ and $\gamma \in \Lambda(n) = \left[g_1(n), n^{d-3+\delta}\right]$ we have

$$F^d_n(\gamma) \geq P^d_n(\gamma).$$

Proof. By Statement 2, we have that $F^d_{k+1}(\gamma) \geq P^d_{k+1}(\gamma)$ for all γ from

$$\Delta_1(k+1) = \left[g_1(k+1), g_2(k+1,1)\frac{k}{k-1}\right] = \left[g_1(k+1), g_2\left(k+1, \frac{k}{k-1}\right)\right].$$

Then we apply Statement 2 to the interval $\Delta_1(k+1)$ and obtain the same inequality for all γ from the interval $\Delta_2(k+2) = \left[g_1(k+2), g_2\left(k+2, \frac{k+1}{k-1}\right)\right]$, and so on. After m steps we obtain that $F^d_{k+m}(\gamma) \geq P^d_{k+m}(\gamma)$ for all γ from $\Delta_m(k+m) = \left[g_1(k+m), g_2\left(k+m, \frac{k+m-1}{k-1}\right)\right]$.

In case $d > 3$ find m such that $\frac{k+m-1}{k-1} \geq (k+m)^{\delta-\sigma}$, and in case $d = 3$ find m such that $\frac{k+m-1}{k-1} \geq \frac{(k+m)^\delta}{\ln(k+m)}$. Then $g_2\left(k+m, \frac{k+m-1}{k-1}\right) \geq (k+m)^{d-3+\delta}$, and we have that

$$F^d_n(\gamma) \geq P^d_n(\gamma)$$

for all $n \geq k+m$ and for all $\gamma \in \Lambda(n)$. □

To obtain now the main result it is sufficient to prove that $F^d_n(\gamma) \geq P^d_n(\gamma)$ for all $\gamma \in [g_1(n), g_2(n,1)]$ and for all sufficiently large n. For this purpose we use some extension of the following statement:

Statement 4. *Let A be a 2-dimensional (0,1)-matrix of order n and let γ_i be the number of 1's in the ith row. Then*

$$\mathrm{per}A \leq \prod_{i=1}^{n} \gamma_i!^{1/\gamma_i}.$$

Corollary 2. *Let A be a nonnegative 2-dimensional matrix of order n whose entries are not greater than 1. Suppose that $\sum\limits_{i,j=1}^{n} a_{i,j} = \gamma n$. Then*

$$\mathrm{per}A \leq (\gamma+1)^n e^{-n} (e\sqrt{\gamma+1})^{\frac{n}{\gamma+1}}.$$

Proof. Let v_i be the ith row of A and let $\gamma_i = w(v_i)$. Construct recursively nonnegative 2-dimensional matrices $A = A^0, A^1, \ldots, A^n$ such that their entries are not greater than 1 and $\mathrm{per}A^i \leq \mathrm{per}A^{i+1}$ for all $i \in \{0, \ldots, n-1\}$.

Assume that the matrix A^i is constructed. Let us construct A^{i+1}. Rearrange the columns of the matrix A^i so that $\mathrm{per}(A^i|(i+1,k)) \geq \mathrm{per}(A^i|(i+1,k+1))$ for all k. Call the resulting matrix B^i. Let $A^{i+1} = (a^{i+1}_{j,k})^n_{j,k=1}$ and $B^i = (b^i_{j,k})^n_{j,k=1}$. Put

$a^{i+1}_{j,k} = b^i_{j,k}$ for $j \neq i+1$,

$a^{i+1}_{i+1,k} = 1$ for $k \leq \lceil \gamma_{i+1} \rceil$,

and $a^{i+1}_{i+1,k} = 0$ for $k > \lceil \gamma_{i+1} \rceil$.

Then A^n is a (0,1)-matrix with $\lceil\gamma_i\rceil$ ones in the ith row and $\sum\limits_{i=1}^{n}\lceil\gamma_i\rceil \le \sum\limits_{i=1}^{n}(\gamma_i+1) = (\gamma+1)n$. By construction and by Statement 4, we have

$$\mathrm{per}A \le \mathrm{per}A^n \le \prod_{i=1}^{n}\lceil\gamma_i\rceil!^{\frac{1}{\lceil\gamma_i\rceil}}.$$

Using the approximation of a factorial

$$x! \le e x^{x+1/2}e^{-x},$$

we obtain

$$\mathrm{per}A \le \prod_{i=1}^{n} e^{-1+1/\lceil\gamma_i\rceil}\lceil\gamma_i\rceil^{1+\frac{1}{2\lceil\gamma_i\rceil}}.$$

It can be proved that $e^{1/x}x^{1+1/2x}$ is a concave function for $x>1$. Therefore,

$$\mathrm{per}A \le \prod_{i=1}^{n} e^{-1+\frac{1}{\gamma+1}}(\gamma+1)^{1+\frac{1}{2(\gamma+1)}} = (\gamma+1)^n e^{-n}(e\sqrt{\gamma+1})^{\frac{n}{\gamma+1}}.$$

□

We are now ready to prove the main theorem.

Theorem 2. *Let $d\ge 3$. For all $\delta\in(0,1]$ and $\gamma = n^{d-3+\delta}$, the maximal permanent of the matrices from the set $M^d_{n,\gamma}$ is equal to $\gamma^n e^{-(d-1)n+o(n)}$ as $n\to\infty$:*

$$P^d_n(\gamma) = \gamma^n e^{-(d-1)n+o(n)}.$$

Proof. Proceed by induction on the dimension of matrices.

Basis: $d=3$.

For arbitrary $\delta,\varepsilon\in(0,1]$ and $\gamma\in\Lambda(n) = \left[\ln n - 3, n^{\delta}\right]$, consider the function

$$F^3_n(\gamma) = \gamma^n e^{-n\left(1+\frac{\ln\gamma}{\delta\ln n}\right)(1-\varepsilon)}.$$

Let $\gamma\in\Delta_0(n) = [\ln n - 3, \ln n]$ and let A be a matrix from $M^3_{n,\gamma}$ such that $\mathrm{per}A = P^3_n(\gamma)$. Denote by $\tilde{A}$ the projection of A on one of its hyperplanes. Recall that $\mathrm{per}A \le \mathrm{per}\tilde{A}$. By Corollary 2, we have

$$P^3_n(\gamma) = \mathrm{per}A \le \mathrm{per}\tilde{A} \le (\gamma+1)^n e^{-n}(e\sqrt{\gamma+1})^{\frac{n}{\gamma+1}}.$$

Since ε is nonzero, there exists N such that for all $n > N$ and $\gamma \in \Delta_0(n)$

$$F_n^3(\gamma) = \gamma^n e^{-n\left(1+\frac{\ln\gamma}{\delta\ln n}\right)(1-\varepsilon)} \geq (\gamma+1)^n e^{-n}(e\sqrt{\gamma+1})^{\frac{n}{(\gamma+1)}} \geq P_n^3(\gamma).$$

Therefore, $F_n^3(\gamma) \geq P_n^3(\gamma)$ for all sufficiently large n and $\gamma \in \Delta_0(n)$. Using Statement 3, we get this inequality for all $n \geq N(\varepsilon)$ and for all $\gamma \in \Lambda(n)$.

Put $\gamma = n^\delta$. Then

$$P_n^3(\gamma) \leq F_n^3(\gamma) = \gamma^n e^{-2n(1-\varepsilon)}$$

starting from a certain $N(\varepsilon)$. Since ε can be chosen arbitrarily close to zero, it follows that

$$P_n^3(\gamma) \leq \gamma^n e^{-2n+o(n)} \text{ as } n \to \infty.$$

On the other hand, by the proof of Property 3

$$P_n^3(\gamma) \geq \gamma^n e^{-2n+o(n)}.$$

Finally, for all $\delta \in (0,1]$ and $\gamma = n^\delta$

$$P_n^3(\gamma) = \gamma^n e^{-2n+o(n)} \text{ as } n \to \infty.$$

Inductive step: Assume that for all $\delta \in (0,1]$ and $\gamma = n^{d-4+\delta}$ the function $P_n^{d-1}(\gamma)$ is equal to $\gamma^n e^{-(d-2)n+o(n)}$ as $n \to \infty$. Let us prove an analogous statement for $P_n^d(\gamma)$.

As before, we fix arbitrary $\delta \in (0,1]$ and $0 < \varepsilon < \frac{1}{d-2}$. Suppose that $g_2(n) = n^{d-3+\sigma}$, where $\sigma = \frac{\varepsilon\delta(d-3)}{1-\varepsilon} < \delta$.

Let $\gamma \in \Delta_0(n) = [g_1(n), n^{d-3+\sigma}]$ and let A be a matrix from $M_{n,\gamma}^d$ such that $\mathrm{per}A = P_n^d(\gamma)$. Project the matrix A on one of its hyperplanes and divide the result by n. We obtain the $(d-1)$-dimensional matrix $\tilde{A} \in M_{n,\bar{\gamma}}^{d-1}$, where $\bar{\gamma} = \gamma/n$, and $\mathrm{per}A \leq n^n \mathrm{per}\tilde{A}$.

By the inductive assumption and Property 5, there exists N such that

$$P_n^d(\gamma) = \mathrm{per}A \leq n^n \mathrm{per}\tilde{A} \leq n^n \bar{\gamma}^n e^{-n(d-2-\varepsilon)} \leq F_n^d(\gamma)$$

for all $n > N$ and $\gamma \in \Delta_0(n)$, where $F_n^d(\gamma)$ is defined by (1).

The application of Statement 3 yields $F_n^d(\gamma) \geq P_n^d(\gamma)$ for all sufficiently large n and $\gamma \in \Lambda(n)$. As in the 3-dimensional case, it follows that

$$P_n^d(\gamma) = \gamma^n e^{-(d-1)n+o(n)} \text{ as } n \to \infty$$

for all $\delta \in (0,1]$ and $\gamma = n^{d-3+\delta}$.

□

Corollary 3. *Let $d \geq 3$. Denote by Ω_n^d the set of d-dimensional polystochastic matrices of order n. Then*

$$\max_{A \in \Omega_n^d} \operatorname{per} A = P_n^d(n^{d-2}) = n^{(d-2)n} e^{-(d-1)n+o(n)} \text{ as } n \to \infty.$$

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