Stable Bundles on Irregular Vaisman Manifolds

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Abstract

A locally conformally Kähler (LCK) manifold is a complex manifold whose universal cover is Kähler with monodromy group acting on the universal cover by holomorphic homotheties. A Vaisman manifold \( M \) is a compact non-Kähler LCK manifold admitting a holomorphic conformal action of a group \( G = \mathbb{C} \) lifting to an action on a Kähler cover by nontrivial homotheties. When the orbits of the action on \( M \) are compact, it is known that every stable holomorphic vector bundle over \( M, \dim(M) \geq 3 \), is \( G \)-equivariant and filtrable. In the present paper we generalize this result to irregular Vaisman manifolds.

1 Introduction

1.1 Overview and statement of the problem

Let \( E \) be a holomorphic vector bundle over a compact complex manifold \( X \). In order to investigate properties of \( E \), it is often convenient to construct filtrations \( 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = E \) by coherent subsheaves and study the successive quotients \( \mathcal{E}_{i+1}/\mathcal{E}_i \). Well-known examples of this kind include the Harder-Narasimhan filtration with semistable quotients and the Jordan-Hölder filtration of a semistable sheaf (see Chapter 1 of [HL10]). Another example is given by the following definition.

Definition 1.1.1. A holomorphic vector bundle \( E \) is filtrable if there exists a filtration \( 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = E \) such that quotients \( \mathcal{E}_{i+1}/\mathcal{E}_i \) have rank 1.

Remark 1.1.1. In the above case the reflexive hulls \( \mathcal{L}_i = (\mathcal{E}_{i+1}/\mathcal{E}_i)^{**} \) are line bundles.

Every vector bundle on a projective variety is filtrable (see e. g. [B96], p. 91). It's no longer true for nonalgebraic manifolds. For instance, the tangent bundle of a generic K3 surface has no rank 1 subsheaves. As for non-Kähler examples, R. Moraru [M] proved that on a Hopf surface a generic vector bundle of rank 2 is non-filtrable. The presence of non-filtrable bundles makes the study of moduli spaces of polystable vector bundles on surfaces very difficult. Hence, nonalgebraic manifolds admitting only filtrable stable bundles deserve a lot of attention.

Indeed, the theorem of M. Verbitsky states that on a diagonal Hopf manifolds of dimension at least 3 every stable bundle is filtrable. The same result holds for positive principal elliptic fibrations ([V05, V06]). In fact, the argument in [V06]
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is valid for a larger class of non-Kähler manifolds, the so-called quasiregular Vaisman manifolds.

The goal of this paper is to establish an analogous result for irregular Vaisman manifolds. Sections 2 and 3 are expository; we present there some relevant facts about Vaisman geometry and stability of vector bundles. Our argument is carried out in Section 4.

2 Complex geometry of Vaisman manifolds

For a general information on LCK and Vaisman geometry see [DO98] and references therein.

2.1 Basic definitions

Definition 2.1.1. A complex Hermitian manifold $(M, \omega, J)$ is locally conformally Kähler (LCK) if $d\omega = \omega \wedge \theta$, where $\theta$ is a closed 1-form called the Lee form.

Equivalently, a compact complex manifold is LCK if it is covered by a Kähler manifold $\tilde{M}$ and $\pi_1(M)$ acts on $\tilde{M}$ by homotheties. The corresponding representation of $\pi_1(M)$ defines a flat real line bundle $L$. Its complexification $L_C := L \otimes \mathbb{C}$ is called the weight bundle of $M$; the covering $p: \tilde{M} \to M$ is also called the weight covering.

Definition 2.1.2. A compact LCK manifold $M$ is Vaisman if $\theta$ is parallel with respect to the Levi-Civita connection on $M$: $\nabla \theta = 0$.

The geometry of Vaisman manifolds is extensively studied in [OV05]. This class contains diagonal Hopf manifolds; conversely, every Vaisman manifold can be holomorphically immersed in diagonal Hopf. The weight covering of a Vaisman manifold has monodromy $\mathbb{Z}$ and $\tilde{M}$ is a conical Kähler manifold.

2.2 The fundamental foliation

Definition 2.2.1. Let $M$ be a Vaisman manifold. Consider the dual vector field to the Lee form $\theta$. Then $\theta^\sharp$ and $J(\theta^\sharp)$ generate a one-dimensional holomorphic Riemannian foliation $\Sigma$, called the fundamental foliation of $M$.

The fundamental foliation together with the form $\omega_0 := \omega - \theta \wedge J(\theta) = d(J\theta)$ give the so-called transversely Kähler structure. This means that $\Sigma$ is the kernel of $\omega_0 := \omega - \theta \wedge J(\theta)$ and $\omega_0$ is $\theta^\sharp$-equivariant: $\mathcal{L}_{\theta^\sharp} \omega_0 = d(\omega(\theta^\sharp, \cdot)) + d\omega_0(\theta^\sharp, \cdot) = 0$ by Cartan formula. Thus the leaf space of the foliation $\Sigma$ is locally a Kähler manifold. The existence of such a structure has a lot of consequences for the geometry of $M$ (see [V06] for details).

Definition 2.2.2. A Vaisman manifold $M$ is called quasiregular if the leaves of $\Sigma$ are compact; otherwise $M$ is irregular.

Remark 2.2.1. In quasiregular case the space of leaves $Q := M/\Sigma$ is a compact Kähler orbifold and $M$ is elliptically fibered over $Q$.

The proposition below (Proposition 4.6 in [OV05]) plays a crucial role in the proof of our main result.
Theorem 2.2.1. Every irregular Vaisman structure on a given manifold \( M \) can be approximated by quasiregular ones. More precisely, there exist arbitrarily small quasiregular deformations \( M' \) with the same weight cover \( \tilde{M} \). 

2.3 Automorphism group of a Vaisman manifold

Vaisman manifolds can be described in terms of their automorphisms, as the following theorem indicates (for a proof see [KO05]):

Theorem 2.3.1. An LCK manifold is conformally isomorphic to a Vaisman manifold if and only if it admits an action of a holomorphic flow such that this action lifts to non-trivial homotheties of the weight covering.

Another remarkable property of \( \text{Aut}(M) \) easily follows from the results stated in Section 3:

Theorem 2.3.2. The group \( \text{Aut}(M) \) of conformal holomorphic automorphisms of a Vaisman manifold coincides with the group of holomorphic isometries \( \text{Isom}_H(M, g) \) and, as a consequence, is compact.

Proof. By Remark 3.1.1 a Vaisman metric is always Gauduchon, hence unique in its conformal class. Therefore a conformal holomorphic automorphism must preserve the metric.

A holomorphic one-dimensional Lie group \( G \) (isomorphic to a quotient of \( \mathbb{C} \)) generated by the flow of \( \theta^\sharp \) is called the complex Lee flow. The orbits of \( G \)-action on \( M \) are precisely the leaves of the fundamental foliation. We denote by \( \overline{G} \) the closure of \( G \) in \( \text{Aut}(M) \); \( \overline{G} \) is a closed connected abelian subgroup of a compact Lie group, hence \( \overline{G} \simeq (S^1)^k \). If \( M \) is quasiregular we have \( k = 2 \), on the other hand, in the case of \( M \) irregular the dimension of \( \overline{G} \) can be anything.

3 Stability in non-Kähler geometry

In this short section we briefly recall the notions of stability in the non-Kähler setting. The general references for this section are [LT95, Br05].

3.1 Stability, degree and slope

Definition 3.1.1. A Hermitian metric \( g \) on \( n \)-dimensional complex manifold \( M \) is called Gauduchon if the corresponding form \( \omega \) satisfies \( \partial \overline{\partial} (\omega^{n-1}) = 0 \).

Remark 3.1.1. On a Vaisman manifold the metric is automatically Gauduchon [DO98].

As it was proved in [Ga84], a Gauduchon metric always exists and is unique in any given conformal class. This allows to define the degree and slope of a given holomorphic Hermitian bundle \( (E, h) \) or a torsion-free coherent sheaf \( \mathcal{E} \) on a compact manifold \( X \):

\[
\deg_g(E) := \int_M c_1(\det(E)) \wedge \omega^{n-1}
\]

\[
\mu(E) := \frac{\deg_g(E)}{\text{rk}(E)}
\]
Definition 3.1.2. A holomorphic vector bundle $E$ is \textbf{stable} (resp. \textbf{semistable}) if for any coherent subsheaf $\mathcal{F} \subset E$ we have $\mu(\mathcal{F}) \leq \mu(E)$ (resp. $\mu(\mathcal{F}) < \mu(E)$); \textbf{polystable} if $E$ is a direct sum of stable bundles of the same slope.

3.2 Hermitian-Einstein metrics

Definition 3.2.1. A metric $h$ on a bundle $E$ is \textbf{(weakly) Hermitian-Einstein} if the curvature $\Theta_{E,h}$ of the Chern connection satisfies $\sqrt{-1} \Lambda \omega \Theta_{E,h} = C \cdot \text{Id}_E$ for some real constant (resp. a function) $C$. Here $\Lambda$ is the formal dual to the operator of multiplication by $\omega$.

The proof of the following theorem can be found in [UY86, LT95]:

Theorem 3.2.1 (The Kobayashi - Hitchin Correspondence). A holomorphic vector bundle $E$ over a compact Hermitian manifold admits a Hermitian-Einstein metric if and only if it is polystable. □

4 Equivariance and filtrability of stable bundles

Everywhere in this section $M$ stands for a compact Vaisman manifold of dimension $n \geq 3$.

4.1 The action of the flow $G$

Theorem 4.1.1. Let $(M, \omega, \theta)$ be a Vaisman manifold with fundamental foliation $\Sigma$ and $E \to M$ be a stable holomorphic vector bundle endowed with a Hermitian metric $h$. Then the curvature of $E$ satisfies $\Theta_{E,h}(v, \cdot) = 0$ for any $v \in \Sigma$. As a consequence, $E$ admits a natural $G$-equivariant structure.

\textbf{Proof.} See [V06], Theorem 4.1 and Remark 4.3. □

Remark 4.1.1. We give an outline of the proof of filtrability in quasiregular case. Let $E$ be a stable bundle of degree $\lambda$; then by Theorem 4.1.1 $E \otimes L_{-\lambda}$ is equivariant and flat on the leaves of $\Sigma$. Hence, $E \otimes L_{-\lambda}$ is isomorphic to a pullback $p^*(E_0)$ of some orbibundle $E_0$ on the leaf space $Q$. By Proposition 3.6 of [OV05] $p_*L_C$ an ample line orbibundle on $Q$. By Baily's generalization of Kodaira embedding theorem [Ba57] $Q$ is projective. Consequently, $E_0$ is filtrable, as an algebraic bundle on a projective orbifold; therefore $E \simeq E_0 \otimes L_\lambda$ is also filtrable.

4.2 Extension of the action to the closure

The preceding argument has to be slightly modified to deal with irregular case.

Theorem 4.2.1. Suppose that a Vaisman manifold $M$ is irregular. Then under the assumptions of Theorem 4.1.1 $E$ is $\overline{G}$-equivariant.

\textbf{Proof.} Indeed, the group $\overline{G}$ acts on $M$ by holomorphic isometries and a dense subgroup $G \subset \overline{G}$ acts on the total space $\text{Tot}(E)$ by isometries. Consider the bundle of unit vectors in the fibers of $E$; its total space $N$ is a compact Riemannian manifold acted on by $G$. As the isometry group of $N$ is compact, the action of $G$ extends to $\overline{G}$ by the universal property. Taking conjugations
by dilations \((x, v) \mapsto (x, v/||v||)\), we see that \(\overline{G}\) acts on \(\text{Tot}(E)\). Moreover, the action of a dense subgroup \(G\) preserves the metric, therefore the same is true for \(\overline{G}\)-action. Finally, it remains to prove that \(\overline{G}\) acts on \(E\) holomorphically. As the question is local, it suffices to show that for every element \(g \in \overline{G}\) and a holomorphic section \(s\) of \(E\) on an open ball \(U \subset M\) a section \(g(s)\) is also holomorphic. We represent \(g\) as the limit of a sequence \((g_k)\) of holomorphic transformations. Therefore, \((g_k(s))\) is a sequence of holomorphic sections which has a limit in topology of the ambient space of continuous sections. By Montel’s theorem, \(g(s) = (\lim g_k)(s) = \lim(g_k(s))\) must be holomorphic. □

**Theorem 4.2.2.** Let us denote by \(\text{Pst}(M)\) the category of polystable holomorphic vector bundles over a Vaisman manifold \(M\). Also let \(\overline{G}\) denote the lift of \(G\) to the weight cover. Consider the category \(\text{Bun}_{\overline{G}}(\overline{M})\) of \(\overline{G}\)-equivariant holomorphic Hermitian vector bundles on \(\overline{M}\) satisfying the condition \(\sqrt{-1}\Lambda \Theta_{E,h} = C \cdot \text{Id}_E\); here \(\Lambda = \Lambda_{\rho, \omega}\) is a formal dual to multiplication by the pullback of the Hermitian form on \(M\) and \(C\) is a real constant. The map \(E \to p^*(\overline{E})\) gives an equivalence of categories between \(\text{Pst}(M)\) and \(\text{Bun}_{\overline{G}}(\overline{M})\). For \(M\) quasiregular we have \(G = \overline{G} = T\) where \(T\) is a two-dimensional compact torus.

**Proof.** The correspondence \(E \to p^*(E)\) is clearly functorial; it suffices to prove that it is fully faithful and essentially surjective. Note that \(M \simeq \overline{M}/\Gamma\) where \(\Gamma \simeq \mathbb{Z}\) is the monodromy group of the weight cover. It is clear that the category of complex vector bundles over \(M\) is equivalent to that of \(\mathbb{Z}\)-equivariant complex vector bundles over \(\overline{M}\). Suppose that \(\overline{E}\) comes from \(M\) by pullback; then it clearly satisfies the above Hermite-Einstein condition. In [OV05] it was proved that the forgetful map gives an exact sequence \(0 \to \Gamma \to \overline{G} \to G \to 0\). Hence, \(\Gamma \subset \overline{G}\) and thus any \(\overline{G}\)-equivariant holomorphic vector bundle over \(\overline{M}\) satisfying the Hermite-Einstein condition is a pullback of a polystable bundle on \(M\). The same argument applies to morphisms:

\[
\text{Hom}(E_1, E_2) \simeq \text{Hom}(p^*(E_1), p^*(E_2))^{\overline{G}} \simeq \text{Hom}_{\text{Bun}_{\overline{G}}(\overline{M})}(p^*(E_1), p^*(E_2))
\]

and this completes the proof. □

**Corollary 4.2.1.** There exists a quasiregular deformation \(M'\) of \(M\) such that the category of polystable vector bundles \(\text{Pst}(M)\) is equivalent to a subcategory of \(\text{Pst}(M')\).

**Proof.** Let \(\gamma\) be a generator of \(\Gamma\) and denote by \(\Gamma'\) the group generated by some other element \(\gamma' \in \overline{G}\). Then by [OV05], Proposition 4.6 we can take \(\gamma'\) such that the quotient \(M' := \overline{M}/\Gamma'\) is a quasiregular Vaisman manifold. Then by Theorem 4.2.2 we have an equivalence of categories \(\text{Bun}_G(M) \to \text{Bun}_{\overline{G}}(\overline{M})\). As \(\Gamma' \subset \overline{G}\), we have \(\overline{T} \subset \overline{G}\) which means that \(\text{Bun}_{\overline{G}}(\overline{M}) \subset \text{Bun}_{\overline{G}}(\overline{M})\). Then composition with equivalences constructed in Theorem 4.2.2 gives the desired embedding of \(\text{Pst}(M)\) into \(\text{Pst}(M')\). □

Finally, we can prove the main theorem:

**Theorem 4.2.3.** Let \(E\) be a stable holomorphic Hermitian vector bundle of degree \(\lambda\) over an irregular Vaisman manifold \(M\). Then \(E\) is filtrable.
Proof. Again, by Theorem 4.1.1 $E \otimes L^{-\lambda}$ is equivariant and flat on the leaves of $\Sigma$. By Theorem 4.2.1 $E$ is also $G$-equivariant. Choose a quasiregular deformation of $M$ as constructed in Corollary 4.2.1. Then the pullback $p^*(E)$ is $C^*$-equivariant and trivial on the fibers of the map $r: \tilde{M} \to \tilde{M}/C^*$. Hence by Corollary 4.2.1 $p^*(E)$ is isomorphic to a pullback of some bundle from $\tilde{M}/C^*$ and Remark 4.1.1 implies that $p^*(E)$ is filtrable. □

References


