Stable Bundles on Irregular Vaisman Manifolds

A. Golota

Abstract

A locally conformally Kähler (LCK) manifold is a complex manifold whose universal cover is Kähler with monodromy group acting on the universal cover by holomorphic homotheties. A Vaisman manifold M is a compact non-Kähler LCK manifold admitting a holomorphic conformal action of a group $G = \mathbb{C}$ lifting to an action on a Kähler cover by nontrivial homotheties. When the orbits of the action on M are compact, it is known that every stable holomorphic vector bundle over M, $\dim(M) \geq 3$, is G-equivariant and filtrable. In the present paper we generalize this result to irregular Vaisman manifolds.

1 Introduction

1.1 Overview and statement of the problem

Let E be a holomorphic vector bundle over a compact complex manifold X. In order to investigate properties of E, it is often convenient to construct filtrations $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = E$ by coherent subsheaves and study the successive quotients $\mathscr{E}_{i+1}/\mathscr{E}_i$. Well-known examples of this kind include the Harder-Narasimhan filtration with semistable quotients and the Jordan-Hölder filtration of a semistable sheaf (see Chapter 1 of [HL10]). Another example is given by the following definition.

Definition 1.1.1. A holomorphic vector bundle E is **filtrable** if there exists a filtration $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = E$ such that quotients $\mathscr{E}_{i+1}/\mathscr{E}_i$ have rank 1.

Remark 1.1.1. In the above case the reflexive hulls $L_i = (\mathscr{E}_{i+1}/\mathscr{E}_i)^{**}$ are line bundles.

Every vector bundle on a projective variety is filtrable (see e. g. [B96], p. 91). It's no longer true for nonalgebraic manifolds. For instance, the tangent bundle of a generic K3 surface has no rank 1 subsheaves. As for non-Kähler examples, R. Moraru [M] proved that on a Hopf surface a generic vector bundle of rank 2 is non-filtrable. The presence of non-filtrable bundles makes the study of moduli spaces of polystable vector bundles on surfaces very difficult. Hence, nonalgebraic manifolds admitting only filtrable stable bundles deserve a lot of attention.

Indeed, the theorem of M. Verbitsky states that on a diagonal Hopf manifolds of dimension at least 3 every stable bundle is filtrable. The same result holds for positive principal elliptic fibrations ([V05, V06]). In fact, the argument in [V06] is valid for a larger class of non-Kähler manifolds, the so-called quasiregular Vaisman manifolds.

The goal of this paper is to establish an analogous result for irregular Vaisman manifolds. Sections 2 and 3 are expository; we present there some relevant facts about Vaisman geometry and stability of vector bundles. Our argument is carried out in Section 4.

2 Complex geometry of Vaisman manifolds

For a general information on LCK and Vaisman geometry see [DO98] and references therein.

2.1 Basic definitions

Definition 2.1.1. A complex Hermitian manifold (M, ω, J) is **locally conformally Kähler (LCK)** if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form called the **Lee form**.

Equivalently, a compact complex manifold is LCK if it is covered by a Kähler manifold $\tilde{M}$ and $\pi_1(M)$ acts on $\tilde{M}$ by homotheties. The corresponding representation of $\pi_1(M)$ defines a flat real line bundle L. Its complexification $L_{\mathbb{C}} := L \otimes_{\mathbb{R}} \mathbb{C}$ is called the **weight bundle** of M; the covering $p \colon \tilde{M} \to M$ is also called the **weight covering**.

Definition 2.1.2. A compact LCK manifold M is **Vaisman** if θ is parallel with respect to the Levi-Civita connection on M: $\nabla\theta = 0$.

The geometry of Vaisman manifolds is extensively studied in [OV05]. This class contains diagonal Hopf manifolds; conversely, every Vaisman manifold can be holomorphically immersed in diagonal Hopf. The weight covering of a Vaisman manifold has monodromy $\mathbb{Z}$ and $\tilde{M}$ is a conical Kähler manifold.

2.2 The fundamental foliation

Definition 2.2.1. Let M be a Vaisman manifold. Consider the dual vector field to the Lee form θ. Then $\theta^\sharp$ and $J(\theta^\sharp)$ generate a one-dimensional holomorphic Riemannian foliation Σ, called the **fundamental foliation** of M.

The fundamental foliation together with the form $\omega_0 := \omega - \theta \wedge J(\theta) = d(J\theta)$ give the so-called **transversely Kähler structure**. This means that Σ is the kernel of ω_0 and ω_0 is $\theta^\sharp$-equivariant: $\mathscr{L}_{\theta^\sharp}\omega_0 = d(\omega(\theta^\sharp, \cdot)) + d\omega_0(\theta^\sharp, \cdot) = 0$ by Cartan formula. Thus the leaf space of the foliation Σ is locally a Kähler manifold. The existence of such a structure has a lot of consequences for the geometry of M (see [V06] for details).

Definition 2.2.2. A Vaisman manifold M is called **quasiregular** if the leaves of Σ are compact; otherwise M is **irregular**.

Remark 2.2.1. In quasiregular case the space of leaves $Q := M/\Sigma$ is a compact Kähler orbifold and M is elliptically fibered over Q.

The proposition below (Proposition 4.6 in [OV05]) plays a crucial role in the proof of our main result.

Theorem 2.2.1. Every irregular Vaisman structure on a given manifold M can be approximated by quasiregular ones. More precisely, there exist arbitrarily small quasiregular deformations M' with the same weight cover $\tilde{M}$. □

2.3 Automorphism group of a Vaisman manifold

Vaisman manifolds can be described in terms of their automorphisms, as the following theorem indicates (for a proof see [KO05]):

Theorem 2.3.1. An LCK manifold is conformally isomorphic to a Vaisman manifold if and only if it admits an action of a holomorphic flow such that this action lifts to non-trivial homotheties of the weight covering. □

Another remarkable property of $\mathrm{Aut}(M)$ easily follows from the results stated in Section 3:

Theorem 2.3.2. The group $\mathrm{Aut}(M)$ of conformal holomorphic automorphisms of a Vaisman manifold coincides with the group of holomorphic isometries $\mathrm{Isom}_H(M, g)$ and, as a consequence, is compact.

Proof. By Remark 3.1.1 a Vaisman metric is always Gauduchon, hence unique in its conformal class. Therefore a conformal holomorphic automorphism must preserve the metric. □

A holomorphic one-dimensional Lie group G (isomorphic to a quotient of $\mathbb{C}$) generated by the flow of $\theta^\sharp$ is called the **complex Lee flow**. The orbits of G-action on M are precisely the leaves of the fundamental foliation. We denote by $\overline{G}$ the closure of G in $\mathrm{Aut}(M)$; $\overline{G}$ is a closed connected abelian subgroup of a compact Lie group, hence $\overline{G} \simeq (S^1)^k$. If M is quasiregular we have $k = 2$, on the other hand, in the case of M irregular the dimension of $\overline{G}$ can be anything.

3 Stability in non-Kähler geometry

In this short section we briefly recall the notions of stability in the non-Kähler setting. The general references for this section are [LT95, Br05].

3.1 Stability, degree and slope

Definition 3.1.1. A Hermitian metric g on n-dimensional complex manifold M is called **Gauduchon** if the corresponding form ω satisfies $\partial\overline{\partial}(\omega^{n-1}) = 0$.

Remark 3.1.1. On a Vaisman manifold the metric is automatically Gauduchon [DO98].

As it was proved in [Ga84], a Gauduchon metric always exists and is unique in any given conformal class. This allows to define the **degree** and **slope** of a given holomorphic Hermitian bundle (E, h) or a torsion-free coherent sheaf $\mathscr{E}$ on a compact manifold X:

$$\deg_g(E) := \int_M c_1(\det(E)) \wedge \omega^{n-1}$$

$$\mu(E) := \frac{\deg_g(E)}{\mathrm{rk}(E)}$$

Definition 3.1.2. A holomorphic vector bundle E is **stable**(resp. **semistable**) if for any coherent subsheaf $\mathscr{F} \subset E$ we have $\mu(\mathscr{F}) \leq \mu(E)$(resp. $\mu(\mathscr{F}) < \mu(E)$); **polystable** if E is a direct sum of stable bundles of the same slope.

3.2 Hermitian-Einstein metrics

Definition 3.2.1. A metric h on a bundle E is **(weakly) Hermitian-Einstein** if the curvature $\Theta_{E,h}$ of the Chern connection satisfies $\sqrt{-1}\Lambda_\omega\Theta_{E,h} = C \cdot \mathrm{Id}_E$ for some real constant (resp. a function) C. Here Λ is the formal dual to the operator of multiplication by ω.

The proof of the following theorem can be found in [UY86, LT95]:

Theorem 3.2.1 (**The Kobayashi - Hitchin Correspondence**)**.** A holomorphic vector bundle E over a compact Hermitian manifold admits a Hermitian-Einstein metric if and only if it is polystable. □

4 Equivariance and filtrability of stable bundles

Everywhere in this section M stands for a compact Vaisman manifold of dimension $n \geq 3$.

4.1 The action of the flow G

Theorem 4.1.1. Let (M, ω, θ) be a Vaisman manifold with fundamental foliation Σ and $E \to M$ be a stable holomorphic vector bundle endowed with a Hermitian metric h. Then the curvature of E satisfies $\Theta_{E,h}(v, \cdot) = 0$ for any $v \in \Sigma$. As a consequence, E admits a natural G-equivariant structure.

Proof. See [V06], Theorem 4.1 and Remark 4.3. □

Remark 4.1.1. We give an outline of the proof of filtrability in quasiregular case. Let E be a stable bundle of degree λ; then by Theorem 4.1.1 $E \otimes L_{-\lambda}$ is equivariant and flat on the leaves of Σ. Hence, $E \otimes L_{-\lambda}$ is isomorphic to a pullback $p^*(E_0)$ of some orbibundle E_0 on the leaf space Q. By Proposition 3.6 of [OV05] $p_* L_{\mathbb{C}}$ an ample line orbibundle on Q. By Baily's generalization of Kodaira embedding theorem [Ba57] Q is projective. Consequently, E_0 is filtrable, as an algebraic bundle on a projective orbifold; therefore $E \simeq E_0 \otimes L_\lambda$ is also filtrable.

4.2 Extension of the action to the closure

The preceding argument has to be slightly modified to deal with irregular case.

Theorem 4.2.1. Suppose that a Vaisman manifold M is irregular. Then under the assumptions of Theorem 4.1.1 E is $\overline{G}$-equivariant.

Proof. Indeed, the group $\overline{G}$ acts on M by holomorphic isometries and a dense subgroup $G \subset \overline{G}$ acts on the total space $\mathrm{Tot}(E)$ by isometries. Consider the bundle of unit vectors in the fibers of E; its total space N is a compact Riemannian manifold acted on by G. As the isometry group of N is compact, the action of G extends to $\overline{G}$ by the universal property. Taking conjugations by dilations $(x, v) \mapsto (x, v/||v||)$, we see that $\overline{G}$ acts on $\mathrm{Tot}(E)$. Moreover, the action of a dense subgroup G preserves the metric, therefore the same is true for $\overline{G}$-action. Finally, it remains to prove that $\overline{G}$ acts on E holomorphically. As the question is local, it suffices to show that for every element $g \in \overline{G}$ and a holomorphic section s of E on an open ball $U \subset M$ a section $g(s)$ is also holomorphic. We represent g as the limit of a sequence (g_k) of holomorphic transformations. Therefore, $(g_k(s))$ is a sequence of holomorphic sections which has a limit in topology of the ambient space of continuous sections. By Montel's theorem, $g(s) = (\lim g_k)(s) = \lim(g_k(s))$ must be holomorphic. $\square$

Theorem 4.2.2. Let us denote by $\mathrm{Pst}(M)$ the category of polystable holomorphic vector bundles over a Vaisman manifold M. Also let $\tilde{\overline{G}}$ denote the lift of G to the weight cover. Consider the category $\mathrm{Bun}_{\tilde{\overline{G}}}(\tilde{M})$ of $\tilde{\overline{G}}$-equivariant holomorphic Hermitian vector bundles on $\tilde{M}$ satisfying the condition $\sqrt{-1}\Lambda\Theta_{E,h} = C \cdot \mathrm{Id}_E$; here $\Lambda = \Lambda_{p^*\omega}$ is a formal dual to multiplication by the pullback of the Hermitian form on M and C is a real constant. The map $E \to p^*(E)$ gives an equivalence of categories between $\mathrm{Pst}(M)$ and $\mathrm{Bun}_{\tilde{\overline{G}}}(\tilde{M})$. For M quasiregular we have $G = \overline{G} = T$ where T is a two-dimensional compact torus.

Proof. The correspondence $E \to p^*(E)$ is clearly functorial; it suffices to prove that it is fully faithful and essentially surjective. Note that $M \simeq \tilde{M}/\Gamma$ where $\Gamma \simeq \mathbb{Z}$ is the monodromy group of the weight cover. It is clear that the category of complex vector bundles over M is equivalent to that of $\mathbb{Z}$-equivariant complex vector bundles over $\tilde{M}$. Suppose that $\tilde{E}$ comes from M by pullback; then it clearly satisfies the above Hermite-Einstein condition. In [OV05] it was proved that the forgetful map gives an exact sequence $0 \to \Gamma \to \tilde{\overline{G}} \to \overline{G} \to 0$. Hence, $\Gamma \subset \tilde{\overline{G}}$ and thus any $\tilde{\overline{G}}$-equivariant holomorphic vector bundle over $\tilde{M}$ satisfying the Hermite-Einstein condition is a pullback of a polystable bundle on M. The same argument applies to morphisms:

$$\mathrm{Hom}(E_1, E_2) \simeq \mathrm{Hom}(p^*(E_1), p^*(E_2))^{\tilde{\overline{G}}} \simeq \mathrm{Hom}_{\mathrm{Bun}_{\tilde{\overline{G}}}(\tilde{M})}(p^*(E_1), p^*(E_2))$$

and this completes the proof. $\square$

Corollary 4.2.1. There exists a quasiregular deformation M' of M such that the category of polystable vector bundles $\mathrm{Pst}(M)$ is equivalent to a subcategory of $\mathrm{Pst}(M')$.

Proof. Let γ be a generator of Γ and denote by Γ' the group generated by some other element $\gamma' \in \tilde{\overline{G}}_{\mathbb{C}}$. Then by [OV05], Proposition 4.6 we can take γ' such that the quotient $M' := \tilde{M}/\Gamma'$ is a quasiregular Vaisman manifold. Then by Theorem 4.2.2 we have an equivalence of $\mathrm{Pst}(M)$ and $\mathrm{Bun}_{\tilde{T}'}(M)$. As $T' \subset \overline{G}$, we have $\tilde{T} \subset \tilde{\overline{G}}$ which means that $\mathrm{Bun}_{\tilde{\overline{G}}}(\tilde{M}) \subset \mathrm{Bun}_{\tilde{T}'}(\tilde{M})$. Then composition with equivalences constructed in Theorem 4.2.2 gives the desired embedding of $\mathrm{Pst}(M)$ into $\mathrm{Pst}(M')$. $\square$

Finally, we can prove the main theorem:

Theorem 4.2.3. Let E be a stable holomorphic Hermitian vector bundle of degree λ over an irregular Vaisman manifold M. Then E is filtrable.

Proof. Again, by Theorem 4.1.1 $E \otimes L_{-\lambda}$ is equivariant and flat on the leaves of Σ. By Theorem 4.2.1 E is also $\overline{G}$-equivariant. Choose a quasiregular deformation of M as constructed in Corollary 4.2.1. Then the pullback $p^*(E)$ is $\mathbb{C}^*$-equivariant and trivial on the fibers of the map $r\colon \tilde{M} \to \tilde{M}/\mathbb{C}^*$. Hence by Corollary 4.2.1 $p^*(E)$ is isomorphic to a pullback of some bundle from $\tilde{M}/\mathbb{C}^*$ and Remark 4.1.1 implies that $p^*(E)$ is filtrable. $\square$

References

[Ba57] W. L. Baily, *On the imbedding of V-manifolds in projective space,* American Journal of Mathematics, vol. 79 no. 2 (1957) 403 - 430

[B96] V. Brînzănescu, *Holomorphic Vector Bundles over Compact Complex Surfaces,* Lect. Notes in Math. 1624, Springer (1996)

[Br05] L. Bruasse, *Harder-Narasimhan filtration on non-Kähler manifolds* International Journal of Mathematics, Vol. 12, No. 5 (2001) 579-594

[DO98] S. Dragomir, L. Ornea, *Locally Conformal Kähler Geometry,* Progress in Mathematics, 155. Birkhäuser, Boston, MA, 1998

[Ga84] P. Gauduchon, La 1-forme de torsion d'une variete hermitienne compacte, Math. Ann., 267 (1984), 495-518

[HL10] D. Huybrechts, M. Lehn, *The Geometry of Moduli Spaces of Sheaves,* 2nd Edition, CUP, 2010

[KO05] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformally Kähler manifolds,* Tohoku Math. J. 57 (2005), 201 - 221

[LT95] M. Lübke and A. Teleman, *The Kobayashi-Hitchin correspondence,* World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[M] R. Moraru, *Stable bundles on Hopf manifolds,* arXiv:math/0408439

[MPPS97] A. Madsen, H. Pedersen, Y. Poon, A. Swann, *Compact Einstein-Weyl manifolds with large symmetry group,* Duke Math. J. 88 (1997) 407 - 434

[OV05] L. Ornea, M. Verbitsky, *An immersion theorem for Vaisman manifolds,* Math. Ann. 332 (2005), no. 1, 121–143.

[UY86] K. Uhlenbeck, S.-T. Yau, *On the Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundle,.* Communications in Pure and Applied Mathematics, Vol. XXXIX, 1986, Supplement, pp S257–S293

[V05] M. Verbitsky, *Stable bundles on positive principal elliptic fibrations,* Math. Res. Lett. 12 (2005), no. 2-3, 251–264

[V06] M. Verbitsky, *Holomorphic bundles on diagonal Hopf manifolds,* Izvestiya Math., 2006, 70, no. 5, pp. 13-31