Upon the fibrations with hyperkähler fibers

Rodion N. Déev

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Abstract

In the present paper we prove that any family of hyperkähler manifolds over a compact simply connected base can be pulled back from a family over a curve.

Introduction

Let $\mathbb{H}$ be the skew-field of quaternions.

Definition 1. A manifold M with a left $\mathbb{H}$-action in the tangent bundle is called *quaternionic*. Any quaternion q with $q^2 = -1$ defines an almost complex structure on M, and if all these complex structure are integrable, M is called *hypercomplex*.

One does not need to check the integrability condition for all the 2-shpere $q^2 = -1$, but only for three quaternions spanning the space of imaginary ones.

Any hypercomplex structure admits a unique torsion-free connection preserving the hypercomplex structure, called *Obata connection* [**Ka**].

Definition 2. A Riemannian hypercomplex manifold (M, g) is called *hyperkähler*, if its Obata and Levi-Civita connections are equal.

One can think of hyperkähler manifolds as of Riemannian manifolds with three integrable almost complex structures I, J, K such that $IJ = -JI = K$ which are Kähler with respect to the Riemannian metric, or as of Riemannian manifolds with holonomy in the group $\mathrm{Sp}(n)$. When $n = 1$, $\mathrm{Sp}(1) = \mathrm{SU}(2)$ and all the hyperkähler manifolds are either K3 surfaces or tori.

One can wonder what are the families of hyperkähler manifolds, i. e. submersions of complex manifolds with hyperkähler fibers. This question is ultimately closely related to the variations of Hodge structures, because any family of complex manifolds give rise to a variation of Hodge structures. Some geometric structures on complex manifolds are completely determined by their Hodge structures; statements of such type are known as "Torelli theorems". Although the global Torelli theorem for hyperkähler manifolds was proved in 2009 [**V**], its local version (for Kähler manifolds with vanishing first Chern class) was established in 1964 by G. N. Tjurina [**T**] and its global version for K3 surfaces was obtained in 1977 by Vik. S. Kulikov [**Ku**] (earlier proceedings in this topic include a 1971 paper by I. I. Pyatetski-Shapiro and I. R. Shafarevich [**PShSh**]). Because of that study of the families of complex manifolds was reduced to study of variations of Hodge structures since olden times. The first proceeding in this direction is due to Griffiths.

Fact (Ph. A. Griffiths, 1970)**.** *Any variation of Hodge structures over a compact simply connected base is trivial.*

Of course, Griffiths has proved much stronger statement, called "Theorem of the Fixed Part" [**G**, Ch. II, Application 7], but we shall not need it in its full generality. It follows from this fact that any polarized family of hyperkähler manifolds over a compact simply connected base is trivial. The following definition asserts that one cannot drop out the polarization condition in the Griffiths' statement.

Definition 3. If X is a hypercomplex (for example, hyperkähler) manifold, then any imaginary unit quaternion q defines a complex structure. That gives rise to a nontrivial almost complex structure on the space $X \times \mathbb{C}P^1$, where we identify $\mathbb{C}P^1$ with the unit sphere in the space of imaginary quaternions: namely, tangent space at point (x, q) splits as $T_xX \oplus T_q\mathbb{C}P^1$, and one can put the complex structure q on T_xX and the standard one on $T_q\mathbb{C}P^1$.

Theorem (M. Obata, 1953 [**Ob**], S. Salamon, 1982 [**S**], D. Kaledin, 1996 [**Ka**])**.** *This almost complex structure is integrable.*

We shall call the manifold $X \times \mathbb{C}P^1$ with this complex structure *the twistor space.*

The projection $X \times \mathbb{C}P^1 \to \mathbb{C}P^1$ is holomorphic and defines a nontrivial family of hyperkähler manifolds. Clearly, the twistor space cannot bear any Kähler form. The concept of twistors has appeared in physics, in works of R. Penrose and M. A. H. MacCallum, in the beginning of 1970ies [**PMC**], and has been trasferred to geometry by M. F. Atiyah, N. J. Hitchin and I. M. Singer [**AHS**] (in context of 4-dimensional Riemannian geometry) and S. Salamon [**S**] (in context of quaternionic Kähler geometry).

Seeking for a way to generalize a well-known fact about isotriviality of the complete families of elliptic curves, R. E. Borcherds, L. Katzarkov, T. Pantev and N. I. Shepherd-Barron proved in 1997 the following theorem, which allows to drop out the condition of simply connectedness in Griffiths' statement at least for fibrations with fibers K3 surfaces.

Fact (R. E. Borcherds, L. Katzarkov, T. Pantev and N. I. Shepherd-Barron, 1997, [**BKPShB**])**.** *Any complete family of minimal Kähler surfaces of Kodaira dimension* 0 *and constant Picard number is isotrivial.*

They dealt separately with the cases of hyperelliptic, Enriques and K3 fiber, and the latter was the essential one. Their technique heavily uses the theory of automorphic forms, namely the properties of the Borcherds' automorphic form

Φ_{12} on the Cartan symmetric space for the group $\mathrm{O}(\mathrm{II}_{2,26})$, and one cannot prove similar results for fibrations with hyperkähler fibers of dimension greater than 28 in the same way. It seems that the only extent of their result to arbitrary hyperkähler manifolds is the following theorem of K. Oguiso.

Fact (K. Oguiso, 2000, [**Og**]). *Let $\mathfrak{X} \to \Delta$ be a nontrivial family of hyperkähler manifolds over a disk. Then the set of points where the Picard number of the fiber jumps is a dense countable subset of a disk.*

We shall deal with non-polarized case. The global Torelli theorem is known for curves, tori, hyperkähler manifolds and some exotic cases such as cubic threefolds [**G**, Ch. XII and XIII]. Curves and threefolds are automatically polarized, so we are not interested in them. The tori are also out of the scope of our paper. The main theorem we are going to prove is the following.

Theorem. *Let $\mathfrak{X} \to B$ be a smooth fibration of compact complex manifolds with smooth hyperkähler fibers and simply connected base. Then there exists a smooth fibration $\mathfrak{X}' \to C$ with hyperkähler fibers over a curve together with a map $B \xrightarrow{f} C$ such that $\mathfrak{X} = f^*(\mathfrak{X}')$.*

Here is a brief outline of the paper. In Section 1 we give some well-known facts about hyperkähler manifolds, such as the global Torelli theorem. In Section 2 we describe some geometry of the moduli space of hyperkähler manifolds, which is also widely known. Following those facts, we prove the main theorem. After all, in Section 3 we give some other relevant observations.

1 Preliminaries

It follows from a straightforward calculation and is well-known that a form

$$\Omega_I(u,v) = g(Ju,v) + \sqrt{-1}g(Ku,v)$$

on a compact hyperkähler manifold (M,g,I,J,K) is holomorphic with respect to the complex structure I.

One has $h^{2,0}(M,I) = 1$, and the cohomology class $[\Omega_I]$ spans the line $H^{2,0}(M,I) \subset H^2(M,\mathbb{C})$. Consider a complex family $(\mathfrak{X},\mathcal{I})$ of hyperkähler manifolds over a simply connected base B (so that $R^2\pi_*\mathbb{C}$ is a trivial bundle), and let X_b denote the fiber over a point $b \in B$. Then we can define the *period map* $B \xrightarrow{per} \mathbb{P}(H^2(X,\mathbb{C}))$ which sends the point b to the line spanned by the class $[\Omega_{\mathcal{I}|_{X_b}}]$.

If the base B is not simply connected, the period map is defined as a map from the universal cover $\widetilde{B}$. The fundamental group of the base acts on the universal cover and on the period space (as the monodromy group of the local system $R^2\pi_*\mathbb{C}$), and the period map is equivariant with respect to these two actions. One can try to obtain a period map from the base B into the quotient of the period space by this action, but this quotient can have very poor topology. It is known to be an orbifold at least in the case of the polarized fibration with fiber K3 surface. However, in other cases the topology on the factor can be somewhat like codiscrete. The necessity of consideration an action of the fundamental group is the source of automorphic forms in this science.

1.1 Bogomolov–Beauville–Fujiki form

For a K3 surface X the intersection form is an inner product on the space $H^2(X,\mathbb{R})$; the Hodge index theorem states that its signature is $(3,19)$. We shall need a similar inner product on the second cohomology space of a hyperkähler manifold, which would have the signature $(3,b_2-3)$.

Fact (F. A. Bogomolov, 1978 [**Bo**], A. Beauville, 1983 [**Be**], A. Fujiki, 1985, 1987 [**F**])**.** *Let X be a hyperkähler manifold of real dimension $4n$. There exists a unique primitive quadratic form $q\colon H^2(X,\mathbb{Z})\to\mathbb{Z}$ and a constant c such that for any $\alpha\in H^2(X)$ one has*

$$\int_X \lambda^{2n} = cq(\alpha)^n$$

and for non-zero $\sigma\in H^{2,0}(X)$ one has $q(\sigma+\overline{\sigma})>0$.

For more details, see [**OG**].

This form is uniquely determined by this condition. One can write down an explicit formula for q (here we use same letter q for the polarization of the Bogomolov–Beauville–Fujiki quadratic form):

$$cq(\alpha,\beta)=2\int_X \alpha\wedge\beta\wedge\Omega_I^{n-1}\wedge\overline{\Omega_I}^{n-1}- \\ -\frac{n-1}{n}\frac{(\int_X\alpha\wedge\Omega_I^{n-1}\wedge\overline{\Omega_I}^{n})(\int_X\beta\wedge\Omega_I^{n}\wedge\overline{\Omega_I}^{n-1})}{\int_X\Omega_I^n\wedge\overline{\Omega_I}^n},$$

where the positive constant on the left-hand side is needed for the form q to be integer.

It is positive definite on the real part of the space spanned by Ω_I, $\overline{\Omega_I}$ and the Kähler form ω, and negative definite on the primitive forms (i. e. it has signature $(3,b_2-3)$). The image of the period map lies in the set of lines spanned by the classes α such that $q(\alpha)=0$ and $q(\alpha+\overline{\alpha})>0$ (or, equivalently, $q(\alpha,\overline{\alpha})>0$).

1.2 Global Torelli theorem

The main reference for this section is [**V**].

Definition 4. Let (X,g) be a Riemannian manifold, and $\mathfrak{I}$ be the set of complex structures on X which can be extended to hyperkähler structures. The group $\mathrm{Diff}_0(X)$ of oriented diffeomorphisms of X act on $\mathfrak{I}$. The *Teichmüller space* $\mathrm{Teich}(X)$ of X is the factor $\mathfrak{I}/\mathrm{Diff}_0(X)$.

The space $\mathrm{Teich}(X)$ admits a period map into $\mathbb{P}(H^2(X,\mathbb{C}))$ in the same way as the base of any fibration with hyperkähler fibers (provided that the base is simply connected). The image of the period map lies in the *period space* $\mathrm{Per}(X) = \{[\alpha] \in \mathbb{P}(H^2(X,\mathbb{C})) \mid q(\alpha) = 0, q(\alpha,\overline{\alpha}) > 0\}$. Actually, Teich is not Haussdorff, but its non-Haussdorff points correspond to the bimeromorphically equivalent hyperkähler manifolds, so there exists a Haussdorff space Teich_b with map $\mathrm{Teich} \to \mathrm{Teich}_b$ such that any map from Teich to a Haussdorff space factorizes through this map (so the period map $\mathrm{Teich}_b(X) \to \mathrm{Per}(X)$ is well-defined).

Fact (M. Verbitsky, 2009 [**V**]). *The map* $\mathfrak{per}\colon \mathrm{Teich}_b \to \mathrm{Per}$ *is a diffeomorphism on each connected component of* Teich_b.

2 Geometry of the period space

2.1 Positive Grassmannians

Definition 5. Let V be an $\mathbb{R}$-vector space with non-degenerate inner product $\langle\cdot\|\cdot\rangle$. The subset in the oriented Grassmannian $\mathrm{Gr}_2(V)$ consisting of oriented 2-planes with positive definite restriction of inner product (we shall call such planes *positive*) is called *the positive Grassmannian* and denoted $\mathrm{Gr}_{++}(V)$.

Lemma (C. LeBrun, 1993 [**LB**]). *The set* $\mathrm{Gr}_{++}(V)$ *is in one-to-one correspondence with the projectivization of the set of vectors* $v \in V\otimes\mathbb{C}$ *such that* $\langle v\|v\rangle = 0$ *and* $\langle v\|\overline{v}\rangle > 0$ *(we shall call them* positive null-vectors*).*

Proof. If $W \subseteq V$ is a point in $\mathrm{Gr}_{++}(V)$, then the cone of null-vectors in $W\otimes\mathbb{C}$ consists of two lines, which are interchanged by the complex conjugation. If w is a vector spanning one of the lines, then one needs to be $\langle w\|\overline{w}\rangle > 0$ for the inner product on W to be positive definite, and vectors w and $\overline{w}$ correspond to two copies of the plane W coming with different orientations: if the basis $\{w+\overline{w}, i(w-\overline{w})\}$ is positively oriented, then the null-vector corresponding to the plane W is w, and $\overline{w}$ otherwise.

Conversely, if $v \in V\otimes\mathbb{C}$ is such that $\langle v\|v\rangle = 0$ and $\langle v\|\overline{v}\rangle > 0$, then vectors $v+\overline{v}$ and $i(v-\overline{v})$ are both real and linearly independent, so they span a plane in V with an orientation. It is easy to see that the metric on this plane is positive definite. ☐

Proposition 1. *Let* $v \in V$ *be a non-zero vector. Then the subset of positive 2-planes orthogonal to* V *is exactly* $\mathrm{Gr}_{++}(v^\perp) \subset \mathrm{Gr}_{++}(V)$.

Proof. Indeed, the positive 2-plane W is orthogonal to v if and only if both w and $\overline{w}$ are. ☐

The tangent space to $\mathrm{Gr}_{++}(V)$ at the point W is the same as to the Grassmannian, $\mathrm{Hom}(W, V/W)$. But W is an oriented plane with positive definite metric, so it can be regarded as a one-dimensional complex vector space, turning the tangent space into a complex one. It gives an almost complex structure on $\mathrm{Gr}_{++}(V)$, which is the same as the restriction of the complex structure from the complex quadric $\{\langle v \| v\rangle = 0\} \subset \mathbb{P}(V \otimes \mathbb{C})$.

Proposition 2. $\mathrm{Gr}_{++}(\mathbb{R}^{2,n})$ *is a Stein manifold.*

Proof. $\mathrm{Gr}_{++}(\mathbb{R}^{2,n}) = \mathrm{SO}(2,n)/\mathrm{SO}(2) \times \mathrm{SO}(n)$, so it is a symmetric domain of non-compact type. Due to a theorem of É. Cartan [**C**], it can be holomorphically embedded into a complex space as a bounded domain.

See [**H**, Theorem 7.1] for a complete proof. □

Proposition 3. $\mathrm{Gr}_{++}(\mathbb{R}^{2,1})$ *is a topological disk.*

Proof. Let u, v, w be the orthogonal basis of $\mathbb{R}^{2,1}$ such that $\|v\|^2 = \|w\|^2 = 1$ and $\|u\|^2 = -1$. Any 2-plane W with positive definite restriction of the metric projects along u onto the plane $W_0 = \langle v, w\rangle$ isomorphically (just because W cannot contain the kernel of this projection, the line spanned by u). So $W = \langle v + au, w + bu\rangle$ for some real numbers a and b, and different pairs of numbers define different planes. The restriction of metric on W is positive definite iff for any real numbers x, y (at least one of which is not equal to zero) one has $0 < \|x(v + au) + y(w + bu)\|^2 = x^2 + y^2 - (ax + by)^2$, which is equivalent to the condition $a^2 + b^2 < 1$.

One can also send a positive plane in $\mathbb{R}^{2,1}$ into its orthogonal, which is a line spanned by a negative vector, and obtain a representation of $\mathrm{Gr}_{++}(\mathbb{R}^{2,1})$ as the projectivization of the negative cone in $\mathbb{R}^{2,1}$, which is precisely the Cayley–Klein model for the Bolyai–Lobachevskian plane. □

Proposition 4. $\mathrm{Gr}_{++}(\mathbb{R}^{n,m})$ *can be retracted onto* $\mathrm{Gr}_{++}(\mathbb{R}^{n,0})$. *In particular,* $\mathrm{Gr}_{++}(\mathbb{R}^{2,n})$ *is contractible.*

Proof. Let $l \subset V$ be a line spanned by a negative vector. Then if W is a positive 2-plane, then its projection along l in $l^{\perp}$ is also a positive 2-plane. This defines a fibration $\mathrm{Gr}_{++}(V) \to \mathrm{Gr}_{++}(l^{\perp})$, and its fiber over a plane $W' \subseteq l^{\perp}$ is a set of positive 2-planes in the linear hull of W' and l, i. e. $\mathrm{Gr}_{++}(\mathbb{R}^{2,1})$, which is contractible due to the previous Proposition. □

2.2 Period space

Now we shall study the geometry of the period space of a hyperkähler manifold X itself, which is, thanks to the global Torelli theorem, the space $\mathrm{Gr}_{++}(H^2(X,\mathbb{R}))$. From now onwards the letter V stands for a real vector space with metric of signature $(3, n)$.

Let $U \subset V$ be a 3-dimensional space with positive definite restriction of the metric on it. Then any 2-plane in it is positive, and they constitute a rational curve $\mathbb{C}P^1 = \mathrm{Gr}_{++}(U) \subset \mathrm{Gr}_{++}(V)$. In the case $V = H^2(X,\mathbb{R})$ there exist a natural positive 3-subspace in V spanned by the Kähler forms ω_I, ω_J and ω_K, and the corresponding family is the twistorial family. Because of that we shall call such curves *twistorial lines*, and denote the line consisting of 2-planes in a positive 3-subspace U by Tw_U. Twistorial lines are parametrized by the manifold $\mathrm{Gr}_{+++}(V)$ of positive 3-subspaces in the space V. Unlike the positive Grassmannian of 2-planes $\mathrm{Gr}_{++}(V)$, the Grassmannian of positive 3-subspaces $\mathrm{Gr}_{+++}(V)$ do not carry a natural complex structure (for example, in the case of period space of $K3$ surfaces its real dimension equals 57). On the other hand, the rational curves in $\mathrm{Gr}_{++}(V)$ are parametrized by the Hilbert scheme, which is a scheme over $\mathbb{C}$, so twistorial lines cannot be the only rational curves in $\mathrm{Gr}_{++}(V)$. It is also clear that the normal bundles of the twistorial lines are ample. One can map into, say twistorial line, a curve of any genus via the ramified covering, and then deform it into an embedded curve (that is possible because of amplitude of the normal bundle due to a theorem of J. Kollár [**Ko**]). That gives examples of many families of hyperkähler manifolds over curves.

Let $v \in V$ be a positive vector. Then the space $v^{\perp}$ has signature $(2, n)$, and $\mathrm{Gr}_{++}(v^{\perp})$ can be identified with a contractible bounded domain in a complex vector space. On the other hand, it is a divisor in the space $\mathrm{Gr}_{++}(V)$. We shall call it *a Cauchy divisor* (because of reasons explained in Section 3) and denote as Cau_v. For any 2-plane $W \in \mathrm{Gr}_{++}(V)$ the set of Cauchy divisors passing through W is the projectivization of the positive cone in $W^{\perp}$. It is easy to see that twistorial lines intersect the Cauchy divisors at one point.

Proposition 5. *The period space contains no compact submanifolds of dimension greater than* 1*.*

Proof. If $X \subset \mathrm{Per}$ is a compact submanifold, then $\mathrm{Cau}_v \cap X$ is a divisor on X. If it is not empty, it is a compact submanifold of the Stein manifold Cau_v, i. e. a set of points. Therefore the dimension of X needs to be equal to 1.

One can prove this theorem the other way around: the period space Per is a subset of a quadric in a complex projective space, so it carries a positive $(1,1)$-form ω. Due to the positivity one has $\int_X (\omega|_X)^{\dim X} > 0$. But the period space Per retracts onto the twistorial line, so $\omega^{\dim X} = d\eta$ for some $(2\dim X - 1)$-form η and the integral needs to vanish unless $\dim X > 1$. □

Now we can prove the main statement.

Proposition 6. *Any family of hyperkähler manifold over a compact simply connected base can be obtained as a pullback of a family of hyperkähler manifolds over a curve.*

Proof. The image of the period map $\mathfrak{per}(B)$ is a compact submanifold in the period space Per, thus a curve or a point. The period map factorizes through the normalization of the image $\widetilde{\mathfrak{per}(B)}$. □

3 Lorentzian Kähler metric on the period space

In the present section, we shall also deal with the fibrations over noncompact or non-simply connected base. All fibrations are assumed to be such that the corresponding period map is an immersion (so that the universal covers of their bases are subvarieties in the period space Per).

Definition 6. Let (X, g, I) be a complex manifold equipped with a metric of signature $(+, +, -, -, \dots)$. If the 2-form $\omega(u, v) = g(Iu, v)$ is closed, then we shall call such manifold a *Lorentzian Kähler manifold.*

Due to the LeBrun's lemma stated in the Section 2, the period space Per is in fact the positive Grassmannian $\mathrm{Gr}_{++}(\mathbb{R}^{3,b_2-3})$. Its tangent space at the point $W \subset V$ is equal to $\mathrm{Hom}(W, W^\perp)$ and carries a natural (and hence $\mathrm{SO}(3, b_2-3)$-invariant) metric of indefinite signature $(2, 2b_2 - 6)$. We shall denote in by g_{Per} and put $\omega_{\mathrm{Per}}(u, v) = g_{\mathrm{Per}}(Iu, v)$. This is a non-degenerate 2-form.

Proposition 7. $d\omega_{\mathrm{Per}} = 0$.

Proof. Stabilizer of a point W is a subgroup $\mathrm{SO}(W) \times \mathrm{SO}(W^\perp) \subset \mathrm{SO}(3, b_2 - 3)$. The group $\mathrm{SO}(W) = \mathrm{SO}(2)$ contains an operator $-\mathrm{Id}$, so the $\mathrm{SO}(3, b_2 - 3)$-invariant form $d\omega_{\mathrm{Per}}$ would be invariant under the fiberwise operator $-\mathrm{Id}$. However, it is a 3-form and thence vanishes. ☐

Nevertheless, this form is not exact because of the following reason:

Proposition 8. *Let $U \subset V = \mathbb{R}^{3,b_2-3}$ be a 3-subspace with positive definite metric, and $\mathrm{Gr}_{++}(U) \subset \mathrm{Gr}_{++}(V)$ be the corresponding twistorial line. Then the restriction $\omega_{\mathrm{Per}}|_{\mathrm{Gr}_{++}(U)}$ is the Fubini–Study form.*

Proof. Actually, one can naturally associate such a form to any positive Grassmannian Gr_{++}, and it would be compatible to its inclusions obtained from ones of vector spaces. The fact that this form on the $\mathbb{C}P^1 = \mathrm{Gr}_{++}(\mathbb{R}^{3,0})$ is its Fubini–Study form may be regarded as its definition. ☐

As $\mathrm{Gr}_{++}(\mathbb{R}^{3,n})$ retracts onto any twistorial line, its second cohomology is one-dimensional, and it is spanned by the cohomology class $[\omega_{\mathrm{Per}}]$. Moreover, one holds the following

Proposition 9. *The form ω_{Per} is the unique up to rescaling* $\mathrm{SO}(3, b_2 - 3)$-*invariant 2-form on the period space* Per.

Proof. One needs to check that the form $g_{\mathrm{Per}}|_W$ is a unique $\mathrm{SO}(2) \times \mathrm{SO}(1, b_2-3)$-invariant form on the space $\mathrm{Hom}(W, W^\perp)$. This 2-form defines a representation homomorphism $\mathrm{Hom}(W, W^\perp) \to \mathrm{Hom}(W^\perp, W)$. But $\mathrm{Hom}(W, W^\perp) = W \otimes (W^\perp)^*$. W is an irreducible representation of $\mathrm{SO}(2)$, and $(W^\perp)^*$ is an irreducible representation of $\mathrm{SO}(1, b_2 - 3)$, so $\mathrm{Hom}(W, W^\perp)$ is an irreducible representation of $\mathrm{SO}(2) \times \mathrm{SO}(1, b_2-3)$. Schur's lemma implies that the space of homomorphisms between $\mathrm{Hom}(W, W^\perp)$ and its dual is one-dimensional. This proves the Proposition. ☐

Because of invariance of the form ω_{Per} on the period space, its pullback $\mathfrak{per}^*(\omega_{\mathrm{Per}}) \in \Omega^2(\widetilde{B})$ is invariant under the $\pi(B)$-action, so it descends to the base and thus defines an invariant of a family $\mathfrak{X} \to B$ of hyperkähler manifolds in the space of 2-forms on B. We shall denote the 2-form on B obtained via this construction as $\varpi_{\mathfrak{X}} \in \Omega^2(B)$.

Proposition 10. *The 2-form $\varpi_{\mathfrak{X}}$ corresponding to a family $\mathfrak{X}$ is non-degenerate.*

Proof. The form $\varpi_{\mathfrak{X}}$ is closed, so the distribution of its kernels is integrable. As the form ω_{Per} is non-degenerate, the leaves of this distribution map into points in the period space. If these leaves are not 0-dimensional (i. e., the form $\varpi_{\mathfrak{X}}$ is degenerate), then the period map cannot be an immersion. □

Due to the Proposition 8, if $\mathfrak{X} \to \mathbb{C}P^1$ is the twistorial family, then $\varpi_{\mathfrak{X}}$ is the Fubini–Study form on $\mathbb{C}P^1$. One may hope that this form $\varpi_{\mathfrak{X}}$ is the proper substitute for the Fubini–Study form for the families other than twistorial in the statements like [**KV**, Proposition 8.15].

Proposition 11. *Base of any family of hyperkähler manifolds carries either a Kähler or a Lorentzian Kähler structure.*

Proof. The 2-form ϖ respects the complex structure on the base because of naturality of its construction. The corresponding pseudo-Riemannian metric has the positive index of inertia either two or zero. If latter, then $-\varpi$ is a Kähler form. If former, ϖ is a Lorentzian Kähler form. □

One can find some similarities between the Lorentzian Kähler geometry of the positive Grassmannian $\mathrm{Gr}_{++}(\mathbb{R}^{3,n})$ and geometry of usual Lorentzian manifolds. For example, twistorial lines resemble timelike geodesics, whilst the divisors Cau_v are similar to Cauchy hypersurfaces. Nevertheless, this similarity is not complete: for example, while on a Lorentzian manifold X containing a Cauchy hypersurface M a bunch of timelike geodesics startled orthogonally from M defines a decomposition $X = M \times \mathbb{R}$ [**ChN**, Section 3], twistorial lines orthogonal to a Cauchy divisor Cau_v massively intersect, forming somewhat like a Lefschetz pencil.

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