

A new randomized algorithm for the Erdős–Hajnal problem

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Abstract

In 1961 Erdős and Hajnal introduced the quantity $m(n)$ as the minimum number of edges in an n -uniform hypergraph with chromatic number at least 3. The best known lower and upper bounds for $m(n)$ are $c_1 \sqrt{\frac{n}{\ln n}} 2^n$ and $c_2 n^2 2^n$ respectively. The lower bound is due to Radhakrishnan and Srinivasan (see [6]). A natural generalization for $m(n)$ is the quantity $m(n, r)$, which is the minimum number of edges in an n -uniform hypergraph with chromatic number at least $r + 1$. In this work, we present a new randomized algorithm yielding a bound $m(n, r) \geq cn^{\frac{r-1}{r}} r^{n-1}$, which improves upon all the previous bounds in a wide range of the parameters n, r . Moreover, for $r = 2$, we get exactly the same bound as in the work [6] of Radhakrishnan and Srinivasan, and our proof is simpler.

Keywords: hypergraph colorings, randomized algorithms.

1 Introduction

In 1961 Erdős and Hajnal (see [1]) introduced the quantity $m(n)$, which is the minimum number of edges in an n -uniform hypergraph H with chromatic number greater than 2. The problem of finding the right order of $m(n)$ is one of the most important in extremal combinatorics, and substantial work has been done in this direction (see, e.g., [3], [7]). In particular, the best known lower bound for the value $m(n)$ is due to Radhakrishnan and Srinivasan (see [6]) who proposed in 2000 an ingenious randomized algorithm resulting in the following theorem.

Theorem 1. *For any $q < \sqrt{2}$, there exists n_0 such that for every $n \geq n_0$, the inequality holds*

$$m(n) \geq q \left(\frac{n}{\ln n} \right)^{1/2} 2^{n-1}. \quad (1)$$

An important generalization of the quantity $m(n)$ is the quantity $m(n, r)$, which is the minimum number of edges in an n -uniform hypergraph H with chromatic number greater than r (see [2]). For $m(n, r)$, several different lower bounds are found, each of them being currently the strongest for some range of the parameters n, r . Let us briefly describe the whole picture. First, we mention the work [8] by Shabanov who just noticed that the randomized algorithm by Radhakrishnan and Srinivasan can be generalized to give the bound

$$m(n, r) \geq (\sqrt{3} - 1) \left(\frac{n}{\ln n} \right)^{1/2} r^{n-1} \quad (2)$$

for any $n \geq 2, r \geq 2$. Second, there was an interesting and rather involved algorithm by Kostochka (see [4]) proving that for $r = O(\sqrt{\ln \ln n})$, one has

$$m(n, r) \geq e^{-4r^2} \left(\frac{n}{\ln n} \right)^{a/(a+1)} r^n, \quad (3)$$

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where $a = \lfloor \log_2 r \rfloor$. Finally, there were some improvements by Pluhár and Shabanov (see [5], [9]):

$$m(n, r) \geq \frac{1}{4} n^{1/2} r^{n-1}, \quad n \geq 2, \quad r \geq 3, \quad (4)$$

and

$$m(n, r) \geq \left(\pi^{\frac{1}{r}} e^{-\frac{1}{12(n-1)}} \right) \frac{1}{e\sqrt{2\pi}} (n-1)^{\frac{1}{2}-\frac{1}{2r}} r^{n+2/r}. \quad (5)$$

It is worth noting here that sometimes we write in the bounds r^n and sometimes we write r^{n-1} . The difference becomes substantial if r is a growing function of n .

In this paper we propose a new randomized algorithm, and using it we prove the following theorem.

Theorem 2. *Let $r = o\left(\frac{\ln n}{\ln \ln n}\right)$. Let*

$$q < \frac{(r!)^{\frac{1}{r}}}{2^{\frac{1}{r}} \left(1 - \frac{1}{r}\right)^{\frac{r-1}{r}} r^{\frac{r-2}{r}}} =: q'.$$

Then there exists n_0 such that for every $n \geq n_0$, the inequality holds

$$m(n, r) \geq q \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1}. \quad (6)$$

On the one hand, estimate (6) improves substantially upon all the previous estimates (2)–(5), provided r is not too large. On the other hand, if we put $r = 2$ in Theorem 2, then we get *exactly* the result of Theorem 1. As we will see from the proof in the next section, our proof is even shorter than that of Theorem 1.

Note that the condition $r = o\left(\frac{\ln n}{\ln \ln n}\right)$ is sufficient, but not necessary. It is just taken for simplicity of the proof.

To complete this section we present one more result, which will be obtained automatically from the proof of Theorem 2.

Theorem 3. *Let $r = o\left(\frac{\ln n}{\ln \ln n}\right)$. Then there exists a constant $c > 0$ such that whenever an n -uniform hypergraph has maximum edge degree*

$$D < c \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1},$$

its chromatic number is not greater than r .

2 Proofs of new results

2.1 An auxiliary result

In this section, we obtain a result, which is suggested by Pluhár's work [5]. First, we briefly describe Pluhár's assertion. Let $H = (V, E)$ be a hypergraph (not necessarily uniform). Let σ denote an ordering of the vertices V . A family of edges $A_1, \dots, A_r \in E$ is called *ordered r -chain* in σ , if the following conditions hold:

1. for any $i \in \{1, \dots, r-1\}$, we have $|A_i \cap A_{i+1}| = 1$;
2. for any i, j such that $|i - j| > 1$, we have $A_i \cap A_{i+1} = \emptyset$;

3. for any $i \in \{1, \dots, r-1\}$ and any $v \in A_i, u \in A_{i+1}$, we have $\sigma(v) \leq \sigma(u)$.

Proposition 1 below was proved by Pluhár.

Proposition 1. *The chromatic number of a hypergraph $H = (V, E)$ does not exceed r if and only if there exists an ordering σ of V such that there are no ordered r -chains in H .*

Here we introduce the notion of a *strong ordered r -chain*. To this end we assume that there is a map f from the set of edges E of a hypergraph to the set $\{1, \dots, r\}$, i.e., to each edge a label is assigned. A family of edges $A_1, \dots, A_r \in E$ is called *strong ordered r -chain* in σ , if it is subject to conditions 1–3 and, moreover, for any i , we have $f(A_i) = i$. Note that both notions of r -chains have sense even in the cases of hypergraphs with multiple edges and with edges of length 1. The only subtlety here is as follows: if we have coinciding edges of length 1, then condition 2 is not completely correct; for example, a chain $\{1, 2\}, \{2\}, \{2\}, \{2, 3\}$ is admissible.

Proposition 2. *Consider a hypergraph $H = (V, E)$ and a map f from E to $\{1, \dots, r\}$. Then the following statements are equivalent: (i) there is a coloring of V in r colors such that no edge A in E consists only of vertices of color $f(A)$; (ii) there is an order of elements of V without strong ordered r -chains.*

Proof of Proposition 2. *(ii) implies (i).* Fix an ordering σ of V . Make a greedy coloring. First, color all the vertices in color 1. Then consider the vertices in order σ . If the i th vertex is the first vertex of some edge, then recolor it in the color with the minimal number such that there is no conflict with edges that contain only vertices from the set $1, \dots, i$. If this is impossible, then recolor the i th vertex with color r .

Clearly if the final coloring is not good, then any bad edge has color r . Consider an arbitrary bad edge A_r . By definition $f(A_r) = r$. The first vertex of A_r could not be colored $r-1$. Hence it is the last vertex of an edge A_{r-1} all of whose vertices except for one are colored $r-1$; moreover, $f(A_{r-1}) = r-1$. Going this way we get a strong ordered r -chain.

(i) implies (ii). Assume we have a good coloring. Then we order the vertices by growth of color. If there is a strong ordered r -chain, then it is easy to prove by induction that the i th edge in it has a vertex whose color is greater than i , which is impossible for the last edge. It is a contradiction.

2.2 Proof of Theorem 2

Fix a hypergraph $H = (V, E)$ with

$$|E| \leq q \left(\frac{n}{\ln n} \right)^{\frac{r-1}{r}} r^{n-1}.$$

We want to show that its chromatic number is not greater than r . As it is usual in this type of proofs, we make a random coloring. The procedure consists of two steps. At the first step, we color every vertex (independently) into color $i \in \{1, \dots, r\}$ with probability $\frac{1-p}{r}$ for some p . So with probability p a vertex is colorless. Suppose there are no monochromatic, almost monochromatic, and fully colorless edges (an edge is almost monochromatic, if exactly one of its vertices is colorless and all the other vertices have same color). At the second step, we take the set $W \subseteq V$ of colorless vertices. Then for every $A \in E$, we take $B = A \cap W$: $|B| \notin \{0, 1, |A|\}$. We get a hypergraph $H' = (W, F)$. If an edge $B \in F$ is such that $A \setminus B$ is not monochromatic after the first step, we may forget about it, since it is already colored in a right way. Thus, we consider only those $B \in F$, for which $A \setminus B$ is completely colored with some color $i \in \{1, \dots, r\}$. Let $f(B) = i$ and take a random ordering of W . If there are no strong ordered r -chains, then by Proposition 2 we have a good coloring.

We must prove that the probability of a failure is less than 1. Put

$$p = \frac{(r-1) \ln \left(r^{\frac{r}{r-1}} n \ln^2 n \right)}{rn}.$$

The probability that a monochromatic edge appears after the first step is bounded from above by

$$|E| \cdot r \cdot \left(\frac{1-p}{r}\right)^n \leq |E| r^{1-n} n^{\frac{1-r}{r}} r^{-1} (\ln^2 n)^{\frac{1-r}{r}} \leq q \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1} r^{1-n} n^{\frac{1-r}{r}} r^{-1} (\ln^2 n)^{\frac{1-r}{r}} \rightarrow 0, \quad n \rightarrow \infty.$$

The probability that an almost monochromatic edge appears after the first step is bounded from above by

$$|E| \cdot rnp \cdot \left(\frac{1-p}{r}\right)^{n-1} \leq (1+o(1))q \left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1} r^{2-n} (\ln n) n^{\frac{1-r}{r}} r^{-1} (\ln^2 n)^{\frac{1-r}{r}} \sim q(\ln n)^{1-3\frac{r-1}{r}} \rightarrow 0.$$

The probability that a fully colorless edge appears after the first step does not exceed $|E|p^n$, which is tiny.

Fix an r -chain in H' . Let $a_1, \dots, a_r \geq 2$ be the cardinalities of its edges. The probability that this chain is ordered equals

$$2 \frac{(a_1 - 1)!(a_r - 1)! \prod_{i=2}^{r-1} (a_i - 2)!}{(\sum a_i - r + 1)!} =: M(a_1, \dots, a_r).$$

The probability that our chain is strong equals

$$p^{r-1} p^{a_1-1} \left(\frac{1-p}{r}\right)^{n-a_1} C_{n-1}^{a_1-1} \left(\prod_{i=2}^{r-1} p^{a_i-2} \left(\frac{1-p}{r}\right)^{n-a_i} C_{n-2}^{a_i-2}\right) p^{a_r-1} \left(\frac{1-p}{r}\right)^{n-a_r} C_{n-1}^{a_r-1} =: N(a_1, \dots, a_r).$$

The number of r -chains is less than or equal to $|E|^r/r!$. Thus the final probability does not exceed the value

$$\frac{|E|^r}{r!} \sum_{a_1, \dots, a_r} M(a_1, \dots, a_r) N(a_1, \dots, a_r) \leq \frac{2|E|^r}{r!} \sum_{a_1, \dots, a_r} \frac{p^{\sum a_i - r + 1} n^{\sum a_i - 2r + 2} \left(\frac{1-p}{r}\right)^{nr - \sum a_i}}{(\sum a_i - r + 1)!}.$$

Fix $t \in \mathbb{N}$. The number of collections of a_i 's with $\sum a_i - 2r + 2 = t$ is less than t^{r-1} . Hence, we may write the estimate

$$\begin{aligned} & \frac{2|E|^r}{r!} \sum_{t=0}^{\infty} t^{r-1} \frac{p^{t+r-1} n^t \left(\frac{1-p}{r}\right)^{nr-t-2r+2}}{(t+r-1)!} \leq \frac{2|E|^r}{r!} p^{r-1} \left(\frac{1-p}{r}\right)^{nr-2r+2} \sum_{t=0}^{\infty} \frac{p^t n^t \left(\frac{1-p}{r}\right)^{-t}}{t!} \leq \\ & \leq \frac{q}{q'} (1+o(1)) \left(1 - \frac{1}{r}\right)^{1-r} r^{2-r} \left(\frac{n}{\ln n}\right)^{r-1} r^{nr-r} \left(1 - \frac{1}{r}\right)^{r-1} n^{1-r} (\ln n)^{r-1} e^{-pnr} r^{2r-2-nr} e^{\frac{pnr}{1-p}} = \frac{q}{q'} (1+o(1)). \end{aligned}$$

Theorem 2 is proved.

2.3 Proof of Theorem 3

Let us proceed with the same random coloring as in the proof of Theorem 2. We have the two following types of events. On the one hand, at the first step of our algorithm, we can get a monochromatic, an almost monochromatic, or a fully colorless edge. Let P_1 be the probability of any such event (corresponding to a fixed edge). On the other hand, at the second step, we can get a strong ordered r -chain. Let P_2 be the probability of any such event (corresponding to a fixed r -chain). The values of P_1, P_2 are given in the previous section.

We have to prove that none of these events occurs with positive probability. It is quite standard to use here Lovász Local Lemma (LLL). Clearly any edge intersects at most D other edges and D^r different r -chains. Also any r -chain intersects at most rD edges and rD^r other r -chains. Thus, to utilize LLL we just need to find such x and y that simultaneously satisfy the inequalities

$$P_1 \leq x(1-x)^D(1-y)^{D^r}, \quad P_2 \leq y(1-x)^{rD}(1-y)^{rD^r}.$$

A tedious and standard calculation that we omit here shows that taking $x = \frac{a}{D}$ with an arbitrary positive a and $y \asymp \frac{1}{rD^r}$ one eventually gets the result of Theorem 3.

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