Tau function and moduli of spin curves

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Abstract

The goal of the paper is to give an analytic proof of the formula of G. Farkas for the divisor class of spinors with multiple zeros in the moduli space of odd spin curves. We make use of the technique developed by Korotkin and Zograf that is based on properties of the Bergman tau function.

1 The moduli space of odd spin curves.

Let $\mathcal{M}_g$ be the moduli space of smooth genus g algebraic curves, assume that $g \geq 3$. Let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. The boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ consists of $\left[\frac{g}{2}\right] + 1$ irreducible divisors $\Delta_0, \ldots, \Delta_{\left[\frac{g}{2}\right]}$ where Δ_0 is the closure of the locus of irreducible curves with one node and Δ_j for $j \geq 1$ is the closure of the locus of reducible one-nodal curves.

The moduli space $\mathcal{S}_g^-$ of *smooth* odd spin curves is $2^{g-1}(2^g - 1)$ cover of $\mathcal{M}_g$. The cover is extended to a branched cover of $\overline{\mathcal{M}}_g$ by the Cornalba compactification $\overline{\mathcal{S}}_g^-$ of $\mathcal{S}_g^-$ ramified over Δ_0.

Cornalba compactification. A nodal curve C is called *quasi-stable* if it satisfies two conditions:

1) A rational component E of C intersects $\overline{C \setminus E}$ at two or more points;

2) Any two rational components E_1, E_2 of C such that $\# E_i \cap \overline{C \setminus E_i} = 2$ are disjoint.

Rational component E of C intersecting $\overline{C \setminus E}$ at exactly two points is called *exceptional*.

Following [2] we define a *spin curve* as a triple (C, η, β) consisting of a quasi-stable curve C, a line bundle η of degree $g-1$ on it and a homomorphism $\beta : \eta^{\otimes 2} \to \omega_C$ with the following properties:

1) η is of degree one on every exceptional component of C;

2) β is not a zero on every non-exceptional component of C.

The parity of the spin curve (C, η, β) is the parity of $\dim H^0(C, \eta)$. The parity is invariant under continuous deformations (see [11] or [1]).

An isomorphism between (C, η, β) and (C', η', β') is an isomorphism $\sigma : C \to C'$ such that $\sigma^* \eta'$ and η are isomorphic and the following diagram

$$\begin{array}{ccc} \eta^2 & \xrightarrow{\phi \otimes \phi} & \sigma^*(\eta')^2 \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \sigma^* \beta'} \\ \omega_C & \xrightarrow{\simeq} & \sigma^* \omega_{C'} \end{array}$$

is commutative, where ϕ is an isomorphism between η and $\sigma^* \eta'$. The moduli space $\overline{\mathcal{S}}_g^-$ consists of all equivalence classes of odd spin curves under such isomorphisms. The projection

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$\rho : \overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ maps (an equivalence class of) a triple (C, η, β) to (an equivalence class of) a curve $\tilde{C}$ which is obtained from C by contracting all exceptional components to points.

Rational Picard group of $\overline{\mathcal{S}}_g^-$. We follow notations of [4] in the description of the Picard group here.

The boundary $\overline{\mathcal{S}}_g^- \setminus \mathcal{S}_g^-$ is the union of irreducible divisors $A_0, \dots, A_{[g/2]}, B_0, \dots, B_{[g/2]}$ such that $\rho(A_j) = \rho(B_j) = \Delta_j$ for $j = 0, \dots, \left[\frac{g}{2}\right]$.

Description of A_j and B_j for $j \neq 0$. Note that there are no spin curves (C, η, β) with a reducible one-nodal base curve C, since the relative dualizing sheaf ω_C on a reducible curve with one node being restricted to each component must be of odd degree (see [2], [4, p.5] for more details).

Let (C, η, β) be a spin curve such that $C = C_1 \cup E \cup C_2$ where C_1 and C_2 are smooth curves of genus j and $g - j$ respectively and E is an exceptional component. The divisor A_j parametrizes the closure of the locus of such curves with the property that η restricted to C_1 is odd. The divisor B_j is the closure of the locus of the same type spin curves such that η restricted to C_1 is even.

Description of A_0 and B_0. Unlike the case $j \neq 0$ a spin curve (C, η, β) such that $\rho(C, \eta, \beta)$ is an irreducible one-nodal curve, does not necessary have exceptional components. Let A_0 parametrize the closure of the locus of spin curves with one-nodal irreducible underlying curve and B_0 parametrize the closure of the locus of spin curves mapping to Δ_0 under ρ and having an exceptional component.

Recall that ρ has a two-order branching along B_0 and is unramified on $\overline{\mathcal{S}}_g^- \setminus B_0$.

Denote by α_j and β_j the classes of A_j and B_j in the rational Picard group $\mathrm{Pic}(\overline{\mathcal{S}}_g^-) \otimes \mathbb{Q}$ respectively. Let λ be the pullback of the Hodge class on $\overline{\mathcal{M}}_g$ under ρ. The Picard group is generated by the classes

$$\mathrm{Pic}(\overline{\mathcal{S}}_g^-) \otimes \mathbb{Q} = span_{\mathbb{Q}}(\lambda, \alpha_0, \dots, \alpha_{\left[\frac{g}{2}\right]}, \beta_0, \dots, \beta_{\left[\frac{g}{2}\right]}). \tag{1.1}$$

Consider the divisor $\mathcal{Z}_g$ on $\overline{\mathcal{S}}_g^-$ parametrizing the closure of the locus of smooth spin curves (C, η) such that sections of η has multiple zeros. The class of $\mathcal{Z}_g$ in the rational Picard group $\mathrm{Pic}(\overline{\mathcal{S}}_g^-) \otimes \mathbb{Q}$ can be expressed as a linear combination of generators (1.1) (see (4.13)). G. Farkas determined the coefficients in this expansion and used it for the birational classification of moduli spaces of odd spin curves (see [4]). The goal of this paper is to show how this coefficients can be computed analytically from properties of the Bergman tau function on the moduli space of abelian differentials.

The paper is organized as follows: we introduce the Bergman tau function and list its basic properties in Section 2. In Section 3 we study the asymptotics of the theta function under a degeneration of a curve; this asymptotics is well-known (see [13]) but we write it down to fix notations. Then in Section 4 we construct an odd spinor using the theta function and analyze the behavior of the tau function on the space of squares of these odd spinors. This results in the Farkas formula for $\mathcal{Z}_g$. Finally in Section 5 we study the theta-null divisor on the moduli space of even spin curves. The goal is to show how to express the theta-null in terms of standard generators of the rational Picard group in the framework of the classical theory of theta functions. This expression was also obtained by G. Farkas in his work [5] by different methods. G. Farkas used this expression for the birational classification of the moduli space of even spin curves.

2 The Bergman tau function on moduli spaces of holomorphic differentials with double zeros.

Let $\mathcal{H}_g$ denote the moduli space of holomorphic differentials on smooth genus g curves (see [8]). This space admits a natural stratification according to multiplicities of zeros of the differential. Denote by $\mathcal{H}_g([2^{g-1}])$ the stratum corresponding to differentials with $g-1$ distinct zeros of multiplicity two. Let C be a genus g curve and ω be a differential on C such that $(C,\omega) \in \mathcal{H}_g([2^{g-1}])$. If $\operatorname{div}\omega = 2D$ then the linear system $|D|$ corresponds to a spin bundle on $L \to C$. Let $\mathcal{H}_g^-([2^{g-1}])$ be the connected component of $\mathcal{H}_g([2^{g-1}])$ corresponding to the case when L is an odd spin bundle (see [9]).

Homological coordinates. Consider the (non-holomorphic) vector bundle $H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{C})$ over $\mathcal{H}_g^-([2^{g-1}])$ whose fiber over a pont (C,ω) is the relative cohomology group $H^1(C,\{p_1,\dots,p_{g-1}\},\mathbb{C})$, where $p_1,\dots,p_{g-1}$ are zeros of ω. We have a natural map $\mathcal{H}_g^-([2^{g-1}]) \to H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{C})$ which sends (C,ω) to the cohomology class of ω. The bundle $H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{C})$ has a lattice $H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{Z})$ in it. Take an open coordinate (in the sense of orbifold) subset of $\mathcal{H}_g^-([2^{g-1}])$ and consider a trivialization $H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{C})|_{\mathcal{U}} \to \mathcal{U}\times\mathbb{C}^{3g-2}$ such that $H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{Z})$ maps to the lattice $\mathbb{Z}^{3g-2}\subset\mathbb{C}^{3g-2}$. The composition of the map $\mathcal{H}_g^-([2^{g-1}]) \to H^1(\cdot,\{p_1,\dots,p_{g-1}\},\mathbb{C})$ and such trivialization gives a set of holomorphic local coordinates called *homological* (see [8]). Let us study this construction in more datails.

Denote by $\mathcal{T}_g$ the moduli space of Torelli marked curves (i. e. curves with a fixed symplectic basis in $H_1(C)$), and let $\tilde{\mathcal{H}}_g^-([2^{g-1}])$ be the cover of $\mathcal{H}_g^-([2^{g-1}])$ induced by the forgetful map $\mathcal{T}_g \to \mathcal{M}_g$.

Fix an arbitrary point $(C,\nu,\omega) \in \tilde{\mathcal{H}}_g^-([2^{g-1}])$, where we denote the Torelli marking by ν. Let $p_1,\dots,p_{g-1}\in C$ be the zeros of ω. Consider simple non-intersecting paths l_j connecting p_{g-1} with p_j for $j=1,\dots,g-2$. Let $a_1,\dots,a_g,b_1,\dots,b_g$ be simple loops on $C\setminus\{p_1,\dots,p_{g-1}\}$ representing that do not intersect $\{l_j\}_{j=1}^{g-2}$. Then homological coordinates coordinates at $(C,\{a_j,b_j\}_{j=1}^{g},\omega)$ are given by:

$$\begin{aligned} z_j &= \int_{a_j^\circ}\omega, \quad j=1,\dots,g,\\ z_{j+g} &= \int_{b_j^\circ}\omega, \quad j=1,\dots,g,\\ z_{j+2g} &= \int_{l_j}\omega, \quad j=1,\dots,g-2. \end{aligned}$$

Denote by $s_1,\dots,s_{3g-2}$ the basis in $H_1(C\setminus\{p_1,\dots,p_{g-1}\})$ dual to the basis represented by $a_1,\dots,a_g,b_1,\dots,b_g,l_1,\dots,l_{g-2}$ in the relative homology group $H_1(C,\{p_1,\dots,p_{g-1}\})$; we have $s_j=-b_j, s_{g+j}=a_j$ and s_{2g+j} is homologous to a small positive oriented circle around p_j. We will use this notations until the end of the paper.

Projective connections. Let $f: U\to V$ be a holomorphic map between two domains $U,V\subset\mathbb{C}P^1$. Recall that the Shcwarzian derivative of f with respect to a local parameter $z\in U$ is defined as

$$S_z^f = \frac{\frac{d^3f}{dz^3}}{\frac{df}{dz}} - \frac{3}{2}\left(\frac{\frac{d^2f}{dz^2}}{\frac{df}{dz}}\right)^2.$$

If $z=h(w)$ is a change of the parameter then

$$S_w^{f\circ h}\,dw^2 = S_z^f\,dz^2 + S_w^h\,dw^2.$$

We also have

$$S_z^f\,dz^2 = -S_f^z\,df^2.$$

Note that two relations above implies that S_z^f is invariant under mobius transformations of f and z.

Let C be a smooth curve of genus g and U_j, z_j be a coordinate covering of C. A meromorphic *projective connection* on C is a collection of meromorphic functions $f_j : U_j \to \mathbb{C}P^1$ such that

$$f_j\, dz_j^2 = f_k\, dz_k^2 - S_{z_j}^{z_k}\, dz_j^2.$$

It is clear from the definition and properties of the Schwarzian derivative that all meromorphic projective connections form an affine space over the space of meromorphic quadratic differentials on C.

Denote the diagonal of $C \times C$ by Δ. Let π_1 and π_2 be projections to the first and the second factors. We call a symmetric holomorphic section of the sheaf $(\pi_1^* K_C \otimes \pi_2^* K_C)(2\Delta)$ *symmetric bideffirential of the second kind*. Consider such a section B and its expansion near Δ with respect to some local coordinate ζ on C:

$$B(x,y) = \left(\frac{\alpha}{(\zeta(x) - \zeta(y))^2} + \frac{S(\zeta(x))}{6} + O(\zeta(x) - \zeta(y))^2 \right), \quad \text{as } x \to y.$$

The number α is called *biresidue* of B. It does not depend on the choice of a local coordinate. One can directly compute that $S(\zeta(x))$ behaves as a projective connection.

Definition of the tau function and its basic properties. Let ζ be a local coordinate on a curve C. For any differential ω on C introduce the meromorphic projective connection $S_\omega = \frac{\omega''}{\omega} - \frac{3}{2}\left(\frac{\omega'}{\omega}\right)^2$ (that is, the Schwarzian derivative of the abelian integral $\int^x \omega$ with respect to a local parameter ζ on C). The *canonical bidifferential* is a symmetric bideffirential of the second kind with biresidue 1 whose a-periods with respect to each coordinate are zero. Denote the canonical bidifferential on C by $\mathcal{B}(x,y)$. It has the following expansion in terms of a local parameter ζ:

$$\mathcal{B}(x,y) = \left(\frac{1}{(\zeta(x) - \zeta(y))^2} + \frac{S_B(\zeta(x))}{6} + O(\zeta(x) - \zeta(y))^2 \right) d\zeta(x) d\zeta(y) \quad \text{as } x \to y.$$

The projective connection S_B is called the *Bergman projective connection*. The difference of the two projective connections $S_B - S_\omega$ is a meromorphic quadratic differential on C. Introduce a connection on the trivial line bundle on $\tilde{\mathcal{H}}_g^-([2^{g-1}])$ by the formula

$$d_B = d + \frac{6}{\pi i} \sum_{j=1}^{3g-2} \left(\int_{s_j} \frac{S_B - S_\omega}{\omega} \right) dz_j.$$

As it was shown in [6] this connection is flat. The *tau function* $\tau = \tau(C, \{a_j, b_j\}_{j=1}^g, \omega)$ is defined up to a constant factor[1] as a horizontal (covariant constant) section of the trivial line bundle on $\tilde{\mathcal{H}}_g^-([2^{g-1}])$. In other words, $\tau : \tilde{\mathcal{H}}_g^-([2^{g-1}]) \to \mathbb{C}$ is a holomorphic function such that

$$d_B \tau = 0. \tag{2.1}$$

A solution of (2.1) was explicitly constructed in [6].

The group $Sp(2g, \mathbb{Z}) \times \mathbb{C}^*$ acts naturally on $\tilde{\mathcal{H}}_g^-([2^{g-1}])$ by changing the Torelli marking and multiplying the differential by a nonzero complex number. Note that $\tilde{\mathcal{H}}_g^-([2^{g-1}])/Sp(2g, \mathbb{Z})$ coincides with $\mathcal{H}_g^-([2^{g-1}])$.

Consider a natural map $\pi : \mathcal{H}_g^-([2^{g-1}])/\mathbb{C}^* \to \mathcal{S}_g^-$ which assigns to a differential the spin bundle associated with the square root of the differential. The map π is generally one-to-one, since an odd spin bundle generically has one-dimensional space of holomorphic sections. The image of π is $\mathcal{S}_g^- \setminus \mathcal{Z}_g$.

[1] In fact this is the 72-th power of the Bergman tau function studied in [6]. The name "tau function" is due to the relation of τ to isomonodromic Jimbo-Miwa tau function in the case of Hurwitz spaces.

Consider the tautological line bundle $\mathcal{L} \to \mathcal{H}_g^-([2^{g-1}])/\mathbb{C}^*$ with respect to the action of $\mathbb{C}^*$. Let $\mathbb{E}_g$ be the pullback of the Hodge vector bundle on $\mathcal{M}_g$ to $\mathcal{H}_g^-([2^{g-1}])/\mathbb{C}^*$. Denote by Λ the corresponding determinant bundle $\bigwedge^g \mathbb{E}_g$.

Lemma 2.1 (see [6] for the proof)**.** *The tau function has the following properties:*

1) τ is a nowhere vanishing holomorphic function on $\tilde{\mathcal{H}}_g^-([2^{g-1}])$.

2) For any $t \in \mathbb{C}^*$

$$\tau(C, \{a_j, b_j\}_{j=1}^g, t\,\omega) = t^{16(g-1)}\,\tau(C, \{a_j, b_j\}_{j=1}^g, \omega).$$

3) For any symplectic transformation σ in $H_1(C)$

$$\tau(C, \{\sigma(a_j), \sigma(b_j)\}_{j=1}^g, \omega) = \det(b\Omega + a)^{72}\,\tau(C, \{a_j, b_j\}_{j=1}^g, \omega),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the basis $\{a_j, b_j\}_{j=1}^g$.

Equivalently, τ is a section of a bundle $\mathrm{Hom}(\mathcal{L}^{16(g-1)}, \Lambda^{72})$ *on $\mathcal{H}_g^-([2^{g-1}])/\mathbb{C}^*$.*

3 Asymptotic behavior of the theta function under a curve degeneration.

All facts written down in this section are well-known and can be found in classical literature.

Theta characteristics. Let F be a vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension $2g$ with non-degenerate skew-symmetric pairing. Fix a symplectic basis $e_1, \ldots, e_g, f_1, \ldots, f_g \in F$. The set of all quadratic forms on F is in natural bijection with points from $(\mathbb{Z}/2\mathbb{Z})^{2g}$: given $(\eta_1, \ldots, \eta_{2g}) \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ we construct a quadratic form by the rule

$$\sum_{j=1}^{g}(a_j e_j + b_j f_j) \mapsto \sum_{j=1}^{g}(\eta_{2j-1} a_j + \eta_{2j} b_j) + \sum_{j=1}^{g} a_j b_j.$$

It is convenient for us to define *theta characteristic* to be a vector $\eta \in (\mathbb{Z}/2\mathbb{Z})^{2g}$. The parity of a theta characteristic η is given by the Arf invariant of the corresponding quadratic form (recall that the Arf invariant is equal to $\sum_{j=1}^{g} \eta_{2j}\eta_{2j-1}$). We call $0 \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ *zero characteristic.* The action of $Sp(g, \mathbb{Z})$ on F pulls back to the action on the set of all theta characteristics.

Consider a smooth curve C of genus g. Any spin bundle L over C defines a quadratic form q_L on the $\mathbb{Z}/2\mathbb{Z}$-vector space $J_2(C) := \{X \in \mathrm{Jac}(C) \mid 2X = 0\}$ (the symplectic pairing on $J_2(C)$ is induced from the Jacobian) by the following rule (see [11]):

$$q_L(X) = h^0(C, L \otimes X) + h^0(C, L) \mod 2.$$

The correspondence $L \mapsto q_L$ is a bijection between the set of isomorphism classes of spin bundles over C and the set of quadratic forms on $J_2(C)$. The parity of L coincides with the Arf invariant of q_L. It we fix a Torelli marking of C, then we obtain a basis in $J_2(C)$. Therefore *a choice of a Torelli marking induces a natural correspondence between spin bundles and theta characteristics. This correspondence respects the parity and commutes with the action of the symplectic group.*

Plumbing families. We introduce families in $\overline{\mathcal{S}}_g^-$ whom intersect the boundary of $\overline{\mathcal{S}}_g^-$ transversally at generic points.

For $0 \leq j \leq \left[\frac{g}{2}\right]$ consider a one-nodal curve C^j representing a generic point in Δ_j. Let p_1, p_2 be points in the normalization of C^j which are identified to form a node of and ζ_1, ζ_2 be local coordinates in neighborhoods U_1, U_2 of p_1 and p_2 respectively such that C^j is give locally by the equation $\zeta_1\zeta_2 = 0$. For small $t \in \mathbb{C}$ consider a family of curves

$$C_t^j = (C^j \smallsetminus (U_1 \cup U_2)) \cup \{(x_1, x_2) \in U_1 \times U_2 \mid \zeta_1(x_1)\,\zeta_2(x_2) = t\}. \tag{3.1}$$

We call C_t^j a *plumbing family.* It is well-known that C_t^j defines a smooth family in $\overline{\mathcal{M}}_g$ and this family intersects the boundary transversally.

Consider $j > 0$. Given Torelli markings ν_1 and ν_2 of irreducible components of C^j we can form a Torelli marking $\nu_1 \cup \nu_2$ of C_t^j in natural way: take a collection of loops $a_1, \ldots, a_g, b_1, \ldots, b_g$ such that $a_1, \ldots, a_j, b_1, \ldots, b_j$ represents ν_1 and $a_{j+1}, \ldots, a_g, b_{j+1}, \ldots, b_g$ represents ν_2; then classes of $a_1, \ldots, a_g, b_1, \ldots, b_g$ in the first homology group give a Torelli marking of C_t^j for all small t. We will consider Torelli markings of C_t^j formed only in such way.

Fix now a Torelli marking of C_t^j. Let $\eta = \eta_1 \oplus \eta_2$ be some odd theta characteristic such that $\eta_1 \in (\mathbb{Z}/2\mathbb{Z})^{2j}$ and $\eta_2 \in (\mathbb{Z}/2\mathbb{Z})^{2(g-j)}$. The family C_t^j equipped with η defines a family of in $\overline{\mathcal{S}}_g^-$ (recall the correspondence between theta characteristics and spin bundles). By the definition of A_j, B_j this family intersects A_j if η_1 is odd and B_j is η_1 is even. The intersection is transversal because the map $\overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ is unbranched over a generic point of Δ_j for $j > 0$. We call this family *plumbing family for A_j (resp. B_j)* if η_1 is odd (resp. even).

Consider now the case $j = 0$. The cover $\mathcal{T}_g \to \mathcal{M}_g$ has an infinite branching when we turn around the bundary divisor Δ_0, thus we cannot trivialize the bundle $H_1(C_t^0, \mathbb{Z})$, $t \neq 0$ as we did in the reducible case. Let us restrict C_t^0 to the family C_t^0, $t \in \mathbb{C} \smallsetminus \mathbb{R}_{\geq 0}$. Consider loops $a_1, \ldots, a_{g-1}, b_1, \ldots, b_{g-1}$ representing a Torelli marking of the normalization of C_0^0. Let a_g be a small positive oriented loop around p_1 and b_g be a path from p_2 to p_1 which does not intersect $a_1, \ldots, a_{g-1}, b_1, \ldots, b_{g-1}$. Then $a_1, \ldots, a_g, b_1, \ldots, b_g$ induces a Torelli marking of C_t^0, $t \notin \mathbb{R}_{\geq 0}$. We will consider only such Torelli markings of C_t^0.

Consider an odd theta characteristic $\eta = \eta_1 \oplus \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$ such that $\eta_1 \in (\mathbb{Z}/2\mathbb{Z})^{2(g-1)}$ and $\varepsilon, \delta \in \mathbb{Z}/2\mathbb{Z}$. The family C_t^0, $t \in \mathbb{C} \smallsetminus \mathbb{R}_{\geq 0}$ equipped with η and a Torelli marking gives us a family in $\overline{\mathcal{S}}_g^-$.

If $\delta = 1$ then this family extends to a family over all small $t \in \mathbb{C}$. The extended family intersects the boundary at A_0 and the intersection is transversal (since the cover $\overline{\mathcal{S}}_g^- \to \overline{\mathcal{M}}_g$ is unbranched at a generic point of A_0). We call the extended family *plumbing family for A_0*.

If $\delta = 0$ then we have to take the double cover $r = \sqrt{t}$ of the parameter space and then the family $C_{r^2}^0$, $r \in \mathbb{C} \smallsetminus \mathbb{R}_{\geq 0}$ equipped with η pulls back to the family in $\overline{\mathcal{S}}_g^-$ which intersects B_0 transversally. We call this family *plumbing family for B_0* and denote it simply by C_r^0.

Let us now describe the asymptotic behavior of the theta function with respect to degenerations described above. We refer to [13] for more information.

The case of reducible curves. Fix $j > 0$, consider the plumbing family C_t^j equipped with a Torelli marking and a theta characteristic η. Denote the matrix of b-periods for C_t^j by Ω_t. Let

$$\theta[\eta](\cdot, \Omega_t) : \mathbb{C}^g \to \mathbb{C}$$

be the theta function corresponding to Ω_t with the characteristic η. Let C_1, C_2 be irreducible components of C_0^j. Denote matrices of b-periods on C_1 and C_2 by Ω_1 and Ω_2 respectively.

Proposition 3.1. *Let $W_1 = (w_1, \ldots, w_j) \in \mathbb{C}^j$ and $W_2 = (w_{j+1}, \ldots, w_g) \in \mathbb{C}^{g-j}$. Put $R_i = \frac{v_i}{d\zeta_1}|_{p_1}$ if $i \leq j$ and $R_i = \frac{v_i}{d\zeta_2}|_{p_2}$ if $i > j$. Then one has*

$$\begin{aligned}\theta[\eta](W, \Omega_t) = \;& \theta[\eta_1](W_1, \Omega_1)\,\theta[\eta_2](W_2, \Omega_2) \\ & -\tfrac{t}{2\pi i}\left[\sum_{i=1}^{j} \tfrac{\partial}{\partial w_i}\theta[\eta_1](W_1, \Omega_1) R_i\right]\left[\sum_{k=1}^{g-j} \tfrac{\partial}{\partial w_k}\theta[\eta_2](W_2, \Omega_2) R_k\right] + O(t^2)\end{aligned}$$

as $t \to 0$ uniformly on compact subsets of $\mathbb{C}^g$, where $W = W_1 \oplus W_2 \in \mathbb{C}^g$.

Proof. Proposition immediately follows from the expansion (see [13])

$$\Omega_t = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} - t \begin{pmatrix} 0 & R_1^T R_2 \\ R_2^T R_1 & 0 \end{pmatrix} + O(t^2).$$

□

The case of irreducible curves. Consider a plumbing family C_t^0 equipped with a Torelli marking and a theta characteristic $\eta = \eta_1 \oplus \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$ such that $\eta_1 \in (\mathbb{Z}/2\mathbb{Z})^{2(g-1)}$ and $\varepsilon, \delta \in \mathbb{Z}/2\mathbb{Z}$.

Denote by Ω_t the matrix of b-periods of C_t^0, and consider the corresponding theta function with the characteristic η:

$$\theta[\eta](\cdot, \Omega_t) : \mathbb{C}^g \to \mathbb{C}.$$

Denote by Ω the matrix of b-periods on C_0^0.

Proposition 3.2. *Assume that $\delta = 1$. Then $\theta[\eta](\cdot, \Omega_t)$ has the following asymptotics on every compact subset of $\mathbb{C}^g$*

$$\theta[\eta](w_1, \dots, w_g, \Omega_t) = t^{1/8}\Big(e^{-cw_g + r}\, \theta[\eta_1](w_1, \dots, w_{g-1}, \Omega) + e^{cw_g}\, \theta[\eta_1](w_1 + c_1, \dots, w_{g-1} + c_{g-1}, \Omega) + O(t)\Big),$$

where c, r, c_j are independent on $\{w_j\}$ but depend on moduli of curve and $c \neq 0$ and $\theta[\eta_1](c_1, \dots, c_{g-1}, \Omega) \neq 0$ outside of some divisor in the moduli space $\overline{\mathcal{M}}_{g-1,2}$.

Proposition 3.3. *Assume that $\delta = 0$. Then $\theta[\eta](\cdot, \Omega_t)$ depends on the choice of a branch of $\sqrt{t}$ and has the following asymptotics uniformly on compact subsets of $\mathbb{C}^g$:*

$$\begin{aligned} \theta[\eta](w_1, \dots, w_g, \Omega_t) = \; & \theta[\eta_1](w_1, \dots, w_{g-1}, \Omega) \\ & + \sqrt{t}\, e^{cw_g + r}\, \theta[\eta_1](w_1 + c_1, \dots, w_{g-1} + c_{g-1}, \Omega) \\ & + \sqrt{t}\, e^{-cw_g - r}\, \theta[\eta_1](w_1 - c_1, \dots, w_{g-1} - c_{g-1}, \Omega) + O(t), \end{aligned}$$

where c, r, c_j are moduli-dependent constants and $c \cdot \theta[\eta_1](c_1, \dots, c_{g-1}, \Omega) \neq 0$ outside of some divisor in the moduli space.

The two propositions above follow directly from the asymptotics of Ω_t (see [13]):

$$\Omega_t = \begin{pmatrix} \Omega & R^T \\ R & \frac{1}{2\pi i}\log t + c \end{pmatrix} + O(t), \tag{3.2}$$

where $R \in \mathbb{C}^{g-1}$ and $c \in \mathbb{C}$ are moduli-dependent constants. The eight root of the parametr t in Proposition 3.2 is determined by the branch of the logarithm in the asymptotics above.

4 Farkas' formula for $\mathcal{Z}_g$.

4.1 Odd spinors.

Consider a point in $\tilde{\mathcal{H}}_g^-([2^{g-1}])$ represented by a triple (C, ν, ω) as above. Then $\sqrt{\omega}$ is a section of an odd spin bundle L. Denote by Ω the matrix of b-periods for C with respect to ν. Let $\theta[\eta](\cdot, \Omega) : \mathbb{C}^g \to \mathbb{C}$ be the theta function with the odd characteristic η given by L and ν. Introduce the differential

$$\varsigma_C(p) = d_x\, \theta[\eta](\mathcal{A}(x - p), \Omega)|_{x=p}\,,$$

where $\mathcal{A}$ is the Abel map (note that $\varsigma_C(p)$ does not depend on a lift of $\mathcal{A}(x-p)$ to $\mathbb{C}^g$ since $\theta[\eta](0,\Omega)=0$). This differential is non-zero if and only if $\dim H^0(C,L)=1$ and is the square of a section of L. Therefore,

$$\varsigma_C = c\,\omega$$

for some (moduli-dependent) constant c.

Let us describe the asymptotics of ς under a degeneration of a curve.

The case of reducible curve. Fix $j>0$ and consider the plumbing family C_t^j,η for A_j. Denote by C_1 and C_2 irreducible components of C_0^j.

Let $K_i\subset C_i\smallsetminus\{p_i\}$ be a compact subset. We may assume that $K_i\subset C_t$ for all sufficiently small t. Then Proposition 3.1 implies that

$$\varsigma_{C_t^j}(p)=v_1(p)+tv_2(p), \tag{4.1}$$

where v_1 is a non-zero holomorphic differential on C_1 and v_2 is a holomorphic differential on K_1;

$$\varsigma_{C_t^j}(p)=t\,w_1(p)+t^2w_2(p), \tag{4.2}$$

where w_1 is a non-zero meromorphic differential on C_2 having double pole at p_2 and no other poles, and w_2 is a holomorphic differential on K_2.

The case of B_j is completely analogous.

The case of irreducible curves. Consider first the plumbing family C_t^0,η for A_0. Let K be a compact subset of C_0^0 disjoint from the node. Proposition 3.2 implies that ς_{C_t} is determined up to an 8th root of unity and has the following asymptotics:

$$\varsigma_{C_t^0}(p)=t^{1/8}(v_1(p)+tv_2(p)), \tag{4.3}$$

where v_1 is a non-zero meromorphic differential on C having simple poles at p_1 and p_2 and no other poles, and v_2 is a holomorphic differential on K.

Case 2. Consider now the plumbing family C_r^0,η for B_0. Let K be a compact subset of C_0^0 disjoint from the node. Proposition 3.3 implies that ς_{C_r} is well-defined for all $r\neq 0$ and has the asymptotics

$$\varsigma_{C_r^0}(p)=v_1(p)+rv_2(p)+r^2v_3(p) \tag{4.4}$$

where v_1 is a non-zero holomorphic differential, v_2 is a meromorphic differential on C having simple poles at p_1,p_2 and no other poles, and v_3 is a holomorphic differential on K.

Let us analyze the global behavior of ς. Let $f:\tilde{\mathcal{H}}_g^-([2^{g-1}])\to\mathcal{H}_g^-([2^{g-1}])$ be the forgetful projection. We first consider ς as a section of the tautological line bubdle $f^*\mathcal{L}\to\tilde{\mathcal{H}}_g^-([2^{g-1}])/\mathbb{C}^*$.

Recall that the group $Sp(g,\mathbb{Z})$ acts on $\tilde{\mathcal{H}}_g^-([2^{g-1}])$ by changing a Torelli marking, and we have $\tilde{\mathcal{H}}_g^-([2^{g-1}])/Sp(g,\mathbb{Z})=\mathcal{H}_g^-([2^{g-1}])$. The fact that the theta function is a modular form of the weight 1/2 can be restated in the following way:

Proposition 4.1. *Let (C,ν,L) be a Torelli marked curve, and σ be a $Sp(g,\mathbb{Z})$ - transformation acting on $H_1(C)$. Denote by $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ the matrix of σ with respect to the basis ν. Then*

$$\varsigma_{\sigma_*C}=\gamma\sqrt{\det(b\Omega+a)}\cdot\varsigma_C,$$

where $\gamma^8=1$.

The proposition follows directly from the transformation properties of theta functions (see [10]).

Corollary 4.1. *ς^8 can be considered as a section of the line bundle $\mathcal{L}^8\otimes\Lambda^4\to\mathcal{H}_g^-([2^{g-1}])/\mathbb{C}^*$.*

We finalize with the following remark:

Remark 4.1. *Let $\mu : \mathcal{C}_g^- \to \overline{\mathcal{S}}_g^-$ be the universal spinor curve and ω^s be the line bundle on $\mathcal{C}_g^-$ such that ω^s is the corresponding spin bundle restricted to each fiber of μ. Then $\mu_*\omega^s$ turns out to be a locally-free sheaf of the dimension one. ς^8 induces a section of the line bundle $(\mu_*\omega^s)^{16} \otimes \lambda^4$ restricted to $\mathcal{S}_g^-$. The asymptotics relations (4.3) – (4.2) imply that this section can be extended to a section of $(\mu_*\omega^s)^{16} \otimes \lambda^4$ and the divisor of this section is A_0. But ς^8 considered as a section of $Sym^8\, \mathbb{E}_g^s \otimes \lambda^4$ (where $\mathbb{E}_g^s$ is the Hodge bundle on $\overline{\mathcal{S}}_g^-$) has a bigger zero locus: it consists of A_0 and of the closure of the locus of spin curves $(C, L) \in \mathcal{S}_g^-$ such that* $\dim H^0(C, L) > 1$. *This is connected with the fact that the pushforward functor is not right exact.*

4.2 Asymptotics of the tau function.

We begin with the following technical observation. Let C be a Riemann surface of genus g and v be a holomorphic differential or a meromorphic differential with double poles and zero residues. Denote zeros of v by $p_1, \dots, p_d \in C$. Consider simple paths l_j from p_d to p_j for all $j = 1, \dots, d-1$. Let $a_1, \dots, a_g, b_1, \dots, b_g$ be simple loops on $C \setminus \{p_1, \dots, p_d\}$ which do not intersect l_j and such that their homology classes in $H_1(C)$ form a symplectic basis. Denote by $s_1, \dots, s_{2g+d-1}$ a basis in $H_1(C \setminus \{p_1, \dots, p_d\})$ dual to the basis represented by $a_1, \dots, a_g, b_1, \dots, b_g, l_1, \dots, l_{d-1}$ in the relative homology group $H_1(C; \{p_1, \dots, p_d\})$; we have $s_j = -b_j, s_{g+j} = a_j$ and s_{2g+j} is homologous to a small positive oriented circle around p_j.

Put

$$\begin{aligned} z_j &= \textstyle\int_{a_j} v, \quad j = 1, \dots, g, \\ z_{j+g} &= \textstyle\int_{b_j} v, \quad j = 1, \dots, g, \\ z_{j+2g} &= \textstyle\int_{l_j} v, \quad j = 1, \dots, d-1. \end{aligned}$$

In the case when v is a holomorphic differential with double zeros the set $\{z_1, \dots, z_{2g+d-1}\}$ is the set of homological coordinates introduced above.

Let S_B be the Bergman projective connection with respect to the Torelli marking induced by $a_1, b_1, \dots, a_g, b_g$. Denote by m_k the multiplicity of the zero p_k.

Lemma 4.1. *The following relation holds:*

$$\sum_{k=1}^{2g+d-1} z_k \int_{s_k} \frac{S_B - S_v}{v} = -\pi i \left(d + \sum_{k=1}^{d} (m_k - \frac{1}{1+m_k}) \right) \tag{4.5}$$

Proof. From Riemann bilinear relations we get that

$$\sum_{k=1}^{2g} z_k \int_{s_k} \frac{S_B - S_v}{v} = -2\pi i \sum_{x \in C} \text{Res}_x \left(\frac{S_B - S_v}{v} \int_{p_d} v \right).$$

Computing residues we obtain

$$-2\pi i \sum_{x \in C} \text{Res}_x \left(\frac{S_B - S_v}{v} \int_{p_d} v \right) = - \sum_{k=2g+1}^{2g+d-1} z_k \int_{s_k} \frac{S_B - S_v}{v} - \pi i \left(d + \sum_{k=1}^{d} (m_k - \frac{1}{1+m_k}) \right)$$

which implies (4.5). □

Remark 4.2. *If v is holomorphic differential with double zeros then the right-hand side of* (4.5) *is equal to $\frac{8}{3}(1-g)$. This implies the homogeneity property of the tau function.*

In fact (4.5) *implies that if a function* F *is defined on some open subset* $\mathcal{U} \subset \tilde{\mathcal{H}}_g^-([2^{g-1}])$ *and satisfies differential equations*

$$\partial_{z_j} \log F(C,v) = \frac{-\alpha}{\pi i} \int_{s_j} \frac{S_B - S_v}{v}, \quad j = 1, \ldots, 2g+d-1,$$

for some $\alpha \in \mathbb{Q}$, *then it must satisfy the homogeneity property*

$$F(C,tv) = t^{\alpha\left(d + \sum_{k=1}^{d}(m_k - \frac{1}{1+m_k})\right)} F(C,v).$$

Proposition 4.2. *Consider a family of Torelli marked curves* (C_t, ν_t) *in* $\mathcal{T}_g$ *and an odd theta characteristic* $\eta \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ *such that* C_0 *with the spin bundle* $L_0 \to C_0$ *represents a point in* $\mathcal{Z}_g$ *and* $t \in \mathbb{C}$ *is transversal to* $\mathcal{Z}_g$. *Put* $\varsigma_{C_t} = \varsigma_t$ *for simplicity. Assume that* $\varsigma_0 \neq 0$ *(i. e.* $\dim H^0(C_0, L_0) = 1$*). Then the tau function* τ *has the following asymptotics near* $\mathcal{Z}_g$*:*

$$\tau(C_t, \varsigma_t) = c_0\, t^8 (1 + o(1)) \quad \textit{as } t \to 0. \tag{4.6}$$

Proof. Let $\mu : \mathcal{C}_g^- \to \overline{\mathcal{S}}_g^-$ be the universal spinor curve and ω^s be the line bundle on $\mathcal{C}_g^-$ such that ω^s is the corresponding spin bundle restricted to each fiber of μ. Let $\mathcal{D} \subset \mathcal{C}_g^-$ be the zero locus of ζ^8 considered as a section of $(\omega^s)^{16} \otimes \lambda^4$ and $\tilde{\mathcal{Z}}_g$ be the irreducible component of the singular subvarity of $\mathcal{D}$ such that $\mathcal{Z}_g \subset \mu(\tilde{\mathcal{Z}}_g)$. We claim that $\mathcal{D}$ intersects itself transversally at a generic point of $\tilde{\mathcal{Z}}_g$. It is enough to give an example of such a point to prove our claim. Consider the closure of the locus $Hyp \subset \mathcal{C}_g^-$ consisting of hyperelliptic curves. Then $\mathcal{D} \cap Hyp$ parametrize Weirstrass points of curves and $\tilde{\mathcal{Z}}_g \cap Hyp$ is given by singular points of curves. Since all singular points are simple by the definition we have the desired transversality.

Let us now prove Proposition. We may assume that C_t defines a family of complex structures on a fixed topological surface. Let $p_{g-2}(t), p_{g-1}(t) \in C$ be the zeros of ς_t that coalesce when $t \to 0$. Introduce a local coordinate $z_t : U \to \mathbb{C}$ on C_t near $p_{2g-2}(0)$ such that $z_t(p_{g-2}(t)) = \sqrt{t}$ and such that $z_t(p_{g-1}(t)) = -\sqrt{t}$ (it is possible because $\mathcal{D}$ intersects itself transversally at a generic point of $\tilde{\mathcal{Z}}_g$). Note that the point (C_t, ς_t) in $\mathcal{H}_g^-([2^{g-1}])$ does not depend on a labeling of zeros, so in our case we have a double cover on which $\sqrt{t}$ make sense. Then one has $\varsigma_t \circ z_t^{-1}(x) = (x^2 - t)^2 (c + O(t))\, dx$ for some $c \neq 0$ and therefore

$$\int_{p_{g-2}(t)}^{p_{g-1}(t)} \varsigma_t = t^{5/2}(c_1 + O(t)),$$

where the path of integration is chosen such that $\int_{p_{g-2}(t)}^{p_{g-1}(t)} \varsigma_t \to 0$.

Let $z_1(t), \ldots, z_{3g-2}(t)$ be the homological coordinates associated with the triple $(C_t, \nu_t, \varsigma_t)$ for $t \neq 0$. We may assume that $z_{3g-2}(t) = t^{5/2}(c_1 + O(t))$. Consider a small open neighborhood $\mathcal{U} \subset \mathcal{S}_g^-$ of (C_0, L_0). Then calculations above imply that the map

$$\mathcal{U} \xrightarrow{[z_1 : \cdots : z_{3g-3} : z_{3g-2}^{2/5}]} \mathbb{C}P^{3g-3}$$

is an embedding and the image of $\mathcal{Z}_g \cap \mathcal{U}$ is given by the intersection with the hyperplane $\{z_{3g-2} = 0\}$.

Denote the image of $\mathcal{U}$ in $\mathbb{C}P^{3g-3}$ by $\mathcal{V}$ and the pullback of $\mathcal{V}$ to $\mathbb{C}^{3g-2}$ by $\tilde{\mathcal{V}}$. The function τ written in local coordinates $z_1, \ldots, z_{3g-2}$ can be considered as a function on the two-sheeted cover of $\tilde{\mathcal{V}} \setminus \{z_{3g-2} = 0\}$ which is defined by the square root $\sqrt{z_{3g-2}}$. The relation (2.1) implies that

$$\tau(z_1, \ldots, z_{3g-2}) = c(z_{3g-2}^{2/5})\tilde{\tau}(z_1, \ldots, z_{3g-3})\,(1 + o(1)) \tag{4.7}$$

as $z_{3g-2} \to 0$ where c is a meromorphic function having a singularity at the origin and $\tilde{\tau}(z_1, \dots, z_{3g-3})$ is a holomorphic function ($\tilde{\tau}$ is nothing but 72th power of the Bergman tau function considered on the stratum of holomorphic differentials on genus g surfaces having $g-3$ double zero and one zero of order 4. This stratum projects to a dense open subset of $\mathcal{Z}_g$). A simple estimate shows that $z\frac{d}{dz}\log c(z)$ is bounded and therefore c must be meromoprhic near the origin.

Lemma 4.1 applied to the function $\tilde{\tau}$ implies that $\tilde{\tau}$ is homogenous with the degree of homogeneity equal to $16(g-1) - \frac{16}{5}$. Therefore comparing the degree of homogeneity of the left-hand side and the right-hand side of (4.7) one concludes that $c(z) = z^8\,(c_0 + o(1))$. □

Fix $j > 0$ and consider a plumbing family C_t^j, η for A_j. Denote C_t^j and $\varsigma_{C_t^j}$ by C_t and ς_t for simplicity.

Proposition 4.3. *The tau function τ has the following asymptotics near A_j, $j > 0$:*

$$\tau(C_t, \varsigma_t) = c\,t^{16(g-j)}(1 + o(1)) \quad \textit{as } t \to 0. \tag{4.8}$$

Proof. Recall that on any compact subset of $C_2 \setminus \{x_2\}$ one has

$$t^{-1}\varsigma_t \to v_2$$

as $t \to 0$, where v_2 is a meromorphic differential on C_2 with a double pole at x_2 and no other poles (see (4.2)). Fix some enumeration $p_1(t), \dots, p_{g-1}(t)$ of zeros of ς_t such that $p_1(t), \dots, p_{g-j-1}(t), p_{g-1}(t)$ C_2. Let $z_1, \dots, z_{3g-2}$ be homological coordinates constructed with respect to $\nu_1 \cup \nu_2$ and the chosen numeration of zeros. Then direct computations using the differential equation (2.1) and asymptotics relations (4.1) and (4.2) give

$$\frac{d}{dt}\log\tau(C_t, \varsigma_t) = -t^{-1}\cdot\frac{6}{\pi i}\sum_{k=1}^{d} z_k \int_{s_k}\frac{S_B - S_{v_2}}{v_2} + O(1) \quad \text{as } t \to 0,$$

where S_B is the Bergman projective connection, $d = 3g - j - 1$ and $s_1, \dots, s_d$ is the basis in $H_1(C_2 \setminus \{p_1(0), \dots, p_{g-j-1}(0), p_{g-1}(0)\})$ dual to the basis in the relatives homology group defining homological coordinates. By Lemma 4.1 one sees that

$$\frac{d}{dt}\log\tau(C_t, \varsigma_t) = \frac{16(g-j)}{t} + O(1),$$

which implies (4.9).

□

Proposition 4.4. *The tau function τ has the following asymptotics near B_j, $j > 0$:*

$$\tau(C_t^j, \varsigma_{C_t^j}) = c\,t^{16j}(1 + o(1)) \quad \textit{as } t \to 0. \tag{4.9}$$

The proof is completely analogously to the previous one.

Consider the plumbing family C_t^0, η for A_0 and the corresponding differential $\varsigma_{C_t^0}$. Fix some branch of $t^{1/8}$. Recall that $\frac{1}{t^{1/8}}\varsigma_{C_t} \to v$ as $t \to 0$ for some (generically not identically vanishing) meromorphic differential v on the normalization of C_0^0 having simple poles at p_1, p_2 (where p_1 and p_2 projects to the node) and no other poles (see (4.3)). We denote $\frac{1}{t^{1/8}}\varsigma_{C_t}$ by $\tilde{\varsigma}_t$ and C_t^0 by C_t.

Proposition 4.5. *The tau function τ has the following asymptotics near A_0:*

$$\tau(C_t, \tilde{\varsigma}_t) = c\,t^6(1 + o(1)) \quad \textit{as } t \to 0. \tag{4.10}$$

Proof. Let $z_1(t) = \int_a \tilde\varsigma_t$ and $z_2(t) = \int_b \tilde\varsigma_t$. Consider the parameter

$$\tilde t = \exp\left(2\pi i \frac{z_2(t)}{z_1(t)}\right).$$

Recall that when t goes around zero then b changes to $b + a$ and $\tilde\varsigma_t$ to $\gamma\tilde\varsigma_t$ where $\gamma^8 = 1$. This implies that $\tilde t$ can be naturally extended as a function of t for all $t \in \mathbb{D}$. The asymptotics $\int_b \tilde\varsigma_t = \frac{z_1(0)}{2\pi i}\log t + O(1)$ (see (3.2)) implies that $\tilde t(0) = 0$ and $\tilde t(t)$ is one-to-one map near the origin.

We fix some labeling of zeros of $\tilde\varsigma_t$ and introduce the corresponding homological coordinates. Note that $\tau(C_t, \tilde\varsigma_t)$ is correctly defined for all sufficiently small $t \in \mathbb{C}$. Using the equation (2.1) defining the tau function we compute by the chain rule that

$$-\frac{\pi i}{6}\cdot\frac{d}{d\tilde t}\log\tau(C_{\tilde t}, \tilde\varsigma_{\tilde t}) = \frac{z_1(0)}{2\pi i\,\tilde t}\int_a \frac{S_B - S_{\tilde\varsigma_{\tilde t}}}{\tilde\varsigma_{\tilde t}}\cdot(1 + o(1))$$

as $t \to 0$. Computing the residue $\operatorname{Res}_{p_1}\frac{S_B - S_{\tilde\varsigma_0}}{\tilde\varsigma_0}$ we obtain

$$\frac{d}{d\tilde t}\log\tau(C_{\tilde t}, \tilde\varsigma_{\tilde t}) = \frac{6}{\tilde t}(1 + o(1)),$$

which implies (4.10). $\square$

Consider now the plumbing family C_r^0, η for B_0. Recall that by (4.4) there exists a holomorphic differential v on the normalization C of C_0^0 and a meromorphic differential w on C having simple poles at p_1 and p_2 (where p_1 and p_2 projects to the node) and no other poles such that $\varsigma_{C_r^j} = v + rw + O(r^2)$. Put $\varsigma_r = \varsigma_{C_r^0}$ and $C_r = C_r^0$ to simplify notations.

Proposition 4.6. *The tau function τ has the following asymptotics near B_0:*

$$\tau(C_r, \varsigma_r) = c\,r^{16}(1 + o(1)) \qquad \textit{as } r \to 0.$$

Proof. Let $\mathcal{U}$ be a small open polydisc in $\overline{\mathcal{S}}_g^-$ centered at (C_0, L_0) and let $\mathcal{V}$ be a connected component of the pullback of $\mathcal{U}$ to $\tilde{\mathcal{H}}_g^-([2^{g-1}])$. Introduce homological coordinates $z_1, \ldots, z_{3g-2}$ on $\mathcal{V}$ that are numbered as follows:

$$z_g\left(C_r, \{a, b\} \cup \nu, \varsigma_{C_r}\right) = \int_a \varsigma_r, \quad z_{2g}\left(C_r, \{a, b\} \cup \nu, \varsigma_r\right) = \int_b \varsigma_r$$

and the $(g-1)$th zero of the differential ς_r tends to the node under a degeneration of the underlying curve. Note that by the asymptotics (4.4)

$$z_g\left(C_r, \{a, b\} \cup \nu, \varsigma_r\right) = c\,r(1 + o(1))$$

for some generically non-zero constant c. The asymptotics (4.4) implies that

$$r\int_b \frac{S_B - S_{\varsigma_r}}{\varsigma_r} = O(1) \tag{4.11}$$

as $r \to 0$.

Computing the derivatives of τ with respect to z_j for all $j \neq g, 2g$ by (2.1), we obtain the asymptotics

$$\tau(z_1, \ldots, z_{3g-2}) = c(z_g, z_{2g})\,\tilde\tau(z_1, \ldots, \hat z_g, \ldots, \hat z_{2g}, \ldots, z_{3g-3})(1 + o(1)) \quad \text{as } z_g \to 0, \tag{4.12}$$

where $\tilde{\tau}$ is the tau function on $\tilde{\mathcal{H}}^-_{g-1}([2^{g-2}])$.

The factor $c(z_g, z_{2g})$ is a holomorphic function in some punctured neighborhood of the line $\{(0,z),\ z\ \in\ \mathbb{C}\}$ in $\mathbb{C}^2$. The estimate (4.11) shows that $\frac{\partial}{\partial z_g}\log\tau$ has at most simple pole at $z_g = 0$, hence the function $c(z_g, z_{2g})$ is meromorphic at $z_g = 0$. Consider the Laurent series

$$c(z_g, z_{2g}) = \sum_{j=N}^{+\infty} c_j(z_{2g}) z_g^j,$$

It follows from the differential equation defining τ that $\frac{\partial}{\partial z_{2g}}\log\tau = O(z_g)$; therefore c_N does not depend on z_{2g}. According to Lemma 2.1 the degree of homogeneity of τ under the $\mathbb{C}^*$-action on differentials is equal to $16(g-1)$, whereas the degree of homogeneity of $\tilde{\tau}$ is equal to $16(g-2)$. Thus, comparing the orders of homogeneity of the right-hand side and the left-hand side of (4.12) one sees that $N = 16$.

□

4.3 The formula.

Now we can prove the following statement originally obtained by G. Farkas [4]:

Theorem. *The class $[\mathcal{Z}_g]$ has the following expression via the standard basis of the rational Picard group of $\overline{\mathcal{S}}^-_g$:*

$$[\mathcal{Z}_g] = (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 2\beta_0 - \sum_{j=1}^{[g/2]} 2(g-j)\alpha_j - \sum_{j=1}^{[g/2]} 2j\beta_j. \tag{4.13}$$

Proof. Note that $\varsigma^{16(g-1)}$ is a section of the line bundle

$$\begin{array}{c} \mathcal{L}^{16(g-1)} \otimes \Lambda^{8(g-1)} \\ \big\downarrow \\ \mathcal{H}^-_g([2^{g-1}])/\mathbb{C}^* \end{array}$$

as it was shown in Corollary 4.1 (see Subsection 4.1 for the definition of ς). By Lemma 2.1 the tau function defines a homomorphism from $\mathcal{L}^{16(g-1)}$ to Λ^{72}. Applying this homomorphism to the section $\varsigma^{16(g-1)}$ we obtain a section of Λ^{8g+64} which we denote by $\tilde{\psi}$.

Consider the locus $\mathcal{X} = \{(C,\omega) \in \mathcal{H}^-_g([2^{g-1}])\ :\ \dim|\mathrm{div}\sqrt{\omega}| > 0\}$ (that is, the locus of abelian differentials with double zeros such that the dimension of the space of holomorphic sections of the corresponding spin bundle is larger than one). Note that $\pi|_{\mathcal{H}^-_g([2^{g-1}])\setminus\mathcal{X}}$ is one-to-one and $\pi(\mathcal{H}^-_g([2^{g-1}]) \setminus \mathcal{X}) = \mathcal{S}^-_g \setminus \mathcal{Z}_g$, where π is the map from $\mathcal{H}^-_g([2^{g-1}])$ to $\mathcal{S}^-_g$ which maps a differential to the corresponding spin bundle. We also have $\pi(\mathcal{X}) \subset \mathcal{Z}_g$.

Put $\psi = \pi_*(\tilde{\psi}|_{\mathcal{H}^-_g([2^{g-1}])\setminus\mathcal{X}})$. We have $\pi_*\Lambda^{8g+64} \simeq \lambda^{8g+64}$, therefore ψ is a holomorphic section of $\lambda^{8g+64}|_{\mathcal{S}^-_g\setminus\mathcal{Z}_g}$. Let $\mathcal{U} \subset \overline{\mathcal{S}}^-_g$ be an open contractible subset. Choosing a trivialization $\phi : \lambda|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}$ we obtain a holomorphic function $\phi^{\otimes 8g+64} \circ \psi : \mathcal{U} \cap (\mathcal{S}^-_g \setminus \mathcal{Z}_g) \to \mathbb{C}$. Propositions 4.2 – 4.6 and asymptotics (4.3) – (4.2) imply that this function can be holomorphicaly extended to $\mathcal{U}$. Therefore we can extend the section ψ to $\overline{\mathcal{S}}^-_g$. Propositions 4.2 – 4.6 and asymptotics (4.3) – (4.2) also imply that

$$[\mathrm{div}\,\psi] = 16\beta_0 + (4+2g)\alpha_0 + 16\sum_{j=2}^{[g/2]}(g-j)\alpha_j + 16\sum_{j=2}^{[g/2]} j\beta_j + 8[\mathcal{Z}_g].$$

On the other hand,

$$[\operatorname{div}\psi] = (8g+64)\lambda$$

in the rational Picard group of $\overline{\mathcal{S}}_g^-$ by definition of ψ. Hence

$$(8g+64)\lambda = 16\beta_0 + (4+2g)\alpha_0 + 16\sum_{j=2}^{[g/2]}(g-j)\alpha_j + 16\sum_{j=2}^{[g/2]} j\beta_j + 8[\mathcal{Z}_g].$$

which implies Formula (4.13).

□

5 A formula for the theta-null divisor.

The purpose of this Section is to show that the Farkas' formula for the class of the theta-null divisor (see [5]) in the rational Picard group of the moduli space of even spin curves can be obtained by using the modular properties of the theta function.

Let $\overline{\mathcal{S}}_g^+$ be the moduli space of even spin curves of genus g and let $\mathcal{S}_g^+ \subset \overline{\mathcal{S}}_g^+$ be the subspace consisting of smooth spin curves. Consider the theta-null divisor:

$$\Theta_{\text{null}} = Cl\{(C,L)\in\mathcal{S}_g^+ \;:\; \dim\, H^0(C,L)>0\},$$

where the closure is taken in $\overline{\mathcal{S}}_g^+$.

The rational Picard group of $\overline{\mathcal{S}}_g^+$. We follow notations of [5] in the description of the Picard group here. Let $\rho : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ be the natural projection. The boundary $\overline{\mathcal{S}}_g^+ \smallsetminus \mathcal{S}_g^+$ is a union of irreducible divisors $A_0, B_0, \ldots, A_{[g/2]}, B_{[g/2]}$ such that $\rho(A_j) = \rho(B_j) = \Delta_j$ for all $j = 0, \ldots, \left[\frac{g}{2}\right]$.

If $j \neq 0$ then a generic point in A_j is represented by an even spin bundle on each of the two irreducible components of a reducible genus g curve with one node. Generic points in B_j are similarly represented by odd spin bundles. In these cases we also replace the node by an exceptional component.

Pulling back a one-nodal curve from Δ_0 to $\overline{\mathcal{S}}_g^-$ we may have two possibilities, either the the obtained spin curve has an exceptional component or not. Let the divisor B_0 parametrizes such spin curves with exceptional component and A_0 parametrizes one-nodal irreducible spin curves.

Let $\mathbb{E}_g$ be the pullback of the Hodge vector bundle from $\overline{\mathcal{M}}_g$ to $\overline{\mathcal{S}}_g^+$ and let λ be the class in $\operatorname{Pic}(\overline{\mathcal{S}}_g^+)\otimes\mathbb{Q}$ of the determinant bundle $\bigwedge^g \mathbb{E}_g$. Denote by α_j and β_j the classes of A_j and B_j in the rational Picard group respectively. The group $\operatorname{Pic}(\overline{\mathcal{S}}_g^+)\otimes\mathbb{Q}$ is generated by $\lambda, \alpha_0, \ldots, \alpha_{[g/2]}, \beta_0, \ldots, \beta_{[g/2]}$.

Theta function as a modular form. Consider a smooth spin curve (C,L) representing some point in $\mathcal{S}_g^+$ and let ν be a Torelli marking of C. Denote by η the theta characteristic of L and by Ω the matrix of b-periods induced by ν. It is well-known that $\Big(\theta[\eta](0,\Omega)\Big)^8$ is a modular form of weight 4 on the level 2 cover of $\mathcal{M}_g$ (note that the action of $Sp(g,\mathbb{Z})$ on theta characteristics projects to the action of $Sp(g,\mathbb{Z}/2\mathbb{Z})$). Therefore $\Big(\theta[\eta](0,\Omega)\Big)^8$ pulls back to $\mathcal{S}_g^+$ as a section of $(\bigwedge^g \mathbb{E}_g)^{\otimes 4}|_{\mathcal{S}_g^+}$. We denote this section by ϑ. From the classical Riemann theorem we get that the divisor of ϑ on $\mathcal{S}_g^+$ is equal to $n\cdot(\Theta_{\text{null}}\cap\mathcal{S}_g^+)$ for some $n\in\mathbb{Z}_{>0}$. It is well-known that the order of vanishing of theta constants along Θ_{null} is equal to 2, thus we have $n = 16$ (see [12]).

The section ϑ can be extended to the compactified space $\overline{\mathcal{S}}_g^+$. To show this consider a trivialization $\phi : \bigwedge^g \mathbb{E}_g|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}$ over some open subset $\mathcal{U} \subset \overline{\mathcal{S}}_g^+$. Then well-known boundary asymptotics of theta constants near the boundary (see [3]) implies that the holomorphic function $\phi \circ \vartheta : \mathcal{U} \cap \mathcal{S}_g^+ \to \mathbb{C}$ can be extended to the holomorphic function on $\mathcal{U}$ (see also computations in the Section 3). Therefore the section ϑ can be extended to $\overline{\mathcal{S}}_g^+$ and we have the following relation in the rational Picard group of $\overline{\mathcal{S}}_g^+$:

$$[\mathrm{div}_{\overline{\mathcal{S}}_g^+} \vartheta] = 16[\Theta_{\mathrm{null}}] + \alpha_0 + 8 \sum_{j=1}^{[g/2]} \beta_j, \tag{5.1}$$

where [ ] denotes the class of a divisor in the rational Picard group.

Theorem. *We have*

$$[\Theta_{\mathrm{null}}] = \frac{1}{4}\lambda - \frac{1}{16}\alpha_0 - \frac{1}{2} \sum_{j=1}^{[g/2]} \beta_j. \tag{5.2}$$

Proof. Since ϑ is a section of $(\bigwedge^g \mathbb{E}^g)^{\otimes 4}$,

$$[\mathrm{div}_{\overline{\mathcal{S}}_g^+}]\, \vartheta = 4\lambda,$$

from where (5.2) immediately follows. □

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