THE CLASSIFICATION OF CERTAIN LINKED 3-MANIFOLDS IN 6-SPACE

S. AVVAKUMOV

Abstract. We work entirely in the smooth category. An embedding $f : (S^2 \times S^1) \sqcup S^3 \to \mathbb{R}^6$ is *Brunnian*, if the restriction of f to each component is isotopic to the standard embedding. For each triple of integers k, m, n such that $m \equiv n \pmod 2$, we explicitly construct a Brunnian embedding $f_{k,m,n} : (S^2 \times S^1) \sqcup S^3 \to \mathbb{R}^6$ such that the following theorem holds.

Theorem. *Any Brunnian embedding $f : (S^2 \times S^1) \sqcup S^3 \to \mathbb{R}^6$ is isotopic to $f_{k,m,n}$ for some integers k, m, n such that $m \equiv n \pmod 2$. Two embeddings $f_{k,m,n}$ and $f_{k',m',n'}$ are isotopic if and only if $k = k'$, $m \equiv m' \pmod{2k}$ and $n \equiv n' \pmod{2k}$.*

We use Haefliger's classification of embeddings $S^3 \sqcup S^3 \to \mathbb{R}^6$ in our proof. The following corollary shows that the relation between the embeddings $(S^2 \times S^1) \sqcup S^3 \to \mathbb{R}^6$ and $S^3 \sqcup S^3 \to \mathbb{R}^6$ is not trivial.

Corollary. *There exist embeddings $f : (S^2 \times S^1) \sqcup S^3 \to \mathbb{R}^6$ and $g, g' : S^3 \sqcup S^3 \to \mathbb{R}^6$ such that the componentwise embedded connected sum $f \# g$ is isotopic to $f \# g'$ but g is not isotopic to g'.*

1. Introduction.

Statement of the main result.

We work entirely in the smooth category.

An embedding[1,2] $f : S^2 \times S^1 \sqcup S^3 \to S^6$ is *Brunnian*, if restriction of f to each component is isotopic to the standard embedding. The standard embedding $S^2 \times S^1 \to S^6$ is defined later in this section.

For each integer k, we explicitly construct a Brunnian embedding $f_{k,0,0} : S^2 \times S^1 \sqcup S^3 \to S^6$. The construction is given later in this section.

An embedding $g : S^3 \sqcup S^3 \to S^6$ is *Brunnian*, if the restriction of g to each component is isotopic to the standard embedding. For each pair m, n of integers such that $m \equiv n \pmod 2$ we recall an explicit construction of a Brunnian embedding $g_{m,n} : S^3 \sqcup S^3 \to S^6$ later in this section.

We define the *embedded connected sum* later in this section. The (componentwise) embedded connected sum is denoted by #.

For each triple of integers k, m, n such that $m \equiv n \pmod 2$, let $f_{k,m,n} := f_{k,0,0} \# g_{m,n}$. Each embedding $f_{k,m,n}$ is Brunnian since both $f_{k,0,0}$ and $g_{m,n}$ are Brunnian.

Theorem 1. *Any Brunnian embedding $f : S^2 \times S^1 \sqcup S^3 \to S^6$ is isotopic to $f_{k,m,n}$ for some integers k, m, n such that $m \equiv n \pmod 2$. Two embeddings $f_{k,m,n}$ and $f_{k',m',n'}$ are isotopic if and only if $k = k'$, $m \equiv m' \pmod{2k}$ and $n \equiv n' \pmod{2k}$.*

The invariants necessary for the proof of Theorem 1 are defined in Section 2.

---

[1]Here and below $S^2 \times S^1 \sqcup S^3$ means $(S^2 \times S^1) \sqcup S^3$.

[2]For $m \geq n + 2$ classifications of embeddings of n-manifolds into S^m and into $\mathbb{R}^m$ are the same (see [Atl1, 1.1 Sphere and Euclidean space]). It is more convenient to us to consider embeddings into S^6 instead of $\mathbb{R}^6$.

The classification of Brunnian embeddings $S^3 \sqcup S^3 \to S^6$ is given by the following Haefliger Theorem proved in [Ha66].

Theorem 2 (Haefliger)**.** *Any Brunnian embedding* $g : S^3 \sqcup S^3 \to S^6$ *is isotopic to* $g_{m,n}$ *for some integers* m, n *such that* $m \equiv n \pmod 2$*. Two embeddings* $g_{m,n}$ *and* $g_{m',n'}$ *are isotopic if and only if* $m = m'$ *and* $n = n'$*.*

Theorem 1 together with Theorem 2 implies the following Corollary 1.

Corollary 1. *There exist embeddings* $f : S^2 \times S^1 \sqcup S^3 \to S^6$ *and* $g, g' : S^3 \sqcup S^3 \to S^6$ *such that* $f\#g$ *is isotopic to* $f\#g'$ *but* g *is not isotopic to* g'*.*

Corollary 1 is deduced from the theorems above at the end of this section.

Discussion of the main result.
A lot of interesting results on links were obtained during the rapid advancement of topology in 20th century. However, it was soon realized that the complete isotopy classification of links is unachievable even for the links with unknotted components. The same is true for the high-dimensional links in codimension 2.

The isotopy classification of high dimensional linked spheres in codimension greater than 2 was obtained by A. Haefliger. The solution is given in homotopy theoretic terms. Due to the complexity of the corresponding homotopy theoretic problems in the general case, the classification is a reduction rather than a complete solution. However, in many important special cases the corresponding problems are solvable. For instance, when

$$m \geq \frac{3n+4}{2},$$

the generalized linking numbers (for instance see [Atl2, definition of λ_{12}]) constitute a complete system of invariants of linked n-spheres in $\mathbb{R}^m$. In this case any two embeddings of the n-sphere in $\mathbb{R}^m$ are isotopic.

A. Haefliger also obtained the isotopy classification of linked n-spheres in $\mathbb{R}^m$ in the case $n + 3 \leq m = \frac{3n+3}{2}$. The smallest pair (m, n) of numbers satisfying this condition is $(6, 3)$.

A. Haefliger proved also that the generalized linking numbers and the isotopy classes of the restrictions to the components constitute a complete system of invariants of linked n-manifolds in $\mathbb{R}^m$ for $m \geq \frac{3n+4}{2}$. If $m \geq \frac{3n+4}{2}$, then the isotopy classes of the restrictions to the components are classified by the *Haefliger-Wu invariant* [Sk08, §5].

For some $m \leq \frac{3n+3}{2}$ the simpler problem of the *link homotopy* classification of linked n-manifolds in $\mathbb{R}^m$ was solved by A. Skopenkov [Sk00]. However, Theorem 1 is the first result on the isotopy classification of some linked n-manifolds (different from the homology spheres) in $\mathbb{R}^m$ for $m \leq \frac{3n+3}{2}$.

To prove Theorem 1, we define three isotopy invariants $\varkappa$, μ, and ν which constitute a complete system of invariants for our problem (see Section 2). The definition of the last two invariants μ and ν is rather similar, albeit more complicated, to the definition of the Haefliger's invariants of embeddings $S^3 \sqcup S^3 \to S^6$. It is harder to prove, however, that μ and ν are well defined. The first invariant $\varkappa$ is very simple but has no analogue in the case of embeddings $S^3 \sqcup S^3 \to S^6$.

We partly reduce the injectivity of our invariants to the Haefliger Theorem (Theorem 2). Another important part of the proof of the injectivity involves explicit construction of isotopies of embeddings $S^2 \times S^1 \sqcup S^3 \to S^6$ (see Section 7, Lemmas 20 and 23).

Finally, the surjectivity of the invariants is proved by a rather simple explicit construction of the embeddings $f_{k,m,n}$ and Lemma 6.

Embedded connected sum.
Let M and N be compact oriented connected manifolds, $\dim M = \dim N = m$. Let $r : D^m \to D^m$ be a mirror symmetry. Define the connected sum $M\#N$ as

$$M_0 \cup_{\partial M_0 = S^{m-1}\times 0} S^{m-1} \times I \cup_{S^{m-1}\times 1 = r(\partial N_0)} N_0,$$

where M_0 and N_0 are complements of M and N to embedded m-disks, respectively.

Let $f : M \to S^k$ and $g : N \to S^k$ be embeddings with disjoint images. Take an embedding $l : D^m \times I \to S^k$, such that $l|_{D^m\times 0} = f|_{M\setminus M_0}$ and $l|_{D^m\times 1} = g|_{N\setminus N_0} \circ r$; moreover, the image of l does not intersect elsewhere the union of $f(M)$ and $g(N)$.

The *embedded connected sum* of the embeddings f and g is the embedding $f\#g : M\#N \to S^k$, defined by $f\#g|_{M_0} = f$, $f\#g|_{N_0} = g$, and $f\#g|_{S^{m-1}\times I} = l$ (compare with [Ha62, §2] and [Ha66, §3]). We may choose l so that the map $f\#g$ is smooth.

The isotopy class of $f\#g$ is well defined if $k - m \geq 3$, i.e., it does not depend on the choice of l. It is not true, however, if $k - m < 3$.

Definition of standard embeddings. Let $x_1, x_2, \dots, x_m$ be the coordinates in $\mathbb{R}^m$.

Denote by S^{p-1} and D^p the unit sphere and the unit disk in $\mathbb{R}^p$, respectively. For $p < m$ identify $\mathbb{R}^p$ with the hyperplane of $\mathbb{R}^m$ given by the equations $x_{p+1} = x_{p+2} = \ldots = x_m = 0$. Thus S^{p-1} and D^p are identified with subsets of S^{m-1} and D^m, respectively. The obtained inclusions are called the *standard embeddings* $S^{p-1} \to S^{m-1}$ and $D^p \to D^m$.

The *standard embedding* $D^3 \times S^1 \to S^4$ is given by the formula

$$(\mathbf{x}, (y_1, y_2)) \mapsto (\mathbf{x}, y_1\sqrt{2 - \mathbf{x}^2}, y_2\sqrt{2 - \mathbf{x}^2})/\sqrt{2} \in S^4,$$

where $\mathbf{x}$ denotes $(x_1, x_2, x_3) \in D^3$. The image of the embedding is the tubular neighborhood $\{x_1^2 + x_2^2 + x_3^2 \leq \frac{1}{2}, x_4^2 + x_5^2 \geq \frac{1}{2}\} \subset S^4$ of the circle[3] $\{x_4^2 + x_5^2 = 1\} \subset S^4$.

The *standard embedding* $\mathrm{i} : D^3 \times S^1 \to S^6$ is the composition of standard embeddings $D^3 \times S^1 \to S^4$ and $S^4 \to S^6$.

The *standard embedding* $S^2 \times S^1 \to S^6$ is the restriction $\mathrm{i}|_{S^2\times S^1}$. The image of the embedding is $\{x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}, x_4^2 + x_5^2 = \frac{1}{2}\} \subset S^6$.

Construction of the embeddings $f_{k,0,0}$ and $g_{m,n}$. Proof of Corollary 1.

Construction of the embeddings $f_{k,0,0}$. Take any embedding $t : S^3 \to S^6$ isotopic to the standard embedding and such that $t(S^3)$ lies in a 6-disk disjoint from $\mathrm{i}(S^2\times S^1)$. Let $f_{0,0,0} : S^2\times S^1 \sqcup S^3 \to S^6$ be the embedding such that $f_{0,0,0}|_{S^2\times S^1} = \mathrm{i}|_{S^2\times S^1}$ and $f_{0,0,0}|_{S^3} = t$.

For $\frac{1}{10} \geq \epsilon \geq 0$ define the embeddings $z_{3,\epsilon} : S^3 \to S^6$ and $\overline{z}_{3,\epsilon} : S^3 \to S^6$ by the formulae:

$$z_{3,\epsilon}(x_1, x_2, x_3, x_4) = (\epsilon, 0, 0, (x_1, x_2, x_3, x_4)\sqrt{1 - \epsilon^2})$$

$$\overline{z}_{3,\epsilon}(x_1, x_2, x_3, x_4) = (\epsilon, 0, 0, (x_1, x_2, x_3, -x_4)\sqrt{1 - \epsilon^2})$$

Spheres $z_{3,\epsilon}(S^3)$ and $\overline{z}_{3,\epsilon}(S^3)$ are isotopic to the standard 3-sphere and are linked with $\mathrm{i}(S^2\times\cdot)$. The respective linking numbers are 1 and -1.

For an integer $k > 0$ let $f_{k,0,0} : S^2 \times S^1 \sqcup S^3 \to S^6$ be the embedding such that

$$f_{k,0,0}|_{S^3} = z_{3,\frac{1}{10k}}\#z_{3,\frac{2}{10k}}\#\ldots\#z_{3,\frac{k}{10k}} \quad \text{and} \quad f_{k,0,0}|_{S^2\times S^1} = \mathrm{i}|_{S^2\times S^1}.$$

[3]From the equation $x_4^2 + x_5^2 = 1$ it follows that $x_1 = x_2 = x_3 = 0$. To shorten the notation, we do not include the equation $x_1 = x_2 = x_3 = 0$ in the formula of the circle. The same goes for rest of the text.

For an integer $k<0$ let $f_{k,0,0}:S^2\times S^1\sqcup S^3\to S^6$ be the embedding such that

$$f_{k,0,0}|_{S^3}=\overline{z}_{3,\frac{1}{10|k|}}\#\overline{z}_{3,\frac{2}{10|k|}}\#\dots\#\overline{z}_{3,\frac{|k|}{10|k|}}\quad\text{and}\quad f_{k,0,0}|_{S^2\times S^1}=\mathrm{i}|_{S^2\times S^1}.$$

The embedding $f_{k,0,0}$ is Brunnian for each k.

Construction of the embeddings $g_{m,n}$. Let $g_{0,0}:S^3\sqcup S^3\to S^6$ be some Brunnian embedding such that the image of the first component lies in an open 6-ball disjoint from the image of the second component. We call $g_{0,0}$ a *trivial link.*

The Zeeman map $\tau:\pi_3(S^2)\to\mathrm{Emb}^6(S^3\sqcup S^3)$ is defined in [Atl2]. Let $\eta:S^3\to S^2$ be the Hopf map.

The Whitehead link $\omega:S^3\sqcup S^3\to S^6$ is also defined in [Atl2].

Both embeddings $\tau(\eta),\omega:S^3\sqcup S^3\to S^6$ are Brunnian[4]. For $m\equiv n \pmod 2$ let $g_{m,n}:=m\tau(\eta)\#\frac{(n-m)}{2}\omega$. Here both $0\tau(\eta)$ and 0ω are trivial links.

Proof of Corollary 1. Consider the trivial link $g_{0,0}:S^3\sqcup S^3\to S^6$ and the Whitehead link $g_{0,2}=\omega:S^3\sqcup S^3\to S^6$.

By Theorem 1, the embeddings $f_{1,0,0}\#g_{0,0}$ and $f_{1,0,0}\#\omega$ are isotopic, because $f_{1,0,0}\#g_{0,0}$ is isotopic to $f_{1,0,0}$ and $f_{1,0,0}\#\omega=f_{1,0,2}$. However, by the Haeifliger Theorem, the Whitehead link $g_{0,2}=\omega$ is not trivial, i.e., is not isotopic to $g_{0,0}$. □

2. Plan of the proof of the main result.

In this section we prove Theorem 1 modulo Lemmas 1 through 8. Proofs of all the Lemmas depend on this section. Lemmas 2 and 3 are proved in this section. Lemmas 4 and 5 are proved in Section 3, their proofs depend on Section 2. Lemma 7 is proved in Section 4, its proof depends on Section 2. Lemma 1 is proved in Section 6, its proof depends on Section 2 and Section 5 (which uses Lemma 7). Lemma 8 is proved is Section 8, its proof depends on Sections 2, 4, 5, and 7. Finally, Lemma 6 is proved is Section 9, its proof depends on Sections 2, 5, and 7.

Denote by $\mathrm{Emb}^6_B(S^2\times S^1\sqcup S^3)$ the set of isotopy classes of Brunnian embeddings $S^2\times S^1\sqcup S^3\to S^6$. For each m denote by 1 the point $(1,0,\dots,0)\in S^m$.

Definition of $\varkappa$. Define a map

$$\varkappa:\mathrm{Emb}^6_B(S^2\times S^1\sqcup S^3)\to\mathbb{Z}\quad\text{by the formula}\quad \varkappa([f]):=\mathrm{lk}(f(S^3),f(S^2\times 1))\in\mathbb{Z}.$$

Here $\mathrm{lk}(f(S^3),f(S^2\times 1))$ is the algebraic number of points of intersection[5] $f(S^3)\cap q(D^3)$, where $q:D^3\to S^6$ is a general position w.r.t. $f(S^3)$ map such that $q|_{S^2}=f|_{S^2\times 1}$. Clearly, $\varkappa$ is well defined.

To shorten the notation, we shall write $\varkappa(f)$ instead of $\varkappa([f])$.

In this text *framing* means *normal framing.*

---

[4] The image of $\tau(\eta)$ is as follows. Let $S^3\times D^3\subset S^6$ be the *standard tubular neighborhood* of the standard sphere $S^3\subset S^6$. Then the image of $\tau(\eta)$ is the union of the standard sphere $S^3\subset S^6$ and the graph of the Hopf map $\eta:S^3\to S^2$ in the boundary $S^3\times S^2\subset S^6$ of $S^3\times D^3\subset S^6$.

[5] We define the *algebraic number of points of intersection* in the usual way. Let $M^m,N^n\subset S^{m+n}$ be oriented submanifolds intersecting transversally. Let A be a point of $M\cap N$. Let $u_1,u_2,\dots,u_m$ and $v_1,v_2,\dots,v_n$ be positive tangent frames of M and N at A, respectively. The point A is a *positive* point of the intersection $M\cap N$ if the frame $u_1,u_2,\dots,u_m,v_1,v_2,\dots,v_n$ is a positive tangent frame of S^{m+n} at A. Note that according to this definiton the algebraic number of points of intersection $M^3\cap N^3\subset S^6$ equals *negative* algebraic number of points of intersection $N^3\cap M^3$.

Definition of a framed intersection. Let $M, N \subset S^k$ be transversal submanifolds. Assume that N is framed. At each point $P \in M \cap N$ the framing of the submanifold $M \cap N$ of M is the projection of the framing of N at P onto the tangent space of M at P.

In this text any intersection of a submanifold and a framed submanifold is assumed to be framed.

Definition of a framed preimage. Let $f : M \to N$ be an embedding and let $L \subset N$ be a framed submanifold. Then the preimage $f^{-1}(L)$ is a framed submanifold of M whose framing is the df-preimage of the framing of L.

In this text any preimage of a framed submanifold is assumed to be framed.

Definition of standard framings. Denote by e_k the vector $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^m$ where 1 is at the k-th position.

The *standard framing* of the standard $S^{p-1} \subset S^{m-1}$ at any point is $(e_{p+1}, e_{p+2}, \dots, e_m)$. The *standard framing* of $\mathrm{i}(D^3 \times S^1) \subset S^6$ is the restriction of the standard framing of $S^4 \supset \mathrm{i}(D^3 \times S^1)$.

Notation for framed submanifolds. By $M_{e_{i_1}, e_{i_2}, \dots, e_{i_k}}$ we denote the framed submanifold $M \subset S^m$ with the framing $e_{i_1}, e_{i_2}, \dots, e_{i_k}$. E.g., $\mathrm{i}(D^3 \times S^1)_{e_6, e_7}$ denotes the standardly framed $\mathrm{i}(D^3 \times S^1) \subset S^6$.

An embedding $f : S^2 \times S^1 \sqcup S^3 \to S^6$ is called *simple* if

$$f \quad \text{is Brunnian and} \quad f|_{S^2 \times S^1} = \mathrm{i}|_{S^2 \times S^1}.$$

Definition of $C_{\partial \mathrm{i}}$ and $\mathring{f}$. Denote by $C_{\partial \mathrm{i}}$ the closure of the complement to a small tubular neighborhood of $\mathrm{i}(S^2 \times S^1)$ in S^6. For a simple embedding $f : S^2 \times S^1 \sqcup S^3 \to S^6$, denote by

$$\mathring{f} : S^3 \to C_{\partial \mathrm{i}} \quad \text{the abbreviation}^6 \text{ of} \quad f.$$

Definition of ρ and $\equiv$. For integers a and c such that $c|a$, let $\rho_{a,c} : \mathbb{Z}_a \to \mathbb{Z}_c$ be reduction modulo c [7]. Let b be an integer such that $c|b$. Then for any $x \in \mathbb{Z}_a$ and $y \in \mathbb{Z}_b$

$$x \equiv y \pmod{c} \quad \text{indicates that} \quad \rho_{a,c}(x) = \rho_{b,c}(y).$$

Notation for the Hopf invariant. Given a map $\phi : S^3 \to S^2$, denote by $h(\phi) \in \mathbb{Z}$ its Hopf invariant. Given a framed 1-submanifold $a \subset S^3$, denote by $h(a) \in \mathbb{Z}$ its Hopf invariant [8].

Definition of $\widehat{\nu}$ and ν. Let $f : S^2 \times S^1 \sqcup S^3 \to S^6$ be a simple embedding such that $\mathrm{i}(D^3 \times S^1)$ and $\mathring{f}(S^3)$ are in general position. Let $\widehat{\nu}(f) \in \mathbb{Z}$ be the value of the Hopf invariant of the framed 1-submanifold $\mathring{f}^{-1}(\mathrm{i}(D^3 \times S^1)_{e_6, e_7})$ of S^3. I.e.,

$$\widehat{\nu}(f) := h\mathring{f}^{-1}(\mathrm{i}(D^3 \times S^1)_{e_6, e_7}) \in \mathbb{Z}.$$

Let

$$\nu([f]) := \rho_{0, 2\varkappa(f)}(\widehat{\nu}(f)) \in \mathbb{Z}_{2\varkappa(f)}.$$

Lemma 1. *The map $\nu : \varkappa^{-1}(k) \to \mathbb{Z}_{2k}$ is well defined for each k.*

---

[6]If $A \subset X$ and $B \subset Y$, then for every $f : X \to Y$ we have a map $g : A \to B, g(x) = f(x)$, which is called the *abbreviation* of f. Here we follow the definition of the abbreviation given in [Vi, Chapter II, Submaps, p.58].

[7]Here $\mathbb{Z}_a$ denotes $\mathbb{Z}/a\mathbb{Z}$. Thus $\mathbb{Z}_0$ is $\mathbb{Z}$.

[8]Slightly shift a along the first vector of its framing. Denote by a' the obtained 1-submanifold of S^3. The Hopf invariant of a is the linking number $\mathrm{lk}(a, a')$. Equivalently, the Hopf invariant of a is the Hopf invariant of the map $S^3 \to S^2$ obtained from a by the Pontryagin-Thom construction.

Denote

$$D^q_+ := \{y \in S^q \big| y_1 \geq 0\} \quad \text{and} \quad D^q_- := \{y \in S^q \big| y_1 \leq 0\}.$$

An embedding $f : S^2 \times S^1 \sqcup S^3 \to S^6$ is called *strictly simple* if

$$f \quad \text{is simple and} \quad f(S^3) \subset \mathrm{Int} D^6_+.$$

Lemma 2. *Let f be a simple embedding. Then there exists an isotopy fixed on $S^2 \times S^1$ between f and some strictly simple embedding.*

Proof. Denote by Δ^2 the 2-disk $\{x_1 \leq -\frac{1}{\sqrt{2}}, x_2 = x_3 = x_6 = x_7 = 0\}$ in S^6. Then $\Delta^2 \subset D^6_-$, $\mathrm{Int}\Delta^2 \cap \mathrm{i}(S^2 \times S^1) = \emptyset$ and $\partial\Delta^2 = \mathrm{i}((-1, 0, 0) \times S^1)$.

By general position, we may assume that $\mathring{f}(S^3)$ is disjoint from Δ^2. Denote by B^6 a small tubular neighborhood of Δ^2 disjoint from $\mathring{f}(S^3)$. Consider the framing e_2, e_3 of $\partial\Delta^2_{e_2,e_3} \subset \mathrm{i}(S^2 \times S^1)$. This framing extends to a framing of $\Delta^2_{e_2,e_3} \subset S^6$ since the vectors e_2, e_3 are orthogonal to Δ^2 at each point of Δ^2.

Hence there is an isotopy $H : C_{\partial\mathrm{i}} \times I \to C_{\partial\mathrm{i}} \times I$ mapping $B^6 \cap C_{\partial\mathrm{i}}$ to $\Delta^6_- \cap C_{\partial\mathrm{i}}$ and $C_{\partial\mathrm{i}} - B^6$ to $\Delta^6_+ \cap C_{\partial\mathrm{i}}$. Let $f' : S^2 \times S^1 \sqcup S^3 \to S^6$ be the simple embedding such that $\mathring{f}' = H_1\mathring{f}$. Then f' is as required. ☐

Lemma 3. *Let $f : S^3 \to S^6$ be an embedding isotopic to the standard embedding and such that $f(S^3) \subset \mathrm{Int}(D^6_+)$. Then there exists an embedded disk $\Delta \subset \mathrm{Int}(D^6_+)$ such that $\partial\Delta = f(S^3)$.*

Proof. Since f is isotopic to the standard embedding, there exists an embedded disk $\Delta \subset S^6$ such that $\partial\Delta = f(S^3)$. Disk D^6_- is a tubular neighborhood of a point. Therefore, by general position, we may assume that D^6_- and Δ are disjoint. Hence Δ is as required. ☐

Definition of $\theta : \mathrm{Int}D^2_+ \times S^1 \to S^3$. The embedding $\theta : \mathrm{Int}D^2_+ \times S^1 \to S^3$ is given by the formula:

$$\theta((x_1, x_2, x_3), (y_1, y_2)) := (x_2, x_3, y_1\sqrt{2 - x_2^2 - x_3^2}, y_2\sqrt{2 - x_2^2 - x_3^2})/\sqrt{2}.$$

The image of θ is the open tubular neighborhood $\{x_1^2 + x_2^2 < \frac{1}{2}\} \subset S^3$ of the circle $\{x_3^2 + x_4^2 = 1\} \subset S^3$.

Definition of $\widehat{\mu}$. Let f be a strictly simple embedding. Lemma 3 asserts that there is a 4-disk $\Delta \subset \mathrm{Int}D^6_+$ such that $\partial\Delta = \mathring{f}(S^3)$. Frame Δ arbitrarily. We may assume that Δ is in general position w.r.t. $\mathrm{i}(S^2 \times S^1)$.

Note that the framed intersection $\mathrm{i}(S^2 \times S^1) \cap \Delta$ lies in $\mathrm{i}(\mathrm{Int}D^2_+ \times S^1)$. Therefore, we have $\mathrm{i}^{-1}|_{S^2 \times S^1}(\Delta) \subset \mathrm{Int}D^2_+ \times S^1$ and $\theta\mathrm{i}^{-1}|_{S^2 \times S^1}(\Delta) \subset S^3$. Define

$$\widehat{\mu}(f) := h\theta\mathrm{i}^{-1}|_{S^2 \times S^1}(\Delta) \in \mathbb{Z}.$$

Lemma 4. *The number $\widehat{\mu}(f)$ is well defined, i.e., does not depend on the choice of Δ.*

Definition of μ. Consider an isotopy class from $\mathrm{Emb}^6_B(S^2 \times S^1 \sqcup S^3)$. By Lemma 2, this class contains a strictly simple embedding f. Let

$$\mu([f]) := \rho_{0,2\varkappa(f)}(\widehat{\mu}(f)) \in \mathbb{Z}_{2\varkappa(f)}.$$

Lemma 5. *The map $\mu : \varkappa^{-1}(k) \to \mathbb{Z}_{2k}$ is well defined for each k.*

To shorten the notation, we shall write $\nu(f)$ and $\mu(f)$ instead of $\nu([f])$ and $\mu([f])$, respectively. It is important to remember, however, that $\nu(f)$ and $\mu(f)$ depend only on the isotopy class of f while $\widehat{\nu}(f)$ and $\widehat{\mu}(f)$ depend also on the choice of the map f.

Lemma 6. *For any integers k, m, n such that $m \equiv n \pmod 2$:*

- $\varkappa(f_{k,m,n}) = k$
- $\mu(f_{k,m,n}) \equiv m \pmod{2k}$
- $\nu(f_{k,m,n}) \equiv n \pmod{2k}$

Lemma 7. *Let $f : S^2 \times S^1 \sqcup S^3 \to S^6$ be a simple embedding. Then $\mu(f) \equiv \widehat{\nu}(f) \pmod 2$.*

Lemma 8. *Let $f, g : S^2 \times S^1 \sqcup S^3 \to S^6$ be Brunnian embeddings. Suppose that $\varkappa(f) = \varkappa(g)$, $\mu(f) = \mu(g)$ and $\nu(f) = \nu(g)$. Then f is isotopic to g.*

Proof of Theorem 1. Let $f : S^2 \times S^1 \sqcup S^3 \to S^6$ be a Brunnian embedding. Let m and n be integers such that $\rho_{0,2\varkappa(f)}(m) = \mu(f)$ and $\rho_{0,2\varkappa(f)}(n) = \nu(f)$. Lemma 7 asserts that $m \equiv n \pmod 2$. By Lemma 6 embeddings f and $f_{\varkappa(f),m,n}$ have the same $\varkappa$, μ and ν invariants. Hence f is isotopic to $f_{\varkappa(f),m,n}$ by Lemma 8.

By Lemma 6, the embeddings $f_{k,m,n}$, $f_{k,m+2k,n}$, and $f_{k,m,n+2k}$ have the same $\varkappa$, μ and ν invariants. Hence they are isotopic to each other by Lemma 8.

Suppose that the embeddings $f_{k,m,n}$ and $f_{k',m',n'}$ are isotopic. Then they have the same $\varkappa$, μ, and ν invariants. Hence by Lemma 6 it follows that $k = k'$, $m \equiv m' \pmod{2k}$ and $n \equiv n' \pmod{2k}$. □

Notation for isotopies. Let $H : M \times I \to N \times I$ be an isotopy. For any $t \in I$ let $\mathrm{pr}_{M,t} : M \times t \to M$ and $\mathrm{pr}_{N,t} : N \times t \to N$ be the projections on the first factor. Denote by H_t the map $\mathrm{pr}_{N,t} \circ H|_{M \times t} \circ \mathrm{pr}^{-1}_{M,t} : M \to N$.

A relative framed cobordism and the relative Pontryagin-Thom isomorphism. Let M^m be a compact orientable manifold with boundary. Let $N_0, N_1 \subset M$ be properly embedded framed submanifolds. Framed submanifolds N_0 and N_1 are *relative framed cobordant* if there exists a properly embedded framed submanifold $W \subset M \times I$ such that $W \cap (M \times 0) = N_0$, $W \cap (M \times 1) = N_1$ and the framings of N_0 and N_1 are the restrictions of the framing of W. Framed submanifold $W \subset M \times I$ is called a *relative framed cobordism* between N_0 and N_1.

The *relative Pontryagin-Thom isomorphism* is a bijection between the homotopy classes of maps $M^m \to S^n$ and the relative framed cobordism classes of the properly embedded framed $(m-n)$-submanifolds of M^m. The bijection is given by the relative Pontryagin-Thom contruction analogous to the absolute Pontryagin-Thom construction for closed manifolds.

Let $N^n \subset M$ be a properly embedded framed submanifold. We may use the relative Pontryagin-Thom construction to produce a map $f : M \to S^{m-n}$ such that N is the framed preimage of a regular point of f. We shall say that under the relative Pontryagin-Thom isomorphism, the framed submanifold N *corresponds* to the map f.

Clearly, the existence of the relative Pontryagin-Thom isomorphism implies that relative framed cobordant framed submanifolds correspond to homotopic maps and vice versa.

3. Proof of Lemmas 4 and 5 (that $\widehat{\mu}$ and μ are well-defined).

Lemma 9. *Let $f : S^3 \to S^6$ be an embedding. Denote by C_f the closure of the complement to a small tubular neighborhood of $f(S^3)$ in S^6. Let $\Delta \subset S^6$ and $\Delta' \subset S^6$ be framed 4-disks such that $\partial\Delta = \partial\Delta' = f(S^3)$.*

Then the framed submanifolds $\Delta \cap C_f$ and $\Delta' \cap C_f$ are relative framed cobordant in C_f.

Proof. Under the relative Pontryagin-Thom isomorphism, the framed submanifolds $\Delta \cap C_f$ and $\Delta' \cap C_f$ of C_f correspond to some maps $\delta : C_f \to S^2$ and $\delta' : C_f \to S^2$, respectively. To prove the Lemma, it suffices to prove that δ is homotopic to δ'.

Let $G : D^3 \to S^6$ be an embedding such that the disk $G(D^3)$ intersects $f(S^3)$ transversally and the intersection $f(S^3) \cap G(D^3)$ is a single positive point. Let $g : S^2 \to C_f$ be the abbreviation of $G|_{S^2}$. Clearly, $\mathrm{lk}(f(S^3), g(S^2)) = 1$.

By Alexander duality, $[g(S^2)]$ is a generator of $H_2(C_f;\mathbb{Z}) \cong \mathbb{Z}$ and the rest of the homology groups of C_f are trivial. The space C_f is simply connected. Hence by the Hurewicz and Whitehead Theorems, $g : S^2 \to C_f$ is a homotopy equivalence. Therefore, $g_* : [C_f, S^2] \to [S^2, S^2]$ is a bijection.

By the definiton of g, the algebraic number of points of $g(S^2) \cap \Delta$ is 1, so $\delta \circ g : S^2 \to S^2$ is a degree one map. Likewise, $\delta' \circ g : S^2 \to S^2$ is also a degree one map. Hence $\delta \circ g$ is homotopic to $\delta' \circ g$. Since $g_* : [C_f, S^2] \to [S^2, S^2]$ is a bijection, it follows that δ is homotopic to δ'. □

Proof of Lemma 4. Let $\Delta \subset \mathrm{Int}D^6_+$ and $\Delta' \subset \mathrm{Int}D^6_+$ be framed 4-disks such that $\partial\Delta = \partial\Delta' = \mathring{f}(S^3)$. Denote by $C_{\mathring{f}}$ the closure of the complement to a small tubular neighborhood of $\mathring{f}(S^3)$ in S^6.

Lemma 9 asserts that $\Delta \cap C_{\mathring{f}}$ and $\Delta' \cap C_{\mathring{f}}$ are relative framed cobordant in $C_{\mathring{f}}$. Disk D^6_- is a tubular neighborhood of a point. Hence, by general position, we may assume that $\Delta \cap C_{\mathring{f}}$ and $\Delta' \cap C_{\mathring{f}}$ are relative framed cobordant in $C_{\mathring{f}} \cap \mathrm{Int}D^6_+$. The framed intersection of $\mathrm{i}(\mathrm{Int}D^2_+ \times S^1)$ and this relative framed cobordism is a framed cobordism between $K := \mathrm{i}(\mathrm{Int}D^2_+ \times S^1) \cap \Delta$ and $K' := \mathrm{i}(\mathrm{Int}D^2_+ \times S^1) \cap \Delta'$. Therefore, the framed intersections K and K' are framed cobordant. Hence $h\theta\mathrm{i}^{-1}(K) = h\theta\mathrm{i}^{-1}(K')$. The Lemma follows. □

The following Lemma 10 is a corollary of [Ce07, Theorem 1 (L.S. Pontryagin)]. We follow the notation of [Ce07] in the statement and the proof of the Lemma.

Lemma 10. *Let L_1 and L_2 be framed 1-submanifolds of $\mathrm{Int}D^2_+ \times S^1 \subset S^2 \times S^1$ homologous to $k[1 \times S^1]$ for some integer k. Suppose that L_1 is framed cobordant to L_2 in $S^2 \times S^1$.*

Then $h(\theta(L_1)) \equiv h(\theta(L_2)) \pmod{2k}$.

Proof. In codimension 2 any integer homology between codimension 3 submanifolds can be realized by a submanifold. Since L_1 and L_2 are homologous, it follows that there is a (not framed) cobordism L in $\mathrm{Int}D^2_+ \times S^1 \times I$ between them. The definition of the *relative normal Euler class* $\overline{e}(L) \in \mathbb{Z}$ is given in [Ce07, §2]. We have

$$h(\theta(L_2)) - h(\theta(L_1)) = \overline{e}(\theta(L)) = \overline{e}(L) \equiv 0 \pmod{2k},$$

where the first two equations are obvious and the last follows by (2) in the proof of Theorem 1 in [Ce07]. □

Proof of Lemma 5. Suppose that f and f' are isotopic strictly simple embeddings.

Let $\Delta, \Delta' \subset \mathrm{Int}D^6_+$ be framed 4-disks such that $\partial\Delta = \mathring{f}(S^3)$ and $\partial\Delta' = \mathring{f}'(S^3)$.

We shall prove that $\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta)$ is framed cobordant to $\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta')$ in $S^2 \times S^1$. By Lemma 10 this would imply the assertion of the Lemma.

Let $H : S^6 \times I \to S^6 \times I$ be an isotopy between f and f'. Notation $X \underset{fr}{=} Y$ means that X is framed cobordant to Y. We have

$$\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta) \underset{fr}{\overset{(1)}{=}} (H_1 \circ \mathrm{i}|_{S^2\times S^1})^{-1}(H_1\Delta) \underset{fr}{\overset{(2)}{=}} \mathrm{i}^{-1}|_{S^2\times S^1}(H_1\Delta) \underset{fr}{\overset{(3)}{=}} \mathrm{i}^{-1}|_{S^2\times S^1}(\Delta').$$

(1) Here the framed cobordism is the $\mathrm{i}|_{S^2\times S^1}$-preimage of $H\Delta$.
(2) Both f and f' are simple, so the map $H_1 : S^6 \to S^6$ is identical on $\mathrm{i}(S^2 \times S^1)$.

(3) Denote by $C_{\mathring{f}'}$ the closure of the complement to a small tubular neighborhood of $\mathring{f}'(S^3)$ in S^6. The equation holds since $(H_1\Delta)\cap C_{\mathring{f}'}$ and $\Delta'\cap C_{\mathring{f}'}$ are relative framed cobordant in $C_{\mathring{f}'}$ by Lemma 9.

□

4. Additional lemmas, proof of Lemma 7 (that μ and ν have the same parity).

By S^3_1 and S^3_2 we denote two distinct copies of S^3.

Definition of μ and ν for embeddings $S^3_1 \sqcup S^3_2 \to S^6$. Consider a Brunnian embedding $g : S^3_1 \sqcup S^3_2 \to S^6$. Let Δ^4 be an arbitrary framed general position embedded 4-disk such that $\partial\Delta^4 = g(S^3_2)$. Define

$$\mu([g]) := h(g|_{S^3_1}{}^{-1}(\Delta^4)).$$

The definition of $\nu([g])$ is analogous and is obtained by exchanging the components.

To shorten the notation, we shall write $\nu(g)$ and $\mu(g)$ instead of $\nu([g])$ and $\mu([g])$, respectively.

Our definition of μ and ν is equivalent to the definition in [Atl2] of $\lambda_{12} : \mathrm{Emb}^6(S^3 \sqcup S^3) \to \mathbb{Z}$ and $\lambda_{21} : \mathrm{Emb}^6(S^3 \sqcup S^3) \to \mathbb{Z}$, respectively.

Recall that $\lambda_{12}(\omega) = 0$, $\lambda_{21}(\omega) = 2$, $\lambda_{12}(\tau(\eta)) = 1$ (see [Atl2]) and $\lambda_{21}(\tau(\eta)) = 1$ (see [Sk07, proof of the τ-relation, end of §4]). Hence, $\mu(\omega) = 0$, $\nu(\omega) = 2$, $\mu(\tau(\eta)) = 1$ and $\nu(\tau(\eta)) = 1$

Lemma 11. *Let m and n be numbers such that $m \equiv n \pmod 2$. Then $\mu(g_{m,n}) = m$ and $\nu(g_{m,n}) = n$.*

Proof. Obviously, invariants μ and ν are additive with respect to #. Hence, $\mu(g_{m,n}) = \mu(m\tau(\eta)\#\frac{(n-m)}{2}\omega) = m$ and $\nu(g_{m,n}) = \nu(m\tau(\eta)\#\frac{(n-m)}{2}\omega) = n$. □

Let $\sigma : S^3 \to S^6$ be an embedding with the image

$$\{x_1 \geq 0, x_1^2 + x_2^2 + x_3^2 = \frac{1}{2}, x_4^2 + x_5^2 = \frac{1}{2}\} \cup \{x_1 < 0, x_1^2 + x_4^2 + x_5^2 = \frac{1}{2}, x_2^2 + x_3^2 = \frac{1}{2}\}.$$

Informally, $\sigma(S^3)$ is the result of an embedded 1-surgery on the circle $\mathrm{i}((-1)\times S^1)$ of $\mathrm{i}(S^2 \times S^1)$.

We shall use that in $S^4 \subset S^6$ the sphere $\sigma(S^3) \subset S^4$ bounds the 4-disk

$$\Delta_\sigma := \{x_1 \geq 0, x_1^2 + x_2^2 + x_3^2 \leq \frac{1}{2}, x_4^2 + x_5^2 \geq \frac{1}{2}\} \cup \{x_1 < 0, x_1^2 + x_4^2 + x_5^2 \geq \frac{1}{2}, x_2^2 + x_3^2 \leq \frac{1}{2}\}.$$

Hence the embedding σ is isotopic to the standard embedding. Also note that $\sigma(S^3)\cap D^6_+ = \mathrm{i}(D^2_+ \times S^1)$.

Lemma 12. *Let f be a strictly simple embedding. Denote by g the embedding $\sigma \sqcup f|_{S^3} : S^3_1 \sqcup S^3_2 \to S^6$. Then g is Brunnian and $\widehat{\mu}(f) = \mu(g)$ and $\widehat{\nu}(f) = \nu(g)$.*

Proof. By Lemma 3, there exists an arbitrary framed general position embedded 4-disk $\Delta^4 \subset \mathrm{Int}D^6_+$ such that $\partial\Delta^4 = f(S^3) = g(S^3_2)$. We have

$$\widehat{\mu}(f) \overset{(1)}{=} h\theta\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta^4) \overset{(2)}{=} h\sigma^{-1}\mathrm{i}|_{\mathrm{Int}D^2_+\times S^1}\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta^4) \overset{(3)}{=} h\sigma^{-1}(\Delta^4) \overset{(4)}{=} \mu(g), \text{ where}$$

(1) This is the definition of $\widehat{\mu}(f)$.
(2) Holds, because $\theta|_{\mathrm{Int}D^2_+\times S^1}$ is isotopic to $\sigma^{-1}\mathrm{i}|_{\mathrm{Int}D^2_+\times S^1} : \mathrm{Int}D^2_+ \times S^1 \to S^3$.
(3) Holds, because $\sigma^{-1}\mathrm{i}|_{\mathrm{Int}D^2_+\times S^1}\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta^4) = \sigma^{-1}(\Delta^4)$.
(4) This is the definition of $\mu(g)$.

Consider the framed submanifold Δ_{σ,e_6,e_7}. The framings of Δ_{σ,e_6,e_7} and of $\mathrm{i}(D^3\times S^1)_{e_6,e_7}$ coincide on $\Delta_\sigma\cap D^6_+=\mathrm{i}(D^3\times S^1)\cap D^6_+$.

$$\widehat{\nu}(f)\overset{(1)}{=}h\mathring{f}^{-1}(\mathrm{i}(D^3\times S^1)_{e_6,e_7})\overset{(2)}{=}h\mathring{f}^{-1}(\Delta_{\sigma,e_6,e_7})\overset{(3)}{=}hg|_{S^3_2}^{-1}(\Delta_{\sigma,e_6,e_7})\overset{(4)}{=}\nu(g),\ \text{where}$$

(1) This is the definition of $\widehat{\nu}(f)$.
(2) Holds, because the framings of Δ_{σ,e_6,e_7} and of $\mathrm{i}(D^3\times S^1)_{e_6,e_7}$ coincide on $\Delta_\sigma\cap D^6_+=\mathrm{i}(D^3\times S^1)\cap D^6_+$.
(3) Holds, because $\mathring{f}=g|_{S^3_2}$ by definition of g.
(4) Since $\partial\Delta_\sigma=\sigma(S^3)$, the last equation is the definition of $\nu(g)$.

□

Lemma 13. *Let f and f' be simple embeddings. Suppose that there is an isotopy between f and f' fixed on $S^2\times S^1$. Then $\widehat{\nu}(f)=\widehat{\nu}(f')$.*

Proof. Since every smooth isotopy is ambient, there is an ambient isotopy $H:S^6\times I\to S^6\times I$ between f and f' fixed on $S^2\times S^1$.

We may assume that $\mathrm{i}(D^3\times S^1)\times I$ and $H(\mathring{f}(S^3)\times I)$ are in general position. Then the framed intersection $H(\mathring{f}(S^3)\times I)\cap(\mathrm{i}(D^3\times S^1)\times I)_{e_6,e_7}$ is a framed cobordism between framed 1-submanifolds $\mathring{f}(S^3)\cap\mathrm{i}(D^3\times S^1)_{e_6,e_7}$ and $\mathring{f}'(S^3)\cap\mathrm{i}(D^3\times S^1)_{e_6,e_7}$ of $H(\mathring{f}(S^3)\times I)$.

So, $\mathring{f}^{-1}(\mathrm{i}(D^3\times S^1)_{e_6,e_7})$ and $\mathring{f}'^{-1}(\mathrm{i}(D^3\times S^1)_{e_6,e_7})$ are framed cobordant in S^3. Hence the values of their Hopf invariants $\widehat{\nu}(f)$ and $\widehat{\nu}(f')$ are equal. □

Proof of Lemma 7. Lemma 2 asserts that there exists an isotopy fixed on $S^2\times S^1$ between f and some strictly simple embedding f'. Denote by g the embedding $\sigma\sqcup f'|_{S^3}:S^3\sqcup S^3\to S^6$. Then g is Brunnian and

$$\mu(f)\overset{(1)}{\equiv}\mu(f')\overset{(2)}{\equiv}\widehat{\mu}(f')\overset{(3)}{\equiv}\mu(g)\overset{(4)}{\equiv}\nu(g)\overset{(5)}{\equiv}\widehat{\nu}(f')\overset{(6)}{\equiv}\widehat{\nu}(f)\pmod 2,\ \text{where}$$

(1) Holds, because μ is well-defined and f is isotopic to f'.
(2) Holds by the definition of μ.
(3,5) Hold by Lemma 12.
(4) By Theorem 2, g is isotopic to $g_{m,n}$ for some integers $m\equiv n\pmod 2$. Lemma 11 asserts that $\mu(g)=m$ and $\nu(g)=n$. Hence the equality holds.[9]
(6) Holds by Lemma 13, since the isotopy between f and f' is fixed on $S^2\times S^1$.

□

5. On the homotopy type of $C_{\partial\mathrm{i}}$.

Under the relative Pontryagin-Thom isomorphism, the framed submanifolds $\mathrm{i}(D^3\times S^1)_{e_6,e_7}\cap C_{\partial\mathrm{i}}$ and $\mathrm{i}(D^3\times 1)_{e_5,e_6,e_7}\cap C_{\partial\mathrm{i}}$ of $C_{\partial\mathrm{i}}$ correspond to some maps

$$p_2:C_{\partial\mathrm{i}}\to S^2\quad\text{and}\quad p_3:C_{\partial\mathrm{i}}\to S^3,$$

respectively.

Define the embedding $z_2':S^2\to C_{\partial\mathrm{i}}$ by the formula:

$$z_2'(x_1,x_2,x_3)=(0,0,0,0,x_1,x_2,x_3).$$

[9]Here it is convinient to us to deduce the fact that $\mu(g)\equiv\nu(g)\pmod 2$ from Theorem 2. However, the proof of this fact is an essential part of the proof of Theorem 2.

Define the map $z_2 : S^3 \to C_{\partial \mathrm{i}}$ as the composition:

$$S^3 \xrightarrow{\text{Hopf map}} S^2 \xrightarrow{z_2'} C_{\partial \mathrm{i}}.$$

Define the embedding $z_3 : S^3 \to C_{\partial \mathrm{i}}$ by the formula:

$$z_3(x_1, x_2, x_3, x_4) = (0, 0, 0, x_1, x_2, x_3, x_4).$$

Given a map $\phi : S^3 \to C_{\partial \mathrm{i}}$ we denote by $[\phi] \in \pi_3(C_{\partial \mathrm{i}})$ its homotopy class[10].

Lemma 14. *Homotopy classes $[z_2]$ and $[z_3]$ form a basis of $\pi_3(C_{\partial \mathrm{i}})$. For each map $\phi : S^3 \to C_{\partial \mathrm{i}}$ the coefficients of its homotopy class $[\phi]$ in the basis are $h(p_2\phi)$ and $\deg(p_3\phi)$, i.e., $[\phi] = h(p_2\phi)[z_2] + \deg(p_3\phi)[z_3]$.*

Proof. The framed 0-submanifold $z_3^{-1}(\mathrm{i}(D^3 \times 1)_{e_5,e_6,e_7})$ is the framed preimage of a regular point of $p_3z_3 : S^3 \to S^3$. This framed 0-submanifold consists of a single positive point $(1, 0, 0, 0)_{e_2,e_3,e_4} \subset S^3$. Therefore, $\deg(p_3z_3) = 1$.

Similarly $\deg(p_2z_2') = 1$. Map $p_2z_2 : S^3 \to S^2$ is the composition of $p_2z_2' : S^2 \to S^2$ and the Hopf map $S^3 \to S^2$, hence $h(p_2z_2) = 1$.

The framed 0-submanifold $z_2^{-1}(\mathrm{i}(D^3 \times 1)_{e_5,e_6,e_7})$ is the framed preimage of a regular point of $p_3z_2 : S^3 \to S^3$. This submanifold is empty. Therefore $\deg(p_3z_2) = 0$.

The framed submanifold $z_3^{-1}(\mathrm{i}(D^3 \times S^1)_{e_6,e_7}) \subset S^3$ is the framed preimage of a regular point of $p_2z_3 : S^3 \to S^2$. This framed submanifold is the framed circle $\{x_1^2 + x_2^2 = 1\}_{e_3,e_4} \subset S^3$. The Hopf invariant of this framed circle is zero. Therefore $h(p_2z_3) = 0$.

Define a map $\xi : \pi_3(C_{\partial \mathrm{i}}) \to \mathbb{Z} \oplus \mathbb{Z}$ by the formula $\xi([\phi]) = \deg(p_3\phi) \oplus h(p_2\phi)$. Clearly, ξ is a homomorphism. As we have just shown, $\xi([z_3]) = 1 \oplus 0$ and $\xi([z_2]) = 0 \oplus 1$, hence ξ is surjective.

The manifold $C_{\partial \mathrm{i}}$ is homotopy equivalent to $S^2 \vee S^3 \vee S^4$. Therefore $\pi_3(C_{\partial \mathrm{i}}) \cong \mathbb{Z} \oplus \mathbb{Z}$ by the Hilton Theorem on the homotopy groups of wedge product of spheres. So ξ is an isomorphism, and the Lemma follows. ☐

Lemma 15. *Let f be a simple embedding. Then $[\mathring{f}] = \widehat{\nu}(f)[z_2] + \varkappa(f)[z_3]$.*

Proof. Lemma 14 asserts that $[\mathring{f}] = h(p_2\mathring{f})[z_2] + \deg(p_3\mathring{f})[z_3]$. It remains to prove that $h(p_2\mathring{f}) = \widehat{\nu}(f)$ and $\deg(p_3\mathring{f}) = \varkappa(f)$.

We have,

$$h(p_2\mathring{f}) = h(\mathring{f}^{-1}(\mathrm{i}(D^3 \times S^1)_{e_6,e_7})) = \widehat{\nu}(f).$$

Here the first equation holds since $\mathring{f}^{-1}(\mathrm{i}(D^3 \times S^1)_{e_6,e_7})$ is the framed preimage of a regular point of $p_2\mathring{f} : S^3 \to S^2$. The second equation is the definition of $\widehat{\nu}$.

We also have,

$$\deg(p_3\mathring{f}) = \#(\mathring{f}^{-1}(\mathrm{i}(D^3 \times 1)_{e_5,e_6,e_7})) = \#(\mathring{f}(S^3) \cap \mathrm{i}(D^3 \times 1)_{e_5,e_6,e_7}) = \varkappa(f).$$

Here $\#(\cdot)$ denotes the algebraic number of points[11] of a framed 0-submanifold. The first equation holds since $\mathring{f}^{-1}(\mathrm{i}(D^3 \times 1)_{e_5,e_6,e_7})$ is the framed preimage of a regular point of $p_3\mathring{f} : S^3 \to S^3$. The last equation is the definition of $\varkappa$. ☐

Lemma 16. *Let $H : S^6 \times I \to S^6 \times I$ be an isotopy such that $H_0|_{S^2 \times S^1} = H_1|_{S^2 \times S^1} = \mathrm{i}|_{S^2 \times S^1}$. Suppose that $H_1(C_{\partial \mathrm{i}}) = C_{\partial \mathrm{i}}$. Let $\widetilde{H} : C_{\partial \mathrm{i}} \to C_{\partial \mathrm{i}}$ be the abbreviation of H_1. Then*

[10]The space $C_{\partial \mathrm{i}}$ is simply connected so we can ignore the base points.

[11]See footnote 5.

(I) $[\widetilde{H}z_2] = [z_2]$.
(II) $[\widetilde{H}z_3] = 2k[z_2] + [z_3]$ *for some integer* k.

Proof. (I). Map $\pi_2(C_{\partial \mathrm{i}}) \to \mathbb{Z}$ given by the formula $[\phi] \mapsto \mathrm{lk}(\phi(S^2), \mathrm{i}(S^2 \times S^1))$ is an isomorphism. The linking number is an isotopy invariant. Therefore $\widetilde{H}$ induces the identity isomorphism of $\pi_2(C_{\partial \mathrm{i}})$.

So, the maps $z_2' : S^2 \to C_{\partial \mathrm{i}}$ and $\widetilde{H}z_2' : S^2 \to C_{\partial \mathrm{i}}$ are homotopic. Hence, by the definition of z_2, the maps $z_2 : S^3 \to C_{\partial \mathrm{i}}$ and $\widetilde{H}z_2 : S^3 \to C_{\partial \mathrm{i}}$ are homotopic as well, i.e., $[\widetilde{H}z_2] = [z_2]$.

(II). Consider the simple embedding $f : S^2 \times S^1 \sqcup S^3 \to S^6$ such that $\mathring{f} = z_3$. We have,

$$[\widetilde{H}z_3] = [\widetilde{H}\mathring{f}] \overset{(1)}{=} \widehat{\nu}(\widetilde{H}f)[z_2] + \varkappa(\widetilde{H}f)[z_3] \overset{(2)}{=} \widehat{\nu}(\widetilde{H}f)[z_2] + [z_3].$$

(1) Holds by Lemma 15.
(2) The linking number is an isotopy invariant, so $\varkappa(\widetilde{H}f) = \varkappa(f) = 1$.

The integer $\widehat{\nu}(\widetilde{H}f)$ is even since

$$\widehat{\nu}(\widetilde{H}f) \overset{(1)}{\equiv} \mu(\widetilde{H}f) \overset{(2)}{\equiv} \mu(f) \overset{(3)}{\equiv} \widehat{\nu}(f) \overset{(4)}{\equiv} 0 \pmod 2.$$

(1,3) Follow from Lemma 7.
(2) Holds, because μ is well defined (Lemma 4) and $\widetilde{H}f$ is isotopic to f.
(4) Follows from Lemmas 14 and 15 since $\mathring{f} = z_3$.

□

6. Proof of Lemma 1 (that $\widehat{\nu}$ is well-defined).

Let f and f' be isotopic simple embeddings. We shall prove that $\widehat{\nu}(f') - \widehat{\nu}(f)$ is divisible by $2\varkappa(f) = 2\varkappa(f')$. This will imply that $\nu(f)$ is well defined.

Let $H : S^6 \times I \to S^6 \times I$ be an isotopy between f and f'. Since the embeddings f and f' are simple, it follows that $H_0|_{S^2 \times S^1} = H_1|_{S^2 \times S^1} = \mathrm{i}|_{S^2 \times S^1}$. Therefore we may assume that $H_1(C_{\partial \mathrm{i}}) = C_{\partial \mathrm{i}}$. Let $\widetilde{H} : C_{\partial \mathrm{i}} \to C_{\partial \mathrm{i}}$ be the abbreviation of H_1. Let $\widetilde{H}^* : \pi_3(C_{\partial \mathrm{i}}) \to \pi_3(C_{\partial \mathrm{i}})$ be the homomorphism induced by $\widetilde{H}$. Then for some integer k

$$\widehat{\nu}(f')[z_2] + \varkappa(f)[z_3] \overset{(1)}{=} [\mathring{f}'] = [\widetilde{H}\mathring{f}] = \widetilde{H}^*[\mathring{f}] \overset{(2)}{=} \widetilde{H}^*\big(\widehat{\nu}(f)[z_2] + \varkappa(f)[z_3]\big) =$$
$$\widehat{\nu}(f)[\widetilde{H}z_2] + \varkappa(f)[\widetilde{H}z_3] \overset{(3)}{=} \big(\widehat{\nu}(f) + 2\varkappa(f)k\big)[z_2] + \varkappa(f)[z_3].$$

(1) Holds by Lemma 15.
(2) Holds, because $[\mathring{f}] = \widehat{\nu}(f)[z_2] + \varkappa(f)[z_3]$ by Lemma 15.
(3) Holds, because $[\widetilde{H}z_2] = [z_2]$ and $[\widetilde{H}z_3] = 2k[z_2] + [z_3]$ by Lemma 16.

The homotopy classes $[z_2]$ and $[z_3]$ form a basis of $\pi_3(C_{\partial \mathrm{i}})$ by Lemma 14. So, $\widehat{\nu}(f') - \widehat{\nu}(f) = 2\varkappa(f)k =$. □

7. Framing an embedded connected sum. Definition of a framed isotopy.

Lemma 17. *Let $f : M \to S^k$ and $g : N \to S^k$ be embeddings. Suppose that $f(M)$ and $g(N)$ are framed. Then the framing of $f(M_0) \sqcup g(N_0)$ extends to a framing of $(f\#g)(M\#N)$.*

Proof. The framing of $f(\partial M_0)$ extends to the framing of the disk $f(M \setminus \mathrm{Int} M_0)$. Likewise, the framing of $g(\partial N_0)$ extends to the framing of the disk $g(N \setminus \mathrm{Int} N_0)$.

Identify the disks $f(M \setminus \mathrm{Int} M_0)$ and $g(N \setminus \mathrm{Int} N_0)$ by an ambient isotopy. All the framings of an embedded disk are equivalent. Therefore, the framings of $f(\partial M_0)$ and $g(\partial N_0)$ are also equivalent. Hence, the framing of $f(M_0) \sqcup g(N_0)$ extends to a framing of $(f\#g)(M\#N)$. □

Definition of a framed isotopy. Let $f, g : M^m \to S^k$ be embeddings. Suppose that $f(M)$ and $g(M)$ are framed. An isotopy $H : S^k \times I \to S^k \times I$ between f and g is a *framed isotopy* if for each point $x \in M$ and for each integer $1 \le i \le k - m$ the dH_1-image of the i-th vector of the framing of $f(M)$ at $f(x)$ is the i-th vector of the framing of $g(M)$ at $g(x)$.

The isotopy H is also called a *framed isotopy between the framed submanifolds* $f(M)$ and $g(M)$.

Lemma 18. *Let $f, g : D^n \times S^m \to S^k$ be embeddings. Suppose that $f(D^n \times S^m)$ and $g(D^n \times S^m)$ are framed. Let $B^n \subset D^n$ be an n-disk. Suppose that $f|_{B^n \times S^m} = g|_{B^n \times S^m}$ and that the framings of $f(B^n \times S^m)$ and of $g(B^n \times S^m)$ are the same.*

Then f is framed isotopic to g.

Proof. Using the framing of $f(D^n \times S^m)$ we may naturally construct an embedding $\widetilde{f} : (D^n \times S^m) \times D^{k-m-n} \to S^k$ such that $\widetilde{f}|_{(D^n \times S^m)\times 0} = f$ and for each $x \in D^n \times S^m$ and each $1 \le i \le k - m - n$ the $d\widetilde{f}$-image of the vector $x \times e_i$ is the i-th vector of the framing of $f(D^n \times S^m)$.

An embedding $\widetilde{g} : (D^n \times S^m) \times D^{k-m-n} \to S^k$ is constructed analogously.

It suffices to prove that $\widetilde{f}$ is isotopic to $\widetilde{g}$.

Without loss of generality we may assume that $B^n = \frac{1}{2}D^n$. Let $s : D^n \to B^n$ be the map given by the formula $s(x) = \frac{1}{2}x$.

Clearly, $\widetilde{f}$ is isotopic to $\widetilde{f} \circ (s \times \mathrm{id}_{S^m} \times \mathrm{id}_{D^{k-m-n}}) : D^n \times S^m \times D^{k-m-n} \to S^k$. Likewise, $\widetilde{g}$ is isotopic to $\widetilde{g} \circ (s \times \mathrm{id}_{S^m} \times \mathrm{id}_{D^{k-m-n}}) : D^n \times S^m \times D^{k-m-n} \to S^k$.

By the assumption of the Lemma $\widetilde{f}$ and $\widetilde{g}$ coincide on $\mathrm{Im}(s \times \mathrm{id}_{S^m} \times \mathrm{id}_{D^{k-m-n}}) = B^n \times S^m \times D^{k-m-n}$. Hence, $\widetilde{f}$ is isotopic to $\widetilde{g}$. □

8. Proof of Lemma 8.

The following three lemmas are proved later in this section.

Lemma 19. *Let f and f' be strictly simple embeddings. Suppose that $\varkappa(f) = \varkappa(f')$, $\widehat{\mu}(f) = \widehat{\mu}(f')$ and $\widehat{\nu}(f) = \widehat{\nu}(f')$. Then f is isotopic to f'.*

Lemma 20. *Let f be a strictly simple embedding. Then for any integer c there exists a strictly simple embedding f' isotopic to f and such that $\widehat{\mu}(f') = \widehat{\mu}(f) + 2\varkappa(f)c$ and $\widehat{\nu}(f') = \widehat{\nu}(f)$.*

Lemma 21. *Let f be a simple embedding. Then for any integer c there exists a simple embedding f' isotopic to f and such that $\widehat{\nu}(f') = \widehat{\nu}(f) + 2\varkappa(f)c$.*

Proof of Lemma 8. Without loss of generality, we may assume that both f and g are simple.

By Lemma 21, there are simple embeddings f' and g' isotopic to f and g, respectively, and such that $\widehat{\nu}(f') = \widehat{\nu}(g')$.

By Lemma 2, there is an isotopy fixed on $S^2 \times S^1$ between f' and some strictly simple embedding f''. Likewise, there is an isotopy fixed on $S^2 \times S^1$ between g' and some strictly simple embedding g''. Since the isotopy between f' and f'' is fixed on $S^2 \times S^1$, it follows by Lemma 13 that $\widehat{\nu}(f') = \widehat{\nu}(f'')$. Likewise, $\widehat{\nu}(g') = \widehat{\nu}(g'')$. Therefore, $\widehat{\nu}(f'') = \widehat{\nu}(g'')$.

By Lemma 20, there are strictly simple embeddings f''' and g''' isotopic to f'' and g'', respectively, and such that $\widehat{\nu}(f''') = \widehat{\nu}(g''')$ and $\widehat{\mu}(f''') = \widehat{\mu}(g''')$. Clearly, $\varkappa(f''') = \varkappa(f) = \varkappa(g) = \varkappa(g''')$.

By Lemma 19, the embeddings f''' and g''' are isotopic. Hence f and g are isotopic as well. □

To prove Lemma 19, we need the following Lemma 22. See its proof in [Ha66, proof of Theorem 7.1] or in [MSk13, proof of Claim 3.1].

Lemma 22. *Let $g, g' : S^3_1 \sqcup S^3_2 \to S^6$ be isotopic embeddings. Suppose that $g|_{S^3_1} = g'|_{S^3_1}$. Then there exists an isotopy between g and g' fixed on S^3_1.*

Proof of Lemma 19. The idea of the proof is to construct an isotopy between $\mathring{f}$ and $\mathring{f}'$. The existence of such an isotopy will follow from Theorem 2.

Denote by g the embedding $\sigma \sqcup f|_{S^3} : S^3_1 \sqcup S^3_2 \to S^6$. Denote by g' the embedding $\sigma \sqcup f'|_{S^3} : S^3_1 \sqcup S^3_2 \to S^6$. Pairs g and g' of linked spheres are isotopic because

$$\mu(g) \overset{(1)}{=} \widehat{\mu}(f) \overset{(2)}{=} \widehat{\mu}(f') \overset{(3)}{=} \mu(g').$$

(1,3) Hold by Lemma 12.

(2) Holds by the assumption of the Lemma.

Similarly, $\nu(g) = \nu(g')$. So, by Lemma 11 and Theorem 2, it follows that g is isotopic to g'.

Denote by C_σ the closure of the complement to a small tubular neighborhood of $\sigma(S^3)$ in S^6. Let $g_2 : S^3 \to C_\sigma$ and $g'_2 : S^3 \to C_\sigma$ be the abbreviations of $g|_{S^3_2}$ and $g'|_{S^3_2}$, respectively. By Lemma 22 there is an isotopy between g and g' fixed on S^3_1. Hence g_2 is isotopic to g'_2.

Let $H : S^3 \times I \to C_\sigma \times I$ be an isotopy between g_2 and g'_2. The image $\mathrm{Im}(H)$ is disjoint from $\mathrm{i}(D^2_+ \times S^1) \subset \sigma(S^3)$ since $\mathrm{Im}(H) \subset C_\sigma \times I$ (however $\mathrm{Im}(H)$ is not necessarily disjoint from $\mathrm{i}(D^2_- \times S^1)$. We shall "push" $\mathrm{Im}(H)$ out of $(C_\sigma \cap D^6_-) \times I$ into $(C_\sigma \cap D^6_+) \times I$).

Denote

$$\Delta^2 := \{x_1 \leq -\frac{1}{\sqrt{2}}, x_4 = x_5 = x_6 = x_7 = 0\} \subset S^6.$$

Note that Δ^2 is a 2-disk, $\Delta^2 \subset D^6_-$ and $\mathrm{Int}\Delta^2 \cap \sigma(S^3) = \emptyset$.

Let $C \subset \sigma(S^3) \cap D^6_-$ be an embedded cylinder diffeomorphic to $S^1 \times I$ with the boundary $\partial\Delta^2 \sqcup \partial(\mathrm{i}(D^2_+ \times 1))$ (it is not assumed that $\mathrm{Int}\Delta^2 \cap \mathrm{Int}C = \emptyset$ or that $\mathrm{Int}\,\mathrm{i}(D^2_+ \times 1) \cap \mathrm{Int}C = \emptyset$). Let $[\Delta^2]$, $[C]$, and $[\mathrm{i}(D^2_+ \times 1)]$ be 2-chains in $C_2(S^6; \mathbb{Z})$ with supports Δ^2, C, and $\mathrm{i}(D^2_+ \times 1)$, respectively, and such that $\partial([\Delta^2] + [C] + [\mathrm{i}(D^2_+ \times 1)]) = 0$. Denote $\Sigma^2 := [\Delta^2] + [C] + [\mathrm{i}(D^2_+ \times 1)]$.

Clearly,

$$\mathrm{lk}(\mathrm{Im}(H_0), \Sigma^2 \times 0) = \mathrm{lk}(g_2(S^3), \Sigma^2) = \mathrm{lk}(f(S^3), \Sigma^2) = \varkappa(f) = \varkappa(f') =$$
$$= \mathrm{lk}(f'(S^3), \Sigma^2) = \mathrm{lk}(g'_2(S^3), \Sigma^2) = \mathrm{lk}(\mathrm{Im}(H_1), \Sigma^2 \times 1).$$

Hence, the algebraic number of points of the intersection[12] $\mathrm{Im}(H) \cap \Sigma^2 \times I$ is zero[13]. Also $\mathrm{Im}(H) \cap (C \cup \mathrm{i}(D^2_+ \times 1)) \times I = \emptyset$ since $C \cup \mathrm{i}(D^2_+ \times 1) \subset \sigma(S^3)$. Therefore, the algebraic number of

[12]See footnote 5.

[13]This statement does not depend on the choice of the orientations of $\mathrm{Im}(H)$ and $\Sigma^2 \times I$.

points of the intersection $\mathrm{Im}(H)\cap\Delta^2\times I$ is zero as well. The space $C_\sigma\times I$ is simply connected, $\dim(C_\sigma\times I)-\dim(\mathrm{Im}(H))=3$ and $\dim(C_\sigma\times I)-\dim(\Delta^2\times I)=4>3$. So, by the Whitney Trick Theorem ([Fr, Theorem 0.1]), there is a concordance $H':S^3\times I\to C_\sigma\times I$ between g_2 and g_2', and disjoint from $\Delta^2\times I$. Since in codimension 3 concordance implies isotopy ([Hu70, Theorem 2.1]), we may assume that H' is an isotopy.

Denote by B^6 a small tubular neighborhood of Δ^2 such that $B^6\times I$ is disjoint from $\mathrm{Im}(H')$. Consider the framing e_4,e_5 of $\partial\Delta^2$ in $\sigma(S^3)$. This framing extends to a framing of Δ^2 in S^6 since the vectors e_4,e_5 are orthogonal to Δ^2 at each point of Δ^2.

Therefore, there is an isotopy $G:C_\sigma\times I\to C_\sigma\times I$ such that

- $G_1(B^6\cap C_\sigma)=D^6_-\cap C_\sigma$
- $G_t(D^6_+\cap C_\sigma)\subseteq D^6_+\cap C_\sigma$ for each t

Consider the isotopy $F:S^3\times 3I\to C_\sigma\times 3I$ given by the formula

$$F_t=\begin{cases} G_t g_2, & 0\le t\le 1\\ G_1 H'_{t-1}g_2, & 1<t\le 2\\ G_{3-t}H'_1 g_2, & 2<t\le 3\end{cases}$$

Clearly, $F_0=g_2$, $F_3=g_2'$ and $F_t(S^3)\subset C_\sigma\cap D^6_+$ for each t. So $F_t(S^3)$ is disjoint from $\mathrm{i}(S^2\times S^1)$ for each t. Hence $\mathring{f}=g_2$ and $\mathring{f}'=g_2'$ are isotopic to each other. Therefore f and f' are also isotopic to each other (the isotopy is even fixed on $S^2\times S^1$).

□

Proof of Lemma 20. It suffices to consider only the case $c=1$. Since f is strictly simple, there exists $\epsilon>0$ such that $f(S^3)\subset\{x_1>2\epsilon\}\subset D^6_+$.

Denote $\bar\epsilon:=\sqrt{1-\epsilon^2}$. Define an embedding $w:S^3\to S^6$ by the formula:

$$w(x_1,x_2,x_3,x_4):=(\epsilon,\bar\epsilon x_1,\bar\epsilon x_2,0,0,\bar\epsilon x_3,\bar\epsilon x_4).$$

Clearly, $w(S^3)\cap\mathrm{i}(S^2\times S^1)=\emptyset$. Since $w(S^3)\subset\{x_1=\epsilon\}$ and $f(S^3)\subset\{x_1>2\epsilon\}$, it follows that $w(S^3)\cap f(S^3)=\emptyset$.

Let $f':S^2\times S^1\sqcup S^3\to S^6$ be the strictly simple embedding such that $f'|_{S^3}=f|_{S^3}\#w$. We shall prove that f' is as required.

The sphere $w(S^3)$ bounds the disk $\{x_1\le\epsilon,x_4=0,x_5=0\}$ in S^6. Clearly, this disk is disjoint from $\mathrm{i}(S^2\times S^1)$. This disk is also disjoint from $f(S^3)$, since it lies in $\{x_1\le\epsilon\}$ and since $f(S^3)\subset\{x_1>2\epsilon\}$. Therefore, there is an isotopy between f' and f fixed on $S^2\times S^1$. By Lemma 13, it follows that $\widehat\nu(f')=\widehat\nu(f)$.

It remains to prove that $\widehat\mu(f')-\widehat\mu(f)=2\varkappa(f)$.

The sphere $w(S^3)$ also bounds the disk $W:=\{x_1=\epsilon,x_4\ge 0,x_5=0\}$ in S^6. By Lemma 3, there is a 4-disk $\Delta\subset\{x_1>2\epsilon\}$ with boundary $f(S^3)$. The disk Δ is disjoint from W since $W\subset\{x_1=\epsilon\}$. Frame Δ arbitrarily.

Denote $a:=\mathrm{i}^{-1}|_{S^2\times S^1}(W_{e_1,e_5})$ and $b:=\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta)$. Then a and b are framed 1-submanifolds of $S^2\times S^1$. We have

$$\begin{aligned}\widehat\mu(f')-\widehat\mu(f)&=h(\theta(a\sqcup b))-h(\theta(b))=\\ &=h(\theta(a))+h(\theta(b))+2\mathrm{lk}(\theta(a),\theta(b))-h(\theta(b))=\\ &=h(\theta(a))+2\mathrm{lk}(\theta(a),\theta(b))=0+2\varkappa(f).\end{aligned}$$

It remains to prove the last equation, i.e., that $h(\theta(a))=0$ and $\mathrm{lk}(\theta(a),\theta(b))=\varkappa(f)$.

The framed intersection $\mathfrak{i}(S^2 \times S^1) \cap W_{e_1,e_5}$ is the framed circle

$$\{x_1 = \epsilon,\ x_2^2 + x_3^2 = \frac{1}{2} - \epsilon^2,\ x_4 = \frac{1}{\sqrt{2}}\}_{e_1,e_5} \subset S^6.$$

So, $a = \mathfrak{i}^{-1}|_{S^2 \times S^1}(W_{e_1,e_5})$ is the framed circle

$$(\{x_1 = \sqrt{2}\epsilon, x_2^2 + x_3^2 = 1 - 2\epsilon^2\} \times 1)_{e_1 \times 0, 0 \times e_2} \subset S^2 \times S^1.$$

Then $\theta(a)$ is the framed circle

$$\{x_1^2 + x_2^2 = \frac{1}{2} - \epsilon^2, x_3 = \sqrt{\frac{1}{2} + \epsilon^2}, x_4 = 0\}_{v,e_4} \subset S^3,$$

for a certain normal field v. Clearly, $h(\theta(a)) = 0$.

Consider the handlebody $T := \{x_1^2 + x_2^2 \leq \frac{1}{2}, x_3^2 + x_4^2 \geq \frac{1}{2}\}$ in S^3 diffeomorphic to $D^2 \times S^1$. The circle $\theta(a)$ is "almost" the meridian $S^1 \times 1$ of T. The circle $\theta(b)$ lies in Int$T$ and is "further away" from ∂T than $\theta(a)$. Moreover, $\theta(b)$ is homologous to $\varkappa(f)[0 \times S^1]$ in T. Therefore, $\mathrm{lk}(\theta(a), \theta(b)) = \varkappa(f)$.

□

To prove Lemma 21, we need the following Lemma 23.

Lemma 23. *There exists an isotopy $H : S^6 \times I \to S^6 \times I$ such that $H_1|_{\mathfrak{i}(S^2 \times S^1)} = \mathrm{id}$, $H_1(C_{\partial\mathfrak{i}}) = C_{\partial\mathfrak{i}}$, and $[\widetilde{H} z_3] = 2[z_2] + [z_3]$, where $\widetilde{H} : C_{\partial\mathfrak{i}} \to C_{\partial\mathfrak{i}}$ is the abbreviation of H_1.*

Proof. Define an embedding $z_4 : S^4 \to S^6$ by the formula

$$z_4(x_1, x_2, x_3, x_4, x_5) := (x_1, x_2, x_3, 0, 0, x_4, x_5).$$

Note that $z_4(S^4)$ is a sphere isotopic to the standard sphere and linked with $\mathfrak{i}(0 \times S^1)$.

Frame $\mathfrak{i}(D^3 \times S^1)$ with vectors e_6, e_7 at each point (this is the standard framing). Frame $z_4(S^4)$ with vectors e_4, e_5 at each point. By Lemma 17, we may extend the framings of $\mathfrak{i}((D^3 \times S^1)_0)_{e_6,e_7}$ and $z_4(S^4_0)_{e_4,e_5}$ to a framing ζ of $(\mathfrak{i}\# z_4)(D^3 \times S^1)$.

We have framed the images of the embeddings $\mathfrak{i}\# z_4 : D^3 \times S^1 \to S^6$ and $\mathfrak{i} : D^3 \times S^1 \to S^6$. Take a 3-disk $B^3 \subset D^3$ such that $B^3 \times S^1 \subset (D^3 \times S^1)_0$. Clearly, $(\mathfrak{i}\# z_4)|_{B^3 \times S^1} = \mathfrak{i}|_{B^3 \times S^1}$ and the framings of $(\mathfrak{i}\# z_4)(B^3 \times S^1)$ and $\mathfrak{i}(B^3 \times S^1)$ are the same.

So, by Lemma 18, there is a framed isotopy $H : S^6 \times I \to S^6 \times I$ between the embeddings $\mathfrak{i}\# z_4$ and $\mathfrak{i}$ (the isotopy H is not assumed to be fixed on the boundary $\partial(D^3 \times S^1)$). We may assume that $H_1(C_{\partial\mathfrak{i}}) = C_{\partial\mathfrak{i}}$. We shall prove that H is as required.

Lemmas 14 and 16 assert that

$$[\widetilde{H} z_3] = h(p_2 \widetilde{H} z_3)[z_2] + \deg(p_3 \widetilde{H} z_3)[z_3] \quad \text{and} \quad [\widetilde{H} z_3] = 2k[z_2] + [z_3], \quad \text{for some integer } k.$$

By Lemma 14, the homotopy classes $[z_2]$ and $[z_3]$ form a basis of $\pi_3(C_{\partial\mathfrak{i}})$, so $\deg(p_3 \widetilde{H} z_3) = 1$. It remains to prove that $h(p_2 \widetilde{H} z_3) = 2$.

By the definition of p_2, the framed submanifold $\mathfrak{i}(D^3 \times S^1)_{e_6,e_7} \cap C_{\partial\mathfrak{i}}$ is the framed preimage of a regular point of $p_2 : C_{\partial\mathfrak{i}} \to S^2$. Therefore, the framed submanifold $(\mathfrak{i}\# z_4)(D^3 \times S^1)_\zeta \cap C_{\partial\mathfrak{i}}$ is the framed preimage of a regular point of $p_2 \widetilde{H} : C_{\partial\mathfrak{i}} \to S^2$.

By general position, we may assume that

$$z_3(S^3) \cap (\mathfrak{i}\# z_4)(D^3 \times S^1) = (z_3(S^3) \cap \mathfrak{i}(D^3 \times S^1)) \sqcup (z_3(S^3) \cap z_4(S^4)).$$

So, the value of the Hopf invariant $h(p_2\widetilde{H}z_3)$ is equal to the value of the Hopf invariant of the union of framed circles

$$z_3^{-1}(\mathrm{i}\#z_4(D^3\times S^1)_\zeta) = z_3^{-1}(\mathrm{i}(D^3\times S^1)_{e_6,e_7}) \sqcup z_3^{-1}(z_4(S^4)_{e_4,e_5}) = \\ = \{x_1^2+x_2^2=1\}_{e_3,e_4} \sqcup \{x_3^2+x_4^2=1\}_{e_1,e_2} \subset S^3.$$

I.e,

$$h(p_2\widetilde{H}z_3) = h(\{x_1^2+x_2^2=1\}_{e_3,e_4} \sqcup \{x_3^2+x_4^2=1\}_{e_1,e_2}) = \\ = h(\{x_1^2+x_2^2=1\}_{e_3,e_4}) + h(\{x_3^2+x_4^2=1\}_{e_1,e_2}) + 2\mathrm{lk}(\{x_1^2+x_2^2=1\},\{x_3^2+x_4^2=1\}) = \\ = 0+0+2=2.$$

□

Proof of Lemma 21. It suffices to consider only the case $c=1$.

Lemma 23 asserts that there is an isotopy $H : S^6\times I \to S^6\times I$ such that $H_1|_{\mathrm{i}(S^2\times S^1)} = \mathrm{id}$, $H_1(C_{\partial\mathrm{i}}) = C_{\partial\mathrm{i}}$, and $[\widetilde{H}z_3] = 2[z_2]+[z_3]$, where $\widetilde{H} : C_{\partial\mathrm{i}} \to C_{\partial\mathrm{i}}$ is the abbreviation of H_1.

We shall prove that the simple embedding $f' := H_1 f$ is as required, i.e, that $\widehat{\nu}(f') = \widehat{\nu}(f) + 2\varkappa(f)$. Let $\widetilde{H}^* : \pi_3(C_{\partial\mathrm{i}}) \to \pi_3(C_{\partial\mathrm{i}})$ be the homomorphism induced by $\widetilde{H}$. We have

$$\widehat{\nu}(f')[z_2] + \varkappa(f')[z_3] \overset{(1)}{=} [\mathring{f}'] = [\widetilde{H}\mathring{f}] = \widetilde{H}^*[\mathring{f}] \overset{(2)}{=} \widetilde{H}^*(\widehat{\nu}(f)[z_2] + \varkappa(f)[z_3]) = \\ = \widehat{\nu}(f)[\widetilde{H}z_2] + \varkappa(f)[\widetilde{H}z_3] \overset{(3)}{=} (\widehat{\nu}(f)+2\varkappa(f))[z_2] + \varkappa(f)[z_3].$$

(1,2) Hold by Lemma 15.
(3) Holds because $[\widetilde{H}z_2] = [z_2]$ by Part (I) of Lemma 16, and $[\widetilde{H}z_3] = 2[z_2]+[z_3]$ by the choice of H.

By Lemma 14, the homotopy classes $[z_2]$ and $[z_3]$ form a basis of $\pi_3(C_{\partial\mathrm{i}})$, so $\widehat{\nu}(f') = \widehat{\nu}(f) + 2\varkappa(f)$.

□

9. Proof of Lemma 6.

To prove Lemma 6, we need the following Lemma 24.

Lemma 24. *Let f be a strictly simple embedding. Let $g : S_1^3 \sqcup S_2^3 \to S^6$ be a Brunnian embedding. Suppose that the image of g lies in a small 6-disk in $\mathrm{Int}D_+^6$ disjoint from the image of f and $\mathrm{i}(D^3\times S^1)$. Then:*

(I) $\varkappa(f\#g) = \varkappa(f)$
(II) $\mu(f\#g) = \mu(f) + \rho_{0,2\varkappa(f)}(\mu(g))$
(III) $\nu(f\#g) = \nu(f) + \rho_{0,2\varkappa(f)}(\nu(g))$

Proof. Part (I) is obvious.

The following construction is needed to prove parts (II) and (III).

Let $\Delta_f \subset \mathrm{Int}D_+^6$ be an embedded 4-disk such that $\partial\Delta_f = f(S^3)$. Since g is a simple embedding, there are embedded 4-disks $\Delta_1, \Delta_2 \subset \mathrm{Int}D_+^6$ such that $\partial\Delta_1 = g(S_1^3)$ and $\partial\Delta_2 = g(S_2^3)$. We may assume that Δ_1 and Δ_2 lie in a small 6-disk in $\mathrm{Int}D_+^6$ disjoint from Δ_f and $\mathrm{i}(D^3\times S^1)$.

The definition of the *boundary embedded connected sum* $M\natural N \subset S^k$ of the embedded compact orientable manifolds $M \subset S^k$ and $N \subset S^k$ with boundary is analogous to the definition of the (absolute) embedded connected sum. Clearly, any framing of $M \sqcup N$ can be extended to a framing of $M \sharp N$.

Frame Δ_f, Δ_1, and Δ_2 arbitrarily. Extend the framings of $\mathrm{i}(D^3 \times S^1)_{e_6,e_7}$, Δ_f, Δ_1, and Δ_2 to framings of $\mathrm{i}(D^3 \times S^1)\sharp\Delta_1$ and $\Delta_f\sharp\Delta_2$.

By Lemma 18, there is a framed isotopy $H : S^6\times I \to S^6\times I$ between the framed submanifolds $\mathrm{i}(D^3 \times S^1)\sharp\Delta_1$ and $\mathrm{i}(D^3 \times S^1)_{e_6,e_7}$.

Then $H_1(f\#g) : S^2 \times S^1 \sqcup S^3 \to S^6$ is a strictly simple embedding and $H_1(\mathrm{i}(D^3 \times S^1)\sharp\Delta_1) = \mathrm{i}(D^3 \times S^1)_{e_6,e_7}$.

Completion of the proof of (II). By the definition of μ

$$\begin{aligned}\mu(f\#g) &\equiv \mu(H_1(f\#g)) \equiv h(\theta\mathrm{i}^{-1}|_{S^2\times S^1}(H_1(\Delta_f\sharp\Delta_2))) \equiv \\ &\equiv h(\theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_f\sharp\Delta_2)) \equiv h(\theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_f) \sqcup \theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_2)) \overset{(1)}{\equiv} \\ &\overset{(1)}{\equiv} h(\theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_f)) + h(\theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_2)) \equiv \mu(f) + \mu(g) \pmod{2\varkappa(f)}.\end{aligned}$$

Here (1) holds since framed 1-submanifolds $\theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_f)$ and $\theta(\mathrm{i}|_{S^2\times S^1}\#g|_{S^3_1})^{-1}(\Delta_2)$ of S^3 are unlinked (they lie in disjoint 3-disks).

Completion of the proof of (III). By the definition of ν

$$\begin{aligned}\nu(f\#g) &\equiv \nu(H_1(f\#g)) \equiv h((H_1\mathring{f}\#H_1g|_{S^3_2})^{-1}\mathrm{i}(D^3 \times S^1)) \overset{(1)}{\equiv} \\ &\overset{(1)}{\equiv} h((H_1\mathring{f})^{-1}\mathrm{i}(D^3 \times S^1)) + h((H_1g|_{S^3_2})^{-1}\mathrm{i}(D^3 \times S^1)) \equiv \\ &\equiv h(\mathring{f}^{-1}\mathrm{i}(D^3 \times S^1)) + h(g|_{S^3_2}{}^{-1}(\Delta_1)) \equiv \nu(f) + \nu(g) \pmod{2\varkappa(f)}.\end{aligned}$$

Here (1) holds since framed 1-submanifolds $(H_1\mathring{f})^{-1}\mathrm{i}(D^3 \times S^1)$ and $(H_1g|_{S^3_2})^{-1}\mathrm{i}(D^3 \times S^1)$ of S^3 are unlinked (they lie in disjoint 3-disks).

□

Proof of Lemma 6. By Lemma 24, it suffices to consider only the case $m = n = 0$.

Clearly, $\varkappa(f_{k,0,0}) = \mathrm{lk}(f_{k,0,0}(S^3), f_{k,0,0}(S^2 \times 1)) = k$.

Let $k = 0$. By definition of $f_{0,0,0}$, submanifolds $f_{0,0,0}(S^2 \times S^1) = \mathrm{i}(S^2 \times S^1) \subset S^6$ and $f_{0,0,0}(S^3) \subset S^6$ lie in disjoint 6-disk. So, there is 4-disk bounded by $f_{0,0,0}(S^3)$ and disjoint from $f_{0,0,0}(S^2 \times S^1)$, meaning that $\mu(f_{0,0,0}) = 0$. Also, we may assume that $f_{0,0,0}(S^3)$ is disjoint from $\mathrm{i}(D^3 \times S^1)$, meaning that $\nu(f_{0,0,0}) = 0$.

Let us now consider the case $k = 1$. The case $k \neq 0, k \neq 1$ is analogous.

Consider the framed 4-disk

$$\Delta := \{x_1 \geq \frac{1}{10}, x_2 = x_3 = 0\}_{e_2,e_3} \subset S^6.$$

Clearly, $\Delta \subset \mathrm{Int}D^6_+$ and $\partial\Delta = z_{3,\frac{1}{10}}(S^3) = f_{1,0,0}(S^3)$. So,

$$\widehat{\mu}(f_{1,0,0}) = h\theta\mathrm{i}^{-1}|_{S^2\times S^1}(\Delta_{e_2,e_3}) = h(\{x_3^2 + x_4^2 = 1\}_{e_1,e_2}) = 0.$$

Hence, $\mu(f_{1,0,0}) = 0$.

The map $\mathring{f}_{1,0,0}$ is homotopic to z_3. Therefore, by Lemmas 14 and 15 it follows that $\widehat{\nu}(f_{1,0,0}) = 0$, so $\nu(f_{1,0,0}) = 0$. □

References

[Ce07] M. Cencelj, D. Repovs, M. Skopenkov, *Classification of framed links in 3-manifolds*, Proc. Indian Acad. Sci. (Math. Sci.) 117:3 (2007), 301–306, http://arxiv.org/pdf/0705.4166v2.pdf

[Fr] J. Francis, notes by H. Tanaka, *Math 465, Lecture 11: The Whitney trick, first part*, http://math.northwestern.edu/~jnkf/classes/mflds/11whitneytrick1.pdf

[Ha62] A. Haefliger, *Differentiable links*, Topology, Volume 1, Issue 3, JulySeptember 1962, 241–244.

[Ha66] A. Haefliger, *Enlacements de sphères en codimension supériure à 2*, Comment. Math. Helv. 41 (1966–67), 51–72.

[Hu70] J. F. P. Hudson, *Concordance, isotopy, and diffeotopy*, Ann. of Math. (2) 91 (1970), 425–448.

[Atl1] Manifold Atlas, *High codimension embeddings: classification*, http://www.map.mpim-bonn.mpg.de/High_codimension_embeddings:_classification#Sphere_and_Euclidean_space

[Atl2] Manifold Atlas, *High codimension links*, http://www.map.mpim-bonn.mpg.de/High_codimension_links

[Ma] Y. Matsumoto, *An Introduction to Morse Theory*, Translations of Mathematical Monographs, Vol. 208.

[Sk00] A. Skopenkov, *On the generalized MasseyRolfsen invariant for link maps*, Fund. Math. 165 (2000), 1–15.

[Sk07] A. Skopenkov, *Classification of embeddings below the meatstable dimension*, http://arxiv.org/pdf/math/0607422.pdf

[Sk08] A. Skopenkov, *Embedding and Knotting of Manifolds in Euclidean Spaces*, London Math. Soc. Lect. Notes, 347 (2008) 248–342, http://arxiv.org/pdf/math/0604045.pdf

[MSk13] M. Skopenkov, *When is the set of embeddings finite?*, http://arxiv.org/pdf/1106.1878v2.pdf

[Vi] O. Ya. Viro, O. A. Ivanov, N. Yu. Netsvetaev, V. M. Kharlamov, *Elementary Topology: Problem Textbook*, AMS, 2008.