DIFFERENTIAL-GRADED LIE ALGEBRA TRIVIALITY CRITERION

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Abstract. In this paper we prove (under some technical assumptions) that any differential-graded lie algebra which has adjoint module quasi-isomorphic to the trivial module is homotopy abelian, i.e. itself quasi-isomorphic to abelian lie superalgebra.

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1. Introduction

The adjoint module of a lie algebra contains all the information about it - it's almost meaningless that if adjoint module is trivial, then the lie algebra is abelian. However, for the differential-graded lie algebras, the homotopy analogue of this statement is not trivial at all. The theorem presented in this paper suffers from some technical assumptions, but we hope that they can be avoided (considering some finiteness conditions on the cohomology) and this fact could be useful in deformation theory.

Theorem 1.1. *Let L be a differential-graded lie algebra with finite-dimensional grading components. Let ad_L be its adjoint module. Then if ad_L is quasi-isomorphic to the module with trivial action, then L is quasiisomorphic to the abelian algebra.*

Our proof uses the heavy machinery of L_∞-algebras, but, in our opinion, is quite straightforward from the $\infty-$point of view.

2. Recall of L_∞ things

We prefer to work with coalgebras rather than with algebras, but under the technical assumptions we use these two categories are antiequivalent (by the functor of graded dual). Every time we are talking about vector spaces we mean graded finite dimensional in each grading component supervector spaces.

Let us denote $\overline{S(V)}$ the algebra of formal series on the vector space V with the standard topology (the sequence converges iff it becomes stable in each degree component).

Definition 2.1. L_∞-coalgebra structure on the vector space V is the continuous odd derivation Q on $\overline{S(V[1])}$, such that $Q^2 = 0$, preserving augmentation.

$V[1]$ topologically generates $\overline{S(V[1])}$, so Q can be given in terms of its action on the first component, i.e. infinite series of operations

$$Q_k : V \to \Lambda^k(V)[k-1]$$

The fact that $Q^2 = 0$ in this form is expressed in terms of some explicit quadratic relations between these operations.

Definition 2.2. Homomorphisms in the category of L_∞-coalgebras are the augmentation-preserving homomorphisms of the underlying algebras, commuting with Q.

So, the category of L_∞-coalgebras can be defined as the full subcategory of the category of topological dg-algebras with augmentation consisting of the completed free algebras.

Example 2.3. Let L be the differential-graded lie coalgebra with a differential ∂ and cocommutator Δ, and let us denote the sign shifting operator by σ. Then setting $Q_1 = \sigma \circ \partial$ and $Q_2 = (1 \otimes \sigma) \circ \Delta$ we get the so-called Chevalley-Eilenberg complex, which is the simplest example of L_∞-coalgebra.

Theorem 2.4 (Minimal model theorem)**.** *Any L_∞-coalgebra is isomorphic to the direct sum of it's minimal model - coalgebra with $Q_1 = 0$ and acyclic coalgebra - coalgebra with Q_1 acyclic and $Q_k = 0$ for $k > 0$. Minimal model is unique up to L_∞ morphism.*

This theorem is proved in [2], but because this book is unpublished, in sake of completeness of the exposition we recall the proof of this theorem in the appendix.

Theorem 2.5 (Functorial replacement theorem)**.** *Any L_∞-coalgebra can be functorially replaced with the quasiisomorphic differential-graded lie coalgebra.*

This theorem is proved in [1], Corollary 1.6.

Corrolary 2.6. *Two dg-lie coalgebras sharing the same minimal model are quasiisomorphic.*

Definition 2.7. Let us call a $\mathbb{Z}$-graded completed symmetric algebra $\overline{S(V)}$ contained in some degrees iff V is contained in these degrees.

It is independent of picking the generators of our algebra, because V can be canonically identified with $\mathbf{m}/\mathbf{m}^2$, where $\mathbf{m}$ is the unique maximal ideal of $\overline{S(V)}$.

Definition 2.8. Category of coalgebra-comodule pairs $L_\infty[\varepsilon]$ is the category of L_∞-coalgebras with additional $\mathbb{Z}$-grading, contained in the degrees 0 and 1.

Having this structure on the space $\overline{S(L \oplus M)}$ (where L is of additional degree 0, M is of additional degree 1), we denote it as (L, M) and call M a comodule over coalgebra L.

Now let (L, M) be an object of $L_\infty[\varepsilon]$. Then we can define operations $Q_k : L[1] \to S^k(L[1])$ and $A_k : M[1] \to M[1] \otimes S^{k-1}(L[1])$, where the last space is viewed as the component of additional degree 1 in $S^k(L \oplus M[1])$.

Definition 2.9. The polarization morphism $Pol_k : S^k(V) \to V \otimes S^{k-1}$

is defined as

$$Pol_k(P) = \sum_i e^i \otimes \frac{\partial}{\partial e^i} P$$

where $e^1, ..., e^n$ is a basis in V.

Definition 2.10. Adjoint comodule (L, ad_L) is the pair (L, L) with the operations $A_k = Pol_k \circ Q_k$.

This notion also makes sense for any algebra over any operad.

3. Adjoint comodule as a tangent bundle

In this section we are going to prove that adjoint comodule is a functor from L_∞ to $L_\infty[\varepsilon]$. In order to do this we present a canonical construction of it.

Let $A_i = \overline{S(V_i)}$ be the algebra of formal power series over V_i. Then we will denote by $A_1 \otimes A_2 = \overline{S(V_1 \oplus V_2)}$. It is clearly a bifunctor (i.e. any augmentation preserving continuous morphisms $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ gives rise to the morphism $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$).

Obviously, taking $\overline{S(V)} \otimes \overline{S(W)} = \overline{S(V \oplus W)}$ corresponds to the direct sum of L_∞ structures.

Definition 3.1. The diagonal ideal $I_\Delta \subset A \otimes A$ is the ideal generated by vectors $v \otimes 1 - 1 \otimes v$, $v \in V$ (equivalently, it's the kernel of the multiplication morphism $m : A \otimes A \to A$).

Let us assume that A is endowed with a differential Q, providing the L_∞-structure on $V[-1]$. Then $A \otimes A$ also has a structure $Q_k = Q_k \otimes 1 + 1 \otimes Q_k$. It is easy to see that I_Δ is invariant under the action of this derivation.

Definition 3.2. The tangent bundle to the L_∞ structure on A is the induced differential on the completed associated graded algebra for the filtration given by $F_k = I_\Delta^k$ (which turns out to be the same as an algebra, but with completely different Q).

Theorem 3.3. *The pair (L, ad_L) is isomorphic to the tangent bundle of L.*

Proof. Let us pick a basis $t^1, t^2, ...$ in a space L. Then for algebra $\overline{S(L \oplus L)}$ let us pick the basis in $L \oplus L$ as $x^i = 1 \otimes t^i$, $y^i = t^i \otimes 1 - 1 \otimes t^i$. The ideal I_Δ is generated by y^i's, and so the completed associated graded algebra can be naturally identified with algebra $\overline{S(x^1,...,y^1,...)}$ with x^i's having degree 0 and y^i's having degree 1 (and shifted parity).

Also we will make use of another basis $\tilde{x}^i = x^i, \tilde{y}^i = x^i + y^i$

Now let us calculate the differential on the tangent bundle:

$$Q = \sum_k [Q_k]^j_{i_1...i_k} \tilde{x}^{i_1}...\tilde{x}^{i_k} \frac{\partial}{\partial \tilde{x}^j} + [Q_k]^j_{i_1...i_k} \tilde{y}^{i_1}...\tilde{y}^{i_k} \frac{\partial}{\partial \tilde{y}^j} =$$

$$= \sum_k [Q_k]^j_{i_1...i_k} x^{i_1}...x^{i_k} (\frac{\partial}{\partial x^j} - \frac{\partial}{\partial y^j}) + \sum_k [Q_k]^j_{i_1...i_k} (x^{i_1}+y^{i_1})...(x^{i_k}+y^{i_k}) \frac{\partial}{\partial y^j} =$$

$$= \sum_k [Q_k]^j_{i_1...i_k} x^{i_1}...x^{i_k} \frac{\partial}{\partial x^j} + \sum_k \sum_l [Q_k]^j_{i_1...i_k} x^{i_1}...y^{i_l}...x^{i_k} \frac{\partial}{\partial y^j} + o(y)$$

and the answer tautologically correspond to the $\sum_k Q_k + A_k$ for the adjoint comodule.

□

Corrolary 3.4. *Adjoint comodule is canonical (i.e. for any L_∞-morphism $f : L_1 \to L_2$) there is $Df : (L_1, ad_{L_1}) \to (L_2, ad_{L_2})$ with obvious functoriality properties.*

Note 3.5. Adjoint module can be made into a functor explicitly. If we start with picking a basis $x^1, ..., x^n$ in an L_∞ coalgebra L_1, $y^1, ..., y^n$ in its adjoint module, analogous $\tilde{x}^1, ..., \tilde{x}^n$, $\tilde{y}^1, ..., \tilde{y}^n$ for L_2 and assume a morphism $F : L_2 \to L_1$ such that $F(\tilde{x}^i) = f_i(x^1, ..., x^n)$, then $adF(\tilde{y}^i) = \frac{\partial f^i}{\partial x^j} y^j$.

4. Main theorem

In this section we are going to prove the following theorem:

Theorem 4.1. *Let the pair (L, ad_L) be quasiisomorphic to the pair (L, M), where the action of L on M is trivial. Then L is quasiisomorphic to the abelian lie dg-algebra.*

Proof. Let us assume that the statement $(L, ad_L) \simeq^{qsi} (L, M)$ holds.

Step 1: By the minimal model theorem $L \simeq H_L \oplus A$, where A is acyclic. So, by functoriality (Corollary 3.4) the pair $(L, ad_L) \simeq (H_L \oplus A, ad_{H_L} \oplus ad_A) = (H_L, ad_{H_L}) \oplus (A, ad_A)$, and, hence, (H_L, ad_{H_L}) is a minimal model for (L, ad_L). On the other hand, the pair (L, M) is just $(L, 0) \oplus (0, M)$ and, hence, its minimal model for (L, M) is just (H_L, H_M), where H_M is a cohomology of Q_1 on M.

Note 4.2. This step can be done explicitly using the sum-over-trees formula, avoiding the subtle argument based on the canonicity of the adjoint comodule. This point of view is exposed in the Appendix 2.

Step 2: So, $(H_L, ad_{H_L}) \simeq (H_L, H_M)$. Let us pick the first nontrivial higher operation Q_k.

Step 3: L_∞-morphisms act on a first nontrivial operation by linear transformations.

Step 4: But the first nontrivial operation has the component A_k which is the polarization of Q_k (hence nonzero if Q_k is nonzero) and it can't be killed by linear transformations. □

Corrolary 4.3. *Theorem 1.1 holds.*

5. Appendix: Minimal model theorem

The proofs in this section mostly follow the book [2].

At first, let us assume that $F : \overline{S(X)} \to \overline{S(Y)}$ is a continuous morphism of algebras. Then the **formal inverse function theorem** holds:

Theorem (Inverse function theorem)**.** *The map F is invertible iff $F_1 : X \to Y$ is invertible.*

Proof. Taking $F_1^{-1} \circ F$ we get an endomorphism of $\overline{S(X)}$, which, as a linear operator, has the form $Id + N$, where N increases the degree. Then $1 - N + N^2 - ...$ converges and gives the inverse endomorphism. □

Now, let us consider the L_∞-algebra with acyclic Q_1. Then the simple case of MMT, the **acyclicity theorem** states that

Theorem (Acyclicity theorem)**.** $(\overline{S(V)}, Q) \simeq (\overline{S(V)}, Q_1)$

Proof. Let us pick the basis $x^1, ..., x^n, y^1, ..., y^n$ such that $Q_1(x^i) = y^i$ (which can be done because Q_1 is acyclic). Then the change of coordinates $\tilde{x}^i \to x^i$, $\tilde{y}^i \to Q(x^i)$ gives us the invertible morphism from $(\overline{S(V)}, Q_1)$ to the $(\overline{S(V)}, Q)$. □

Now we are going to consider the general case.

Theorem (Minimal model theorem)**.** *Any L_∞-coalgebra is isomorphic to the direct sum of acyclic coalgebra and minimal model - coalgebra with $Q_1 = 0$.*

Proof. Analogously, we consider the basis $x^1, ..., x^n, y^1, ..., y^n, z^1, ..., z^m$ such that $Q(x^i) = y^i$, $Q(y^i) = Q(z^i) = 0$. Now we change the variables to the $x^i \to x^i$, $y^i \to Q(x^i)$, $z^i \to z^i$. In these variables we have

$$Q = \sum_k [P_k]^i(x, y, z)\frac{\partial}{\partial z^i} + y^i \frac{\partial}{\partial x^i}$$

where $[P_k]^i$ are homogeneous polynomials of degree k in supercommuting variables x^i, y^i, z^i.

We proceed by induction. Let us consider the minimal l such that $[P_l]$ depends not only of z. From $Q^2 = 0$ we instantly get $\forall j$

$$y^i \frac{\partial [P_l]^j}{\partial x^i} = 0$$

$[P_l]^j = [Z_l]^j + [S_l]^j$, where $[Z_l]$ depends only on z, $[S_l] \in I = \langle x, y \rangle$.

Then $[S_l]^j \in \overline{S^{>0}(x,y)} \otimes \overline{S(z)}$, which is acyclic under the action of $y^i \frac{\partial}{\partial x^i}$. Now let us consider M^j such that

$$y^i \frac{\partial M^j}{\partial x^i} = [S_l]^j$$

Now we perform the transformation $x^i \to x^i, y^i \to y^i, z^i \to z^i - M^i$. Then, $Q(z^i - M^i) = Q(z^i) - [P_l]^i + [Z_l]^i$ which depends only of z in the degrees $\leq l$ and, as $M^i = o(z)$, depends only of $z^i - M^i$ in the degrees $\leq l$.

Hence, doing this transformations step by step, we converge to the transformation which turns Q into a form

$$Q = \sum_k [P_k]^i(z) \frac{\partial}{\partial z^i} + y^i \frac{\partial}{\partial x^i}$$

□

6. Appendix 2: The sum-over-trees formula

The goal of this appendix is to give the direct proof of the fact that minimal model for the adjoint module (L, ad_L) is the adjoint module of the minimal model of L: (H_L, ad_{H_L}), using the explicit formula for the higher operations on the minimal model.

Theorem (The sum-over-trees formula)**.** *Let us use the notation from the previous theorem.* $\overline{S(z)}$ *is canonically identified with* $\frac{\overline{S(x,y,z)}}{\langle x, Q(x) \rangle}$. *Let us denote by* P *the differential* Q *induced on* $S(z)$ *and by the* h *an operator on* V, *given by* $h(y^i) = x^i, h(x^i) = h(z^i) = 0$.

Then, the sum-over-trees formula states that

$$[P_k]^j_{i_1,..,i_k} = \sum_T T(Q)$$

where the sum is taken over all planar trees T *with* k *outgoing leaves, and by* $T(Q)$ *we denote the following convolution: on each vertex with* l *outgoing edges we put a tensor* $[Q_l]$, *and on each internal edge we put an operator* $-h$.

The resulting tensor is automatically symmetric.

Proof. Direct application of implicit function theorem. □

Theorem (Step 1 of the main theorem explicitly)**.** (H_L, ad_{H_L}) *is a minimal model for* (L, ad_L)

Proof. The chain homotopy operator h is assumed to be $h \oplus h$, acting on the (L, ad_L)

We are going to use trees with colored edges: each edge can be "red" or "black". If it is "black" it corresponds to the convolution in the coalgebra, and if it is "red" it corresponds to the convolution in the comodule. Obviously, contraction of "red" and "black" indices equals zero.

Then, full $[\widetilde{Q}_k]$ in (L, ad_L) can be represented as a sum of $k+1$ copies of the vertex, corresponding to $[Q_k]$ itself and $[Q_k]$-s with one red incoming and red outcoming edges.

Each tree T in the sum-over-trees formula for (L, ad_L), then, expands in the sum of colored trees. Only nonzero summands are black T and its recolored copies with the red "paths", going from one of the incoming edges to the outcoming. This sum is, obviously, the polarization of its "black" part, i.e. original $T(Q)$, because there is only one path for each incoming edge of a tree.

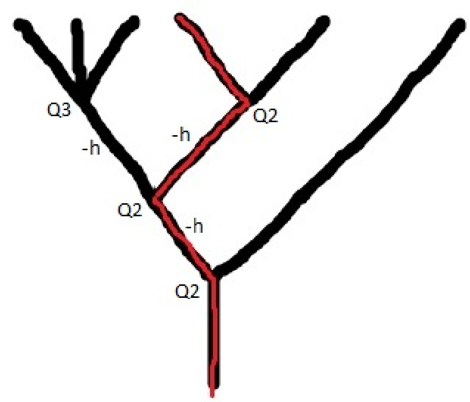

FIGURE 1. One of the correct summands in the expansion

□

References

[1] I. Kriz, J.P. May, *Operads, algebras, modules, and motives*, Asterisque, 233 (1995)
[2] M. Kontsevich, Y. Soibelman, *Deformation theory vol. 1*, unpublished, www.math.ksu.edu/ soibel/Book-vol1.ps