Arithmetic of $3-$valent graphs and equidissections of flat surfaces

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Contents

Abstract

Our main object of study is a $3-$valent graph with a vector function on its edges. The function assignes to an edge a pair of $2-$adic integer numbers and satisfies additional condition: the sum of its values on three edges, terminating in the same vertex, is equal to 0. For each vertex of the graph three vectors corresponding to these edges generate a lattice over the ring of $2-$adic integers. In this paper we study the restrictions, imposed on these lattices by the combinatorics of the graph.

As an application we obtain the following fact: a rational balanced polygon cannot be cut into an odd number of triangles of equal areas. First result of this type was obtained by Paul Monsky in 1970. He proved that a square cannot be cut into an odd number of triangles of equal areas. In 2000 Sherman Stein conjectured that the same holds for any balanced polygon. We prove this conjecture in the case, when coordinates of all vertices of the cut are rational numbers.

1 Introduction

This paper is motivated by author's attempts to find a new proof of Monsky theorem, which claims that a square can not be cut into an odd number of triangles of equal areas. The only known proof of this theorem, which we will sketch in section 2, has a few drawbacks.

The main one is that while the statement is, obviously, invariant under the group of affine transformations of the plane, the proof is not. It is based on a construction of a coloring of the plane in such a way that a color of a point depends on the $2-$adic valuations of its coordinates. But after applying an affine transformation, the $2-$adic valuations of the coordinates change in an uncontrollable way.

Another drawback is that this proof seems not to be generalizable on a wider class of polygons, for which the statement holds.

The third drawback is that not of the proof, but of the theorem itself. The statement of Monsky theorem is rather restricted, it just claims nonexistance of a triangulation with some bizarre property. It seems to be more interesting to find a property of any triangulation, from which Monsky theorem would follow.

This paper is the result of an attempt to find a proof of Monsky theorem and its generalisations free of these defects. In the next section we will formulate the exact generalization of Monsky theorem that we will prove — Rational Stein Conjecture.

Instead of working with a triangulation of a polygon, we will work with a pair, consisting of a 3−valent graph and a vector function on its edges. The graph will be morally a dual graph of the triangulation, and the function will assign to each edge a vector in the plane, which represents the side, shared by two triangles, corresponding to the vertices of the edge. The function, constructed in this way, will have a property that the sum of the three vectors, corresponding to the three edges with the same terminal vertex, is 0. We will call a 3−valent graph with such a function — a *balanced* graph. For each vertex of a balanced graph one can define its *multiplicity.* It is equal to a 2−adic valuation of a determinant, constructed from the values of the balancing function on the edges, terminating in the vertex. In original terms it is the 2−adic valuation of the area of the triangle in the triangulation.

The main result of our paper, proved in section 3, is a theorem about balanced graphs. It claims that the number of vertices of a balanced graph with minimal multiplicity among its all vertices is even. Rational Stein Conjecture is a corollary of this fact, we prove it in section 4.

The original Stein conjecture is formulated for any balanced polygon with any triangulation, while we will restrict ourselfs to the case, when all coordinates of vertices of a triangulation are rational numbers. Moreover, the theorem about balanced graphs will be formulated for a 2−adic-valued balancing function. This restriction is justified by the following reasons. Firstly, the proof of the main theorem is based on the analysis of lattices over a nonarchemedian field. Secondly, the classical proof of Monsky theorem for rational triangulations is as hard as for real triangulations.

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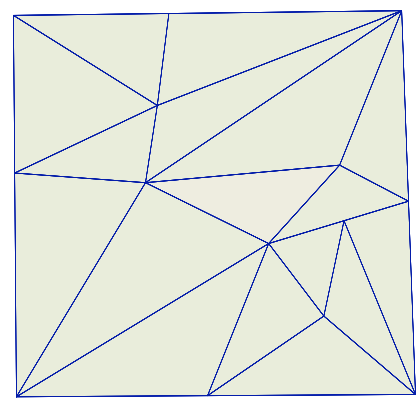

Figure 1: A square is cut into triangles

2 Equidissection Problems

The history of Equidissection Problems[1] started with both beautiful and puzzling proof of the following fact:

Theorem 2.1 (P. Monsky, 1970)**.** *A square cannot be cut[2] into an odd number of triangles[3] of equal areas.*

The main idea of the proof is to color the plane in three different colors in such a way that each line contains vertices of at most two colors. Though this coloring can be simply defined by formulas, one can view it more conceptually from the point of view of Tropical Geometry. Let us first restrict ourselfs to the case of a triangulation, in which all coordinates of vertices are rational numbers. Then we can apply to the triangulation $2-$adic[4] tropicalisation

[1]By *Equidissection Problems* we mean various results in the spirit of Monsky Theorem.

[2]By the phrase *polygon B is cut into triangles* we mean that B can be presented as a union of a finite number of triangles so that the interiors of the triangles have an empty intersection with each other. Fig. 1 illustrates this.

[3]Throughout this article "triangle" is taken to include the degenerate case.

[4]By $2-$ adic valuation of rational number we mean a maximal number of 2 deviding its nomirator minus maximal number of 2 deviding its denominator. It will be denoted by ν_2. For example, $\nu_2(20) = 2, \nu_2(\frac{7}{32}) = -5, \nu_2(0) = \infty$.

map T :

$$T : (x, y) \in \mathbb{Q}^2 \longrightarrow (\nu_2(x), \nu_2(y)) \in \mathbb{R}^2.$$

One can color points with rational coordinates in the plane in three colors according to the rule described below. Firstly, consider the image of the line $x + y + 1 = 0$ under the map T. It is a union of three rays: two of them are parallel to the coordinate lines and the third is parallel to the diagonal. This image, called a tropical line, cuts the plane into three parts. We will color points of these parts in three colors acording to the scheme presented in Fig.2.

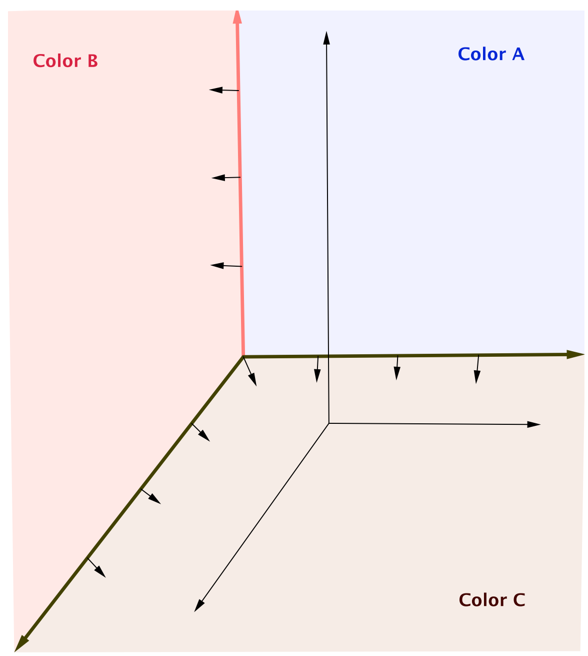

Figure 2: Tropical coloring of the plane and the image of the line

The pullback of this coloring under map T is the coloring of the plane, satisfying the desired property, which can be shown as follows. An image of any line is a tropical line as well and can be obtained by a parallel transport from the image of the line $x + y + 1 = 0$. It is visual frome Fig.2 that any tropical line contains points of two colors only. Furthermore, a stronger property holds: any triangle, which has vertices of three different colors, has negative 2$-$adic valuation. The latter statement can be verified by a direct computation.

Now we are ready to finish the proof. Suppose that a square can be cut into odd number of triangles of equal areas. Since this statement is affine invariant, without loss of generality we can suppose that such cut exists for a square, whose coordinates of vertices are $(0,0),(0,1),(1,0),(1,1)$. Since its area is equl to 1, areas of all triangles are equal to the inverse odd integer, so have the $2-$adic valuation equal to 0. From the property, stated above, each of the triangles is colored in two colors only. One can show that this contradicts Sperners Lemma, which finishes the proof of Monsky theorem.

After that, several generalizations of Monsky's results appear. The first generalization was conjectured by Stein and proved by Monsky in 1990 [5]. It claims that a centrally symmetric polygon cannot be cut into an odd number of triangles of equal areas. Although it is based on the same idea of $3-$coloring, this proof is technically more challenging than the proof in the case of a square and uses a non-trivial homological technique.

In 1994 Bekker and Netsvetaev proved a similar statement in higher dimensions [2].

To state another generalization we need a definition. Let us call a finite union of squares of area 1 with integer coordinates of vertices a *polyomino.* First, Stein proved in 1999 [9] that a polyomino of an odd area can not be cut into an odd number of triangles of equal areas, and in 2002 Praton [6] proved the same for an even-area polyomino.

These results rise the following natural questions:

Problem 1. *Find an algorithm, which proves or disproves the existence of an odd equidissection for a given polygon.*

Problem 2. *Prove that a polygon from some wide class cannot be cut into an odd number of triangles of equal areas.*

Problem 3. *For a given polygon let us fix a combinatorial type of its triangulation. Find restrictions on areas of triangles in a triangulation of this type.*

A general algorithm, solving Problem 1, is not known yet. On the other hand, in the next section we will describe a necessery condition for a polygon to have an odd equidissection. It is not algorithmically verifiable, and the question, whether this condition is sufficient or not, is still open.

Some results on problem 3 were obtained by Aaron Abrams and James Pommersheim, see [1]. The set of possible areas is, in general, an algebraic variety, depending on combinatorial type of the triangulation.

Our main result is the solution of problem 2 for a class of balanced rational polygons, which we will now describe.

Let P be a plane polygon with clockwise oriented boundary. P is called *balanced* if its edges can be divided into pairs so that in each pair edges are parallel, equal in length and have opposite orientation (the edges are oriented, their orientation comes from the orientation of the boundary). P is called *rational* if it can be drawn in the plane in a way that all its vertices have rational coordinates. Its triangulation is called *rational* if all its vertices have rational coordinates. For an example of a balanced rational polygone, see Fig.3.

In 2000 Stein [7] made a conjectural generalization of Theorem 1, see also [8].

Conjecture 2.2 (S. Stein, 2000)**.** *A balanced polygon cannot be cut into an odd number of triangles of equal areas.*

In this note we will present a proof of a partial case of Conjecture 2.2. Namely, we will prove the following theorem, which we will call Rational Stein Conjecture.

Theorem 2.3 (Rational Stein Conjecture)**.** *Let $\mathcal{P}$ be a balanced polygon in the plane. Then it is not possible to cut it into an odd number of triangles having equal areas in a way that all the coordinates of the vertices of the triangulation are rational numbers.*

Remark 2.4. From Stein Conjecture it follows that a flat orientable surface cannot be cut into an odd number of triangles of equal areas. If such a cut would be possible then a ballanced polygon, obtained as an unfolding of the flat surface would have such a cut as well, which contradicts Stein Conjecture.

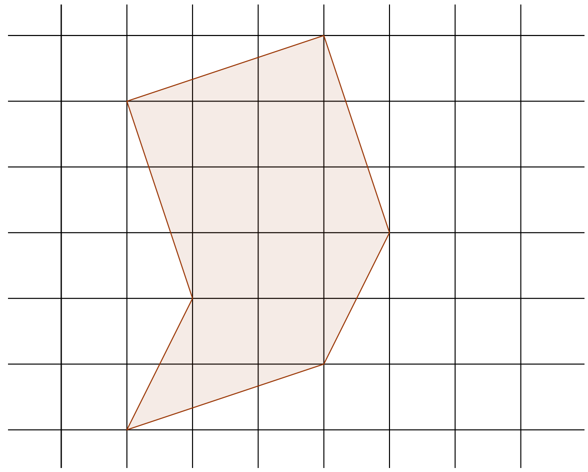

Figure 3: Balanced rational polygon

3 Balanced graphs and primitive lattices

Our main object of study in the remaining part of the paper will be a pair, consisting of a $3-$valent graph and a function, assigning to each edge a pair of $2-$adic integers and subject to some conditions. We will call the function "balancing", and a graph with such a function — a "balanced graph".

Firstly, we would like to specify terminology connected with a $3-$valent graph. Sometimes we would prefer to think of it as of undirected. Specifically, talking about cycles, degrees of vertices, etc. But in the definition of a balanced graph it is easier to think of it as of directed, simply substituting each unoriented edge with a pair of oriented edges going in the opposite directions. We hope that this little ambiguity won't lead to any misunderstanding.

Definition 3.1. *Let Γ be a $3-$valent graph. We will call a function B, assigning a pair of $2-$adic integers to each of $\Gamma's$ oriented edges, a balancing function if it satisfies the following two properties:*

- $B(\mathbf{e}^+) = -B(\mathbf{e}^-)$, *where $\mathbf{e}^+$ and $\mathbf{e}^-$ are the two directed edges, corresponding to the same undirected edge $\mathbf{e}$.*
- $B(\mathbf{e_1})+B(\mathbf{e_2})+B(\mathbf{e_3}) = \mathbf{0}$ *for any three directed edges $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ sharing the same terminal vertex.*

We think of its values as of vectors lying in the lattice $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and denote their coordinates by B_x and B_y. The pair, consisting of a graph Γ and a balancing function B is denoted by $\{\Gamma, B\}$.

Now let us introduce two notions: a *multiplicity* of a vertex and a *lattice* of a vertex. We need the former to state the main result of our paper, while keeping track of the latter will be the main ingredient of the proof of the main result.

From now on we will suppose that our graph carries a balancing function.

Let a vertex $\mathbf{v}$ be terminal for three edges $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$. Then we know that $B(\mathbf{e_1})+B(\mathbf{e_2})+B(\mathbf{e_3}) = \mathbf{0}$. Therefore, the following definitions make sense:

Definition 3.2. *A multiplicity of a vertex is the $2-$adic valuation of the value of the determinant built from any two of the balancing vectors. More concretly,*

$$m(\mathbf{v}) = \nu_{\mathbf{2}}(\mathbf{B_x}(\mathbf{e_1})\mathbf{B_y}(\mathbf{e_2}) - \mathbf{B_y}(\mathbf{e_1})\mathbf{B_x}(\mathbf{e_2})).$$

One should bare in mind that the multiplicity of a vertex could be infinite.

Definition 3.3. *A lattice of a vertex is a sublattice of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated over $\mathbb{Z}_2$ by any two of the balancing vectors. More conctetly,*

$$L(\mathbf{v}) = \langle \mathbf{B}(\mathbf{e_1}), \mathbf{B}(\mathbf{e_2}) \rangle = \mathbb{Z}_{\mathbf{2}}\mathbf{B}(\mathbf{e_1}) + \mathbb{Z}_{\mathbf{2}}\mathbf{B}(\mathbf{e_2}).$$

The lattice of a vertex is of rank 1 if its multiplicity is infinite and of rank 2 if its multiplicity is finite.

From the balancing condition it is clear that in both definitions neither the choice of the pair of vectors nor their order matter. One should bare in mind that the notion of a lattice in a vertex is sharper than that of multiplicity. Moreover, the multiplicity $m(\mathbf{v})$ is just a $2-$adic valuation of an index of $L(v)$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Now we are ready to formulate the main result.

Theorem 3.4. *Let Γ be a balanced $3-$valent graph. Then the number of its vertices, whose multiplicity is minimal among the vertices of Γ, is even.*

For each pair $\{\Gamma, B\}$ we will denote by $M(\{\Gamma, B\})$ the minimal multiplicity of a vertex of Γ.

Our proof will be organized in the following way. We are going to prove Theorem 3.4 by induction on M. Firstly, we will prove the base case $M = 0$. In the proof of the induction step we will modify the balancing function B, keeping track of the parity of the number of vertices with multiplicity M. Eventually, we will come to the balancing function B', whose x and y coordinates on each edge are even $2-$adic integers. Dividing the coordinate function by 2, we will construct a balanced graph $\{\Gamma, B''\}$, to which the induction hypothesis can be applyed.

Let us call a vector $(u_x, u_y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ primitive if at least one of its coordinates is an odd $2-$adic integer. An edge of a balanced graph will be called primitive, if the corresponding vector is primitive. Analogically, we will call a sublattice of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ primitive if it contains a primitive vector. A vertex of a balanced graph will be called primitive if the corresponding lattice is primitive. The main advantage of this notion comes from the following fact:

Lemma 3.5. *Let* $\mathbf{v}$ *be a vertex of a balanced graph* Γ. *Then either* $m(\mathbf{v}) = \mathbf{0}$ *and all the three edges terminating in this vertex are primitive or* $m(\mathbf{v}) > \mathbf{0}$ *and the number of primitive edges, terminating in this vertex, is even.*

The proof is a simple computation, we will state it after the following corollaries:

Corollary 3.6. *(Base of induction.)*

If $M(\{\Gamma, B\}) = 0$ *then Theorem 3.4 holds for* $\{\Gamma, B\}$.

Proof. Consider a subgraph $\mathcal{P}$ of Γ, consisting of primitive edges only. By Lemma 3.5, it has only $3-$valent and $2-$valent vertices. We need to show that the number of the $3-$valent vertices is even. Let us denote the number of $2-$valent vertices of $\mathcal{P}$ by $\mathcal{V}_2(\mathcal{P})$, $3-$valent vertices of $\mathcal{P}$ by $\mathcal{V}_3(\mathcal{P})$ and the number of edges of $\mathcal{P}$ by $\mathcal{E}(\mathcal{P})$. Obviously,

$$2\mathcal{V}_2(\mathcal{P}) + 3\mathcal{V}_3(\mathcal{P}) = 2\mathcal{E}(\mathcal{P}).$$

So, $\mathcal{V}_3(\mathcal{P})$ is even. $\square$

Corollary 3.7. *If* $M(\{\Gamma, B\}) > 0$, *then the primitive edges form a system of nonintersecting cycles of* Γ.

Proof. The proof is obvious. $\square$

These cycles will be called *primitive.* Further, we will work with these cycles separately, modifying the balancing function on each of them. Eventually, we will get rid of all the primitive edges and apply the induction hypothesis. But before that we return to the proof of Lemma 3.5.

Proof. Suppose that the three edges $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ terminate in the vertex $\mathbf{v}$. Let $B(\mathbf{e_i})= (x_i, y_i)$. If any two of these vectors are equivalent modulo 2, then $m(\mathbf{v})> 0$. In this case either they are both primitive and then the third one is not, thanks to the balancing condition, or none of them is primitive. In the latter case both coordinates of the third vector are even $2-$adic integers by the same reason.

It remains to analyze the case when all the three vectors are different modulo 2. Applying the balancing condition again, we see that none of them can have two even coordinates. Therefore, these vectors equal $(0, 1), (1, 0)$ and $(1, 1)$ modulo 2. Obviously, in this case

$$m(\mathbf{v}) = \nu_{\mathbf{2}} \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) = \mathbf{0}.$$

$\square$

In the following we suppose that $M(\Gamma) > 0$ and we are in a position to apply Corollary 3.7. From now on we will concentrate on the structure of primitive cycles. The main issue for us will be to understand which lattices can correspond to the vertices of such cycle. Obviously, all these lattices are primitive. The main observation is that the primitive lattices over $\mathbb{Z}_2$ form some sort of a tree. We will describe their structure in the following two lemmas.

Lemma 3.8. *Let* L *be a primitive sublattice of* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *of multiplicity* d. *Then for each* $0 \leq i \leq d \leq \infty$ *there exists exactly one primitive sublattice of* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *which contains* L *and whose multiplicity is equal to* i.

If L contains a primitive vector u with odd first coordinate and i is finite, then the lattice of multiplicity i containing L is generated by the vectors u and $(0, 2^i)$.

If L contains a primitive vector w with odd second coordinate and i is finite, then the lattice of multiplicity i containing L is generated by the vectors w and $(2^i, 0)$.

If L contains a primitive vector w and $i = d$ is infinite, then the lattice of multiplicity i containing L is equal to L. In this case L is generated by w.

Proof. Let M be any primitive lattice between $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and L. Let u be a primitive vetor in L. Then it will be a primitive vector in M as well. Without loss of generality suppose that the first coordinate of u is odd. It is well known, and, essentially, a special case of the classification theorem of abelian groups, that there exists a vector $u' \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ such that $\langle u, u' \rangle$ form a basis of M. But then $\langle u, u' - \frac{u'_x}{u_x} u \rangle$ is a basis as well. Dividing the second vector by invertible $2-$adic integer we see that M has a basis $\langle u, (0, 2^i) \rangle$ for some i. Obviously, $i = m(M)$. First two statements of the lemma follow from that. The last statement is obvious. □

In the next lemma we will explain which sublattices of minimal index a primitive lattice might have.

Lemma 3.9. *Let L be a primitive lattice of multiplicity $d < \infty$. Then it has exactly three sublattices of multiplicity $d+1$. Two of them are primitive (we will denote them L^+ and L^-) and one is not (it will be called L^0). The last one consists of all nonprimitive vectors in L. Every primitive vector in L lies either in L^+ or in L^-.*

Proof. It is easy to show that there are only three sublattices of multiplicity $d+1$. Leaving the proof to the reader, we will just construct them. Without loss of generality let us suppose that L contains a primitive vector u with an odd first coordinate. By previous lemma, L has a basis of the form $\langle u, (0, 2^d) \rangle$. Its primitive sublattices of multiplicity $d+1$ are $\langle u, (0, 2^{d+1}) \rangle$ and $\langle u + (0, 2^d), (0, 2^{d+1}) \rangle$. A nonprimitive one is $\langle 2u, (0, 2^d) \rangle$. □

So, primitive lattices form a 3−valent tree under inclusion with the root $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In this interpretation, multiplicity of a lattice is simply the distace to the root.

Lemma 3.10. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two vertices of a balanced graph connected by a primitive edge. If* $m(\mathbf{v}) \geq m(\mathbf{w})$*, then* $L(\mathbf{v}) \subseteq L(\mathbf{w})$*.*

Proof. Let us denote the edge, which connects the two vertices, by $\mathbf{e}$. By Lemma 3.8, $L(\mathbf{v})$ has a basis of the form $\langle B(\mathbf{e}), (\mathbf{0}, \mathbf{2^{m(v)}})\rangle$ and $L(\mathbf{w})$ has a basis of the form $\langle B(\mathbf{e}), (\mathbf{0}, \mathbf{2^{m(w)}})\rangle$. From this the statement follows. □

The following lemma contains information about lattices corresponding to vertices of a primitive cycle, which is essential for our proof. As it has been stated before, we suppose that $M\{\Gamma, B\} > 0$.

Lemma 3.11. *Suppose that vertices* $\mathbf{v_n} = \mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ *form a cycle* $\mathcal{C}$ *in* Γ *and are all primitive. If at least one of them is of finite multiplicity, then the following is true:*

1. *Among the lattices* $L(\mathbf{v_0}), \ldots, L(\mathbf{v_{n-1}})$ *there exists one which contains all the others. We will call it maximal and denote by* $L(\mathcal{C})$*.*

2. *The number of the vertices of the cycle* $\mathcal{C}$*, whose lattices are equal to* $L(\mathcal{C})$*, is even.*

3. *All the vectors, corresponding to the edges, which connect a vertex in the cycle with a vertex not in the cycle, are contained in the lattice* $L(\mathcal{C})^0$*.*

Proof. 1. Let us form an abstract graph $S(\mathcal{C})$, whose vertices correspond to $\mathbf{v_0}, \ldots, \mathbf{v_{n-1}}$. We will use the same symbols to denote the vertices of $\mathcal{C}$ and of $S(\mathcal{C})$.Two vertices $\mathbf{a}$ and $\mathbf{b}$ will be connected by an edge if either $L(\mathbf{a}) \subseteq L(\mathbf{b})$ or $L(\mathbf{b}) \subseteq L(\mathbf{a})$. From the previous lemma we know that $\mathbf{v_i}$ is connected by an edge with $\mathbf{v_{i+1}}$, so this graph is connected.

Let us take a vertex $\mathbf{m}$ in the cycle, whose lattice $L(\mathbf{m})$ is maximal by inclusion among the lattices $L(\mathbf{v_i})$. We will show that it contains all other lattices of the cycle. Let us suppose the opposite and take any vertex $\mathbf{t}$, for which $L(\mathbf{t}) \nsubseteq L(\mathbf{m})$.

Since $S(\mathcal{C})$ is connected, $\mathbf{m}$ and $\mathbf{t}$ can be connected in $S(\mathcal{C})$ by a path of minimal length. If the length of the path is equal to 1, then we come to a contradiction. We know that $L(\mathbf{t})\nsubseteq L(\mathbf{m})$ by the suggestion about $\mathbf{t}$ and $L(\mathbf{m})\nsubseteq L(\mathbf{t})$ by the maximality of $\mathbf{m}$.

We are going to show that the path can be made shorter, which contradicts to its choice. Let us denote its vertices by $\mathbf{w_0} = \mathbf{m}, \mathbf{w_1}, \ldots, \mathbf{w_l} = \mathbf{t}$. For each j either $L(\mathbf{w_j}) \subseteq L(\mathbf{w_{j+1}})$ or $L(\mathbf{w_j}) \supseteq L(\mathbf{w_{j+1}})$. If for all j the case is the same, then we have either $L(\mathbf{m}) \subseteq L(\mathbf{t})$ or $L(\mathbf{m}) \supseteq L(\mathbf{t})$, none of which is possible. Moreover, by maximality of $\mathbf{m}$ we know that $L(\mathbf{w_0}) \supseteq L(\mathbf{w_1})$. Therefore, there exists j such that $L(\mathbf{w_{j-1}}) \supseteq L(\mathbf{w_j}) \subseteq L(\mathbf{w_{j+1}})$. Since all these lattices are primitive, it follows from Lemma 3.10 that $L(\mathbf{w_{j-1}}) \supseteq L(\mathbf{w_{j+1}})$ or $L(\mathbf{w_{j-1}}) \subseteq L(\mathbf{w_{j+1}})$. So $\mathbf{w_{j-1}}$ and $\mathbf{w_{j+1}}$ are connected in $S(\mathcal{C})$ by an edge and the chosen path is not minimal.

2. Let's denote the edge of Γ connecting $\mathbf{v_i}$ and $\mathbf{v_{i+1}}$ by $\mathbf{e_i}$. For each edge $\mathbf{e_i}$ we know that $B(\mathbf{e_i})$ is a primitive vector, $B(\mathbf{e_i}) \in L(\mathcal{C})$. Since at least one vertex of the cycle had finite multiplicity, $L(\mathcal{C})$ has finite multiplicity. First we would like to show that either $B(\mathbf{e_i}) \in L(\mathcal{C})^+$ or $B(\mathbf{e_i}) \in L(\mathcal{C})^-$. If $B(\mathbf{e_i})$ lies in both $L(\mathcal{C})^+$ and $L(\mathcal{C})^-$, then $L(\mathcal{C})^+ = L(\mathcal{C})^-$ by Lemma 3.10 which contradicts Lemma 3.9. At the same time, by Lemma 3.9, any primitive vector of $L(\mathcal{C})$ is contained in $L(\mathcal{C})^+$ or $L(\mathcal{C})^-$.

   Therefore, we can divide the edges of the cycle in two groups: those for which $B(\mathbf{e_i}) \in L(\mathcal{C})^+$ or $L(\mathcal{C})^-$. The eveness of the number of the vertices $\mathbf{v}$ for which $L(\mathbf{v}) = L(\mathcal{C})$ will follow from the following fact: $L(\mathbf{v_{i+1}}) = L(\mathcal{C})$ if and only if $B(\mathbf{e_i}) \in L(\mathcal{C})^+$ and $B(\mathbf{e_{i+1}}) \in L(\mathcal{C})^-$, or $B(\mathbf{e_i}) \in L(\mathcal{C})^-$ and $B(\mathbf{e_{i+1}}) \in L(\mathcal{C})^+$.

   The if-part follows from the fact that $B(\mathbf{e_i})$ and $B(\mathbf{e_{i+1}})$ form a basis of $L(\mathbf{v_{i+1}})$, so if they both are contained in $L(\mathcal{C})^+$ or $L(\mathcal{C})^-$, then the whole lattice $L(\mathbf{v_{i+1}})$ is.

   The only-if-part is also easy to show. If $L(\mathbf{v_{i+1}}) \neq L(\mathcal{C})$ then by Lemma 3.9 we have $L(\mathbf{v_{i+1}}) \subseteq L(\mathcal{C})^+$ or $L(\mathbf{v_{i+1}}) \subseteq L(\mathcal{C})^-$. In the first case $B(\mathbf{e_i})$ and $B(\mathbf{e_{i+1}})$ are contained in $L(\mathcal{C})^+$, in the second case they are contained in $L(\mathcal{C})^-$.

   So the vertices, whose lattices are equal to $L(\mathcal{C})$, are exactly those, at which a change of the type of the edge happens. Therefore, there number is even.

3. An edge going from a vertex in the cycle to a vertex not in the cycle is not primitive by Corollary 3.7 so it is contained in L^0 by Lemma 3.9.

$\square$

Now we can finish the proof of the Theorem 3.4.

If $M\{\Gamma, B\} = \infty$, we need to show that the number of vertices of the graph is even. But this is true for any 3−valent graph. So we can suppose that $M\{\Gamma, B\}$ is finite.

We prove the statement for a pair $\{\Gamma, B\}$ by induction on $M\{\Gamma, B\}$. Base follows from Corollary 3.6, so we can suppose that $M\{\Gamma, B\} > 0$. By Corollary 3.7, the nonprimitive edges form a number of nonintersecting cycles. Each primitive cycle either does not contain any vertices of multiplicity M, if its maximal vertex is of greater multiplicity, or contains even number of them, if its maximal vertex is of multiplicity exactly M.

Now we will change B on each edge of each primitive cycle in such a way that all the edges become nonprimitive and all the vertices have multiplicity at least $m + 1$. This can be done separately for each primitive cycle. Let us take a cycle $\mathcal{C}$ with vertices $\mathbf{v_n} = \mathbf{v_0}, \ldots, \mathbf{v_{n-1}}$ and edges $\mathbf{f_n} = \mathbf{f_0}, \ldots, \mathbf{f_{n-1}}$, going out of the cycle. Let $L(\mathcal{C})$ be a maximal lattice of the cycle. Then $m(L(\mathcal{C})) \geq M$ and $m(L(\mathcal{C})^0) \geq M+1$. All the vectors $B(\mathbf{f_i})$ are contained in $L(\mathcal{C})^0$ by Lemma 3.11. We can modify B on the edges of the cycle, assigning to the edge, which connects $\mathbf{v_i}$ and $\mathbf{v_{i+1}}$, a vector

$$-\sum_{j=1}^{i} B(\mathbf{f_j}).$$

It is easy to check that if B was a balancing function, then the modified function will also be balancing. Now for each edge of the cycle the corresponding vector is inside L^0, so it is nonprimitive and all the lattices, corresponding to the vertices, are contained in L^0, so they have multiplicity greater than $M + 1$. We can do it for all the primitive cycles consequently and eventually construct a new balancing $B^{‘}$ with the desired property.

If all the vertices of $\{\Gamma, B\}$, having multiplicity $M\{\Gamma, B\}$, were primitive, then theorem 1 is proved for $\{\Gamma, B\}$, since we know that in each cycle the number of vertices of multiplicity $M\{\Gamma, B\}$ is even. If not, there exist a nonprimitive vertex of multiplicity $M\{\Gamma, B\}$.

Let us consider a function $B'' = \frac{B'}{2}$, which is also balancing by the fact that all the vectors of $B^{`}$ are nonprimitive: both their coordinates are even. We know that $M\{\Gamma, B''\} = M\{\Gamma, B\} - 2$, since in $\{\Gamma, B\}$ there was a nonprimitive vertex of multiplicity $M\{\Gamma, B\}$. So by the induction hypothesis, the number of vertices in $\{\Gamma, B''\}$ of multiplicity $M\{\Gamma, B\} - 2$ is even. But the number of vertices in $\{\Gamma, B\}$ of multiplicity $M\{\Gamma, B\}$ has the same parity, from which Theorem 3.4 follows.

4 Rational Stein's conjecture

Now we are ready to give a proof of Rational Stein Conjecture.

Theorem 4.1 (Rational Stein Conjecture)**.** *Let P be a balanced polygon in the plane. Then it is not possible to cut it into an odd number of triangles having equal areas in a way that all coordinates of vertices of the triangulation are rational.*

Proof. Let's suppose that such a cut exists and come to a contradiction. Since the statement is invariant under affine transformations of the plane, we can suppose that all coordinates of vertices of the triangulation are integer numbers.

Some triangles in the cut can intersect not in the proper way: a vertex of a triangle can lie in the interior of a side of another triangle. By adding additional degenerate triangles of area 0 we can make the triangulation proper. In this modified triangulation there will be an odd number of triangles of equal areas and several triangles of area 0.

From a triangulation of the balanced polygon P we can form a $3-$valent graph $\Gamma(P)$ in a natural way. First we take a dual graph of the triangulation. Then we add an extra edge for each pair of the corresponding sides of the boundary of the polygon P.

The inclusion of polygon P in the plane determines a balancing function $B(P)$. On each edge $\mathbf{e}$ from the dual triangulation balancing B is defined to be a vector of the common side of the two triangles, corresponding to the ends of $\mathbf{e}$. For extra edges we can take the corresponding vector of the side of P. Two triangles, corresponding to the ends of the edge have the same vector of the side, because P is balanced. Coordinates of such a vector will be integer numbers, and one can think of them as of $2-$adic integers.

Let's suppose that all nondegenerate triangles in the triangulation have the same area S. Multiplicity of a vertex, corresponding to a nondegenerate triangle of $\Gamma(P)$ equals $1 + \nu_2(S)$, while that of a vertex, correponding to a degenerate triangle is infinite. So, by Theorem 3.4 applied to the ballanced graph $\{\Gamma(P), B(P)\}$, the number of triangles of area A is even. This leads to the contradiction. $\square$

5 References

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