Moscow State University

On the Blaschke-Bol problem in the plane

Nilov Fedor, a first-year PhD student

Scientific advisers
prof. V.Yu. Protasov, prof. V.A. Timorin

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1. Introduction

There are many theorems in geometry, in which some natural construction closures with period 6; e.g., see [1, p. 104–107]. For example, the famous Pappus and Brianchon theorems can be stated in this way; we give these statements a bit later. It turns out that such closure theorems come from a beautiful general theory called *web geometry*. It was founded by W. Blaschke and his collaborators in 1920s; see a nice introduction and references in [7, §18]. Since then many interesting results have been obtained in the area (see references in [16]) but many natural questions remained open. In this paper we give several new examples of webs of circles, which is an advance in one of such open questions.

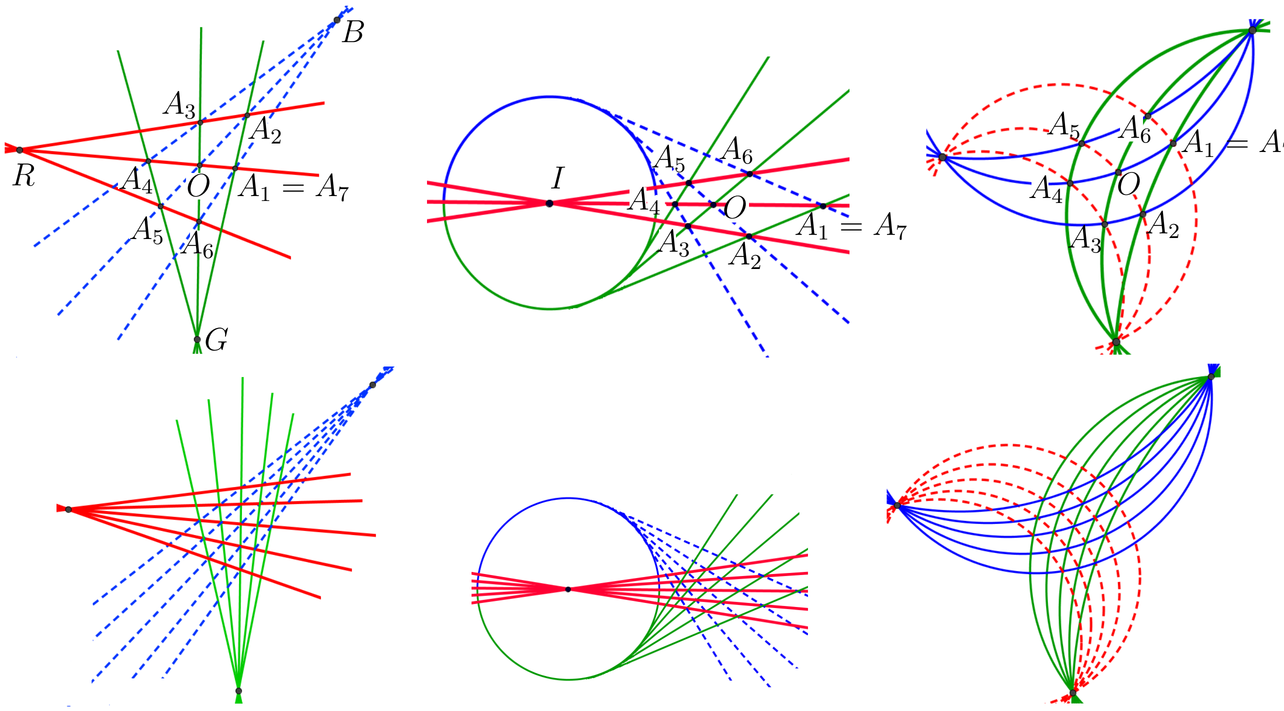


Figure 1. Top: Left: *The Pappus theorem.* Middle: *The Brianchon theorem.* Right: *The Blaschke theorem.* Bottom: Left: *The Pappus web* formed by three pencils of lines. Middle: *The Brianchon web* formed by the set of tangent lines to a circle counted twice and a pencil of lines. Right: *The Blaschke web* formed by three elliptic pencils of circles with the vertices (R, G), (G, B), and (B, G) (see the definition of pencils in §1).

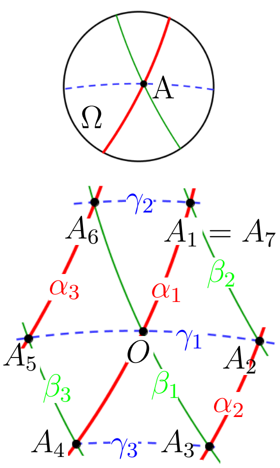
Let us give the statements of the Pappus and Brianchon theorems as closure theorems.

The Pappus Theorem. *A red (R), a green (G), and a blue (B) points are marked in the plane (see Figure 7 to the left). Each line passing through exactly one of the marked points is painted the same color as the point. Take an arbitrary point O inside the triangle RGB. Draw the red, the green, and the blue line through the point. On the red line take an arbitrary point A_1 inside the triangle RGB. Draw the green line through the point A_1. Suppose that the green line intersects the blue line through the point O at a point A_2. The green and the blue line through the point A_2 have already been drawn; draw the red line through A_2. The intersection point of the obtained red line with the green line through the point O is denoted by A_3. Continuing this construction we get the points A_4, A_5, A_6, A_7. Then $A_7 = A_1$.*

The Brianchon Theorem. *A circle with a point I inside and a point O outside are given; see Figure 7 to the middle. The lines passing through I are painted red. The rays starting at the points of this circle, tangent to it, and looking clockwise or counterclockwise are painted green or blue, respectively. Construct the points $A_1 \ldots A_7$ as in the Pappus Theorem. Then $A_7 = A_1$.*

In this paper we consider a general construction generating theorems of this kind; e.g., see Figure 7 to the right. In what follows by a *circular arc* we mean either a circular arc or a line segment, or a ray, or a line. We assume that all circular arcs do not contain their endpoints.

Suppose that some circular arcs contained in a domain $\Omega \subset \mathbb{R}^2$ with the endpoints contained in the boundary of Ω are painted red, green, and blue. We say that they have *hexagonal property*, if the following 2 conditions hold (see figures to the left):



- **Foliation condition:** For each point $A \in \Omega$ there is exactly one circular arc of each color passing through A. The arcs of distinct colors either are disjoint or intersect once transversely.
- **Closure condition:** Consider an arbitrary point $O \in \Omega$. Let α_1, β_1, and γ_1 be the red, green, and blue circular arcs passing through O, respectively. Consider an arbitrary point $A_1 \in \alpha_1$. Let β_2 and γ_2 be the green and blue circular arcs, respectively, passing through A_1. Let A_2 be the intersection point of β_2 and γ_1. Consider the red circular arc α_2 passing through A_2. Let A_3 be the intersection point of α_2 and β_1. Analogously define the points A_4, A_5, A_6, and A_7. The *hexagonal closure condition* asserts that if all the above points exist, then $A_7 = A_1$.

A trivial example of lines having the hexagonal property is the lines parallel to the sides of a fixed triangle painted red, green, and blue. Now if each of the lines intersects a domain $\Omega \subset \mathbb{R}^2$ by at most one line segment then these segments have the hexagonal closure property. If a real analytic diffeomorphism $f\colon \Omega \to \Omega' \subset \mathbb{R}^2$ maps these segments to circular arcs (painted the same color) then the circular arcs have the hexagonal property as well. This is a motivation for the following definition.

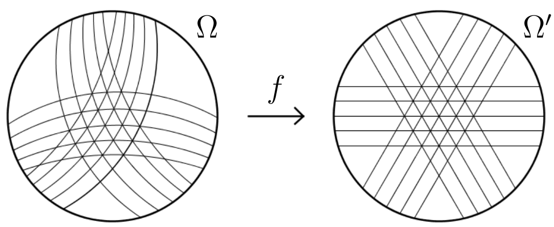


FIGURE 2. A definition of a hexagonal 3-web.

Definition. Three sets of circular arcs in a domain Ω is a *hexagonal* 3*-web of circular arcs* (or simply a *web of circular arcs*), if there is a real analytic diffeomorphism $f\colon \Omega \to \Omega' \subset \mathbb{R}^2$ which takes the sets of arcs to the intersections of the sets of lines parallel to the sides of a fixed triangle with the domain Ω', and all the nonempty intersections of these lines with Ω' are connected; see Figure 2.

W. Blaschke proved under some regularity assumptions that if some circular arcs have the hexagonal property then they is a hexagonal 3-web.

We say that three sets of circles in the plane *contain a hexagonal* 3*-web*, if their appropriate arcs (possibly empty) is a hexagonal 3-web in an appropriate domain. We allow two of the three sets of circles to coincide; this means that we take two disjoint arcs from each circle of such set. In the latter case we say that the set is *counted twice*; see the example in the Figure 3 and the Brianchon Theorem above.

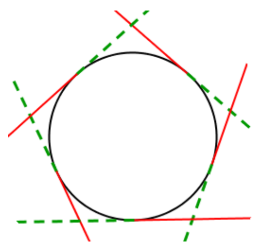


FIGURE 3. The set of tangent lines to a circle is counted twice. We take two disjoint rays from each tangent line to the circle.

For instance, the following sets of circles contain a hexagonal 3-web; see [3, p. 19–20] and Figure 7 to the bottom:

(a) *The Pappus web.* Three sets of lines passing through three distinct points R, G, and B, respectively;

(b) *The Brianchon web.* The set of tangent lines to a conic counted twice and a set of lines passing through a fixed point;

(c) *The Blaschke web.* A set of circles passing through R and G, a set of circles passing through G and B, and a set of circles passing through B and G.

Hexagonal 3-webs from circular arcs are rare; it is always a luck to find an example.

In 1938 W. Blaschke and G. Bol stated the following problem which is still open.

The Blaschke–Bol Problem. *(See [3, p. 31].) Find all hexagonal 3-webs from circular arcs.*

Let us outline the state of the art; we give precise statements of all known results on such webs in Section 2. All hexagonal 3-webs of straight line segments were found by H. Graf and R. Sauer [3]. All hexagonal 3-webs of circles belonging to one bundle were found by O. Volk and K. Strubecker. A class of webs of circles generated by a one-parameter group of Möbius transformations was considered by W. Wunderlich [21]. We give an elementary restatement of his result; see Theorem 2.4 below. A highly nontrivial example of webs of circles doubly tangent to a cyclic was found by W. Wunderlich [21]. For many decades there were no new examples of hexagonal 3-webs of circular arcs except several webs formed by pencils of circles by V.B. Lazareva, R.S. Balabanova, and H. Erdogan; see [18]. Recently, A.M. Shelekhov discussed the classification of all hexagonal 3-webs formed by pencils of circles [18].

Webs of circular arcs on all surfaces distinct from a plane or a sphere are classified in [16]: N. Lubbes proved that any surface in 3-space containing ≥ 3 circles through each point is a so-called Darboux cyclide, and the webs on the latter are classified in [16]; see Figure 4 to the left. There are many examples of webs of conics; e.g., see Figure 4 to the right and references in [16].

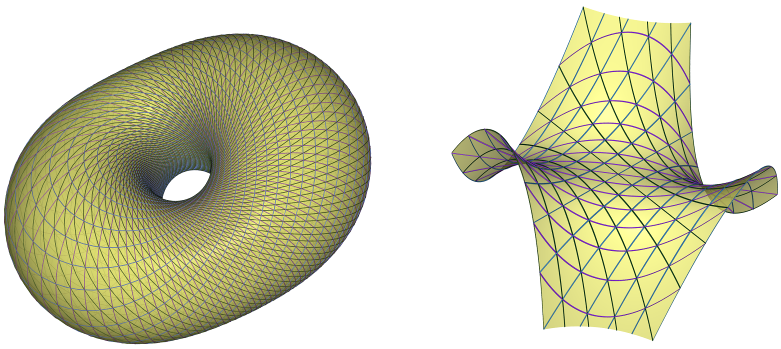


FIGURE 4. Left: A hexagonal 3-web of circular arcs on the Darboux cyclide [16]. Right: The surface $z = xy(y - x)$ contains a web of conics, which are in fact isotropic circles; see Problem 5.4.

Main results of the paper are new examples of webs of circular arcs (see Theorem 1.1 below). They involve pencils of lines, circles and double tangent circles of conics.

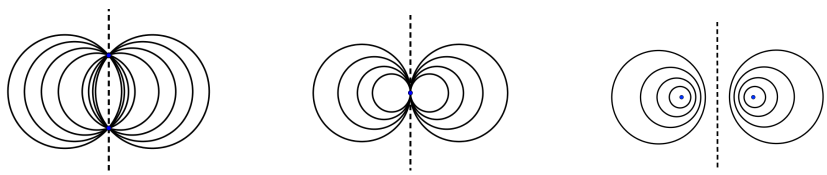


FIGURE 5. Left: An elliptic pencil of circles. Middle: A parabolic pencil of circles. Right: A hyperbolic pencil of circles.

Let $c_1(x, y) = 0$ and $c_2(x, y) = 0$ be equations of degree 2 or 1 of two distinct circles c_1 and c_2, respectively. A *pencil of circles* is the set of all the circles having the equation of the form $\alpha c_1(x, y) + \beta c_2(x, y) = 0$, where α and β are real numbers not vanishing simultaneously. So a *pencil of lines* is a set of all the lines passing through a fixed point (the *vertex* of the pencil) or parallel to a fixed line. If c_1 and c_2 are circles with two distinct common points (the *vertices* of the pencil), then all the circles in the pencil pass through these points and the pencil is called *elliptic*; see Figure 5 to the left. If the circles c_1 and c_2 are tangent (the tangency point then is called the *vertex* of the pencil), then each circle in the pencil is tangent to c_1 and c_2 at the same point and the pencil is called *parabolic*; see Figure 5 to the middle. If the circles c_1 and c_2 have no common points, then the pencil is called *hyperbolic*; see Figure 5 to the right. Any hyperbolic pencil contains "circles" degenerating to points (bold points in the figure). They are called *limiting points* of the pencil. A pencil of intersecting (respectively, parallel) lines is considered as an elliptic (respectively, parabolic) pencil of circles. If we take three circles $c_1(x,y) = 0$, $c_2(x,y) = 0$, and $c_3(x,y) = 0$ not belonging to one pencil then a *bundle of circles* is the set of all the circles having the equation of the form $\alpha c_1(x,y) + \beta c_2(x,y) + \gamma c_3(x,y) = 0$. By a *general conic* we mean either an ellipse distinct from a circle or a hyperbola.

Theorem 1.1. [12] *The following sets of circles contain a hexagonal 3-web of circular arcs:*

(a) The tangent lines to a circle counted twice and a parabolic pencil of circles with the vertex at the center of the circle;

(b) The tangent lines to a general conic counted twice and the hyperbolic pencil of circles with limiting points at the foci of the conic;

(c) The tangent lines to a general conic (counted once), a pencil of lines with the vertex at a focus of the conic, and circles doubly tangent to the conic such that their centers lie on the minor axis of the conic;

(d) The tangent lines to a parabola counted twice and a hyperbolic pencil of circles with limiting points at the focus and an arbitrary point on the directrix;

(e) The circles doubly tangent to an ellipse with the eccentricity $\frac{1}{\sqrt{2}}$ counted twice such that their centers lie on the major axis of the ellipse and the elliptic pencil of circles with vertices at the foci of the ellipse.

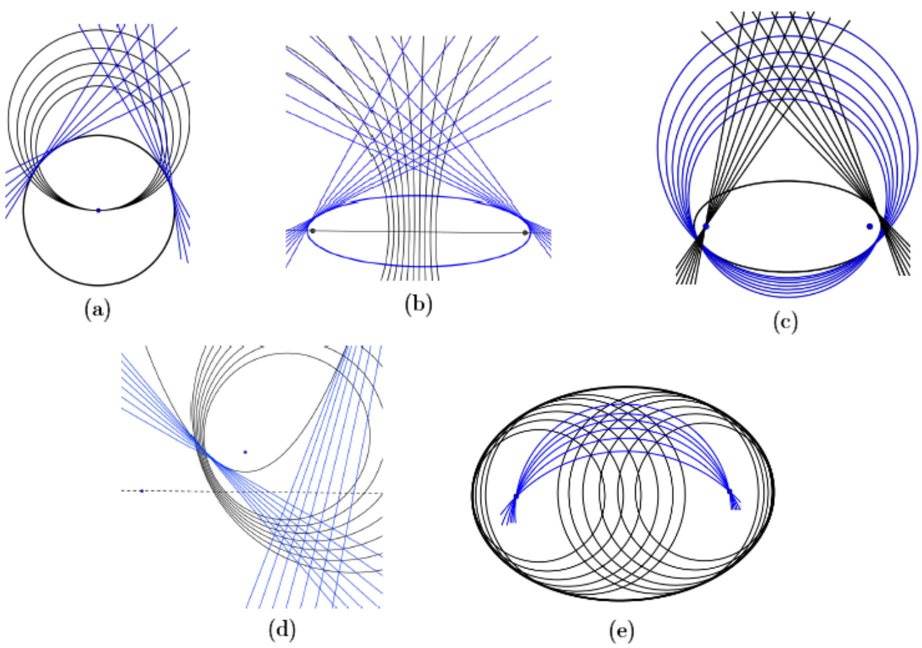


FIGURE 6. New examples of webs of circular arcs in the plane.

We prove Theorem 1.1 in Section 3.

The second part of our paper is based on joint work with M.B. Skopenkov [13]. Surfaces generated by simplest curves (lines and circles) are popular subject in pure mathematics and have applications to design and architecture [15, 4]. If a surface contains two such curves through each point then we get a mesh on the surface. Famous examples of such meshes are V. G. Shukhov's hyperboloid structures. A natural question is which other surfaces can be constructed from straight and circular beams.

It is well-known that a surface containing two lines through each point (*doubly ruled surface*) must be a quadric. We show that a smooth surface containing both a line and a circle through each point still must be a quadric; see Figure 7 to the left.

Theorem 1.2. [13] *If through each point of a smooth surface in $\mathbb{R}^3$ one can draw both a straight line segment and a circular arc transversal to each other and fully contained in the surface (and continuously depending on the point) then the surface is a piece of either a one-sheeted hyperboloid, or a quadratic cone, or an elliptic cylinder, or a plane.*

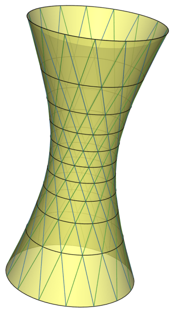


FIGURE 7. A one-sheeted hyperboloid contains both a line and a circle through each point. To find all surfaces with this property (Theorem 1.2), we prove that the planes of the generating circles are parallel (Lemma 3.6) and intersect the surface only at the points of the circles (Lemma 3.11).

In what follows a line (circle) continuously depending on a real parameter is called a *family* of lines (circles). Note that Theorem 1.2 is more tricky than the classical description of doubly ruled surfaces. Let us illustrate the difference. First, the classical result does not really require 2 lines through *each* point: a surface covered by 1 family of lines and containing just 3 more lines intersecting them all must already be a quadric. Second, the classical result remains true in *complex* 3-space. However, similar generalizations of Theorem 1.2 are far from being true; see Examples 4.1–4.4 below.

The next natural problem, which seems to be still open (and is going to be studied in detail in a subsequent publication), is to describe all surfaces containing several circles through each point.

An example of such surface is a *cyclide*, i.e., the surface given by the equation of the form

$$a(x^2+y^2+z^2)^2+(x^2+y^2+z^2)(bx+cy+dz)+Q(x,y,z)=0, \tag{1}$$

where a, b, c, d are constants and $Q(x, y, z)$ is a polynomial of degree at most 2; see Figure 7 to the right. Such a surface is also called a *Darboux cyclide*, not to be confused with a *Dupin cyclide* being a particular case. An introduction to cyclides and circles on them can be found in the work of Pottmann et al. [16]. Any cyclide (besides some degenerate cases) contains at least 2 circles through each point [16]. Conversely, a surface containing 2 cospherical or 2 orthogonal circles through each point must be a cyclide; see [9, Theorems 1 and 2]. However, this is not true without the assumption of either cosphericity or orthogonality; see Example 4.5 below. Recently, N. Lubbes proved that any surface containing ≥ 3 circles through each point is a cyclide [11].

A torus is an example of a cyclide with 4 circles through each point: a meridian, a parallel, and 2 Villarceau circles. There are cyclides with 6 circles through each point [16]. It is known that a surface with 7 circles through each point must be a sphere; see [19].

Further generalizations concern *conic bundles*, in particular, surfaces containing a conic through each point. Surfaces containing both a line and a conic through each point were classified by Brauner [5]. Such a surface has degree at most 4. Notice that it is much more difficult to deduce Theorem 1.2 from this classification than to prove Theorem 1.2 itself. Surfaces containing several conics through each point were classified by Schicho [17]. Such surfaces have degree at most 8 and admit a biquadratic rational parametrization.

The paper is organized as follows. In Section 2 we give a survey on webs of circular arcs in the plane. In Section 3 we prove Theorem 1.1, Theorem 1.2 and the equivalence of Theorem 2.4 and the Wunderlich Theorem 2.3. In Section 4 we give several illustrative examples related to Theorem 1.2. In Section 5 we state some open problems.

2. Known examples of webs of circles

Let us give precise statements of all known examples of hexagonal 3-webs of circular arcs in the plane. Three sets of circles are *transversal*, if for each point from some domain we can find three circles from distinct sets intersecting transversely at the point. Although we always consider webs of *real* circular arcs, in Theorems 2.4, 2.5, and 2.6 we use some auxiliary *complex* points or circles (we skip their formal definition because it is not used in the proof of main results).

Let $F(a,b,c)$ be a homogeneous polynomial of degree 3. The set of lines $ax+by+c=0$ such that $F(a,b,c)=0$ is called a *set of lines tangent to a curve of class* 3.

The following theorem characterizes all hexagonal 3-webs of straight line segments.

The Graf–Sauer Theorem 2.1. [3, §3] *If the lines tangent to a curve of class* 3 *counted triply are transversal then they contain a hexagonal* 3*-web; see Figure 8 to the left. Conversely, any hexagonal* 3*-web of line segments is contained in such set of lines.*

In the particular case when the polynomial $F(a,b,c)$ above is reducible we get either the Pappus or the Brianchon web.

The following theorem characterizes all hexagonal 3-webs of circular arcs belonging to one bundle. The *Darboux transformation* is the composition of a central projection from a plane in space to a sphere and another central projection from the sphere to the plane such that the center of the second projection belongs to the sphere. A.G. Khovanskii proved that Darboux transformations are the only maps of planar domains that take all line segments to circular arcs (see [20, p. 562]).

The Volk–Strubecker Theorem 2.2. *The image of any hexagonal* 3*-web of straight line segments under a Darboux transformation is a hexagonal* 3*-web of circular arcs. Conversely, any hexagonal* 3*-web of circular arcs belonging to one bundle can be obtained by this construction.*

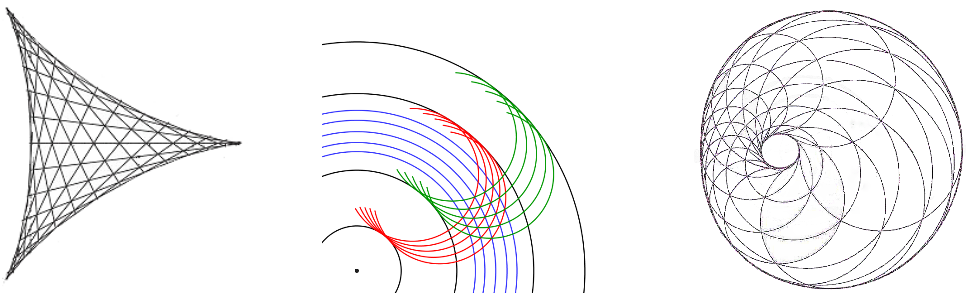


Figure 8. Left: A hexagonal 3-web of lines tangent to a deltoid which is a particular case of a curve of class 3. Middle: Generation of a hexagonal 3-web of circles using a one-parametric group of rotations. Right: A hexagonal 3-web of circles doubly tangent to a cyclic.

W. Wunderlich considered the following example of a hexagonal 3-web of circular arcs.

The Wunderlich Theorem 2.3. *Let two circles ω_1 and ω_2 in the plane have a common point O of transversal intersection. Let a one-parametric group $\mathcal{M}_t$ of Möbius transformations of the plane be such that each orbit is either a circle or a point. Suppose that the orbit of O intersects transversely ω_1 and ω_2. Then the circles $\{\mathcal{M}_t(\omega_1)\}$, $\{\mathcal{M}_t(\omega_2)\}$, and the orbits of $\mathcal{M}_t$ contain a hexagonal* 3*-web; see Figure 8 to the middle.*

In Section 3 we show that this construction is equivalent to the following elementary one.

An *Apollonian set* of a pencil of circles is either a set of circles tangent to two distinct (possibly complex or null) circles from the pencil, or a parabolic pencil with the vertex at a vertex of the pencil, or a hyperbolic pencil with limiting points at the vertices of the pencil.

Theorem 2.4. *If a pencil of circles and two Apollonian sets of this pencil are transversal then they contain a hexagonal* 3*-web.*

Let us give the most wonderful example of webs of circular arcs.

A *cyclic* is the curve given by an equation of the form

$$a(x^2+y^2)^2+(x^2+y^2)(bx+cy)+Q(x,y)=0, \tag{2}$$

where a, b, c are constants and $Q(x, y)$ is a polynomial of degree at most 2 not vanishing simultaneously. Note that conics, limaçons of Pascal, and Cartesian ovals are particular cases of cyclics.

We say that a circle is *tangent* to a cyclic, if the circle either has a real tangency point with the cyclic, or a complex one, or passes through a singular point of the cyclic. The circles doubly tangent to a cyclic naturally split into ≤ 4 *families*: the centers of circles from one family lie on one conic or line; see [16, Remark 16].

The Wunderlich Theorem 2.5. [21] *If three distinct families of circles doubly tangent to a cyclic are transversal then they contain a hexagonal* 3*-web; see Figure 8 to the right.*

A particular case (stated by Blaschke in 1953) of the Blashke-Bol problem was to find all triples of pencils of circles containing webs.

Theorem 2.6. [18] *The following pencils of circles contain a hexagonal* 3*-web:*

(a) (Volk-Strubecker) *Three pencils of circles belonging to the same bundle.*

(b) (Lazareva) *Three hyperbolic pencils with a common complex circle such that in each of the pencils there is a circle orthogonal to all the circles of the other two pencils.*

(c) (Lazareva) *Two elliptic pencils and one hyperbolic pencil with a common real circle such that in each of the pencils there is a circle orthogonal to all the circles of the other two pencils.*

(d) (Balabanova) *Two orthogonal pencils and the third pencil having a common circle with each of the two orthogonal pencils.*

(e) (Balabanova) *Two orthogonal parabolic pencils and one hyperbolic pencil; one of the limiting points of the hyperbolic pencil coincides with the common vertex of the parabolic pencils.*

(f) (Blaschke) *Three elliptic pencils of circles with the vertices (A, B), (B, C), and (C, A).*

(g) (Erdogan) *Two elliptic pencils with the vertices (A, B) and (B, C) and the hyperbolic pencil with the limiting points C and A.*

(h) (Lazareva) *Two parabolic pencils and an elliptic pencil with the vertices at the vertices of the parabolic pencils.*

(j) (Erdogan) *An elliptic pencil with the vertices A and B, the hyperbolic pencil with the limiting points B and C, and a parabolic pencil with the vertex at A such that the common circle of the elliptic and hyperbolic pencils is orthogonal to the circle passing through the points A, B, and C.*

According to A.M. Shelekhov [18] these are all the possible triples of pencils of circles containing a hexagonal 3-web. (He divides some of examples (a)–(j) into several subclasses.)

We see that our examples of hexagonal 3-webs of circular arcs in Theorem 1.1(a)-(e) are indeed new; see Table 1.

Table 1.

Known examples	Difference from the new examples
Theorem 2.1, 2.2	the circles in each example of Theorem 1.1(a)-(e) do not belong to one bundle
Theorem 2.4	Theorem 1.1(a): the lines tangent to the circle do not belong to the given parabolic pencil; Theorem 1.1(b)-(e): there is a family of circles enveloping a conic distinct from a circle
Theorem 2.5	the envelope of all the circles in examples Theorem 1.1(a)-(e) is not one cyclic
Theorem 2.6	the circles in each example Theorem 1.1(a)-(e) do not belong to 3 pencils

3. Proofs

In the proofs below we construct a real analytic diffeomorphism $f : \Omega \to \Omega' \subset \mathbb{R}^2$ which maps the intersections of circles from the given sets with an appropriate domain Ω to the segments of the lines $x = \text{const}$, $y = \text{const}$ and $x + y = \text{const}$.

We need the notions of left and right tangent lines from a point to a conic. Let a point and a conic be given. Consider a line passing through the point not intersecting the conic. Let us start to rotate this line around the given point counterclockwise. Suppose that there are two moments when this line is either tangent to the conic or is an asymptotic line. We say that the lines at the first and the second moments (if such moments exist) are called the *left* and the *right* tangent lines, respectively. If the point lies on the conic then by definition the left and the right tangent lines coincide with the ordinary tangent line.

Let $d(X, \lambda)$ be the distance from a point X to a line λ. By $\angle(\alpha, \beta) \in [0, \pi)$ we denote the oriented angle between lines α and β.

Proof of Theorem 1.1(a). This assertion is a limiting case of Theorem 1.1(b) in which the foci of the conic converge to each other. The given conic converges to a circle. The hyperbolic pencil of circles with limiting points at the foci converges to a parabolic pencil of circles with the vertex at the center of the circle. Thus point (a) follows from (b) because the foliation condition is clearly satisfied and passing to the limit respects the hexagonal closure condition. Theorem 1.1(a) is proved modulo Theorem 1.1(b).

Proof of Theorem 1.1(b). Denote by γ the given general conic. Denote by F_1 and F_2 the foci of γ. Let U be an appropriate domain such that for each point $A \in U$ the left and the right tangent lines $\alpha(A)$ and $\beta(A)$ to the conic γ passing through A, and the circle $\omega(A)$ from the pencil passing through A exist and intersect transversely.

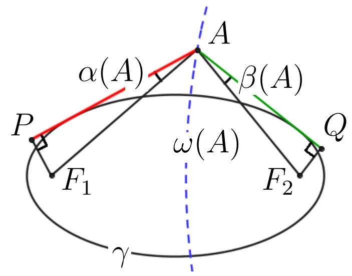


FIGURE 9. To the proof of Lemma 3.1.

Lemma 3.1. *For a fixed circle ω from the pencil the ratio $\frac{d(F_1,\alpha(A))}{d(F_2,\beta(A))}$ does not depend on the point $A \in \omega \cap U$.*

Proof. Let P be the orthogonal projection of F_1 onto the line $\alpha(A)$. Let Q be the orthogonal projection of F_2 onto the line $\beta(A)$; see Figure 9. By the *isogonal property of conics* (see [2, §1.4]), we have $\angle PAF_1 = \angle F_2AQ$. Thus the right triangles F_1AP and F_2AQ are similar. By the well-known *geometric characterization of a hyperbolic pencil of circles* (see [2, Theorem 2.12]) the ratio $\frac{|F_1A|}{|F_2A|}$ does not depend on the point A lying on a circle ω from the pencil with limiting points F_1 and F_2. Hence

$$\frac{d(F_1, \alpha(A))}{d(F_2, \beta(A))} = \frac{|F_1P|}{|F_2Q|} = \frac{|F_1A|}{|F_2A|} = \text{const}.$$

□

Consider the map $f\colon U \to \mathbb{R}^2$ such that for each point $A \in U$

$$f(A) := (\ln d(F_1, \alpha(A)), -\ln d(F_2, \beta(A))).$$

Choose a subdomain (still denoted by U) in which the differential of f is nonzero. From Lemma 3.1 it follows that f maps the intersections of the left tangent lines to γ, the right tangent lines to γ, and the circles from the pencil with U to the segments of the lines $x = \text{const}$, $y = \text{const}$, and $x + y = \text{const}$. In particular, three transversal curves $\alpha(A)$, $\beta(A)$, and $\omega(A)$ have transversal f-images. So the differential of f in U is nondegenerate because it is nonzero. Thus the restriction of the map f to an appropriate subdomain $\Omega \subset U$ is a diffeomorphism. Theorem 1.1(b) is proved.

Proof of Theorem 1.1(c). Denote by γ the given conic. Denote by F_1 the given focus of γ. For each point A denote by $\alpha(A)$ the line belonging to the pencil and passing through A. Denote by $\beta(A)$ the left tangent line to γ passing through A.

Let $(\phi_1, \phi_2) \subset [0, \pi/2)$ be an interval such that for each $\phi \in (\phi_1, \phi_2)$ there is a point T on the conic γ satisfying $\angle(\alpha(T), \beta(T)) = \phi$. Let U be an appropriate domain such that for each point $A \in U$ there exist $\alpha(A)$, $\beta(A)$ and $\angle(\alpha(A), \beta(A)) \in (\phi_1, \phi_2)$.

Lemma 3.2. *The locus of all the points $A \in U$ such that $\angle(\alpha(A), \beta(A)) = \phi$, where $\phi \in (\phi_1, \phi_2)$ is fixed, is an arc of the circle doubly tangent to the conic γ such that the center of this circle lies on the minor axis of the conic.*

Proof. Denote by P the projection of the focus F_1 onto the line containing $\beta(A)$; see Figure 10 to the left. By *pedal circle property of conics* (see [1, §11.8]) the projections of the focus F_1 of the conic γ onto the tangent lines of γ lie on one circle ω_0. The center of the circle ω_0 coincides with the center of the conic.

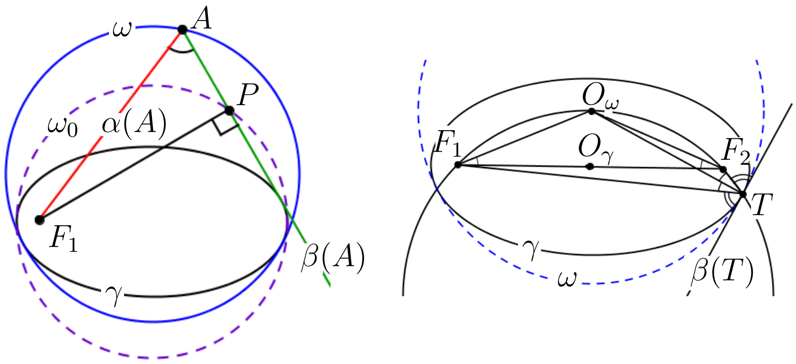


FIGURE 10. To the proof of Lemma 3.2.

Note that $|F_1A|/|F_1P| = 1/|\sin\angle(\alpha(A), \beta(A))| = 1/|\sin\phi|$ and $\angle(F_1P, F_1A) = \frac{\pi}{2} - \angle(\alpha(A), \beta(A)) = \frac{\pi}{2} - \phi$. The image of the circle ω_0 under the composition of the counterclockwise rotation about F_1 through the angle $\frac{\pi}{2} - \phi$ and the homothety with the center F_1 and the coefficient of dilation $1/|\sin\phi|$ is a circle ω. Evidently, the given locus is $\omega \cap U$.

Let us prove that ω is doubly tangent to γ and the center of ω lies on the minor axis of γ. Denote by O_γ the center of γ and by O_ω the center of ω; see Figure 10 to the right. Since $\phi \in (\phi_1, \phi_2)$, it follows that there is $T \in \gamma$ such that $\angle(\alpha(T), \beta(T)) = \phi$. Let F_2 be the other focus of γ. Since ω is the image of ω_0 under the above composition, we have $\angle F_1O_\gamma O_\omega = \angle F_1PA = \frac{\pi}{2}$ and $\phi = \angle F_1AP = \angle F_1O_\omega O_\gamma$. So the point O_ω lies on the bisector of the segment F_1F_2, $\angle(F_1F_2, F_1O_\omega) = \angle(F_2O_\omega, F_2F_1)$, and $\angle F_1O_\omega F_2 = 2\phi$. By the construction of the point T and the *optical property of conics* (See [2, §1.3]) we have $\phi = \angle(TF_1, \beta(T)) = \angle(\beta(T), TF_2)$. Thus $\angle(O_\omega F_1, O_\omega F_2) = \angle(TF_1, TF_2)$. So the points F_1, O_ω, F_2, and T are cocyclic. Since $\angle(F_1F_2, F_1O_\omega) = \angle(F_2O_\omega, F_2F_1)$ and the points F_1, O_ω, F_2, and T are cocyclic we have $\angle(TO_\omega, TF_1) = \angle(TF_2, TO_\omega)$. So $O_\omega T$ is perpendicular to $\beta(T)$. Hence, $\beta(T)$ is the tangent line to ω at T. Thus T is a tangency point of ω and γ. By reflection symmetry, the point T' symmetric to T with respect to the minor axis of γ is another tangency point of ω and γ. Thus ω and γ are doubly tangent. □

Denote by $\omega(A)$ the locus of all the points $X \in U$ such that $\angle(\alpha(X), \beta(X)) = \angle(\alpha(A), \beta(A))$. Let λ be the major axis of the conic γ.

Consider the map $f\colon U \to \mathbb{R}^2$ such that for each point $A \in U$

$$f(A) := (\angle(\alpha(A), \lambda), \angle(\lambda, \beta(A))).$$

Choose a subdomain (still denoted by U) in which f is continuous, the differential of f is nonzero, and for each point $A \in U$ the curves $\alpha(A)$, $\beta(A)$, and $\omega(A)$ are transversal. By Lemma 3.2 it follows that f maps the intersections of lines from the pencil, the left tangent lines to γ, and considered circles doubly tangent to γ with U to the segments of the lines $x = \text{const}$, $y = \text{const}$ and $x + y = \text{const}$. In particular, three transversal curves $\alpha(A)$, $\beta(A)$, and $\omega(A)$ have transversal f-images. So the differential of f in U is nondegenerate because it is nonzero. Thus the restriction of the map f to an appropriate domain $\Omega \subset U$ is a diffeomorphism. Theorem 1.1(c) is proved.

Proof of Theorem 1.1(d). Denote by γ the given parabola. Denote by F and δ the focus and the directrix of the parabola. Consider the hyperbolic pencil with the limiting points F and $L \in \delta$. Let U be an appropriate domain such that for each point $A \in U$ the left and the right tangent lines $\alpha(A)$ and $\beta(A)$ to the parabola γ passing through A, and the circle $\omega(A)$ from the pencil passing through A exist and intersect transversely.

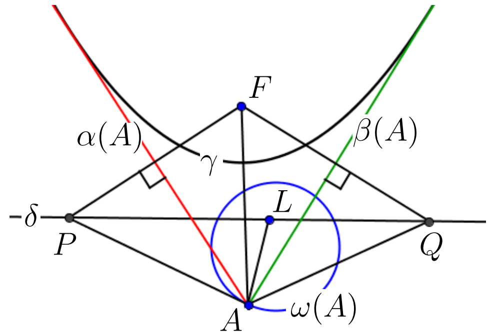


FIGURE 11. To the proof of Lemma 3.3.

Consider the line passing through F and perpendicular to $\alpha(A)$; see Figure 11. Denote by P the intersection of this line and δ. Consider the line passing through F and perpendicular to $\beta(A)$. Denote by Q the intersection of this line and δ.

Set

$$s(A) := \frac{|PL| \cdot |\cos\angle(\alpha(A), \delta)|}{|FP|} \quad \text{and} \quad t(A) := \frac{|QL| \cdot |\cos\angle(\delta, \beta(A))|}{|FQ|}.$$

Lemma 3.3. *For a fixed circle ω from the pencil the product $s(A) \cdot t(A)$ does not depend on the point $A \in \omega \cap U$.*

Proof. By a well-known property of a parabola, *the point A is the center of the circle circumscribed about $\triangle FPQ$* (see [2, Lemma 1.2]). So $|AF| = |AP| = |AQ| = R$, where R is a radius of the circle circumscribed about $\triangle FPQ$. By the sine theorem we have

$$2R = \frac{|FP|}{|\sin\angle(QF, QP)|} = \frac{|FP|}{|\cos\angle(\delta, \beta(A))|} \text{ and } 2R = \frac{|FQ|}{|\sin\angle(PQ, PF)|} = \frac{|FQ|}{|\cos\angle(\alpha(A), \delta)|}.$$

By the well-known property of a power of a point with respect to a circle we have $|AL|^2 = R^2 - |PL| \cdot |QL|$. Then

$$s(A) \cdot t(A) = \left(\frac{|PL| \cdot |\cos\angle(\alpha(A), \delta)|}{|FP|}\right) \cdot \left(\frac{|QL| \cdot |\cos\angle(\delta, \beta(A))|}{|FQ|}\right) =$$
$$= \frac{|PL| \cdot |QL|}{4R^2} = \frac{1}{4}\left(1 - \left(\frac{|AL|}{|AF|}\right)^2\right) = \text{const},$$

where the fourth equality follows from the geometric characterization of a hyperbolic pencil of circles (see the proof of Theorem 1.1(b)).

□

Consider the map $f \colon U \to \mathbb{R}^2$ such that for each point $A \in U$

$$f(A) := (\ln s(A), \ln t(A)).$$

Choose a subdomain (still denoted by U) in which the differential of f is nonzero. From Lemma 3.3 it follows that f maps the intersections of the left tangent lines to γ, the right tangent lines to γ, and the circles from the pencil with U to the segments of the lines $x = \text{const}$, $y = \text{const}$, and

$x+y=$ const. In particular, three transversal curves $\alpha(A)$, $\beta(A)$, and $\omega(A)$ have transversal f-images. So the differential of f in U is nondegenerate. Thus the restriction of the map f to an appropriate subdomain $\Omega \subset U$ is a diffeomorphism. Theorem 1.1(d) is proved.

Proof of Theorem 1.1(e). Denote by γ the given ellipse. Denote by F_1, F_2, and O_γ the foci and the center of γ. Consider the elliptic pencil of circles with the vertices F_1 and F_2. Consider the Cartesian coordinate system such that O_γ is the origin and the line containing the major axis λ is the Ox-axis. Without loss of generality assume that the minor axis of γ has length 2. Let a circle with the center on the major axis λ be doubly tangent to γ. The line passing through the tangency points separates the circle into two circular arcs: the "left" tangent circular arc and the "right" tangent circular arc. Let U be an appropriate domain such that for each point $A \in U$ the left and the right tangent circular arcs $\alpha(A)$ and $\beta(A)$ to γ passing through A, and the circle $\omega(A)$ from the pencil passing through A intersect transversely, the center O_α of the circle containing $\alpha(A)$ "lies to the left" of O_γ, the center O_β of the circle containing $\beta(A)$ "lies to the right" of O_γ, and also $s(A) > t(A)$, where $s(A) := |O_\alpha O_\gamma|$ and $t(A) := |O_\beta O_\gamma|$; see Figure 12.

Lemma 3.4. *For a fixed circle ω the product $\frac{1-s^2(A)}{s^2(A)} \cdot \frac{1-t^2(A)}{t^2(A)}$ does not depend on the point $A \in \omega \cap U$.*

Proof. Let P and Q be points on the circles containing $\alpha(A)$ and $\beta(A)$, respectively, such that $O_\alpha P$ and $O_\beta Q$ are perpendicular to λ. Consider the circle ω_0 such that the minor axis of the ellipse γ is a diameter of ω_0. Since the eccentricity of γ is equal to $\frac{1}{\sqrt{2}}$ we get that the foci F_1 and F_2 lie on ω_0.

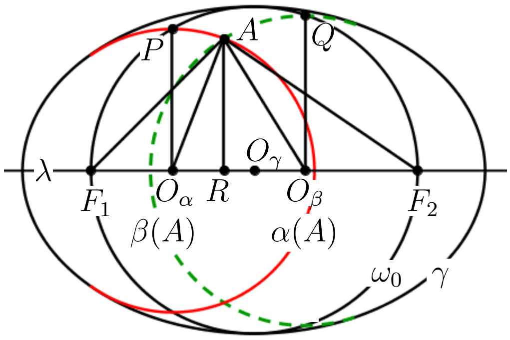


FIGURE 12. To the proof of Lemma 3.4.

Lemma 3.5. *The points P and Q lie on the circle ω_0.*

Proof. We are going to prove that for each point $Y \in \omega_0$ the circle ω with the center at the projection X of Y onto λ and the radius $|XY|$ is doubly tangent to γ. Let $(a, 0)$ be coordinates of X. Then $|XY| = \sqrt{1-a^2}$. So the circle ω has the equation of the form $(x-a)^2 + y^2 = 1 - a^2$. We can rewrite this equation in the form $g(x, y, a) = 0$, where $g(x, y, a) = (x-a)^2 + y^2 + a^2 - 1$. Let us compute the envelope of this family of circles. It is well-known that the equation of such envelope can be obtained from the following system

$$(1)\ g(x, y, a) = 0 \text{ and } \ (2)\ \frac{\partial}{\partial a} g(x, y, a) = 0$$

by eliminating a. We have $\frac{\partial}{\partial a} g(x, y, a) = -2x + 4a$. By (2) we have $a = x/2$. Substituting $a = x/2$ in (1), we have the envelope equation $\frac{x^2}{2} + y^2 = 1$. This is the equation of γ. Thus ω is tangent to γ. Since the center of ω lies on the major axis, ω is doubly tangent to γ. $\square$

Let us continue the proof of Lemma 3.4. Further we denote $s(A)$ and $t(A)$ simply by s and t.

From Lemma 3.5 it follows that $|AO_\alpha| = |PO_\alpha| = \sqrt{1-s^2}$ and $|AO_\beta| = |QO_\beta| = \sqrt{1-t^2}$. Let R be the projection of A onto λ. Then $|RO_\alpha|^2 - |RO_\beta|^2 = |AO_\alpha|^2 - |AO_\beta|^2 = t^2 - s^2$. Since $|O_\alpha O_\beta| = s + t$ we have $|RO_\alpha| = t$ and $|RO_\beta| = s$. Hence $|AR| = \sqrt{O_\alpha A^2 - O_\alpha R^2} = \sqrt{O_\alpha P^2 - O_\alpha R^2} = \sqrt{1 - s^2 - t^2}$, $|F_1 R| = 1 - s + t$, and $|F_2 R| = 1 + s - t$. Thus $|F_1 A|^2 = 1 - s^2 - t^2 + (1 - s + t)^2 = 2(1 + (t - s) - st)$ and $|F_2 A|^2 = 1 - s^2 - t^2 + (1 + s - t)^2 = 2(1 + (s - t) - st)$.

For each point $A \in \omega \cap U$ we have $\angle F_1AF_2 = \text{const}$. By the cosine theorem we have

$$\cos\angle F_1AF_2 = \frac{|F_1F_2|^2 - |F_1A|^2 - |F_2A|^2}{2|F_1A|\cdot|F_2A|} = \frac{4 - 2(1+(t-s)-st) - 2(1+(s-t)-st)}{\sqrt{4(1+(t-s)-st)\cdot(1+(s-t)-st)}} =$$
$$= \frac{2st}{\sqrt{(1-s^2)(1-t^2)}}.$$

Thus $\frac{1-s^2(A)}{s^2(A)}\cdot\frac{1-t^2(A)}{t^2(A)} = \frac{4}{\cos^2\angle F_1AF_2} = \text{const}$. Lemma 3.4 is proved.

Consider the map $f\colon U \to \mathbb{R}^2$ such that for each point $A \in U$

$$f(A) := \left(\ln\frac{1-s^2(A)}{s^2(A)}, \ln\frac{1-t^2(A)}{t^2(A)}\right).$$

Choose a subdomain (still denoted by U) in which the differential of f is nonzero. From Lemma 3.4 it follows that f maps the intersections of the left tangent circular arcs to γ, the right tangent circular arcs to γ, and the circles from the pencil with U to the segments of the lines $x = \text{const}$, $y = \text{const}$, and $x + y = \text{const}$. In particular, three transversal curves $\alpha(A)$, $\beta(A)$, and $\omega(A)$ have transversal f-images. So the differential of f in U is nondegenerate. Thus the restriction of the map f to an appropriate subdomain $\Omega \subset U$ is a diffeomorphism. Theorem 1.1(e) is proved. □

Now let us prove Theorem 1.2. We work over the field of complex numbers except otherwise is explicitly indicated. Denote by $\mathbb{P}^3$ the 3-dimensional complex projective space with homogeneous coordinates $x : y : z : w$. The *infinitely distant plane* is the plane $w = 0$. The *absolute conic* is given by the equations $x^2 + y^2 + z^2 = 0$, $w = 0$. A (*nondegenerate*) *complex circle* is an irreducible conic in $\mathbb{P}^3$ having two distinct common points with the absolute conic. Clearly, a circle in $\mathbb{R}^3$ is a subset of a complex circle.

The set of projective lines in $\mathbb{P}^3$ can be naturally identified with the *Plücker quadric* $Gr(2,4)$ in $\mathbb{P}^5$: the line passing through points $x_1 : y_1 : z_1 : w_1$ and $x_2 : y_2 : z_2 : w_2$ is identified with the point $x_1y_2 - x_2y_1 : x_1z_2 - x_2z_1 : x_1w_2 - x_2w_1 : y_1z_2 - y_2z_1 : y_1w_2 - y_2w_1 : z_1w_2 - z_2w_1$.

Proof of Theorem 1.2. Let $\Phi \subset \mathbb{R}^3$ be a surface covered by a family of real line segments and a family of real circular arcs simultaneously. The complex lines and complex circles containing the members of these families are called *generating lines* and *generating circles*, respectively. Hereafter assume that $\Phi \subset \mathbb{R}^3$ is not a plane.

Lemma 3.6. *The planes of the generating circles are parallel to each other.*

To prove the lemma, we need several auxiliary propositions. The first one is essentially known.

Proposition 3.7. *Let $\gamma_1, \gamma_2, \gamma_3 \subset \mathbb{P}^3$ be pairwise distinct irreducible algebraic curves.*
(1) *The set $J(\gamma_1) \subset Gr(2,4)$ of all the lines passing through the curve γ_1 is an algebraic subset of $Gr(2,4)$.*
(2) *The set $J(\gamma_1,\gamma_2) \subset Gr(2,4)$ of all the lines passing through each of the curves γ_1,γ_2 but not passing through their intersection $\gamma_1\cap\gamma_2$ is a piece of a 2-dimensional algebraic surface in $Gr(2,4)$.*
(3) *The union $J(\gamma_1,\gamma_2,\gamma_3) \subset \mathbb{P}^3$ of all the lines passing through each of the curves $\gamma_1,\gamma_2,\gamma_3$ but not passing through their pairwise intersections is a piece of an algebraic surface in $\mathbb{P}^3$.*

Remark. Moreover, one can check that $J(\gamma_1,\gamma_2)$ and $J(\gamma_1,\gamma_2,\gamma_3)$ are quasi-projective varieties.

Proof. (1) The set S of pairs $(P,\lambda) \in \mathbb{P}^3 \times Gr(2,4)$ such that the point P belongs to the line λ is algebraic. The set $J(\gamma_1)$ is the image of the projection $S \cap (\gamma_1 \times Gr(2,4)) \to Gr(2,4)$ and by [8, Theorem 3.12] is also algebraic.

(2) Consider the polynomial map $(\gamma_1 - \gamma_1\cap\gamma_2)\times(\gamma_2 - \gamma_1\cap\gamma_2) \to Gr(2,4)$ taking a pair of points (P,Q) to the line passing through P and Q. Then $J(\gamma_1,\gamma_2)$ is the image of this map. The map $(\gamma_1 - \gamma_1\cap\gamma_2)\times(\gamma_2 - \gamma_1\cap\gamma_2) \to J(\gamma_1,\gamma_2)$ is a finite covering by a piece of a 2-dimensional irreducible surface. Thus $J(\gamma_1,\gamma_2)$ is a piece of a 2-dimensional irreducible algebraic surface $\bar{J}(\gamma_1,\gamma_2) \subset Gr(2,4)$.

(3) By (2) the set $\bigcap_{i\neq j} J(\gamma_i,\gamma_j)$ is a (possibly empty) piece of the algebraic set $\bigcap_{i\neq j}\bar{J}(\gamma_i,\gamma_j)$. Since for each $i \neq j$ the surface $\bar{J}(\gamma_i,\gamma_j)$ is 2-dimensional and irreducible it follows that $\dim\bigcap_{i\neq j}\bar{J}(\gamma_i,\gamma_j) \leq$

2. If $\dim \bigcap_{i\neq j} \bar{J}(\gamma_i, \gamma_j) = 2$ then $\bar{J}(\gamma_1, \gamma_2) = \bar{J}(\gamma_2, \gamma_3) = \bar{J}(\gamma_3, \gamma_1)$ and by [10, Theorem 1, $n = 3$] it follows that $J(\gamma_1, \gamma_2, \gamma_3)$ is a piece of a plane. If $\dim \bigcap_{i\neq j} \bar{J}(\gamma_i, \gamma_j) \leq 1$ then $J(\gamma_1, \gamma_2, \gamma_3)$ is a piece of an algebraic surface as the image of the projection $S \cap (\mathbb{P}^3 \times \bigcap_{i\neq j} J(\gamma_i, \gamma_j)) \to \mathbb{P}^3$. □

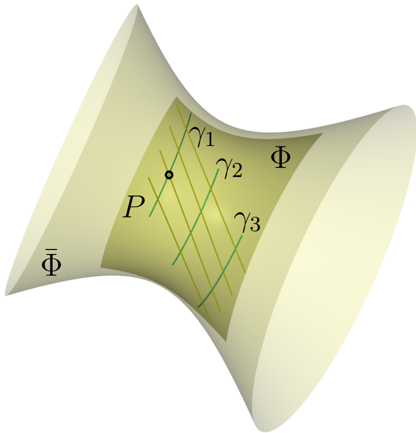


FIGURE 13. To the proof of Proposition 3.8.

Proposition 3.8. (Cf. [17, Theorems 1 and 2]) *The surface $\Phi \subset \mathbb{R}^3$ is contained in an irreducible ruled algebraic surface $\bar{\Phi} \subset \mathbb{P}^3$. The family of generating lines is a piece of an irreducible algebraic curve in $Gr(2, 4)$.*

Proof. Take a point $P \in \Phi$; see Figure 13. Draw a circular arc $\gamma_1 \subset \Phi$ through the point P. Draw the line segments in Φ from our continuous family through each point of γ_1. Since the drawn segments are transversal to γ_1, the circular arcs contained in Φ form a continuous family, and Φ is smooth it follows that there are arcs $\gamma_2, \gamma_3 \subset \Phi$ (sufficiently close to γ_1) which intersect each of the drawn segments. Let $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$ be the complex circles in $\mathbb{P}^3$ containing the arcs $\gamma_1, \gamma_2, \gamma_3$. Let $\Phi' \subset \Phi$ be the union of those drawn segments, whose lines do not pass through the intersections $\bar{\gamma}_1 \cap \bar{\gamma}_2$, $\bar{\gamma}_2 \cap \bar{\gamma}_3$, $\bar{\gamma}_3 \cap \bar{\gamma}_1$. By construction $\Phi' \subset J(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$. Thus by Proposition 3.7(3) the collar Φ' is a piece of an algebraic surface. Take $\bar{\Phi}$ to be an irreducible component of the surface containing a closed 2-dimensional subset of the initial surface Φ including the point P. (If there are no such components, e.g., $\Phi' = \emptyset$, then the drawn segments sufficiently close to the point P form a quadratic cone with vertex at one of the intersection points of the circles $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3$; in this case set $\bar{\Phi}$ to be this cone.) The algebraic surface $\bar{\Phi} \subset \mathbb{P}^3$ does not depend on the point P because the smooth surface $\Phi \subset \mathbb{R}^3$ cannot jump from one irreducible algebraic surface to another.

By Proposition 3.7(2) the lines containing the drawn segments form a piece of the algebraic set $\bigcap_{i\neq j} \bar{J}(\gamma_i, \gamma_j)$. Since Φ is not a plane it follows that the latter set is an algebraic curve [10, Theorem 1, $n = 3$]. Take $\alpha \subset Gr(2, 4)$ to be an irreducible component of this curve containing the lines sufficiently close to the point P. Clearly, the union of the lines of the whole curve α covers $\bar{\Phi}$, i.e., $\bar{\Phi}$ is ruled. It remains to show that the curve α does not depend on the choice of the point P. Indeed, assume that the generating lines through a neighborhood of another point P' form a curve $\alpha' \subset Gr(2, 4)$ distinct from α. Then $\bar{\Phi}$ is doubly ruled and hence it is a quadric. Thus $\alpha \cap \alpha' = \emptyset$ and hence the generating lines cannot form a continuous family. This contradiction proves the proposition. □

Hereafter any line belonging to the irreducible algebraic curve in $Gr(2, 4)$ containing the generating lines is also called a *generating line*. No confusion will arise from this.

Proposition 3.9. (Cf. [14, Lemma 1.3]) *If $\gamma \subset \bar{\Phi}$ is an irreducible algebraic curve distinct from a generating line then each generating line intersects γ.*

Proof. Since $\bar{\Phi} \subset \mathbb{P}^3$ is ruled it follows that there is a generating line through each point of γ. Thus infinitely many generating lines belong to $J(\gamma)$. By Proposition 3.7(1) the set $J(\gamma)$ is algebraic, hence the whole irreducible algebraic curve in $Gr(2, 4)$ formed by generating lines is contained in $J(\gamma)$. □

Proposition 3.10. *The surface $\bar{\Phi} \subset \mathbb{P}^3$ does not contain the absolute conic.*

Proof. Assume that $\bar{\Phi}$ contains the absolute conic γ. Then by Proposition 3.9 all the generating lines intersect γ. Since γ has no real points it follows that there are no real generating lines (except infinitely distant). This contradiction proves the proposition. □

Proof of Lemma 3.6. By Propositions 3.8 and 3.10 the intersection of the surface $\bar{\Phi} \subset \mathbb{P}^3$ with the absolute conic is a finite set I. The plane of each generating circle intersects the infinitely distant plane by a line joining two points of the set I. Since the set I is finite and the family of generating circles is continuous it follows that all these lines coincide, that is, all the planes of the circles are parallel. □

Lemma 3.11. *There are infinitely many generating circles γ such that the projective plane $\Pi \subset \mathbb{P}^3$ of γ intersects the surface $\bar{\Phi} \subset \mathbb{P}^3$ only at the points of γ.*

To prove the lemma, we need the following auxiliary proposition.

Proposition 3.12. *The projective planes $\Pi \subset \mathbb{P}^3$ of infinitely many generating circles γ do not contain generating lines.*

Proof. Assume that the projective planes of only finitely many generating circles do not contain generating lines. Thus the projective planes $\Pi \subset \mathbb{P}^3$ of infinitely many generating circles γ contain generating lines λ_γ. By Lemma 3.6 all the projective planes Π intersect the absolute conic by the same 2-point set $I = \{P, Q\}$. It suffices to consider the following 3 cases.

Case 1: *For some γ we have $\lambda_\gamma \cap I = \emptyset$.* Take a generating circle $\gamma' \not\subset \Pi$. Then $\Pi \cap \gamma' = I$ by Lemma 3.6. Then by Proposition 3.9 we have $\emptyset \neq \lambda_\gamma \cap \gamma' \subset \Pi \cap \gamma' - I = \emptyset$, a contradiction.

Case 2: *For infinitely many γ the intersection $\lambda_\gamma \cap I$ consists of a single point.* All the lines λ_γ with this property are pairwise distinct because by Lemma 3.6 they are contained in the planes through I. We get infinitely many generating lines intersecting I. Thus by Proposition 3.8 *each* generating line must intersect I. Then the generating lines through each point of a generating circle γ must intersect I, a contradiction.

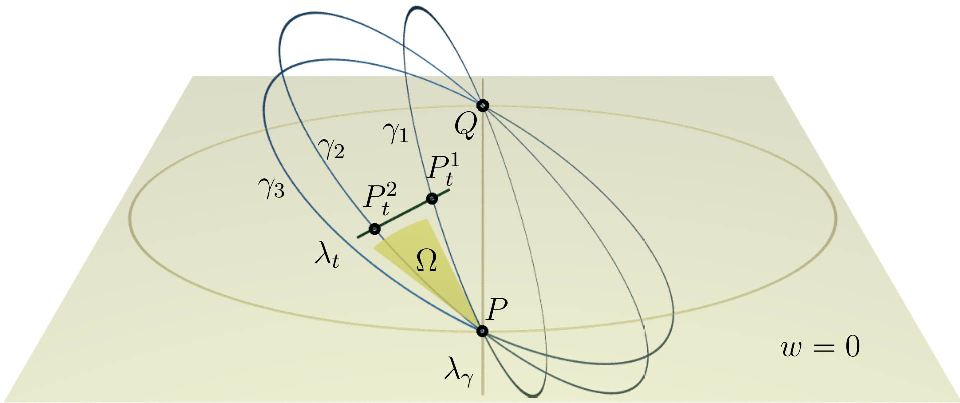


FIGURE 14. To the proof of Proposition 3.12 case (3).

Case 3: *For some γ we have $\lambda_\gamma \cap I = I$*; see Figure 14. Then λ_γ is the infinitely distant line of the projective plane Π. Since the generating lines form an algebraic curve in $Gr(2,4)$ it follows that there is a sequence of generating lines $\lambda_t \neq \lambda_\gamma$ converging to λ_γ. Since there are only finitely many generating lines crossing I, we may assume that $\lambda_t \cap I = \emptyset$. Take 3 pairwise noncoplanar generating circles γ_1, γ_2, and γ_3. By Proposition 3.9 for each $i = 1, 2, 3$ the line λ_t intersects the circle γ_i at some point P_t^i. Each of the 3 points P_t^i converges to one of the 2 points of the set I. By the pigeonhole principle we may assume that, say, P_t^1, P_t^2 converge to P. Then the plane $PP_t^1P_t^2$ converges to the projective plane Ω containing projective tangent lines to γ_1 and γ_2 at the point P (the tangent lines are distinct because γ_1 and γ_2 are not coplanar). The projective plane Ω has a unique common point with γ_1 while $\lambda_\gamma \subset \Omega$ intersects γ_1 by the 2-point set I. This contradiction proves the proposition. □

Proof of Lemma 3.11. By Proposition 3.12 there are infinitely many generating circles γ such that $\bar{\Phi} \cap \Pi$ does not contain generating lines. Then $\bar{\Phi} \cap \Pi = \gamma$ because by Proposition 3.9 the generating line through each point of $\bar{\Phi} \cap \Pi$ crosses γ. □

Proof of Theorem 1.2. By Lemma 3.11 we have $\bar{\Phi} \cap \Pi = \gamma$ for infinitely many generating circles γ. So there is a generating circle γ with this property which is not a singular curve of the surface $\bar{\Phi}$, because $\bar{\Phi}$ contains only finitely many singular curves. By Proposition 3.12 the plane Π does not touch the surface $\bar{\Phi}$ along the curve γ. Thus the circle γ has multiplicity 1 in the curve $\bar{\Phi} \cap \Pi$. By the Bezout theorem [8, Theorem 18.3] the degree of the surface $\bar{\Phi} \subset \mathbb{P}^3$ equals to the degree of its planar section (with multiplicity), and thus equals to 2. Since $\bar{\Phi}$ contains both real lines and real circles, it is either a one-sheeted hyperboloid, or a quadratic cone, or an elliptic cylinder. Theorem 1.2 is proved. □

Proof that Theorem 2.4 is equivalent to the Wunderlich Theorem 2.3. It is well-known that *any one-parameter group of Möbius transformations is conjugate to a one-parameter group of either dilations, or rotations, or translations, or loxodromic transformations*

The orbits of a one-parameter group of loxodromic transformations are not circular arcs. Thus it suffices to consider the following three cases.

Case 1. *$\mathcal{M}_t$ is conjugate to a one-parameter group of dilations.* The orbits of the group are arcs of circles belonging to one elliptic pencil. It's easy to verify that there are the following three possibilities: 1) there are two distinct (possibly complex) circles from the pencil tangent to ω_i; 2) ω_i passes through a vertex of the pencil; 3) ω_i belongs to a hyperbolic pencil with limiting points at the vertices of the pencil. If a circle ω_i passes through a vertex of the pencil then the images of ω_i are circles belonging to the parabolic pencil with the same vertex. If ω_i belongs to a hyperbolic pencil with limiting points at the vertices of the pencil then the images of ω_i are circles belonging to the hyperbolic pencil. Overwise the images of ω_i are circles tangent to two distinct (possibly complex) fixed circles from the pencil.

Case 2. *$\mathcal{M}_t$ is conjugate to a one-parameter group of rotations.* The orbits of the group are arcs of circles belonging to one hyperbolic pencil. The images of ω_i are circles tangent to two distinct (possibly null) fixed circles from the pencil.

Case 3. *$\mathcal{M}_t$ is conjugate to a one-parameter group of translations.* The orbits of the group are arcs of circles belonging to one parabolic pencil. If a circle ω_i passes through the vertex of the pencil then the images of ω_i are circles belonging to the parabolic pencil with the same vertex. If ω_i does not pass through a vertex of the pencil then the images of ω_i are circles tangent to two distinct fixed circles from the pencil.

□

4. Examples

Let us give several illustrative examples to Theorem 1.2. We begin with examples over the field of complex numbers.

Example 4.1. The irreducible complex cyclide $(x^2+y^2+z^2)^2 + (x+iy)^2 - z^2 = 0$, which can be parametrized in $\mathbb{P}^3$ as $t^2-1 : i(t^2-1-2st) : s(t^2+1) : s(t^2-1)+4t$, is covered by a family of complex lines $t = \text{const}$ and a family of complex circles $s = \text{const}$ simultaneously.

Example 4.2. A general position degree 3 complex cyclide is covered by a family of complex circles and contains 27 complex lines; however, the surface contains no families of complex lines.

Proof. Any cyclide is covered by at least one family of complex circles [6, Chapter VII]. A general position degree 3 cyclide is nonsingular and hence contains exactly 27 complex lines. □

Example 4.3. A general position ruled complex cubic surface is covered by a family of complex lines and contains 15 complex circles; however, the surface contains no families of complex circles.

Proof of Example 4.2. Consider the intersection I of the ruled cubic surface with the absolute conic. In general position it consists of six distinct points. Let P, Q be two of the intersection points. Let λ_1 be the line passing through P, Q, and let R be the third common point of λ_1 and the surface. In general position $R \neq P, Q$. Consider the ruling λ_2 passing through R. Take the plane Π containing the lines λ_1 and λ_2. The intersection of Π and the surface consists of the ruling λ_2 and a curve γ of degree 2. The curve γ is irreducible once the plane Π contains neither the singular line nor the isolated line of the surface, i.e., the points P, Q, R do not belong to these lines (this follows from the classification of ruled cubic surfaces [14, Section 2]). Thus in general position γ is a conic through P and Q, i.e., a complex circle. There are 15 ways to choose two distinct points $P, Q \in I$ leading to 15 complex circles on the surface. $\square$

Finally, let us proceed to examples over the field of real numbers. Their obvious proofs are omitted.

Example 4.4. (See Figure 15 to the left.) The surface $(x^2-z^2)(3z-2)+(y-z)(3yz-2y-4z+2)=0$ is covered by a family of circles in the planes $z=$ const and contains 4 lines: $l_1(t)=(t,t,t)$, $l_2(t)=(-t,t,t)$, $l_3(t)=(t,1-t,2t)$, $l_4(t)=(-t,1-t,2t)$; however, the surface contains no families of lines.

Example 4.5. (See [16, Section 1] and Figure 15 in the middle.) The surface $(x^2+y^2+z^2+3)^2-4y^2z^2-16x^2-12y^2=0$ obtained by translation of a circle along another one is covered by 2 families of circles in the planes $y=$ const and $z=$ const but it is not a cyclide.

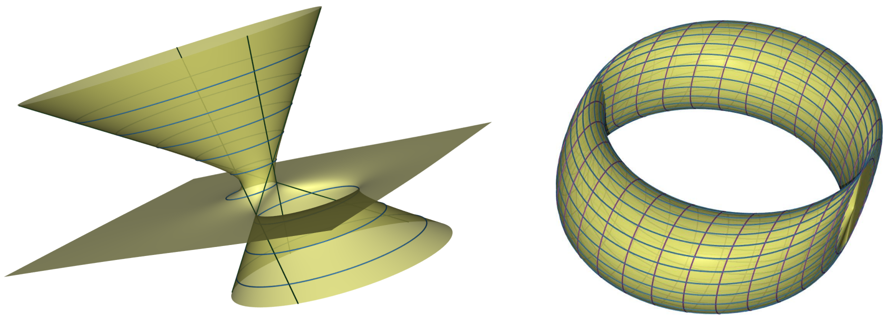


FIGURE 15. Left: a surface covered by 1 family of circles and containing 4 lines (Example 4.4). Right: a surface covered by 2 families of circles (Example 4.5).

5. Open problems

The following open problems may be a good warm-up before attacking the Blaschke–Bol Problem.

Problem 5.1. *Web Transformation Problem.* Prove that in Theorem 1.1(b) the hyperbolic pencil can be replaced by the elliptic one with the vertices at the limiting points of the initial pencil. Analogous replacement is not possible for Theorem 1.1(d).

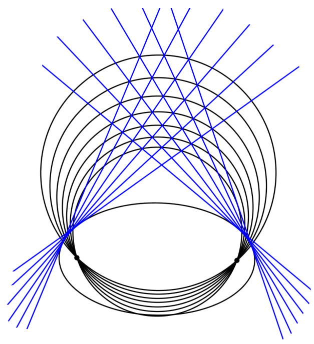


FIGURE 16. To Problem 5.1.

A *cubic series* is a family of circles $a(t)(x^2+y^2)+b(t)x+c(t)y+d(t)=0$, where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are polynomials of degree ≤ 3.

Problem 5.2. *Cubic Series.* Prove that the set of circles $(1-t^3)(x^2+y^2)+2(1+t)x+2(t^2+t^3)y-1-t^3=0$, where $t\in\mathbb{R}$, counted triply, contains a web. Find all cubic series that contain a web.

It is easy to see that examples of webs in Problem 5.1 and Problem 5.2 are not particular cases of the examples considered in §1, 2.

Problem 5.3. *Complement Problem.* (A.A. Zaslavsky and I.I. Bogdanov, private communication) Given two sets of lines $x = \text{const}$ and $y = \text{const}$, find all sets of circles which together with them contain a web.

Our last problem concerns 3-dimensional isotropic geometry; see references in [16]. An *isotropic circle* is either a parabola with the axis parallel to Oz or an ellipse whose projection onto the plane Oxy is a circle. An *isotropic plane* is a plane parallel to Oz. An *isotropic sphere* is a paraboloid of revolution with the axis parallel to Oz.

Problem 5.4. *Webs of isotropic circles.* Find all webs of isotropic circles in the isotropic plane and on all surfaces except planes and isotropic spheres; see Figure 4 to the right.

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