Maximization of the first nontrivial eigenvalue on the surface of genus two

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Abstract

The first nontrivial eigenvalue of the Laplacian can be considered as a functional on the space of all Riemannian metrics of unit volume on a fixed surface. In this paper we prove that for the surface of genus 2 the supremum of this functional is equal to 16π. This provides a positive answer to the conjecture by Jakobson, Levitin, Nadirashvili, Nigam and Polterovich.
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1 Introduction

Let M be a closed surface and g be a Riemannian metric on M. Let us consider the associated Laplace-Beltrami operator Δ acting on the space of smooth functions on M,

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right).$$

It is well-known that the spectrum of Δ is non-negative and consists only of eigenvalues, each eigenvalue has a finite multiplicity and the eigenfunctions are smooth. Let us denote the eigenvalues of Δ by

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \lambda_3(M,g) \leqslant \dots,$$

where eigenvalues are written with multiplicities.

Let us fix M and consider eigenvalues as functionals on the space of all Riemannian metrics. These functionals possess the following rescaling property,

$$\lambda_i(M,tg) = \frac{\lambda_i(M,g)}{t}.$$

Thus, in order to get scale-invariant functionals on the space of Riemannian metrics one has to normalize the eigenvalue functionals. The most natural way to do it is to multiply by the area, so we consider the following functionals

$$\Lambda_i(M,g) = \mathrm{Area}_g(M)\lambda_i(M,g).$$

What is the supremum of these functionals on the space of all Riemannian metrics on M?

It is known that the functionals $\Lambda_i(M,g)$ are bounded from above. In 1980 Yang and Yau proved in the paper [13] that for an orientable surface of genus γ the following inequality holds,

$$\Lambda_1(M,g) \leqslant 8\pi\left[\frac{\gamma+3}{2}\right]. \qquad (1)$$

Later this result was generalized for an arbitrary i. More precisely, Korevaar proved in 1993 in the paper [8] that there exists a universal constant C such that

$$\Lambda_i(M,g) \leqslant C\left[\frac{\gamma+3}{2}\right] i.$$

However, the list of known sharp upper bounds for particular surfaces is quite short.

• In 1970 Hersch proved in the paper [4] that

$$\Lambda_1(\mathbb{S}^2,g) \leqslant 8\pi$$

and the equality is achieved only for the standard metric on $\mathbb{S}^2$.

• In 1982 Li and Yau proved a similar result for the projective plane. It is proved in their paper [9] that

$$\Lambda_1(\mathbb{RP}^2,g) \leqslant 12\pi$$

and the equality is achieved only for the standard metric on $\mathbb{RP}^2$.

• In 1996 Nadirashvili proved in the paper [10] that for the two-dimensional torus

$$\Lambda_1(\mathbb{T}^2,g) \leqslant \frac{8\pi^2}{\sqrt{3}}$$

and the equality is achieved only for the metric on the flat equilateral torus.

• In 2002 Nadirashvili proved in the paper [11] that for the second nontrivial eigenvalue on the sphere one has

$$\Lambda_2(\mathbb{S}^2,g) \leqslant 16\pi.$$

The equality is achieved on the singular metric obtained by gluing together two copies of the standard sphere.

• Finally, the result is also known for $\Lambda_1(\mathbb{K},g)$ where $\mathbb{K}$ denotes the Klein bottle. In 2006 Jakobson, Nadirashvili and Polterovich constructed the extremal metric for the first eigenvalue on the Klein bottle in the paper [5]. Later in the same year El Soufi, Giacomini and Jazar proved the uniqueness of this extremal metric in [3]. Thus, one has the inequality

$$\Lambda_1(\mathbb{K},g) \leqslant 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$$

where the equality achieved on the metric constructed in [5]. Here $E(k)$ is an elliptic integral of the second kind given by the formula

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx.$$

And these are all known results concerning the exact value of $\sup \Lambda_i(M,g)$. However, one has to mention the conjecture made by Jakobson, Levitin, Nadirashvili, Nigam and Polterovich in the paper [6]. They conjectured that the right hand side of Yang-Yau inequality (1) gives the value of the supremum for the surface of genus 2. They presented a candidate for a maximal metric and reduced the proof of their conjecture to an inequality on the first eigenvalue of a mixed Dirichlet-Neuman problem on the quater-sphere. This inequality was verified by numerical calculations. Since the proof used computer calculations, the result was stated as a conjecture. The main result of the present paper is a rigorous proof of the JLNNP-conjecture.

Theorem 1 (JLNNP-conjecture [6])**.** *Let Σ_2 be the surface of genus 2, then one has*

$$\sup \Lambda_1(\Sigma_2, g) = 16\pi.$$

Using the same methods we can prove another interesting theorem.

Definition 1. *Let I be a smooth involution on a manifold M. A function $f \in C^\infty(M)$ is called I-odd (I-even) if $I^* f = -f$ ($I^* f = f$).*

Let the involution I be an isometry. Then I^* commutes with the Laplacian and one can choose the common basis of eigenfunctions, i.e. a basis consisting of I-odd and I-even eigenfunctions of Laplacian. Let us denote the first nontrivial eigenvalue corresponding to I-even eigenfunction by $\lambda_1^{\mathrm{even}}$ and the first eigenvalue corresponding to I-odd eigenfunction is denoted by λ_1^{odd}. The following theorem is a straightforward corollary of our method.

Theorem 2. *Let (M, g) be a Riemannian manifold such that M is diffeomorphic to the 2-dimensional sphere. Let I be a free isometric involution. Then one has the following inequality*

$$\lambda_1^{\mathrm{odd}}(M, g) < \lambda_1^{\mathrm{even}}(M, g)$$

Question. *Does Theorem 2 hold for high-dimensional spheres?*

The paper is organized in the following way. In Section 2 we describe a candidate for a maximal metric. Sections 3.1 and 3.2 contain elementary facts from the nodal geometry used in the proof. In Sections 4 and 5 we complete the proof of Theorem 1. Section 6 is dedicated to the proof of Theorem 2. Finally, in Section 7 we discuss a family of maximal metrics for the functional $\Lambda_1(\Sigma_2, g)$.

2 Bolza surface

We start by introducing the candidate for a maximal metric presented in the paper [6]. The Bolza surface is a hyperelliptic Riemann surface $\mathcal{P}$ given by the equation

$$w^2 = z\frac{(z - e^{\frac{\pi i}{4}})(z - e^{\frac{3\pi i}{4}})}{(z + e^{\frac{\pi i}{4}})(z + e^{\frac{3\pi i}{4}})}. \tag{2}$$

We stick to the notations from the paper [6], so we denote the projection onto Riemann sphere $\bar{\mathbb{C}}$ by Π and the corresponding hyperelliptic involution $(z, w) \mapsto (z, -w)$ by T. We consider $\bar{\mathbb{C}}$ equipped with the metric g_0 induced by a stereographic projection from $\mathbb{S}^2 \subset \mathbb{R}^3$,

$$g_0 = \frac{4dz\, d\bar{z}}{(1 + |z|^2)^2}.$$

Let us endow $\mathcal{P}$ with the pullback $g = \Pi^* g_0$ of the metric g_0. Metric g is not a smooth Riemannian metric but a metric with isolated conical singularities.

Definition 2. *A spectrum of a manifold M endowed with a metric with isolated conical singularities is the spectrum of the Friedrichs extension of the Laplacian, where domain of the Laplacian is the space of smooth functions supported on the complement of the singularities.*

Remark. For any manifold with isolated conical singularities one can always construct a sequence of smooth Riemannian manifolds such that its area as well as its eigenvalues converge to the area and eigenvalues of the initial surface, see e.g. [12].

On the one hand, due to this remark there exists a sequence of surfaces (M_n, g_n) such that $\Lambda_1(M_n, g_n)$ tend to $\Lambda_1(\mathcal{P}, g)$ as n tends to infinity. Therefore one has

$$\sup \Lambda_1(\Sigma_2, g) \geqslant \Lambda_1(\mathcal{P}, g).$$

On the other hand, by Yang-Yau inequality (1)

$$\sup \Lambda_1(\Sigma_2, g) \leqslant 16\pi.$$

Thus, in order to prove Theorem 1 it is sufficient to prove the following theorem.

Theorem 3. *For the first eigenvalue of the Bolza surface endowed with the metric g one has*

$$\Lambda_1(\mathcal{P}, g) = 16\pi.$$

Since T is an isometry there is a decomposition of eigenvalues into the ones corresponding to T-odd eigenfunctions and the ones corresponding to T-even eigenfunctions. Now suppose that λ_1 corresponds to an even eigenfunction. This eigenfunction descends to the eigenfunction on $\bar{\mathbb{C}}$. Then the equalities $\lambda_1(\bar{\mathbb{C}}, g_0) = 2$ and $\mathrm{Area}_g(\mathcal{P}) = 2\,\mathrm{Area}_{g_0}(\bar{\mathbb{C}}) = 8\pi$ imply that $\Lambda_1(\Sigma_2, g) = 16\pi$. Thus, Theorem 3 can be derived from the following theorem.

Theorem 4. *The first eigenfuction of Δ is even with respect to T. In particular for the first eigenvalue λ_1^{odd} corresponding to T-odd eigenfuction one has*

$$\lambda_1^{\mathrm{odd}}(\mathcal{P}, g) > 2.$$

This theorem is stated as a conjecture in the paper [6].

3 Nodal set

In this section we collect all facts from nodal geometry necessary for the proof.

3.1 Nodal domains. For an eigenfunction u on a Riemannian manifold M its nodal set $\mathcal{N}(u)$ is defined as $\mathcal{N}(u) = \{x \in M \,|\, u(x) = 0\}$. A nodal domain of u is a connected component of $M \backslash \mathcal{N}(u)$. One of the most powerful theorems of nodal geometry is the following theorem.

Courant's nodal domain theorem. *The n-th eigenfunction cannot have more than $n+1$ nodal domains.*

Remark. Note that we start our numeration of eigenvalues from λ_0. That is why our statement of Courant's nodal domain theorem differs from the classical one.

Remark. Courant's nodal domain theorem still holds for metrics with isolated conical singularities. The proof stays essentially the same.

If an eigenfunction on $\mathcal{P}$ corresponding to λ_1^{odd} has more than two nodal domains, then λ_1^{odd} can not be the first eigenvalue of the Laplacian on $\mathcal{P}$. Therefore Theorem 4 is an immediate corollary of the following proposition.

Proposition 1. *The space of T-odd functions has a basis consisting of eigenfunctions of the Laplace-Beltrami operator such that any element of this basis has at least three nodal domains.*

In order to prove this proposition we need more properties of nodal set, namely the graph structure.

3.2 Nodal graph. Let u be an eigenfunction of the Laplacian on a surface M. Let us define the vanishing order $\mathrm{ord}_p(u)$ of u at a point $p \in M$ as the lowest number n such that not all of the derivatives $\dfrac{\partial^n u}{\partial^i x \partial^{n-i} y}(p)$ are equal to zero.

Proposition 2 (Bers [1]). *Let (M, g) be a compact 2-dimensional closed Riemannian manifold, and u be an eigenfunction of the Laplacian on M. Then for any point $x_0 \in M$ there exist its neighbourhood chart U with coordinates $x \in U \subset \mathbb{R}^2$ and a non-trivial homogeneous harmonic polynomial P_n of degree $n = \mathrm{ord}_{x_0}(u)$ on the Euclidean plane $\mathbb{R}^2$ such that*

$$u(x) = P_n(x - x_0) + O(|x - x_0|^{n+1}),$$

where $x \in U$.

It is a known fact that in the polar coordinates (r, φ) in $\mathbb{R}^2$ any homogeneous harmonic polynomial P_n has the form $P_n(r, \varphi) = r^n(a \cos n\varphi + b \sin n\varphi)$. The zeroes of such polynomials form n straight lines intersecting at origin at equal angles.

For a given l we introduce the following notation

$$\mathcal{N}^l(u) = \{x \in M |\, \mathrm{ord}_x(u) \geqslant l\}.$$

Note that $\mathcal{N}^1(u)$ is exactly the nodal set $\mathcal{N}(u)$.

Using Proposition 2, Cheng proved in the paper [2] that there exists such a neighbourhood of x_0 that in this neighbourhood the nodal set is diffeomorphic to the nodal set of P_n. As a result the nodal set carries a natural structure of finite graph, the vertices are the points of $\mathcal{N}^2(u)$ (they are isolated according to Theorem 2) and the edges are C^∞-arcs from $\mathcal{N}(u) \backslash \mathcal{N}^2(u)$. Moreover any vertex of this graph is of even degree.

For a graph Γ on a surface M a face of Γ is defined as a connected component of $M \backslash \Gamma$. Note that nodal domains are exactly faces of a nodal graph. One of the most basic theorems about graphs on surfaces is the following theorem.

The Euler inequality. *Let Γ be a finite graph on a closed surface M, and V_Γ, E_Γ, and F_Γ be the number of its vertices, edges, and faces respectively. Then the following inequality holds,*

$$V_\Gamma - E_\Gamma + F_\Gamma \geqslant \chi(M), \tag{3}$$

where $\chi(M)$ is the Euler characteristic of M. Besides, the equality occurs if and only if Γ is the 1-skeleton of a cell decomposition of M.

In the case of surfaces with isolated conical singularities we cannot apply inequality (3) straight away since the points of $\mathcal{N}^2(u)$ can accumulate towards singularities. However this possibility can be ruled out using resolution procedure from Section 3.1 of the paper [7]. Let us give a short description of the resolution procedure, we refer to the paper [7] for more details. For every vertex $x \in \mathcal{N}^2(u)$ consider a neighbourhood U in which $\mathcal{N}(u)$ is diffeomorphic to $k = \mathrm{ord}_x(u)$ straight lines intersecting at x. Let us numerate intersection points $\partial U \cap \mathcal{N}(u)$ by numbers $1, \ldots, 2k$ in a clockwise direction. We denote by l_i a smooth curve connecting $2i-1$-th point with $2i$-th point, $i = 1, \ldots, k$ such that l_i are pairwise non-intersecting curves lying inside of U. The resolution procedure at x is a replacement of the nodal graph $\mathcal{N}(u)$ by the graph $(\mathcal{N}(u) \backslash U) \cup \{l_1, \ldots, l_k\}$. This procedure decreases the number of vertices by 1 and does not increase the number of faces. Now we are ready to prove the following proposition.

Proposition 3. *A nodal graph of an eigenfunction u on a surface with isolated conical singularities is finite.*

Proof. Suppose that there are infinitely many points in $\mathcal{N}^2(u)$, it is easy to see that in this case the set $\mathcal{N}^2(u)$ is countable. Then the only possible accumulation points of $\mathcal{N}^2(u)$ are conical singularities. For each conical singularity p_j let us choose a base of neighbourhoods $V_i^{(j)}$ such that $\bar{V}_{i+1}^{(j)} \subset V_i^{(j)}$ and $\mathcal{N}^2(u) \cap \bigcup_{i=1}^{\infty} \partial V_i^{(j)} = \emptyset$. Hence for the sets $V_i = \cup_j V_i^{(j)}$ we have $\bar{V}_{i+1} \subset V_i$, $\mathcal{N}^2(u) \cap \bigcup_{i=1}^{\infty} \partial V_i = \emptyset$ and $M \backslash V_i$ contains only finite quantity of elements of $\mathcal{N}^2(u)$. For any i for the points of $\mathcal{N}^2(u)$ in $V_i \backslash \bar{V}_{i+1}$ one can choose a collection of disjoint neighbourhoods U_{ki} such that $\bar{U}_{ki} \subset V_i \backslash \bar{V}_{i+1}$. Thus we constructed a collection of disjoint neighbourhoods of all points in $\mathcal{N}^2(u)$.

Next we apply resolution procedure at all but finite number of vertices. Choosing this finite number big enough and applying Euler's inequality we arrive at contradiction with Courant's nodal domain theorem. □

Thus we can apply Euler inequality to the nodal graph on a surface with isolated conical singularities. One can also consult [7] for more details on nodal graphs theory and applications of nodal graphs to the eigenvalue multiplicity problem.

4 Symmetries of Bolza surface

Consider the following three involutions,

$$\begin{aligned} s &\colon (z,w) \mapsto \left(\frac{1}{\bar{z}}, \frac{\bar{w}}{\bar{z}}\right); \\ T &\colon (z,w) \mapsto (z,-w); \\ \sigma &\colon (z,w) \mapsto \left(-\frac{1}{\bar{z}}, -i\frac{1}{\bar{w}}\right). \end{aligned}$$

In the paper [6] s is denoted by s_3 and σ is equal to $s_3 s_2 s_1$. Let us also denote by d_j, $j = 1,2,3,4$ the arcs $d_k = \{z = e^{\frac{t\pi i}{4}}, t \in (2k-3, 2k-1)\}$ on $\bar{\mathbb{C}}$, the arcs d_k correspond to a_{k+4} in [6]. Let us denote the preimages $\Pi^{-1} d_k \subset \mathcal{P}$ of d_k by c_k.

Let us denote by $Fix(A)$ the fixed points of an involution A.

Lemma 1. *1) T, s, σ commute pairwise.*

2) $Fix(\sigma) = Fix(T\sigma) = \emptyset$.

3)

$$\begin{aligned} \sigma(c_2) &= T\sigma(c_2) = c_4; \\ \sigma(c_4) &= T\sigma(c_4) = c_2; \\ \sigma(c_1) &= T\sigma(c_1) = c_3; \\ \sigma(c_3) &= T\sigma(c_3) = c_1. \end{aligned}$$

4)

$$\begin{aligned} Fix(s) &= c_2 \cup c_4; \\ Fix(Ts) &= c_1 \cup c_3. \end{aligned}$$

Proof. Statement 1) is a consequence of an obvious computation. The proofs of statements 2) and 3) can be obtained by looking on z-coordinate of the corresponding involutions. The proof of statement 4) can be found in Proposition 2.4 in the paper [6]. □

5 Proof of Proposition 1

The involutions s, T, σ are isometries, therefore they commute with Δ. By Lemma 1 s, T, σ, Δ commute pairwise, one can consider a common basis of eigenfunctions for these operators.

Suppose f is a T-odd function from such a basis. Let us consider the case $s^*f = -f$ and $\sigma^*f = f$. All other cases can be treated in a similar way by replacing $\sigma \leftrightarrow T\sigma$, $s \leftrightarrow Ts$, $c_2 \cup c_4 \leftrightarrow c_1 \cup c_3$ where applicable. Since $s^*f = -f$, by Lemma 1 we have $Fix(s) = c_2 \cup c_4$, then $f \equiv 0$ on $c_2 \cup c_4$. By Proposition 2, the function f changes sign on c_2, therefore f has at least two nodal domains. The final part of the proof is to prove that f cannot have exactly two nodal domains. Suppose the contrary, i.e. f has two nodal domains.

Let us consider f as a function on the surface $\mathcal{P} \backslash (c_2 \cup c_4)$ diffeomorphic to the interior of a sphere with 4 holes, which we denote by N. Indeed $\mathcal{P}$ is obtained by gluing two copies of $\bar{\mathbb{C}}$ along 3 cuts with ends at the ramification points of Π. Now the claim follows from the fact that c_2 and c_4 are two of those cuts, so $\mathcal{P} \backslash (c_2 \cup c_4)$ is obtained by gluing two copies of $\bar{\mathbb{C}}$ with three cuts by only one cut.

The next our claim is that σ can be continouosly extended to the boundary of the holes in N. It easily follows from the fact that σ was initially defined on c_2 and c_4. Then by Lemma 1 this extension acts freely on boundary components of holes. Therefore σ descends to a free involution on $\tilde{N}$, where $\tilde{N}$ is obtained from N by a consequent contracting of boundary components.

Let Γ be the image of the nodal graph of f in $\tilde{N}$. Let us note that Γ is σ-invariant. Moreover since f is σ-invariant and has only two nodal domains, the faces of Γ are also σ-invariant. The final ingredient of the proof is the following purely topological lemma.

Lemma 2. *Suppose a surface M is homeomorphic to a two-dimensional sphere and σ is an involution on M without fixed points. Let Γ be a finite σ-invariant graph, such that any its face is also σ-invariant and any vertex has degree at least* 2. *Then Γ cannot have two faces.*

Proof. Suppose the contrary, that is $F_\Gamma = 2$. Then the Euler formula for graphs implies that $V_\Gamma - E_\Gamma \geqslant 0$. Moreover the degree of every vertex is at least 2, this implies $V_\Gamma \leqslant E_\Gamma$. Therefore we have equality in Euler formula, so any face is a disk. Then the restriction of σ on any face is a free involution on a disc. We arrive at contradiction with the Brouwer's fixed point theorem. □

6 Proof of Theorem 2

Let I be a free isometric involution on a 2-dimensional sphere $(\mathbb{S}^2, g)$. Suppose f is an eigenfunction corresponding to $\lambda_1^{\mathrm{even}}$. Since f is orthogonal to a constant function, f has at least 2 nodal domains. Suppose it has exactly two nodal domains. Then the nodal graph of f satisfies the conditions of Lemma 2. Thus f has at least three nodal domains. Therefore f cannot be the first nontrivial eigenfunction by Courant's nodal domain theorem. □

7 A family of maximal metrics for $\Lambda_1(\Sigma_2, g)$

In the paper [6] the authors derive from Theorem 4 existance of a family of maximal metrics for $\Lambda_1(\Sigma_2, g)$. This family is constructed as follows. Instead of the surface $\mathcal{P}$ given by equation (2) we consider surfaces $\mathcal{P}_t$ given the following family of equations

$$w^2 = z\frac{(z - e^{it})(z + e^{-it})}{(z - e^{-it})(z + e^{it})},$$

where parameter t lies in $\left(0, \frac{\pi}{2}\right)$. Let Π be a hyperelliptic projection $\mathcal{P}_t \to \bar{\mathbb{C}}$, we endow $\mathcal{P}_t$ with the metric $\Pi^* g_0$. Then in Section 3.5 of the paper [6] it is shown that given Theorem 4 there exist subinterval $(a,b) \subset \left(0, \frac{\pi}{2}\right)$, such that for any $t \in (a,b)$ one has $\Lambda_1(\mathcal{P}_t, g) = 16\pi$. In this section we show how using our method provide even wider family of maximal metrics.

Consider two distinct points $\xi, \eta \in \{w \in \mathbb{C}, |w| = 1\}$ such that $\xi \neq -\eta$. Then let us consider a hyperelliptic surface $\mathcal{P}_{\xi,\eta}$ given by the equation

$$w^2 = z\frac{(z-\xi)(z-\eta)}{(z+\xi)(z+\eta)}.$$

We once again denote the hyperelliptic projection by Π and endow $\mathcal{P}_{\xi,\eta}$ with the metric $\Pi^* g_0$. Then the same arguments as for $\mathcal{P}$ show that $\Lambda_1(\mathcal{P}_{\xi,\eta}, g) = 16\pi$. Indeed the reduction to Proposition 1 goes without changes. The involutions s, T, σ are still isometries of $\mathcal{P}_{\xi,\eta}$. The only difference is the definitions of d_i and c_i. We denote by d_i the connected components of $\{w \in \mathbb{C}, |w| = 1\} \backslash \{\xi, \eta, -\xi, -\eta\}$ numerated in a clockwise direction such that d_2 is the short arc between ξ and η. The cycles c_i are defined as preimages of d_i under Π. Then one checks that Lemma 1 holds in this notations. Then the arguments of Section 5 carry over without changes.

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