SINGULARITIES OF INTEGRABLE HAMILTONIAN SYSTEMS: A CRITERION FOR NON-DEGENERACY, WITH AN APPLICATION TO THE MANAKOV TOP

DMITRY TONKONOG

Abstract. Let (M,ω) be a symplectic $2n$-manifold and $h_1,\dots,h_n$ be functionally independent commuting functions on M. We present a geometric criterion for a singular point $P \in M$ (i.e. such that $\{\mathrm{d}h_i(P)\}_{i=1}^n$ are linearly dependent) to be non-degenerate in the sence of Eliasson–Vey.

The criterion is applied to find non-degenerate singularities in the Manakov top system (aka the 4-dimensional rigid body). Then we apply Fomenko's theory to study the neighborhood U of the singular Liouville fiber containing saddle-saddle singularities of the Manakov top. Namely, we describe the singular Liouville foliation and behavior of action variables on U. A relation with the quantum Manakov top studied by Sinitsyn and Zhilinskii (SIGMA 3 2007, arXiv:math-ph/0703045) is discussed.

Contents

1. Introduction and a criterion for non-degeneracy

This paper is on singularities of Liouville integrable Hamiltonian systems.

First we briefly present basic definitions used in the paper. A *Liouville integrable Hamiltonian system (IHS)* $(M,\omega,h_1,\dots,h_n)$ is a symplectic $2n$-manifold (M,ω) with functionally independent commuting functions $h_1,\dots,h_n : M \to \mathbb{R}$ traditionally called *integrals*. (For our purposes it is not important which of them is the actual Hamiltonian and which are additional integrals.) The *momentum map* $\mathcal{F} : M \to \mathbb{R}^n$ is given by $\mathcal{F}(x) := (h_1(x),\dots,h_n(x))$. For a function g on M, its Hamiltonian vector field is denoted by sgrad g. A point $x \in M$ is called a *singular (critical) point of rank* r, $0 \le r < n$, if $\operatorname{rk} \mathrm{d}\mathcal{F}(x) = r$. The $\mathcal{F}$-image of all singular points is called the *bifurcation diagram*. For singular points, there is a natural notion of non-degeneracy [12], [6, Definition 1.23]. Recall this definition for zero-rank critical points. The general definition is given below.

Definition 1.1. Let $(M,\omega,h_1,\dots,h_n)$ be an IHS and $P \in M$ be a zero-rank singular point, i.e. $\mathrm{d}h_i(P) = 0$ for each i. The point $P \in M$ is called *non-degenerate* if the commutative subalgebra K of $sp(2n,\mathbb{R})$ generated by linear parts of Hamiltonian vector fields sgrad $h_1,\dots,$ sgrad h_n at point P [1] is a Cartan subalgebra of $sp(2n,\mathbb{R})$.

Structure of the paper. In this section we present Theorem 1.2 (main result), which is a geometric criterion for non-degeneracy of zero-rank singularities *of elliptic-hyperbolic type* (see Remark 1.3),

The author was partially supported by Federal Target Program grant 02.740.11.5213 "Bi-Hamiltonian structures and singularities of integrable systems" (research group of Prof. A.V. Bolsinov) in 2010/2011, and by Dobrushin Scholarship at the Independent University of Moscow in 2011.

Comment for the Moebius contest version. A version of the work is available at http://arxiv.org/abs/1009.0863 (currently the arxiv version is older; the only significant difference is the absence of Proposition 3.4 from the arxiv version). There is a poster presentation available at http://wmii.uwm.edu.pl/~panas/school/tonkonog.pdf which briefly explains main results of the article.

[1] Equivalently, K is generated by linear operators $\{\omega^{-1}\mathrm{d}^2 h_i(P)\}_{i=1}^n$. The commutativity of K is implied by the fact that h_i commute.

and Theorem 1.3 extending Theorem 1.2 on singularities of arbitrary rank. We prove both theorems in §2. In §3 we study the Manakov top system, aka the 4-dimensional rigid body. Namely, we apply Theorem 1.2 to find non-degenerate singularities of the Manakov top (Proposition 3.1) in terms of the bifurcation diagram. After that we study the 4-dimensional neighborhood U of the singular Liouville fiber containing *saddle-saddle* (see Definition 1.2) singularities of the Manakov top. The proved non-degeneracy allows us to describe in Proposition 3.3 the singular *Liouville foliation* (i.e. foliation on level sets of $\mathcal{F}$) on U very easily, just by finding the correct alternative from the list of singularities in [6, Tables 9.1 and 9.3]. In Proposition 3.4 we describe behavior of action variables on U. Proofs of statements from §3 are given in §4.

Relations with other results. Singularities of the Manakov top were previously studied in [28, 29, 2, 13, 32, 3, 4], see §3 for details. In particular, the recent paper [3] obtains a comprehensive description of non-degenerate singularities of the Manakov top, from which Proposition 3.1 could be deduced. However, the proofs in [3] involve rather long computation; the proof of Proposition 3.1 using Theorem 1.2 is considerably shorter.

The problem to describe the structure of saddle-saddle singularities of the Manakov top was raised in [32] during analysis of the quantum Manakov top. In particular, [32, figures 1 and 13] show the joint spectrum lattice of the two quantum operators. On the other hand, Proposition 3.4 describes the 'Bohr-Sommerfeld' lattice of the Manakov top (which we define as the momentum map image of those Liouville tori on which the action variables take values in $2\pi h\mathbb{Z}$, ignoring for simplicity any Maslov-type correction). The two lattices are very similar; we discuss this in the end of §3.

Now we briefly discuss the notion of non-degeneracy to motivate Theorem 1.2.

In general, non-degenerate singularities are important because they are generic and because the local structure of integrable systems in their neighborhood is well understood, see Theorem 1.1. Global structure of non-degenerate singularities is being extensively studied, see survey [7], book [6] and papers [26, 27, 20, 19, 30]. The following is the fundamental fact about non-degenerate singularities, cf. Remark 1.5.

Theorem 1.1 (on Normal Form)**.** [31, 33, 15]. *Let $P \in M$ be a non-degenerate zero-rank singular point of an analytic IHS $(M, \omega, h_1, \dots, h_n)$. Then there exist a local system of coordinates $(p_1, \dots, p_n, q_1 \dots, q_n)$ at point P and nonnegative integers m_1, m_2, m_3 with $m_1 + m_2 + 2m_3 = 2n$ such that $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ and for each $i = 1, \dots, n$ we get $h_i = h_i(G_1, \dots, G_n)$ where*

$$G_j = p_j^2 + q_j^2 \qquad \textit{(elliptic type)} \qquad j = 1, \dots, m_1$$

$$G_j = p_j q_j \qquad \textit{(hyperbolic type)} \qquad j = m_1 + 1, \dots, m_2$$

$$\left.\begin{aligned} G_j &= p_j q_{j+1} - p_{j+1} q_j \\ G_{j+1} &= p_j q_j + q_{j+1} p_{j+1} \end{aligned}\right\} \quad \textit{(focus-focus type)} \quad \begin{aligned} j = m_1 + m_2 + 1,\ m_1 + m_2 + 3, \dots \\ \dots, m_1 + m_2 + 2m_3 - 1. \end{aligned}$$

Definition 1.2. The triple (m_1, m_2, m_3) is called the *Williamson type* of K, cf. [35]. In the case of two degrees of freedom ($n = 2$) these types are also called: *center-center* $(2, 0, 0)$, *center-saddle* $(1, 1, 0)$, *saddle-saddle* $(0, 2, 0)$, *focus-focus* $(0, 0, 1)$.

If P is a non-degenerate zero-rank singular point of an analytic IHS then the bifurcation diagram around $\mathcal{F}(P)$ *looks in the canonical way*, i.e. is locally (at point $\mathcal{F}(P)$) diffeomorphic to the *canonical* bifurcation diagram corresponding to functions G_j [6, 1.8.4], [7, p.9]. Figure 1 shows these canonical bifurcation diagrams for $n = 2$. The canonical bifurcation diagram for Williamson type $(s, n - s, 0)$ consists of n hypersurfaces: $n - s$ hyperplanes and s half-hyperplanes. For example, bifurcation diagrams on figure 2(1) look in the canonical way.

Analogous statement exists if we replace the bifurcation diagram by the image $\mathcal{F}(K \cup \{P\})$ where K is the set of all singularities of rank 1. The 'canonical' image $\mathcal{F}(K \cup \{P\})$ for Williamson type $(s, n - s, 0)$ consists of $n - s$ lines and s rays.

The converse is false: a point $P \in M$ can be a degenerate zero-rank singular point such that the bifurcation diagram still looks in the canonical way around $\mathcal{F}(P)$. A trivial example is as follows. Denote $M := \mathbb{R}^4$ with coordinates (p_1, p_2, q_1, q_2), $\omega := dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, $h_i := p_i^4 + q_i^4$ for $i = 1, 2$. Then $(\mathbb{R}^4, \omega, h_1, h_2)$ is an IHS, $P := 0 \in \mathbb{R}^4$ is a degenerate zero-rank point, but the bifurcation diagram consists of two lines $x = 0$ and $y = 0$ on the plane $\mathbb{R}^2(x, y)$, thus looks in the canonical way.

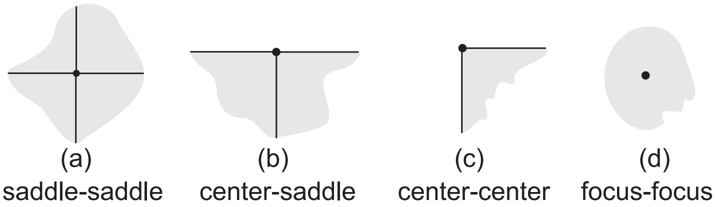

FIGURE 1. Canonical bifurcation diagrams in the neighborhood of $\mathcal{F}(P)$ corresponding to functions G_j, $n = 2$. The image of the momentum map is shaded gray.

In this example we get $\mathrm{d}^2 h_i(P) = 0$. A natural question arises: does the condition that the bifurcation diagram looks in the canonical way *plus* some condition on $\mathrm{d}^2 h_i(P)$ (which holds for non-degenerate singularities and which can be readily checked in real examples) guarantee non-degeneracy of P? Theorem 1.2 gives the positive answer.

To prove that a singular point $P \in M$ is non-degenerate by definition, one usually applies Lemma 2.1 below. This requires comparison of eigenvalues which is a tricky computational task (papers following this strategy are e.g. [25, 3]). Theorem 1.2 is intended to simplify computation. It is more effective for IHSs of 2 and 3 degrees of freedom: the geometric condition (b) can be effectively visualized then.

Theorem 1.2. *Consider a completely integrable Hamiltonian system* $(M, \omega, h_1, \dots, h_n)$. *Let* $\mathcal{F} : M \to \mathbb{R}^n$ *be the momentum map and* $P \in M$ *be a zero-rank singular point of the system. Denote by* K *the set of all singular points of rank 1 in a neighborhood of* P.

If the following conditions hold, then P *is non-degenerate:*
(a) $\bigcap_{i=1}^n \ker \mathrm{d}^2 h_i(P) = \{0\}$.
(b) The image $\mathcal{F}(K \cup \{P\})$ *contains* n *smooth curves* $\gamma_1, \dots, \gamma_n$, *each curve having* P *as its end point or its inner point.* [2] *The vectors tangent to* $\gamma_1, \dots, \gamma_n$ *at* $\mathcal{F}(P)$ *are independent in* $\mathbb{R}^n$.
(c) K *is a smooth submanifold of* M *or, at least,* $K \cup \{P\}$ *coincides with the closure of the set of all points* $x \in K$ *having a neighborhood* $V(x) \subset M$ *for which* $K \cap V(x)$ *is a smooth submanifold of* M.

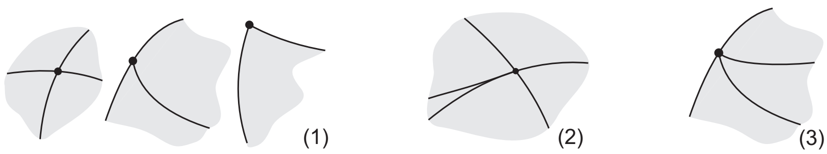

FIGURE 2. Images $\mathcal{F}(K \cup \{P\})$ diagrams satisfying condition (b) of Theorem 1.2, $n = 2$. The diagram (2) appears in the non-analytic case and (3) when the zero-rank point is degenerate. The image of the momentum map is shaded gray.

Remark 1.1. Condition (c) is very weak. For example, it automatically holds if the integrals h_i are polynomials (in a suitable system of local coordinates at point P) because in this case each D_i is given by a system of algebraic equations. It also holds if K consists of non-degenerate singular points of rank 1 (in this case K is smooth [6, Proposition 1.18]).

Remark 1.2. By Lemma 2.3 below, condition (a) is equivalent to the following condition (a'): *There exists a non-degenerate linear combination of forms* $\{\mathrm{d}^2 h_i(P)\}_{i=1}^n$.

Remark 1.3 (on the converse of Theorem 1.2)**.** In this remark we consider analytic IHSs for simplicity. If P satisfies Theorem 1.2, then it automatically has elliptic-hyperbolic type, i.e. its Williamson type is $(s, n-s, 0)$ for some s, see Definition 1.2. Indeed, for a non-degenerate point of type $(s, 2n-2k-s, 2k)$, $k > 0$, the image $\mathcal{F}(K \cup \{P\})$ does not satisfy condition (b) by Theorem 1.1, see discussion above and fig. 1(3). So Theorem 1.2 does not cover focus-focus singularities. The converse of Theorem 1.2 is true for elliptic-hyperbolic singularities: *Let* P *be a non-degenerate zero-rank singular point of an IHS, and suppose* P *has Williamson type* $(s, n-s, 0)$ *for some* s. *Then it satisfies conditions (a),(b),(c) of Theorem 1.2.*

[2]Figure 2 (1),(2),(3) shows examples for $n = 2$.

This is well known. Conditions (a)–(c) can be verified in normal coordinates of Theorem 1.1. Condition (a) follows from the fact that $\mathrm{d}^2 h_i(P)$ are independent. The sets of critical points of rank r for functions h_i and G_i coincide. Hence condition (c) follows [7, Theorem 3]; condition (b) also follows as already stated above.

Remark 1.4. In Theorem 1.2 we do not demand that the image $\mathcal{F}(K \cup \{P\})$ coincides with the union of γ_i. It may contain additional curves as on fig. 2(2),(3). As discussed above, only n curves appear in the non-degenerate analytic case. So Theorem 1.2 implies the following interesting corollary. *If P is a zero-rank singular point of an algebraic IHS $(M, \omega, h_1, \ldots, h_n)$ and $\mathcal{F}(K \cup \{P\})$ contains more than n curves with pairwise independent tangent vectors as on fig. 2(3) then all linear combinations of forms $\mathrm{d}^2 h_1(P)$, $\mathrm{d}^2 h_2(P)$ are degenerate.* This can be observed in a wide range of examples, for instance, in the Jukowsky integrable case of rigid body dynamics [29, 6]. Here the assumption that IHS is algebraic is used to guarantee condition (c), see Remark 1.1.

Remark 1.5. In the C^∞ case, Theorem 1.1 is proved for singularities of Williamson type $(s, n-s, 0)$ [12, 24] and very recently for focus-focus singularities $(0, 0, 1)$ [34]. Remark 1.3 is true in the non-analytic case, but now the bifurcation diagram near the image $\mathcal{F}(P)$ of a non-degenerate singularity may split as shown on fig. 2(2) (one curve splits into two curves with infinite order of tangency). This example is found in [6, 1.8.4].

We now turn to a criterion for non-degeneracy of r-rank singularities. The definition of non-degeneracy [6, Definition 1.23] is as follows. [3]

Definition 1.3. Let $(M, \omega, f_1, \ldots, f_n)$ be an IHS and $P \in M$ be a singular point of rank r. Find any regular linear change of integrals $f_1, \ldots, f_n$ so that the new functions, which we denote $h_1, \ldots, h_n$, satisfy the property: $\mathrm{d}h_{r+1}(P) = \ldots = \mathrm{d}h_n(P) = 0$. Consider the space $L \subset T_P M$ generated by $\operatorname{sgrad} h_1, \ldots, \operatorname{sgrad} h_r$ and its ω-orthogonal complement $L' \supset L$. Denote by $A_{r+1}, \ldots, A_n$ the linear parts of vector fields $\operatorname{sgrad} h_{r+1}, \ldots, \operatorname{sgrad} h_n$. They are commuting operators in $sp(2n, \mathbb{R})$. By [6, Lemma 1.8] the subspace L belongs to the kernel of every operator $A_{r+1}, \ldots, A_n$ and their image lies in L'. Thus they can be regarded as operators on L'/L. By [6, Lemma 1.9] L'/L admits a natural symplectic structure and $A_{r+1}, \ldots, A_n \in sp(L'/L, \mathbb{R}) \cong sp(2n-2r, \mathbb{R})$. The point $P \in M$ is called *non-degenerate* if $A_{r+1}, \ldots, A_n$ generate a Cartan subalgebra in $sp(2n-2r, \mathbb{R})$.

Remark 1.6. Clearly, the definition does not depend on a regular $C^\infty(M)$-linear change of the integrals. In Theorem 1.3 we will consider integrals such that that $\mathrm{d}h_{r+1}(P) = \ldots = \mathrm{d}h_n(P) = 0$. To apply Theorem 1.3 for a general integrable system $(M, \omega, f_1, \ldots, f_n)$ it is sufficient to obtain integrals h_i satisfying this property by a regular $C^\infty(M)$-linear change of f_i.

Theorem 1.3. *Consider a completely integrable Hamiltonian system $(M, \omega, h_1, \ldots, h_n)$. Let $\mathcal{F} : M \to \mathbb{R}^n$ be the momentum map and $P \in M$ be a singular point of rank r. Denote by K the set of all singular points of rank $r+1$ in a neighborhood of P. Suppose that $\mathrm{d}h_{r+1}(P) = \ldots = \mathrm{d}h_n(P) = 0$ and $h_i(P) = 0$ for all i.*

If the following conditions hold, then P is non-degenerate:
(a) There exist a number $k \in \{r+1, \ldots, n\}$ and a $(2n-2r)$-dimensional subspace $F \subset T_P M$ such that
(a_1) $F \subset \bigcap_{j=1}^{r} \ker \mathrm{d}h_j(P)$,
(a_2) $F \cap \operatorname{Lin}\{\operatorname{sgrad} h_1(P), \ldots, \operatorname{sgrad} h_r(P)\} = \{0\}$ and
(a_3) $\bigcap_{i=r+1}^{n} \ker \mathrm{d}^2 h_i(P)|_F = \{0\}$.
(b) The intersection of the closure of $\mathcal{F}(K) \subset \mathbb{R}^n$ with the submanifold $\{h_1 = \ldots = h_r = 0\}$ contains $n-r$ smooth curves, each curve having P as its end point or its inner point. The vectors tangent to these curves at $\mathcal{F}(P)$ are independent in $\mathbb{R}^n$.
(c) K is an analytic submanifold of M or, at least, the closure of $K' := K \cap \{x \in M : h_1(x) = \ldots = h_r(x) = 0\}$ coincides with the closure of the set of all points $x \in K'$ having a neighborhood $V(x) \subset M$ for which $K' \cap V(x)$ is a smooth submanifold of M.

[3] This definition is equivalent to P being a non-degenerate *zero-rank* singular point of Marsden-Weinstein symplectic reduction of the given system by the local action of $\mathbb{R}^{n-r}$ generated by flows of Hamiltonian vector fields of $n-r$ independent integrals. This helps to deduce Theorem 1.3 easily from Theorem 1.2.

As in the case of Theorem 1.2, the converse of Theorem 1.3 is true for non-degenerate points of Williamson type $(s, n-r-s, 0)$.

2. Proofs of Theorems 1.2 and 1.3

We will need the following well-known lemmas. We prove Lemma 2.3 at the end of this section since we do not have a reference for it.

Lemma 2.1. (Cf. [6, 1.10.2]) *A commutative subalgebra $K \subset sp(2n,\mathbb{R})$ is a Cartan subalgebra if and only if K is n-dimensional and it contains an element whose eigenvalues are all different.*

Lemma 2.2. (Cf. [23, Lemma 2.20]) *Suppose $A \in sp(2n,\mathbb{R})$ or $sp(2n,\mathbb{C})$. If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.*

Lemma 2.3. *Suppose $A_1, \dots, A_n \in GL(k,\mathbb{R})$ commute pairwise. Then there exist $\mu_i \in \mathbb{R}$ such that* $\ker \sum_{i=1}^n \mu_i A_i = \bigcap_{i=1}^n \ker A_i$. [4]

Proof of Theorem 1.2. Step 1. Introducing new integrals. Denote $D_i := \mathcal{F}^{-1}(\gamma_i) \cap K$. Condition (b) enables us to construct a new set $\{f_i\}_{i=1}^n$ of independent commuting integrals such that $f_j|_{D_i} \equiv 0$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Indeed, let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism taking γ_i to the i-axis and $\mathcal{F}(P)$ to $0 \in \mathbb{R}^n$; then define $f_i := g h_i$. Below we work with the new integrals f_i. Although the corresponding momentum maps for $\{h_i\}$ and $\{f_i\}$ are different, the critical set K remains the same. Moreover, $\{\mathrm{d}^2 f_i(P)\}$ are obtained from $\{\mathrm{d}^2 h_i(P)\}$ by a regular linear change given by the operator $\mathrm{d}g(\mathcal{F}(P))$, so we can verify Definition 1.1 for $\{f_i\}$ as well as for $\{h_i\}$. Below we write $\mathrm{d}^2 f_i$ instead of $\mathrm{d}^2 f_i(P)$ (and the same for other functions). Denote

$$T_i := \bigcap_{\substack{j=1,\dots,n \\ j\neq i}} \ker \mathrm{d}^2 f_j.$$

Denote by $A_i \in sp(T_P M) \cong sp(2n,\mathbb{R})$ the linear part of the vector field $\operatorname{sgrad} f_i$ (equivalently, $A_i = \omega^{-1}\mathrm{d}^2 f_i$). Clearly, $\ker A_i = \ker \mathrm{d}^2 f_i$ and $\{A_i\}_{i=1}^n$ commute pairwise. Thus T_j is A_i-invariant for each i, j.

Step 2. Proof that $T_i \neq \{0\}$ for each i. [5] Suppose to the contrary that $T_j = \{0\}$ for some $j \in \{1, \dots, n\}$. Then by Lemma 2.3 some linear combination of $\{A_i\}_{i\neq j}$ is non-degenerate, and thus the same combination of the forms $\{\mathrm{d}^2 f_j\}_{i\neq j}$ is non-degenerate. Let F be the linear combination of functions $\{f_i\}_{i\neq j}$ with the same coefficients. We obtain: (1°) $\mathrm{d}^2 F$ is non-degenerate and (2°) $F|_{D_j} \equiv 0$ since $f_i|_{D_j} \equiv 0$ for $i \neq j$. By (1°) and the Morse lemma there exists a punctured neighborhood $U'(P) \subset M$ of point P such that $\mathrm{d}F(x) \neq 0$ for all $x \in U'(P)$. Now suppose $x \in D_j$ has a neighborhood $V(x)$ such that $V(x) \cap D_j$ is a smooth submanifold. By (2°) we get $\mathrm{d}(F|_{D_j})(x) = 0$ for all $x \in U'(P)$, meaning that $\mathrm{d}F(x) \perp T_x D_j$. But x is a point of rank 1, so $\mathrm{d}F(x)$ and $\mathrm{d}f_j(x)$ are linearly dependent. Since $\mathrm{d}F(x) \neq 0$, this implies that $\mathrm{d}f_j(x) \perp T_x D_j$, thus $f_j|_{D_j}(x) = 0$. By (c), this holds for almost all $x \in U'(P)$ so $f_j|_{D_j} \equiv \text{const}$. On the other hand, $f_j|_{D_j}$ is not a constant function since the image $f_j(D_j)$ is a line segment and not a point. This contradiction shows that $T_j \neq \{0\}$.

Step 3. Proof that $\dim T_i \geq 2$ for each i. Suppose to the contrary that $\dim T_j = 1$ for some j. Without loss of generality, assume $j = 1$. Take $x \in T_1$, $x \neq 0$. By definition, $A_i(x) = 0$ for $i = 2, \dots, n$. Then $A_1(x) \neq 0$, because otherwise $x \in \bigcap_{i=1}^n \ker A_i = \bigcap_{i=1}^n \ker \mathrm{d}^2 f_i = \bigcap_{i=1}^n \ker \mathrm{d}^2 h_i$, which contradicts to condition (a). But T_1 is A_1-invariant, and we obtain $A_1(x) = \lambda x$ for some $\lambda \neq 0$. Lemma 2.2 implies that $(-\lambda)$ is also an eigenvalue of A_1, meaning that there exists $y \in T_P M$, $y \neq 0$, such that $A_1(y) = -\lambda y$. The subspace $L := \operatorname{Lin}(\{x, y\})$ is A_i-invariant for each $i = 1, \dots, n$ and in its basis $\{x, y\}$ we get

$$A_1|_L = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad A_i|_L = \begin{pmatrix} 0 & b_i \\ 0 & c_i \end{pmatrix} \quad \text{for } i \geq 2.$$

---

[4] The following matrices: $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ show that the commutativity condition is indeed necessary.

[5] If we were given that $\overline{D_i} = D_i \cup \{P\}$ is a smooth submanifold, then $T_i \neq \{0\}$ follows from the obvious inclusion $T_P \overline{D_i} \subset T_i$. We use condition (c) in this step only.

Since A_1 commutes with A_i, we obtain that $a_i = b_i = 0$ for $i \geq 2$, so $A_i(y) = 0$. By definition this means $y \in T_1$.

Step 4. Proof that $\dim T_i = 2$ *and* $\bigoplus_{i=1}^n T_i = T_P M$. By condition (a), any n non-zero vectors $v_i \in T_i$, $1 \leq i \leq n$, are independent. (Indeed, suppose to the contrary that v_1 is a linear combination of $\{v_2, \dots, v_n\}$. Then by construction $v_1 \in \bigcap_{i=1}^n \ker \mathrm{d}^2 f_i = \bigcap_{i=1}^n \ker \mathrm{d}^2 h_i$, which contradicts to condition (a).) Combining this with Step 3 we obtain that $\dim T_i = 2$ and $\bigoplus_{i=1}^n T_i = T_P M$ for each $i \in \{1, \dots, n\}$.

Step 5. Final step. By construction, $\ker A_i = \ker \mathrm{d}^2 f_i = \bigcup_{j \in \{1 \dots n\} \setminus \{i\}} T_j$. This means that for all $i, j \in \{1, \dots, n\}$, $i \neq j$, we obtain $A_i|_{T_j} \equiv 0$. Condition (a) now implies that $\ker A_i|_{T_i} = \{0\}$. So the eigenvalues of $A_i|_{T_i}$ are $\{\pm\lambda_i \neq 0\}$ for some $\lambda_i \in \mathbb{C}$. Let us prove that P is non-degenerate. Clearly, $\{A_i\}_{i=1}^n$ are independent. The eigenvalues of a linear combination $\sum_{i=1}^n \mu_i A_i$ are $\{\pm\mu_i\lambda_i\}_{i=1}^n$ which are obviously all different for well-chosen coefficients μ_i. Thus P is non-degenerate by Definition 1.1, Lemma 2.1 and the argument in Step 1. Proof of Theorem 1.2 is finished. ■

Proof of Theorem 1.3. By the Darboux theorem, we can complete functions $p_1 := h_1, \dots, p_r := h_r$ up to a coordinate system $\{p_i, q_i\}_{i=1}^n$ at point P such that $\{p_i, p_j\} = 0$, $\{p_i, q_j\} = \delta_{ij}$ for all $1 \leq i, j \leq n$. Denote $\Pi := \mathrm{Lin}\ \{\partial/\partial p_i, \partial/\partial q_i\}_{i=r+1}^n \subset T_P M$. Consider the symplectic submanifold $Q \subset M$ in a neighborhood of P given by equations $\{p_i = 0, q_i = 0\}_{i=1}^r$; then $T_P Q = \Pi$. By Definition 1.2, P is non-degenerate if the restricted operators $\{\omega^{-1}\mathrm{d}^2 h_{r+1}|_\Pi, \dots, \omega^{-1}\mathrm{d}^2 h_n|_\Pi\}$ generate a Cartan subalgebra of $sp(2n-2r, \mathbb{R})$. Clearly, this is equivalent to P being a non-degenerate zero-rank singular point of the reduced IHS $(Q, \omega|_Q, \{h_i|_Q\}_{i=r+1}^n)$. We can apply Theorem 1.2 to this reduced system by verifying the three conditions of Theorem 1.2.

By (a_3) there is a linear combination H of $h_{r+1}, \dots, h_n$ such that $\mathrm{d}^2 H|_F$ is non-degenerate. By (a_1) $F \subset \mathrm{Lin}\ (\Pi \cup \{\partial/\partial q_i\}_{i=1}^r)$; by ($a_2$) the projection $F \xrightarrow{\mathrm{pr}} \Pi$ has zero kernel and is an isomorphism since $\dim\ F = \dim\ \Pi$. Since h_i commute, it follows that $\{h_i\}_{i=r+1}^n$ do not depend on $\{q_1, \dots, q_r\}$, so $\mathrm{d}^2 H(v) = \mathrm{d}^2 h_k(\mathrm{pr}\, v)$ for $v \in \mathrm{Lin}\ (\Pi \cup \{\partial/\partial q_i\}_{i=1}^r)$. Together with ($a_3$) this implies that $\mathrm{d}^2 H|_\Pi$ is non-degenerate. Condition (a) of Theorem 1.2 is verified.

Let $\tilde{K}$ and $\tilde{\mathcal{F}}$ denote respectively the set of 1-rank points near P and the momentum map of the restricted system. Condition (b) of Theorem 1.2 follows from the given condition (b) because $\tilde{\mathcal{F}}(\tilde{K}) = \mathcal{F}(K) \cap \{x \in M :\ h_1 = \dots = h_r = 0\}$. Condition (c) of Theorem 1.2 follows from the given condition (c). Indeed, $\tilde{K} = K \cap Q$ and since all gradients $\{\mathrm{d}h_i\}_{i=1}^n$ are independent of $\{q_i\}_{i=1}^r$, K' is a cylinder over $\tilde{K}$. So if K is an analytic submanifold or K' is 'almost everywhere regular' in the sence of condition (c) then $\tilde{K}$ also 'almost everywhere regular', i.e. satisfies condition (c) in Theorem 1.2. ■

Proof of Lemma 2.3. Let us first prove the lemma for $n = 2$; denote $A := A_1$, $B := A_2$. Consider a basis $(e_1, \dots, e_k)$ for $\mathbb{R}^k$ such that $(e_1, \dots, e_j)$ spans $\ker A$ for some j. In this basis we get $A = \begin{pmatrix} 0_{j\times j} & A'' \\ 0 & A' \end{pmatrix}$ and $B = \begin{pmatrix} B''' & B'' \\ 0 & B' \end{pmatrix}$. Here $0_{j\times j}$ and B''' are $j\times j$-matrices and A', B' are $(k-j)\times(k-j)$-matrices. By construction A' is non-degenerate. Clearly $\ker(A + \varepsilon B) = \ker A \cap \ker B$ for sufficiently small ε.

The general case is proved by induction on n. Let us prove the step. Given $A_1, \dots, A_n$, we can find by the induction hypothesis a linear combination B of $A_1, \dots A_{n-1}$ whose kernel is $\bigcap_{i=1}^{n-1} \ker A_i$. By the $n = 2$ case, there is a linear combination of B and A_n whose kernel is $\ker B \cap \ker A_n = \bigcap_{i=1}^n \ker A_i$. ■

3. Application to the classical and quantum Manakov top

3.1. **A short introduction to the Manakov top system.** The Manakov top integrable system (also known as the geodesic flow on $so(4)$ and the 4-dimensional rigid body) was introduced in [21]. Oshemkov [28, 29] [6] studied the topology and bifurcation diagrams of the system; we reproduce the bifurcation diagrams below. For certain parameters, the Manakov top contains a focus-focus point. The corresponding *Hamiltonian monodromy* [11] was calculated by Audin [2] using algebraic technique which allowed not to check non-degeneracy of the point.

[6] The result from these references are also found in [6, vol. 2, 5.10]

Let us recall the Manakov top system following [29]. Consider $\mathbb{R}^6$ with coordinates $p_1, p_2, p_3, m_1, m_2, m_3$. Define the Lie-Poisson bracket on $\mathbb{R}^6$:

$$\{m_i, m_j\} = \epsilon_{ijk} m_k, \quad \{m_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = \epsilon_{ijk} m_k.$$

Here $\epsilon_{ijk} = (i-j)(j-k)(k-i)$. This bracket has two Casimir functions

$$f_1 = m_1^2 + m_2^2 + m_3^2 + p_1^2 + p_2^2 + p_3^2, \quad f_2 = m_1 p_1 + m_2 p_2 + m_3 p_3.$$

Fix three numbers $0 < b_1 < b_2 < b_3$. Functions

$$h_1 = b_1 m_1^2 + b_2 m_2^2 + b_3 m_3^2 - (b_1 p_1^2 + b_2 p_2^2 + b_3 p_3^2),$$
$$h_2 = (b_1 + b_2)(b_1 + b_3) p_1^2 + (b_2 + b_1)(b_2 + b_3) p_2^2 + (b_3 + b_1)(b_3 + b_3) p_3^2$$

commute with respect to the defined bracket and thus define an IHS on a symplectic leaf

$$M^4_{d_1, d_2} := \{x \in \mathbb{R}^6 : f_1(x) = d_1, \ f_2(x) = d_2\}$$

of the Lie-Poisson bracket, $|2d_2| < d_1$. This system is called the Manakov top. Its parameters are $(b_1, b_2, b_3, d_1, d_2)$.

For a certain (open) set of parameters b_i, d_i, the bifurcation diagram has one of the three types shown on fig. 3; see [29] for details. The diagram of the third type separates the image of the momentum map into three domains. The $\mathcal{F}$-preimage of each point of the inner domain consists of 4 tori. The preimage of each point of the two other domains consists of 2 tori. Let Q be the intersection point of the two inner curves on the bifurcation diagram, see fig. 3. The preimage $\mathcal{F}^{-1}(Q)$ contains two zero-rank points [29]. It is natural to expect that they are non-degenerate saddle-saddle singularities. The proof becomes simple with the help of Theorem 1.2.

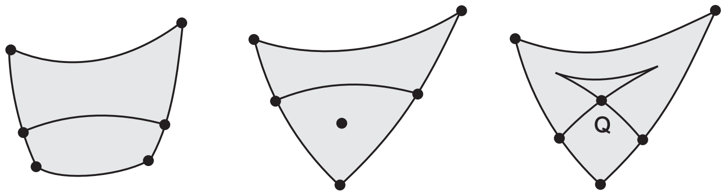

FIGURE 3. Three types of generic bifurcation diagrams of the Manakov top. Point Q is the image of two saddle-saddle singularities.

3.2. **Non-degenerate singularities of the Manakov top.** The explicit parameters of the Manakov top under which the system contains degenerate singularities were recently obtained in [3, Theorem 5.3], cf. [4]. In the following proposition, the description of non-degenerate singularities is very natural: it essentially says that all degenerate singularities are easily seen to be degenerate by looking at the bifurcation diagrams. As already mentioned, the proof of Proposition 3.1 using Theorem 1.2 is considerably shorter than the proofs in [3]. [7] Recall the Williamson type of a non-degenerate singularity was introduced in Definiton 1.2.

Proposition 3.1. *Let $P \in M$ be a zero-rank singular point of the Manakov top with parameters (b_i, d_i). Then P is non-degenerate and not of focus-focus type if and only if for each set of parameters (b_i', d_i') sufficiently close to (b_i, d_i), the bifurcation diagram of the Manakov top with parameters (b_i', d_i') can be transformed by a diffeomorphism of a neighborhood of $\mathcal{F}(P)$ to one of the three diagrams shown on fig. 1(a,b,c).* [8]

(The 'only if' part of Proposition 3.1 is trivial.) Degenerate singularities thus do not appear when the bifurcation diagram has one of the generic types shown on fig. 3,

[7] In the proof of Proposition 3.1 we essentially determine the parameters (b_i, d_i) which contain degenerate singularities. They seem to agree with those from [3], although that paper uses different notation. Also, Proposition 3.1 could be deduced from [3] and even is in part implicitly stated there, see [3, text after Theorem 5.3].

[8] As previously mentioned, Theorem 1.2 does not cover focus-focus singularities, so we have to exclude them from this proposition as well.

Corollary 3.2. *Let $P \in M$ be a zero-rank singular point of the Manakov top with parameters (b_i, d_i). Then P is a non-degenerate saddle-saddle singular point if and only if the bifurcation diagram of the Manakov top with parameters (b_i, d_i) can be transformed by a diffeomorphism of a neighborhood of $\mathcal{F}(P)$ to the diagram shown on fig. 1(a).*

In this case, $\mathcal{F}^{-1}(\mathcal{F}(P))$ contains two zero-rank points, both of saddle-saddle type.

Proof of Corollary 3.2 modulo Proposition 3.1. By looking at the types of bifurcation diagrams in [29] it easily seen that hypothesis of the Corollary 3.2 is stable under parameter perturbation and thus implies the hypothesis of Proposition 3.1. The fact that $\mathcal{F}^{-1}(\mathcal{F}(P))$ contains two zero-rank points is proved in [29] and is easy; it also follows from the proof of Proposition 3.1. ■

For example, if $Q \in \mathbb{R}^2$ is the point from fig. 3 or fig. 5, the two zero-rank points in the preimage $\mathcal{F}^{-1}(Q)$ are nondegenerate and of saddle-saddle type.

Remark 3.1. There are higher-dimensional versions of the Manakov top system, called the n-dimensional rigid body. For $n \geq 5$ it should be explored using a different approach because it is hard to study the bifurcation diagrams of this system. Remarkably, an approach using the bi-Hamiltonian structure provides the complete answer (A. Izosimov, preprint).

3.3. **Semilocal structure of saddle-saddle singularities of the Manakov top.** Recall that an IHS (M^4, ω, f_1, f_2) defines the singular *Liouville foliation* on M whose fibers are common level sets of functions (f_1, f_2), i.e. level sets of the momentum map $\mathcal{F}$. Regular fiber of this foliation is a disjoint union of tori (under certain assumptions which hold in the Manakov top) [1].

Definition 3.1. We will call a diffeomorphism preserving Liouville foliation a *Liouville equivalence* or a ($\mathcal{F}$-)fiberwise diffeomorphism.

In Proposition 3.3 below we describe *semilocal structure* of the saddle-saddle singularities of the Manakov top, i.e. describe the (singular) Liouville foliation on $\mathcal{F}^{-1}(V)$ where $V \subset \mathbb{R}^2$ is a small neighborhood of Q. [9]

To state Proposition 3.3, we have to introduce some notation (cf. [6, 18]). Let C_2 be the fibered 2-manifold with boundary shown on fig. 4. Formally, C_2 is the preimage $h^{-1}(-\varepsilon, \varepsilon)$ of a certain Morse function $h : \mathbb{R}^2 \to \mathbb{R}$ having two singular points at one critical value 0. Level sets of h define the singular fibration on C_2. Two shades on fig. 4 show the areas below and over the critical value of h. A regular fiber on C_2 is a disjoint union of two circles. The circles in ∂C_2 are distrubuted between two fibers. The direct product $C_2 \times C_2$ is a 4-manifold with boundary equipped with the product fibration. [10] Its regular fiber is a disjoint union of four tori. Let α be rotation by 180°, the free fiberwise involution on C_2. The involution (α, α) preserves fibration on $C_2 \times C_2$ and thus defines the fibered 4-manifold $(C_2 \times C_2)/(\alpha, \alpha)$.

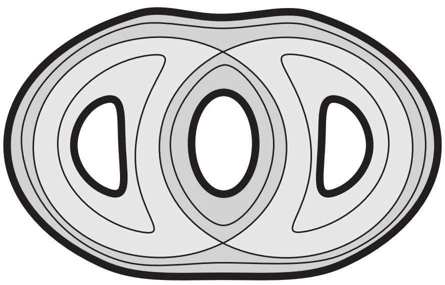

FIGURE 4. The fibered 2-manifold C_2.

Proposition 3.3. *Let $Q \in \mathbb{R}^2$ be the point on the bifurcation diagram of the Manakov top as on fig. 3 or fig. 5 and V its neighborhood such that $\mathcal{F}^{-1}(V)$ retracts onto $\mathcal{F}^{-1}(Q)$. Then $\mathcal{F}^{-1}(V)$ is Liouville equivalent to $(C_2 \times C_2)/(\alpha, \alpha)$.*

[9] The word 'semilocal' is used since the preimage $\mathcal{F}^{-1}(V)$, even $\mathcal{F}^{-1}(Q)$, is not at all local, i.e. does not belong to small neighborhood in M^4. It contains two distant zero-rank singularities.

[10] This fibration comes from an integrable system on $C_2 \times C_2$ [6, 9.6] so can be called Liouville foliation.

There is a complete list of the Liouville equivalence classes of neighborhoods of singular fibers containing two saddle-saddle singular points for all integrable systems with two degrees of freedom: [6, 9.6, Tables 9.1 and 9.3], compare [5, 22]. Since Q is non-degenerate by Corollary 3.2, $\mathcal{F}^{-1}(V)$ is Liouville equivalent to one of the 39 items from these tables. To prove Proposition 3.3, we just have to identify the correct item. It is very easy, see the proof in §4.

Note there is a general theorem by Nguen Tien Zung stating that all neighborhoods of fibers containing saddle-saddle singularities can be obtained as a quotient of a direct product of certain fibered 2-manifolds [26].

3.4. **Action variables around saddle-saddle singularities of the Manakov top and relations to the quantum Manakov top.** Our last goal is to describe the structure of action variables around the singular fiber containing saddle-saddle singularities of the Manakov top. First, we recall [32, Appendix A] that under some parameters, the Manakov top has rich symmetry which is expressed by the following Condition 3.1. Recall that $\mathcal{F}$ is the momentum map $M \to \mathbb{R}^2$, where (M, ω) is the phase space of the Manakov top system. Each regular fiber of $\mathcal{F}$ is a disjoint union of 2 or 4 tori.

Condition 3.1. Every Liouville torus can be mapped onto any other torus on the same regular $\mathcal{F}$-fiber via an $\mathcal{F}$-fiberwise symplectomorphism of (M, ω).

More precisely, the Manakov top system (with certain parameters) satisfies Condition 3.1 when its bifurcation diagram looks as shown on fig. 5. Condition 3.1 implies that action variables on a regular torus are the same on the other tori of the same $\mathcal{F}$-fiber, which means they can be regarded as functions over the image of the momentum map, a domain in $\mathbb{R}^2$. In the following proposition, part (c) is most interesting and is discussed below. This proposition is an easy corollary of Proposition 3.3.

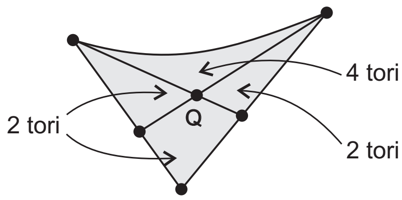

FIGURE 5. Bifurcation diagram of the Manakov top system satisfying Condition 3.1

Proposition 3.4. *Consider the Manakov top system (M, ω, h_1, h_2) with parameters satisfying Condition 3.1 and containing a saddle-saddle singularity P. Let $V \subset \mathbb{R}^2$ be a small neighborhood of $Q := \mathcal{F}(P)$ such that $U := \mathcal{F}^{-1}(V)$ retracts onto $\mathcal{F}^{-1}(Q)$. There is a 1-form θ on U such that $d\theta = \omega|_U$ and two continuous functions $a_1, a_2 : U \to \mathbb{R}$ such that:*

(a) a_1, a_2 are smooth at regular values of $\mathcal{F}$. For each Liouville torus $T \subset U$, there is a basis (ρ_1, ρ_2) of $H_1(T; \mathbb{Z})$ such that

$$a_1(\mathcal{F}(T)) = \int_{\rho_1} \theta, \quad a_2(\mathcal{F}(T)) = \int_{\rho_2} \theta$$

if $\mathcal{F}^{-1}(\mathcal{F}(T))$ consists of two tori and

$$1/2(a_1 - a_2)(\mathcal{F}(T)) = \int_{\rho_1} \theta, \quad 1/2(a_1 + a_2)(\mathcal{F}(T)) = \int_{\rho_2} \theta$$

if $\mathcal{F}^{-1}(\mathcal{F}(T))$ consists of four tori.

(b) The map $\psi := (a_1; a_2)$ is a homeomorphism from V to a neighborhood of point $(0; 0) \in \mathbb{R}^2$. It takes Q to $(0; 0)$. Moreover, ψ is C^∞ outside the bifurcation diagram. The image of the bifurcation diagram is a union of two curves intersecting at $(0; 0)$. They are C^∞ everywhere except maybe for point $(0; 0)$, are C^1 at $(0; 0)$ and intersect transversally.

(c) Let L_h, $h \in \mathbb{R}_+$, be the union of all Liouville tori in U satisfying the following condition: the values of all action functions (with respect to the 1-form θ) on the torus belong to $2\pi h\mathbb{Z}$. [11] *For each*

[11] If T is the torus in question, this is equivalent to $a_1(\mathcal{F}(T))$, $a_2(\mathcal{F}(T)) \in 2\pi h\mathbb{Z}$ if $\mathcal{F}^{-1}(\mathcal{F}(T))$ consists of 2 tori and $1/2(a_1 - a_2)(\mathcal{F}(T))$, $1/2(a_1 + a_2)(\mathcal{F}(T)) \in 2\pi h\mathbb{Z}$ if $\mathcal{F}^{-1}(\mathcal{F}(T))$ consists of 4 tori.

$h \in \mathbb{R}_+$ the homeomorphism ψ takes the set $\mathcal{F}(L_h)$ to the following set (see fig. 6):

$$\{(x,y) \in \psi(V) \quad \textit{such that} \quad \begin{cases} x, y \in 2\pi h\mathbb{Z}, & \textit{if } \mathcal{F}^{-1}(\phi^{-1}(x,y)) \textit{ consists of 2 tori} \\ x-y, x+y \in 4\pi h\mathbb{Z}, & \textit{if } \mathcal{F}^{-1}(\phi^{-1}(x,y)) \textit{ consists of 4 tori} \end{cases} \}.$$ [12]

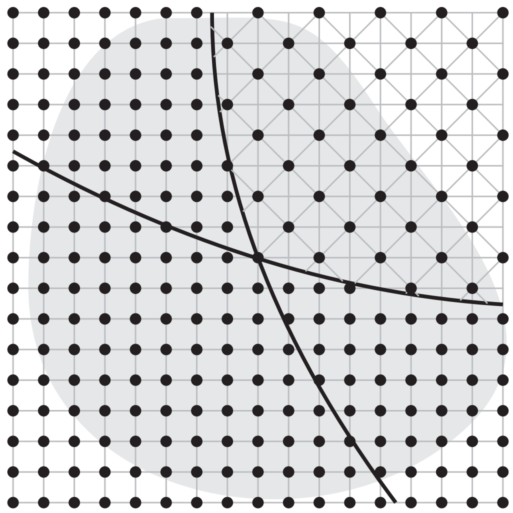

FIGURE 6. An example of the lattice $\psi\mathcal{F}(L_h)$ from Proposition 3.4. The two curves are the ψ-image of the bifurcation diagram. The shaded area is $\psi(V)$.

Part (c) is most interesting in the context of quantization of the Manakov top system. Roughly speaking, it predicts the qualitative view of the joint spectrum lattice of a quantized Manakov top, see below.

For the quantum Manakov top described in [17], the joint spectrum of the two quantum operators was numerically computed and visualized by Sinitsyn and Zhilinskii [32, figures 1 and 13]. The reader is invited to compare these figures to fig. 6 here: they are very similar! For convenience, we reproduce [32, figures 1 and 13] here in fig. 7.

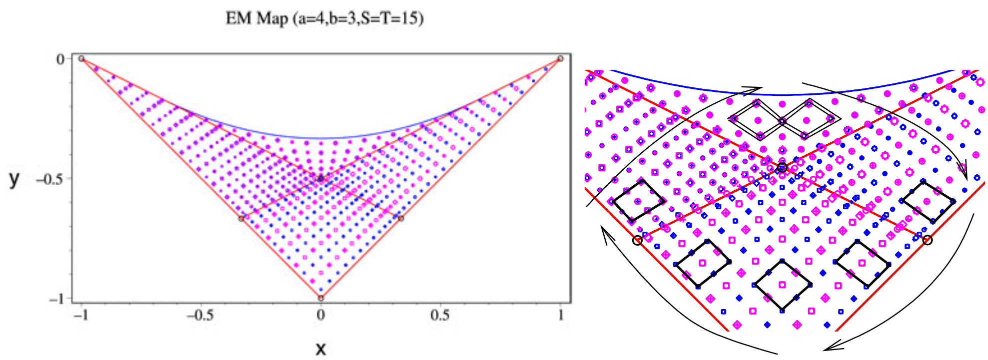

FIGURE 7. Exact joint spectrum of the quantum Manakov top reproduced from [32, figures 1 and 13].

Fig. 6 grasps the main features of the lattice from [32]. Note that this figure is obtained by general arguments, without any computation. An analogue of Proposition 3.4 is true for other integrable systems having saddle-saddle singularities of the same type, including the Clebsch system.

Remark 3.2. When we say that fig. 6 is similar to [32], we ignore different symmetry types of the eigenvalues pictured by different shapes and colors in [32, figures 1 and 13] (i.e. consider all the points on these figures as black points). The author is grateful to Professor B.I. Zhilinskii for indicating that there is an important feature of rearrangement of different types of eigenvalues near the bifurcation diagram. It would be very interesting to find a classical description of this phenomenon as well.

[12] Recall that the bifurcation diagram splits V into four domains. On one of these domains, the $\mathcal{F}$-preimage of a point consists of 4 tori, and on the other ones it consists of 2 tori.

Proposition 3.4 describes the lattice $\mathcal{F}(L_h)$ 'up to homeomorphism'. There are general results stating that $\mathcal{F}(L_h)$ (or its modification, e.g. a Maslov correction) approximates the spectral lattice of the quantum system for different quantization schemes including Toeplitz quantization [9], Maslov asymptotic quantization [16], pseudo-differential quantization (the first two are applicable to the Manakov top). Unfortunately, the author was not able to find any general result of this kind in the framework of quantization used in [32]. Thus we *do not prove* that $\mathcal{F}(L_h)$ does indeed approximate the spectrum of the quantum Manakov top from [32]. Discussion above shows this is very likely to be true.

4. Proofs of Propositions 3.1, 3.3 and 3.4

Proof of Proposition 3.1. Denote $M := M_{d_1,d_2}$. We will check the three conditions of Theorem 1.2. Condition (b) is obvious. Condition (c) holds automatically, see Remark 1.1. It is left to check condition (a). We will check the equivalent condition (a') from Remark 1.2 instead. Denote $H_i = h_i|_M$. For each $i = 1, 2$ we get $\mathrm{d}H_i(P) = 0$. This is equivalent to the fact that

$$\mathrm{d}h_i(P) = \lambda_i \mathrm{d}f_1(P) + \mu_i \mathrm{d}f_2(P)$$

for some $\lambda_i, \mu_i \in \mathbb{R}$. It is easy to check [29] that the equation $\mathrm{d}H_1(P) = 0$ has exactly twelve solutions for $P \in M$:

$$\begin{aligned} &(\pm A, 0, 0, \pm B, 0, 0), \quad (\pm B, 0, 0, \pm A, 0, 0),\\ &(0, \pm A, 0, 0, \pm B, 0), \quad (0, \pm B, 0, 0, \pm A, 0),\\ &(0, 0, \pm A, 0, 0, \pm B), \quad (0, 0, \pm B, 0, 0, \pm A), \end{aligned}$$

where $2A = \sqrt{d_1 + 2d_2} + \sqrt{d_1 - 2d_2}$, $2B = \sqrt{d_1 + 2d_2} - \sqrt{d_1 - 2d_2}$. At these points we also get $\mathrm{d}h_2(P) = 0$, so they are of zero rank. We can assume that $P = (\pm A, 0, 0, \pm B, 0, 0)$ (other points are considered analogously). Let us find a combination $h_1 + \alpha h_2$ such that $\mathrm{d}(h_1 + \alpha h_2)(P) = \beta \mathrm{d}f_1(P)$. Easy calculation shows that

$$\begin{pmatrix} \mathrm{d}h_1(P) \\ \mathrm{d}h_2(P) \\ \mathrm{d}f_1(P) \end{pmatrix} = \begin{pmatrix} 2b_1 m_1, & 0, & 0, & -2b_1 p_1, & 0, & 0 \\ 0, & 0, & 0, & 2(b_1 + b_2)(b_1 + b_3)p_1, & 0, & 0 \\ m_1, & 0, & 0, & p_1, & 0, & 0 \end{pmatrix}$$

so we can take $\beta = 2b_1$, $\alpha = 2b_1/(b_1 + b_2)(b_1 + b_3)$. Let us prove that $\mathrm{d}^2(H_1 + \alpha H_2)(P)$ is a non-degenerate form on $T_P M$. Clearly

$$\mathrm{d}^2(H_1 + \alpha H_2)(P) = (\mathrm{d}^2(h_1 + \alpha h_2 - \beta f_1)(P))|_{T_P M}.$$

In the basis $(\partial/\partial p_2, \partial/\partial p_3, \partial/\partial q_2, \partial/\partial q_3)$ for $T_P M$ we get

$$\mathrm{d}^2(H_1 + \alpha H_2)(P) = 2\,\mathrm{diag}\,(b_2 - b_1, b_3 - b_1, c/(b_1 + b_3), c/(b_1 + b_2))$$

where $c = b_1 b_2 + b_1 b_3 - b_2 b_3 - b_1^2$. If $c \neq 0$, then condition (a) is satisfied and P is non-degenerate by Theorem 1.2.

Suppose $c = 0$; we will come to a contradiction. Let $\{b'_1, b'_2, b'_3\}$ be some parameters close to $\{b_1, b_2, b_3\}$. Let h'_1, h'_2 be the integrals of the system corresponding to parameters $\{b'_i, d_1, d_2\}$ and $H'_i = h'_i|_M$. Define α', c' analogously to α, c replacing b_i by b'_i. We can choose b'_i such that $c' \neq 0$, hence condition (a) of Theorem 1.2 is satisfied for the system with parameters b'_i. By the hypothesis, the bifurcation diagram for b'_i also satisfies condition (b). Thus P is non-degenerate for the system with parameters b'_i. Moreover, by the hypothesis and Theorem 1.1 point P has the same Williamson type (see §1) for each b'_i. Thus any linear combination of Hessians of the integrals, including combination $\mathrm{d}^2(H'_1 + \alpha' H'_2)(P)$, has the same signature for each b'_i. On the other hand, c' can be of arbitrary sign when b'_i are arbitrarily close to b_i. Thus there are two sets of parameters b'_i arbitrarily close to b_i such that $\mathrm{d}^2(H'_1 + \alpha' H'_2)(P)$ has different signatures. This contradiction proves that $c \neq 0$. ∎

Proof of Proposition 3.3. By [6, Theorems 9.7, 9.8], $\mathcal{F}^{-1}(V)$ is Liouville equivalent to one of the 39 items from [6, 9.6, Table 9.1]. It is easy to identify the correct item. We know that the numbers of tori in the preimage of a point in V are 2/2/2/4 depending on one of the four domains containing the point. The only two items in the table [6, Table 9.1] satisfying this condition have numbers 12 and 17. However, item 12 is different because it contains a non-orientable separatrix, and by [29] the Manakov top does not. Thus $\mathcal{F}^{-1}(V)$ is Liouville equivalent to item 17 from [6, Table 9.1] which corresponds by [6, Table 9.3] to $(C_2 \times C_2)/(\alpha, \alpha)$. ■

Proof of Proposition 3.4. The symplectic form ω is exact on U because U retracts onto the fiber $\mathcal{F}^{-1}(\mathcal{F}(P))$ which is Lagrangian. Let θ be a 1-form on U such that $d\theta = \omega$. Recall Proposition 3.3 stating there is a fibered 2-covering $\pi : C_2 \times C_2 \to U$. We will denote the lift of θ to $C_2 \times C_2$ by Θ.

Let us define two functions A_1, A_2 on $C_2 \times C_2$ as follows: $A_1(p, q) := 1/2 \int_{L(p)\times\{q\}} \Theta$ and $A_2(p, q) := 1/2 \int_{\{p\}\times L(q)} \Theta$ where $(p, q) \in C_2 \times C_2$ and $L(x)$ denotes the fiber through $x \in C_2$ on C_2. Recall that all fibers on C_2 except for the singular one are a disjoint union of two circles.

Let $L \subset C_2$ be a regular fiber on C_2. It is a disjoint union of two circles S_1, S_2. Then by [6, Lemma 4.4], for each $q \in C_2$ the fiberwise symplectomorphism from Condition 3.1 which takes the torus containing $\pi(S_1 \times \{q\})$ to the torus containing $\pi(S_2 \times \{q\})$ must take $\pi(S_1 \times \{q\})$ to a cycle homologous to $\pi(S_2 \times \{q\})$. Thus $\int_{S_1\times\{q\}} \Theta = \int_{S_2\times\{q\}} \Theta$. Consequently, for each $(p, q) \in C_2 \times C_2$ on a regular fiber we get $A_1 = \int_{S(p)\times\{q\}} \Theta$, $A_2 = \int_{\{p\}\times S(q)} \Theta$ where $S(x)$ denotes the connected component of the fiber on C_2 through x which contains x. This notation will be used further.

Now, A_1, A_2 are invariant under the involution (α, α) from Proposition 3.3 and thus can be pushed forward to functions a_1, a_2 on U. Next, a_1, a_2 are constant on the $\mathcal{F}$-fibers and thus can be regarded as functions of h_1, h_2 defined over $V = \mathcal{F}(U) \subset \mathbb{R}^2$.

Suppose $(p, q) \in C_2 \times C_2$ are such that $\mathcal{F}^{-1}(\mathcal{F}(\pi(p, q)))$ is a union of two tori T_1, T_2. Let T_1 be the torus containing $\pi(p, q)$ (the other is considered by changing p, q). For each $j = 1, 2$ the preimage $\pi^{-1}(T_j)$ is a union of two tori. In this case, cycles $\pi(S(p) \times \{q\})$, $\pi(\{p\} \times S(q))$ define are a basis of $H_1(T_1; \mathbb{Z})$. This means that $a_1(\pi(p, q)), a_2(\pi(p, q))$ satisfy the desired equality in Proposition 3.4(a) for the case of 2 tori.

Now suppose $(p, q) \in C_2 \times C_2$ are such that $\mathcal{F}^{-1}(\mathcal{F}(\pi(p, q)))$ is a union of four tori T_j. Then for each $j = 1, \dots 4$ the preimage $\pi^{-1}(T_j)$ is a unique torus. Let T_1 be the torus containing $\pi(p, q)$ (others are considered by changing p, q). If we denote $S_a := S(p) \times \{q\}$ and $S_b := \{p\} \times S(q)$, then $\pi^{-1}(T_1) = S_a \times S_b \subset C_2 \times C_2$. The involution (α, α) acts on it as simultaneous rotation of S_a and S_b by 180°. In this case $\pi(S_a)$ and $\pi(S_b)$ generate an index 2 subgroup in $H_1(T_1; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Moreover, $\pi(S_a + S_b)$ and $\pi(S_a - S_b)$ generate a subgroup of type $2\mathbb{Z} \oplus 2\mathbb{Z}$ which implies the desired equality in Proposition 3.4(a) for the case of 4 tori. Proposition 3.4(a) is proved.

Proposition 3.4(b) is immediately implied by the following local structure of functions (a_1, a_2). This map is a local diffeomorphism at the regular values of $\mathcal{F}$ by the Arnold-Liouville theorem. If $R \in V$ belongs to one of the branches of the bifurcation diagram, then in suitable smooth local coordinates (u, v) at R the map ψ can be written as $(u \ln u + f(u, v), v + \text{const})$ with the bifurcation diagram given locally by $\{u = 0\}$. Here f is a smooth function. This is well-known, see e.g. [10, Theorem 1.11]. Finally, if R is the point Q at which two branches of the bifurcation diagram intersect, ψ can be written as $(u \ln u + f(u, v), v \ln v + g(u, v))$ with the bifurcation diagram given by $\{u = 0\} \cup \{v = 0\}$. Here f, g are smooth functions. Proposition 3.4(c) follows from parts (a), (b).

Acknowledgements. The author is grateful to A.V. Bolsinov and A.T. Fomenko for fruitful discussions and constant support, and to San Vũ Ngọc and B.I. Zhilinskii for encouraging comments.

The work was presented at conferences “Geometry, Dynamics, Integrable systems” GDIS 2010 (Belgrade, Serbia), “Finite Dimensional Integrable Systems in Geometry and Mathematical Physics” FDIS 2011 (Jena, Germany) and “Bihamiltonian Geometry and Integrable Systems” BGIS 2011 (Bedlewo, Poland). The author is grateful to the organizers of the above conferences for their hospitality.

References

[1] Arnol'd V.I., Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1978.

[2] Audin M., Hamiltonian Monodromy via Picard-Lefschetz Theory, Comm. Math. Phys. **229**:3 (2002), 459-489.

[3] Birtea P., Caşu I., Ratiu T. S., Turhan M., Stability of equilibria for the $\mathfrak{so}(4)$ free rigid body, preprint, arXiv:0812.3415v3 [math.DS].

[4] Birtea P., Caşu I., Energy methods in the stability problem for the $\mathfrak{so}(4)$ free rigid body, preprint, arXiv:1010.0295v1 [math.DS].

[5] Bolsinov A.V., Methods of calculation of Fomenko-Zieschang topological invariant, In: Advances in Soviet Mathematics vol. 6 (1991), editor A.T. Fomenko, 147-183.

[6] Bolsinov A.V., Fomenko A.T., Integrable Hamiltonian systems. Geometry, topology, classification, Chapman and Hall/CRC, Boca Raton, Florida, 2004.

[7] Bolsinov A.V., Oshemkov A.A., Singularities of integrable Hamiltonian systems, In: Topological Methods in the Theory of Integrable Systems, Cambridge Scientific Publ., 2006, 1-67.

[8] Cavicchioli A., Repovš D., Skopenkov A.B., An Extension of the Bolsinov-Fomenko Theorem on Orbital Classification of Integrable Hamiltonian Systems, Rocky Mountain J. Math. **30**:2 (2000), 447-476.

[9] Charles L., Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators. Comm. Partial Differential Equations 28 (2003), no. 9-10,

[10] Colin de Verdière Y., Vũ Ngọc San, Bohr-Sommerfeld rules for 2D integrable systems, Ann. Sci. Ecole Norm. Sup. **36** (2003), 1-55. arXiv:math/0005264v1 [math.AP]

[11] Duistermaat J.J., On global action angle coordinates, Comm. Pure Appl. Math. **33** (1980), 687-706.

[12] Eliasson L.H., Normal forms for Hamiltonian systems with Poisson commuting integrals – elliptic case, Comment. Math. Helvetici **65** (1990), 4-35.

[13] Feher L., Marshall I., Stability analysis of some integrable Euler equations for $SO(n)$, J. Nonlin. Math. Phys. **10** (2003), 304-317. arXiv:math-ph/0203053v2.

[14] Fomenko A.T., Topological invariants of Liouville integrable Hamiltonian systems, Functional Analysis and Its Applications **22**:4 (1990), 286-296.

[15] Ito H., Convergence of Birkhoff normal forms for integrable systems, Comment. Math. Helvetici **64** (1989), 412-461.

[16] Karasev M.V., Maslov V.P., Nonlinear Poisson Brackets: Geometry and Quantization. American Mathematical Society, Providence (1993).

[17] Komarov I.V., Kuznetsov V.B., Quantum Euler–Manakov top on the 3-sphere S_3, J. Phys. A: Math. Gen. **24** (1991), L737–L742.

[18] Kudryavtseva E.A., Realization of smooth functions on surfaces as height functions, Sbornik: Mathematics **190**:3 (1999), 349-405.

[19] Lerman L.M., Isoenergetical structure of integrable Hamiltonian systems in an extended neighborhood of a simple singular point: three degrees of freedom, Methods of qualitative theory of differential equations and related topics, Supplement, 219-242, Amer. Math. Soc. Transl. Ser. 2, 200, Amer. Math. Soc., Providence, RI, 2000.

[20] Lerman L.M., Umanskiy Ya.L., Four-Dimensional Integrable Hamiltonian Systems with Simple Singular Points (Topological Aspects), Translations of mathematical monographs, volume 176, AMS, 1998.

[21] Manakov S.V., Note on the integration of Euler's equation of the dynamics of an N dimensional rigid body, Funct. Anal. Appl. **11** (1976), 328-329.

[22] Matveev V.S., Computation of values of the Fomenko invariant for a point of the type "saddle-saddle" of an integrable Hamiltonian system, Tr. Semin. Vektorn. Tenzorn. Anal. **25** (1993), 75-105 (in Russian).

[23] MacDuff D., Salamon D., Introduction to symplectic topology, Clarendon Press, Oxford (1998).

[24] Miranda E., On symplectic linearization of singular Lagrangian foliations, Ph. D. thesis, Universitat de Barcelona, June, 2003.

[25] Morozov P.V., The Liouville classification of integrable systems of the Clebsch case, Sbornik: Mathematics **193:10** (2002), 1507-1533.

[26] Nguen Tien Zung, Decomposition of nondegenerate singularities of integrable Hamiltonian systems, Lett. Math. Phys. **33** (1995), 187-193.

[27] Nguen Tien Zung, Symplectic topology of integrable Hamiltonian systems, I: Arnold-Liouville with singularities, Compositio Math. **101** (1996), 179-215. arXiv:math/0106013v1 [math.DS]

[28] Oshemkov A.A., Topology of isoenergy surfaces and bifurcation diagrams for integrable cases of rigid body dynamics on $so(4)$, Uspekhi Mat. Nauk **42** (1987), 199-200.

[29] Oshemkov A.A., Fomenko invariants for the main integrable cases of the rigid body motion equations, In: Advances in Soviet Mathematics vol. 6 (1991), editor A.T. Fomenko, 67-146.

[30] Oshemkov A.A., Classification of hyperbolic singularities of rank zero of integrable Hamiltonian systems, Sbornik: Mathematics **201**:8 (2010), 1153-1191.

[31] Rüssmann H., Über das Verhalten analytischer Hamiltonscher Differentialgleichungen in der Nähe einer Gleichgewichtslösung, Math. Ann. **154** (1964), 285-300.

[32] Sinitsyn E., Zhilinskii B., Qualitative Analysis of the Classical and Quantum Manakov Top, SIGMA 3 (2007), 046, 23 pages. arXiv:math-ph/0703045v1

[33] Vey J., Sur certains systemes dynamiques separables, Amer. J. Math. **100** (1978), 591-614.

[34] Vũ Ngọc San, Wacheux C., Smooth normal forms for integrable hamiltonian systems near a focus-focus singularity, preprint (2011). arXiv:1103.3282v1 [math.SG]

[35] Williamson J., On the algebraic problem concerning the normal forms of linear dynamical systems, Amer. J. Math. **58**:1 (1936), 141-163.

Department of Differential Geometry and Applications, Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119991, Russia.
E-mail address: dtonkonog@gmail.com