Incompressible limit of the linearized Navier–Stokes equations

N.A. Gusev*

Abstract

Initial–boundary value problem for the linearized equations of viscous barotropic fluid motion in a bounded domain is considered. Existence, uniqueness and estimates of weak solutions to this problem are derived. Convergence of the solutions towards the incompressible limit when compressibility tends to zero is studied.

Contents

*PhD student, Moscow Institute of Physics and Technology; e-mail: n.a.gusev@gmail.com

1 Introduction

In many cases mathematical treatment of liquids is done in the framework of *incompressible* fluid. However, from the physical point of view, all the liquids existing in nature are *low compressible*. Therefore it is reasonable to study asymptotic properties of solutions to equations of low compressible fluid, in particular, convergence to the corresponding incompressible limit.

To formalize the notion of the incompressible limit one should introduce a parameter α, which represents a measure of compressibility of a fluid. Having this done, one can study passage to the limit when $\alpha \to 0$. We will call such a parameter α the *compressibility* of a fluid.

It appears that the first mathematical treatment of the incompressible limit was carried out in [1] for the barotropic Euler equations. The equation of state in [1] had the form $p = k\varrho^\gamma$, where $k = \text{const}$, and passage to the limit when $\gamma \to \infty$ was studied. In this case the compressibility can be defined as $\alpha = (k\gamma)^{-1}$. Later, in [2] the barotropic Euler (and Navier–Stokes) equations with equation of state $p = \lambda^2 P(\varrho)$ were considered and passage to the limit when $\lambda \to \infty$ was studied. In this case the compressibility can be defined as $\alpha = \lambda^{-2}$.

Slightly different approach to the introduction of α was used in [3, 4, 5, 6]. That approach is based on putting the equations of fluid motion in a dimensionless form which contains the Mach number ε, and on the argument that when ε is small, the flow is nearly incompressible. In barotropic case the resulting system is equivalent to the equations of motion of a fluid with equation of state $p = P(\varrho)/\varepsilon^2$, and the compressibility can be defined as $\alpha = \varepsilon^2$.

Another approach was suggested in [7, 8]. That approach is based on the idea that the incompressible model can be formally obtained from the compressible model with the equation of state $\varrho = F(p)$ if we let $F = F_0$, where $F_0(p) \equiv \varrho_0 > 0$. Therefore one might expect that when F is close to F_0 the flow is nearly incompressible. For instance, in [7] the function F was given by $F(p) = \varrho_0 + \alpha R(p)$ and the parameter α was called the *compressibility factor*. In this case α is proportional to $d\varrho/dp$, so it measures the response of the density to variations of the pressure; when $\alpha = 0$ there is no response.

In spite of the observed differences, all the approaches used in [1, 2, 3, 4, 5] lead to the convergence of the velocity of the compressible fluid towards the velocity of the incompressible fluid when $\alpha \to 0$. The topology of this convergence, however, depends on the problem setting. In case of strong solutions (which are local in time) and divergence-free initial data (see e.g. [1, 2]) such convergence holds in $C(0, T; H^s)$-norm, where H^s denotes the Sobolev space. (In non-barotropic case convergence of local solutions with general initial data was studied in [9] in the low Mach number framework.)

Convergence of weak solutions (which are global in time) with general initial data was studied in [3, 4, 5, 6]. In particular, it was proved (see [3, 4]) that there exists a sequence of weak solutions to the compressible Navier–Stokes(–Fourier) equations such that the velocity converges *weakly* (in $L^2(0, T; H^1_0)$) to the velocity of the incompressible fluid. However from the physical consideration one could desire *strong* convergence of the velocity (and pressure as well) to yield better approximation of compressible fluid by incompressible one.

Strong convergence of the velocity was established in [10, III, §8] for the solutions of "compressible" system arising in the method of artificial compressibility. It was also proved that the gradient of the pressure converges weakly, but convergence of the pressure itself was not examined.

In this paper we study the incompressible limit of weak solutions to the *linearized* equations of compressible fluid motion. These equations describe the first order correction to the equations of the incompressible fluid motion, which arises due to compressibility. The linearized equations are considerably easier than the original nonlinear equations, but have similar structure. This allows us to carry out a more comprehensive study of the passage to the limit when $\alpha \to 0$. In particular, we study the convergence of the pressure. (Also note that the linearized equations of compressible fluid are of interest on they own. For instance, spectral properties of the operator corresponding to linearized steady equations of compressible fluid were examined in [11].)

Linearization of the compressible Navier–Stokes equations near a state with zero reference velocity was studied in [12, 13]. Estimate of strong solution to an initial–boundary value problem for these equations in a bounded three-dimensional domain was derived in [13], and existence of strong solution to Cauchy problem in the whole space for them was established in [12]. Estimates of strong solutions to linearization of the compressible Navier–Stokes equations near a state with non-zero smooth reference velocity was studied in [9]. It appears that existence of weak solutions to these equations has not been addressed, especially when the reference velocity is non-smooth.

In this paper we derive existence, uniqueness and estimates of weak solutions to the initial–boundary value problem for the linearized Navier–Stokes equations. We examine convergence of these solutions to the incompressible limit when the compressibility tends to zero. Briefly, we prove that

- in general case the velocity field converges *weakly*;
- if the initial condition for the velocity is divergence-free then the velocity converges *strongly* and the pressure converges $*$-*weakly*;
- if, in addition, the *initial condition* for the pressure is compatible with the *initial value* of the pressure in the incompressible system then the convergence of the pressure is *strong*.

1.1 Overview of the main results

Let $D \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain with a piecewise-smooth boundary ∂D. In Section 3 we show that the homogeneous linearization of the Navier–Stokes equations in D near a state with velocity $\overline{\mathfrak{u}}$ and constant density can be written in the form

$$\rho_t + \operatorname{div}(\mathbf{u} + \rho\overline{\mathrm{u}}) = 0, \tag{1.1}$$

$$\mathbf{u}_t + (\mathbf{u}, \nabla)\overline{\mathrm{u}} + \operatorname{div}(\overline{\mathrm{u}} \otimes \mathbf{u}) + \nabla p = \nu\Delta\mathbf{u} + \kappa\nabla \operatorname{div} \mathbf{u} + \rho\mathbf{f}, \tag{1.2}$$

$$\rho = \alpha p, \tag{1.3}$$

where $\alpha > 0$ is the *compressibility*; the unknowns ρ, p and $\mathbf{u}$ are proportional to the variations of density, pressure and velocity respectively; $\nu > 0$, $\kappa \geq 0$; $\mathbf{f}$ is a fixed vector field. We assume that $\overline{\mathrm{u}}|_{\partial D} = 0$ and consider the following initial and boundary conditions for the system (1.1)–(1.3):

$$\mathbf{u}|_{t=0} = \mathbf{u}^\circ, \quad \mathbf{u}|_{\partial D} = 0, \quad p|_{t=0} = p^\circ. \tag{1.4}$$

When $\alpha = 0$, the system (1.1)–(1.3) formally takes the form

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.5}$$

$$\boldsymbol{v}_t + (\boldsymbol{v}, \nabla)\overline{\mathrm{u}} + \operatorname{div}(\overline{\mathrm{u}} \otimes \boldsymbol{v}) + \nabla q = \nu\Delta\boldsymbol{v}, \tag{1.6}$$

and for this system we consider the following initial and boundary conditions:

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}^\circ, \quad \boldsymbol{v}|_{\partial D} = 0. \tag{1.7}$$

Our goal is to study the behavior of the solutions to (1.1)–(1.4) when $\alpha \to 0$.

Let $T > 0$. To present our results in the shortest form let us suppose that $\overline{\mathrm{u}}$ is smooth (i.e. $\overline{\mathrm{u}} \in C^\infty(\overline{D \times (0,T)})^d$), $\overline{\mathrm{u}}|_{\partial D} = 0$ and

$$\{\boldsymbol{v}, q\} \in C^\infty(\overline{D \times (0,T)})^d \times C^\infty(\overline{D \times (0,T)})$$

is a smooth solution to (1.5)–(1.7). (When $\partial D \in C^\infty$ and $\overline{\mathrm{u}} \equiv 0$, such solution exists if, for instance, $\boldsymbol{v}^\circ$ is an eigenfunction of the Stokes problem, see e.g. [10], I.2.6.)

Theorem 1.1. *For all $\alpha \in (0,1)$, $p^\circ \in L^2(D)$ and $\mathbf{u}^\circ \in L^2(D)^d$ the problem (1.1)–(1.4) has a unique weak solution*

$$\{\mathbf{u}, p\} \equiv \{\mathbf{u}_\alpha, p_\alpha\} \in L^2(0,T; H_0^1(D)^d) \times L^\infty(0,T; L^2(D)).$$

(Theorem 1.1 follows from Corollary 4.5. Similar existence results for the Navier–Stokes equations were obtained in [3, 4, 5].)

Let $H(D)$ denote the divergence-free subspace of $L^2(D)^d$ and P_H denote the orthogonal projector of $L^2(D)^d$ onto $H(D)$ (see Section 2).

Theorem 1.2. *If $P_H\mathbf{u}^\circ = \boldsymbol{v}^\circ$ then*

$$\mathbf{u}_\alpha \rightharpoonup \boldsymbol{v} \;\; \text{in} \;\; L^2(0,T; H_0^1(D)^d),$$

$$\nabla p_\alpha \overset{*}{\rightharpoonup} \nabla q \;\; \text{in} \;\; H^{-1}(D \times (0,T))^d$$

as $\alpha \to 0$.

(Theorem 1.2 follows from Theorem 5.3. Similar results for the Navier–Stokes equations were obtained in [3, 4, 5].)

Theorem 1.3. *If* $\mathbf{u}^\circ = \boldsymbol{v}^\circ$ *then*

$$\mathbf{u}_\alpha \to \boldsymbol{v} \quad \text{in} \quad L^2(0,T;H_0^1(D)^d),$$
$$p_\alpha \overset{*}{\rightharpoonup} \widehat{q} \quad \text{in} \quad L^\infty(0,T;L^2(D))$$

as $\alpha \to 0$*, where* $\{\boldsymbol{v}, \widehat{q}\}$ *is the solution to* (1.5)–(1.7) *such that*

$$\int_D \widehat{q}(t)\, dx = \int_D p^\circ \, dx \tag{1.8}$$

for a.e. $t \in [0,T]$*.*

(This Theorem follows from Theorems 5.5 and 5.10. Similar result concerning convergence of the velocity was obtained in [10] for the system coming from the artificial compressibility method.)

Integrating (1.1) over D we observe that for any $\alpha \in (0,1)$ the pressure p_α satisfies (1.8), i.e. $\int_D p_\alpha(t)\, dx = \int_D p^\circ \, dx$ for a.e. $t \in [0,T]$. However the pressure q in the incompressible system may not satisfy (1.8), since q is defined up to an additive function of time. Nevertheless Theorem 1.3 shows that (1.8) is conserved during the passage to the limit when $\alpha \to 0$.

Theorem 1.3 also shows that the compatibility condition $\mathbf{u}^\circ = \boldsymbol{v}^\circ$ is sufficient for the strong convergence of the velocity. The following result shows that there is a similar compatibility condition for p° and $q|_{t=0}$ which is sufficient for the strong convergence of the pressure:

Theorem 1.4. *If* $\mathbf{u}^\circ = \boldsymbol{v}^\circ$ *and* $\nabla p^\circ = \nabla q|_{t=0}$ *then*

$$\mathbf{u}_\alpha \to \boldsymbol{v} \quad \text{in} \quad L^2(0,T;H_0^1(D)^d),$$
$$p_\alpha \to \widehat{q} \quad \text{in} \quad L^\infty(0,T;L^2(D))$$

as $\alpha \to 0$*, where* $\widehat{q}$ *is defined in Theorem 1.3.*

(This Theorem follows from Theorem 5.11.)

In Section 5.3 we give an explicit solution to the problem (1.1)–(1.4) in simplified one-dimensional setting. Using this solution we demonstrate that the *sufficient conditions* (for the convergence of velocity and pressure) given in Theorems 1.2–1.4 are also *necessary*. Therefore the results of these theorems can be summarized in the following table:

Condition ↓ Result →	$\mathbf{u}_\alpha \rightharpoonup \boldsymbol{v}$	$\mathbf{u}_\alpha \to \boldsymbol{v}$	$p_\alpha \overset{*}{\rightharpoonup} \widehat{q}$	$p_\alpha \to \widehat{q}$
$P_H \mathbf{u}^\circ = \boldsymbol{v}^\circ$, $\mathbf{u}^\circ \neq \boldsymbol{v}^\circ$	+	−	−	−
$\mathbf{u}^\circ = \boldsymbol{v}^\circ$, $\nabla p^\circ \neq \nabla q|_{t=0}$	+	+	+	−
$\mathbf{u}^\circ = \boldsymbol{v}^\circ$, $\nabla p^\circ = \nabla q|_{t=0}$	+	+	+	+

1.2 Methods of proofs

The proof of the existence part of Theorem 1.1 relies on the Galerkin method, energy estimates for the approximate solutions and Banach–Alaoglu theorem. In the proof of the uniqueness part we first use the DiPerna–Lions theory to establish the energy equality (4.13). From it we derive energy estimates (see Theorem 4.3) which imply uniqueness since (1.1)–(1.3) is linear.

The energy estimates and Banach–Alaoglu theorem also allow us to select a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$: $\alpha_n \to 0$, $n \to \infty$, such that $\mathbf{u}_{\alpha_n}$ converges weakly as $n \to \infty$. We show that the limit of $\mathbf{u}_{\alpha_n}$ is $\boldsymbol{v}$. Since $\boldsymbol{v}$ is unique, $\mathbf{u}_\alpha \rightharpoonup \boldsymbol{v}$ as $\alpha \to 0$, which completes the proof of Theorem 1.2.

The general strategy of proving Theorems 1.3 and 1.4 is very simple and can be roughly formulated as follows. We rewrite the problem (1.1)–(1.4) in the form $(A+\alpha B)u_\alpha = 0$ and rewrite the problem (1.5)–(1.7) in the form $Av = 0$, where A and B are some differential operators. The assumptions on the regularity of the solution to (1.5)–(1.7) allow us to conclude that the difference $u_\alpha - v$ satisfies $(A+\alpha B)(u_\alpha - v) = -\alpha Bv$. Then we generalize the energy estimates for the problem (1.1)–(1.4) with additional terms on the right-hand side and apply these estimates to the system for the difference $u_\alpha - v$. We get an estimate of the following sort: $\|u_\alpha - v\| \leq C_\alpha \alpha \|Bv\|$, which implies the strong convergence if $C_\alpha = o(1/\alpha)$ when $\alpha \to 0$.

Under the assumptions of Theorem 1.3 we can show only that the pressure is bounded (the corresponding constant $C_\alpha \sim 1/\alpha$ is not $o(1/\alpha)$), but then we use Banach–Alaoglu theorem to show weak* convergence of the pressure and pass to the limit in (1.8).

Under the assumptions of Theorem 1.4 we can use an operator similar to the Bogovskii operator (which is right-inverse to $\operatorname{div}\colon H_0^1 \to L^2$) to improve our estimates (see Theorem 4.8) and show that the corresponding constant C_α is $o(1/\alpha)$.

2 Notation and preliminaries

2.1 Common functional spaces

Let $E \subset \mathbb{R}^n$ be a bounded domain, $n \in \mathbb{N}$. We will use the following standard spaces:

- $L^p(E)$ is the Lebesgue space of real-valued functions on E summable with p-th power, $1 \leq p < \infty$, or essentially bounded when $p = \infty$;
- $\widehat{L}^p(E) = \{ u \mid u \in L^p(E),\ \int_E u\,dx = 0 \}$;
- $H^s(E) = W^{s,2}(E)$, where $W^{s,p}(E)$ is the Sobolev space of real-valued functions whose weak derivatives up to order $s \in \mathbb{N}$ belong to $L^p(E)$;
- $C_0^\infty(E)$ is the space of smooth real-valued functions on E with compact support;

- $\mathscr{D}(\mathbb{R}^n)$ is the space of test functions on $\mathbb{R}^n$, i.e. $\mathscr{D}(\mathbb{R}^n)$ is $C_0^\infty(\mathbb{R}^n)$ where the following definition of convergence is introduced:
  $\mathscr{D}(\mathbb{R}^n) \supset \{\varphi_m\}_{m\in\mathbb{N}} \to \varphi \in \mathscr{D}(\mathbb{R}^n)$, $m \to \infty$, if and only if

  1. $\exists R > 0$: $\operatorname{supp}\varphi_m \subset \{\, x \in E \mid |x| < R\}$, $m \in \mathbb{N}$;
  2. $\forall (k_1, k_2, ..., k_n) \in (\mathbb{N} \cup 0)^n$
  $$\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}...\partial_{x_n}^{k_n}\varphi_m \to \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}...\partial_{x_n}^{k_n}\varphi \quad \text{in} \quad C(U_R), \quad m \to \infty.$$

  $\mathscr{D}'(\mathbb{R}^n)$ is the space of distributions on E, i.e. the space dual to $\mathscr{D}(\mathbb{R}^n)$;
  $\mathscr{D}(E) = \{\, \varphi \in \mathscr{D}(\mathbb{R}^n) \mid \operatorname{supp}\varphi \cap E \text{ is compact in } E\}$;

- $H_0^1(E)$ and $W_0^{s,p}(E)$ denote the closures of $C_0^\infty(E)$ in $H^1(E)$-norm and $W^{s,p}(E)$-norm respectively;

- $H^{-1}(E)$ denotes the dual space of $H_0^1(E)$;

(For $E = (a, b) \subset \mathbb{R}$ we will omit undue brackets, i.e. $L^2(a, b) = L^2((a, b))$.)

Let $D \subset \mathbb{R}^d$ be a bounded domain with a piecewise-smooth boundary ∂D, $d \in \mathbb{N}$. Let $T > 0$.

For $\mathbb{R}^k$-valued functions ($k \in \mathbb{N}$) we will use Cartesian products of these spaces, e.g. $L^2(D)^k$. Since $H^{-1}(D)^k$ is linearly and continuously isomorphic to the dual space of $H_0^1(D)^k$, let $H^{-1}(D)^k$ *denote* the latter.

Vector-valued functions will be denoted by bold letters. We will use the following spaces for such functions:

- $\mathcal{V}(D) = \big\{\mathbf{u} \,\big|\, \mathbf{u} \in C_0^\infty(D)^d,\ \operatorname{div}\mathbf{u} = 0\big\}$;

- $H(D)$ is the closure in $L^2(D)^d$-norm of $\mathcal{V}(D)$;

- $V(D)$ is the closure in $H_0^1(D)^d$-norm of $\mathcal{V}(D)$.

2.2 Scalar products and duality

Let $\|\cdot\|_X$ denote the norm of a Banach space X (with dual space X^*) and let $\langle\cdot,\cdot\rangle_X$ denote the duality brackets for the pair (X^*, X). We will use the following notation:

- $x_n \rightharpoonup x$ means that the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to x weakly;

- $f_n \overset{*}{\rightharpoonup} f$ means that the sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ converges to f $*$-weakly.

Let $(\cdot,\cdot)$ denote the standard dot product in $\mathbb{R}^k$, and let $(\cdot,\cdot)_D$ denote the dot product in $L^2(D)^k$, i.e. $(\mathbf{u}, \mathbf{v})_D = \sum_{i=1}^k \int_D u_i v_i \, dx$.

2.3 Spaces of Banach space valued functions

Consider an arbitrary closed interval $[0,T]$ where $T > 0$. Let X be a Banach space and $s \in \mathbb{N}$. Let $L^p(0,T;X)$ and $W^{s,p}(0,T;X)$ denote accordingly Lebesgue–Bochner and Sobolev–Bochner spaces of X-valued functions of real variable $t \in [0,T]$ (see, e.g., [14]), $1 \le p \le \infty$. (These spaces are separable provided that X is separable [14].) Let f_t denote the weak derivative of a function $f \in W^{1,p}(0,T;X)$ with respect to t. Let $C(0,T;X)$ denote the space of continuous functions $f\colon [0,T] \to X$. Finally, let H be a Hilbert space with the dot product $(\cdot,\cdot)_H$.

Now we recall the following well-known properties of Sobolev space of Banach space valued functions (see, e.g., [14]):

Proposition 2.1. *For any $u \in W^{1,1}(0,T;X)$ there exists unique $\bar{u} \in C(0,T;X)$ such that $\bar{u} = u$ a.e. on $[0,T]$ and*

$$\|\bar{u}\|_{C(0,T;X)} \le C\|u\|_{W^{1,1}(0,T;X)}.$$

Moreover, $\bar{u}$ is absolutely continuous and for a.e. $t \in [0,T]$ it has strong derivative $\partial_t\bar{u}(t)$ (in Fréchet sense) which is for a.e. $t \in [0,T]$ equal to the distributional derivative $u_t(t)$.

Remark 2.2. We will call $\bar{u}$ the *continuous version* of u and denote

$$u|_{t=\tau} := \bar{u}(\tau)$$

for $\forall\tau \in [0,T]$.

Remark 2.3. Conversely, if $u \in C(0,T;X)$ is absolutely continuous and X is reflexive then $u \in W^{1,1}(0,T;X)$. Hence the strong derivative and the distributional derivative of u are equal a.e. on $[0,T]$.

Remark 2.4. For $\forall\varphi \in C^\infty([0,T])$ and $\forall a,b \in [0,T]$ we also have

$$\int_a^b u_t\varphi\,dt = \bar{u}\varphi|_a^b - \int_a^b u\varphi_t\,dt$$

Remark 2.5. If $X = H$ and $u \in W^{1,2}(0,T;H)$ then the mapping $t \mapsto \|\bar{u}(t)\|_H^2$ is absolutely continuous with

$$\partial_t\|\bar{u}\|_H^2 = 2(u_t,u)_H$$

a.e. on $[0,T]$ (hence also in $\mathscr{D}'(0,T)$ by Remark 2.3).

Let X be a reflexive Banach space (with dual space X^*) and let H be a Hilbert space for which there exists a linear bounded dense embedding $\kappa\colon X \to H$. Let $\pi\colon H \to X^*$ be the embedding given by $\pi\colon h \mapsto (h,\kappa(\cdot))_H$, where $(\cdot,\cdot)_H$ is the dot product in H. Then embeddings π and $\imath = \pi\circ\kappa$ are linear, bounded and dense. Triple (X,H,X^*) (with embeddings $\kappa,\pi,\imath$) is said to be a *Gelfand triple*, or *evolution triple* [14]. For given evolution triple let

$$\widetilde{W}^{1,2}(0,T;X) = \left\{ f \;\middle|\; f \in L^2(0,T;X),\ \imath(f) \in W^{1,2}(0,T;X^*)\right\}.$$

This space is sometimes referred to as *Sobolev–Lions* space [15]. It is a reflexive Banach space with norm given by

$$\|f\|_{\widetilde{W}^{1,2}(0,T;X)} = \|f\|_{L^2(0,T;X)} + \|\imath(f)_t\|_{L^2(0,T;X^*)}.$$

Embedding $\imath$ is often omitted and the space $\widetilde{W}^{1,2}(0,T;X)$ is then introduced as the space of functions belonging to $L^2(0,T;X)$ whose weak derivative belongs to $L^2(0,T;X^*)$. In this paper $\imath$, κ and π will be omitted when they are not the subject matter.

The introduced space has the following property (see, e.g., [14]):

Proposition 2.6. *For any $u \in \widetilde{W}^{1,2}(0,T;X)$ there exists unique $\overline{u} \in C(0,T;H)$ such that $\overline{u} = u$ a.e. on $[0,T]$ and*

$$\|\overline{u}\|_{C(0,T;X)} \leq C\|u\|_{\widetilde{W}^{1,2}(0,T;X)}.$$

Remark 2.7. We will call $\overline{u}$ the *continuous version* of u and denote

$$u|_{t=\tau} := \overline{u}(\tau)$$

for $\forall\tau \in [0,T]$.

Remark 2.8. For $\forall a,b \in [0,T]$, $\forall v \in X$ and $\forall\varphi \in C^\infty([0,T])$

$$\int_a^b \langle u_t, v\rangle\, \varphi\, dt = (\overline{u}(t)\varphi(t), v)_H\Big|_a^b - \int_a^b (u,v)_H \varphi_t\, dt.$$

Consequently, *if $\forall v \in X$ and $\forall\varphi \in C_0^\infty([0,T))$*

$$\int_0^T \langle u_t, v\rangle\, \varphi\, dt = -(u_0\varphi(0), v)_H - \int_0^T (u,v)_H \varphi_t\, dt,$$

where $u_0 \in H$, then $\overline{u}(0) = u_0$. (Similar procedure can be used to identify the initial value of a function from $W^{1,2}(0,T;H)$.)

Remark 2.9. The mapping $t \mapsto \|\overline{u}(t)\|_H^2$ is absolutely continuous with

$$\partial_t\|\overline{u}(t)\|_H^2 = 2\,\langle u_t(t), u(t)\rangle$$

for a.e. $t \in [0,T]$ (hence also in $\mathscr{D}'(0,T)$ by Remark 2.3).

In this paper we consider evolution triples $(H_0^1(D)^k, L^2(D)^k, H^{-1}(D)^k)$ ($k \in \mathbb{N}$) and $(V(D), H(D), V(D)^*)$. In both cases the embeddings κ, π and $\imath$ are given by $\kappa\colon \mathbf{u} \mapsto \mathbf{u}$ (natural embedding), $\pi\colon \mathbf{u} \mapsto \int_D (\mathbf{u}, \cdot)\, dx$ and $\imath = \pi\circ\kappa$. For brevity let us also denote $\mathcal{W}(0,T) := \widetilde{W}^{1,2}(0,T;H_0^1(D)^d)$.

The following theorem describes the space which is dual to a Lebesgue–Bochner space [14]:

Proposition 2.10. *If X is a reflexive Banach space, then*

$$L^p(0,T;X)^* = L^{p'}(0,T;X^*),$$

where $1 \leq p < \infty$, $1/p + 1/p' = 1$ and duality is given by $\langle f,g\rangle = \int_0^T fg\, dt$, $f \in L^{p'}(0,T;X^*)$, $g \in L^p(0,T;X)$.

Different constants which are not dependent on the principal parameters (such as initial conditions) will be denoted by the same letter C. The dependence of such constant on some parameter will be indicated in the subscript.

2.4 Compactness

We will use the following well-known sequential version of Banach–Alaoglu theorem:

Proposition 2.11. *Let X be a separable Banach space. Then for any bounded sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ there exist a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f \in X^*$ such that $f_{n_k} \xrightarrow{*} f$ in X^* as $k \to \infty$. In addition, $\|f\|_{X^*} \le \liminf_{k\to\infty} \|f_{n_k}\|_{X^*}$.*

Remark 2.12. If the sequence $\{f_n\}$ from Proposition 2.11 has *no more than one* cluster point in weak* topology, then $f_n \xrightarrow{*} f$ in X^* as $n \to \infty$.

In other words, if a bounded sequence cannot have two different cluster points in weak* topology then *the whole* sequence converges weakly* to some element of X^*.

2.5 Auxiliary inequalities

Let us recall two well-known statements:

Proposition 2.13. *Let $a \ge 0$, $b \ge 0$ and J be real numbers. If $J^2 \le a + bJ$ then $J \le b + \sqrt{a}$.*

Proposition 2.14 (Gronwall's inequality)**.** *Let I be an absolutely continuous nonnegative function of a variable $t \in [0,T]$ and let $\varphi, \psi \in L^1(0,T)$ be nonnegative functions. If the derivative $I'(t)$ satisfies*

$$I'(t) \le \varphi(t)I(t) + \psi(t) \quad \text{a.e. on } [0,T],$$

then for a.e. $t \in [0,T]$

$$I(t) \le e^{\int_0^t \varphi(\tau)\,d\tau} \left(I(0) + \int_0^t \psi(\tau)\,d\tau \right).$$

The proof of Proposition 2.13 is elementary and the proof of Proposition 2.14 can be found e.g. in [16].

We will use the following mix of Propositions 2.13 and 2.14:

Lemma 2.15. *Let I and J be absolutely continuous nonnegative functions of a variable $t \in [0,T]$, $J \in L^2(0,T)$. Let $a, c \in L^1(0,T)$ and $b \in L^2(0,T)$ be nonnegative functions. If for a.e. $t \in [0,T]$*

$$I'(t) + J^2(t) \le a(t)I(t) + b(t)J(t) + c(t) \tag{2.1}$$

then

$$\|J\|_{L^2(0,T)} \le C_a \left(\sqrt{I(0)} + \sqrt{\|c\|_{L^1(0,T)}} + \|b\|_{L^2(0,T)}\right),$$
$$\|I\|_{L^\infty(0,T)} \le C_a \left(I(0) + \|c\|_{L^1(0,T)} + \|b\|^2_{L^2(0,T)}\right),$$

where constant C_a *depends only on* $A = \|a\|_{L^1(0,T)}$

Proof. From (2.1) for a.e. $t \in [0,T]$

$$I'(t) \le a(t)I(t) + b(t)J(t) + c(t)$$

and then, by Proposition 2.14 and Cauchy–Bunyakovsky inequality,

$$I(t) \le \exp\left(\int_0^t a\,d\tau\right)\left(I(0) + \int_0^t (bJ+c)d\tau\right) \le$$
$$\le e^{\|a\|_{L^1(0,t)}}\left(I(0) + \|b\|_{L^2(0,t)}\|J\|_{L^2(0,t)} + \|c\|_{L^1(0,t)}\right)$$

and hence

$$\|I\|_{L^\infty(0,T)} \le e^{\|a\|_{L^1(0,T)}}\left(\|b\|_{L^2(0,T)}\|J\|_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0)\right).$$

Integrating the inequality $J^2 \le aI + bJ + c - I'$ and noting that $I(T) \ge 0$ we obtain

$$\int_0^T J^2 dt \le \int_0^T aI\,d\tau + \int_0^T bJ\,d\tau + \int_0^T c\,d\tau + I(0) \le$$
$$\le \|I\|_{L^\infty(0,T)}\|a\|_{L^1(0,T)} + \|b\|_{L^2(0,T)}\|J\|_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) \le$$
$$\le C_a\left(\|b\|_{L^2(0,T)}\|J\|_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0)\right) \quad (2.2)$$

where

$$C_a = 1 + \|a\|_{L^1(0,T)}e^{\|a\|_{L^1(0,T)}}.$$

Applying Proposition 2.13 to (2.2) we get

$$\|J\|_{L^2(0,T)} \le C_a\left(\sqrt{\|c\|_{L^1(0,T)}} + \sqrt{I(0)} + \|b\|_{L^2(0,T)}\right)$$

Finally, by Young's inequality

$$\|I\|_{L^\infty(0,T)} \le e^{\|a\|_{L^1(0,T)}}\left(\frac{\|b\|^2_{L^2(0,T)} + \|J\|^2_{L^2(0,T)}}{2} + \|c\|_{L^1(0,T)} + I(0)\right) \le$$
$$\le e^{\|a\|_{L^1(0,T)}}\Big(\|b\|^2_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0) +$$
$$+ \frac{3}{2}C_a^2\left(\|b\|^2_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0)\right)\Big) =$$
$$= \tilde{C}_a\left(\|b\|^2_{L^2(0,T)} + \|c\|_{L^1(0,T)} + I(0)\right),$$

where

$$\tilde{C}_a = e^{\|a\|_{L^1(0,T)}}\left(1 + \frac{3}{2}C_a^2\right). \qquad \square$$

2.6 On the transport equation in a bounded domain

Let $\mathbf{b}\colon \overline{D} \times [0,T] \to \mathbb{R}^d$ be a vector field such that $\mathbf{b}|_{\partial D} = 0$. Consider the following problem:

$$u_t - (\mathbf{b}, \nabla)u + cu = f \qquad \text{in} \quad D \times (0,T), \tag{2.3}$$

$$u|_{t=0} = u^\circ \qquad \text{in} \quad D, \tag{2.4}$$

where $u, c, f\colon D \times [0,T] \to \mathbb{R}$ and $u^\circ\colon D \to \mathbb{R}$ is the initial condition for the unknown function $u = u(x,t)$, $x \in D$, $t \in [0,T]$.

Definition 2.16. Let $1 < p < \infty$ and $u^\circ \in L^p(D)$, $\mathbf{b} \in L^1(0,T; W_0^{1,p'}(D))$, $c \in L^1(0,T; L^{p'}(D))$, $f \in L^1(0,T; L^1(D))$. Function $u \in L^\infty(0,T; L^p(D))$ is said to be a *weak solution* to the problem (2.3), (2.4) if for all $\Phi \in \mathscr{D}(\mathbb{R}^d \times [0,T))$ the following equality holds:

$$-\int_0^T \int_D u\Phi_t \, dx\, dt - \int_D u^\circ \Phi(\cdot, 0)\, dx + \\ + \int_0^T \int_D u[(c + \operatorname{div} \mathbf{b})\Phi + (\mathbf{b}, \nabla)\Phi]\, dx\, dt = \int_0^T \int_D f\Phi\, dx\, dt$$

Note that in Definition 2.16 $\Phi|_{\partial D}$ can be nonzero.

Theorem 2.17. *Let $u \in L^\infty(0,T; L^p(D))$ be a weak solution to* (2.3), (2.4)*. If, in addition, $\{c, \operatorname{div} \mathbf{b}\} \subset L^1(0,T; L^\infty(D))$ and $f \in L^1(0,T; L^p(D))$, then for all $\psi \in \mathscr{D}([0,T))$ the following equality holds:*

$$-\int_0^T \psi_t \int_D |u|^p \, dx\, dt - \int_D |u^\circ|^p \psi(0)\, dx + \\ + \int_0^T \psi \int_D \big(|u|^p \operatorname{div} \mathbf{b} + pc|u|^p - p|u|^{p-1} f \operatorname{sign} u\big)\, dx\, dt = 0. \tag{2.5}$$

The proof of Theorem 2.17 is based on some minor modifications of the technique introduced in [17], where a similar result was established in case when $f = 0$ and $D = \mathbb{R}^d$ (see [17], equality (26) on page 520). We skip this proof here because it is quite long and goes beyond the scope of the present paper.

Remark 2.18. As in [17] one can show that if the assumptions of Theorem 2.17 are satisfied, then there exists a unique function $\overline{u} \in C(0,T; L^p(D))$ which is a weak solution to (2.3), (2.4) such that $\overline{u}(t) = u(t)$ for a.e. $t \in [0,T]$ and $\overline{u}|_{t=0} = u^\circ$.

3 Linearization of the Navier–Stokes equations

In this section we derive the linearized equations of fluid motion which we are going to study. Let us consider the barotropic Navier–Stokes equations in the cylinder $D \times (0,T)$:

$$\varrho_t + \operatorname{div}(\varrho\mathfrak{u}) = 0, \tag{3.1}$$

$$(\varrho\mathfrak{u})_t + \operatorname{div}(\varrho\mathfrak{u}\otimes\mathfrak{u}) + \nabla \mathrm{p} - (\mu\Delta\mathfrak{u} + \lambda\nabla\operatorname{div}\mathfrak{u} + \varrho\mathbf{g}) = 0, \tag{3.2}$$

$$\varrho - F(\mathrm{p}) = 0, \tag{3.3}$$

where ϱ, $\mathfrak{u}$ and p — are the density, velocity and pressure respectively; $\mu > 0$ and $\lambda \geq 0$ are the coefficients of viscosity, $\mathbf{g}$ is the external force (per unit volume), F is a function of state.

Let $E_k(\varrho,\mathfrak{u},\mathrm{p})$, $k = 1,2,3$, denote the left-hand sides of the equations (3.1), (3.2) and (3.3) respectively. We will look for the solution of (3.1), (3.2), (3.3) in the form

$$\varrho = \overline{\varrho} + \tau\varrho', \quad \mathfrak{u} = \overline{\mathfrak{u}} + \tau\mathfrak{u}', \quad \mathrm{p} = \overline{\mathrm{p}} + \tau\mathrm{p}',$$

where $\overline{\varrho}, \overline{\mathfrak{u}}, \overline{\mathrm{p}}$ are some given fields of density, velocity and pressure respectively, $\tau \in \mathbb{R}$. We require that (3.1), (3.2) and (3.3) hold when $\tau = 1$.

First we calculate $LE_k := \dfrac{d}{d\tau}E_k(\overline{\varrho} + \tau\varrho', \overline{\mathfrak{u}} + \tau\mathfrak{u}', \overline{\mathrm{p}} + \tau\mathrm{p}')\Big|_{\tau=0}$:

$$\begin{gathered} LE_1 = \varrho'_t + \operatorname{div}(\overline{\varrho}\mathfrak{u}' + \varrho'\overline{\mathfrak{u}}), \\ LE_2 = (\overline{\varrho}\mathfrak{u}' + \varrho'\overline{\mathfrak{u}})_t + \operatorname{div}(\overline{\varrho}\overline{\mathfrak{u}}\otimes\mathfrak{u}' + \overline{\varrho}\mathfrak{u}'\otimes\overline{\mathfrak{u}} + \varrho'\overline{\mathfrak{u}}\otimes\overline{\mathfrak{u}}) + \\ + \nabla\mathrm{p}' - \mu\Delta\mathfrak{u}' - \lambda\nabla\operatorname{div}\mathfrak{u}' - \varrho'\mathbf{g}, \\ LE_3 = \varrho' - F'(\overline{\mathrm{p}})\mathrm{p}' \end{gathered}$$

Then we can write

$$E_k(\overline{\varrho} + \tau\varrho', \overline{\mathfrak{u}} + \tau\mathfrak{u}', \overline{\mathrm{p}} + \tau\mathrm{p}') - E_k(\overline{\varrho}, \overline{\mathfrak{u}}, \overline{\mathrm{p}}) = \tau LE_k + \tau^2 DE_k,$$

$k = 1,2,3$, where

$$\begin{gathered} DE_1 := \operatorname{div}(\varrho'\mathfrak{u}'), \\ DE_2 := (\varrho'\mathfrak{u}')_t + \operatorname{div}(\overline{\varrho}\mathfrak{u}'\otimes\mathfrak{u}' + \varrho'\overline{\mathfrak{u}}\otimes\mathfrak{u}' + \varrho'\mathfrak{u}'\otimes\overline{\mathfrak{u}} + \varrho'\mathfrak{u}'\otimes\mathfrak{u}'\tau), \\ DE_3 := -(F(\overline{\mathrm{p}} + \tau\mathrm{p}') - F(\overline{\mathrm{p}}) - \tau F'(\overline{\mathrm{p}})\mathrm{p}')/\tau^2 \end{gathered}$$

Hence the "variations" $\varrho', \mathfrak{u}', \mathrm{p}'$ of density, velocity and pressure satisfy the equations

$$LE_k = S_k, \quad k = 1,2,3, \tag{3.4}$$

where $S_k := -(E_k(\overline{\varrho}, \overline{\mathfrak{u}}, \overline{\mathrm{p}}) + DE_k|_{\tau=1})$ since $E_k(\overline{\varrho} + \varrho', \overline{\mathfrak{u}} + \mathfrak{u}', \overline{\mathrm{p}} + \mathrm{p}') = 0$.

Let us consider the system (3.4) when the terms S_k are given in advance. In this case the system (3.4) is linear with respect to $\varrho', \mathfrak{u}', \mathrm{p}'$. Let us simplify (3.4). First note that

$$\begin{aligned} (\varrho'\overline{\mathfrak{u}})_t = \varrho'_t\overline{\mathfrak{u}} + \varrho'\overline{\mathfrak{u}}_t &= (S_1 - \operatorname{div}(\overline{\varrho}\mathfrak{u}' + \varrho'\overline{\mathfrak{u}}))\overline{\mathfrak{u}} + \varrho'\overline{\mathfrak{u}}_t = \\ &= S_1\overline{\mathfrak{u}} - \operatorname{div}(\overline{\varrho}\mathfrak{u}'\otimes\overline{\mathfrak{u}} + \varrho'\overline{\mathfrak{u}}\otimes\overline{\mathfrak{u}}) + (\overline{\varrho}\mathfrak{u}', \nabla)\overline{\mathfrak{u}} + \varrho'((\overline{\mathfrak{u}}, \nabla)\overline{\mathfrak{u}} + \overline{\mathfrak{u}}_t), \end{aligned}$$

hence for $k = 2$ we get

$$(\overline{\varrho}\mathfrak{u}')_t + (\overline{\varrho}\mathfrak{u}', \nabla)\overline{\mathfrak{u}} + \operatorname{div}(\overline{\varrho}\overline{\mathfrak{u}} \otimes \mathfrak{u}') + \\ + \nabla \mathrm{p}' - \mu\Delta\mathfrak{u}' - \lambda\nabla \operatorname{div}\mathfrak{u}' - \varrho'(\mathbf{g} - \overline{\mathfrak{u}}_t - (\overline{\mathfrak{u}}, \nabla)\overline{\mathfrak{u}}) = S_2 - S_1\overline{\mathfrak{u}}. \quad (3.5)$$

Denote

$$\varrho'' = F'(\overline{\mathrm{p}})\mathrm{p}',$$

then $\varrho' = S_3 + \varrho''$ and when $k = 1$ the equation (3.4) takes the form

$$\varrho''_t + \operatorname{div}(\overline{\varrho}\mathfrak{u}' + \varrho''\overline{\mathfrak{u}}) = S_1 - (S_3)_t - \operatorname{div}(S_3\overline{\mathfrak{u}}), \quad (3.6)$$

and the equation (3.5) takes the form

$$(\overline{\varrho}\mathfrak{u}')_t + (\overline{\varrho}\mathfrak{u}', \nabla)\overline{\mathfrak{u}} + \operatorname{div}(\overline{\varrho}\overline{\mathfrak{u}} \otimes \mathfrak{u}') + \\ + \nabla \mathrm{p}' - \mu\Delta\mathfrak{u}' - \lambda\nabla \operatorname{div}\mathfrak{u}' - \varrho''\mathbf{f} = S_2 - S_1\overline{\mathfrak{u}} + S_3\mathbf{f}, \quad (3.7)$$

where $\mathbf{f} = \mathbf{g} - \overline{\mathfrak{u}}_t - (\overline{\mathfrak{u}}, \nabla)\overline{\mathfrak{u}}$.

We will study the linearization of the equations (3.1), (3.2) and (3.3) near the state with *constant* density $\varrho_0 > 0$, so let $\overline{\varrho} = \varrho_0$.

In addition we will assume that the function of state $F(\cdot)$ is linear and denote $\alpha = dF/d\mathrm{p}$. Alternatively, we could consider arbitrary $F(\cdot)$ but *constant* reference pressure $\overline{\mathrm{p}} = \mathrm{const}$ and then we would denote $\alpha = F'(\overline{\mathrm{p}})$. In both cases from the physical point of view we have $1/\alpha = d\mathrm{p}/d\varrho = c^2 > 0$, where c is the speed of sound.

Under these assumptions the equations (3.6) and (3.7) take the form

$$\rho_t + \operatorname{div}(\mathbf{u} + \rho\overline{\mathfrak{u}}) = \sigma, \quad (3.8)$$

$$\mathbf{u}_t + (\mathbf{u}, \nabla)\overline{\mathfrak{u}} + \operatorname{div}(\overline{\mathfrak{u}} \otimes \mathbf{u}) + \nabla p = \nu\Delta\mathbf{u} + \kappa\nabla \operatorname{div}\mathbf{u} + \rho\mathbf{f} + \mathbf{s}, \quad (3.9)$$

$$\rho = \alpha p, \quad (3.10)$$

where

$$\rho = \varrho'', \quad \mathbf{u} = \varrho_0\mathfrak{u}', \quad p = \mathrm{p}',$$
$$\nu = \mu/\varrho_0, \quad \kappa = \lambda/\varrho_0,$$
$$\sigma = S_1 - (S_3)_t - \operatorname{div}(S_3\overline{\mathfrak{u}}), \quad \mathbf{s} = S_2 - S_1\overline{\mathfrak{u}} + S_3\mathbf{f}.$$

Note that in the homogeneous case (i.e. when $S_1 = S_2 = S_3 = 0$) we have $\mathbf{s} = 0$ and $\sigma = 0$.

4 Initial–boundary value problem for the linearized Navier–Stokes equations

In this section we study the following initial–boundary value problem for a slightly generalized version of the system (3.8)–(3.10):

$$\rho_t - (\mathbf{b}, \nabla)\rho + c\rho + \operatorname{div} \mathbf{u} = \sigma, \tag{4.1}$$
$$\mathbf{u}_t + \nabla p = -A\mathbf{u} + \rho \mathbf{f} + \mathbf{s}, \tag{4.2}$$
$$\rho = \alpha p, \tag{4.3}$$
$$\mathbf{u}|_{t=0} = \mathbf{u}^\circ, \tag{4.4}$$
$$p|_{t=0} = p^\circ, \tag{4.5}$$
$$\mathbf{u}|_{\partial D} = 0, \tag{4.6}$$

where

$$-A\mathbf{u} \equiv \nu \Delta \mathbf{u} + \kappa \nabla \operatorname{div} \mathbf{u} - (\mathbf{a}, \nabla)\mathbf{u} + M\mathbf{u}, \tag{4.7}$$

where $\mathbf{b}, \mathbf{u}, \mathbf{f}, \mathbf{s} \colon \overline{D \times (0,T)} \to \mathbb{R}^d$ are vector fields, $\rho, c, \sigma, p, \colon \overline{D \times (0,T)} \to \mathbb{R}$ are scalar fields, $\mathbf{u}^\circ \colon \overline{D} \to \mathbb{R}^d$, $p^\circ \colon \overline{D} \to \mathbb{R}$, $\alpha > 0$, $M \colon \overline{D \times (0,T)} \to \mathbb{R}^{d \times d}$ is a square matrix of size $d \times d$ dependent on $x \in \overline{D}$ and $t \in [0,T]$. In what follows we assume that $\mathbf{b}|_{\partial D} = 0$, so there is no need in boundary conditions for ρ. If

$$\mathbf{b} = -\overline{\mathrm{u}}, \qquad c = -\operatorname{div} \mathbf{b} = \operatorname{div} \overline{\mathrm{u}},$$
$$\mathbf{a} = \overline{\mathrm{u}}, \qquad M_{ij} = -\operatorname{div}(\overline{\mathrm{u}})\delta_{ij} - \partial_j \overline{\mathrm{u}}_i,$$

where δ_{ij} is the Kronecker symbol, $i, j = 1, 2, ..., d$, then the system (4.1), (4.2), (4.3) coincides with (3.8), (3.9), (3.10).

There are three unknowns in the problem (4.1)–(4.6): density ρ, velocity $\mathbf{u}$ and pressure p; the rest quantities are given in advance. Since it is easy to exclude the equation (4.3) from the system (4.1)–(4.3), we will consider only $\mathbf{u}$ and p as the unknowns. In what follows by default we assume that ρ is determined by p via (4.3).

Throughout this paper we will assume that the following assumptions are satisfied:

1° $\nu > 0$, $\kappa \geq 0$, $\quad M \in L^\infty(D \times (0,T); \mathbb{R}^{d \times d})$,
$\mathbf{a} \in L^\infty(D \times (0,T); \mathbb{R}^d)$, $\quad \mathbf{f} \in L^2(0,T; L^\infty(D)^d)$;

2° $\mathbf{b} \in L^1(0,T; H^1_0(D)^d)$, $\quad \operatorname{div} \mathbf{b} \in L^1(0,T; L^\infty(D))$,
$c \in L^1(0,T; L^\infty(D))$;

3° $\mathbf{u}^\circ \in L^2(D)^d$, $\quad \sigma \in L^2(0,T; L^2(D))$,
$p^\circ \in L^2(D)$, $\quad \mathbf{s} \in L^2(0,T; H^{-1}(D)^d)$.

Now let us give a definition of weak solution to the problem (4.1)–(4.6).

Definition 4.1. A pair $\{\mathbf{u}, p\} \in L^2(0,T; H^1_0(D)^d) \times L^\infty(0,T; L^2(D))$ is called *weak solution* to the problem (4.1)–(4.6) if for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and $\Phi \in \mathscr{D}(D)^d$ the functions $t \mapsto (\rho(t), \varphi)_D$ and $t \mapsto (\mathbf{u}(t), \Phi)_D$ satisfy

$$\partial_t (\rho, \varphi)_D + (\rho[c + \operatorname{div} \mathbf{b}], \varphi)_D + (\rho \mathbf{b}, \nabla \varphi)_D + (\operatorname{div} \mathbf{u} - \sigma, \varphi)_D = 0, \tag{4.8}$$
$$\partial_t (\mathbf{u}, \Phi)_D - (p, \operatorname{div} \Phi)_D = -\langle A\mathbf{u}, \Phi \rangle + (\rho \mathbf{f}, \Phi)_D + \langle \mathbf{s}, \Phi \rangle \tag{4.9}$$

in sense of distributions $\mathscr{D}'(0,T)$ and moreover

$$(\rho(t),\varphi)_D\,|_{t=0} = (\rho^\circ,\varphi)_D\,, \tag{4.10}$$

$$(\mathbf{u}(t),\Phi)_D\,|_{t=0} = (\mathbf{u}^\circ,\Phi)_D\,. \tag{4.11}$$

Equations (4.10) and (4.11) should be read in accordance with Remark 2.2.

Before turning the question of well-posedness of the problem (4.1)–(4.6) let us discuss some properties of the operator A Let $t\in[0,T]$. Consider an operator $A(t)\colon \mathscr{D}(D)^d \to H^{-1}(D)^d$, which maps $\mathbf{v}\in\mathscr{D}(D)^d$ to the functional

$$\ell_{\mathbf{v}}(\mathbf{h}) = \int_D (A(t)\mathbf{v},\mathbf{h})\,dx, \quad \mathbf{h}\in H_0^1(D)^d.$$

By the assumption 1° this mapping is defined for a.e. $t\in[0,T]$. Since ∂D is piecewise-smooth, by Ostrogradsky–Gauss theorem

$$\ell_{\mathbf{v}}(\mathbf{h}) = \int_D \big(\nu(\nabla\otimes\mathbf{u},\nabla\otimes\mathbf{h}) + \\ + \kappa\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{h} + (\mathbf{h},(\mathbf{a}(t),\nabla)\mathbf{u}) - (M(t)\mathbf{u},\mathbf{h})\big)\,dx, \tag{4.12}$$

where $(\nabla\otimes\mathbf{u},\nabla\otimes\mathbf{h}) = \sum_{i,k=1}^d(\partial_i u_k)(\partial_i h_k)$.

Lemma 4.2. *1. There exists a constant $C>0$, dependent only on $\nu,\kappa,\mathbf{a},M$ and D such that for a.e. $t\in[0,T]$ for all $\mathbf{v}\in\mathscr{D}(D)^d$ and $\mathbf{h}\in H_0^1(D)^d$ the following inequality holds:*

$$|\langle A(t)\mathbf{v},\mathbf{h}\rangle| \le C\|\mathbf{v}\|_{H_0^1(D)^d}\|\mathbf{h}\|_{H_0^1(D)^d}.$$

2. *For a.e. $t\in[0,T]$ the operator $A(t)\colon H_0^1(D)^d\ni\mathbf{v}\mapsto\ell_{\mathbf{v}}\to H^{-1}(D)^d$ where $\ell_{\mathbf{v}}$ is defined in (4.12) is linear and bounded (uniformly with respect to $t\in[0,T]$).*

3. *There exist positive constants γ and β, dependent only on $\nu,\kappa,\mathbf{a},M$ and D, such that for a.e. $t\in[0,T]$ for any $\mathbf{v}\in H_0^1(D)^d$ the following inequality holds:*

$$\beta\|\mathbf{v}\|_{H_0^1(D)^d} \le \langle A(t)\mathbf{v},\mathbf{v}\rangle + \gamma\|\mathbf{v}\|_{L^2(D)^d}.$$

The proof is almost identical to the proof of well-known similar statement from the theory of parabolic equations (see e.g. 6.2.2 in [16], p. 300). □

Lemma 4.2 shows that $A(t)\in B(H_0^1(D)^d;H^{-1}(D)^d)$ for a.e. $t\in[0,T]$, where $B(X,Y)$ denotes the space of bounded linear operators from a Banach space X to another Banach space Y.

4.1 Uniqueness of weak solution

Theorem 4.3. *If $\{\mathbf{u},p\}$ is a weak solution to the problem (4.1)–(4.6) then*

1. $\{\mathbf{u}, p\}$ *satisfies the* energy equality*:*

$$\frac{1}{2}\left(\|\mathbf{u}\|^2_{L^2(D)^d} + \alpha\|p\|^2_{L^2(D)}\right)_t + \alpha\left(p\left[\frac{1}{2}\operatorname{div}\mathbf{b} + c\right], p\right)_D + \\ + \langle A\mathbf{u}, \mathbf{u}\rangle = (p, \sigma)_D + (\rho\mathbf{f}, \mathbf{u})_D + \langle \mathbf{s}, \mathbf{u}\rangle \quad \text{in} \quad \mathscr{D}'(0, T); \tag{4.13}$$

2. *for any $\alpha_1 > 0$ there exists a constant $C > 0$ (dependent only on α_1, T, domain D, coefficients of the operator A and fields $\mathbf{b}$, c and $\mathbf{f}$) such that for any $\alpha \in (0, \alpha_1)$ the following estimates of $\{\mathbf{u}, p\}$ hold:*

$$\|\mathbf{u}\|_{L^2(0,T;H^1_0(D)^d)} \le C \cdot E,$$
$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p\|_{L^\infty(0,T;L^2(D))} \le C \cdot E,$$

where

$$E \equiv \|\mathbf{u}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \\ + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))}.$$

Proof. Let $\{\mathbf{u}, p\}$ be a weak solution to (4.1)–(4.6).

Theorem 2.17 implies that the density ρ satisfies

$$\left(\int_D |\rho|^2\,dx\right)_t + \int_D \left(|\rho|^2 \operatorname{div}\mathbf{b} + 2c|\rho|^2 + 2\rho(\operatorname{div}\mathbf{u} - \sigma)\right) dx = 0 \tag{4.14}$$

in $\mathscr{D}'(0,T)$. Definition 4.1 and Assumption 2° together with (4.14) imply that the function $t \mapsto \int_D |\rho(t)|^2\,dx$ belongs to $W^{1,1}(0,T;\mathbb{R})$. Hence (2.5) and Remark 2.8 imply that

$$\left.\int_D |\rho(t)|^2\,dx\right|_{t=0} = \int_D |\rho^\circ|^2\,dx. \tag{4.15}$$

Definition 4.1 implies that $\forall \mathbf{v} \in H^1_0(D)^d$ and $\forall \psi \in \mathscr{D}(0,T)$

$$-\int_0^T \langle \imath(\mathbf{u}), \mathbf{v}\rangle\,\psi_t\,dt = \int_0^T \left(-\langle \nabla p, \mathbf{v}\rangle - \langle A\mathbf{u}, \mathbf{v}\rangle + (\rho\mathbf{f}, \mathbf{v})_D + \langle \mathbf{s}, \mathbf{v}\rangle\right)\psi\,dt,$$

where the embedding $\imath\colon H^1_0(D)^d \to H^{-1}(D)^d$ is defined on page 9. Then

$$-\left\langle \int_0^T \imath(\mathbf{u})\psi_t\,dt, \mathbf{v}\right\rangle = \left\langle \int_0^T \left(-\nabla p - A\mathbf{u} + \rho\mathbf{f} + \mathbf{s}\right)\psi\,dt, \mathbf{v}\right\rangle,$$

hence, since $\mathbf{v}$ is arbitrary,

$$-\int_0^T \imath(\mathbf{u})\psi_t\,dt = \int_0^T \left(-\nabla p - A\mathbf{u} + \rho\mathbf{f} + \mathbf{s}\right)\psi\,dt$$

in $H^{-1}(D)^d$. Lemma 4.2 implies that A is a bounded linear operator from $L^2(0,T;H_0^1(D)^d)$ to $L^2(0,T;H^{-1}(D)^d)$. But ∇ is a bounded linear operator which maps $L^2(D)$ to $H^{-1}(D)^d$ (see e.g. [10]), hence ∇ maps $L^\infty(0,T;L^2(D))$ to $L^\infty(0,T;H^{-1}(D)^d)$. Therefore the assumption 1° implies that $\imath(\mathbf{u}) \in W^{1,2}(0,T;H^{-1}(D)^d)$ and

$$\imath(\mathbf{u})_t + \nabla p = -A\mathbf{u} + \rho\mathbf{f} + \mathbf{s}. \tag{4.16}$$

Hence $\mathbf{u} \in \mathcal{W}(0,T)$, consequently Proposition 2.6 and equality (4.11) imply that $\mathbf{u}|_{t=0} = \mathbf{u}^\circ$ and hence

$$\int_D |\mathbf{u}(t)|^2\,dx\bigg|_{t=0} = \int_D |\mathbf{u}^\circ|^2\,dx \tag{4.17}$$

in sense of Remark 2.2.

For brevity let us introduce the following notation:

$$\mathbf{u}_t \equiv \imath(\mathbf{u})_t,$$

$$|\cdot| \equiv \|\cdot\|_{L^2(D)^k},$$

$$\|\cdot\| \equiv \|\cdot\|_{H_0^1(D)^k},$$

$$\|\cdot\|_{-1} \equiv \|\cdot\|_{H^{-1}(D)^k},$$

where the natural number k is uniquely defined by "$\cdot$".

Equation (4.14) and equation of state (4.1) imply that

$$\frac{1}{2}\alpha|p|_t^2 + \alpha\left(p\left[\frac{1}{2}\operatorname{div}\mathbf{b} + c\right], p\right)_D + (p, \operatorname{div}\mathbf{u} - \sigma)_D = 0. \tag{4.18}$$

Since $\mathbf{u}(t) \in H_0^1(D)^d$ for a.e. $t \in [0,T]$ we have

$$\langle \mathbf{u}_t, \mathbf{u}\rangle + \langle \nabla p, \mathbf{u}\rangle = -\langle A\mathbf{u}, \mathbf{u}\rangle + (\rho\mathbf{f}, \mathbf{u})_D + \langle \mathbf{s}, \mathbf{u}\rangle \tag{4.19}$$

a.e. on $[0,T]$. By Remark 2.9 $\frac{1}{2}|\mathbf{u}|_t^2 = \langle \mathbf{u}_t, \mathbf{u}\rangle$, so adding (4.19) to (4.18) and using the identity $\langle \nabla p, \mathbf{u}\rangle = -(p, \operatorname{div}\mathbf{u})_D$ we obtain

$$\left(|\mathbf{u}|^2 + \alpha|p|^2\right)_t + \alpha\left(p\left[\frac{1}{2}\operatorname{div}\mathbf{b} + c\right], p\right)_D + \\ + \langle A\mathbf{u}, \mathbf{u}\rangle = (p,\sigma)_D + (\rho\mathbf{f}, \mathbf{u})_D + \langle \mathbf{s}, \mathbf{u}\rangle \tag{4.20}$$

in $\mathscr{D}'(0,T)$, i.e. the energy identity (4.13) holds.

Let us fix $t \in [0,T]$. Lemma 4.2 implies that

$$\langle A\mathbf{u}, \mathbf{u}\rangle \geq \beta\|\mathbf{u}\|^2 - \gamma|\mathbf{u}|^2 \geq \beta\|\mathbf{u}\|^2 - \gamma\left(|\mathbf{u}|^2 + \alpha|p|^2\right),$$

where γ and β are positive constants dependent only on $\nu, \kappa, \mathbf{a}, M$ and D.

Let us denote

$$I(t) = \frac{1}{2}\left(|\mathbf{u}(t)|^2 + \alpha|p(t)|^2\right),$$

$$J(t) = \sqrt{\beta}\|\mathbf{u}(t)\|,$$

then

$$\langle A\mathbf{u},\mathbf{u}\rangle \geq J^2 - 2\gamma I.$$

Denoting $\|\cdot\|_\infty \equiv \|\cdot\|_{L^\infty(D)^k}$ we obtain

$$-\alpha\left(p\left[\frac{1}{2}\operatorname{div}\mathbf{b}+c\right],p\right)_D \leq \alpha\left\|\frac{1}{2}\operatorname{div}\mathbf{b}+c\right\|_\infty |p|^2 \leq \|\operatorname{div}\mathbf{b}+2c\|_\infty I.$$

The inequalities of Young and Cauchy–Bunyakovski imply that

$$(p,\sigma)_D = \left(\sqrt{\alpha}p, \frac{1}{\sqrt{\alpha}}\sigma\right)_D \leq \frac{1}{2}\alpha|p|^2 + \frac{1}{2\alpha}|\sigma|^2 \leq I + \frac{1}{2\alpha}|\sigma|^2,$$

$$(\rho\mathbf{f},\mathbf{u})_D \leq \sqrt{\alpha}\|\mathbf{f}\|_\infty\sqrt{\alpha}|p|\cdot|\mathbf{u}| \leq \sqrt{\alpha}\|\mathbf{f}\|_\infty\frac{1}{2}\left(\alpha|p|^2+|\mathbf{u}|^2\right) = \sqrt{\alpha}\|\mathbf{f}\|_\infty I.$$

Finally, $\langle\mathbf{s},\mathbf{u}\rangle \leq \|\mathbf{s}\|_{-1}\cdot\|\mathbf{u}\| = \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{-1}J$.

The calculations above are valid for a.e. $t\in[0,T]$, hence (4.20) implies that

$$I_t + J^2 \leq \tilde{a}I + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{-1}J + \frac{1}{2\alpha}|\sigma|^2, \tag{4.21}$$

where

$$\tilde{a} = 1 + 2\gamma + \|\operatorname{div}\mathbf{b}\|_\infty + 2\|c\|_\infty + \alpha\|\mathbf{f}\|_\infty.$$

Equalities (4.15) and (4.17) imply that $I|_{t=0} = \frac{1}{2}\left(|\mathbf{u}^\circ|^2 + \alpha|p^\circ|^2\right)$. Hence (4.21) implies that the continuous version $\bar{I}$ of I satisfies

$$\bar{I}' + J^2 \leq \tilde{a}\bar{I} + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{-1}J + \frac{1}{2\alpha}|\sigma|^2,$$

a.e. in $[0,T]$. Consequently by Lemma 2.15 we have

$$\|J\|_{L^2(0,T)} \leq C_{\tilde{a}}\left(\sqrt{\bar{I}(0)} + \sqrt{\frac{1}{2\alpha}\|\sigma\|^2_{L^2(0,T;L^2(D))}} + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)}\right),$$

$$\|\bar{I}\|_{L^\infty(0,T)} \leq C_{\tilde{a}}\left(\bar{I}(0) + \frac{1}{2\alpha}\|\sigma\|^2_{L^2(0,T;L^2(D))} + \frac{1}{\beta}\|\mathbf{s}\|^2_{L^2(0,T;H^{-1}(D)^d)}\right),$$

where constant $C_{\tilde{a}}$ depends only on $\int_0^T \tilde{a}\,dt$. Using simple inequality $\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$, $x,y\geq 0$, we obtain

$$\sqrt{\beta}\|\mathbf{u}\|_{L^2(0,T;H^1_0(D)^d)} \leq C_{\tilde{a}}\Big(\|\mathbf{u}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \\ + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)}\Big),$$

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p\|_{L^\infty(0,T;L^2(D))} \leq C_{\tilde{a}}\Big(\|\mathbf{u}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \\ + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)}\Big).$$

When $\alpha\in(0,\alpha_1)$ we have $\tilde{a} \leq \hat{a} \equiv 1+2\gamma+\|\operatorname{div}\mathbf{b}\|_\infty + 2\|c\|_\infty + \alpha_1\|\mathbf{f}\|_\infty$. From the proof of Lemma 2.15 one can see that $C_{\tilde{a}} \leq C_{\hat{a}}$. □

4.2 Existence of weak solution

Theorem 4.4. *The problem* (4.1)–(4.6) *has a weak solution* $\{\mathbf{u}, p\}$.

Proof. Let $\{\pi_i\}_{i=1}^{\infty}$ be a orthonormal basis of polynomials in $L^2(D)$. Then $p^\circ = \lim_{m\to\infty} p_m^\circ$, where

$$p_m^\circ = \sum_{j=1}^{m} P_j^\circ \pi_j,$$

$P_j^\circ = (p^\circ, \pi_j)_D$. Let $\{\mathbf{e}_i\}_{i=1}^{\infty}$ be an orthonormal basis of $H_0^1(D)^d$. Since $H_0^1(D)$ is dense in $L^2(D)$, the linear span of $\{\mathbf{e}_i\}_{i=1}^{\infty}$ is also dense in $L^2(D)^d$. Hence $\mathbf{u}^\circ$ can be approximated by linear combinations of $\{\mathbf{e}_i\}_{i=1}^{\infty}$. Without loss of generality we may assume that $\mathbf{u}^\circ = \lim_{m\to\infty} \mathbf{u}_m^\circ$, where

$$\mathbf{u}_m^\circ \equiv \sum_{j=1}^{m} U_{m,j}^\circ \mathbf{e}_j,$$

$U_{m,j}^\circ \in \mathbb{R}$, $j = 1, 2, ..., m$. Let us look for absolutely continuous functions $P_{m,j} = P_{m,j}(t)$ and $U_{m,j} = U_{m,j}(t)$ such that

$$\mathbf{u}_m = \sum_{j=1}^{m} U_{m,j}\mathbf{e}_j, \qquad p_m = \sum_{j=1}^{m} P_{m,j}\pi_j$$

and $\rho_m \equiv \alpha p_m$ satisfy the "approximate" system

$$((\rho_m)_t - (\mathbf{b}, \nabla)\rho_m + c\rho_m + \operatorname{div} \mathbf{u}_m, \pi_i)_D = (\sigma, \pi_i)_D\,, \quad i = 1, ..., m, \tag{4.22}$$

$$\langle (\mathbf{u}_m)_t + \nabla p_m, \mathbf{e}_i\rangle = \langle -A\mathbf{u}_m + \rho_m \mathbf{f} + \mathbf{s}, \mathbf{e}_i\rangle\,, \quad i = 1, ..., m \tag{4.23}$$

and initial conditions

$$p_m|_{t=0} = p_m^\circ, \tag{4.24}$$

$$\mathbf{u}_m|_{t=0} = \mathbf{u}_m^\circ. \tag{4.25}$$

(The equalities (4.22) and (4.23) are understood in sense of $\mathscr{D}'(0, T)$.)

The equalities (4.22) and (4.23) together with the initial conditions (4.24) and (4.25) represent a Cauchy problem for a linear system of ordinary differential equations. This system has a unique (weak) solution $\{P, U\} \in W^{1,2}(0, T; \mathbb{R}^m)^2$, which can be considered to be absolutely continuous by Proposition 2.1.

Since for all $t \in [0, T]$ $\mathbf{u}_m(t)$ and $p_m(t)$ belong to the linear spans of $\{\mathbf{e}_i\}_{i=1}^{m}$ and $\{\pi_i\}_{i=1}^{m}$ respectively, from (4.22) and (4.23) we obtain

$$(\alpha(p_m)_t - (\mathbf{b}, \nabla)\alpha p_m + c\alpha p_m + \operatorname{div} \mathbf{u}_m, p_m)_D = (\sigma, p_m)_D\,, \tag{4.26}$$

$$\langle (\mathbf{u}_m)_t + \nabla p_m, \mathbf{u}_m\rangle = \langle -A\mathbf{u}_m + \rho_m \mathbf{f} + \mathbf{s}, \mathbf{u}_m\rangle\,. \tag{4.27}$$

Since $p_m(t)$ is a polynomial, by Ostrogradsky–Gauss theorem

$$-\left((\mathbf{b}, \nabla)p_m, p_m\right)_D = -\int_D (\mathbf{b}, \tfrac{1}{2}\nabla p_m^2)\, dx = \int_D \tfrac{1}{2} p_m^2 \operatorname{div} \mathbf{b}\, dx.$$

By Remark 2.5

$$((p_m)_t, p_m)_D = \tfrac{1}{2}|p_m|_t^2,$$
$$((\mathbf{u}_m)_t, \mathbf{u}_m)_D = \tfrac{1}{2}|\mathbf{u}_m|_t^2$$

a.e. in $[0,T]$, where $|\cdot| \equiv \|\cdot\|_{L^2(D)^k}$, $k \in \mathbb{N}$ being determined by the argument of $|\cdot|$. Adding (4.26) to (4.27) and using the identity

$$\langle \nabla p_m, \mathbf{u}_m \rangle = -\,(p_m, \operatorname{div} \mathbf{u}_m)_D$$

we get

$$\left(|\mathbf{u}_m|^2 + \alpha |p_m|^2\right)_t + \alpha \left(p_m \left[\frac{1}{2} \operatorname{div} \mathbf{b} + c \right], p_m \right)_D + \\ + \langle A\mathbf{u}_m, \mathbf{u}_m \rangle = (p_m, \sigma_m)_D + (\rho_m \mathbf{f}, \mathbf{u}_m)_D + \langle \mathbf{s}, \mathbf{u}_m \rangle \quad (4.28)$$

Repeating for $\{\mathbf{u}_m, p_m\}$ the calculations we carried out for $\{\mathbf{u}, p\}$ on pages 18–19 we obtain the estimates

$$\sqrt{\beta}\|\mathbf{u}_m\|_{L^2(0,T;H_0^1(D)^d)} \leq C_{\tilde{a}} \Big(\|\mathbf{u}_m^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p_m^\circ\|_{L^2(D)} + \\ + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} \Big), \quad (4.29)$$

$$\|\mathbf{u}_m\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p_m\|_{L^\infty(0,T;L^2(D))} \leq C_{\tilde{a}} \Big(\|\mathbf{u}_m^\circ\|_{L^2(D)^d} + \\ + \sqrt{\alpha}\|p_m^\circ\|_{L^2(D)} + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} + \frac{1}{\sqrt{\beta}}\|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} \Big), \quad (4.30)$$

where the constant $C_{\tilde{a}}$ depends only on $\int_0^T \tilde{a}\, dt$,

$$\tilde{a} = 1 + 2\gamma + \|\operatorname{div} \mathbf{b}\|_\infty + 2\|c\|_\infty + \alpha\|\mathbf{f}\|_\infty.$$

When m is sufficiently large the following inequalities hold:

$$\|\mathbf{u}_m^\circ\|_{L^2(D)^d} \leq 2\|\mathbf{u}^\circ\|_{L^2(D)^d} \quad \text{and} \quad \|p_m^\circ\|_{L^2(D)} \leq 2\|p^\circ\|_{L^2(D)}.$$

Then the estimates (4.29) and (4.30) imply that the sequence $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0,T;H_0^1(D)^d)$ (and in $L^\infty(0,T;L^2(D)^d)$), and the sequence $\{p_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(D))$. By Proposition 2.11 (we also take into account Proposition 2.10) there exists a sequence $\{m_k\}_{k\in\mathbb{N}}$ such that

$$\mathbf{u}_{m_k} \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(0,T;H_0^1(D)^d),$$
$$p_{m_k} \rightharpoonup p \quad \text{in} \quad L^2(0,T;L^2(D))$$

when $k \to \infty$. Then $\rho_{m_k} \rightharpoonup \rho \equiv \alpha p$ in $L^2(0,T;L^2(D))$ when $k \to \infty$.

Let us write (4.22) and (4.23) in the integral form: for all $\psi \in \mathscr{D}([0,T))$, $\varphi \in \operatorname{span}(\{\pi_i\}_{i=1}^m)$ and $\Phi \in \operatorname{span}(\{\mathbf{e}_i\}_{i=1}^m)$

$$
-\int_0^T (\rho_m, \varphi)_D\, \psi_t\, dt - (\rho_m^\circ, \varphi)_D\, \psi(0) + \int_0^T (\rho_m[c + \operatorname{div}\mathbf{b}], \varphi)_D\, \psi\, dt + \\
+ \int_0^T (\rho_m \mathbf{b}, \nabla\varphi)_D\, \psi\, dt + \int_0^T (\operatorname{div}\mathbf{u}_m - \sigma, \varphi)_D\, \psi\, dt = 0,
$$

$$
-\int_0^T (\mathbf{u}_m, \Phi)_D\, \psi_t\, dt - (\mathbf{u}_m^\circ, \Phi)_D\, \psi(0) - \int_0^T (p_m, \operatorname{div}\Phi)_D\, \psi\, dt = \\
= -\int_0^T \langle A\mathbf{u}_m, \Phi\rangle\, \psi\, dt + \int_0^T (\rho_m \mathbf{f}, \Phi)_D\, \psi\, dt + \int_0^T \langle \mathbf{s}, \Phi\rangle\, \psi\, dt.
$$

If we pass to the limit in the equalities above when $m = m_k$ and $k \to \infty$, then we will see that ρ, $\mathbf{u}$ and p satisfy (4.8) and (4.9) for all $\psi \in \mathscr{D}([0,T))$, $\varphi \in \operatorname{span}(\{\pi_i\}_{i=1}^\infty)$ and $\Phi \in \operatorname{span}(\{\mathbf{e}_i\}_{i=1}^\infty)$. Since $\operatorname{span}(\{\pi_i\}_{i=1}^\infty)$ is dense in $C^1(\overline{D})$, and $\operatorname{span}(\{\mathbf{e}_i\}_{i=1}^\infty)$ is dense in $H_0^1(D)^d$, then (4.8) and (4.9) hold for all $\varphi \in C^1(\overline{D})$ and $\Phi \in H_0^1(D)^d$. Then by Definition 4.1 the pair $\{\mathbf{u}, p\}$ is a weak solution to the problem (4.1)–(4.6). □

Combining Theorems 4.3, 4.4 and Remark 2.18 we obtain

Corollary 4.5. *The problem* (4.1)–(4.6) *has a unique weak solution* $\{\mathbf{u}, p\}$. *Besides,*

1. *there exists unique* $\bar{p} \in C(0,T; L^2(D))$ *such that*

$$\bar{p}(0) = p^\circ \quad \text{and} \quad \bar{p}(t) = p(t) \text{ for a.e. } t \in [0,T];$$

2. $\mathbf{u} \in \mathcal{W}(0,T)$;

3. *there exists unique* $\bar{\mathbf{u}} \in C(0,T; L^2(D)^d)$ *such that*

$$\bar{\mathbf{u}}(0) = \mathbf{u}^\circ \quad \text{and} \quad \bar{\mathbf{u}}(t) = \mathbf{u}(t) \text{ for a.e. } t \in [0,T].$$

4.3 Enhanced estimates of weak solutions

Theorem 4.6. *Suppose that* $\{\mathbf{u}, p\}$ *is a weak solution to the problem* (4.1)–(4.6) *and* $\alpha \in (0, \alpha_1)$, *where* $\alpha_1 > 0$. *Assume that the term* σ *in the equation* (4.1) *has the form*

$$\sigma = \sigma_1 + \sigma_2, \tag{4.31}$$

where

$$
\begin{gathered}
\sigma_1 \in L^2(0,T; L^2(D)), \\
\sigma_2 = \operatorname{div}\mathbf{w}, \quad \text{where} \quad \mathbf{w} \in \mathcal{W}(0,T).
\end{gathered} \tag{4.32}
$$

Then there exists a constant $C > 0$ (dependent only on α_1, T, domain D, coefficients of A and fields $\mathbf{b}$, c and $\mathbf{f}$) such that $\{\mathbf{u}, p\}$ satisfies the following estimates:

$$\|\mathbf{u}\|_{L^2(0,T;H_0^1(D)^d)} \le C \cdot E', \tag{4.33}$$
$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p\|_{L^\infty(0,T;L^2(D))} \le C \cdot E', \tag{4.34}$$

where

$$E' \equiv \|\mathbf{u}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \\ + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} + \frac{1}{\sqrt{\alpha}}\|\sigma_1\|_{L^2(0,T;L^2(D))} + \|\mathbf{w}\|_{\mathcal{W}(0,T)}.$$

Proof. Let us look for the solution to (4.1)–(4.6) in the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is the weak solution to

$$\begin{gathered}
\rho_t - (\mathbf{b}, \nabla)\rho + c\rho + \operatorname{div} \mathbf{v} = \sigma_1, \\
\mathbf{v}_t + \nabla p = -A\mathbf{v} + \rho \mathbf{f} + \mathbf{s} - \mathbf{w}_t - A\mathbf{w}, \\
\rho = \alpha p, \\
\mathbf{v}|_{t=0} = \mathbf{u}^\circ - \mathbf{w}|_{t=0}, \\
p|_{t=0} = p^\circ, \\
\mathbf{v}|_{\partial D} = 0,
\end{gathered}$$

From Lemma 4.2 one can see that

$$\|\mathbf{w}_t + A\mathbf{w}\|_{L^2(0,T;H^{-1}(D)^d)} \le \|\mathbf{w}_t\|_{L^2(0,T;H^{-1}(D)^d)} + \|A\|\|\mathbf{w}\|_{L^2(0,T;H_0^1(D)^d)} \le \\ \le \|\mathbf{w}\|_{\mathcal{W}(0,T)} + C_1\|\mathbf{w}\|_{\mathcal{W}(0,T)} \le C\|\mathbf{w}\|_{\mathcal{W}(0,T)},$$

where $C_1, C = \text{const}$ and $\|A\| = \operatorname{ess\,sup}_{t\in[0,T]} \|A(t)\|_{B(H_0^1(D)^d;H^{-1}(D)^d)}$. By Proposition 2.6 $\|\mathbf{w}|_{t=0}\|_{L^2(D)^d} \le C\|\mathbf{w}\|_{\mathcal{W}(0,T)}$, where $C = \text{const}$. Theorem 4.3 provides the following estimate of $\mathbf{v}$:

$$\|\mathbf{v}\|_{L^2(0,T;H_0^1(D)^d)} + \|\mathbf{v}\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p\|_{L^\infty(0,T;L^2(D))} \le C \cdot E,$$

where

$$E \equiv \|\mathbf{u}^\circ - \mathbf{w}|_{t=0}\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \\ + \|\mathbf{s} - \mathbf{w}_t - A\mathbf{w}\|_{L^2(0,T;H^{-1}(D)^d)} + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))} \le \\ \le \|\mathbf{u}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \\ + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} + C\|\mathbf{w}\|_{\mathcal{W}(0,T)} + \frac{1}{\sqrt{\alpha}}\|\sigma\|_{L^2(0,T;L^2(D))}.$$

To complete the proof we write

$$
\begin{aligned}
\|\mathbf{u}\|_{L^2(0,T;H^1_0(D)^d)} + \|\mathbf{u}\|_{L^\infty(0,T;L^2(D)^d)} \le \\
\le \|\mathbf{u}-\mathbf{w}\|_{L^2(0,T;H^1_0(D)^d)} + \|\mathbf{u}-\mathbf{w}\|_{L^\infty(0,T;L^2(D)^d)} + \\
+ \|\mathbf{w}\|_{L^2(0,T;H^1_0(D)^d)} + \|\mathbf{w}\|_{L^\infty(0,T;L^2(D)^d)} \le \\
\le \|\mathbf{v}\|_{L^2(0,T;H^1_0(D)^d)} + \|\mathbf{v}\|_{L^\infty(0,T;L^2(D)^d)} + C\|\mathbf{w}\|_{\mathcal{W}(0,T)}
\end{aligned}
$$

and use the estimate of $\mathbf{v}$ obtained above. □

Lemma 4.7. *If $f \in W^{1,2}(0,T;\widehat{L}^2(D))$ then there exists a vector field $\mathbf{v} \in W^{1,2}(0,T;H^1_0(D)^d)$ such that*

1) $\operatorname{div}\mathbf{v}(t) = f(t)$ *a.e. in D for a.e. $t \in [0,T]$,*

2) $\|\mathbf{v}\|_{W^{1,2}(0,T;H^1_0(D)^d)} \le C\|f\|_{W^{1,2}(0,T;\widehat{L}^2(D))}$,

where the constant C depends only on the domain D.

Proof. Let us consider the operator ∇ as a mapping from $\widehat{L}^2(D)$ to $H^{-1}(D)^d$ given by

$$\langle \nabla p, \mathbf{u}\rangle = -\left(p, \operatorname{div}\mathbf{u}\right)_D, \quad \mathbf{u} \in H^1_0(D)^d,\ p \in \widehat{L}^2(D).$$

The operator ∇ defined above is adjoint to the linear bounded operator $\operatorname{div}\colon H^1_0(D)^d \to \widehat{L}^2(D)$. Then by Fredholm's theorem

$$\ker(\nabla) = \ker(\operatorname{div}^*) = \operatorname{Im}(\operatorname{div})^\perp,$$

where ker and Im denote the kernel and the image of the operator respectively, and $M^\perp$ denotes the orthogonal complement of a subspace M of $\widehat{L}^2(D)$.

By Nečas inequality (see e.g. [10], I §1, proposition 1.2)

$$\|p\|_{\widehat{L}^2(D)} \le C\|\nabla p\|_{H^{-1}(D)^d},$$

where constant C doesn't depend on p. Hence $\ker(\nabla) = \{0\}$, consequently $\operatorname{Im}(\operatorname{div}) = \widehat{L}^2(D)$.

Let $\Lambda(D)$ denote the orthogonal complement of $V(D)$ in $H^1_0(D)^d$. Since $\ker(\operatorname{div}) = V(D)$ the operator div is a bijection from $\Lambda(D)$ onto $\widehat{L}^2(D)$. By Banach's bounded inverse theorem there exists a bounded linear operator $\mathcal{R}\colon \widehat{L}^2(D) \to \Lambda(D)$ such that

$$\operatorname{div}\mathcal{R}(p) = p \quad \forall p \in \widehat{L}^2(D).$$

Definition of weak derivative implies that $\mathbf{v} := \mathcal{R}(f) \in W^{1,2}(0,T;H^1_0(D)^d)$ and $\mathbf{v}_t = \mathcal{R}(f_t)$. From the definition of $\mathcal{R}$ it is clear that $\mathbf{v}$ also satisfies 1) and 2). □

The idea of the proof given above belongs to A.A. Ilyin. An operator similar to $\mathcal{R}$ was explicitly constructed by M.E. Bogovskii [18].

Theorem 4.8. *Let $\{\mathbf{u}, p\}$ be a weak solution to (4.1)–(4.6) and $\alpha \in (0, \alpha_1)$, $\alpha_1 > 0$. Suppose that the term σ in the equation (4.1) has the form*

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3,$$

where

$$\sigma_1 \in L^2(0,T;L^2(D)),$$
$$\sigma_2 = \operatorname{div} \mathbf{w}, \quad \textit{where} \quad \mathbf{w} \in \mathcal{W}(0,T),$$
$$\sigma_3 \in W^{1,2}(0,T;\widehat{L}^2(D)).$$

Then there exists a constant $C > 0$ (dependent only on α_1, T, domain D, coefficients of A and fields $\mathbf{b}$, c and $\mathbf{f}$) such that $\{\mathbf{u}, p\}$ satisfies the following estimates:

$$\|\mathbf{u}\|_{L^2(0,T;H_0^1(D)^d)} \leq C \cdot E'',$$
$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p\|_{L^\infty(0,T;L^2(D))} \leq C \cdot E'',$$

where

$$E'' \equiv \|\mathbf{u}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ\|_{L^2(D)} + \|\mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} + \\ + \frac{1}{\sqrt{\alpha}}\|\sigma_1\|_{L^2(0,T;L^2(D))} + \|\mathbf{w}\|_{\mathcal{W}(0,T)} + \|\sigma_3\|_{W^{1,2}(0,T;\widehat{L}^2(D))}.$$

Proof. By Lemma 4.7 there exists $\mathbf{v} \in W^{1,2}(0,T;H_0^1(D)^d)$ such that $\operatorname{div} \mathbf{v} = \sigma_3$ and $\|\mathbf{v}\| \leq C\|\sigma_3\|_{W^{1,2}(0,T;\widehat{L}^2(D))}$.

Using Poincaré inequality one can show that $W^{1,2}(0,T;H_0^1(D)^d)$ is continuously embedded into $\mathcal{W}(0,T)$, hence

$$\|\mathbf{v}\|_{\mathcal{W}(0,T)} \leq C\|\mathbf{v}\|_{W^{1,2}(0,T;H_0^1(D)^d)}$$

(where $C = \text{const}$), consequently $\sigma = \sigma_1 + \operatorname{div}(\mathbf{v} + \mathbf{w})$ and all that remains to complete the proof is to use Theorem 4.6. ☐

5 Incompressible limit

Let us consider a family of the initial–boundary problems (4.1)–(4.6) where the terms $\mathbf{s}$, σ and the initial data p°, $\mathbf{u}^\circ$ depend on the compressibility α:

$$\sigma = \sigma_\alpha \in L^2(0,T;L^2(D)), \quad \mathbf{s} = \mathbf{s}_\alpha \in L^2(0,T;H^{-1}(D)^d),$$
$$\mathbf{u}^\circ = \mathbf{u}_\alpha^\circ \in L^2(D)^d, \quad p^\circ = p_\alpha^\circ \in L^2(D).$$

We will also assume that $\alpha \in (0,1)$. Then Corollary 4.5 implies that for any $\alpha \in (0,1)$ the corresponding problem (4.1)–(4.6) has a unique weak solution $\{\mathbf{u}, p\}$, which we denote by $\{\mathbf{u}_\alpha, p_\alpha\}$. In this section we focus on passage to the limit when $\alpha \to 0$.

If $\sigma_\alpha \to 0$ and $\mathbf{s}_\alpha \to \mathbf{s}$ (we will specify the topology of this convergence later) as $\alpha \to 0$, then the equations (4.1) and (4.2) formally turn into the equations

$$\operatorname{div} \boldsymbol{v} = 0, \tag{5.1}$$
$$\boldsymbol{v}_t + \nabla q = -A\boldsymbol{v} + \mathbf{s} \tag{5.2}$$

respectively. For these equations we will consider the following initial and boundary conditions:

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}^\circ, \tag{5.3}$$
$$\boldsymbol{v}|_{\partial D} = 0, \tag{5.4}$$

Let us give a definition of the solution to the initial–boundary value problem (5.1)–(5.4):

Definition 5.1. A pair $\{\boldsymbol{v}, q\} \in L^2(0,T; V(D)) \times \mathscr{D}'(D \times (0,T))$ is called a *weak solution* to (5.1)–(5.4) if (5.1), (5.2) hold in sense of $\mathscr{D}'(D \times (0,T))$ and for any $\Phi \in \mathcal{V}(D)$

$$\partial_t \left(\boldsymbol{v}, \Phi\right)_D = \langle -A\boldsymbol{v} + \mathbf{s}, \Phi \rangle \tag{5.5}$$

in $\mathscr{D}'(0,T)$, and also $\left(\boldsymbol{v}(t), \Phi\right)_D|_{t=0} = \left(\boldsymbol{v}^\circ, \Phi\right)_D$ (see Remark 2.2).

Theorem 5.2. *Let $\{\boldsymbol{v}, q\}$ be a weak solution to* (5.1)–(5.4)*. Then*

$$\frac{1}{2}|\boldsymbol{v}|_t^2 = -\langle A\boldsymbol{v}, \boldsymbol{v}\rangle + \langle \mathbf{s}, \boldsymbol{v}\rangle \tag{5.6}$$

in $\mathscr{D}'(0,T)$. If $\{\boldsymbol{v}_1, q_1\}$ is another weak solution to the problem (5.1)–(5.4) *then $\boldsymbol{v}_1(t) = \boldsymbol{v}(t)$ for a.e. $t \in [0,T]$ and $\nabla q_1 = \nabla q$.*

When $A = -\nu\Delta$ this theorem is proved in [10], III §1. Lemma 4.2 shows that in fact the proof from [10] is also valid when the operator A given by (4.7). □

Our goal is to study the convergence of the solutions $\{\mathbf{u}_\alpha, p_\alpha\}$ as $\alpha \to 0$. Such passage to the limit is *singular* because the equation (4.1) has the term αp_t which vanishes when $\alpha = 0$.

5.1 Convergence of velocity

First of all we prove the following analog of the results obtained in [3, 4]:

Theorem 5.3. *If when $\alpha \to 0$ we have*

$$\|\sigma_\alpha\|_{L^2(0,T;L^2(D))} = O(\sqrt{\alpha}), \qquad \mathbf{s}_\alpha \xrightarrow{*} \mathbf{s} \quad in \;\; L^2(0,T; H^{-1}(D)^d),$$
$$\mathbf{u}_\alpha^\circ \rightharpoonup \mathbf{u}^\circ \quad in \;\; L^2(D)^d, \qquad \|p_\alpha^\circ\|_{L^2(D)} = O\left(\frac{1}{\sqrt{\alpha}}\right),$$

then

$$\begin{aligned} \mathbf{u}_\alpha &\rightharpoonup \boldsymbol{v} \quad in \quad L^2(0,T;H_0^1(D)^d), \\ \mathbf{u}_\alpha &\overset{*}{\rightharpoonup} \boldsymbol{v} \quad in \quad L^\infty(0,T;L^2(D)^d), \\ \nabla p_\alpha &\overset{*}{\rightharpoonup} \nabla q \quad in \quad H^{-1}(D\times(0,T))^d \end{aligned} \tag{5.7}$$

as $\alpha\to 0$*, where* $\{\boldsymbol{v}, q\}$ *is a weak solution to* (5.1)–(5.4) *with initial condition*

$$\boldsymbol{v}^\circ = P_H \mathbf{u}^\circ. \tag{5.8}$$

Proof. From Theorem 4.3 and the hypotheses of Theorem 5.3 the solutions $\{\mathbf{u}_\alpha, p_\alpha\}$ to the problem (4.1)–(4.6) are bounded when $\alpha\in(0,1)$:

$$\|\mathbf{u}_\alpha\|_{L^2(0,T;H_0^1(D)^d)} + \|\mathbf{u}_\alpha\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p_\alpha\|_{L^\infty(0,T;L^2(D))} \le \text{ const} \tag{5.9}$$

By Proposition 2.11 $\exists\mathbf{u}\in L^2(0,T;H_0^1(D)^d)$ and $\exists\{\alpha_n\}_{n\in\mathbb{N}}\subset(0,1)$ such that $\alpha_n\to 0$ as $n\to 0$ and

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(0,T;H_0^1(D)^d).$$

By Definition 4.1 for $\forall\alpha\in(0,1)$ and $\forall\varphi\in\mathscr{D}(\mathbb{R}^d\times[0,T))$

$$-\int_0^T\int_D \alpha p_\alpha\varphi_t\,dx\,dt - \int_D \alpha p_\alpha^\circ\varphi(\cdot,0)\,dx + \int_0^T\int_D \alpha p_\alpha[c+\operatorname{div}\mathbf{b}]\varphi\,dx\,dt + \\ + \int_0^T\int_D \alpha p_\alpha(\mathbf{b},\nabla)\varphi\,dx\,dt + \int_0^T\int_D(\operatorname{div}\mathbf{u}-\sigma)\varphi\,dx\,dt = 0,$$

Passing to the limit in this equality when $\alpha=\alpha_n$ and $n\to 0$ (and using boundedness of $\sqrt{\alpha}p_\alpha$ in $L^\infty(0,T;L^2(D))$) we obtain

$$\int_0^T\int_D(\operatorname{div}\mathbf{u})\varphi\,dx\,dt = 0.$$

Since φ is arbitrary, the du Bois-Reymond lemma implies that $\operatorname{div}\mathbf{u}=0$ for a.e. $(x,t)\in D\times(0,T)$, hence

$$\mathbf{u}\in L^2(0,T;V(D)).$$

From Definition 4.1 we see that for $\forall\psi\in\mathscr{D}([0,T))$ and $\forall\mathbf{h}\in\mathcal{V}(D)$

$$-\int_0^T\int_D(\mathbf{u}_\alpha,\mathbf{h})\psi_t\,dx\,dt - \int_D(\mathbf{u}_\alpha^\circ,\mathbf{h})\psi(0)\,dx - 0 = \\ = -\int_0^T\langle A\mathbf{u}_\alpha,\mathbf{h}\rangle\,\psi\,dt + \int_0^T\int_D\alpha p_\alpha(\mathbf{f},\mathbf{h})\psi\,dx\,dt + \int_0^T\langle\mathbf{s}_\alpha,\mathbf{h}\rangle\,\psi\,dt.$$

Passing to the limit when $\alpha = \alpha_n$ and $n \to \infty$ we obtain

$$-\int_0^T \int_D (\mathbf{u}, \mathbf{h})\psi_t \, dx\, dt - \int_D (\mathbf{u}^\circ, \mathbf{h})\psi(0)\, dx = \\ = -\int_0^T \langle A\mathbf{u}, \mathbf{h}\rangle \, \psi \, dt + 0 + \int_0^T \langle \mathbf{s}, \mathbf{h}\rangle \, \psi \, dt.$$

Since $\mathbf{h} \in V(D)$ we have $(\mathbf{u}^\circ, \mathbf{h})_D = (\mathbf{u}^\circ, P_H\mathbf{h})_D = (P_H\mathbf{u}^\circ, \mathbf{h})_D$. Hence by Definition 5.1 $\boldsymbol{v} = \mathbf{u}$ is a solution to the problem (5.1)–(5.4) with the initial condition $\boldsymbol{v}^\circ = P_H\mathbf{u}^\circ$. By Theorem 5.2 the velocity $\boldsymbol{v}$ is unique, hence Remark 2.12 implies that when $\alpha \to 0$

$$\mathbf{u}_\alpha \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(0, T; H_0^1(D)^d).$$

By Proposition 2.11 $\exists \mathbf{w} \in L^\infty(0, T; L^2(D)^d)$ and $\exists \{\alpha'_n\}_{n\in\mathbb{N}} \subset (0, 1)$ such that when $n \to 0$ we have $\alpha'_n \to 0$ and

$$\mathbf{u}_{\alpha'_n} \overset{*}{\rightharpoonup} \mathbf{w} \quad \text{in} \quad L^\infty(0, T; L^2(D)^d).$$

Hence $\forall\psi \in \mathscr{D}(0, T)$ and $\forall\Phi \in \mathscr{D}(D)^d$

$$\int_0^T \int_D (\mathbf{u}_{\alpha_n}, \Phi)\psi \, dx\, dt \to \int_0^T \int_D (\mathbf{w}, \Phi)\psi \, dx\, dt, \quad n \to \infty.$$

On the other hand, since $\mathbf{u}_\alpha \rightharpoonup \mathbf{u}$ in $L^2(0, T; H_0^1(D)^d)$ as $\alpha \to 0$, we have

$$\int_0^T \int_D (\mathbf{u}_{\alpha_n}, \Phi)\psi \, dx\, dt \to \int_0^T \int_D (\mathbf{u}, \Phi)\psi \, dx\, dt, \quad n \to \infty.$$

Hence $\mathbf{u} = \mathbf{w}$ for a.e. $(x, t) \in D \times (0, T)$. Then Remark 2.12 implies that

$$\mathbf{u}_\alpha \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in} \quad L^\infty(0, T; L^2(D)^d).$$

as $\alpha \to 0$. Then we introduce the pressure q as described in [10], III, §1.5.

Finally we express the gradient of the pressure from the equation (4.2) and pass to the limit when $\alpha \to 0$ using the established convergence of the velocity and boundedness of $\sqrt{\alpha}p_\alpha$ in $L^\infty(0, T; L^2(D))$. □

Remark 5.4. The proof of theorem 5.3 can be considered as a proof of existence of weak solution to the problem (5.1)–(5.4).

We have shown that when $\alpha \to 0$ the solutions of the problem (4.1)–(4.6) converge to the *incompressible limit*, i.e. to the solution of the problem (5.1)–(5.4) with $\boldsymbol{v}^\circ = P_H\mathbf{u}^\circ$. Now let us fix $\{\boldsymbol{v}, q\}$ and analyze the established convergence in more detail.

In what follows and till the end of Section 5.2 $\{\boldsymbol{v}, q\}$ will denote a weak solution of the problem (5.1)–(5.4) with *fixed* $\boldsymbol{v}^\circ \in H(D)$.

Theorem 5.5. *Let*

$$\mathbf{b} \in L^\infty(0,T;L^\infty(D)^d), \quad c \in L^2(0,T;L^\infty(D)), \tag{5.10}$$
$$q \in W^{1,2}(0,T;L^2(D)) \cap L^2(0,T;H^1(D)). \tag{5.11}$$

If when $\alpha \to 0$ we have

$$\|\sigma_\alpha\|_{L^2(0,T;L^2(D))} = o(\sqrt{\alpha}), \qquad \mathbf{s}_\alpha \to \mathbf{s} \quad \text{in } L^2(0,T;H^{-1}(D)^d),$$
$$\mathbf{u}^\circ_\alpha \to \mathbf{u}^\circ \quad \text{in } L^2(D)^d, \qquad \|p^\circ_\alpha\|_{L^2(D)} = o\left(\frac{1}{\sqrt{\alpha}}\right),$$

and

$$\mathbf{u}^\circ = \boldsymbol{v}^\circ \tag{5.12}$$

then

$$\mathbf{u}_\alpha \to \boldsymbol{v} \quad \text{in} \quad L^2(0,T;H^1_0(D)^d),$$
$$\mathbf{u}_\alpha \to \boldsymbol{v} \quad \text{in} \quad L^\infty(0,T;L^2(D)^d),$$
$$\nabla p_\alpha \to \nabla q \quad \text{in} \quad H^{-1}(D\times(0,T))^d.$$

as $\alpha \to 0$.

Proof. The hypotheses of Theorem 5.5 and the equation (5.2) imply that $\boldsymbol{v} \in \mathcal{W}(0,T)$ and the difference $\{\mathbf{u}_\alpha - \boldsymbol{v}, p_\alpha - q\}$ is a weak solution to

$$(\alpha(p_\alpha - q))_t - (\mathbf{b},\nabla)(\alpha(p_\alpha - q)) + c\alpha(p_\alpha - q) + \operatorname{div}(\mathbf{u}_\alpha - \boldsymbol{v}) =$$
$$= \sigma_\alpha - \alpha q_t + \alpha(\mathbf{b},\nabla)q - \alpha c q \equiv \sigma'_\alpha, \tag{5.13}$$

$$(\mathbf{u}_\alpha - \boldsymbol{v})_t + \nabla(p_\alpha - q) = -A(\mathbf{u}_\alpha - \boldsymbol{v}) + \alpha(p_\alpha - q)\mathbf{f} + \alpha q\mathbf{f} + \mathbf{s}_\alpha - \mathbf{s}, \tag{5.14}$$
$$(\mathbf{u}_\alpha - \boldsymbol{v})|_{t=0} = \mathbf{u}^\circ_\alpha - \boldsymbol{v}^\circ, \tag{5.15}$$
$$(p_\alpha - q)|_{t=0} = p^\circ_\alpha - q^\circ, \tag{5.16}$$
$$(\mathbf{u}_\alpha - \boldsymbol{v})|_{\partial D} = 0, \tag{5.17}$$

where $q^\circ = q|_{t=0}$ (see Remark 2.2). Then by Theorem 4.3 the difference $\{\mathbf{u}_\alpha - \boldsymbol{v}, p_\alpha - q\}$ satisfies

$$\|\mathbf{u}_\alpha - \boldsymbol{v}\|_{L^2(0,T;H^1_0(D)^d)} + \|\mathbf{u}_\alpha - \boldsymbol{v}\|_{L^\infty(0,T;L^2(D)^d)} + \sqrt{\alpha}\|p_\alpha - q\|_{L^\infty(0,T;L^2(D))} \le$$
$$\le C\Bigg(\|\mathbf{u}^\circ_\alpha - \boldsymbol{v}^\circ\|_{L^2(D)^d} + \sqrt{\alpha}\|p^\circ_\alpha - q^\circ\|_{L^2(D)} + \alpha\|q\mathbf{f}\|_{L^2(0,T;H^{-1}(D)^d)} +$$
$$+ \|\mathbf{s}_\alpha - \mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} + \frac{\|\sigma'_\alpha\|_{L^2(0,T;L^2(D))}}{\sqrt{\alpha}}\Bigg) \equiv E(\alpha) \tag{5.18}$$

when $0 < \alpha < 1$. By the hypotheses of Theorem 5.5 $\mathbf{u}^\circ = \boldsymbol{v}^\circ$, so

$$\|\mathbf{u}^\circ_\alpha - \boldsymbol{v}^\circ\| = o(1)$$

when $\alpha \to 0$. Besides,

$$\sqrt{\alpha}\|p_\alpha^\circ - q^\circ\|_{L^2(D)} \le \sqrt{\alpha}\|p_\alpha^\circ\|_{L^2(D)} + \sqrt{\alpha}\|q^\circ\|_{L^2(D)} = o(1) + o(\sqrt{\alpha}) = o(1)$$

when $\alpha \to 0$. We also have

$$\begin{aligned} \|(\mathbf{b},\nabla)q\|_{L^2(0,T;L^2(D))} &\le C\|\mathbf{b}\|_{L^\infty(0,T;L^\infty(D)^d)}\|q\|_{L^2(0,T;H^1(D))}, \\ \|cq\|_{L^2(0,T;L^2(D))} &\le C\|c\|_{L^2(0,T;L^\infty(D))}\|q\|_{L^\infty(0,T;L^2(D))} \le \\ &\le C\|c\|_{L^2(0,T;L^\infty(D))}\|q\|_{W^{1,2}(0,T;L^2(D))}, \\ \|q_t\| &\le \|q\|_{W^{1,2}(0,T;L^2(D))}, \\ \|q\mathbf{f}\|_{L^2(0,T;L^2(D))} &\le \|q\|_{L^\infty(0,T;L^2(D))}\|\mathbf{f}\|_{L^2(0,T;L^\infty(D))} \le \\ &\le C\|q\|_{W^{1,2}(0,T;L^2(D))}\|\mathbf{f}\|_{L^2(0,T;L^\infty(D))} \end{aligned}$$

hence

$$\|\sigma'_\alpha\|_{L^2(0,T;L^2(D))} = o(\sqrt{\alpha}) + O(\alpha) = o(\sqrt{\alpha})$$

when $\alpha \to 0$. Hence the convergence $\mathbf{s}_\alpha \to \mathbf{s}$ implies that $E(\alpha) = o(1)$ when $\alpha \to 0$. This means that the velocity converges strongly. Finally we express the gradient of the pressure ∇p_α from (4.2) and pass to the limit when $\alpha \to 0$ using the established strong convergence of the velocity and the boundedness of $\sqrt{\alpha}p_\alpha$ in $L^\infty(0,T;L^2(D))$. $\square$

Remark 5.6. The estimate (5.18) indicates that when $\mathbf{u}_\alpha^\circ \equiv \boldsymbol{v}^\circ$, $\mathbf{s}_\alpha \equiv \mathbf{s}$, $p_\alpha^\circ \equiv p^\circ$ and $\sigma_\alpha \equiv 0$

$$\|\mathbf{u}_\alpha - \boldsymbol{v}\|_{L^2(0,T;H_0^1(D)^d)} = O(\sqrt{\alpha}),$$

i.e. the rate of the convergence of the velocity is $\sqrt{\alpha}$.

Remark 5.7. The pressure q has the regularity demanded in (5.11) when, for instance, $\partial D \in C^2$, $\mathbf{b} = 0$, $c = 0$, $-A \equiv \nu\Delta$, $\mathbf{f} = 0$, $\mathbf{s}_\alpha = 0$, $\sigma_\alpha = 0$ and $\boldsymbol{v}^\circ$ satisfies

1. $\boldsymbol{v}^\circ \in V(D) \cap H^3(D)$;
2. $P_H\Delta\boldsymbol{v}^\circ \in V(D)$.

Remark 5.8. In fact condition (5.12) is not only *sufficient* but also *necessary* for the convergence of the velocity $\mathbf{u}_\alpha$ to be strong. More precisely, if when $\alpha \to 0$ we have

$$\mathbf{u}_\alpha \to \boldsymbol{v} \ \text{ in } \ L^2(0,T;H_0^1(D)^d),$$
$$p_\alpha \quad \text{bounded} \quad \text{in} \quad L^\infty(0,T;L^2(D)),$$

then $\mathbf{u}^\circ = \boldsymbol{v}^\circ$.

Proof. By Corollary 4.5 without loss of generality we can assume that $\forall\alpha \in (0,1)$

$$\{\mathbf{u}_\alpha, p_\alpha\} \in C(0,T;L^2(D)^d) \times C(0,T;L^2(D)).$$

By Proposition 2.6 we can also assume that $\boldsymbol{v} \in C(0,T;H(D))$.

Since $\mathbf{u}_\alpha \to \boldsymbol{v}$ in $L^2(0,T;H_0^1(D)^d)$ as $\alpha \to 0$, there exists a sequence $\alpha \equiv \alpha_n$, $n \in \mathbb{N}$ such that $\alpha_n \to 0$, $n \to \infty$ and when $\alpha \to 0$

$$\mathbf{u}_\alpha(t) \to \boldsymbol{v}(t) \quad \text{in} \quad H_0^1(D)^d \tag{5.19}$$

for a.e. $t \in [0,T]$ (see e.g. [19], III.3.6, III.6.13).

Let us denote $|\cdot| \equiv \|\cdot\|_{L^2(D)^k}$ and $\|\cdot\| \equiv \|\cdot\|_{H_0^1(D)^k}$, the value of k being defined by the argument. Consider a sequence of real numbers

$$X_\alpha = |\mathbf{u}_\alpha(t) - \boldsymbol{v}(t)|^2 + \alpha|p_\alpha(t)|^2 + \\ + \int_0^t \alpha\,(p_\alpha[\operatorname{div}\mathbf{b} + 2c], p_\alpha)_D\ dt + 2\int_0^t \langle A(\mathbf{u}_\alpha - \boldsymbol{v}), \mathbf{u}_\alpha - \boldsymbol{v}\rangle\ dt.$$

(Recall that $\alpha \equiv \alpha_n$.) Expanding the parentheses we write

$$X_\alpha = X^{(1)} + X_\alpha^{(2)} + X_\alpha^{(3)},$$

where

$$X^{(1)} = |\boldsymbol{v}(t)|^2 + 2\int_0^t \langle A\boldsymbol{v}, \boldsymbol{v}\rangle\ dt,$$

$$X_\alpha^{(2)} = -2(\mathbf{u}_\alpha(t), \boldsymbol{v}(t)) - 2\int_0^t \langle A\mathbf{u}_\alpha, \boldsymbol{v}\rangle\ dt - 2\int_0^t \langle A\boldsymbol{v}, \mathbf{u}_\alpha\rangle\ dt,$$

$$X_\alpha^{(3)} = |\mathbf{u}_\alpha(t)|^2 + \alpha|p_\alpha(t)|^2 + \\ + \int_0^t \alpha\,(p_\alpha[\operatorname{div}\mathbf{b} + 2c], p_\alpha)_D\ dt + 2\int_0^t \langle A\mathbf{u}_\alpha, \mathbf{u}_\alpha\rangle\ dt$$

Integrating (5.6) with respect to time t we obtain

$$X^{(1)} = |\boldsymbol{v}^\circ|^2 + 2\int_0^t \langle \mathbf{s}, \boldsymbol{v}\rangle\ dt.$$

Then (5.19) and Theorem 5.3 imply

$$X_\alpha^{(2)} \to -2(\boldsymbol{v}(t), \boldsymbol{v}(t)) - 2\int_0^t \langle A\boldsymbol{v}, \boldsymbol{v}\rangle\ dt - 2\int_0^t \langle A\boldsymbol{v}, \boldsymbol{v}\rangle\ dt, \\ -2|\boldsymbol{v}(t)|^2 - 4\int_0^t \langle A\boldsymbol{v}, \boldsymbol{v}\rangle\ dt = -2X^{(1)}.$$

Now let us rewrite $X^{(3)}$ using the energy equality (4.13) and pass to the limit when $\alpha \to 0$:

$$X_\alpha^{(3)} = |\mathbf{u}^\circ|^2 + \alpha|p^\circ|^2 + 2\int_0^t (p_\alpha, \sigma_\alpha)_D\ dt + 2\int_0^t \langle \alpha p_\alpha \mathbf{f} + \mathbf{s}_\alpha, \mathbf{u}_\alpha\rangle \to \\ \to |\mathbf{u}^\circ|^2 + 2\int_0^t \langle \mathbf{s}, \boldsymbol{v}\rangle = |\mathbf{u}^\circ|^2 - |\boldsymbol{v}^\circ|^2 + X^{(1)}.$$

Thus, when $\alpha \to 0$ we have

$$X_\alpha \to |\mathbf{u}^\circ|^2 - |\boldsymbol{v}^\circ|^2.$$

On the other hand it is clear from our assumptions and the definition of X_α that $X_\alpha \to 0$ as $\alpha \to 0$. Hence $|\mathbf{u}^\circ| = |\boldsymbol{v}^\circ|$. This equality holds only when $\mathbf{u}^\circ = \boldsymbol{v}^\circ$ since $\boldsymbol{v}^\circ = P_H \mathbf{u}^\circ$. □

5.2 Convergence of pressure

Observe that Definition 4.1 implies that any weak solution $\{\mathbf{u}, p\}$ to the problem (4.1)–(4.6) satisfies the equality

$$\partial_t \int_D p(t)\,dx + \int_D (c(t) + \operatorname{div}\mathbf{b}(t))p(t)\,dx = \frac{1}{\alpha}\int_D \sigma\,dx.$$

in sense of distributions. When $\sigma \in L^2(0,T;\widehat{L}^2(D))$ the equality above takes the form

$$\partial_t \int_D p(t)\,dx + \int_D (c(t) + \operatorname{div}\mathbf{b}(t))p(t)\,dx = 0. \tag{5.20}$$

The compressibility coefficient α is not contained in (5.20) explicitly. Hence if we could pass to the limit when $\alpha \to 0$ in (5.20), then the limit of the pressure should satisfy (5.20) as well.

However the pressure q in the incompressible fluid doesn't satisfy the equality (5.20) in general case. Even when $c = -\operatorname{div}\mathbf{b}$ the corresponding equality can fail to hold because the pressure q is defined up to an additive function of time. Nevertheless, this function can be redefined in a unique way so that (5.20) *will* hold. More precisely, the following statement is true:

Lemma 5.9. *If $q \in L^\infty(0,T;L^2(D))$ and $\int_D q\,dx \in W^{1,2}(0,T)$ then for any $M \in \mathbb{R}$ there exists a unique function $Q \in W^{1,2}(0,T)$ such that $p := q - Q$ satisfies (5.20) $t \in [0,T]$ and $\int_D p\,dx\big|_{t=0} = M$.*

Proof. Indeed, substituting $p = q - Q$ into (5.20) we obtain

$$\int_D Q_t\,dx + \int_D (c + \operatorname{div}\mathbf{b})Q\,dx = \left(\int_D q\,dx\right)_t + \int_D (c + \operatorname{div}\mathbf{b})q\,dx,$$

Then the initial condition $\int_D p\,dx\big|_{t=0} = M$ is equivalent to the condition $\int Q(0)\,dx = \int_D q\,dx\big|_{t=0} - M$, hence Q must be a solution to the Cauchy problem

$$\begin{aligned} Q_t + kQ &= s, \\ Q|_{t=0} &= \left(\int_D q\,dx\Big|_{t=0} - M\right)\Big/\int_D dx, \end{aligned} \tag{5.21}$$

where

$$k = \int_D c\,dx\Big/\int_D dx,$$

$$s = \left(\left(\int_D q\,dx\right)_t + \int_D (c + \operatorname{div}\mathbf{b})q\,dx\right)\Big/\int_D dx.$$

The hypotheses of Lemma 5.9 imply that $k \in L^2(0,T)$ and $s \in L^2(0,T)$. Hence the problem (5.21) has a unique weak solution $Q \in W^{1,2}(0,T)$. $\square$

By Theorem 5.3 the velocity $\mathbf{u}_\alpha$ converges weakly to the velocity $\boldsymbol{v}$ of the incompressible fluid as $\alpha \to 0$. The following theorem shows that if the solution $\{\boldsymbol{v}, q\}$ has some regularity then the pressure p_α behaves the same way:

Theorem 5.10. *Suppose that the assumptions* (5.10)–(5.11) *hold and* $\forall \alpha \in (0,1)$ $\sigma_\alpha \in L^2(0,T;\widehat{L}^2(D))$. *If when* $\alpha \to 0$ *we have*

$$\|\sigma_\alpha\|_{L^2(0,T;\widehat{L}^2(D))} = O(\alpha), \qquad \|\mathbf{s}_\alpha - \mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} = O(\sqrt{\alpha}),$$
$$\|\mathbf{u}^\circ_\alpha - \mathbf{u}^\circ\|_{L^2(D)^d} = O(\sqrt{\alpha}), \qquad p^\circ_\alpha \rightharpoonup p^\circ \quad \text{in} \;\; L^2(D),$$

and $\mathbf{u}^\circ = \boldsymbol{v}^\circ$ *then*

$$p_\alpha \overset{*}{\rightharpoonup} \widehat{q} \quad \text{in} \quad L^\infty(0,T;L^2(D))$$

as $\alpha \to 0$, *where* $\{\boldsymbol{v}, \widehat{q}\}$ *is the weak solution to* (5.1)–(5.4) *such that*

$$\partial_t \int_D \widehat{q}(t)\,dx + \int_D (c(t) + \operatorname{div} \mathbf{b}(t))\widehat{q}(t)\,dx = 0$$

and $\int_D \widehat{q}\,dx\big|_{t=0} = \int_D p^\circ\,dx$.

Proof. Theorem 5.5 implies that when $\alpha \to 0$

$$\mathbf{u}_\alpha \to \boldsymbol{v} \;\text{ in } L^2(0,T;H^1_0(D)^d),$$
$$\mathbf{u}_\alpha \to \boldsymbol{v} \;\text{ in } L^\infty(0,T;L^2(D)^d).$$

Using our assumptions and the estimates (5.18) we obtain that

$$\|p_\alpha\|_{L^\infty(0,T;L^2(D))} \le \|p_\alpha - q\|_{L^\infty(0,T;L^2(D))} + \|q\|_{L^\infty(0,T;L^2(D))} \le \text{const}$$

when $0 < \alpha < 1$. Then by Proposition 2.11 there exist $\widehat{q} \in L^\infty(0,T;L^2(D))$ and a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that when $n \to \infty$

$$p_{\alpha_n} \overset{*}{\rightharpoonup} \widehat{q} \quad \text{in} \quad L^\infty(0,T;L^2(D)).$$

If we let $\alpha = \alpha_n$ and pass to the limit when $n \to \infty$ in the integral forms of (4.1) and (4.2) then we will see that $\{\boldsymbol{v}, \widehat{q}\}$ is a weak solution to (5.1)–(5.4) (as in the proof of Theorem 5.3).

As it was observed above, for any $\alpha \in (0,1)$ the pressure p_α satisfies (5.20), hence for any $\psi \in \mathscr{D}([0,T))$

$$-\int_0^T \int_D p_\alpha\,dx\psi_t\,dt - \int_D p^\circ_\alpha\,dx\psi(0) + \int_0^T \int_D (c + \operatorname{div} \mathbf{b})p_\alpha\,dx\psi\,dt = 0.$$

Passing to the limit when $\alpha = \alpha_n$ and $n \to \infty$ we obtain

$$-\int_0^T \int_D \widehat{q}\,dx\psi_t\,dt - \int_D p^\circ\,dx\psi(0) + \int_0^T \int_D (c + \operatorname{div} \mathbf{b})\widehat{q}\,dx\psi\,dt = 0,$$

hence $\widehat{q}$ satisfies (5.20) and also the condition $\int_D \widehat{q}\,dx\big|_{t=0} = \int_D p^\circ\,dx$.

In view of Lemma 5.9 the pressure $\widehat{q}$ with the specified properties is *unique*, so by Remark 2.12 $p_\alpha \overset{*}{\rightharpoonup} \widehat{q}$ as $\alpha \to 0$. □

The estimate (5.18) yields only boundedness of the pressure. This is a result of presence of the multiplier $1/\sqrt{\alpha}$ before the term $\|\sigma\|_{L^2(0,T;L^2(D))}$ in the estimate from Theorem 4.3. However Theorem 4.8 shows that if we use a finer norm for the source term, then the multiplier $1/\sqrt{\alpha}$ vanishes, what can be observed from the term $\|\sigma_3\|_{W^{1,2}(0,T;\widehat{L}^2(D))}$ in Theorem 4.8. We use this fact to prove the following result:

Theorem 5.11. *Let*

$$\begin{gathered}\mathbf{b} \in W^{1,2}(0,T;W^{1,\infty}(D)^d), \quad c \in W^{1,2}(0,T;L^\infty(D)), \\ q \in W^{2,2}(0,T;L^2(D)) \cap L^2(0,T;H^1(D)),\end{gathered} \tag{5.22}$$

and suppose that when $\alpha \to 0$ we have

$$\begin{gathered}\|\sigma_\alpha\|_{L^2(0,T;L^2(D))} = o(\alpha), \qquad \|\mathbf{s}_\alpha - \mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)} = o(\sqrt{\alpha}), \\ \|\mathbf{u}^\circ_\alpha - \mathbf{u}^\circ\|_{L^2(D)^d} = o(\sqrt{\alpha}), \qquad p^\circ_\alpha \to p^\circ \quad in\ L^2(D).\end{gathered}$$

If

$$\mathbf{u}^\circ = \boldsymbol{v}^\circ \quad and \quad p^\circ = q|_{t=0} \tag{5.23}$$

then

$$p_\alpha \to \widehat{q} \quad in \quad L^\infty(0,T;L^2(D))$$

where $\widehat{q}$ is defined in Theorem 5.10.

Proof. The real-valued function $Q = q - \widehat{q}$ is the solution to (5.21) with $M = \int_D p^\circ\,dx$. Hypothesis (5.22) implies that $Q \in W^{2,2}(0,T)$. Then $\widehat{q} \in W^{2,2}(0,T;L^2(D)) \cap L^2(0,T;H^1(D))$ and since $\{\boldsymbol{v}, \widehat{q}\}$ is a weak solution to (5.1)–(5.4) without loss of generality we may assume that $q = \widehat{q}$.

Observe that

$$q_t - (\mathbf{b}, \nabla)q + cq = q_t + (c + \operatorname{div}\mathbf{b})q - \operatorname{div}(\mathbf{b}q).$$

Let us denote $\xi = q_t + (c + \operatorname{div}\mathbf{b})q$. The hypotheses of Theorem 5.11 imply that $\xi \in L^2(0,T;\widehat{L}^2(D))$ and

$$\begin{gathered}q_{tt} \in L^2(0,T;L^2(D)), \\ \operatorname{div}\mathbf{b}_t \in L^2(0,T;L^\infty(D)^d), \quad c_t \in L^2(0,T;L^\infty(D)), \\ p, p_t \in L^\infty(0,T;L^2(D)),\end{gathered}$$

hence there exists a weak derivative

$$\xi_t = q_{tt} + (c_t + \operatorname{div}\mathbf{b}_t)q + (c + \operatorname{div}\mathbf{b})q_t \in L^2(0,T;L^2(D)).$$

But $\xi \in L^2(0,T;\widehat{L}^2(D))$, hence (by the definition of weak derivative) $\xi_t \in L^2(0,T;\widehat{L}^2(D))$. Thus $\xi \in W^{1,2}(0,T;\widehat{L}^2(D))$.

Similarly there exists a weak derivative

$$(\mathbf{b}q)_t = \mathbf{b}_t q + \mathbf{b}q_t \in L^2(0,T;L^2(D)^d),$$

hence $\mathbf{b}q \in \mathcal{W}(0,T)$. Then by Theorem 4.8 we have the following estimate

$$\|p_\alpha - q\|_{L^\infty(0,T;L^2(D))} \le C\Bigg(\frac{\|\mathbf{u}_\alpha^\circ - \boldsymbol{v}^\circ\|_{L^2(D)^d}}{\sqrt{\alpha}} + \|p_\alpha^\circ - q^\circ\|_{L^2(D)} + \\ + \sqrt{\alpha}\|q\mathbf{f}\|_{L^2(0,T;H^{-1}(D)^d)} + \frac{\|\mathbf{s}_\alpha - \mathbf{s}\|_{L^2(0,T;H^{-1}(D)^d)}}{\sqrt{\alpha}} + \\ + \frac{1}{\alpha}\|\sigma_\alpha\|_{L^2(0,T;L^2(D))} + \sqrt{\alpha}\|\mathbf{b}q\|_{\mathcal{W}(0,T)} + \sqrt{\alpha}\|\xi\|_{W^{1,2}(0,T;\widehat{L}^2(D))}\Bigg),$$

from which the strong convergence of the pressure follows directly. □

Remark 5.12. The pressure q has the regularity demanded in (5.22) when, for instance, $\partial D \in C^2$, $\mathbf{b} = 0$, $c = 0$, $-A \equiv \nu\Delta$, $\mathbf{f} = 0$, $\mathbf{s}_\alpha = 0$, $\sigma_\alpha = 0$ and $\boldsymbol{v}^\circ$ satisfies

1. $\boldsymbol{v}^\circ \in V(D) \cap H^5(D)$;
2. $P_H\Delta\boldsymbol{v}^\circ \in V(D)$;
3. $(P_H\Delta)^2\boldsymbol{v}^\circ \in V(D)$.

Remark 5.13. Theorem 5.11 still holds if we require only $\nabla p^\circ = \nabla q|_{t=0}$ instead of $p^\circ = q|_{t=0}$ in (5.23). This is a consequence of the fact that only ∇q (but not q) is uniquely determined by $\boldsymbol{v}$.

5.3 Explicit solution in one-dimensional case

Though the one-dimensional incompressible hydrodynamics is quite trivial[1], it is very convenient to demonstrate Theorems 5.3, 5.5, 5.10 and 5.11 in case when $d = 1$. Let $D = [-\pi, \pi]$ and $\nu = 1$. We will assume that $\mathbf{b} = 0$, $c = 0$, $-A \equiv \nu\Delta$, $\mathbf{f} = 0$, $\mathbf{s}_\alpha = 0$ and $\sigma_\alpha = 0$. One can show that the solution to the problem (4.1)–(4.6) with the initial data

$$\mathbf{u}^\circ(x) = A^\circ \sin x, \qquad p^\circ(x) = B^\circ \cos x$$

(where $A^\circ, B^\circ \in \mathbb{R}$) is given by

$$\mathbf{u}_\alpha(x,t) = A(t)\sin x, \qquad p_\alpha(x,t) = B(t)\cos x,$$

[1]When $d = 1$, the equality $\operatorname{div}\boldsymbol{v} = 0$ and the boundary conditions (4.6) imply $\boldsymbol{v} \equiv 0$.

where

$$
\begin{aligned}
A(t) &= e^{-t/2}\left(A^\circ \cos(\omega_\alpha t) + \frac{1}{\omega_\alpha}\left(B^\circ - \frac{1}{2}A^\circ\right)\sin(\omega_\alpha t)\right),\\
B(t) &= \frac{1}{2}A(t) - A^\circ \omega_\alpha e^{-t/2}\sin(\omega_\alpha t) + \left(B^\circ - \frac{1}{2}A^\circ\right) e^{-t/2}\cos(\omega_\alpha t),\\
\omega_\alpha &= \sqrt{1/\alpha - 1/4},
\end{aligned}
\tag{5.24}
$$

provided that $\alpha < 4$. The solution to the limit problem (5.1)–(5.4) with $\boldsymbol{v}^\circ = P_H \mathbf{u}^\circ = 0$ is clearly given by

$$\boldsymbol{v} \equiv 0, \qquad q = Q(t),$$

hence $\widehat{q} \equiv 0$. One can observe from the formulas (5.24) that

1. when $\alpha \to 0$ we have $\mathbf{u}_\alpha \rightharpoonup \boldsymbol{v}$, which is in agreement with Theorem 5.3;
2. $\mathbf{u}_\alpha \to \boldsymbol{v}$ as $\alpha \to 0$ if and only if $A^\circ = 0$, which is equivalent to the condition (5.12) from Theorem 5.5;
3. when $A^\circ = 0$ the rate of the convergence $\mathbf{u}_\alpha \to \boldsymbol{v}$ is $\sqrt{\alpha}$. This is precisely the rate which can be found in the proof of Theorem 5.5 (see Remark 5.6);
4. when $A^\circ = 0$ we also have $p_\alpha \overset{*}{\rightharpoonup} \widehat{q}$ as $\alpha \to 0$, which is in agreement with Theorem 5.10;
5. $p_\alpha \to \widehat{q}$ as $\alpha \to 0$ if and only if $A^\circ = B^\circ = 0$, which is equivalent to the condition (5.23) from Theorem 5.11.

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References

[1] Ebin D.G. Motion of a Slightly Compressible Fluid. *Proc. Nat. Acad. Sci. USA, Vol. 72, No. 2, pp. 539–542*, 1975.

[2] Klainerman S., Majda A. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math., 34:4, 481–524*, 1981.

[3] Lions P.-L., Masmoudi N. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl. 77(6), p. 585–627.*, 1998.

[4] Feireisl E., Novotný A. The Low Mach Number Limit for the Full Navier–Stokes–Fourier System. *Arch. Rational Mech. Anal. 186, p. 77–107*, 2007.

[5] Feireisl E. *Dynamics of viscous compressible fluids.* Oxford Lectures Ser. Math. Appl. – 26, Oxford Univ. Press, Oxford, 2004.

[6] Feireisl E., Novotný A. The Oberbeck–Boussinesq Approximation as a Singular Limit of the Full Navier–Stokes–Fourier System. *J. math. fluid mech., 11, p. 274–302*, 2009.

[7] Shifrin E.G. Unsteady Flows of Viscous Slightly Compressible Fluids: the Condition of Continuous Dependence on Compressibility. *Doklady Physics, vol. 44, No. 3, p. 189–192*, 1999.

[8] Puchnachev V.V. Hierarchy of the models in the theory of convection. *Zapiski Nauchnykh Seminarov POMI, Vol. 288 (in Russian)*, 2002.

[9] Alazard T. Low Mach Number Limit of the Full Navier-Stokes Equations. *Arch. Rational Mech. Anal. 180 1–73*, 2006.

[10] Temam R. *Navier–Stokes equations: theory and numerical analysis.* North Holland Publishing Co., Amsterdam – New York – Oxford, 1979.

[11] Pribyl' M.A. Spectral analysis of linearized stationary equations of viscous compressible fluid in $\mathbb{R}^3$, with periodic boundary conditions. *St. Petersburg Math. J., 20:2, p. 267–288*, 2009.

[12] Ikehata R., Koboyashi T., Matsuyama T. Remark on the L_2 Estimates of the Density for the Compressible Navier–Stokes Flow in R^3. *Nonlinear Analysis, 47*, 2001.

[13] Mucha P.B., Zajaczkowski W.M. On a L_p-estimate for the linearized compressible Navier–Stokes equations with the Dirichlet boundary conditions. *J. Differential Equations, 186, 377–393*, 2002.

[14] Gasinski L., Papageorgiou N.S. *Nonlinear Analysis.* Taylor & Francis Group, LLC, 2005.

[15] Shakhmurov V.B. Embeddings and Separable Differential Operators in Spaces of Sobolev–Lions type. 2008.

[16] Evans L.C. *Partial Differential Equations.* Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.

[17] DiPerna R.J., Lions P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. math. 98, 3, pp. 511–547*, 1989.

[18] Bogovskii M.E. Solution of the first boundary value problem for the equation of continuity of an incompressible medium. *Soviet Math. Dokl., 20, 1094-1098*, 1979.

[19] Dunford N., Schwartz J.T. *Linear Operators, Part 1, General Theory.* Interscience Publishers, Inc., 1958.