EMBEDDINGS OF MANIFOLDS WITH BOUNDARY

DMITRY TONKONOG

Abstract. In this paper we consider two problems on embeddings of manifolds with boundary. There are two parts which are formally independent of each other.

In the first part we study embeddings in Euclidean space. Let N be a closed connected orientable n-manifold. Denote by $\mathrm{Emb}^m N$ the set of isotopy classes of embeddings $N \to \mathbb{R}^m$. Denote $\mathbb{Z}_{(n)} := \mathbb{Z}$ when n is even and $\mathbb{Z}_{(n)} := \mathbb{Z}_2$ when n is odd. Classical results imply that $\mathrm{Emb}^m N = 0$ for $m \geq 2n+1$ and $\mathrm{Emb}^{2n} N = H_1(N; \mathbb{Z}_{(n-1)})$. A complete description of $\mathrm{Emb}^{2n-1} N$ was only known only for $N = \mathbb{R}P^n$and $S^1 \times S^{n-1}$. We estimate $\mathrm{Emb}^{2n-1} N$ in terms of homology of N. Our estimation follows from a theorem by A. Skopenkov (2010) and the following main result.

Theorem. *Denote by N_0 the punctured manifold. For $n \geq 4$ there is a surjection $\mathrm{Emb}^{2n-1} N_0 \twoheadrightarrow H_1(N; \mathbb{Z}_{(n-1)})$ such that the preimage of each element is in 1-1 correspondence with a quotient of $H_1(N; \mathbb{Z})^{\otimes 2}$.*

It was known that $\mathrm{Emb}^m N_0 = 0$ for $m \geq 2n$. The Theorem in some sense extends results of J.C. Becker — H.H. Glover (1971) and O. Saeki (1999).

In the second part we study embeddings into closed 3-manifolds. We prove that there exists an algorithm for recognition of embeddability of 2-polyhedra into some integral homology 3-sphere (the sphere is not fixed in advance). This is a corollary of the following main result.

Theorem. *Let M be a compact connected orientable 3-manifold with boundary. Set $G = \mathbb{Z}$, $G = \mathbb{Z}_p$ or $G = \mathbb{Q}$. If $H_1(M, G) \cong G^k$, then the minimal group $H_1(Q, G)$ for closed 3-manifolds Q containing M is isomorphic to $G^{k - \mathrm{rk}\ H_1(\partial M, \mathbb{Z})}$.*

Another corollary is that for a graph G the minimal number rk $H_1(Q; \mathbb{Z})$ for closed orientable 3-manifolds Q containing $G \times S^1$ is twice the orientable genus of the graph.

I. EMBEDDING PUNCTURED AND CLOSED n-MANIFOLDS INTO $\mathbb{R}^{2n-1}$

1. Introduction and main results

This part of the paper is on the classical Knotting Problem: for a manifold N and a number m describe the set $\mathrm{Emb}^m(N)$ of isotopy classes of embeddings $N \to \mathbb{R}^m$. For recent surveys, see [Sk08, HCEC].

Unless otherwise stated, we work in the PL (piecewise linear) or DIFF (smooth) category and the results are valid in both categories. If (co)homology coefficients are omitted, they are assumed to be $\mathbb{Z}$. We denote $\mathbb{Z}_{(n)} := \mathbb{Z}$ when n is even and $\mathbb{Z}_{(n)} := \mathbb{Z}_2$ when n is odd.

Let N be a closed connected orientable n-manifold. It is known that $\mathrm{Emb}^m N = 0$ for $m \geq 2n+1$ and $\mathrm{Emb}^{2n} N = H_1(N; \mathbb{Z}_{(n-1)})$ [EBSR]. A complete readily calculable description of $\mathrm{Emb}^{2n-1} N$ was only known only for $N = \mathbb{R}P^n$ [Ba75, §9, table on p. 299] and $S^1 \times S^{n-1}$ [Sk08]. We estimate $\mathrm{Emb}^{2n-1} N$ in terms of homology of N (Theorem 1 below). This estimation implies, in particular, the following corollary.

Corollary 1. *Let N be a closed orientable n-manifold, $n \geq 6$. If n is odd, assume that N is spin and the Hurewicz homomorphism $\pi_2(N) \to H_2(N)$ is epimorphic. If $H_1(N)$ and $H_2(N)$ are finite, then $E^{2n-1} N$ is finite.*

For an abelian group G let $G^{\otimes 2} := G \otimes_{\mathbb{Z}} G$. By N_0 we denote N minus the interior of a codimension 0 open ball.

Theorem 1. *Let N be a closed connected orientable n-manifold, $n \geq 6$. If n is odd, assume that N is spin and the Hurewicz homomorphism $\pi_2(N) \to H_2(N)$ is epimorphic. Then there is the* *following diagram whose columns are exact sequences of sets with actions a and b.* [1]

$$\begin{array}{ccc}
H_1(N;\mathbb{Z}_2) & & \\
\big\downarrow{\scriptstyle b} & & \\
\mathrm{Emb}^{2n-1}N & & H_1(N)^{\otimes 2} \\
\big\downarrow{\scriptstyle r\times W_2} & & \big\downarrow{\scriptstyle a} \\
\mathrm{Emb}^{2n-1}N_0 \times H_2(N;\mathbb{Z}_{(n)}) & \xrightarrow{\mathrm{pr}} & \mathrm{Emb}^{2n-1}N_0 \\
\big\downarrow & & \big\downarrow{\scriptstyle W_1\Lambda} \\
\begin{cases} 0, & n \text{ even} \\ H_1(N), & n \text{ odd} \end{cases} & & H_1(N;\mathbb{Z}_{(n-1)}) \\
& & \big\downarrow \\
& & 0
\end{array}$$

The right-hand column of Theorem 1 is new; its existence is implied by the main Theorem 2 below and the fact that the *Whitney invariant* $W_1 : \mathrm{Emb}^{2n}N \to H_1(N;\mathbb{Z}_{(n-1)})$ is a bijection [EBSR]. The left-hand column is [Sk10, Theorem 1.1(c)].

The classical theorem by A. Haefliger-C. Weber [Ha63, We67] reduces the description of $\mathrm{Emb}^m N$ for $2m \geq 3n+4$ to a homotopy problem. In [Ba75] this problem was further reformulated in terms of equivariant cohomology of the *deleted product* $\widetilde{N}$ (defined below). However, 'the descriptions [of $\mathrm{Emb}^{2n-1}N$] are, in general, quite complicated' [Ba75, p. 293]. A classification of $\mathrm{Emb}^{2n-1}N$ is announced in [Ya83]. Although no details are available via Google Scholar, in [Ya83] important preliminaries were set.

Our approach is based on the Haefliger-Weber theorem and the *cone map*

$$\Lambda : \mathrm{Emb}^m(N_0) \to \mathrm{Emb}^{m+1}(N)$$

which adds a cone over ∂N_0, see figure 1. This map is well-defined in the PL category and for $m \geq 3n/2+1$ in the smooth category. [2]

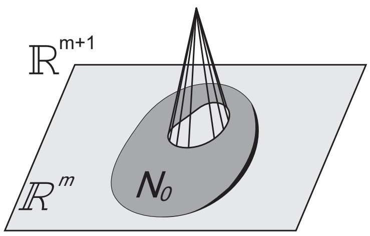

FIGURE 1. The cone map Λ which adds a cone to an embedding of N_0.

Main Theorem 2. *Let N be a closed orientable homologically k-connected n-manifold, $2n \geq 4k+4$, $k \geq 0$. Then there is the following exact sequence of sets with an action a.*

$$H_{k+1}(N)^{\otimes 2} \xrightarrow{\ a\ } \mathrm{Emb}^{2n-2k-1}N_0 \xrightarrow{\ \Lambda\ } \mathrm{Emb}^{2n-2k}N \longrightarrow 0.$$

Main Theorem 2 is an estimation of the 'kernel' of the cone map Λ. The following result is known (compare [Vr89, Corollary 3.3]).

[1]An action a on a set Y is a map $a : X \times Y \to Y$. It induces a map $X \to Y$ if we fix an element in Y. The two exact sequences in Corollary 1 become exact sequences of pointed sets when we fix an arbitrary element in each set from the sequences.

[2]The sphere ∂N_0 is unknotted in $\mathbb{R}^m$ for $m \geq 3n/2+2$, so we can smoothen the cone by changing a neighborhood of the cone's vertex.

Theorem 3. [BG71] *Let N be a closed homologically k-connected n-manifold and $m \geq 3n/2+2$. The cone map $\Lambda : \mathrm{Emb}^m(N_0) \to \mathrm{Emb}^{m+1}(N)$ is one-to-one for $m \geq 2n-2k$ and is surjective for $m = 2n-2k-1$.*

Analogously to Main Theorem 2 we could prove that there is an exact sequence

$$H_1(N)^{\otimes 2} \xrightarrow{a} \mathrm{Emb}^5 N_0 \longrightarrow H_1(N) \longrightarrow 0.$$

in the DIFF category for $n = 3$. This result for $H_1(N)$ torsion free is known [Sa99]. Moreover, in this case each stabilizer of the action a is in 1-1 correspondence with $I_N := H_2(N)/H_2(N)^\perp$, where orthogonal complement is with respect to the intersection $H_2(N) \times H_2(N) \to H_1(N)$. This result of [Sa99] uses a classification of normal bundles of embeddings $N_0 \subset \mathbb{R}^5$ which is much harder for greater dimensions n under conditions of Main Theorem 2.

2. Proof of Main Theorem 2

Denote $m = 2n-2k$. Consider the following commutative diagram of sets, in which the horizontal maps are bijections.

$$\begin{array}{ccccc}
\mathrm{Emb}^m(N) & & \xrightarrow{\quad\alpha\quad} & & \pi^{m-1}_{\mathrm{eq}}(\widetilde{N}) \\
\Lambda\uparrow & & & & \lambda^*\uparrow \\
\mathrm{Emb}^{m-1}(N_0) & \xrightarrow{\alpha_0} & \pi^{m-2}_{\mathrm{eq}}(\widetilde{N_0}) & \xrightarrow{\Sigma} & \pi^{m-1}_{\mathrm{eq}}(\Sigma\widetilde{N_0})
\end{array}$$

Here

- $\widetilde{N}$ is the *deleted product* of N, i.e. N^2 minus an open tubular neighborhood of the diagonal, with standard involution;
- $\pi^i_{\mathrm{eq}}(X)$ is the set of equivariant maps $X \to S^i$ up to equivariant homotopy;
- maps α and α_0 are the *Haefliger-Wu invariants*;
- Σ is the suspension;
- λ^* is induced by an equivariant map $\lambda : \widetilde{N} \to \Sigma\widetilde{N_0}$ defined below.

Construction of λ. We repeat the construction of [BG71]. Represent

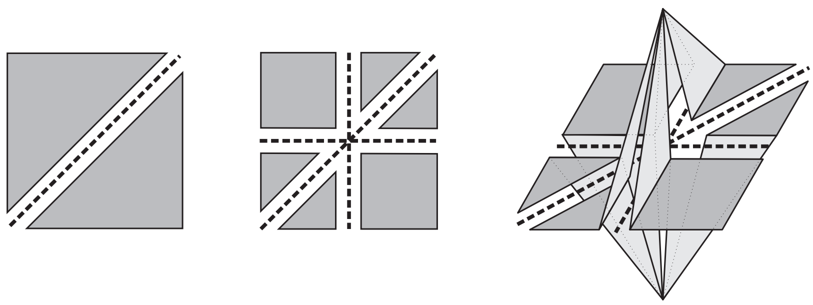

Figure 2. From left to right: $\widetilde{N}$, $\widetilde{N_0}$ and the image $\lambda\widetilde{N} \subset \Sigma\widetilde{N_0}$.

$$\Sigma\widetilde{N_0} = \frac{\widetilde{N_0} \times [-1;1]}{\widetilde{N_0} \times \{-1\},\ \widetilde{N_0} \times \{1\}}. \qquad (1)$$

For $x \in \widetilde{N_0}$ set $\lambda(x) := (x, 0)$. We identify $U_\varepsilon(P)$ with the unit ball in $\mathbb{R}^n$, with P corresponding to $0 \in \mathbb{R}^n$. Now set (see figure 2)

$\lambda(x) := ((x_1, v), t-1)$ for $x = (x_1, tv) \in N_0 \times U_\varepsilon(P)$ where $x_1 \in N_0$; $v \in \partial U_\varepsilon(P)$; $t \in [0;1]$.

Analogously, for $x = (tv, x_1) \in B^n \times N_0$ set $\lambda(x) := ((x_1, v), 1-t)$.

Proof of commutativity of the diagram above. Consider an embedding $f : N_0 \subset \mathbb{R}^{m-1}$. It induces an equivariant map $f_* : \widetilde{N_0} \to S^{m-2}$. By definition of the Haefliger-Wu invariant, $[f_*] = \alpha_0[f]$. [3] Next, Λf induces an equivariant map $(\Lambda f)_* : \widetilde{N} \to S^{m-1}$, $[(\Lambda f)_*] = \alpha\Lambda[f]$. [4] The commutativity of

[3] Square brackets denote a natural class of equivalence which is clear from context. Here these equivalences are: being equivariantly homotopic (for equivariant maps) and being isotopic (for embeddings).

[4] The cone maps from $\mathrm{Emb}^{2m-1}N_0$ and from the set of individual embeddings are both denoted by Λ.

the diagram above is equivalent to the following fact: the map $(\Lambda f)_*$ is equivariantly homotopic to the composition

$$\widetilde{N} \xrightarrow{\lambda} \Sigma\widetilde{N_0} \xrightarrow{\Sigma f_*} S^{m-1}.$$

The maps $(\Lambda f)_*$ and $(\Sigma f_*)\lambda$ coincide on $\widetilde{N_0}$, both map $B^n \times N_0$ to the upper hemisphere of S^{m-1} and $N_0 \times B^n$ to the lower hemisphere. Thus $((\Sigma f_*)\lambda)(x) \in S^{m-1}$ and $(\Lambda f)_*(x) \in S^{m-1}$ are not antipodal for each $x \in \widetilde{N}$, meaning that $(\Sigma f_*)\lambda$ and $(\Lambda f)_*$ are equivariantly homotopic. ■

The map α is one-to-one by the Haefliger-Weber theorem [Ha63, We67], [Sk06, 5.2 and 5.4]. The map α_0 is one-to-one by the the Haefliger theorem for manifolds with boundary (see [Ha63, 6.4], [Sk02, Theorem 1.1$\alpha\partial$] for the DIFF case and [Sk02, Theorem 1.3$\alpha\partial$] for the PL case). [5] Next, Σ is one-to-one by the equivariant version of Freudenthal suspension theorem [CF60, Theorem 2.5].

To prove Theorem 2 we have to show that λ^* is surjective and each preimage ${\lambda^*}^{-1}f_0$ is in 1-1 correspondence with a subset of $H_{k+1}(N)^{\otimes 2}$. We will need the following assertion which is proved below. [6]

Assertion 1. *For a k-connected n-manifold and the constructed λ we get*
(a) $H^{2n-j}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N}) = 0$ *for* $j \le 2k$.
(b) $H^{2n-2k-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N}) \cong H_{k+1}(N)^{\otimes 2}$.

There is a 1-1 correspondence between $\pi^{m-1}_{\text{eq}}\widetilde{N}$ and $\pi^{m-1}_{\text{eq}}\lambda\widetilde{N}$ since λ is not injective only on some cells of dimension $n < m-2$. We will thus work with $\pi^{m-1}_{\text{eq}}\lambda\widetilde{N}$ and $\pi^{m-1}_{\text{eq}}\widetilde{N}$ interchangeably. Take an equivariant map $f_0 : \lambda\widetilde{N} \to S^{m-1}$. It can be extended to a map $f_1 : \Sigma\widetilde{N_0} \to S^{m-1}$ since by Assertion 1(a),

$$H^i_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N}; \pi_{i-1}S^{m-1}) = 0 \quad \text{for each } i.$$

This proves that λ^* is epimorphic.

Fix an extension $f_1 : \Sigma\widetilde{N_0} \to S^{m-1}$ of f_0. Consider the following diagram.

$$\begin{array}{ccccccc} \pi^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N}) & \xrightarrow{j} & \pi^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}) & \xrightarrow{\lambda^*} & \pi^{m-1}_{\text{eq}}(\widetilde{N}) & \longrightarrow & 0 \\ d\downarrow & & & & & & \\ H^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N}) & & & & & & \end{array}$$

Here

- j is the natural map;
- $\pi^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N})$ is the set of equivariant extensions of f_0 on $\Sigma\widetilde{N_0}$ up to equivariant homotopy fixed on $\lambda\widetilde{N}$;
- d is the first obstruction map which is well-defined and bijective by the equivariant analogue of the Hopf-Whitney Theorem [Pr06, Theorem 10.5 p.130] since by Assertion 1(a)

$$H^i_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N}; \pi_i S^{m-1}) = 0 \quad \text{for each } i \ne m-1.$$

(More explicitly, Assertion 1(a) implies that $\pi^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N})$ is in 1-1 correspondence with the set $\pi^{m-1}_{\text{eq}}((\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1})$ where $(\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1}$ is the $(m-1)$-skeleton of $(\Sigma\widetilde{N_0}, \lambda\widetilde{N})$. Next, $\pi^{m-1}_{\text{eq}}((\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1})$ is in a 1-1 correspondence with the set

$$(\pi^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1})_s$$

of equivariant maps $(\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1} \to S^{m-1}$ fixed on the $(m-2)$-skeleton of $(\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1}$ up to relative equivariant homotopy constant on the $(m-3)$-skeleton of $(\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1}$. (The proof is by general position, see [Pr06, Theorem 10.5].) Now let $d_1([f_2]) \in H^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N})$ for $[f_2] \in (\pi^{m-1}_{\text{eq}}(\Sigma\widetilde{N_0}, \lambda\widetilde{N})^{m-1})_s$ be the first obstruction to an existence of equivariant relative homotopy,

[5] The Haefliger-Weber theorem says that the Haefliger-Wu invariant $\alpha : \mathrm{Emb}^m N \to \pi^{m-1}_{\text{eq}}(N)$ is one-to-one for $2m \ge 3n+4$ and the Haefliger theorem for manifolds with boundary says that α is one-to-one if N has $(n-d-1)$-dimensional spine for $2m \ge 3n+1-d$ in the DIFF category and for $2m \ge 3n+2-d$ in the PL category, $d \ge 0$.

[6] Part (a) is essentially proved in [BG71] and part (b) is new. Note that our proof of part (a) is considerably simpler that in [BG71].

constant on the $(m-2)$-skeleton of $(\Sigma\widetilde{N_0},\lambda\widetilde{N})$, between an element $[f_2]$ and the fixed $[f_1]$. This map d_1 is well-defined and is bijective analogous to [Pr06, Theorem 10.5]. Then d_1 is defined on $\pi^{m-1}_{\mathrm{eq}}((\Sigma\widetilde{N_0},\lambda\widetilde{N})^{m-1})$ and on $\pi^{m-1}_{\mathrm{eq}}(\Sigma\widetilde{N_0},\lambda\widetilde{N})$ by the 1-1 correspondences mentioned above.)

The set $\lambda^{*-1}f_0 = \mathrm{Im}\, j$ is in 1-1 correspondence with a subset of $\pi^{m-1}_{\mathrm{eq}}(\Sigma\widetilde{N_0},\lambda\widetilde{N})$ and thus (since d is injective) is in 1-1 correspondence with a subset of $H_{k+1}(N)^{\otimes 2}$. Main Theorem 2 is proved modulo Assertion 1. ■

Proof of Assertion 1. Let us introduce some notation. Take $P \in N \setminus N_0$. We denote by $\boxminus$, $\boxbar$, $\boxslash$ the following submanifolds of $N \times N$, respectively: $N_0\times\{P\}$, $\{P\}\times N_0$, $\{P\}\times N_0 \sqcup \mathrm{diag}\, N_0$, where $\mathrm{diag}\, N_0$ is the diagonal embedding. Let UZ denote a regular neighborhood of an embedded $Z \subset \Sigma\widetilde{N_0}$. Let $\mathrm{con}_+(\partial U\boxminus) \subset \Sigma\widetilde{N_0}$ be the upper cone over $\partial U\boxminus$, and analogously denote the lower cone.

We obtain the following chain of isomorphisms for each $j < n-2$.

$$\begin{aligned}
&H^{2n-j}_{\mathrm{eq}}(\Sigma\widetilde{N_0},\lambda\widetilde{N_0}) \cong && \text{As follows from the definition of } \lambda\\
&H^{2n-j}_{\mathrm{eq}}(\Sigma\widetilde{N_0},\ \widetilde{N_0}\cup\mathrm{con}_+(\partial U\boxminus)\cup\mathrm{con}_-(\partial U\boxbar)) \cong && \\
&H^{2n-j}_{\mathrm{eq}}\left(\frac{(\Sigma\widetilde{N_0})}{\widetilde{N_0}},\ \frac{\widetilde{N_0}\cup\mathrm{con}_+(\partial U\boxminus)\cup\mathrm{con}_-(\partial U\boxbar)}{\widetilde{N_0}}\right) \cong && \text{By equivariant homeomorphism between the two pairs; the induced involution } g' \text{ on } (\Sigma\widetilde{N_0})\vee(\Sigma\widetilde{N_0}) \text{ is described below}\\
&H^{2n-j}_{\mathrm{eq}}((\Sigma\widetilde{N_0})\vee(\Sigma\widetilde{N_0}),\ \Sigma(\partial U\boxminus)\vee\Sigma(\partial U\boxbar)) \cong && \text{Desuspension isomorphism}\\
&H^{2n-j-1}_{\mathrm{eq}}(\widetilde{N_0}\vee\widetilde{N_0},\ \partial U\boxminus\vee\partial U\boxbar) \cong && \text{Because } g' \text{ is a diffeomorhism from one component of } (\Sigma\widetilde{N_0})\vee(\Sigma\widetilde{N_0}) \text{ onto another}\\
&H^{2n-j-1}(\widetilde{N_0},\ \partial U\boxminus) \cong && \text{Excision}\\
&H^{2n-j-1}(N^2-U\boxslash,\ \boxminus) \cong && \text{By exact sequence of pair, } 2n-j-2>n\\
&H^{2n-j-1}(N^2-U\boxslash) \cong && \text{Poincaré duality}\\
&H_{j+1}(N^2-U\boxslash,\partial) \cong && \text{Excision}\\
&H_{j+1}(N^2,\boxslash) \cong && \text{This pair has homological type of the smash product, see below}\\
&H_{j+1}(N\wedge N) \cong && \text{By Künneth formula}\\
&\begin{cases} 0, & 0\le j\le 2k\\ (H_{k+1}N)^{\otimes 2}, & j=2k+1 \end{cases} && .
\end{aligned}$$

The induced involution g' on $(\Sigma\widetilde{N_0})\vee(\Sigma\widetilde{N_0})$ is given the composition of the map trivially changing the two components of $(\Sigma\widetilde{N_0})\vee(\Sigma\widetilde{N_0})$ and of the involution g on $\Sigma\widetilde{N_0}$ applied componentwise.

Here $N\wedge N := N^2/(N\vee N)$, where $N\vee N\subset N^2$ is given by the vertical and horizontal embeddings. The isomorphism $H_{j+1}(N^2,\boxslash)\cong H_{j+1}(N\wedge N)$ is implied by the following easy fact. The map $H_{j+1}N\to H_{j+1}N^2$ induced by diagonal embedding coincides with the composition

$$H_{j+1}N \overset{\mathrm{id}\oplus\mathrm{id}}{\to} H_{j+1}N\oplus H_{j+1}N \overset{\mathrm{v}\oplus\mathrm{h}}{\to} H_{j+1}N^2$$

where v, h are induced by vertical and horizontal embeddings, respectively. Assertion 1 is proved. ■

II. EMBEDDING 3-MANIFOLDS WITH BOUNDARY INTO CLOSED 3-MANIFOLDS

3. Introduction and main results

Numeration of statements will be independent from the previous part.

Corollary 1. *(a) There exists an algorithm that for any given (finite) 2-polyhedron P tells if P is embeddable into some integral homology 3-sphere (the sphere is not fixed in advance).*
(b) Take a field $\mathbb{F}=\mathbb{Z}_p$ or $\mathbb{F}=\mathbb{Q}$. There exists an algorithm that for any given (finite) connected 2-polyhedron P finds the minimal number $\dim H_1(Q,\mathbb{F})$ *among closed orientable 3-manifolds Q containing P, or tells that P is not embeddable into any orientable 3-manifold.*
In particular, there exists an algorithm for recognition of embeddability of 2-polyhedra into some $\mathbb{F}$-homology 3-sphere (the sphere is not fixed in advance).

This corollary is deduced from the main Theorems 1, 2 at the end of this section. The corollary is interesting because the existence of an algorithm recognizing the embeddability of 2-polyhedra into $\mathbb{R}^3$ is unknown [MTW09].

Theorem 1. *Let M be a compact connected 3-manifold with orientable boundary. Denote* $g:=\operatorname{rk}\,H_1(\partial M,\mathbb{Z})/2$. *Take a field $\mathbb{F}=\mathbb{Z}_p$ or $\mathbb{F}=\mathbb{Q}$. Suppose M is orientable or $\mathbb{F}=\mathbb{Z}_2$.*
(a) If M is embedded into a closed 3-manifold Q, then $\dim H_1(Q,\mathbb{F})\geq\dim H_1(M,\mathbb{F})-\ g$.
(b) There is a closed 3-manifold Q containing M such that $\dim H_1(Q,\mathbb{F})=\dim H_1(M,\mathbb{F})-\ g$*, and Q is orientable if M is orientable.*

Part (a) is essentially known, see the proof at the end of the introduction. Part (b) (i.e., the construction of 'minimal' Q) is new. It is presented in §4. The following theorem is an analogue of Theorem 1 for $\mathbb{Z}$-coefficients.

Theorem 2. *Let M be a compact connected orientable 3-manifold with boundary. Denote* $g:=\operatorname{rk}\,H_1(\partial M,\mathbb{Z})/2$.
(a) If M is embedded into a closed 3-manifold Q, then $H_1(Q,\mathbb{Z})$ contains a subgroup which has a quotient isomorphic to

$$C(M):=\mathbb{Z}^{\operatorname{rk}\,H_1(M,\mathbb{Z})-g}\oplus\operatorname{Tors}\,H_1(M,\mathbb{Z}).$$

(b) Suppose $H_1(M,\mathbb{Z})\cong\mathbb{Z}^m$. Then there is a closed orientable 3-manifold Q containing M such that $H_1(Q,\mathbb{Z})\cong C(M)=\mathbb{Z}^{m-g}$.
(c) There is a compact connected orientable 3-manifold M with boundary which is not embeddable into any closed 3-manifold Q such that $H_1(Q,\mathbb{Z})\cong C(M)$.

Again, part (a) is essentially known and part (b) is new; it is proved after Theorem 1(b) in §4. We present an example for part (c) in §5.

Remark 1. It is shown in the proof of Theorems 1(a), 2(a) that $C(M)\cong H_1(M,\mathbb{Z})/iH_1(\partial M,\mathbb{Z})$, where $i:H_1(\partial M,\mathbb{Z})\to H_1(M,\mathbb{Z})$ is inclusion-induced.

Remark 2. Suppose a closed orientable 3-manifold Q contains M and $H_1(Q,\mathbb{Z})\cong C(M)$. Then for *each* field $\mathbb{F}=\mathbb{Z}_p$ and $\mathbb{F}=\mathbb{Q}$ we get $\dim H_1(Q,\mathbb{F})=\dim H_1(M,\mathbb{F})-\operatorname{rk}\,H_1(\partial M,\mathbb{Z})/2$, while Theorem 1(b) generally provides *different* 'minimal' manifolds for different fields.

The following is a straightforward corollary of Theorems 1, 2.

Corollary 2. *Let M be a compact orientable 3-manifold with boundary and suppose $G=\mathbb{Z}$, $G=\mathbb{Z}_p$ or $G=\mathbb{Q}$. Then M embeds into some G-homology 3-sphere if and only if $H_1(M,G)\oplus H_1(M,G)\cong H_1(\partial M,G)$.*

Remark 3. The analogue Theorem 1(b) without the hypothesis that ∂M is orientable is wrong. To show this, it is sufficient to give an example of a 3-manifold M with non-orientable boundary such that $\dim H_1(\partial M,\mathbb{Z}_2)=2\dim H_1(M,\mathbb{Z}_2)$. Such M will not be embeddable into a $\mathbb{Z}_2$-homology 3-sphere (because $\mathbb{Z}_2$-homology spheres are orientable and M is not). Denote $D^2=\{z\in\mathbb{C}:\ |z|\leq1\}$. Consider the manifold $M:=D^2\times[0,1]/(z,0)\sim(\bar{z},1)$. Then $\dim H_1(M,\mathbb{Z}_2)=1$ and ∂M is the Klein bottle, so $\dim H_1(\partial M,\mathbb{Z}_2)=2$.

Corollary 3. *Let L be a connected graph. Set $\mathbb{F}=\mathbb{Z}_p$ or $\mathbb{F}=\mathbb{Q}$.*
(a) The minimal number $\dim H_1(Q,\mathbb{F})$ *for closed orientable 3-manifolds Q containing $L\times S^1$ equals*

to $2g(L)$*, where the* orientable genus $g(L)$ *is the minimal* g *such that* L *embeds into a sphere with* g *handles.*
(b) The minimal number $\dim H_1(Q;\mathbb{Z}_2)$ *for closed 3-manifolds* Q *containing* $L\times S^1$ *equals to the* genus *of* L*, i.e. to the minimal* k *such that* L *is embeddable into a compact closed 2-manifold* Ω *with* $\dim H_1(\Omega,\mathbb{Z}_2)=k$.

The construction of the 'minimal' Q in Corollary 3 is simpler than the general construction in Theorem 1. However, the lower estimation here is harder and is reduced to the lower estimation in Theorem 1 by the following lemma. This lemma is proved in §6.

Lemma 1. *Let* L *be a connected graph. Suppose that the product* $L\times S^1$ *is embedded into a 3-manifold* Q*. Suppose that either* Q *is orientable or* L *is not homeomorphic to* S^1 *or* I*.*
Then the regular neighborhood of $L\times S^1$ *in* Q *is homeomorphic to the product* $K\times S^1$ *for a certain 2-manifold* K *containing* L*. If* Q *is orientable, then* K *is also orientable.*

For instance, let K_5 be the complete graph on 5 vertices. Corollary 3(a) implies that $K_5\times S^1$ is embeddable into a certain closed orientable 3-manifold Q such that $\dim H_1(Q,\mathbb{F})=2$ and is not embeddable into any closed orientable 3-manifold with the first homology group of dimension 0 or 1. This result was obtained by A. Kaibkhanov (unpublished). The non-embeddability of $K_5\times S^1$ into S^3 was stated by M. Galecki and T. Tucker (as far as the author knows, unpublished) and proved by M. Skopenkov in [Sk03]. [7]

Example. *For* $\mathbb{F}=\mathbb{Z}_p$ *or* $\mathbb{F}=\mathbb{Q}$ *denote* $r(M,\mathbb{F}):=\dim H_1(M,\mathbb{F})-\frac{1}{2}\cdot\dim H_1(\partial M,\mathbb{F})$.
(a) Let Ξ *be a sphere with* g *handles and* h *holes. Then* $r(\Xi\times S^1,\mathbb{F})=2g$.
(b) Let Ξ *be a connected sum of* k $\mathbb{R}P^2$*'s with* h *holes. Then* $r(\Xi\times S^1,\mathbb{Z}_2)=k$.

Proof of Corollary 3 modulo Lemma 1. Since $S^1\times S^1\subset S^3$ and $I\times S^1\subset S^3$, it suffices to consider the case when L is not homeomorphic to S^1 or I. Corollaries 3(a), (b) now follow from Example (a), (b), Theorem 1 and Lemma 1. ■

The structure of the paper is as follows. Now we prove Theorems 1(a), 2(a). In this section we also prove Corollary 1 modulo Lemma 2 below. In §4 we prove Theorems 1(b), 2(b). In §5 we prove Theorem 2(c) by providing an example. In §6 we prove Lemmas 1, 2. Remarkably, the proof of these two lemmas uses the same theory developed in [BRS99], though the lemmas look quite different.

Proof of Theorems 1(a), 2(a). Suppose that $M\subset Q$, where M is a 3-manifold with boundary and Q is a closed 3-manifold. In this paragraph, the homology coefficients are $\mathbb{Z}$, $\mathbb{Z}_p$ or $\mathbb{Q}$. Let $i: H_1(\partial M)\to H_1(M)$, $I: H_1(M)\to H_1(Q)$ be the inclusion-induced homomorphisms. From the sequence of pair (Q,M) we obtain that $H_1(Q)$ has a quotient isomorphic to $H_1(M)/\mathrm{Ker}\, I$. Obviously, $\mathrm{Ker}\, I\subset \mathrm{Im}\, i$. So $H_1(M)/\mathrm{Im}\, i$ is a quotient of $H_1(M)/\mathrm{Ker}\, I$.

Let us prove Theorem 1(a); here the coefficients are $\mathbb{F}=\mathbb{Z}_p$ or $\mathbb{F}=\mathbb{Q}$. By the known 'half lives - half dies' lemma, $\dim \mathrm{Im}\, i=\dim H_1(\partial M,\mathbb{F})/2$. It is proved in [FF89, p.158], [Ha, Lemma 3.5] for $\mathbb{F}=\mathbb{Q}$ and an orientable manifold M. But the proof only uses dualities which hold for any coefficient field when M is orientable, and also for $\mathbb{F}=\mathbb{Z}_2$ when M is not necessarily orientable. Thus $\dim H_1(Q,\mathbb{F})\ge \dim H_1(M,\mathbb{F})-\dim H_1(\partial M,\mathbb{F})/2$. The proof of Theorem 1(a) is finished by noting that $\dim H_1(\partial M,\mathbb{F})=2g$ if ∂M is orientable.

To prove Theorem 2(a), it is left to check that $C(M)\cong K:=H_1(M,\mathbb{Z})/iH_1(\partial M,\mathbb{Z})$. Indeed, we obtain that $\mathrm{rk}\, K=g$ by the universal coefficients formula and the argument from the previous paragraph for $\mathbb{Q}$-coefficients, and $\mathrm{Tors}\, K=\mathrm{Tors}\, H_1(M,\partial M;\mathbb{Z})=\mathrm{Tors}\, H_1(M,\mathbb{Z})$ by the exact sequence of pair $(M,\partial M)$ and Poincaré duality. ■

Let P be a (finite) polyhedron. If a 3-manifold M is a regular neighborhood of $P\subset M$ [RoSa72], then M is called a *3-thickening* of P. The following lemma is known to specialists, but the author has not found any proof in the literature. This lemma is proved by combining [BRS99] and [Sk95] (see also [La00]); we prove it in §6.

[7] The non-embeddability of $K_5\times S^1$ into S^3 could be proved in a simpler way using the van Kampen Theorem if we assumed that $S^3\setminus U(K_5\times S^1)$ is homeomorphic to a disjoint union of solid tori. (Here $U(K_5\times S^1)$ denotes the regular neighborhood of $K_5\times S^1$ in S^3.) However, this assumption is not trivial and becomes wrong if we replace K_5 by some other graph G such that $G\times S^1$ embeds into S^3. For example, let G be a point. Take a knotted embedding $S^1\subset S^3$. Then $S^3\setminus U(S^1)$ is not homeomorphic to a solid torus.

Lemma 2. *Each polyhedron P has a finite number of orientable 3-thickenings. There exists an algorithm that for a given polyhedron P constructs all its orientable 3-thickenings (i.e., constructs their triangulations), or tells that the polyhedron has none.*

Proof of Corollary 1 modulo Lemma 2. Clearly, P is embeddable into an orientable 3-manifold Q if and only if there exists an orientable 3-thickening of P which is embeddable into Q. So the algorithm for Corollary 1 is as follows. First, construct all orientable 3-thickenings of P with the help of Lemma 2. Then for each orientable 3-thickening M of P do the following: in the (a) case check the condition of Corollary 2; in the (b) case calculate $\dim H_1(M,\mathbb{F}) - \frac{1}{2}\cdot\dim H_1(\partial M,\mathbb{F})$ (these procedures are algorithmical, given a triangulation of M). In the (a) case, the answer is positive if at least once the condition was fulfilled; in the (b) case, the minimum of the obtained numbers is the required minimal $\dim H_1(Q.\mathbb{F})$ for closed orientable 3-manifolds Q containing M. This assertion follows from Theorem 1. ■

4. Proof of Theorems 1(b), 2(b) (construction of a manifold Q)

In this section give a proof of Theorem 1(b) and then slightly modify it to prove Theorem 2(b).

Proof of Theorem 1(b). Denote $\mathbb{F} := \mathbb{Z}_p$ or $\mathbb{F} := \mathbb{Q}$. In the current proof, if coefficients in a homology group are omitted, they are assumed to be in $\mathbb{F}$.

Let $X \subset \mathbb{R}^3$ be the standardly embedded disjoint union of handlebodies such that $\partial X \cong \partial M$ and let $i : H_1(\partial M) \to H_1(M)$, $i' : H_1(\partial X) \to H_1(X)$ be the inclusion-induced homomorphisms. We construct the required manifold Q as a union of X and M along certain diffeomorphism $f : \partial X \to \partial M$. Consider the Mayer-Vietoris sequence

$$H_1(\partial M) \overset{i\oplus i' f_*^{-1}}{\longrightarrow} H_1(M)\oplus H_1(X) \to H_1(Q) \overset{0}{\to} H_0(\partial M) \to H_0(M)\oplus H_0(X),$$

where the last map is obviously injective. It follows that

$$H_1(Q) \cong \frac{H_1(M)\oplus H_1(X)}{(i\oplus i' f_*^{-1})H_1(\partial M)}.$$

Suppose the map $i\oplus i' f_*^{-1}$ is a monomorphism. Then $\dim H_1(Q) = \dim H_1(M) - g$ as required. So our goal now is to construct the map $f : \partial X \to \partial M$ such that $i\oplus i' f_*^{-1}$ is a monomorphism. We will need the following lemma.

Lemma 3. *Suppose that $\omega : \mathbb{Z}^{2g} \to \mathbb{Z}$ is a nondegenerate unimodular skew-symmetric $\mathbb{Z}$-bilinear form on $\mathbb{Z}^{2g}$. A submodule $B \subset \mathbb{Z}^{2g}$ will be called* Lagrangian *if $\omega|_B \equiv 0$ and $\mathbb{Z}^{2g}/B \cong \mathbb{Z}^g$.*

(a) Take $\mathbb{F} = \mathbb{Z}_p$. Denote by $\phi : \mathbb{Z}^{2g} \to \mathbb{F}^{2g}$ the homomorphism which applies mod p *reduction to each component and by $\omega_\mathbb{F}$ the symplectic form on $\mathbb{F}^{2g}$ which is* mod p *reduction of ω. For each Lagrangian subspace $A \subset \mathbb{F}^{2g}$ there exists a Lagrangian submodule $B \subset \mathbb{Z}^{2g}$ such that $\phi B = A$.*

(b) Take $\mathbb{F} = \mathbb{Q}$. Denote by $\phi : \mathbb{Z}^{2g} \to \mathbb{F}^{2g}$ the inclusion and by $\omega_\mathbb{F}$ the symplectic form on $\mathbb{F}^{2g}$ defined by the restriction $\omega_\mathbb{F}|_{\mathbb{Z}^{2g}} \equiv \omega$. For each Lagrangian subspace $A \subset \mathbb{F}^{2g}$ there exists a Lagrangian submodule $B \subset \mathbb{Z}^{2g}$ such that Lin $\phi B = A$.

Proof of Lemma 3. Part (b) is obvious. Let us prove part (a). Recall that here $\mathbb{F} = \mathbb{Z}_p$ and ϕ is the reduction mod p. Take a set of generators $\{e_i, f_i\}_{i=1}^g$ for $\mathbb{Z}^{2g}$ such that $\omega(e_i, f_i) = \delta_{ij}$. Then $\{\phi e_i, \phi f_i\}$ is a symplectic basis for $\mathbb{F}^{2g}$. There is a transformation $h_\mathbb{F} \in \mathrm{Sp}(2g,\mathbb{F})$ taking Lin $\{\phi e_i\}_{i=1}^g$ to A because $\mathrm{Sp}(2g,\mathbb{F})$ acts transitively on Lagrangians. Since mod p reduction maps $\mathrm{Sp}(2g,\mathbb{Z})$ epimorphically onto $\mathrm{Sp}(2g,\mathbb{F})$ [Ne72, Theorem VII.21], we can find $h \in \mathrm{Sp}(2g,\mathbb{Z})$ such that $\phi h = h_\mathbb{F}$. Then $B := \{hf_i\}_{i=1}^g$ is the required submodule. ■

Continuation of proof of Theorem 1(b). Denote by $\cap : H_1(\partial M,\mathbb{Z}) \times H_1(\partial M,\mathbb{Z}) \to \mathbb{Z}$ the intersection form and by $\cap_\mathbb{F} : H_1(\partial M) \times H_1(\partial M) \to \mathbb{F}$ the induced form (as in Lemma 3); $\cap|_\mathbb{F}$ coincides with the $\mathbb{F}$-coefficients intersection form on $H_1(\partial M)$. It is well-known that $\dim \mathrm{Ker}\ i = g$ and $\cap_\mathbb{F}|_{\mathrm{Ker}\ i} \equiv 0$ (the last assertion is analogous to [FF89 p.158]). In other words, Ker i is Lagrangian with respect to $\cap_\mathbb{F}$. Linear algebra implies that there exists another Lagrangian $A \subset H_1(\partial M)$ such that $\mathrm{Ker}\ i \cap A = \{0\}$. Let ϕ be the homomorphism from Lemma 3. By Lemma 3(a) or Lemma 3(b) (depending on what coefficient field $\mathbb{F}$ we are working with) we obtain a Lagrangian submodule $B \subset H_1(\partial M,\mathbb{Z})$ such that Lin $\phi B = A$ (if $\mathbb{F} = \mathbb{Z}_p$, this is equivalent to $\phi B = A$). Notice that Lin $\phi B = A$ implies that $\mathrm{Ker}\ i \cap \mathrm{Lin}\ \phi B = \{0\}$.

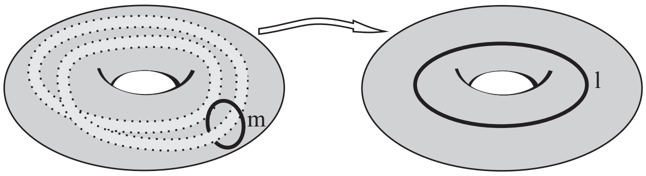

FIGURE 3. Construction of a manifold in Assertion 1.

Recall the Poincaré theorem [P] that for a handlesphere S every automorphism of $(H_1(S,\mathbb{Z}),\cap)$ is induced by some self-diffeomorphism of S.

Denote $i'_{\mathbb{Z}}: H_1(\partial X,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ the inclusion-induced homomorphism; then Ker $i'_{\mathbb{Z}}$ is generated by the meridians and is a Lagrangian submodule in $H_1(\partial X)$. Thus there exists a diffeomorphism $f: \partial X \to \partial M$ such that $f_*\text{Ker } i'_{\mathbb{Z}} = B$. (Indeed, suppose that $\partial X \cong \partial M$ is connected. Pick any diffeomorphism $h_1 : \partial X \to \partial M$. Then $K := h_{1*}\text{Ker } i'_{\mathbb{Z}} \subset H_1(\partial M,\mathbb{Z})$ is a Lagrangian submodule in $H_1(\partial M,\mathbb{Z})$. By the Poincaré theorem and because $\text{Sp}(2g,\mathbb{Z})$ acts transitively on Lagrangian submodules there exists a self-diffeomorphism h_2 of ∂M such that $h_{2*}K = B$. Now take $f := h_2h_1$. If ∂M is not connected, apply this construction componentwise.)

Because X is a disjoint union of handlebodies, Ker $i' =$ Lin ϕKer $i'_{\mathbb{Z}}$ (if $\mathbb{F} = \mathbb{Z}_p$ and not $\mathbb{Q}$, then Ker $i' = \phi$Ker $i'_{\mathbb{Z}}$). So

$$\text{Ker } i'f_*^{-1} = f_*\text{Lin } \phi\text{Ker } i'_{\mathbb{Z}} = \text{Lin } \phi f_*\text{Ker } i'_{\mathbb{Z}} = \text{Lin } \phi B.$$

Recall that Ker $i \cap \text{Lin } \phi B = \{0\}$. Therefore $i \oplus i'f_*^{-1}$ is monomorphic. ■

Proof of Theorem 2(b). We use the notation from the previous proof and work with $\mathbb{Z}$-coefficients here. Recall that Ker i is a Lagrangian submodule (i.e. $\cap|_{\text{Ker } i} \equiv 0$ and $H_1(\partial M)/\text{Ker } i \cong \mathbb{Z}^g$). [FF89 p.158]; thus we can find a set of generators $\{x_1,\dots,x_{2g}\} \in H_1(\partial M)$ such that $\{x_1,\dots,x_g\}$ generate Ker i and $\{x_g,\dots,x_{2g}\}$ also generate a Lagrangian submodule. Then there exists a diffeomorphism $f : \partial X \to \partial M$ such that Ker $i'f_*^{-1}$ is generated by $\{x_{g+1},\dots,x_{2g}\}$. This is done analogously to the proof of Theorem 1(a) using the Poincaré theorem and the fact that $\text{Sp}(2g,\mathbb{Z})$ acts transitively on Lagrangian submodules [8]. By construction we obtain[9]

$$H_1(Q) \cong \frac{H_1(M)\oplus H_1(X)}{(i\oplus i'f_*^{-1})H_1(\partial M)} \cong \frac{H_1(M)}{iH_1(\partial M)} \oplus \frac{H_1(X)}{(i'f_*^{-1})H_1(\partial M)} \cong \frac{H_1(M)}{iH_1(\partial M)} \cong C(M).$$

The second group in the direct sum is obviously zero for X a disjoint union of handlebodies. The last isomorphism is shown in the proof of Theorems 1(a), 2(a). ■

5. PROOF OF THEOREM 2(C)

In this section we work with $\mathbb{Z}$-coefficients. Theorem 2(c) is implied by the following Assertions 1, 2.

Assertion 1. *There exists a connected orientable 3-manifold M such that*
(1) ∂M is a torus and $H_1(M) \cong \mathbb{Z}\oplus\mathbb{Z}_2$.
(2) Let l and m generate $H_1(M)$ and $2m = 0$. For some generators a, b of $H_1(\partial M) \cong \mathbb{Z}\oplus\mathbb{Z}$ the inclusion-induced homomorphism $i : H_1(\partial M) \to H_1(M)$ is given by $i(a) = 2l$, $i(b) = m$.

Proof (construction of the manifold M described). Take a solid torus D. Cut out from D another solid torus which lies inside D and runs twice along the parallel of D (see Figure 1). Unite the result along its outer boundary with another solid torus via the identity (i.e., gluing two meridians and two parallels together). It is easily seen that the orientable 3-manifold M obtained satisfies (1), (2).[10] The generators of $H_1(M)$ as in (1) are shown on Figure 3. ■

Assertion 2. *Consider a manifold M from Assertion 1. Then $C(M) = \mathbb{Z}_2$ (the group $C(M)$ is introduced in Theorem 2) but M is not embeddable into any closed 3-manifold Q such that $H_1(Q) \cong \mathbb{Z}_2$.*

[8]This step is actually easier than in Theorem 1(a) because here we do not need Lemma 3.

[9]In contrast to the proof of Theorem 1(a), now $i\oplus i'f_*^{-1}$ being a monomorphism is not a sufficient condition for $H_1(Q) \cong H_1(M)/iH_1(\partial M)$ since additional torsion can appear.

[10]Clearly, this manifold is also obtained by cutting out one solid torus from $S^1\times S^2$.

Proof. Obviously, $C(M) = \mathbb{Z}_2$. Suppose to the contrary that there is an embedding $M \subset Q$. Denote by X the closure of $Q \setminus M$ and by $i' : H_1(\partial X) = H_1(\partial M) \to H_1(X)$ the inclusion-induced homomorphism. It follows from the Mayer-Vietoris sequence that

$$H_1(Q) \cong \frac{H_1(M) \oplus H_1(X)}{(i \oplus i')H_1(\partial M)}, \quad \text{thus,} \quad H_1(Q) \text{ contains the subgroup } \quad R := \frac{H_1(M)}{i(\text{Ker } i')}.$$

First, suppose Q is orientable. Then the rank of Ker i' is equal to 1, so Ker i' is generated by $pa + qb$ for some $p, q \in \mathbb{Z}$. Notice that $i(pa + qb) = 2pl + qm$. We obtain that R is generated by l and m with the following two relations: $2m = 0$, $2pl + qm = 0$. Clearly, $R \neq 0$ and $R \not\cong \mathbb{Z}_2$ since the determinant of the matrix $\left(\begin{smallmatrix} 0 & 2 \\ 2p & q \end{smallmatrix}\right)$ is divisible by 4 but never equals ± 2 or ± 1, as it should be when $R \cong \mathbb{Z}_2$ or $R = 0$.

The case of non-orientable Q is analogous. We have now to consider the cases rk Ker $i' = 0$ and rk Ker $i' = 2$. In the first case, $R = \mathbb{Z} \oplus \mathbb{Z}_2$. In the second case, the matrix of relations for R: $\left(\begin{smallmatrix} 0 & 2p & 2r \\ 2 & q & s \end{smallmatrix}\right)^t$ is such that all of its 2×2-minors are divisible by 4. This again implies that $R \neq 0$ and $R \not\cong \mathbb{Z}_2$. ■

Remark 4. The manifold M constructed in Assertion 1 is embeddable into a 3-manifold Q with $H_1(Q) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and into $S^1 \times S^2$ with $H_1(S^1 \times S^2) \cong \mathbb{Z}$ (both manifolds are obtained by gluing a solid torus to M appropriately). These two manifolds verify Theorem 1(b) for this particular manifold M: the first manifold Q when $\mathbb{F} \neq \mathbb{Z}_2$, and $S^1 \times S^2$ when $\mathbb{F} = \mathbb{Z}_2$.

6. Proof of Lemmas 1, 2

We will use results from [BRS99]; let us state them here briefly and prove Lemma 1 after that. The proof of Lemma 2 uses the same results and is given at the end of this section. In this section, the (co)homology coefficients are $\mathbb{Z}_2$.

A classification of 3*-thickenings of 2-polyhedra* [BRS99].

Let P be a 2-polyhedron. By P' we will denote the 1-subpolyhedron which is the set of points in P having no neighborhood homeomorphic to 2-disk. By P'' we will denote a (finite) set of points of P' having no neighborhood homeomorphic to a book with n sheets for some $n \geq 1$. Take a point in any component of P' containing no point of P''. Denote by F the union of P'' and these points.

Suppose that $\cup_{A\in F}$ lk A is embeddable into S^2. Take a collection of embeddings $\{g_A : \text{lk}\, A \to S^2\}_{A\in F}$. Take the closure $d \subset P'$ of a connected component of $P' \setminus P''$ and denote its ends by $A, B \in F$ (possibly, $A = B$). Now d meets $\text{lk}\, A \cup \text{lk}\, B$ at two points (distinct, even when $A = B$). If for each such d the maps g_A and g_B give the same or the opposite orders of rotation of the pages of the book at d then the collection $\{g_A\}$ is called *faithful*. Two collections of embeddings $\{f_A : \text{lk}\, A \to S^2\}$, $\{g_A : \text{lk}\, A \to S^2\}$ are called *isopositioned*, if there is a family of homeomorphisms $\{h_A : S^2 \to S^2\}_{A\in F}$ such that $h_A \circ f_A = g_A$ for each $A \in F$. This relation preserves faithfulness. Denote by $E(P)$ the set of faithful collections up to isoposition.

Suppose that M is a 3-thickening of P. Take any point $A \in F$ and consider its regular neighborhood $R_M(A)$. Since $\partial R_M(A)$ is a sphere, we have a collection of embeddings $\{\text{lk}\, A \to \partial R_M(A)\}_{A\in F}$. Since for each closure $d \subset P'$ of a connected component of $P' \setminus P''$ the regular neighborhood of d is embedded into M, this collection of embeddings is faithful. The class $e(M) \in E(P)$ of this collection is called the *e-invariant* of M. By $w_1(M)$ we denote the first Stiefel-Whitney class of M.

Theorem 3. ([BRS99] Theorem 3.1). *Thickenings M_1, M_2 of P are homeomorphic relative to P if and only if $w_1(M_1)|_P = w_1(M_2)|_P$ and $e(M_1) = e(M_2)$.*

Proof of Lemma 1. Without loss of generality we may assume that Q is a regular neighborhood of $L \times S^1$. Due to Theorem 3, it is sufficient to construct a 2-manifold K containing L such that

(a) $K \times S^1$ is a regular neighborhood of $L \times S^1$, $e(K \times S^1) = e(Q)$ and

(b) $w_1(K \times S^1)|_{L\times S^1} = w_1(Q)|_{L\times S^1}$.

First, let us construct a 2-manifold K satisfying *(a).* Take a triangulation of the graph L; we will work with this triangulation only and denote it by the same letter L. For each vertex v in L consider an arbitrarily oriented 2-disk D^2_v. Consider the edges $e_1, \ldots, e_n$ containing v. The embedding $L \times S^1 \subset Q$ defines a cyclic ordering of $e_1, \ldots, e_n$. Take a disjoint union of n arcs in ∂D^2_v (each arc corresponding to an edge e_i) such that the cyclic ordering of the arcs is the same that of the edges.

For each edge e connecting vertices u and v connect D^2_u and D^2_v with a strip $D^1 \times D^1$, gluing it along the two arcs that correspond to e. The strip can be glued in two ways: we can either twist it or not (with respect to the orientations on D^2_u and D^2_v). After gluing a strip for each edge of L, we get a union of disks and strips that is a 2-manifold; denote it by K. The manifold K depends on choosing the twists. However, any such K satisfies *(a)*, no matter what the twists are.

By choosing the twists, let us obtain the property *(b)*.

If Q is orientable, glue all the strips without twists. Then K is orientable, and $w_1(K \times S^1)|_{L\times S^1} = w_1(Q)|_{L\times S^1} = 0$.

Now let us choose the twists in the other case: L is not homeomorphic to S^1 or I (but Q does not need to be orientable). Denote the set of all edges of L by E. Take a point $O \in S^1$. Take a set of cycles $c_1, \dots, c_s \in Z_1(L)$ such that $[c_1], \dots, [c_s] \in H_1(L)$ is a basis. Represent $w_1(Q)|_{L\times\{O\}}$ as a cochain $\{a_e \in \{0,1\}\}_{e\in E}$ so that for all k, $\quad 1 \le k \le s, \quad \sum_{e\in c_k} a_e \bmod 2 = \langle w_1(Q)|_{L\times\{O\}}, c_k\rangle$. For each edge $e \in E$, twist the corresponding strip if $a_e = 1$, and do not twist the corresponding strip if $a_e = 0$. We now obtain $w_1(K \times S^1)|_{L\times\{O\}} = w_1(Q)|_{L\times\{O\}}$ by construction. We claim that the constructed K satisfies *(b)*.

Indeed, take a vertex v of degree at least 3. This can be done, because L is not homeomorphic to S^1 or I. The homology classes of

$$c_i \times \{O\}, \quad 1 \le i \le s, \quad \text{and} \quad \{v\} \times S^1$$

form a basis of $H_1(K \times S^1)$. But

$$\langle w_1(Q), \{v\} \times S^1\rangle = 0 = \langle w_1(K \times S^1), \{v\} \times S^1\rangle$$

because the regular neighborhood of $\{v\} \times S^1$ in Q is orientable (the orientation is defined by the orientation on S^1 and the cyclic ordering of the link of v because $\deg v \ge 3$). Thus we obtain $w_1(K \times S^1)|_{L\times S^1} = w_1(Q)|_{L\times S^1}$, and the proof is finished. ■

Proof of Lemma 2. Let P be a 2-polyhedron. We use the notation from the beginning of §4. Take a faithful collection $\{g_A\}_{A\in F}$ of embeddings. If the phrase from the definition of faithfulness: *'the maps g_A and g_B give the same or the opposite orders of rotation of the pages of the book at d'* is true even in the form *'the maps g_A and g_B always give the opposite orders of rotation of the pages at d'*, then the collection $\{g_A\}$ is called *orientably faithful*. Two collections $\{f_A\}, \{g_A\}$ are called *orientably isopositioned*, if there is a family of orientation-preserving homeomorphisms $\{h_A : S^2 \to S^2\}_{A\in F}$ such that $h_A \circ f_A = g_A$ for each $A \in F$. This relation preserves the property of being orientably faithful. Denote by $SE(P)$ the set of orientably faithful collections up to orientable isoposition.

An orientable 3-thickening M of P induces an *se-invariant* $se(M) \in SE(P)$. It is an orientable version of the e-invariant and is defined analogously. The following is essentially proved in [Sk95] and [La00]: every class $c \in SE(P)$ is an se-invariant of some orientable 3-thickening of P. These papers give an algorithm for construction of such thickening. Moreover, if two orientable 3-thickenings M_1, M_2 of P have the same se-invariants $se(M_1) = se(M_2) \in SE(P)$, they are homeomorphic (this follows from Theorem 3, since the Stiefel-Whitney classes are zeros in the orientable case).

The set $SE(P)$ is obviously finite. Hence the number of orientable 3-thickenings of P is finite. The algorithm for construction of all orientable 3-thickenings of P is as follows. For each class $c \in SE(P)$ build a corresponding orientable 3-thickening using the construction from [Sk95], [La00]. Theorem 3 guarantees that we will obtain all orientable 3-thickenings as result. ■

Remark 5. The following fact also holds. It is stronger than Lemma 2 (because orientability is algorithmically recognizable). *Each polyhedron has a finite number of 3-thickenings. There exists an algorithm that for a given polyhedron P constructs all its 3-thickenings (i.e., constructs their triangulations), or tells that the polyhedron has none.* This assertion implies that the 'non-orientable' version of Corollary 1(b) with $\mathbb{F} = \mathbb{Z}_2$ also holds. We do not prove this assertion. For a proof it is sufficient (similarly to the proof of Lemma 2) to combine Theorem 3 and the fact that every pair of a faithful class $e \in E(P)$ and a cohomology class $w_1 \in H^1(P)$ satisfying certain condition is induced by some 3-thickening of P, and there is an algorithm for construction of such a thickening. In [BRS99] it is proved that every pair (e, w_1) satisfying the condition (which is algorithmically recognizable) is induced by a 3-thickening, but the algorithm for construction is not given there. This algorithm is analogous to the algorithm in the orientable case from [Sk95].

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Department of Differential Geometry and Applications, Faculty of Mechanics and Mathematics, Moscow State University -and- Independent University of Moscow.

E-mail address: dtonkonog@gmail.com