Pfaffian Stochastic Dynamics of Strict Partitions*

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Abstract

We study a family of continuous-time Markov jump processes on strict partitions preserving the distributions introduced by Borodin [Bor99] in connection with projective representations of the infinite symmetric group. The one-dimensional distributions of the processes (i.e., the Borodin's measures) have determinantal structure. We express the dynamical correlation functions of the processes in terms of certain Pfaffians. Both the static and dynamical correlation kernels are expressed through the Gauss hypergeometric function.

The results about the fixed time case were announced in the note [Pet10]. The present paper contains proofs of those results.

Contents

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1 Introduction

We introduce and study a family of continuous-time Markov jump processes on the set of all strict partitions (that is, partitions with distinct parts). These processes preserve the family of probability measures on strict partitions introduced by Borodin [Bor99] in connection with the harmonic analysis of projective representations of the infinite symmetric group. The construction of our dynamics is similar to that of Borodin and Olshanski [BO06a] and is based on a special coherency property[1] of the measures on strict partitions introduced in [Bor99]. Regarding each strict partition $\lambda = (\lambda_1 > \cdots > \lambda_\ell)$, $\lambda_j \in \mathbb{Z}_{>0} := \{1, 2, \dots\}$, as a finite point configuration $\{\lambda_1, \dots, \lambda_\ell\}$ on the half-lattice $\mathbb{Z}_{>0}$, one can say that the state space of our Markov processes is the space of all finite point configurations on $\mathbb{Z}_{>0}$. The fixed time distributions of our dynamics are probability measures on this set of configurations. In other words, in the static (fixed time) picture one sees a random point process on $\mathbb{Z}_{>0}$.

The main result of the paper is the computation of the dynamical (or space-time) correlation functions for our family of Markov processes. We show that these correlation functions can be written in terms of certain Pfaffians and explicitly compute the corresponding kernel. This kernel is expressed through the Gauss hypergeometric function, we call it the *extended* (*Pfaffian*) *hypergeometric-type kernel*. Models with correlation functions of Pfaffian form have already been studied in, e.g., [Fer04], [Mat05], [BR05], [Vul07], [Str10], see also §10 of the survey [Bor09]. Pfaffian structure was also discovered in certain random matrix models in the static picture, e.g., see [Meh04]. Our model is a first example of a continuous-time Pfaffian dynamics.

In the static case the Pfaffian formula for the correlation functions can be reduced to a determinantal one. Thus, in the fixed time picture we have a determinantal point process on $\mathbb{Z}_{>0}$. Its kernel is also expressed through the Gauss hypergeometric function and is called the *hypergeometric-type kernel*. Those of the results that concern the static case were announced in the note [Pet10]. Sections §4–7 of the present paper contain proofs of them.

Our technique is different from that of [BO06a] and is based on computations in the fermionic Fock space involving the so-called Kerov's operators which span a $\mathfrak{sl}(2, \mathbb{C})$-module. The correlation kernels are expressed through certain matrix elements related to this module. This approach is similar to the one invented by Okounkov [Oko01b] to calculate the correlation kernel of the z-measures on ordinary partitions. The use of this method in studying the dynamical model related to the z-measures was suggested in [BO06a] and was carried out by Olshanski [Ols]. The rigorous realization of this approach have required to overcome certain nontrivial technical difficulties. The application of this technique to our model on strict partitions is not straightforward and requires

[1] which has a representation-theoretic meaning.

dealing with the fermionic Fock space instead of the infinite wedge space of [Oko01b], and also with a different type of Clifford algebra.[2]

It is worth noting that the processes of [BO06a], as well as many models that arise in the theory of random matrices and random tilings (e.g., see [NF98], [War07], [JN06], [ANvM10], [Joh02], [Joh05], [BGR09]) are closely related to orthogonal polynomials. For our model this relation seems to be very indirect and does not help to compute the correlation kernels as it was in [BO06a], [BO06b]. Note also that in our situation the static (determinantal) correlation kernel is not a projection operator as it happens in some of the papers cited above.

Organization of the paper. In §2 we give main definitions about our model and state the main results. In §3 we discuss the combinatorial constructions from which our model arises. We also give an argument why our fixed-time random point processes are determinantal. In §4 we study Kerov's operators on strict partitions. These operators provide us with a convenient way of writing expectations with respect to our point processes. Such formulas are used in the computation of both static and dynamical correlation functions. In §5 we recall the formalism of the fermionic Fock space and define an action of a Clifford algebra in it. In §6 we prove that the static correlation functions of our processes can be written as Pfaffians, and express the Pfaffian kernel through matrix elements related to Kerov's operators. In §7 we write the static correlation functions as determinants and explicitly express the determinantal correlation kernel in various forms including double contour integral representations.

The Markov processes on strict partitions are defined in §8 in terms of the jump rates. In §9 we show that the space-time correlation functions of our Markov processes have Pfaffian form, and in §10 we express the dynamical Pfaffian kernel in terms of the Gauss hypergeometric function and also in terms of double contour integrals.

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2 Model and results

2.1 Point processes on the half-lattice

Let us first describe the fixed time picture, that is, the random point processes on the half-lattice $\mathbb{Z}_{>0}$ that we study. They arise from a model of random strict partitions introduced in [Bor99].

[2]One can say that our Clifford algebra is an infinite-dimensional generalization of the Clifford algebra over an odd-dimensional quadratic space. Similar Clifford algebras were used in [DJKM82], [Mat05], [Vul07]. In the latter two papers the fermionic Fock space is also used in computations of certain correlation functions. That approach analogous to the formalism of Schur measures and Schur processes [Oko01a], [OR03] and differs from the one used in the present paper.

By a *strict partition* we mean a partition with distinct nonzero parts, that is, $\lambda = (\lambda_1 > \cdots > \lambda_{\ell(\lambda)} > 0)$, where $\lambda_j \in \mathbb{Z}_{>0}$. The number $|\lambda| := \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$ is called the *weight* of the partition, and the number of nonzero components $\ell(\lambda)$ is the *length* of the partition. By $\mathbb{S}_n$ denote the set of all strict partitions of weight $n = 0, 1, \dots$.[3] Throughout the paper we identify strict partitions and corresponding shifted Young diagrams as in [Mac95, Ch. I, §1, Ex. 9].

The description of the model of [Bor99] starts with the *Plancherel measures* on the sets $\mathbb{S}_n$:

$$\mathsf{Pl}_n(\lambda) := \frac{2^{n-\ell(\lambda)} \cdot n!}{(\lambda_1! \dots \lambda_{\ell(\lambda)}!)^2} \prod_{1 \le k < j \le \ell(\lambda)} \left(\frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j} \right)^2, \qquad \lambda \in \mathbb{S}_n \tag{2.1}$$

(by $\mathsf{Pl}_n(\lambda)$ we denote the measure of a singleton $\{\lambda\}$, and the same agreement for other measures on strict partitions is used throughout the paper). The measure Pl_n is a probability measure on $\mathbb{S}_n$. The set $\mathbb{S}_n$ parametrizes irreducible truly projective representations of the symmetric group $\mathfrak{S}_n$ [Sch11], [HH92], and the measures Pl_n on $\mathbb{S}_n$ are analogues (in the theory of projective representations of $\mathfrak{S}_n$) of the well-known Plancherel measures on ordinary partitions. The system of measures $\{\mathsf{Pl}_n\}$ possesses the coherency property (3.3) that has a representation-theoretic meaning, see §3.2. The Plancherel measures on strict partitions were studied in, e.g., [Bor99], [Iva99], [Iva06], [Pet09a].

We consider the *poissonized Plancherel measure* on the set $\mathbb{S} := \bigsqcup_{n=0}^{\infty} \mathbb{S}_n$ of all strict partitions:

$$\mathsf{Pl}_\theta(\lambda) := \frac{(\theta/2)^{|\lambda|} e^{-\theta/2}}{|\lambda|!} \mathsf{Pl}_{|\lambda|}(\lambda), \qquad \lambda \in \mathbb{S}, \tag{2.2}$$

where $\theta > 0$ is a parameter. In other words, we mix the measures Pl_n on $\mathbb{S}_n$ using the Poisson distribution on the set $\{0, 1, 2, \dots\}$ of indices n. Regarding each strict partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ as a point configuration $\{\lambda_1, \dots, \lambda_{\ell(\lambda)}\}$ on $\mathbb{Z}_{>0}$ (to the empty partition $\varnothing$ corresponds the empty configuration), we view Pl_θ as a random point process on the half-lattice $\mathbb{Z}_{>0}$.[4]

Like for the Plancherel measures on ordinary partitions [Joh01], [Oko00], [BOO00] (see also [BDJ99], [BDJ00]), the poissonization (2.2) of the measures Pl_n leads to a determinantal point process, see Theorem 1 below. In [Mat05] Matsumoto proved that the correlation functions of the poissonized Plancherel measure on strict partitions (2.2) have Pfaffian form. We strengthen this result and show that Pl_θ is a determinantal point process on $\mathbb{Z}_{>0}$, see §7.3 below.

Borodin [Bor99] introduced a deformation of the measures Pl_n (2.1) on $\mathbb{S}_n$ depending on one real parameter $\alpha > 0$ (in [Bor99] this parameter is denoted by x):

$$\mathsf{M}_{\alpha,n}(\lambda) := \mathsf{Pl}_n(\lambda) \cdot \frac{1}{\alpha(\alpha+2)\dots(\alpha+2n-2)} \cdot \prod_{k=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_k - 1} \big(j(j+1) + \alpha\big). \tag{2.3}$$

[3] By agreement, the set $\mathbb{S}_0$ consists of the empty partition $\varnothing$.

[4] Throughout the paper we use this identification of strict partitions with point configurations whenever we speak about the correlation functions.

The deformations $\mathsf{M}_{\alpha,n}$ of the Plancherel measures Pl_n preserve the coherency property (3.3). As $\alpha \to +\infty$, the measure $\mathsf{M}_{\alpha,n}$ on $\mathbb{S}_n$ converges to Pl_n.

Definition 2.1. To simplify certain formulas, instead of the parameter α we will sometimes use another parameter $\nu(\alpha) := \frac{1}{2}\sqrt{1-4\alpha}$. If $0 < \alpha \le \frac{1}{4}$, then $\nu(\alpha)$ is real, $0 \le \nu(\alpha) < \frac{1}{2}$. If $\alpha > \frac{1}{4}$, then $\nu(\alpha)$ can take arbitrary purely imaginary values. The whole picture is symmetric with respect to the replacement of $\nu(\alpha)$ by $-\nu(\alpha)$. Sometimes the argument α in $\nu(\alpha)$ is omitted.

Similarly to the poissonization of the Plancherel measures (2.2), we consider a certain mixing of the measures $\mathsf{M}_{\alpha,n}$.[5] But now as the mixing distribution we take a special case of the *negative binomial distribution*

$$\pi_{\alpha,\xi}(n) := (1-\xi)^{\alpha/2}\frac{(\alpha/2)_n}{n!}\xi^n, \qquad n = 0,1,2,\dots \tag{2.4}$$

with an additional parameter $\xi \in (0,1)$. Here

$$(a)_k := a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \tag{2.5}$$

is the *Pochhammer symbol* and $\Gamma(\cdot)$ is the Euler gamma function. As a result of the mixing we obtain a random point process $\mathsf{M}_{\alpha,\xi}$ on $\mathbb{Z}_{>0}$: $\mathsf{M}_{\alpha,\xi}(\lambda) := \pi_{\alpha,\xi}(|\lambda|)\cdot \mathsf{M}_{\alpha,n}(\lambda)$, where $\lambda \in \mathbb{S}$. The process $\mathsf{M}_{\alpha,\xi}$ is supported by finite configurations. The weight of the configuration $\lambda = \{\lambda_1,\dots,\lambda_\ell\} \subset \mathbb{Z}_{>0}$ has the form

$$\mathsf{M}_{\alpha,\xi}(\lambda) = (1-\xi)^{\alpha/2}\cdot\prod_{k=1}^{\ell} w_{\alpha,\xi}(\lambda_k)\cdot\prod_{1\le k<j\le\ell}\left(\frac{\lambda_k-\lambda_j}{\lambda_k+\lambda_j}\right)^2, \tag{2.6}$$

where

$$w_{\alpha,\xi}(x) := \frac{\xi^x\cos(\pi\nu(\alpha))}{2\pi}\frac{\Gamma(\frac{1}{2}-\nu(\alpha)+x)\Gamma(\frac{1}{2}+\nu(\alpha)+x)}{(x!)^2}, \qquad x \in \mathbb{Z}_{>0}, \tag{2.7}$$

and $(1-\xi)^{\alpha/2}$ is the normalizing constant. Note that in the limit

$$\alpha \to +\infty \text{ and } \xi \to 0 \text{ in such a way that } \alpha\xi \to \theta > 0 \tag{2.8}$$

the measures $\mathsf{M}_{\alpha,\xi}$ converge to the poissonized Plancherel measure Pl_θ. The measure Pl_θ also has the form (2.6) with $w_{\alpha,\xi}$ and $(1-\xi)^{\alpha/2}$ replaced by the limiting values $w_\theta(x) := \frac{\theta^x}{2(x!)^2}$ and $e^{-\theta/2}$, respectively. We call the limit (2.8) the *Plancherel degeneration*.

Our first result is the computation of the correlation functions of the processes $\mathsf{M}_{\alpha,\xi}$ and Pl_θ. By the correlation functions of a point process we mean

$$\rho^{(n)}(x_1,\dots,x_n) := \mathsf{Prob}\,\{\text{the random configuration contains } x_1,\dots,x_n\}, \tag{2.9}$$

[5] The mixing of Plancherel measures Pl_n and the measures $\mathsf{M}_{\alpha,n}$ over n can be viewed as a passage to the grand canonical ensemble, cf. [Ver96].

where $n = 1, 2, \dots$ and $x_1, \dots, x_n$ are pairwise distinct points of $\mathbb{Z}_{>0}$. Under mild assumptions the correlation functions determine the point process uniquely. A point process on $\mathbb{Z}_{>0}$ is called *determinantal* if there exists a function K on $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ (called the (*determinantal*) *correlation kernel*) such that the correlation functions $\rho^{(n)}$ have the following determinantal form:

$$\rho^{(n)}(x_1, \dots, x_n) = \det \left[K(x_k, x_j)\right]_{k,j=1}^{n}.$$

About determinantal point processes see, e.g., the surveys [Sos00], [HKPV06], [Bor09].

Theorem 1. *Both the point processes $\mathsf{M}_{\alpha,\xi}$ and Pl_{θ} on the half-lattice $\mathbb{Z}_{>0}$ are determinantal. The correlation kernel $\mathsf{K}_{\alpha,\xi}$ of $\mathsf{M}_{\alpha,\xi}$ is expressed through the Gauss hypergeometric functions* (7.3), (7.4)*, and, alternatively, in terms of double contour integrals* (*Propositions* 7.6 *and* 7.7)*. The correlation kernel K_{θ} of Pl_{θ} can be written in terms of the Bessel functions of the first kind* (7.16)*,* (7.17)*.*

We call the kernel $\mathsf{K}_{\alpha,\xi}$ the *hypergeometric-type* kernel. The kernel K_{θ} is obtained from $\mathsf{K}_{\alpha,\xi}$ via the Plancherel degeneration (2.8).

In the note [Pet10] two limit regimes for the kernel $\mathsf{K}_{\alpha,\xi}$ as $\xi \nearrow 1$ were considered. The first regime corresponds to studying the asymptotics of smallest parts of the random strict partition distributed according to the measure $\mathsf{M}_{\alpha,\xi}$. We stay on the lattice and let the parameter ξ go to one. As a result we get a point process on $\mathbb{Z}_{>0}$ supported by infinite configurations. This process is determinantal, its correlation kernel is expressed through the Euler gamma function [Pet10, Thm. 3.1].

In the second regime we embed the lattice $\mathbb{Z}_{>0}$ into the half-line $\mathbb{R}_{>0}$, $x \mapsto (1-\xi)x$, where $x \in \mathbb{Z}_{>0}$, and then pass to the limit as $\xi \nearrow 1$. This limit regime corresponds to studying the asymptotics of scaled largest parts of the random strict partition distributed according to the measure $\mathsf{M}_{\alpha,\xi}$. The resulting limit point process lives on infinite configurations on $\mathbb{R}_{>0}$. It is also determinantal, and its correlation kernel is expressed in terms of the Macdonald functions (they are certain versions of the Bessel functions), see [Pet10, Thm. 3.2]. This kernel is called the *Macdonald kernel*. It has already appeared in the recent paper [Lis09, §10.2] and also in [Ols98, §5] in a different context.

Note that in both limits the point processes live on infinite configurations. One cannot describe such processes in terms of probabilities of individual configurations. The description in terms of correlation functions becomes very useful. About determinantal point processes on continuous spaces see, e.g., the survey [Sos00].

We aim to discuss time-dependent extensions of these two limit transitions in a subsequent work.

2.2 Dynamical model

Let us now describe the family of continuous-time Markov jump processes $\boldsymbol{\lambda}_{\alpha,\xi}$ on the state space $\mathbb{S}$ (which is the same as the set of all finite configurations on

$\mathbb{Z}_{>0}$) that preserve the measures $\mathsf{M}_{\alpha,\xi}$. The construction of the processes $\boldsymbol{\lambda}_{\alpha,\xi}$ uses the same ideas as in [BO06a]. The first key ingredient is the continuous-time birth and death process on $\mathbb{Z}_{>0}$ denoted by $\boldsymbol{n}_{\alpha,\xi}$. It depends on our parameters α and ξ and has the following jump rates:

$$\begin{aligned}\mathsf{Prob}\,\{\boldsymbol{n}_{\alpha,\xi}(t+dt)=n+1\,|\,\boldsymbol{n}_{\alpha,\xi}(t)=n\} &= \frac{\xi(n+\alpha/2)}{1-\xi}dt,\\ \mathsf{Prob}\,\{\boldsymbol{n}_{\alpha,\xi}(t+dt)=n-1\,|\,\boldsymbol{n}_{\alpha,\xi}(t)=n\} &= \frac{n}{1-\xi}dt.\end{aligned}$$

The process $(\boldsymbol{n}_{\alpha,\xi})_{t\ge0}$ preserves the negative binomial distribution $\pi_{\alpha,\xi}$ (2.4) on $\mathbb{Z}_{>0}$ and is reversible with respect to it. About birth and death processes in general see, e.g., [KM57], [KM58].

The second key ingredient is the collection of Markov transition kernels $p^{\uparrow}_{\alpha}(n,n+1)$ from $\mathbb{S}_n$ to $\mathbb{S}_{n+1}$ and $p^{\downarrow}(n+1,n)$ from $\mathbb{S}_{n+1}$ to $\mathbb{S}_n$, $n=0,1,2,\dots$, such that

$$\mathsf{M}_{\alpha,n}\circ p^{\uparrow}_{\alpha}(n,n+1)=\mathsf{M}_{\alpha,n+1} \qquad \text{and} \qquad \mathsf{M}_{\alpha,n+1}\circ p^{\downarrow}(n+1,n)=\mathsf{M}_{\alpha,n}. \quad (2.10)$$

These kernels are canonically associated with the system of measures $\{\mathsf{M}_{\alpha,n}\}_{n=0}^{\infty}$, see §3.2 below and also [Bor99], [Pet09a]. Note that the kernels $p^{\uparrow}_{\alpha}(n,n+1)$ depend on the parameter α, and the kernels $p^{\downarrow}(n+1,n)$ do not depend on any parameter. The values $p^{\uparrow}_{\alpha}(n,n+1)_{\mu,\varkappa}$ and $p^{\downarrow}(n+1,n)_{\varkappa,\mu}$, where $\mu\in\mathbb{S}_n$ and $\varkappa\in\mathbb{S}_{n+1}$ (these are the individual transition probabilities, see §3.2), *vanish* unless the shifted Young diagram $\varkappa$ is obtained from μ by adding a box.

We describe the process $\boldsymbol{\lambda}_{\alpha,\xi}$ in terms of its jump rates. The jumps are of two types: one can either add a box to the random shifted Young diagram, or remove a box from it (of course, the result must still be a shifted Young diagram). The events of adding and removing a box are governed by the birth and death process $\boldsymbol{n}_{\alpha,\xi}=|\boldsymbol{\lambda}_{\alpha,\xi}|$. Conditioned on $\boldsymbol{\lambda}_{\alpha,\xi}(t)=\lambda$ and the jump $n\to n+1$ (where $n=|\lambda|$) of the process $\boldsymbol{n}_{\alpha,\xi}$ during the time interval $(t,t+dt)$, the choice of the box to be added to the diagram λ is made according to the probabilities $p^{\uparrow}_{\alpha}(n,n+1)_{\lambda,\varkappa}$, where $\varkappa\in\mathbb{S}_{n+1}$. Similarly, conditioned on $\boldsymbol{\lambda}_{\alpha,\xi}(t)=\lambda$ and the jump $n\to n-1$ of $\boldsymbol{n}_{\alpha,\xi}$ during $(t,t+dt)$, the choice of the box to be removed from λ is made according to the probabilities $p^{\downarrow}(n,n-1)_{\lambda,\mu}$, where $\mu\in\mathbb{S}_{n-1}$.

The fact that the process $\boldsymbol{n}_{\alpha,\xi}$ preserves the mixing distribution $\pi_{\alpha,\xi}$ together with (2.10) implies that the measure $\mathsf{M}_{\alpha,\xi}$ on $\mathbb{S}$ is invariant for the process $\boldsymbol{\lambda}_{\alpha,\xi}$. Moreover, the process is reversible with respect to $\mathsf{M}_{\alpha,\xi}$. In this paper by $(\boldsymbol{\lambda}_{\alpha,\xi}(t))_{t\ge0}$ we mean the equilibrium process, that is, starting from the invariant distribution $\mathsf{M}_{\alpha,\xi}$.

Remark 2.2. A closely related model was considered in [Pet09a], namely, a sequence of discrete-time Markov chains on the sets $\mathbb{S}_n$, $n=0,1,\dots$, with transition operators $p^{\uparrow}_{\alpha}(n,n+1)\circ p^{\downarrow}(n+1,n)$. These chains (called the *up/down Markov chains*) preserve the measures $\mathsf{M}_{\alpha,n}$. Similar models on ordinary partitions with various up and down transition kernels were studied in [Ful05], [Ful09] (spectral properties), and [BO09], [Pet09b], [Ols10] (large n limits).

The nth up/down Markov chain on $\mathbb{S}_n$ can be reconstructed from $\boldsymbol{\lambda}_{\alpha,\xi}$ as follows. Condition the process $\boldsymbol{\lambda}_{\alpha,\xi}$ to stay in the set $\mathbb{S}_n \times \mathbb{S}_{n+1}$ and take its embedded Markov chain, that is, consider the continuous-time process only at the times of jumps. We get a Markov chain on $\mathbb{S}_n \times \mathbb{S}_{n+1}$ that belongs to $\mathbb{S}_n$ at, say, even discrete time moments. Taking this chain at the even moments, we reconstruct the Markov chain on $\mathbb{S}_n$ with transition operator $p^{\uparrow}_{\alpha}(n, n+1) \circ p^{\downarrow}(n+1, n)$.

The finite-dimensional distribution of $\boldsymbol{\lambda}_{\alpha,\xi}$ at, say, time moments $t_1, \ldots, t_n \in \mathbb{R}_{\ge 0}$, can be viewed as a random point process on $\mathbb{Z}_{>0} \sqcup \cdots \sqcup \mathbb{Z}_{>0}$ (n copies). We are interested in the correlation functions of this point process, that is, in the *dynamical* (or *space-time*) *correlation functions* of the Markov process $\boldsymbol{\lambda}_{\alpha,\xi}$:

$$\begin{aligned}&\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \ldots; t_n, x_n)\\ &\quad := \mathsf{Prob}\left\{\text{the configuration } \boldsymbol{\lambda}_{\alpha,\xi}(t) \text{ at time } t = t_j \text{ contains } x_j,\ j = 1, \ldots, n\right\},\end{aligned} \tag{2.11}$$

where the points $(t_1, x_1), \ldots, (t_n, x_n) \in \mathbb{R}_{\ge 0} \times \mathbb{Z}_{>0}$ are pairwise distinct. The dynamical correlation functions uniquely determine the process $\boldsymbol{\lambda}_{\alpha,\xi}$.

The main result of the present paper is the computation of the dynamical correlation functions of $\boldsymbol{\lambda}_{\alpha,\xi}$.

To formulate the result, we need a notation. By $\mathbb{Z}_{\ne 0}$ denote the set of all nonzero integers, and for $x_1, \ldots, x_n \in \mathbb{Z}_{>0}$ put, by definition, $x_{-k} := -x_k$, $k = 1, \ldots, n$.

Theorem 2. *The equilibrium continuous-time process $(\boldsymbol{\lambda}_{\alpha,\xi}(t))_{t\ge 0}$ is Pfaffian, that is, there exists a function $\boldsymbol{\Phi}_{\alpha,\xi}(s, x; t, y)$, $x, y \in \mathbb{Z}_{\ne 0}$, $s \le t$, such that the dynamical correlation functions of $\boldsymbol{\lambda}_{\alpha,\xi}$ have the form*

$$\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \ldots; t_n, x_n) = \mathrm{Pf}\left(\boldsymbol{\Phi}_{\alpha,\xi}[\![T, X]\!]\right), \qquad 0 \le t_1 \le \cdots \le t_n, \tag{2.12}$$

where $\boldsymbol{\Phi}_{\alpha,\xi}[\![T, X]\!]$ is the $2n \times 2n$ skew-symmetric matrix with rows and columns indexed by the numbers $1, -1, \ldots, n, -n$, and the kj-th entry in $\boldsymbol{\Phi}_{\alpha,\xi}[\![T, X]\!]$ above the main diagonal is $\boldsymbol{\Phi}_{\alpha,\xi}(t_{|k|}, x_k; t_{|j|}, x_j)$, where $k, j = 1, -1, \ldots, n, -n$ (thus, $|k| \le |j|$).

The kernel $\boldsymbol{\Phi}_{\alpha,\xi}$ can be expressed through the Gauss hypergeometric function (10.3) *and through the double contour integrals* (§10.2).

Following the common terminology (e.g., see [NF98], [Joh05], [BO06a]), we call the kernel $\boldsymbol{\Phi}_{\alpha,\xi}(s, x; t, y)$ the *extended (Pfaffian) hypergeometric-type kernel*. Pfaffian point processes have already been studied in, e.g., [Fer04], [Mat05], [BR05], [Vul07], [Str10], see also [Bor09, §10]. Our model $(\boldsymbol{\lambda}_{\alpha,\xi}(t))_{t\in\mathbb{R}_{\ge 0}}$ is a first example of a continuous-time Pfaffian dynamics. An additional interesting feature of the model is that its one-dimensional distributions are determinantal.

Numerical computations suggest that the dynamical correlation functions of $\boldsymbol{\lambda}_{\alpha,\xi}$ cannot be written as determinants. We plan to give a rigorous proof of this fact in a subsequent work.

Remark 2.3. If in Theorem 2 we set $t_1 = \cdots = t_n$, then the dynamical correlation functions turn into the (static) correlation functions of the point process $\mathsf{M}_{\alpha,\xi}$ on $\mathbb{Z}_{>0}$. Thus, Theorem 2 implies that the point process $\mathsf{M}_{\alpha,\xi}$ on $\mathbb{Z}_{>0}$ is Pfaffian. To show that it is in fact determinantal requires some work (see Theorem 7.1 and Proposition A.2 from Appendix).

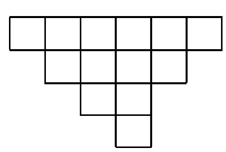
Remark 2.4. Note that in (2.12) we require that the time moments t_j are ordered. However, Theorem 2 allows to compute the correlation functions $\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \ldots; t_n, x_n)$ with arbitrary order of time moments: one should simply permute the space-time points $(t_1, x_1), \ldots, (t_n, x_n)$ (this does no change the value of $\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \ldots; t_n, x_n)$) in such a way that the time moments become nondecreasing, and then apply (2.12).

In §10.3 we consider the Plancherel degeneration (2.8) of the kernel $\mathbf{\Phi}_{\alpha,\xi}$. The resulting kernel $\mathbf{\Phi}_\theta$ is expressed through the Bessel functions of the first kind, see (10.18). The kernel $\mathbf{\Phi}_\theta$ (10.15) has analogues related to the Plancherel measures on ordinary partitions, see [PS02], [BO06c]. We do not focus much on the Plancherel case in the present paper, but formula (10.18) for $\mathbf{\Phi}_\theta$ may also explain the structure of the kernel $\mathbf{\Phi}_{\alpha,\xi}$ which itself has a rather complicated form (10.3).

3 Schur graph and multiplicative measures

3.1 Schur graph

We identify *strict partitions* $\lambda = (\lambda_1 > \cdots > \lambda_{\ell(\lambda)} > 0)$, $\lambda_j \in \mathbb{Z}_{>0}$, and corresponding *shifted Young diagrams* as in [Mac95, Ch. I, §1, Example 9]. The shifted Young diagram of the form λ consists of $\ell(\lambda)$ rows. Each kth row ($k = 1, \ldots, \ell(\lambda)$) has λ_k boxes, and for $j = 1, \ldots, \ell(\lambda) - 1$ the first box of the $(j+1)$th row is right under the second box of the jth row. For example, the shifted Young diagram corresponding to the strict partition $\lambda = (6, 4, 2, 1)$ looks as follows:



Let μ and λ be strict partitions. If $|\lambda| = |\mu| + 1$ and the shifted diagram λ is obtained from the shifted diagram μ by adding a box, then we write $\mu \nearrow \lambda$, or, equivalently, $\lambda \searrow \mu$. The box that is added is denoted by λ/μ.

The set $\mathbb{S} = \bigsqcup_{n=0}^{\infty} \mathbb{S}_n$ of all strict partitions is equipped with a structure of a graded graph: for $\mu \in \mathbb{S}_{n-1}$ and $\lambda \in \mathbb{S}_n$ we draw an edge between μ and λ iff $\mu \nearrow \lambda$. Thus, the edges in $\mathbb{S}$ are drawn only between the consecutive floors. We assume the edges to be oriented from $\mathbb{S}_{n-1}$ to $\mathbb{S}_n$. In this way $\mathbb{S}$ becomes a graded graph. It is called the *Schur graph*.[6] This graph describes

[6]In [Pet09a] the Schur graph had multiple edges, but now it is more convenient for us

the branching of (suitably normalized) irreducible truly projective characters of symmetric groups, e.g., see [Iva99].

Let $\dim_{\mathbb{S}} \lambda$ be the total number of oriented paths from the initial vertex $\varnothing$ to the vertex λ. This number has the form [Mac95, Ch. III, §8, Example 12]:

$$\dim_{\mathbb{S}} \lambda = \frac{|\lambda|!}{\lambda_1! \dots \lambda_{\ell(\lambda)}!} \prod_{1 \le k < j \le \ell(\lambda)} \frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j}, \qquad \lambda \in \mathbb{S}. \tag{3.1}$$

Observe that if the components of λ are not distinct, then $\dim_{\mathbb{S}} \lambda$ vanishes. Clearly, the numbers $\dim_{\mathbb{S}} \lambda$ satisfy the recurrence relations

$$\dim_{\mathbb{S}} \lambda = \sum_{\mu\colon \mu \nearrow \lambda} \dim_{\mathbb{S}} \mu \quad \text{for all } \lambda \in \mathbb{S}, \qquad \text{and } \dim_{\mathbb{S}} \varnothing = 1. \tag{3.2}$$

The number $\dim_{\mathbb{S}} \lambda$ can also be interpreted as the number of shifted standard tableaux of the form λ [Sag87], [Wor84], and as the (suitably normalized) dimension of the irreducible truly projective representation of the symmetric group $\mathfrak{S}_{|\lambda|}$ corresponding to the shifted diagram λ [HH92], [Iva99].

Similarly, by $\dim_{\mathbb{S}}(\mu, \lambda)$ denote the total number of paths from μ to λ in the graph $\mathbb{S}$. Clearly, $\dim_{\mathbb{S}}(\mu, \lambda)$ vanishes unless $\mu \subseteq \lambda$, that is, unless $\mu_k \le \lambda_k$ for all k. If $\mu \subseteq \lambda$, by λ/μ denote the corresponding skew shifted Young diagram, that is, the set difference of λ and μ. In fact, $\dim_{\mathbb{S}}(\mu, \lambda)$ depends only on the shape λ/μ. Clearly, $\dim_{\mathbb{S}} \lambda = \dim_{\mathbb{S}}(\varnothing, \lambda)$. Ivanov [Iva99] obtained an explicit formula for $\dim_{\mathbb{S}}(\mu, \lambda)$ in terms of the factorial Schur's Q-functions, but we do not need it.

3.2 Coherent systems of measures on the Schur graph

Following the general formalism (e.g., see [KOO98]), one can define coherent systems of measures on the Schur graph. The definition starts from the notion of the *down transition probabilities*. For $\lambda, \mu \in \mathbb{S}$, set

$$p^{\downarrow}(\lambda, \mu) := \begin{cases} \frac{\dim_{\mathbb{S}} \mu}{\dim_{\mathbb{S}} \lambda}, & \text{if } \mu \nearrow \lambda; \\ 0, & \text{otherwise.} \end{cases}$$

By (3.2), the restriction of $p^{\downarrow}$ to $\mathbb{S}_{n+1} \times \mathbb{S}_n$ for all $n = 0, 1, \dots$ is a Markov transition kernel. We denote it by $p^{\downarrow}(n+1, n) = \{p^{\downarrow}(n+1, n)_{\lambda,\mu}\}_{\lambda \in \mathbb{S}_{n+1},\, \mu \in \mathbb{S}_n}$ and call it the *down transition kernel*.

Definition 3.1. Let M_n be a probability measure on $\mathbb{S}_n$, $n = 0, 1, \dots$. We call $\{M_n\}$ the *coherent system* of measures iff

$$M_n(\lambda) = \sum_{\varkappa\colon \varkappa \searrow \lambda} M_{n+1}(\varkappa) p^{\downarrow}(\varkappa, \lambda) \quad \text{for all } n \text{ and } \lambda \in \mathbb{S}_n. \tag{3.3}$$

In other words, $M_{n+1} \circ p^{\downarrow}(n+1, n) = M_n$ for all n (cf. (2.10)).

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to consider simple edges as in, e.g., [Bor99]. The difference between these two choices is inessential because the down transition probabilities (§3.2) are the same.

Having a *nondegenerate* coherent system $\{M_n\}$ (that is, $M_n(\lambda) > 0$ for all n and $\lambda \in \mathbb{S}_n$), we can define the corresponding *up transition probabilities*. They depend on a choice of a coherent system. For $\lambda, \varkappa \in \mathbb{S}$, set

$$p^{\uparrow}(\lambda, \varkappa) := \begin{cases} \dfrac{M_{n+1}(\varkappa)}{M_n(\lambda)} p^{\downarrow}(\varkappa, \lambda), & \text{if } \lambda \in \mathbb{S}_n,\ \varkappa \in \mathbb{S}_{n+1} \text{ and } \lambda \nearrow \varkappa, \\ 0, & \text{otherwise.} \end{cases}$$

By (3.3), the restriction of $p^{\uparrow}$ to $\mathbb{S}_n \times \mathbb{S}_{n+1}$ for all $n = 0, 1, \dots$ is a Markov transition kernel. We denote it by $p^{\uparrow}(n, n+1) = \{p^{\uparrow}(n, n+1)_{\lambda,\varkappa}\}_{\lambda \in \mathbb{S}_n,\, \varkappa \in \mathbb{S}_{n+1}}$ and call it the *up transition kernel*. We have $M_n \circ p^{\uparrow}(n, n+1) = M_{n+1}$ (cf. (2.10)).

Let us make a comment about the representation-theoretic meaning of the coherency relation (3.3). The set of all coherent systems of measures on the Schur graph is a convex set. Its extreme points are identified with the points of the infinite-dimensional ordered simplex

$$\Omega_+ := \Big\{(\omega_1, \omega_2, \dots) \colon \omega_1 \ge \omega_2 \ge \dots \ge 0,\ \sum_{k=1}^{\infty} \omega_k \le 1\Big\}.$$

This is the so-called *Martin boundary* of the Schur graph. It was first described by Nazarov [Naz92]. Another proof of this result can be obtained using the general methods of [KOO98] together with the formulas of [Iva99] for dimensions of skew shifted Young diagrams.

Moreover, the following characterization of the coherent systems holds:

Theorem 3.2 ([Naz92])**.** *There is a bijection between coherent systems of measures on the Schur graph $\mathbb{S}$ and Borel probability measures on the simplex Ω_+.*

On the other hand, the points of Ω_+ are in one-to-one correspondence with the indecomposable normalized projective characters of the infinite symmetric group, and any Borel probability measure P on Ω_+ can be viewed as a (possibly decomposable) projective character χ of $\mathfrak{S}_\infty$. This character χ can be restricted to the finite symmetric group $\mathfrak{S}_n \subset \mathfrak{S}_\infty$ (of any order n) and expressed as a linear combination of (suitably normalized) irreducible truly projective characters of $\mathfrak{S}_n$. The coefficients of this expansion are exactly the numbers $\{M_n(\lambda)\}_{\lambda \in \mathbb{S}_n}$.[7] Note that the set $\mathbb{S}_n$ parametrizes irreducible truly projective representations of $\mathfrak{S}_n$ [Sch11], [HH92].

The coherency condition (3.3) for the measures $\{M_n\}$ arises naturally in this context because the restrictions of the character χ to symmetric groups $\mathfrak{S}_n$ (for different n) must be compatible with each other.

3.3 Multiplicative measures

There is a distinguished coherent system on the Schur graph, namely, the Plancherel measures $\{\mathsf{Pl}_n\}$ (2.1). The coherent system $\{\mathsf{Pl}_n\}$ corresponds (in

[7]The coherent system of measures $\{M_n\}$ here is the one that corresponds to the measure P on Ω_+ by Theorem 3.2.

the sense of Theorem 3.2) to the delta measure at the point $(0, 0, \dots) \in \Omega_+$. Using the function $\dim_{\mathbb{S}} \lambda$ defined by (3.1), one can write

$$\mathsf{Pl}_n(\lambda) = \frac{2^{n-\ell(\lambda)}}{n!} \left(\dim_{\mathbb{S}} \lambda\right)^2, \qquad n \in \mathbb{Z}_{>0}, \quad \lambda \in \mathbb{S}_n.$$

The Plancherel measures on strict partitions are analogues (in the theory of projective representations of symmetric groups) of the well-known Plancherel measures on ordinary partitions. The fact that $\sum_{\lambda \in \mathbb{S}_n} \mathsf{Pl}_n(\lambda) = 1$ for all $n \in \mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$ can be established by applying Burnside's theorem to the dimensions of irreducible linear representations of a nontrivial central $\mathbb{Z}_2$-extension of the symmetric group $\mathfrak{S}_n$ which linearizes the projective characters of $\mathfrak{S}_n$, e.g., see [HH92], [Sch11], [Ste89].

Borodin [Bor99] has introduced a deformation $\mathsf{M}_{\alpha,n}$ (2.3) of the measures Pl_n on $\mathbb{S}_n$ depending on one real parameter $\alpha > 0$. Here let us recall the characterization of the measures $\{\mathsf{M}_{\alpha,n}\}$ from [Bor99].

Definition 3.3. A system of probability measures M_n on $\mathbb{S}_n$ is called *multiplicative* if there exists a function $f\colon \{(\mathrm{i}, \mathrm{j})\colon \mathrm{j} \geq \mathrm{i} \geq 1\} \to \mathbb{C}$ such that

$$M_n(\lambda) = c_n \cdot \mathsf{Pl}_n(\lambda) \cdot \prod_{\square = (\mathrm{i},\mathrm{j}) \in \lambda} f(\mathrm{i}, \mathrm{j}) \qquad \text{for all } n \text{ and all } \lambda \in \mathbb{S}_n.$$

Here c_n, $n = 0, 1, \dots$, are the normalizing constants. The product above is taken over all boxes $\square = (\mathrm{i}, \mathrm{j})$ of the shifted Young diagram λ, where i and j are the row and column numbers of the box $\square$, respectively.

Theorem 3.4 ([Bor99])**.** *Let $\{M_n\}$ be a nondegenerate coherent system of measures on the Schur graph. It is multiplicative iff the function f has the form*

$$f(\mathrm{i}, \mathrm{j}) = (\mathrm{j} - \mathrm{i})(\mathrm{j} - \mathrm{i} + 1) + \alpha \tag{3.4}$$

for some parameter $\alpha \in (0, +\infty]$.[8]

Recall that the number $(\mathrm{j} - \mathrm{i})$ is called the *content* of the box $\square = (\mathrm{i}, \mathrm{j})$. For shifted Young diagrams all contents are nonnegative.

We denote by $\{\mathsf{M}_{\alpha,n}\}$ the multiplicative coherent system corresponding to the parameter $\alpha \in (0, +\infty)$. We see that $\mathsf{M}_{\alpha,n}$ tends to Pl_n as $\alpha \to +\infty$. The up transition kernel on $\mathbb{S}_n \times \mathbb{S}_{n+1}$ (§3.2) for the coherent system $\{\mathsf{M}_{\alpha,n}\}$ is denoted by $p_\alpha^\uparrow(n, n+1)$.

Remark 3.5. For certain negative values of α one can also define the measures $\mathsf{M}_{\alpha,n}$ by Definition 3.3 with f given by (3.4). Namely, for $\alpha = -N(N+1)$ (where $N = 1, 2, \dots$) the measures $\mathsf{M}_{\alpha,n}$ are well-defined and nonnegative for $0 \leq n \leq N(N+1)/2$. Moreover, $\mathsf{M}_{\alpha,n}(\lambda) > 0$ iff λ is inside the shifted diagram $(N, N-1, \dots, 1)$. However, we do not focus on this case in the present paper.

[8]If $f(\mathrm{i}, \mathrm{j})$ has the form (3.4), then $c_n = \alpha(\alpha+2)\dots(\alpha+2n-2)$. The case $\alpha = +\infty$ is understood in the limit sense, and $\lim_{\alpha \to +\infty} \frac{1}{c_n} \prod_{\square = (\mathrm{i},\mathrm{j}) \in \lambda} f(\mathrm{i}, \mathrm{j}) = 1$ for all n. This case corresponds to the Plancherel measures $\{\mathsf{Pl}_n\}$.

3.4 Mixing of measures. Point configurations on the half-lattice

For a set $\mathfrak{X}$ by $\mathsf{Conf}(\mathfrak{X})$ denote the space of all (locally finite) point configurations on $\mathfrak{X}$, and by $\mathsf{Conf}_{\mathrm{fin}}(\mathfrak{X}) \subset \mathsf{Conf}(\mathfrak{X})$ denote the subset consisting only of finite configurations. If $\mathfrak{X}$ is discrete, then $\mathsf{Conf}(\mathfrak{X}) \cong \{0,1\}^{\mathfrak{X}}$ is a compact space. A Borel probability measure (with respect to a certain natural topology) on $\mathsf{Conf}(\mathfrak{X})$ is called a random point process on $\mathfrak{X}$. In more detail, e.g., see [Sos00].

As explained in §2.1, we mix the measures $\mathsf{M}_{\alpha,n}$ (2.3) using the negative binomial distribution $\pi_{\alpha,\xi}$ (2.4) on the set $\{0,1,\dots\}$ of indices n. As a result we get a probability measure $\mathsf{M}_{\alpha,\xi}$ (2.6) on the set $\mathbb{S}$ of all strict partitions. Identifying strict partitions with point configurations as in §2.1, we see that the set $\mathbb{S}$ is the same as $\mathsf{Conf}_{\mathrm{fin}}(\mathbb{Z}_{>0})$. Thus, $\mathsf{M}_{\alpha,\xi}$ can be viewed as a point process on $\mathbb{Z}_{>0}$ supported by finite configurations. Under the Plancherel degeneration (2.8) the measures $\mathsf{M}_{\alpha,\xi}$ become the poissonized Plancherel measure Pl_θ (2.2).

Let us now prove that the point processes $\mathsf{M}_{\alpha,\xi}$ and Pl_θ on $\mathbb{Z}_{>0}$ are determinantal. Observe that the processes $\mathsf{M}_{\alpha,\xi}$ and Pl_θ have a general structure described in the following Definition:

Definition 3.6. Let w be a nonnegative function on $\mathbb{Z}_{>0}$ such that

$$\sum_{x=1}^{\infty} w(x) < \infty. \tag{3.5}$$

By $\mathbf{P}^{(w)}$ denote the point process on $\mathbb{Z}_{>0}$ that lives on finite configurations and assigns the weight

$$\mathbf{P}^{(w)}(\lambda) := const \cdot \prod_{k=1}^{\ell} w(\lambda_k) \prod_{1 \le k < j \le \ell} \left(\frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j} \right)^2 \tag{3.6}$$

to every configuration $\lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{Z}_{>0}$, where *const* is the normalizing constant.

The process $\mathsf{M}_{\alpha,\xi}$ has the form $\mathbf{P}^{(w)}$ with $w(x) = w_{\alpha,\xi}(x)$ given by (2.7), and for Pl_θ we have $w(x) = w_\theta(x) = \frac{\theta^2}{2(x!)^2}$, which is the Plancherel degeneration of $w_{\alpha,\xi}(x)$.

Let L be the following $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ matrix:

$$\mathsf{L}(x,y) := \frac{2\sqrt{xyw(x)w(y)}}{x+y}, \qquad x, y \in \mathbb{Z}_{>0}. \tag{3.7}$$

The condition (3.5) implies that the operator in $\ell^2(\mathbb{Z}_{>0})$ corresponding to L is of trace class, and thus the Fredholm determinant $\det(1+\mathsf{L})$ is well defined.

Lemma 3.7. (1) *Let $\lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{Z}_{>0}$ be a point configuration. We have*

$$\mathbf{P}^{(w)}(\lambda) = \frac{\det \mathsf{L}(\lambda)}{\det(1+\mathsf{L})},$$

where $\mathsf{L}(\lambda)$ *denotes the submatrix* $[\mathsf{L}(\lambda_k, \lambda_j)]_{k,j=1}^{\ell}$ *of* L.

(2) *The point process* $\mathbf{P}^{(w)}$ *is determinantal with the correlation kernel* $\mathsf{K} = \mathsf{L}(1+\mathsf{L})^{-1}$.

Proof. The first claim directly follows from the Cauchy determinant identity [Mac95, Ch. I, §4, Ex. 6].

This means that the point process $\mathbf{P}^{(w)}$ is a so-called L-ensemble corresponding to the matrix L defined above (e.g., see [BO00, Prop. 2.1] or [Bor09, §5]). This implies the second claim about the correlation kernel. □

Note that *const* in (3.6) is equal to $\frac{1}{\det(1+\mathsf{L})}$, so the condition (3.5) is necessary for the point process $\mathbf{P}^{(w)}$ to be well defined.

Remark 3.8. The correlation kernel K of the process $\mathbf{P}^{(w)}$ is symmetric, because it has the form $\mathsf{K} = \mathsf{L}(1+\mathsf{L})^{-1}$, where L is symmetric. However, the operator of the form $\mathsf{L}(1+\mathsf{L})^{-1}$ cannot be a projection operator in $\ell^2(\mathbb{Z}_{>0})$. This aspect discriminates our processes from many other determinantal processes appearing in, e.g., random matrix models (see the references given in Introduction).

Lemma 3.7 implies, in particular, that the point processes $\mathsf{M}_{\alpha,\xi}$ and Pl_θ on $\mathbb{Z}_{>0}$ are determinantal. Denote their correlation kernels by $\mathsf{K}_{\alpha,\xi}$ and K_θ, respectively. However, Lemma 3.7 does not give any suggestions on how to calculate these correlation kernels. In the next four sections we compute the correlation kernel $\mathsf{K}_{\alpha,\xi}$ using the fermionic Fock space. The kernel K_θ is obtained from $\mathsf{K}_{\alpha,\xi}$ via the Plancherel degeneration.

4 Kerov's operators

4.1 Definition, characterization and properties

The main tool that we use in the present paper to compute the correlation functions of the point processes $\mathsf{M}_{\alpha,\xi}$ (and also of the associated dynamical models, see §8 below) is a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in the pre-Hilbert space $\ell^2_{\mathrm{fin}}(\mathbb{S})$ given by the so-called Kerov's operators. This approach was introduced by Okounkov [Oko01b] for the z-measures on ordinary partitions.[9]

By $\ell^2_{\mathrm{fin}}(\mathbb{S})$ we denote the space of all finitely supported functions on $\mathbb{S}$ with the inner product

$$(f,g) := \sum_{\lambda\in\mathbb{S}} f(\lambda)g(\lambda).$$

This is a pre-Hilbert space whose Hilbert completion is the usual space $\ell^2(\mathbb{S})$ of all functions on $\mathbb{S}$ which are square integrable with respect to the counting

[9] The z-measures originated from the problem of harmonic analysis for the infinite symmetric group $\mathfrak{S}_\infty$ [KOV93], [KOV04] and were studied in detail by Borodin, Okounkov, Olshanski, and other authors, e.g., see the bibliography in [BO09].

measure on $\mathbb{S}$. The standard orthonormal basis in $\ell^2(\mathbb{S})$ is denoted by $\{\underline{\lambda}\}_{\lambda\in\mathbb{S}}$, that is,

$$\underline{\lambda}(\mu) := \begin{cases} 1, & \text{if } \mu = \lambda; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Definition 4.1. The Kerov's operators in $\ell^2_{\mathrm{fin}}(\mathbb{S})$ depend on our parameter $\alpha > 0$ and are defined as

$$\begin{aligned} \mathsf{U}\underline{\lambda} &:= \sum_{\varkappa\colon \varkappa\searrow\lambda} 2^{-\delta(\mathrm{j}-\mathrm{i})/2}\sqrt{(\mathrm{j}-\mathrm{i})(\mathrm{j}-\mathrm{i}+1)+\alpha}\cdot\underline{\varkappa}, \qquad (\mathrm{i},\mathrm{j}) = \varkappa/\lambda; \\ \mathsf{D}\underline{\lambda} &:= \sum_{\mu\colon \mu\nearrow\lambda} 2^{-\delta(\mathrm{j}-\mathrm{i})/2}\sqrt{(\mathrm{j}-\mathrm{i})(\mathrm{j}-\mathrm{i}+1)+\alpha}\cdot\underline{\mu}, \qquad (\mathrm{i},\mathrm{j}) = \lambda/\mu; \\ \mathsf{H}\underline{\lambda} &:= \Big(2|\lambda| + \frac{\alpha}{2}\Big)\underline{\lambda}. \end{aligned} \tag{4.2}$$

Here $\delta(k) := \delta_{k0}$ is the Kronecker delta. We denote a box by (i,j) if its row number is i and its column number is j. Note that for shifted Young diagrams we always have $\mathrm{j} \ge \mathrm{i}$.

The Kerov's operators in $\ell^2_{\mathrm{fin}}(\mathbb{S})$ are closely related to the measures $\mathsf{M}_{\alpha,n}$ (2.3) on $\mathbb{S}_n$. Namely, it is clear that

$$(\mathsf{U}^n\underline{\varnothing},\underline{\lambda}) = (\mathsf{D}^n\underline{\lambda},\underline{\varnothing}) = \dim_{\mathbb{S}}\lambda\cdot 2^{-\ell(\lambda)/2}\prod_{\square=(\mathrm{i},\mathrm{j})\in\lambda}\sqrt{(\mathrm{j}-\mathrm{i})(\mathrm{j}-\mathrm{i}+1)+\alpha}$$

for all n and $\lambda \in \mathbb{S}_n$, so

$$\mathsf{M}_{\alpha,n}(\lambda) = \frac{1}{Z_n}(\mathsf{U}^n\underline{\varnothing},\underline{\lambda})(\mathsf{D}^n\underline{\lambda},\underline{\varnothing}), \tag{4.3}$$

where Z_n is the normalizing constant. See also the end of this subsection for more connections between Kerov's operators and the measures $\mathsf{M}_{\alpha,n}$.

The Kerov's operators (4.2) satisfy the following four properties:

1. The map

$$U := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to \mathsf{U}, \qquad D := \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \to \mathsf{D}, \qquad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to \mathsf{H} \tag{4.4}$$

defines a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in $\ell^2_{\mathrm{fin}}(\mathbb{S})$. That is, the operators U, D, and H satisfy the commutation relations

$$[\mathsf{H},\mathsf{U}] = 2\mathsf{U}, \qquad [\mathsf{H},\mathsf{D}] = -2\mathsf{D}, \qquad [\mathsf{D},\mathsf{U}] = \mathsf{H}. \tag{4.5}$$

2. The operators U and D are adjoint to each other in the space $\ell^2_{\mathrm{fin}}(\mathbb{S})$.

3. For any $\lambda \in \mathbb{S}$, the vector $\mathsf{U}\underline{\lambda}$ is a linear combination of vectors $\underline{\varkappa}$, where $\varkappa \searrow \lambda$, and the coefficient of $\underline{\varkappa}$ depends only on the box $\varkappa/\lambda$ (through its row and column numbers). Likewise, the vector $\mathsf{D}\underline{\lambda}$ is a linear combination of vectors $\underline{\mu}$, where $\mu \nearrow \lambda$, and the coefficient of $\underline{\mu}$ depends only on the box λ/μ.

4. Each basis vector $\underline{\lambda}$, $\lambda \in \mathbb{S}$, is an eigenvector of the operator H, and the eigenvalue of $\underline{\lambda}$ depends only on $|\lambda|$.

The only property above that is not obvious is the first one:

Lemma 4.2. *The Kerov's operators* U*,* D*, and* H (4.2) *satisfy the commutation relations* (4.5)*.*

Proof. Denote

$$q_\alpha(\square) = q_\alpha(\mathrm{i},\mathrm{j}) := 2^{-\delta(\mathrm{j}-\mathrm{i})/2}\sqrt{(\mathrm{j}-\mathrm{i})(\mathrm{j}-\mathrm{i}+1)+\alpha}, \tag{4.6}$$

where $\square = (\mathrm{i},\mathrm{j})$. The relation $[\mathsf{H},\mathsf{U}] = 2\mathsf{U}$ is straightforward:

$$\begin{aligned}[\mathsf{H},\mathsf{U}]\,\underline{\lambda} = \mathsf{H}\sum_{\varkappa\searrow\lambda} q_\alpha(\varkappa/\lambda)\underline{\varkappa} - \Big(2|\lambda| + \frac{\alpha}{2}\Big)\sum_{\varkappa\searrow\lambda} q_\alpha(\varkappa/\lambda)\underline{\varkappa}& \\ = 2\left(|\lambda|+2-|\lambda|\right)\mathsf{U}\underline{\lambda} = 2\mathsf{U}\underline{\lambda},&\end{aligned}$$

and the same for the relation $[\mathsf{H},\mathsf{D}] = -2\mathsf{D}$.

It remains to prove that $[\mathsf{D},\mathsf{U}] = \mathsf{H}$. The vector $[\mathsf{D},\mathsf{U}]\,\underline{\lambda}$ has the form

$$\sum_{\varkappa\searrow\lambda}\sum_{\rho\nearrow\lambda} q_\alpha(\varkappa/\lambda)q_\alpha(\varkappa/\rho)\underline{\rho} - \sum_{\mu\nearrow\lambda}\sum_{\rho\searrow\mu} q_\alpha(\lambda/\mu)q_\alpha(\rho/\mu)\underline{\rho}. \tag{4.7}$$

This is a linear combination of the vectors $\underline{\rho}$, where $\rho\in\mathbb{S}_n$ and either $\rho=\lambda$, or $\rho = \lambda+\square_1-\square_2$ for some boxes $\square_1\ne\square_2$. In the second case the coefficient by the vector $\underline{\rho}$ with $\rho=\lambda+\square_1-\square_2$ is

$$q_\alpha(\square_1)q_\alpha(\square_2) - q_\alpha(\square_2)q_\alpha(\square_1) = 0.$$

Thus, in (4.7) it remains to consider only the terms with $\rho=\lambda$. Therefore, it remains to prove the combinatorial identity

$$\sum_{\varkappa\colon \varkappa\searrow\lambda} q_\alpha(\varkappa/\lambda)^2 - \sum_{\mu\colon \mu\nearrow\lambda} q_\alpha(\lambda/\mu)^2 = 2|\lambda| + \frac{\alpha}{2}.$$

The proof of this identity (using Kerov's interlacing coordinates of shifted Young diagrams) is essentially contained in §3.1 of the paper [Pet09a] (the arXiv version). ∎

In fact, the Kerov's operators (4.2) are completely characterized by the four properties on p. 15:

Proposition 4.3. *If three operators* U*,* D*, and* H *in the space* $\ell^2_{\mathrm{fin}}(\mathbb{S})$ *satisfy the four properties listed on p. 15, then they have the form* (4.2) *with some parameter* $\alpha\in\mathbb{C}$*.*[10]

Proof. By properties 2 and 3, there exists a (complex-valued) function q on the set of all boxes, that is, on the set $\{(\mathrm{i},\mathrm{j})\colon \mathrm{j}\ge\mathrm{i}\ge1\}$ (where i and j are the row

[10]By agreement, for arbitrary complex α, in the definition of U and D in (4.2) we take the same branches of the square roots $\sqrt{\alpha+c(c+1)}$, $c=0,1,2,\dots$. In fact, it is the square of the function $q_\alpha(\cdot,\cdot)$ (4.6) that really plays the role in the definition of Kerov's operators.

and column numbers of the box, respectively), such that the operators U and D have the form

$$\mathsf{U}\underline{\lambda} = \sum_{\varkappa \searrow \lambda} q(\varkappa/\lambda)\underline{\varkappa}, \qquad \mathsf{D}\underline{\lambda} = \sum_{\mu \nearrow \lambda} q(\lambda/\mu)\underline{\mu}.$$

By property 4, for all $n = 0, 1, \dots$ and all $\lambda \in \mathbb{S}_n$ we have $\mathsf{H}\underline{\lambda} = h_n\underline{\lambda}$ for some (complex) numbers h_n. Using the commutation relation $[\mathsf{H}, \mathsf{U}] = 2\mathsf{U}$ (property 1), it is easy to see that $h_{n+1} = h_n + 2$ for all $n = 0, 1, \dots$. Set $\alpha := 2h_0$ (this is some complex parameter). Thus, we have $h_n = 2n + \frac{\alpha}{2}$.

The function $q(\cdot, \cdot)^2$ can be found by applying the commutation relation $[\mathsf{D}, \mathsf{U}] = \mathsf{H}$ (property 1) to various vectors $\underline{\lambda}$, $\lambda \in \mathbb{S}$. First, applying this relation to $\underline{\varnothing}$ and $\underline{\square}$ (this is the vector corresponding to the one-box shifted diagram), we get

$$q(1,1)^2 = \frac{\alpha}{2} \qquad \text{and} \qquad q(1,2)^2 = 2 + \alpha.$$

Next, apply $[\mathsf{D}, \mathsf{U}] = \mathsf{H}$ to $\underline{(n)}$ for $n \ge 2$:

$$q(1, n+1)^2 - q(1,n)^2 + q(2,2)^2 = 2n + \frac{\alpha}{2}.$$

Solving this recurrence and taking into account the initial value $q(1,2)^2$, we get

$$q(1,n)^2 = -(n-2)q(2,2)^2 + n\left(n - 1 + \frac{\alpha}{2}\right), \qquad n = 2, 3, \dots.$$

To find $q(2,2)^2$, we apply $[\mathsf{D}, \mathsf{U}] = \mathsf{H}$ to $\underline{(2,1)}$:

$$q(1,3)^2 - q(2,2)^2 = 6 + \frac{\alpha}{2} \qquad \Rightarrow \qquad q(2,2)^2 = \frac{\alpha}{2},$$

and so

$$q(1,n)^2 = n(n-1) + \alpha, \qquad n = 2, 3, \dots.$$

To find $q(n,n)^2$ for $n \ge 3$, use the vector $\underline{\lambda}$ with $\lambda = (n, n-1, \dots, 1)$:

$$q(1, n+1)^2 - q(n,n)^2 = 2 \cdot \frac{n(n+1)}{2} + \frac{\alpha}{2} \qquad \Rightarrow \qquad q(n,n)^2 = \frac{\alpha}{2}.$$

Finally, to find $q(\mathrm{i},\mathrm{j})^2$ for arbitrary $\mathrm{j} > \mathrm{i} > 1$ (these are the remaining unknown values of $q(\cdot,\cdot)^2$), we apply the relation $[\mathsf{D}, \mathsf{U}] = \mathsf{H}$ to the vector $\underline{\lambda}$ with $\lambda = (\mathrm{j}, \mathrm{j}-1, \dots, \mathrm{j}-\mathrm{i}+1)$:

$$q(1, \mathrm{j}+1)^2 + q(\mathrm{i}+1, \mathrm{i}+1)^2 - q(\mathrm{i},\mathrm{j})^2 = 2 \cdot \frac{\mathrm{i}(2\mathrm{j} - \mathrm{i} + 1)}{2} + \frac{\alpha}{2}.$$

We thus have $q(\mathrm{i},\mathrm{j})^2 = (\mathrm{j}-\mathrm{i})(\mathrm{j}-\mathrm{i}+1) + \alpha$ for all $\mathrm{j} > \mathrm{i} > 1$.

Putting all together, we see that the function q is identical to the function q_α defined (4.6) (for arbitrary complex α,we take the same branches of the square roots of $q_\alpha(\cdot,\cdot)^2$).

The fact that the commutation relation $[\mathsf{D}, \mathsf{U}]\,\underline{\lambda} = \mathsf{H}\underline{\lambda}$ (with the above choice of $q(\cdot,\cdot)$) holds for all shifted Young diagrams $\lambda \in \mathbb{S}$ follows from Lemma 4.2. This concludes the proof. ∎

Remark 4.4. One can also prove a statement analogous to Proposition 4.3 without property 2 on p. 15. The operator H is still defined uniquely up to a parameter $\alpha \in \mathbb{C}$. The two other operators are equal to U and D (4.2) up to a "gauge transformation" that is written in terms of the matrix elements in the basis $\{\underline{\lambda}\}_{\lambda\in\mathbb{S}}$ as:

$$(\mathsf{U}\underline{\lambda},\underline{\varkappa}) \mapsto f(\varkappa/\lambda)\cdot(\mathsf{U}\underline{\lambda},\underline{\varkappa}), \qquad (\mathsf{D}\underline{\lambda},\underline{\mu}) \mapsto \frac{1}{f(\lambda/\mu)}\cdot(\mathsf{D}\underline{\lambda},\underline{\mu}),$$

where f is some nonzero function on the set of boxes. One possible choice of such operators (which are not adjoint to each other) is (4.8) below.

Remark 4.5. A statement parallel to Proposition 4.3 can be proved for the Young graph. As a result we will get operators similar to those considered in [Oko01b]. This allows one to give a purely combinatorial characterization of the z-measures on the Young graph.

Let us give a couple remarks on how deep is the connection between the measures $\mathsf{M}_{\alpha,n}$ (2.3) and the Kerov's operators (4.2). These remarks are also applicable to the z-measures on the Young graph.

First, using the commutation relations 4.5 for the Kerov's operators, one can compute the normalizing constants Z_n in (4.3):

$$Z_n = \sum_{\lambda\in\mathbb{S}_n}(\mathsf{U}^n\underline{\varnothing},\underline{\lambda})(\mathsf{D}^n\underline{\lambda},\underline{\varnothing}).$$

In the above sum the parameter α is hidden in the definition of the operators U and D (4.2), and one can assume α to be an arbitrary complex number. Write

$$Z_n = \sum_{\lambda\in\mathbb{S}_n}(\mathsf{U}^n\underline{\varnothing},\underline{\lambda})(\mathsf{D}^n\underline{\lambda},\underline{\varnothing}) = \Big(\mathsf{D}^n\sum_{\lambda\in\mathbb{S}_n}(\mathsf{U}^n\underline{\varnothing},\underline{\lambda})\cdot\underline{\lambda},\underline{\varnothing}\Big) = (\mathsf{D}^n\mathsf{U}^n\underline{\varnothing},\underline{\varnothing}).$$

By the commutation relations (4.5),

$$\mathsf{D}\mathsf{U}^n = \mathsf{U}^n\mathsf{D} + \sum_{k=0}^{n-1}\mathsf{U}^{n-k-1}\mathsf{H}\mathsf{U}^k.$$

Using the fact that $\mathsf{D}\underline{\varnothing} = 0$, we get

$$Z_n = \sum_{k=0}^{n-1}(\mathsf{D}^{n-1}\mathsf{U}^{n-k-1}\mathsf{H}\mathsf{U}^k\underline{\varnothing},\underline{\varnothing}) = Z_{n-1}\sum_{k=0}^{n-1}\Big(2k+\frac{\alpha}{2}\Big) = n\Big(n-1+\frac{\alpha}{2}\Big)Z_{n-1}.$$

Taking into account the initial value $Z_0 = (\mathsf{U}^0\mathsf{D}^0\underline{\varnothing},\underline{\varnothing}) = 1$, we see that $Z_n = n!(\alpha/2)_n$.

Thus, the (complex-valued) measures $\mathsf{M}_{\alpha,n}$ are well-defined by (4.3) for all $\alpha \in \mathbb{C}\setminus\{0,-2,-4,\dots\}$ because the normalizing constants Z_n are nonzero for all n. Moreover, for such α the measures $\mathsf{M}_{\alpha,n}$ are nondegenerate in the sense that $\mathsf{M}_{\alpha,n}(\lambda) \neq 0$ for all n and all $\lambda \in \mathbb{S}_n$. Many formulas in the present paper hold in purely algebraic sense for $\alpha \in \mathbb{C} \setminus \{0, -2, -4, \dots\}$.

Now let us present an alternative proof of the coherency condition (3.3) for the measures $\{\mathsf{M}_{\alpha,n}\}$ using Kerov's operators. Here we also assume that $\alpha \in \mathbb{C} \setminus \{0, -2, -4, \dots\}$. Consider slightly different operators:

$$\begin{aligned} \hat{\mathsf{U}}\underline{\lambda} &:= \sum_{\varkappa\colon \varkappa\searrow\lambda} 2^{-\delta(\mathrm{j}-\mathrm{i})}\big((\mathrm{j}-\mathrm{i})(\mathrm{j}-\mathrm{i}+1)+\alpha\big)\cdot\underline{\varkappa}, \qquad (\mathrm{i},\mathrm{j}) = \varkappa/\lambda; \\ \hat{\mathsf{D}}\underline{\lambda} &:= \sum_{\mu\colon \mu\nearrow\lambda} \underline{\mu}. \end{aligned} \tag{4.8}$$

Clearly, $[\hat{\mathsf{D}}, \hat{\mathsf{U}}] = \mathsf{H}$ (see also Remark 4.4), and $(\hat{\mathsf{D}}\underline{\lambda}, \underline{\varnothing}) = \dim_{\mathbb{S}} \lambda$. Moreover,

$$\mathsf{M}_{\alpha,n}(\lambda) = \frac{1}{Z_n}(\hat{\mathsf{U}}\underline{\varnothing}, \underline{\lambda})(\hat{\mathsf{D}}\underline{\lambda}, \underline{\varnothing}) \qquad \text{for all } n \text{ and } \lambda \in \mathbb{S}_n$$

(here $Z_n = n!(\alpha/2)_n$ is the same as in (4.3)) . Fix $n = 1, 2, \dots$ and $\mu \in \mathbb{S}_{n-1}$. Write

$$\begin{aligned} \sum_{\lambda\colon \lambda\searrow\mu} \frac{1}{\dim_{\mathbb{S}}\lambda}(\mathsf{U}^n\underline{\varnothing}, \underline{\lambda})(\mathsf{D}^n\underline{\lambda}, \underline{\varnothing}) &= \sum_{\lambda\colon \lambda\searrow\mu} \frac{1}{\dim_{\mathbb{S}}\lambda}(\hat{\mathsf{U}}^n\underline{\varnothing}, \underline{\lambda})(\hat{\mathsf{D}}^n\underline{\lambda}, \underline{\varnothing}) \\ &= \sum_{\lambda\colon \lambda\searrow\mu} (\hat{\mathsf{U}}^n\underline{\varnothing}, \underline{\lambda}) = (\hat{\mathsf{U}}^n\underline{\varnothing}, \hat{\mathsf{D}}^*\underline{\mu}) = (\hat{\mathsf{D}}\hat{\mathsf{U}}^n\underline{\varnothing}, \underline{\mu}) \\ &= \sum_{k=0}^{n-1}(\hat{\mathsf{U}}^{n-k-1}\mathsf{H}\hat{\mathsf{U}}^k\underline{\varnothing}, \underline{\mu}) = \frac{Z_n}{Z_{n-1}}(\hat{\mathsf{U}}^{n-1}\underline{\varnothing}, \underline{\mu}) \\ &= \frac{Z_n}{Z_{n-1}} \cdot \frac{1}{\dim_{\mathbb{S}}\mu}(\mathsf{U}^{n-1}\underline{\varnothing}, \underline{\mu})(\mathsf{D}^{n-1}\underline{\mu}, \underline{\varnothing}). \end{aligned}$$

The identity that we have obtained is clearly equivalent to the coherency condition (3.3) for the measures $\{\mathsf{M}_{\alpha,n}\}$ written in the form (4.3).

Remark 4.6. The Kerov's operators (4.2) with certain minor modifications fall into the framework of the paper by Fulman [Ful09]. Namely, consider the operators $U_n\colon \mathbb{C}\mathbb{S}_n \to \mathbb{C}\mathbb{S}_{n+1}$ and $D_n\colon \mathbb{C}\mathbb{S}_n \to \mathbb{C}\mathbb{S}_{n-1}$ defined by

$$U_n\underline{\lambda} := \frac{1}{n+\alpha/2}\mathsf{U}\underline{\lambda}, \qquad D_n\underline{\lambda} := \mathsf{D}\underline{\lambda}, \qquad \lambda \in \mathbb{S}_n.$$

Then the operators U_n and D_n satisfy the commutation relations in the form of [Ful09, (1.1)]:

$$D_{n+1}U_n = a_n U_{n-1} D_n + b_n I_n,$$

where $I_n\colon \mathbb{C}\mathbb{S}_n \to \mathbb{C}\mathbb{S}_n$ is the identity operator and

$$a_n = 1 - \frac{1}{n+\alpha/2}, \qquad b_n = 1 + \frac{n}{n+\alpha/2}.$$

4.2 Kerov's operators and averages with respect to our point processes

The weight assigned to a strict partition λ by the measure $\mathsf{M}_{\alpha,\xi}$ (2.6) (which is a mixture of measures $\mathsf{M}_{\alpha,n}$) can be written for small enough ξ as follows:

$$\mathsf{M}_{\alpha,\xi}(\lambda) = (1-\xi)^{\alpha/2}(e^{\sqrt{\xi}\mathsf{U}}\underline{\varnothing},\underline{\lambda})(e^{\sqrt{\xi}\mathsf{D}}\underline{\lambda},\underline{\varnothing}).$$

Here $e^{\sqrt{\xi}\mathsf{D}}\underline{\lambda}$ is clearly an element of $\ell^2_{\mathrm{fin}}(\mathbb{S})$. The fact that the vector $e^{\sqrt{\xi}\mathsf{U}}\underline{\lambda}$ belongs to $\ell^2(\mathbb{S})$ (for small enough ξ) requires a justification (see the proof of Proposition 4.7), because the operator U in $\ell^2(\mathbb{S})$ is unbounded. This makes the above formula for $\mathsf{M}_{\alpha,\xi}(\lambda)$ not very convenient for taking averages with respect to the measure $\mathsf{M}_{\alpha,\xi}$. In this subsection we overcome this difficulty and give a convenient way of writing expectations with respect to $\mathsf{M}_{\alpha,\xi}$. Our approach here is similar to that of Olshanski [Ols] and is also based on the ideas of Okounkov [Oko01b].

Recall that the Kerov's operators U, D, and H (4.2) define (via the map (4.4)) a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in the (complex) pre-Hilbert space $\ell^2_{\mathrm{fin}}(\mathbb{S})$. Consider the real form $\mathfrak{su}(1,1)\subset\mathfrak{sl}(2,\mathbb{C})$ spanned by the matrices $U-D$, $i(U+D)$, and iH (here $i=\sqrt{-1}$). The corresponding operators $\mathsf{U}-\mathsf{D}$, $i(\mathsf{U}+\mathsf{D})$, and $i\mathsf{H}$ act skew-symmetrically in $\ell^2_{\mathrm{fin}}(\mathbb{S})$.

Proposition 4.7. *All vectors of the space $\ell^2_{\mathrm{fin}}(\mathbb{S})$ are analytic for the described above action of the Lie algebra $\mathfrak{su}(1,1)$. Consequently, the action of $\mathfrak{su}(1,1)$ in $\ell^2_{\mathrm{fin}}(\mathbb{S})$ gives rise to a unitary representation of the universal covering group $SU(1,1)^{\sim}$ in the Hilbert space $\ell^2(\mathbb{S})$.*

Proof. Recall [Nel59] that a vector h is analytic for an operator A if the series

$$\sum_{n=0}^{\infty}\frac{\|A^nh\|}{n!}s^n$$

has a positive radius of convergence.

We can use Lemma 9.1 from [Nel59] that guarantees the existence of the desired unitary representation of $SU(1,1)^{\sim}$ in $\ell^2(\mathbb{S})$ if we first prove that for some constant $s_0>0$ we have

$$\|A_{i_1}\dots A_{i_n}h\|\le\frac{n!}{s_0^n} \tag{4.9}$$

for any $h\in\ell^2_{\mathrm{fin}}(\mathbb{S})$, all sufficiently large n (the bound on n depends on h), and any indices $i_1,\dots,i_n$ taking values $1,2,3$, where $A_1=\mathsf{U}-\mathsf{D}$, $A_2=i(\mathsf{U}+\mathsf{D})$, and $A_3=i\mathsf{H}$. Note that this in fact implies that any vector in $\ell^2_{\mathrm{fin}}(\mathbb{S})$ is analytic for the action of $\mathfrak{su}(1,1)$.

It suffices to prove the estimate (4.9) for $\hat{A}_1:=\mathsf{U}$, $\hat{A}_2:=\mathsf{D}$, and $\hat{A}_3:=\mathsf{H}$, this can only affect the value of the constant s_0. Moreover, we can also set $h=\underline{\varkappa}$ for some $\varkappa\in\mathbb{S}$. Because all the matrix elements of the operators U, D, and H are nonnegative in the standard basis $\{\underline{\lambda}\}_{\lambda\in\mathbb{S}}$, we have

$$\|\hat{A}_{i_1}\dots\hat{A}_{i_n}\underline{\varkappa}\|\le\|(\mathsf{U}+\mathsf{D}+\mathsf{H})^n\underline{\varkappa}\|.$$

The desired estimate would follow if we show that the power series expansion of $\exp\left(s(\mathsf{U}+\mathsf{D}+\mathsf{H})\right)\underline{\varkappa}$ converges for some small enough $s>0$. For matrices in $SL(2,\mathbb{C})$ (see (4.4)) we have

$$\exp(s(U+D+H)) = \exp\left(\frac{s}{1-s}U\right)\exp\left(\log\left(\frac{1}{1-s}\right)H\right)\exp\left(\frac{s}{1-s}D\right).$$

Thus, the power series expansion of $\exp\left(s(\mathsf{U}+\mathsf{D}+\mathsf{H})\right)\underline{\varkappa}$ is the same as that of

$$\exp\left(\frac{s}{1-s}\mathsf{U}\right)\exp\left(\log\left(\frac{1}{1-s}\right)\mathsf{H}\right)\exp\left(\frac{s}{1-s}\mathsf{D}\right)\underline{\varkappa}.$$

Since the operator D is locally nilpotent and the operator H acts on each $\underline{\lambda}$ as multiplication by $(2|\lambda|+\alpha/2)$, to obtain the desired estimate (4.9) it remains to show that the series

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}\|\mathsf{U}^n\underline{\mu}\|$$

converges for all $\mu\in\mathbb{S}$ for sufficiently small $s>0$ (the bound on s does not depend on $\mu\in\mathbb{S}$).

Let us fix μ with $|\mu|=k$. We can write by definition of U:

$$\|\mathsf{U}^n\underline{\mu}\|^2=\sum_{\lambda\in\mathbb{S}_{k+n}}(\mathsf{U}^n\underline{\mu},\underline{\lambda})^2=\sum_{\lambda\in\mathbb{S}_{k+n}}\dim_{\mathbb{S}}(\mu,\lambda)^2\prod_{\square\in\lambda/\mu}q_\alpha(\square)^2,$$

where q_α is defined by (4.6). Here the product is taken over all boxes of the skew shifted diagram λ/μ (see the end of §3.1). Since $\dim_{\mathbb{S}}(\mu,\lambda)\le\dim_{\mathbb{S}}\lambda$, we can estimate

$$\begin{aligned}\|\mathsf{U}^n\underline{\mu}\|^2 &\le\Big(\prod_{\square\in\mu}q_\alpha(\square)^{-2}\Big)\cdot\sum_{\lambda\in\mathbb{S}_{k+n}}(\dim_{\mathbb{S}}\lambda)^2\prod_{\square\in\lambda}q_\alpha(\square)^2\\ &=\Big(\prod_{\square\in\mu}q_\alpha(\square)^{-2}\Big)\cdot\sum_{\lambda\in\mathbb{S}_{n+k}}(\mathsf{U}^n\underline{\varnothing},\underline{\lambda})^2=Z_{n+k}\cdot\Big(\prod_{\square\in\mu}q_\alpha(\square)^{-2}\Big).\end{aligned}$$

The factor $\prod_{\square\in\mu}q_\alpha(\square)^{-2}$ is just a constant depending on μ, and the normalizing constants $Z_n=n!(\alpha/2)_n$ were computed in the previous subsection. Putting all together, we get

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}\|\mathsf{U}^n\underline{\mu}\|\le\Big(\prod_{\square\in\mu}q_\alpha(\square)^{-2}\Big)^{\frac12}\cdot\sum_{n=0}^{\infty}\frac{s^n}{n!}\sqrt{(n+k)!(\alpha/2)_{n+k}}.\qquad(4.10)$$

Using [Erd53, 1.18.(5)], we see that

$$\frac{\sqrt{(n+k)!(\alpha/2)_{n+k}}}{n!}\sim\sqrt{\frac{n^{2k+\alpha/2-1}}{\Gamma(\alpha/2)}},$$

so the series (4.10) converges. This concludes the proof of the Proposition. $\square$

By G_ξ denote the matrix

$$G_\xi := \begin{bmatrix} \frac{1}{\sqrt{1-\xi}} & \frac{\sqrt{\xi}}{\sqrt{1-\xi}} \\ \frac{\sqrt{\xi}}{\sqrt{1-\xi}} & \frac{1}{\sqrt{1-\xi}} \end{bmatrix} = \left(\frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\right)^{\frac{U-D}{2}} \in SU(1,1), \qquad 0 \le \xi < 1.$$

Clearly, $(G_\xi)_{0\le\xi<1}$ is a continuous curve in $SU(1,1)$ starting at the unity. By $(\widetilde{G}_\xi)_{0\le\xi<1}$ denote the lifting of this curve to $SU(1,1)^\sim$, again starting at the unity. The unitary operators in $\ell^2(\mathbb{S})$ corresponding (by Proposition 4.7) to $\widetilde{G}_\xi$ are denoted by $\widetilde{\mathsf{G}}_\xi$.

The next thing we need is the weighted ℓ^2 space $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ — the space of functions on $\mathbb{S}$ that are square summable with the weight $\mathsf{M}_{\alpha,\xi}$. This is a Hilbert space with the inner product

$$(f,g)_{\mathsf{M}_{\alpha,\xi}} := \sum_{\lambda\in\mathbb{S}} f(\lambda)g(\lambda)\mathsf{M}_{\alpha,\xi}(\lambda).$$

There is an isometry map $I_{\alpha,\xi}$ from $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ to $\ell^2(\mathbb{S})$:

$$I_{\alpha,\xi} := \text{multiplication by the function } \lambda \mapsto \sqrt{\mathsf{M}_{\alpha,\xi}(\lambda)}. \tag{4.11}$$

The standard orthonormal basis $\{\underline{\lambda}\}_{\lambda\in\mathbb{S}}$ (4.1) of the space $\ell^2(\mathbb{S})$ corresponds to the orthonormal basis $\left\{(\mathsf{M}_{\alpha,\xi}(\lambda))^{-\frac12}\underline{\lambda}\right\}_{\lambda\in\mathbb{S}}$ of $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$. To any operator A in $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ corresponds the operator $I_{\alpha,\xi}AI_{\alpha,\xi}^{-1}$ acting in $\ell^2(\mathbb{S})$.

Now we can prove the main statement of this subsection:

Proposition 4.8. *Let A be a bounded operator in $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$. Then*

$$(A\mathbf{1},\mathbf{1})_{\mathsf{M}_{\alpha,\xi}} = \left(\widetilde{\mathsf{G}}_\xi^{-1}(I_{\alpha,\xi}AI_{\alpha,\xi}^{-1})\widetilde{\mathsf{G}}_\xi\underline{\varnothing}, \underline{\varnothing}\right). \tag{4.12}$$

Here $\mathbf{1} \in \ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ is the constant identity function, and on the right-hand side the inner product is taken in $\ell^2(\mathbb{S})$.

Proof. Let us first show that

$$\widetilde{\mathsf{G}}_\xi\underline{\varnothing} = \sum_{\lambda\in\mathbb{S}} (\mathsf{M}_{\alpha,\xi}(\lambda))^{\frac12}\,\underline{\lambda}. \tag{4.13}$$

In the matrix group $SL(2,\mathbb{C})$ we have

$$G_\xi = \exp\left(\sqrt{\xi}U\right)\exp\left(\frac12\log(1-\xi)H\right)\exp\left(-\sqrt{\xi}D\right).$$

The vector $\underline{\varnothing}$ is analytic for the representation of $\mathfrak{su}(1,1)$ in $\ell^2(\mathbb{S})$ (Proposition 4.7), so on this vector the representation of $SU(1,1)^\sim$ can be extended to a representation of the local complexification of the group $SU(1,1)^\sim$ (see, e.g., the beginning of §7 in [Nel59]). This means that for small enough ξ (when $\widetilde{G}_\xi$ is close to the unity of the group $SU(1,1)^\sim$), we have

$$\widetilde{\mathsf{G}}_\xi \underline{\varnothing} = \exp\left(\sqrt{\xi}\mathsf{U}\right)\exp\left(\frac{1}{2}\log(1-\xi)\mathsf{H}\right)\exp\left(-\sqrt{\xi}\mathsf{D}\right)\underline{\varnothing}.$$

The operator $e^{\sqrt{\xi}\mathsf{D}}$ preserves $\underline{\varnothing}$, and thus

$$\widetilde{\mathsf{G}}_\xi \underline{\varnothing} = (1-\xi)^{\alpha/4}\sum_{\lambda\in\mathbb{S}}\frac{\xi^{|\lambda|}}{|\lambda|!}\dim_{\mathbb{S}}\lambda\bigg(\prod_{\square\in\lambda}q_\alpha(\lambda)\bigg)\underline{\lambda} = \sum_{\lambda\in\mathbb{S}}\left(\mathsf{M}_{\alpha,\xi}(\lambda)\right)^{\frac{1}{2}}\underline{\lambda}.$$

We have established (4.13) for small ξ. The right-hand side of (4.13) is analytic in ξ because $\underline{\varnothing}$ is an analytic vector for the operator $\widetilde{\mathsf{G}}_\xi$. The left-hand side of (4.13) is also analytic in ξ by definition of $\mathsf{M}_{\alpha,\xi}$, see §2.1. Thus, (4.13) holds for all $\xi\in(0,1)$.

It follows that $I_{\alpha,\xi}^{-1}\widetilde{\mathsf{G}}_\xi\underline{\varnothing} = \mathbf{1}\in\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$, see (4.11). Therefore,

$$(\widetilde{\mathsf{G}}_\xi^{-1}I_{\alpha,\xi}AI_{\alpha,\xi}^{-1}\widetilde{\mathsf{G}}_\xi\underline{\varnothing},\underline{\varnothing}) = (\widetilde{\mathsf{G}}_\xi^{-1}I_{\alpha,\xi}(A\mathbf{1}),\underline{\varnothing}) = (I_{\alpha,\xi}\left(A\mathbf{1}\right),\widetilde{\mathsf{G}}_\xi\underline{\varnothing}).$$

because the operator $\widetilde{\mathsf{G}}_\xi$ is unitary and has real matrix elements. We have

$$\begin{aligned}
(I_{\alpha,\xi}\left(A\mathbf{1}\right),\widetilde{\mathsf{G}}_\xi\underline{\varnothing}) &= \bigg(I_{\alpha,\xi}\left(A\mathbf{1}\right),\sum_{\lambda\in\mathbb{S}}\left(\mathsf{M}_{\alpha,\xi}(\lambda)\right)^{\frac{1}{2}}\underline{\lambda}\bigg)\\
&= \sum_{\lambda\in\mathbb{S}}\left(I_{\alpha,\xi}\left(A\mathbf{1}\right),\underline{\lambda}\right)\cdot\left(\mathsf{M}_{\alpha,\xi}(\lambda)\right)^{\frac{1}{2}} = \sum_{\lambda\in\mathbb{S}}\left(I_{\alpha,\xi}\left(A\mathbf{1}\right),I_{\alpha,\xi}(\underline{\lambda})\right)\\
&= \bigg(A\mathbf{1},\sum_{\lambda\in\mathbb{S}}\underline{\lambda}\bigg)_{\mathsf{M}_{\alpha,\xi}} = (A\mathbf{1},\mathbf{1})_{\mathsf{M}_{\alpha,\xi}}.
\end{aligned}$$

This concludes the proof. □

Remark 4.9. The left-hand side of (4.12) can be regarded as an expectation with respect to the measure $\mathsf{M}_{\alpha,\xi}$ of the function $(A\mathbf{1})(\cdot)$ on $\mathbb{S}$. In the special case when the operator A is diagonal, say, $A=A_f$ is the multiplication by a (bounded) function $f(\cdot)$ on $\mathbb{S}$, (4.12) is rewritten as

$$\mathbb{E}_{\alpha,\xi}f := \sum_{\lambda\in\mathbb{S}}f(\lambda)\mathsf{M}_{\alpha,\xi}(\lambda) = \left(\widetilde{\mathsf{G}}_\xi^{-1}A_f\widetilde{\mathsf{G}}_\xi\underline{\varnothing},\underline{\varnothing}\right). \tag{4.14}$$

This case is used in the computation of the static correlation functions, and for the dynamical correlation functions we need to use the more general statement of Proposition 4.8.

5 Fermionic Fock space

In this section we realize the Hilbert space $\ell^2(\mathbb{S})$ as a fermionic Fock space over $\ell^2(\mathbb{Z}_{>0})$, and also define a representation of a Clifford algebra in this Fock space. This Clifford algebra is an infinite-dimensional analogue of a Clifford algebra over an odd-dimensional space (similar Clifford algebras and their Fock representations were considered in, e.g., [DJKM82], [Mat05], [Vul07]). Note that in the case of the z-measures [Oko01b] one should work with an analogue of a Clifford algebra over an even-dimensional space. This difference, in particular, leads to the fact that in our case *a priori* the use of this algebra provides us only with a Pfaffian formula for the correlation functions of the point processes $\mathsf{M}_{\alpha,\xi}$ (the static case). The proof that $\mathsf{M}_{\alpha,\xi}$ is actually a determinantal process requires additional considerations (see §3.4 and Theorem 7.1).

5.1 Wick's theorem

We begin with the definition of a certain Clifford algebra over the Hilbert space $V := \ell^2(\mathbb{Z})$. Denote the standard orthonormal basis of the space V by $\{v_x\}_{x\in\mathbb{Z}}$. Define a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V by

$$\langle v_x, v_y\rangle := \begin{cases} 1, & \text{if } x = -y \neq 0; \\ 2, & \text{if } x = y = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the standard inner product in V (denoted by $(\cdot,\cdot)_V$) is expressed through this bilinear form as

$$(v_x, v_y)_V = \langle v_x, v_{-y}\rangle\, 2^{-\delta(x)},$$

where $\delta(\cdot)$ is the Kronecker delta as in Definition 4.1.

Let V^+ and V^- be the spans of $\{v_x\}_{x\in\mathbb{Z}_{>0}}$ and $\{v_x\}_{x\in\mathbb{Z}_{<0}}$, respectively, and let V^0 denote the space $\mathbb{C}v_0$. Note that the spaces V^+ and V^- are maximal isotropic subspaces for the form $\langle\cdot,\cdot\rangle$, and

$$V = V^- \oplus V^0 \oplus V^+.$$

By $Cl(V)$ denote the *Clifford algebra* over the quadratic space $(V, \langle\cdot,\cdot\rangle)$, that is, $Cl(V)$ is the quotient of the tensor algebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$ of the space V over the ideal generated by the elements

$$\{v\otimes v' + v'\otimes v - \langle v, v'\rangle : v, v' \in V\}.$$

The tensor product of v and v' in $Cl(V)$ is denoted simply by vv'. Thus,

$$vv' + v'v = \langle v, v'\rangle \qquad \text{for all } v, v' \in V. \tag{5.1}$$

Now let us prove a version of Wick's theorem that allows to write certain functionals on $Cl(V)$ as Pfaffians. (In §5.3 below we define a functional on $Cl(V)$ called the vacuum average to which this version of Wick's theorem is applicable.)

Theorem 5.1. *Let* $\mathbf{F}$ *be a linear functional on* $Cl(V)$ *such that* $\mathbf{F}(1) = 1$ *and for any* $p, q, r \in \mathbb{Z}_{\ge 0}$, $f_1^+, \dots, f_p^+ \in V^+$, *and* $f_1^-, \dots, f_q^- \in V^-$, *we have*

$$\mathbf{F}(f_1^+ \dots f_p^+ v_0^r f_1^- \dots f_q^-) = 0 \tag{5.2}$$

if at least one of the numbers p, q *is nonzero.*

Then for any $n \ge 1$ *and any* $2n$ *elements* $f_1, \dots, f_{2n} \in V$ *we have*

$$\mathbf{F}(f_1 \dots f_{2n}) = \mathrm{Pf}(\mathbf{F}[\![f_1, \dots, f_{2n}]\!]),$$

where $\mathbf{F}[\![f_1, \dots, f_{2n}]\!]$ *is the skew-symmetric* $2n \times 2n$ *matrix in which the* kj*-th entry above the main diagonal is* $\mathbf{F}(f_k f_j)$, $k, j = 1, \dots, 2n$, $k < j$.

Proof. **Step 1.** Consider decompositions

$$f_j = f_j^- + f_j^0 + f_j^+, \qquad j = 1, \dots, 2n,$$

where $f_j^\pm \in V^\pm$ and $f_j^0 \in V^0 = \mathbb{C}v_0$. Thus,

$$\mathbf{F}(f_1 \dots f_{2n}) = \sum_{s_1, \dots, s_{2n}} \mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}),$$

where each s_j is a sign, $s_j \in \{-, 0, +\}$, and the sum is taken over all 3^{2n} possible sequences of signs.

Step 2. Fix any particular sequence of signs $(s_1, \dots, s_{2n})$. Consider first the case when all of the s_j's are nonzero. We aim to prove that

$$\mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}) = \mathrm{Pf}(\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]), \tag{5.3}$$

where $\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]$ is the $2n \times 2n$ skew-symmetric matrix in which the kjth entry above the main diagonal is $\mathbf{F}(f_k^{s_k} f_j^{s_j})$.

First, note that if in the sequence $(s_1, \dots, s_{2n})$ all the "+" signs are on the left and all the "−" signs are on the right,[11] then by (5.2) we get (5.3), because in the Pfaffian in the right-hand side of (5.3) each entry is zero.

Next, observe that (5.3) is equivalent to

$$\mathbf{F}(f_1^{s_1} \dots f_{2n}^{s_{2n}}) = \sum_{k=1}^{2n-1} (-1)^{k+1} \mathbf{F}(f_1^{s_1} \dots \widehat{f_k^{s_k}} \dots f_{2n-1}^{s_{2n-1}}) \mathbf{F}(f_k^{s_k} f_{2n}^{s_{2n}}), \tag{5.4}$$

this is just the Pfaffian expansion (here $\widehat{f_k^{s_k}}$ means the absense of $f_k^{s_k}$). It can be readily verified that the right-hand side and the left-hand side of (5.4) vary in the same way under the interchange $f_r^{s_r} \leftrightarrow f_{r+1}^{s_{r+1}}$ for any $r = 1, \dots, 2n-1$. This implies that (5.4) holds because one can always move the "+" signs to the left and the "−" signs to the right. See also the proof of Lemma 2.3 in [Vul07].

Step 3. Now assume that among the sequence of signs $(s_1, \dots, s_{2n})$ there can be zeroes. It is not hard to see that both sides of (5.3) vanish unless the

[11] Including the case when there are only "+" or only "−" signs.

number of zeroes is even. Let the positions of zeroes be $j_1 < \cdots < j_{2k}$. Thus, moving all $f^0_{j_1}, \dots, f^0_{j_{2k}}$ to the left, we have

$$\begin{aligned}\mathbf{F}(&f_1^{s_1} \dots f_{2n}^{s_{2n}}) \\ &= (-1)^{\sum_{m=1}^{2k}(j_m - m)} \mathbf{F}(f^0_{j_1} \dots f^0_{j_{2k}}) \mathbf{F}(f_1^{s_1} \dots \widehat{f^0_{j_1}} \dots \widehat{f^0_{j_{2k}}} \dots f_{2n}^{s_{2n}}).\end{aligned} \tag{5.5}$$

By (5.3), the factor $\mathbf{F}(f_1^{s_1} \dots \widehat{f^0_{j_1}} \dots \widehat{f^0_{j_{2k}}} \dots f_{2n}^{s_{2n}})$ is written as the corresponding Pfaffian of order $(2n-2k)$. Assume that $f^0_{j_m} = c_m v_0$ (where $m = 1, \dots, 2k$), then

$$\mathbf{F}(f^0_{j_1} \dots f^0_{j_{2k}}) = c_1 \dots c_{2k}.$$

Since for any $f \in V^+ \oplus V^-$ we have (using (5.2)) $\mathbf{F}(v_0 f) = \mathbf{F}(f v_0) = 0$, the right-hand side of (5.5) can be interpreted as the Pfaffian of the block $2n \times 2n$ matrix with blocks formed by rows and columns with numbers $j_1, \dots, j_{2k}$ and $\{1, \dots, 2n\} \setminus \{j_1, \dots, j_{2k}\}$, respectively. This skew-symmetric $2n \times 2n$ matrix is exactly $\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]$ for our sequence $(s_1, \dots, s_{2n})$.

This implies that (5.3) holds for any choice of signs $(s_1, \dots, s_{2n})$, $s_j \in \{-, 0, +\}$.

Step 4. Let us now deduce the claim of the Theorem from (5.3). We must prove that

$$\sum_{s_1, \dots, s_{2n}} \mathrm{Pf}(\mathbf{F}[\![f_1^{s_1}, \dots, f_{2n}^{s_{2n}}]\!]) = \mathrm{Pf}(\mathbf{F}[\![f_1, \dots, f_{2n}]\!]).$$

This is done by induction on n. The base is $n = 1$:

$$\mathbf{F}(f_1^- f_2^+) + \mathbf{F}(f_1^0 f_2^0) = \mathbf{F}(f_1 f_2)$$

(all the other combinations of signs in the left-hand side give zero contribution). The induction step is readily verified using the Pfaffian expansion (5.4). This concludes the proof of the Theorem. □

5.2 Fermionic Fock space

Consider the space $\ell^2(\mathbb{Z}_{>0})$ with the standard orthonormal basis $\{\varepsilon_k\}_{k \in \mathbb{Z}_{>0}}$. The exterior algebra $\wedge \ell^2(\mathbb{Z}_{>0})$ is the vector space with the basis

$$\{\mathsf{vac}\} \cup \{\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_\ell} \colon \infty > i_1 > \ldots > i_\ell \geq 1,\ \ell = 1, 2, \dots\}, \tag{5.6}$$

where $\mathsf{vac} \equiv 1$ is called the *vacuum vector*. Define an inner product $(\cdot, \cdot)$ in the exterior algebra $\wedge \ell^2(\mathbb{Z}_{>0})$ with respect to which the basis (5.6) is orthonormal. This inner product turns $\wedge \ell^2(\mathbb{Z}_{>0})$ into a pre-Hilbert space. Its Hilbert completion is called the (*fermionic*) *Fock space* and is denoted by $\mathsf{Fock}(\mathbb{Z}_{>0})$. The space $(\wedge \ell^2(\mathbb{Z}_{>0}), (\cdot, \cdot))$ consisting of finite linear combinations of the basis vectors (5.6) is denoted by $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$.

Clearly, the map

$$\underline{\lambda} \mapsto \varepsilon_{\lambda_1} \wedge \cdots \wedge \varepsilon_{\lambda_{\ell(\lambda)}}, \qquad \lambda \in \mathbb{S}$$

(in particular, $\underline{\varnothing} \mapsto \mathsf{vac}$), defines an isometry between the pre-Hilbert spaces $\ell^2_{\mathrm{fin}}(\mathbb{S})$ and $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ and also between their Hilbert completions $\ell^2(\mathbb{S})$ and $\mathsf{Fock}(\mathbb{Z}_{>0})$. Below we identify $\ell^2(\mathbb{S})$ and $\mathsf{Fock}(\mathbb{Z}_{>0})$, and by $\underline{\lambda}$ we mean the vector $\varepsilon_{\lambda_1} \wedge \cdots \wedge \varepsilon_{\lambda_{\ell(\lambda)}}$.

In the next subsection we describe the structure of $\mathsf{Fock}(\mathbb{Z}_{>0})$ in more detail.

5.3 Creation and annihilation operators. Vacuum average

Let ψ_k, $k = 1, 2, \ldots$, be the *creation operators* in $\mathsf{Fock}(\mathbb{Z}_{>0})$, that is,

$$\psi_k \underline{\lambda} := \varepsilon_k \wedge \underline{\lambda}, \qquad \lambda \in \mathbb{S}.$$

Let ψ_k^*, $k = 1, 2, \ldots$, be the operators that are adjoint to ψ_k. They are called the *annihilation operators* and act as follows:

$$\psi_k^* \underline{\lambda} = \sum_{j=1}^{\ell(\lambda)} (-1)^{j+1} \delta_{k,\lambda_j} \varepsilon_{\lambda_1} \wedge \cdots \wedge \widehat{\varepsilon_{\lambda_j}} \wedge \cdots \wedge \varepsilon_{\lambda_{\ell(\lambda)}}.$$

We also need the operator $\psi_0 = \psi_0^*$ that acts as

$$\psi_0 \underline{\lambda} := (-1)^{\ell(\lambda)} \underline{\lambda}.$$

To simplify certain formulas below, we organize the operators ψ_k, ψ_0 and ψ_k^* into a single family as

$$\boldsymbol{\psi}_m := \begin{cases} \psi_m, & \text{if } m \ge 0; \\ (-1)^m \psi_{-m}^*, & \text{otherwise,} \end{cases} \qquad m \in \mathbb{Z}.$$

It can be readily checked that the operators $\boldsymbol{\psi}_m$ satisfy the following anti-commutation relations:

$$\boldsymbol{\psi}_k \boldsymbol{\psi}_l + \boldsymbol{\psi}_l \boldsymbol{\psi}_k = \begin{cases} 2, & \text{if } k = l = 0; \\ (-1)^l \delta_{k,-l}, & \text{otherwise.} \end{cases} \tag{5.7}$$

In agreement to these definitions, let $\{\boldsymbol{v}_x\}_{x \in \mathbb{Z}}$ be another orthonormal basis in the space $(V, \langle \cdot, \cdot \rangle)$ (considered in §5.1) defined as

$$\boldsymbol{v}_x := \begin{cases} v_x, & \text{if } x \ge 0; \\ (-1)^x v_x, & \text{if } x < 0, \end{cases} \qquad x \in \mathbb{Z}. \tag{5.8}$$

In other words, $\boldsymbol{v}_x = (-1)^{x \wedge 0} v_x$, where by $a \wedge b$ we denote the minimum of two numbers a and b. We have

$$\boldsymbol{v}_x \boldsymbol{v}_y + \boldsymbol{v}_y \boldsymbol{v}_x = \langle \boldsymbol{v}_x, \boldsymbol{v}_y \rangle = \begin{cases} 2, & x = y = 0; \\ (-1)^x \delta_{x,-y}, & \text{otherwise.} \end{cases} \tag{5.9}$$

Definition 5.2. Let $\mathcal{T}$ be a representation of the Clifford algebra $Cl(V)$ in $\mathsf{Fock}(\mathbb{Z}_{>0})$ defined on V by

$$\mathcal{T}(\boldsymbol{v}_x) := \boldsymbol{\psi}_x, \qquad x \in \mathbb{Z},$$

and extended to the whole $Cl(V)$ by (5.1) and by linearity. The fact that $\mathcal{T}$ is indeed a representation follows from (5.7) and (5.9).

Definition 5.3. The representation $\mathcal{T}$ allows to consider the following functional on the Clifford algebra $Cl(V)$:

$$\mathbf{F}_{\mathsf{vac}}(w) := (\mathcal{T}(w)\mathsf{vac}, \mathsf{vac}), \qquad w \in Cl(V)$$

called the *vacuum average*. Here the inner product is taken in $\mathsf{Fock}(\mathbb{Z}_{>0})$.

It can be readily verified that the functional $\mathbf{F}_{\mathsf{vac}}$ on $Cl(V)$ satisfies the hypotheses of Wick's Theorem 5.1.

5.4 The representation R

The space $\ell^2_{\mathrm{fin}}(\mathbb{S})$ is isometric to $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$, and thus the Kerov's operators U, D, and H (4.2) in $\ell^2_{\mathrm{fin}}(\mathbb{S})$ give rise to certain operators in $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$. In this way we obtain a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in $\mathsf{Fock}(\mathbb{Z}_{>0})$. Denote this representation by R.

The pre-Hilbert space $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ serves as a common invariant domain for R. It can be readily verified that the action of the operators $R(U)$, $R(D)$, and $R(H)$ in $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ can be expressed in terms of the creation and annihilation operators as follows:

$$\begin{aligned} R(U) &= \sum_{k=0}^{\infty} 2^{-\delta(k)/2}(-1)^k\sqrt{k(k+1)+\alpha}\cdot\boldsymbol{\psi}_{k+1}\boldsymbol{\psi}_{-k}, \\ R(D) &= \sum_{k=0}^{\infty} 2^{-\delta(k)/2}(-1)^{k+1}\sqrt{k(k+1)+\alpha}\cdot\boldsymbol{\psi}_{k}\boldsymbol{\psi}_{-k-1}, \\ R(H) &= \frac{\alpha}{2}+2\sum_{k=1}^{\infty}(-1)^k k\boldsymbol{\psi}_k\boldsymbol{\psi}_{-k}. \end{aligned} \tag{5.10}$$

Proposition 4.7 can be reformulated for the representation R. Namely, the representation R of $\mathfrak{sl}(2,\mathbb{C})$ restricted to the real form $\mathfrak{su}(1,1)\subset\mathfrak{sl}(2,\mathbb{C})$ (spanned by the matrices $U-D$, $i(U+D)$, and iH) gives rise to a unitary representation of the universal covering group $SU(1,1)^{\sim}$ in the Hilbert space $\mathsf{Fock}(\mathbb{Z}_{>0})$. Denote this representation also by R.

Under the identification of $\ell^2(\mathbb{S})$ with the space $\mathsf{Fock}(\mathbb{Z}_{>0})$ (§5.2), we say that the map $I_{\alpha,\xi}$ (4.11) is an isometry between $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ and $\mathsf{Fock}(\mathbb{Z}_{>0})$. By Proposition 4.8, for any bounded operator A in $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ we have

$$(A\mathbf{1},\mathbf{1})_{\mathsf{M}_{\alpha,\xi}} = \big(R(\widetilde{G}_\xi)^{-1}(I_{\alpha,\xi}AI^{-1}_{\alpha,\xi})R(\widetilde{G}_\xi)\mathsf{vac},\mathsf{vac}\big). \tag{5.11}$$

Here $\widetilde{G}_\xi\in SU(1,1)^{\sim}$, $0\le\xi<1$ is defined in §4.2 and $\mathbf{1}\in\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ is the constant identity function. The inner products on the left and on the right are taken in the spaces $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ and $\mathsf{Fock}(\mathbb{Z}_{>0})$, respectively.

Formula (4.14) for the expectation of a bounded function $f(\cdot)$ on $\mathbb{S}$ with respect to the measure $\mathsf{M}_{\alpha,\xi}$ is rewritten as

$$\mathbb{E}_{\alpha,\xi}f = \big(R(\widetilde{G}_\xi)^{-1}A_f R(\widetilde{G}_\xi)\mathsf{vac},\mathsf{vac}\big), \tag{5.12}$$

where A_f is the operator of multiplication by f.

As we will see below, for certain choices of the operator A the right-hand side of (5.11) can be written as the vacuum average $\mathbf{F}_{\mathsf{vac}}$ on the Clifford algebra $Cl(V)$, that is, the operator $R(\widetilde{G}_\xi)^{-1}(I_{\alpha,\xi}AI_{\alpha,\xi}^{-1})R(\widetilde{G}_\xi)$ has the form $\mathcal{T}(w)$ for some $w \in Cl(V)$.

6 Static correlation functions

Here and in the next section we compute the correlation functions of the point processes $\mathsf{M}_{\alpha,\xi}$ (2.6) and Pl_θ (2.2), and prove Theorem 1.

6.1 Pfaffian formula

By $\mathbb{Z}_{\neq 0}$ denote the set of all nonzero integers, and for $x_1, \dots, x_n \in \mathbb{Z}_{>0}$ put, by definition,

$$x_{-k} := -x_k, \qquad k = 1, \dots, n. \tag{6.1}$$

We use this convention in the formulation of the next Theorem. Let the function $\mathbf{\Phi}_{\alpha,\xi}$ on $\mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\neq 0}$ be defined by

$$\mathbf{\Phi}_{\alpha,\xi}(x,y) := (-1)^{x\wedge 0 + y \wedge 0} \Big(R(\widetilde{G}_\xi)^{-1} \boldsymbol{\psi}_x \boldsymbol{\psi}_y R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac} \Big), \tag{6.2}$$

where the inner product is taken in $\mathsf{Fock}(\mathbb{Z}_{>0})$. In this subsection we prove the following:

Theorem 6.1. *The correlation functions $\rho_{\alpha,\xi}^{(n)}$ (2.9) of the measures $\mathsf{M}_{\alpha,\xi}$ (2.6) are given by the following Pfaffian formula*:

$$\rho_{\alpha,\xi}^{(n)}(x_1, \dots, x_n) = \mathrm{Pf}(\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]), \tag{6.3}$$

where $X = \{x_1, \dots, x_n\} \subset \mathbb{Z}_{>0}$ (here x_j's are distinct), and $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$ is the skew-symmetric $2n \times 2n$ matrix with rows and columns indexed by the numbers $1, 2, \dots, n, -n, \dots, -2, -1$, and the kj-th entry in $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$ above the main diagonal is $\mathbf{\Phi}_{\alpha,\xi}(x_k, x_j)$, where $k, j = 1, \dots, n, -n, \dots, -1$.

In the next section we write $\mathbf{\Phi}_{\alpha,\xi}(x,y)$ in terms of the Gauss hypergeometric function. For this reason, we call $\mathbf{\Phi}_{\alpha,\xi}$ the *Pfaffian hypergeometric-type kernel*.

Remark 6.2. Theorem 6.1 is the same as Proposition 2 in [Pet10], the only difference is that in [Pet10] the product of the factors $(-1)^{x_k \wedge 0}$ (where k runs over $1, 2, \dots, n, -n, \dots, -2, -1$) is put in front of the Pfaffian, and thus in the definition of the Pfaffian kernel in [Pet10] there is no factor of the form $(-1)^{x\wedge 0 + y \wedge 0}$.

Remark 6.3. It is worth noting that in (6.3) instead of the matrix $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$ one can also take another matrix $\mathbf{\Phi}_{\alpha,\xi}[\![X]\!]$ which is defined as the skew-symmetric $2n \times 2n$ matrix with rows and columns indexed by the numbers $1, -1, \dots, n, -n$, and the kjth element above the main diagonal of $\mathbf{\Phi}_{\alpha,\xi}[\![X]\!]$ is $\mathbf{\Phi}_{\alpha,\xi}(x_k, x_j)$,

$k, j = 1, -1, \dots, n, -n$ (and thus here $|k| < |j|$). This fact is readily verified by elementary transformations of Pfaffians, or, alternatively, by comparison of expressions (6.5) and (6.6) below which correspond to $\mathbf{\Phi}_{\alpha,\xi}[\![X]\!]$ and $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$, respectively.

Moreover, it is the matrix $\mathbf{\Phi}_{\alpha,\xi}[\![X]\!]$ (and not $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$) that has a dynamical counterpart (see §9.4 and §10.1), but in the static case it is easier to work with $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$. For example, in terms of $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$ it is easier to see that the Pfaffian in (6.3) can be rewritten as a determinant, see §7.1 and Appendix.

The end of this subsection is devoted to proving Theorem 6.1. Consider the following operators in $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$:

$$\Delta_x \underline{\lambda} := \begin{cases} \underline{\lambda}, & \text{if } x \in \lambda; \\ 0, & \text{otherwise,} \end{cases} \qquad x \in \mathbb{Z}_{>0}.$$

Fix a finite subset $X = \{x_1, \dots, x_n\} \subset \mathbb{Z}_{>0}$ and set $\Delta_{[\![X]\!]} := \Delta_{x_1} \dots \Delta_{x_n}$. This is a diagonal operator of multiplication by a function which is the indicator of the event $\{\lambda \supseteq X\}$. We view $\Delta_{[\![X]\!]}$ as an operator acting in $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$. Since $\Delta_{[\![X]\!]}$ is diagonal, it does not change under the isometry $I_{\alpha,\xi} \colon \ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi}) \to \mathsf{Fock}(\mathbb{Z}_{>0})$ (4.11). Thus, $\Delta_{[\![X]\!]}$ also acts in $\mathsf{Fock}(\mathbb{Z}_{>0})$.

The correlation functions $\rho^{(n)}_{\alpha,\xi}$ (2.9) of the measures $\mathsf{M}_{\alpha,\xi}$ (2.6) can be written as

$$\rho^{(n)}_{(\alpha,\xi)}(x_1, \dots, x_n) = \mathsf{M}_{\alpha,\xi}\left(\lambda \colon \lambda \supseteq \{x_1, \dots, x_n\}\right) = (\Delta_{[\![X]\!]}\mathbf{1}, \mathbf{1})_{\mathsf{M}_{\alpha,\xi}},$$

where $\mathbf{1} \in \ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ is the constant identity function. Using (5.11) (or (5.12)), we can rewrite the correlation functions as

$$\rho^{(n)}_{(\alpha,\xi)}(x_1, \dots, x_n) = \left(R(\widetilde{G}_\xi)^{-1} \Delta_{[\![X]\!]} R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac}\right). \tag{6.4}$$

In this formula the operator $\Delta_{[\![X]\!]}$ acts in $\mathsf{Fock}(\mathbb{Z}_{>0})$. Clearly, $\Delta_{[\![X]\!]}$ is expressed through the creation and annihilation operators as

$$\Delta_{[\![X]\!]} = \prod_{k=1}^{n} \psi_{x_k} \psi^*_{x_k}. \tag{6.5}$$

It is more convenient for us to rewrite $\Delta_{[\![X]\!]}$ using the anti-commutation relations for ψ_x and ψ^*_x (see (5.7)) as follows:

$$\Delta_{[\![X]\!]} = \psi_{x_1} \dots \psi_{x_n} \psi^*_{x_n} \dots \psi^*_{x_1}. \tag{6.6}$$

Our next step is to write (6.4) as the vacuum average $\mathbf{F}_{\mathsf{vac}}$ applied to a certain element of $Cl(V)$. Recall that in §5.3 we have defined the representation $\mathcal{T}$ of $Cl(V)$ in $\mathsf{Fock}(\mathbb{Z}_{>0})$ such that $\mathcal{T}(\boldsymbol{v}_x) = \boldsymbol{\psi}_x$, $x \in \mathbb{Z}$, where $\{\boldsymbol{v}_x\}_{x\in\mathbb{Z}}$ is the basis of V defined by (5.8). One can readily compute the commutators between the operators $\mathcal{T}(\boldsymbol{v}_x)$ and the operators of the representation R (5.10):

$$\begin{aligned} [R(U), \mathcal{T}(\boldsymbol{v}_x)] &= 2^{(\delta(x)-\delta(x+1))/2}\sqrt{x(x+1)+\alpha} \cdot \mathcal{T}(\boldsymbol{v}_{x+1}); \\ [R(D), \mathcal{T}(\boldsymbol{v}_x)] &= 2^{(\delta(x)-\delta(x-1))/2}\sqrt{x(x-1)+\alpha} \cdot \mathcal{T}(\boldsymbol{v}_{x-1}); \\ [R(H), \mathcal{T}(\boldsymbol{v}_x)] &= 2x \cdot \mathcal{T}(\boldsymbol{v}_x), \qquad x \in \mathbb{Z}. \end{aligned} \tag{6.7}$$

Note that if we have used the standard basis $\{v_x\}_{x\in\mathbb{Z}}$ of $V = \ell^2(\mathbb{Z})$ instead of $\{\boldsymbol{v}_x\}_{x\in\mathbb{Z}}$, then formulas (6.7) would be less compact. These formulas motivate the following definition:

Definition 6.4. Let S be the representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in the (pre-Hilbert) space V_{fin} defined as:[12]

$$\begin{array}{lcl} S(U)\boldsymbol{v}_x & := & 2^{(\delta(x)-\delta(x+1))/2}\sqrt{x(x+1)+\alpha}\cdot \boldsymbol{v}_{x+1}; \\ S(D)\boldsymbol{v}_x & := & 2^{(\delta(x)-\delta(x-1))/2}\sqrt{x(x-1)+\alpha}\cdot \boldsymbol{v}_{x-1}; \\ S(H)\boldsymbol{v}_x & := & 2x\cdot \boldsymbol{v}_x, \qquad\qquad x\in\mathbb{Z}. \end{array} \tag{6.8}$$

The representation S is chosen in such a way that for all matrices $M \in \mathfrak{sl}(2,\mathbb{C})$ and vectors $v \in V_{\mathrm{fin}}$ we have

$$[R(M),\mathcal{T}(v)] = \mathcal{T}(S(M)v) \tag{6.9}$$

(the equality of operators in $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$). This fact can be readily verified using the definitions of R, $\mathcal{T}$ and S.

For the representation S one can prove an analogue of Proposition 4.7:

Proposition 6.5. *All vectors of the space V_{fin} are analytic for the action S of the Lie algebra $\mathfrak{su}(1,1) \subset \mathfrak{sl}(2,\mathbb{C})$. Consequently, this action of $\mathfrak{su}(1,1)$ in V_{fin} gives rise to a unitary representation of the universal covering group $SU(1,1)^{\sim}$ in the Hilbert space V.*

Proof. The representation S of $\mathfrak{sl}(2,\mathbb{C})$ in V_{fin} defined by (6.8) is conjugate to the following representation $\check{S}$:

$$\begin{array}{lcl} \check{S}(U)\boldsymbol{v}_x & := & \sqrt{x(x+1)+\alpha}\cdot \boldsymbol{v}_{x+1}; \\ \check{S}(D)\boldsymbol{v}_x & := & \sqrt{x(x-1)+\alpha}\cdot \boldsymbol{v}_{x-1}; \\ \check{S}(H)\boldsymbol{v}_x & := & 2x\cdot \boldsymbol{v}_x, \qquad\qquad x\in\mathbb{Z}. \end{array}$$

That is, $S(U) = Z^{-1}\check{S}(U)Z$ (and the same for D and H), where Z is the diagonal operator that multiplies $\boldsymbol{v}_0 = v_0$ by $\sqrt{2}$ and acts as the identity operator in the orthogonal complement of v_0.

It suffices to prove the claim for the representation $\check{S}$. The case of $\check{S}$ is easier to handle because the operators $\check{S}(U) - \check{S}(D)$, $i\big(\check{S}(U) + \check{S}(D)\big)$, and $i\check{S}(H)$ (where $i = \sqrt{-1}$) act skew-symmetrically in the (complex) pre-Hilbert space V_{fin}. Arguing exactly as in the proof of Proposition 4.7, we see that it is enough to show that the power series expansion of $\exp\big(s(\check{S}(U)+\check{S}(D)+\check{S}(H))\big)\,\boldsymbol{v}_x$ have a positive radius of convergence for all $x \in \mathbb{Z}$ and this radius does not depend on x. In the matrix group $SL(2,\mathbb{C})$ we have

$$\exp(s(U+D+H)) = \exp\left(\frac{s}{1-s}U\right)\exp\left(s(1-s)D\right)\exp\left(\log\left(\frac{1}{1-s}\right)H\right),$$

[12]The space $V_{\mathrm{fin}} = \ell^2_{\mathrm{fin}}(\mathbb{Z})$ consists of all finite linear combinations of the basis vectors $\{\boldsymbol{v}_x\}_{x\in\mathbb{Z}}$. The fact that S is indeed a representation of $\mathfrak{sl}(2,\mathbb{C})$ follows from the appropriate commutation relations between the operators $S(U)$, $S(D)$, and $S(H)$ in V_{fin} (see §4.1) that can be readily checked.

so the power series expansion of $\exp\big(s(\check S(U)+\check S(D)+\check S(H))\big)\,\boldsymbol{v}_x$ is the same as the expansion of

$$\begin{aligned}&\exp\left(\frac{s}{1-s}\check S(U)\right)\exp\big(s(1-s)\check S(D)\big)\exp\left(\log\left(\frac{1}{1-s}\right)\check S(H)\right)\boldsymbol{v}_x\\&\qquad=(1-s)^{-2x}\exp\left(\frac{s}{1-s}\check S(U)\right)\exp\big(s(1-s)\check S(D)\big)\,\boldsymbol{v}_x.\end{aligned}$$

The coefficients $\Big(\exp(\frac{s}{1-s}\check S(U))\exp(s(1-s)\check S(D))\boldsymbol{v}_x,\boldsymbol{v}_r\Big)$ (where $r\in\mathbb{Z}$) can be explicitly computed as in the proof of Proposition 6.7 below, and they decrease exponentially as x is fixed and $|r|\to\infty$ (see also the asymptotics (9.10)). This implies that the power series expansion of $\exp(s(U+D+H))$ has a positive radius of convergence. This concludes the proof. □

Denote the representation of $SU(1,1)^\sim$ in V constructed in Proposition 6.5 by the same symbol S as the corresponding Lie algebra representation (6.8). We now aim to prove the "group level" version of the identity (6.9). The next proposition is due to Olshanski [Ols].

Proposition 6.6. *For all $g\in SU(1,1)^\sim$ and all $v\in V$ we have*

$$R(g)\mathcal{T}(v)R(g)^{-1}=\mathcal{T}(S(g)v)\tag{6.10}$$

(*the equality of operators in* $\mathsf{Fock}(\mathbb{Z}_{>0})$).

Proof. **Step 1.** Since the representation $\mathcal{T}$ is norm preserving, it suffices to take the vector $v\in V$ from the dense subspace V_{fin}. Without loss of generality we can assume that $v=\boldsymbol{v}_x$ for some $x\in\mathbb{Z}$.

Step 2. Rewrite the claim (6.10) as

$$R(g)\mathcal{T}(\boldsymbol{v}_x)=\mathcal{T}(S(g)\boldsymbol{v}_x)R(g).\tag{6.11}$$

This is an equality of operators in the Hilbert space $\mathsf{Fock}(\mathbb{Z}_{>0})$. It is enough to show that these operators agree on $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$, that is, that

$$R(g)\mathcal{T}(\boldsymbol{v}_x)\underline{\lambda}=\mathcal{T}\big(S(g)\boldsymbol{v}_x\big)R(g)\underline{\lambda}\qquad\text{for all } g\in SU(1,1)^\sim,\ x\in\mathbb{Z},\text{ and }\lambda\in\mathbb{S}.\tag{6.12}$$

Step 3. Now let us prove that both sides of (6.12) are analytic functions in $g\in SU(1,1)^\sim$ with values in $\mathsf{Fock}(\mathbb{Z}_{>0})$:

- (left-hand side) The vector $\mathcal{T}(\boldsymbol{v}_x)\underline{\lambda}$ belongs to $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$, and hence is analytic for the representation R, see Proposition 4.7. This means that the function $g\mapsto R(g)\mathcal{T}(\boldsymbol{v}_x)\underline{\lambda}$ is analytic.

- (right-hand side) By Proposition 6.5, the function $g\mapsto S(g)\boldsymbol{v}_x$ is an analytic function with values in the Hilbert space V. Since $\mathcal{T}$ is continuous in the norm topology, $\mathcal{T}\big(S(g)\boldsymbol{v}_x\big)$ is an analytic function with values in the Banach space $\mathrm{End}\big(\mathsf{Fock}(\mathbb{Z}_{>0})\big)$ of bounded operators in the space $\mathsf{Fock}(\mathbb{Z}_{>0})$. On the other hand, the function $R(g)\underline{\lambda}$ is also analytic (with values in $\mathsf{Fock}(\mathbb{Z}_{>0})$). Therefore, the function $g\mapsto\mathcal{T}\big(S(g)\boldsymbol{v}_x\big)R(g)\underline{\lambda}$ is analytic, too.

Step 4. Now it remains to compare the Taylor series expansions of the both sides of (6.11) at $g = e$, the unity element of $SU(1,1)^{\sim}$. That is, we need to establish that for any $M \in \mathfrak{sl}(2,\mathbb{C})$ and any $x \in \mathbb{Z}$:

$$\sum_{k=0}^{\infty} \frac{R(M)^k s^k}{k!} \mathcal{T}(\boldsymbol{v}_x) = \left(\sum_{l=0}^{\infty} \frac{\mathcal{T}\big(S(M)^l \boldsymbol{v}_x\big) s^l}{l!} \right) \left(\sum_{r=0}^{\infty} \frac{R(M)^r s^r}{r!} \right). \tag{6.13}$$

This should be understood as an equality of formal power series in s with coefficients being operators in $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$. Let us divide both sides by the last formal sum, $\sum_{r=0}^{\infty} \frac{R(M)^r s^r}{r!}$. After that it can be readily verified that the identity of formal power series is a corollary of the "Lie algebra level" commutation relations (6.9).

This last step concludes the proof of the Proposition. □

Let us define

$$v_{x,\xi} := S(\widetilde{G}_\xi)^{-1} v_x \in V, \qquad x \in \mathbb{Z}. \tag{6.14}$$

For fixed ξ the operator $S(\widetilde{G}_\xi)^{-1}$ is unitary, and thus $\{v_{x,\xi}\}_{x\in\mathbb{Z}}$ is an orthonormal basis of V which is a deformation of the initial basis $\{v_x\}_{x\in\mathbb{Z}}$. Putting together this definition and Proposition 6.6, we can rewrite the correlation functions (6.4) as the vacuum average (see Definition 5.3):

$$\rho_{\alpha,\xi}^{(n)}(x_1,\dots,x_n) = \mathbf{F}_{\mathsf{vac}}\left(v_{x_1,\xi}\dots v_{x_n,\xi} v_{-x_n,\xi}\dots v_{-x_1,\xi}\right). \tag{6.15}$$

Observe that for $x, y \in \mathbb{Z}_{\neq 0}$ we have

$$\mathbf{F}_{\mathsf{vac}}(v_{x,\xi} v_{y,\xi}) = (-1)^{x\wedge 0 + y\wedge 0} \left(R(\widetilde{G}_\xi)^{-1} \boldsymbol{\psi}_x \boldsymbol{\psi}_y R(\widetilde{G}_\xi) \mathsf{vac}, \mathsf{vac} \right) = \boldsymbol{\Phi}_{\alpha,\xi}(x,y). \tag{6.16}$$

Therefore, formula (6.15) together with Wick's Theorem (Theorem 5.1) immediately implies our Theorem 6.1.

6.2 The representation S and its matrix elements

In this subsection we write matrix elements of the representation S of $SU(1,1)^{\sim}$ in $V = \ell^2(\mathbb{Z})$ in terms of the Gauss hypergeometric function. This computation is similar to the one of Okounkov [Oko01b] and uses explicit formulas (6.8) for the action of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in V_{fin}.

Let us first explain which matrix elements we want to compute. Using the basis $\{v_{x,\xi}\}_{x\in\mathbb{Z}}$ of the Hilbert space V (see (6.14)), we can write the Pfaffian hypergeometric-type kernel of Theorem 6.1 as

$$\boldsymbol{\Phi}_{\alpha,\xi}(x,y) = \mathbf{F}_{\mathsf{vac}}(v_{x,\xi} v_{y,\xi}) = \sum_{k,l\in\mathbb{Z}} (v_{x,\xi}, v_k)_V (v_{y,\xi}, v_l)_V \mathbf{F}_{\mathsf{vac}}(v_k v_l).$$

By definitions of §5.3, we have

$$\mathbf{F}_{\mathsf{vac}}(v_k v_l) = \begin{cases} 1, & \text{if } l = -k \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6.17}$$

Therefore,

$$\mathbf{\Phi}_{\alpha,\xi}(x,y)=\sum_{m=0}^{\infty}(v_{x,\xi},v_{-m})_V\,(v_{y,\xi},v_m)_V\,. \tag{6.18}$$

Thus, we need do compute the matrix elements

$$(v_{x,\xi},v_k)_V=\Big(S(\widetilde{G}_\xi)^{-1}v_x,v_k\Big)_V, \qquad x,k\in\mathbb{Z}.$$

Let us set

$$\epsilon_{x,k}:=\begin{cases}(-1)^{x\wedge 0+k\wedge 0}, & \text{if } x\ge k;\\ (-1)^{x\vee 0+k\vee 0}, & \text{if } x\le k,\end{cases}$$

where for any two numbers a and b, by $a\vee b$ and $a\wedge b$ we denote the maximum and the minimum of a and b, respectively.

Now we can give explicit expressions for the matrix elements $(v_{x,\xi},v_k)_V$:

Proposition 6.7. *For any $x,k\in\mathbb{Z}$ we have*

$$\begin{aligned}(v_{x,\xi},v_k)_V&=\epsilon_{x,k}\cdot 2^{\frac{\delta(x)-\delta(k)}{2}}(1-\xi)^{x\wedge k}\frac{\xi^{\frac{|x-k|}{2}}}{|x-k|!}\times\\ &\times\sqrt{\frac{\Gamma(x\vee k+\nu(\alpha)+\frac12)\Gamma(x\vee k-\nu(\alpha)+\frac12)}{\Gamma(x\wedge k+\nu(\alpha)+\frac12)\Gamma(x\wedge k-\nu(\alpha)+\frac12)}}\times\\ &\times {}_2F_1\left(\begin{matrix}\frac12+\nu(\alpha)-x\wedge k,\ \ \frac12-\nu(\alpha)-x\wedge k\\ |x-k|+1\end{matrix}\,\Big|\,\frac{\xi}{\xi-1}\right).\end{aligned} \tag{6.19}$$

Here ${}_2F_1$ is the Gauss hypergeometric function:

$${}_2F_1\left(\begin{matrix}a,\ \ b\\ c\end{matrix}\,\Big|\,z\right)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r r!}z^r.$$

Note that since $\alpha>0$, the expression under the square root in (6.19) is strictly positive for all x and k (see also Definition 2.1). Moreover, the third parameter of ${}_2F_1$ in (6.19) is a strictly positive integer, so the hypergeometric function is well defined.

Proof. We need a deformation of the basis $\{\boldsymbol{v}_x\}_{x\in\mathbb{Z}}$ (5.8) of V:

$$\boldsymbol{v}_{x,\xi}:=S(\widetilde{G}_\xi)^{-1}\boldsymbol{v}_x, \qquad x\in\mathbb{Z}. \tag{6.20}$$

The family $\{\boldsymbol{v}_{x,\xi}\}_{x\in\mathbb{Z}}$ is also an orthonormal basis of the Hilbert space V.

Fix $x,k\in\mathbb{Z}$. By Proposition 6.5, the function $\xi\mapsto(v_{x,\xi},v_k)_V$ is analytic. The right-hand side of (6.19) is also analytic in ξ. Thus, it suffices to prove (6.19) for small ξ. The vector $\boldsymbol{v}_x\in V_{\text{fin}}$ is analytic for the representation S (by Proposition 6.5), so on this vector the representation S can be extended to a representation of the local complexification of $SU(1,1)^\sim$. This means that for small ξ (when $\widetilde{G}_\xi$ is close to the unity of $SU(1,1)^\sim$) we have:[13]

$$S(\widetilde{G}_\xi)^{-1}\boldsymbol{v}_x = \exp\left(-\sqrt{\xi}S(U)\right)\exp\left(\frac{\sqrt{\xi}}{1-\xi}S(D)\right)\exp\left(\frac{1}{2}\log(1-\xi)S(H)\right)\boldsymbol{v}_x.$$

This follows from the corresponding identity for matrices in $SL(2,\mathbb{C})$.

Denote

$$u_y := 2^{(\delta(y)-\delta(y+1))/2}\sqrt{y(y+1)+\alpha};$$
$$d_y := 2^{(\delta(y+1)-\delta(y))/2}\sqrt{y(y+1)+\alpha},$$

so that the action (6.8) of $S(U)$ and $S(D)$ in V has the form

$$S(U)\boldsymbol{v}_y = u_y\boldsymbol{v}_{y+1} \qquad \text{and} \qquad S(D)\boldsymbol{v}_y = d_{y-1}\boldsymbol{v}_{y-1}, \qquad y\in\mathbb{Z}.$$

Note that

$$u_yd_y = y(y+1)+\alpha = (y+\nu(\alpha)+\tfrac{1}{2})(y-\nu(\alpha)+\tfrac{1}{2}).$$

Also set $a := -\sqrt{\xi}$ and $b := \sqrt{\xi}/(1-\xi)$.

Let us first compute $(\boldsymbol{v}_{x,\xi},\boldsymbol{v}_k)_V$ instead of $(v_{x,\xi},v_k)_V$. We have

$$\begin{aligned}(\boldsymbol{v}_{x,\xi},\boldsymbol{v}_k)_V &= (1-\xi)^x\left(e^{aS(U)}e^{bS(D)}\boldsymbol{v}_x,\boldsymbol{v}_k\right)_V\\ &= (1-\xi)^x\sum_{r=0}^\infty\sum_{l=0}^\infty\frac{a^rb^l}{r!l!}u_{x-l+r-1}\dots u_{x-l}d_{x-l}\dots d_{x-1}(\boldsymbol{v}_{x-l+r},\boldsymbol{v}_k)_V.\end{aligned}$$

Observe that $(\boldsymbol{v}_{x-l+r},\boldsymbol{v}_k)_V = \delta_{x-l+r,k}$. There are two cases: $x\ge k$, and $x\le k$. For $x\ge k$ we perform the above summation on r from 0 to ∞ and set $l = r+x-k$. For $x\le k$ the summation is over l, and we set $r = l+k-x$. After direct calculations we obtain (we omit the argument in $\nu(\alpha)$):

$$\begin{aligned}(\boldsymbol{v}_{x,\xi},\boldsymbol{v}_k)_V &= (1-\xi)^x2^{\frac{\delta(x)-\delta(k)}{2}}\frac{b^{x-k}}{(x-k)!}\sqrt{\frac{\Gamma(x+\nu+\frac12)\Gamma(x-\nu+\frac12)}{\Gamma(k+\nu+\frac12)\Gamma(k-\nu+\frac12)}}\times\\ &\times {}_2F_1\left(\begin{matrix}\frac12+\nu-k,\ \frac12-\nu-k\\ x-k+1\end{matrix}\,\Big|\,ab\right), \qquad \text{if } x\ge k,\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{v}_{x,\xi},\boldsymbol{v}_k)_V &= (1-\xi)^x2^{\frac{\delta(x)-\delta(k)}{2}}\frac{a^{k-x}}{(k-x)!}\sqrt{\frac{\Gamma(k+\nu+\frac12)\Gamma(k-\nu+\frac12)}{\Gamma(x+\nu+\frac12)\Gamma(x-\nu+\frac12)}}\times\\ &\times {}_2F_1\left(\begin{matrix}\frac12+\nu-x,\ \frac12-\nu-x\\ k-x+1\end{matrix}\,\Big|\,ab\right), \qquad \text{if } x\le k.\end{aligned}$$

Observe that

$$(1-\xi)^xb^{x-k} = (1-\xi)^k\xi^{\frac{x-k}{2}}, \qquad \text{and} \qquad (1-\xi)^xa^{k-x} = (-1)^{k+x}(1-\xi)^x\xi^{\frac{k-x}{2}}.$$

[13]See also the proof of Proposition 4.8.

This leads to the appearance of the factor $\epsilon_{x,k}$ in (6.19), because

$$(\boldsymbol{v}_{x,\xi}, \boldsymbol{v}_k)_V = (-1)^{x\wedge 0 + k\wedge 0}(v_{x,\xi}, v_k)_V.$$

Putting both cases together, we see that (6.19) holds for small ξ, and hence for all $\xi \in (0, 1)$. □

Corollary 6.8. *The matrix elements* (6.19) *satisfy the symmetry relation:*

$$(v_{-x,\xi}, v_{-k})_V = (v_{x,\xi}, v_k)_V, \qquad x, k \in \mathbb{Z}.$$

Proof. This is be verified via a direct computation using formula [Erd53, 2.9.(2)] for the Gauss hypergeometric function and the following identity (see (2.5)):

$$\begin{aligned}(-1)^m\Gamma(\tfrac{1}{2} + m \pm \nu) &= \Gamma(\tfrac{1}{2} \pm \nu) \cdot (-1)^m(\tfrac{1}{2} \pm \nu)_m \\ &= \Gamma(\tfrac{1}{2} \pm \nu) \cdot (\tfrac{1}{2} \mp \nu - m)_m = \frac{\Gamma(\frac{1}{2} + \nu)\Gamma(\frac{1}{2} - \nu)}{\Gamma(\frac{1}{2} \mp \nu - m)}.\end{aligned} \tag{6.21}$$

□

This corollary implies, in particular, that the function $\mathbf{\Phi}_{\alpha,\xi}(x, -y)$ is symmetric in $x, y \in \mathbb{Z}_{\neq 0}$ (see 6.18). For this reason in many formulas below we use $\mathbf{\Phi}_{\alpha,\xi}(x, -y)$ instead of $\mathbf{\Phi}_{\alpha,\xi}(x, y)$.

6.3 Reduction formulas for the Pfaffian hypergeometric-type kernel

It is possible to write the Pfaffian hypergeometric-type kernel $\mathbf{\Phi}_{\alpha,\xi}$ (6.18) in a compact form (without the sum):

Proposition 6.9. *For $x, y \in \mathbb{Z}_{\neq 0}$ we have*

$$\begin{aligned}&\mathbf{\Phi}_{\alpha,\xi}(x, -y) = \frac{\sqrt{\alpha\xi/2}}{1-\xi}\times \\ &\times\frac{\left[(v_{x,\xi}, v_1)_V + (v_{x,\xi}, v_{-1})_V\right](v_{y,\xi}, v_0)_V - \left[(v_{y,\xi}, v_1)_V + (v_{y,\xi}, v_{-1})_V\right](v_{x,\xi}, v_0)_V}{x-y}.\end{aligned} \tag{6.22}$$

If $x = y$, the value of the right-hand side is determined by continuity (*according to the L'Hospital's rule*).[14]

Proof. To avoid direct manipulations with the series for $\mathbf{\Phi}_{\alpha,\xi}$ (6.18) involving three-term relations for the Gauss hypergeometric function (see (7.1) and (7.6)), we use an argument introduced in [Oko01b, Proof of Thm. 3] to deal with a similar series in the case of the z-measures.

[14] This means that the function $\mathbf{\Phi}_{\alpha,\xi}(x, -y)$ is defined not only for $x, y \in \mathbb{Z}_{\neq 0}$, but it is also defined and continuous in (x, y) for x and y lying in a suitable neighborhood of $\mathbb{Z}_{\neq 0}$ in $\mathbb{C}$. Thus, for $x = y$ there is no singularity in $\mathbf{\Phi}_{\alpha,\xi}(x, -y)$. This agreement is applicable to all similar formulas below.

We use the same approach as in the proof of Proposition 6.7. Namely, we observe that both sides of (6.22) are analytic in ξ. For the right-hand side this is true because the function $\xi \mapsto (v_{x,\xi}, v_k)_V$ is analytic for fixed $x, k \in \mathbb{Z}$, and for the left-hand side $\mathbf{\Phi}_{\alpha,\xi}(x,-y) = (\mathcal{T}(v_{x,\xi})\mathcal{T}(v_{-y,\xi})\mathsf{vac}, \mathsf{vac})$ the analyticity is established as in the proof of Proposition 6.6 (Step 4). Thus, it suffices to prove (6.22) for small ξ.

From (6.7) it follows that

$$[R(H), \mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})] = 2(x-y)\mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})$$

(this is an equality of operators in $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$). Therefore,

$$\begin{aligned} &2(x-y)\mathbf{\Phi}_{\alpha,\xi}(x,-y) \\ &\quad = (-1)^{x\wedge 0+y\vee 0}\left(R(\widetilde{G}_\xi)^{-1}\cdot 2(x-y)\mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac}\right) \\ &\quad = (-1)^{x\wedge 0+y\vee 0}\left(R(\widetilde{G}_\xi)^{-1}\left[R(H), \mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})\right]R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac}\right). \end{aligned}$$

Now, using Proposition 6.6 we observe that

$$\begin{aligned} &R(\widetilde{G}_\xi)^{-1}\left[R(H), \mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})\right]R(\widetilde{G}_\xi) = [R(H), \mathcal{T}(\boldsymbol{v}_{x,\xi})\mathcal{T}(\boldsymbol{v}_{-y,\xi})] \\ &\quad - [R(H), R(\widetilde{G}_\xi)^{-1}]\mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})R(\widetilde{G}_\xi) - R(\widetilde{G}_\xi)^{-1}\mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})[R(H), R(\widetilde{G}_\xi)] \end{aligned}$$

(this is also an equality of operators in $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$). Since $R(H)$ is diagonal in the basis $\{\underline{\lambda}\}_{\lambda\in\mathbb{S}}$ of $\mathsf{Fock}(\mathbb{Z}_{>0})$, we have

$$([R(H), \mathcal{T}(\boldsymbol{v}_{x,\xi})\mathcal{T}(\boldsymbol{v}_{-y,\xi})]\,\mathsf{vac}, \mathsf{vac}) = 0. \tag{6.23}$$

Thus,

$$\begin{aligned} &2(-1)^{x\wedge 0+y\vee 0}(x-y)\mathbf{\Phi}_{\alpha,\xi}(x,-y) \\ &\quad = -\left([R(H), R(\widetilde{G}_\xi)^{-1}]\mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac}\right) \\ &\qquad -\left(R(\widetilde{G}_\xi)^{-1}\mathcal{T}(\boldsymbol{v}_x)\mathcal{T}(\boldsymbol{v}_{-y})[R(H), R(\widetilde{G}_\xi)]\mathsf{vac}, \mathsf{vac}\right). \end{aligned} \tag{6.24}$$

Now let us compute two commutators in (6.24). The group $SU(1,1)$ acts on the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ by conjugation, and by a simple matrix computation we obtain

$$G_\xi^{-1}HG_\xi = \frac{1+\xi}{1-\xi}H + \frac{2\sqrt{\xi}}{1-\xi}(U+D).$$

For small ξ the operator $R(\widetilde{G}_\xi)^{-1}R(H)R(\widetilde{G}_\xi)$ acts on the analytic (for R) vector vac as $R(G_\xi^{-1}HG_\xi)$, that is, as

$$\frac{1+\xi}{1-\xi}R(H) + \frac{2\sqrt{\xi}}{1-\xi}\big(R(U)+R(D)\big).$$

Therefore, we have

$$\begin{aligned} [R(H), R(\widetilde{G}_\xi)^{-1}]\mathsf{vac} &= -\frac{2\sqrt{\xi}}{1-\xi}\left(\sqrt{\xi}R(H)+R(U)+R(D)\right)R(\widetilde{G}_\xi)^{-1}\mathsf{vac}, \\ [R(H), R(\widetilde{G}_\xi)]\mathsf{vac} &= \frac{2\sqrt{\xi}}{1-\xi}R(\widetilde{G}_\xi)\left(\sqrt{\xi}R(H)+R(U)+R(D)\right)\mathsf{vac}. \end{aligned}$$

Put $A := \frac{2\sqrt{\xi}}{1-\xi}(\sqrt{\xi}H + U + D) \in \mathfrak{sl}(2,\mathbb{C})$. Formula (6.24) for $\mathbf{\Phi}_{\alpha,\xi}(x,-y)$ is rewritten as

$$2(x-y)\mathbf{\Phi}_{\alpha,\xi}(x,-y) = (\mathcal{T}(v_{x,\xi})\mathcal{T}(v_{-y,\xi})\mathsf{vac}, R(A)^*\mathsf{vac}) \\ -(\mathcal{T}(v_{x,\xi})\mathcal{T}(v_{-y,\xi})R(A)\mathsf{vac}, \mathsf{vac}).$$

Applying the symmetric operator $R(A)$ to vac, we obtain

$$R(A)\mathsf{vac} = R(A)^*\mathsf{vac} = \frac{\alpha\xi}{1-\xi}\mathsf{vac} + \frac{\sqrt{2\alpha\xi}}{1-\xi}\varepsilon_1,$$

where $\varepsilon_1 \in \mathsf{Fock}(\mathbb{Z}_{>0})$ is the vector of the standard basis corresponding to the one-box shifted diagram. Thus, we get

$$\mathbf{\Phi}_{\alpha,\xi}(x,-y) = \frac{\sqrt{\alpha\xi/2}}{1-\xi}\frac{(\mathcal{T}(v_{x,\xi})\mathcal{T}(v_{-y,\xi})\mathsf{vac}, \varepsilon_1) - (\mathcal{T}(v_{x,\xi})\mathcal{T}(v_{-y,\xi})\varepsilon_1, \mathsf{vac})}{x-y}.$$

Observe that

$$\mathcal{T}(v_{x,\xi})\mathcal{T}(v_{-y,\xi}) = \sum_{k,l\in\mathbb{Z}} (v_{x,\xi}, v_k)_V (v_{-y,\xi}, v_l)_V \mathcal{T}(v_k)\mathcal{T}(v_l).$$

From the definitions of the representation $\mathcal{T}$ and of the creation and annihilation operators (see §5.3) it follows that

$$(\mathcal{T}(v_k)\mathcal{T}(v_l)\mathsf{vac}, \varepsilon_1) = \begin{cases} -1, & \text{if } k=0 \text{ and } l=1; \\ 1, & \text{if } k=1 \text{ and } l=0; \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mathcal{T}(v_k)\mathcal{T}(v_l)\varepsilon_1, \mathsf{vac}) = \begin{cases} 1, & \text{if } k=0 \text{ and } l=-1; \\ -1, & \text{if } k=-1 \text{ and } l=0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{\Phi}_{\alpha,\xi}(x,-y) = \frac{\sqrt{\alpha\xi/2}}{(1-\xi)(x-y)}\Big[(v_{x,\xi}, v_1)_V (v_{-y,\xi}, v_0)_V - (v_{x,\xi}, v_0)_V (v_{-y,\xi}, v_1)_V \\ + (v_{x,\xi}, v_{-1})_V (v_{-y,\xi}, v_0)_V - (v_{x,\xi}, v_0)_V (v_{-y,\xi}, v_{-1})_V\Big].$$

It remains to apply the symmetry relations (Corollary 6.8) to see that (6.22) holds for small ξ. Thus, (6.22), also holds for all $\xi \in (0,1)$. This concludes the proof. □

The above Proposition together with Corollary 6.8 implies:

Corollary 6.10 (Reduction formulas)**.**
(1) *For all* $x, y \in \mathbb{Z}_{\neq 0}$ *we have* $\mathbf{\Phi}_{\alpha,\xi}(x,-y) = \mathbf{\Phi}_{\alpha,\xi}(y,-x)$;
(2) *For distinct* $x, y \in \mathbb{Z}_{\neq 0}$ *we have* $\mathbf{\Phi}_{\alpha,\xi}(x,-y) = -\mathbf{\Phi}_{\alpha,\xi}(-x,y)$;
(3) *For all* $x, y \in \mathbb{Z}_{\neq 0}$ *we have* $(x+y)\mathbf{\Phi}_{\alpha,\xi}(x,y) = (x-y)\mathbf{\Phi}_{\alpha,\xi}(x,-y)$.

Proof. Claim (1) directly follows from the symmetry relations (Corollary 6.8) and formula (6.2) for $\mathbf{\Phi}_{\alpha,\xi}$.

Claim (2) can be deduced from Proposition 6.9, but it also follows from an independent argument. Namely, for $x, y \in \mathbb{Z}_{\neq 0}$ write (using claim (1)):

$$\begin{aligned}\mathbf{\Phi}_{\alpha,\xi}(x,-y) + \mathbf{\Phi}_{\alpha,\xi}(-x,y) &= \mathbf{\Phi}_{\alpha,\xi}(x,-y) + \mathbf{\Phi}_{\alpha,\xi}(-y,x) \\ &= \Big(R(\widetilde{G}_\xi)^{-1} \mathcal{T}\,(v_x v_{-y} + v_{-y} v_x)\, R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac}\Big).\end{aligned}$$

This is equal to zero for $x \neq y$ by definition of the representation $\mathcal{T}$ of the Clifford algebra $Cl(V)$ in $\mathsf{Fock}(\mathbb{Z}_{>0})$, see §5.

Finally, claim (3) directly follows form the symmetry relations (Corollary 6.8) and Proposition 6.9. ☐

Note that property 3 above implies that $\mathbf{\Phi}_{\alpha,\xi}(x,x) = 0$ for all $x \in \mathbb{Z}_{\neq 0}$. This agrees with the fact that $vv = 0$ in $Cl(V)$ if v is orthogonal to $V^0 = \mathbb{C}v_0$.

7 The hypergeometric-type kernel

7.1 Expressions through the Gauss hypergeometric function

The reduction formulas obtained in the previous subsection (Corollary 6.10) allow to write correlation functions $\rho^{(n)}_{\alpha,\xi}$ of the point processes $\mathsf{M}_{\alpha,\xi}$ in a determinantal form. To give explicit expressions for the (determinantal) correlation kernel $\mathsf{K}_{\alpha,\xi}$ of $\mathsf{M}_{\alpha,\xi}$ we need to introduce some extra notation. Set for $x, m \in \mathbb{Z}$:

$$\begin{aligned}\phi_m(x) &:= {}_2F_1\left(\begin{matrix} -\frac12 - \nu(\alpha) + m, \quad -\frac12 + \nu(\alpha) + m \\ x+m \end{matrix} \;\middle|\; \frac{\xi}{\xi-1}\right), \quad && x+m \geq 1; \\ \widetilde{\phi}(x) &:= {}_2F_1\left(\begin{matrix} \frac32 + \nu(\alpha), \quad -\frac12 - \nu(\alpha) \\ x \end{matrix} \;\middle|\; \frac{\xi}{\xi-1}\right), && x \geq 1.\end{aligned} \tag{7.1}$$

The third parameter of the above hypergeometric functions is a positive integer, therefore, $\phi_m(x)$ and $\widetilde{\phi}(x)$ are well defined. Also set

$$\begin{aligned}\Xi(x,y) := {} & \left\{\Gamma(\tfrac12 - \nu(\alpha) + x)\Gamma(\tfrac12 + \nu(\alpha) + x)\right\}^{\frac12} \times \\ & \times \left\{\Gamma(\tfrac12 - \nu(\alpha) + y)\Gamma(\tfrac12 + \nu(\alpha) + y)\right\}^{\frac12}, \quad x, y \in \mathbb{Z}.\end{aligned} \tag{7.2}$$

Since $\alpha > 0$, the above expressions in the curved brackets are strictly positive, see Definition 2.1.

Theorem 7.1. *For all $\alpha > 0$ and $0 < \xi < 1$ the point process $\mathsf{M}_{\alpha,\xi}$ on $\mathbb{Z}_{>0}$ is determinantal. Its correlation kernel $\mathsf{K}_{\alpha,\xi}$ can be expressed through the Gauss hypergeometric function in two ways (here $x, y \in \mathbb{Z}_{>0}$):*

1. As a series

$$
\begin{aligned}
\mathsf{K}_{\alpha,\xi}(x,y) = {} & \frac{2\,\Xi\,(x,y)\,\sqrt{xy}}{x+y}\times \\
& \times \sum_{m=0}^{\infty} \frac{\xi^{m+\frac{x+y}{2}}(1-\xi)^{-2m}\phi_{m+1}(x)\phi_{m+1}(y)}{2^{\delta(m)}(x+m)!(y+m)!\Gamma(\frac{1}{2}-\nu(\alpha)-m)\Gamma(\frac{1}{2}+\nu(\alpha)-m)};
\end{aligned}
\tag{7.3}
$$

2. In a so-called integrable form

$$
\mathsf{K}_{\alpha,\xi}(x,y) = \frac{\cos(\pi\nu)}{\pi}\frac{\xi^{\frac{x+y}{2}}\Xi(x,y)}{\sqrt{x!y!(x-1)!(y-1)!}}\cdot\frac{P(x)Q(y)-Q(x)P(y)}{x^2-y^2},
\tag{7.4}
$$

where $Q(x) := \phi_1(x)$ and $P(x)$ can be written in one of the two following forms:

- $P^{(1)}(x) := x\,(2\phi_0(x) - \phi_1(x))$;
- $P^{(2)}(x) := \frac{x}{1+\xi}[2\widetilde{\phi}(x) - (1-\xi)\phi_1(x)]$.

Proof. The fact that the process $\mathsf{M}_{\alpha,\xi}$ is determinantal is guaranteed by Lemma 3.7.

On the other hand, the reduction formulas for the Pfaffian kernel $\mathbf{\Phi}_{\alpha,\xi}$ (Corollary 6.10) allow us to apply the argument of Proposition A.2 from Appendix. This implies that

$$
\mathrm{Pf}(\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]) = \det\left[\mathsf{K}_{\alpha,\xi}(x_k,x_j)\right]_{k,j=1}^{n},
$$

where $X = \{x_1,\dots,x_n\} \subset \mathbb{Z}_{>0}$ (with pairwise distinct x_j's), $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$ is the skew-symmetric $2n\times 2n$ matrix introduced in Theorem 6.1, and $\mathsf{K}_{\alpha,\xi}$ is related to $\mathbf{\Phi}_{\alpha,\xi}$ as

$$
\mathsf{K}_{\alpha,\xi}(x,y) = \frac{2\sqrt{xy}}{x+y}\mathbf{\Phi}_{\alpha,\xi}(x,-y), \qquad x,y\in\mathbb{Z}_{>0}.
\tag{7.5}
$$

This gives an argument (independently of Lemma 3.7) that the process $\mathsf{M}_{\alpha,\xi}$ is determinantal. Moreover, this also provides us with explicit formulas for the kernel $\mathsf{K}_{\alpha,\xi}$. Namely, claim 1 of the present Theorem directly follows from the expression of $\mathbf{\Phi}_{\alpha,\xi}$ as a series (6.18) and from the explicit formulas for the matrix elements (Proposition 6.7).

Claim 2 requires certain transformations of the hypergeometric functions. First, a direct computation using (7.5) and Propositions 6.7 and 6.9 shows that

$$
\begin{aligned}
\mathsf{K}_{\alpha,\xi}(x,y) = {} & \frac{\cos(\pi\nu(\alpha))}{\pi}\frac{\xi^{\frac{x+y}{2}}\Xi(x,y)}{(x^2-y^2)\sqrt{x!y!(x-1)!(y-1)!}}\times \\
& \times\left\{\left(x\phi_0(x) - \frac{\xi\alpha}{(1-\xi)^2}\frac{\phi_2(x)}{x+1}\right)\phi_1(y) - \left(y\phi_0(y) - \frac{\xi\alpha}{(1-\xi)^2}\frac{\phi_2(y)}{y+1}\right)\phi_1(x)\right\}.
\end{aligned}
$$

Here have also used the well-known identity

$$\frac{1}{\Gamma(\frac{1}{2}+\nu(\alpha))\Gamma(\frac{1}{2}-\nu(\alpha))} = \frac{\cos(\pi\nu(\alpha))}{\pi}.$$

Next, identities [Erd53, 2.9.(2)] and [Erd53, 2.8.(45)] for the hypergeometric function imply the following three-term relations for $\phi_j(x)$:

$$\begin{aligned} \tfrac{\xi(\alpha+j(j-1))}{1-\xi}\phi_{j+1}(x) + (1-\xi)(j+x)(j+x-1)\phi_{j-1}(x) \\ - (j+x)\big(j+x-1+\xi(j-x-1)\big)\phi_j(x) = 0, \end{aligned} \tag{7.6}$$

where $j = 1, 2, \dots$. When $j = 1$, (7.6) is equivalent to

$$\frac{\xi\alpha}{(1-\xi)^2}\frac{\phi_2(x)}{x+1} = x\left(\phi_1(x) - \phi_0(x)\right).$$

This immediately gives (7.4) with $P^{(1)}$. To obtain (7.4) with $P^{(2)}$, we apply the identity

$$\phi_0(x) = \frac{\widetilde{\phi}(x)}{1+\xi} - \frac{\xi(1+2\nu(\alpha)-x(1-\xi))}{x(1-\xi^2)}\phi_1(x),$$

which is a combination of 2.8(38), 2.8(39) and 2.9(2) in [Erd53]. Therefore, $P^{(2)}(x) = P^{(1)}(x) + c\,Q(x)$, where c does not depend on x, which means that the kernel (7.4) with $P^{(1)}$ is identical to the one with $P^{(2)}$. This concludes the proof. □

We call $\mathsf{K}_{\alpha,\xi}$ the *hypergeometric-type* kernel.

Remark 7.2. The form (7.4) of the kernel $\mathsf{K}_{\alpha,\xi}$ is called *integrable* because the operator (7.4) in $\ell^2(\mathbb{Z}_{>0})$ can be viewed as a discrete analogue of an integrable operator (if we take x^2 and y^2 as variables). About integrable operators, e.g., see [IIKS90], [Dei99]. Discrete integrable operators are discussed in [Bor00] and [BO00, §6].

Remark 7.3. Formulas (7.3) and (7.4) for the kernel $\mathsf{K}_{\alpha,\xi}$ together with relation (7.5) between $\mathsf{K}_{\alpha,\xi}$ and $\mathbf{\Phi}_{\alpha,\xi}$ and with the reduction formulas for $\mathbf{\Phi}_{\alpha,\xi}$ (Corollary 6.10) allow to give explicit expressions for the Pfaffian hypergeometric-type kernel $\mathbf{\Phi}_{\alpha,\xi}(x,y)$ for all $x, y \in \mathbb{Z}_{\ne 0}$ in terms of the Gauss hypergeometric function. We do now write these expressions down because they do not give more information compared to (6.18) and (6.22). The expression (6.18) for $\mathbf{\Phi}_{\alpha,\xi}(x,y)$ as a series also has a dynamical counterpart, see §9.4 and §10.1.

Now let us discuss how Theorem 7.1 is related to Lemma 3.7 that guarantees that the process $\mathsf{M}_{\alpha,\xi}$ is determinantal. Let $\mathsf{L}_{\alpha,\xi}$ be the $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ matrix defined as follows:

$$\mathsf{L}_{\alpha,\xi}(x,y) := \frac{\cos(\pi\nu(\alpha))}{\pi}\frac{\xi^{\frac{x+y}{2}}\,\Xi(x,y)}{(x+y)\sqrt{x!y!(x-1)!(y-1)!}}, \qquad x, y \in \mathbb{Z}_{>0}. \tag{7.7}$$

Clearly, this is the L-kernel (3.7), where $w(x) = w_{\alpha,\xi}(x)$ is given by (2.7). Thus, $\mathsf{L}_{\alpha,\xi}$ corresponds to our point process $\mathsf{M}_{\alpha,\xi}$ in the sense that the probabilities of the configurations under the measure $\mathsf{M}_{\alpha,\xi}$ on $\mathsf{Conf}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ have the form

$$\mathsf{M}_{\alpha,\xi}(\lambda) = \frac{\det \mathsf{L}_{\alpha,\xi}(\lambda)}{\det(1+\mathsf{L}_{\alpha,\xi})}, \qquad \lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{Z}_{>0},$$

where by $\mathsf{L}_{\alpha,\xi}(\lambda)$ we denote the submatrix $[\mathsf{L}_{\alpha,\xi}(\lambda_k, \lambda_j)]_{k,j=1}^{\ell}$.[15]

Proposition 7.4 (cf. Lemma 3.7)**.** *The hypergeometric-type kernel has the form*

$$\mathsf{K}_{\alpha,\xi} = \mathsf{L}_{\alpha,\xi}(1+\mathsf{L}_{\alpha,\xi})^{-1}, \tag{7.8}$$

where $\mathsf{L}_{\alpha,\xi}$ *is given by* (7.7)*.*

Idea of proof. We must show that $\mathsf{K}_{\alpha,\xi} + \mathsf{K}_{\alpha,\xi}\mathsf{L}_{\alpha,\xi} - \mathsf{L}_{\alpha,\xi} = 0$ (the equality of operators in $\ell^2(\mathbb{Z}_{>0})$). Using the expression (7.4) for $\mathsf{K}_{\alpha,\xi}$, we see that this is equivalent to the following system of relations:

$$\begin{aligned}\frac{P(x)Q(y)-Q(x)P(y)}{x-y} &- 1 \\ &+ (x+y)\sum_{k=1}^{\infty} \frac{\xi^k(\frac12+\nu(\alpha))_k(\frac12-\nu(\alpha))_k}{(x+k)(y+k)k!(k-1)!}\,\frac{P(x)Q(k)-Q(x)P(k)}{x-k} = 0,\end{aligned} \tag{7.9}$$

where $x, y \in \mathbb{Z}_{>0}$.

The expression $(P(x)Q(y) - Q(x)P(y))/(x-y)$ is continuous in (x,y) for x and y lying in a suitable neighborhood U of $\mathbb{Z}_{>0}$ in $\mathbb{C}$, and so when $x = y$, we do not get any singularities. Moreover, every summand in the series in (7.9) is also continuous for x and k lying in the same neighborhood. Observe that this series converges rapidly, because the factor $(P(x)Q(k) - Q(x)P(k))/(x-k)$ is bounded in $k \in \mathbb{Z}_{>0}$ and

$$\frac{\xi^k(\frac12+\nu(\alpha))_k(\frac12-\nu(\alpha))_k}{(x+k)(y+k)k!(k-1)!} \sim \mathrm{Const}\cdot\frac{\xi^k}{k^2}, \qquad k \to +\infty.$$

Therefore, the series converges uniformly in (x,y) belonging to compact subsets of $U \times U$, and hence represents a continuous function in (x,y). Thus, it remains to prove (7.9) for distinct nonintegral x and y. We have the following decomposition:

$$\frac{1}{(x+k)(y+k)(x-k)} = -\frac{\big(2x(x-y)\big)^{-1}}{k+x} - \frac{\big(2x(x+y)\big)^{-1}}{k-x} + \frac{\big(x^2-y^2\big)^{-1}}{k+y}.$$

Thus, the sum in (7.9) is reduced to the following three sums which are computed

[15] Note that this implies that $\det(1+\mathsf{L}_{\alpha,\xi}) = (1-\xi)^{-\alpha/2}$.

similarly to the Appendix in [BO00]:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\xi^k(\frac12+\nu(\alpha))_k(\frac12-\nu(\alpha))_k}{k!(k-1)!}\frac{k\phi_0(k)}{u+k} &= u(\phi_1(u)-\phi_0(u));\\
\sum_{k=1}^{\infty}\frac{\xi^k(\frac12+\nu(\alpha))_k(\frac12-\nu(\alpha))_k}{k!(k-1)!}\frac{k\phi_1(k)}{u+k} &= u(\phi_1(u)-1)+\frac{\alpha\xi}{1-\xi};\\
\sum_{k=1}^{\infty}\frac{\xi^k(\frac12+\nu(\alpha))_k(\frac12-\nu(\alpha))_k}{k!(k-1)!}\frac{\phi_1(k)}{u+k} &= 1-\phi_1(u).
\end{aligned}$$

Note that we view the above formulas as equalities of meromorphic functions in u. Using these formulas and the mentioned above identities for the Gauss hypergeometric function (from [Erd53]) it is not hard to see that (7.9) holds for distinct nonintegral x and y, and hence also holds for all $x,y\in\mathbb{Z}_{>0}$. □

7.2 Double contour integral representations

Here we obtain two double contour integral representations for the kernel $\mathsf{K}_{\alpha,\xi}$. Formulas of this type are useful in certain limit transitions, e.g., see [Oko02, BO06a, Ols09]. We use the contour integral representation for the Gauss hypergeometric function which is due to Borodin and Olshanski [BO06a, Lemma 2.2]. Let us rewrite this representation in terms of our functions $\phi_j(x)$ (7.1):

Lemma 7.5. *For all $x,j\in\mathbb{Z}_{>0}$ such that $x+j\ge 1$, we have the following four contour integral representations:*

$$\begin{aligned}
\phi_j(x) &= (1-\xi)^{-\frac12+j\pm\nu(\alpha)}\xi^{\frac12(1-j-x)}\frac{\Gamma(\frac32-j\pm\nu(\alpha))\Gamma(j+x)}{\Gamma(\frac12+x\pm\nu(\alpha))}\times\\
&\times\frac{1}{2\pi\sqrt{-1}}\oint_{\{w\}}\left(1-\sqrt{\xi}w\right)^{-\frac32+j\mp\nu(\alpha)}\left(1-\frac{\sqrt{\xi}}{w}\right)^{\frac12-j\mp\nu(\alpha)}\frac{dw}{w^{j+x}};
\end{aligned}\tag{7.10}$$

$$\begin{aligned}
\phi_j(x) &= (1-\xi)^{-\frac12+j\pm\nu(\alpha)}\xi^{\frac12(1-j-x)}\frac{\Gamma(\frac12-x\pm\nu(\alpha))\Gamma(j+x)}{\Gamma(-\frac12+j\pm\nu(\alpha))}\times\\
&\times\frac{1}{2\pi\sqrt{-1}}\oint_{\{w\}}\left(1-\sqrt{\xi}w\right)^{-\frac12+x\mp\nu(\alpha)}\left(1-\frac{\sqrt{\xi}}{w}\right)^{-\frac12-x\mp\nu(\alpha)}\frac{dw}{w^{j+x}}.
\end{aligned}\tag{7.11}$$

Here all the contours $\{w\}$ are arbitrary simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside (here we assume $0<\xi<1$). The branches of all the functions of the form $(1-z)^\gamma$ above are chosen in such a way that the expansion around $z=0$ looks as $(1-z)^\gamma=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!}z^k$.[16]

Proof. First formulas (7.10) are obtained by a direct application of [BO06a, Lemma 2.2] to the functions $\phi_j(x)$ (7.1). To get second formulas (7.11), one

[16]The same branches are taken in Propositions 7.6 and 7.7 below.

should first apply transformation [Erd53, 2.9.(2)] to $\phi_j(x)$ which results in the identity

$$\phi_j(x) = (1-\xi)^{j-x-1}{}_2F_1\left(\begin{matrix}\frac{1}{2}-\nu(\alpha)+x, \ \frac{1}{2}+\nu(\alpha)+x \\ x+j\end{matrix}\ \middle|\ \frac{\xi}{\xi-1}\right) \\ = (1-\xi)^{j-x-1}\phi_{x+1}(j-1), \tag{7.12}$$

and then use [BO06a, Lemma 2.2] again. ☐

To shorten the notation, set

$$\mathsf{k}(x) := \frac{\sqrt{\Gamma(\frac{1}{2}+\nu(\alpha)+x)\Gamma(\frac{1}{2}-\nu(\alpha)+x)}}{\Gamma(\frac{1}{2}+\nu(\alpha)+x)}, \qquad x\in\mathbb{Z}. \tag{7.13}$$

Here again the expression under the square root is real positive for any $x \in \mathbb{Z}$ (see Definition 2.1). Note that (6.21) implies that $\mathsf{k}(-x) = \frac{(-1)^x}{\mathsf{k}(x)}$ for all $x \in \mathbb{Z}$.

Below in this subsection we will omit the argument α in $\nu(\alpha)$ to shorten the formulas.

Proposition 7.6. *For any $x, y \in \mathbb{Z}_{>0}$ we have*

$$\begin{aligned}\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\mathsf{K}_{\alpha,\xi}(x,y) = \frac{2\sqrt{xy}}{x+y}\frac{1}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu}\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{\frac{1}{2}+\nu}\times \\ \times\left(1-w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu}\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{\frac{1}{2}-\nu}\frac{w_1^{-x}w_2^{-y}}{w_1w_2-1}dw_1dw_2 \\ -\frac{\sqrt{xy}}{x+y}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu}\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu}\times \\ \times\left(1-w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu}\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu}\frac{dw_1dw_2}{w_1^{x+1}w_2^{y+1}}.\end{aligned}$$

The contours $\{w_1\}$ and $\{w_2\}$ go around 0 and $\sqrt{\xi}$ in positive direction leaving $1/\sqrt{\xi}$ outside. Moreover, in the first integral we have to impose an extra condition: the contour $\{w_1^{-1}\}$ lies in the interior of the contour $\{w_2\}$.

Proof. We use the expression for $\mathsf{K}_{\alpha,\xi}$ as a sum (7.3) and write $\phi_{m+1}(x)$ and $\phi_{m+1}(y)$ as (7.10) with "$-$" and the "$+$" signs, respectively. We get

$$\begin{aligned}\mathsf{K}_{\alpha,\xi}(x,y) = \frac{2\Xi(x,y)\sqrt{xy}}{x+y}\frac{1-\xi}{\Gamma(\frac{1}{2}+x-\nu)\Gamma(\frac{1}{2}+y+\nu)}\sum_{m=0}^{\infty}\frac{2^{-\delta(m)}}{(2\pi\sqrt{-1})^2}\Bigg\{ \\ \oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-\sqrt{\xi}w_1\right)^{-\frac{1}{2}+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu}\left(1-\sqrt{\xi}w_2\right)^{-\frac{1}{2}-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu} \\ \times\left(\frac{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}\right)^m\frac{dw_1dw_2}{w_1^{x+1}w_2^{y+1}}\Bigg\}.\end{aligned}$$

We can choose the contours $\{w_1\}$ and $\{w_2\}$ contained in the domain $|w| > 1$ (see Lemma 7.5). For w_1 and w_2 on such contours, we have[17]

$$\left|\frac{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}\right| \le r < 1,$$

so that we can interchange the summation and the integration. We have

$$\sum_{m=0}^{\infty} 2^{-\delta(m)}\left(\frac{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}\right)^m = \frac{(1-\frac{\sqrt{\xi}}{w_1})(1-\frac{\sqrt{\xi}}{w_2})w_1w_2}{(1-\xi)(w_1w_2-1)} - \frac{1}{2}.$$

Thus, we get an expression for $\mathsf{K}_{\alpha,\xi}$ as a difference of two double contour integrals. After that we can relax the assumptions on the contours in those integrals to the assumptions given in the claim of Proposition.

Finally, observe that

$$\begin{aligned}\frac{\sqrt{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+x-\nu)}}{\Gamma(\frac{1}{2}+x-\nu)}& \\ = \frac{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+x-\nu)}{\Gamma(\frac{1}{2}+x-\nu)\sqrt{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+x-\nu)}} &= \frac{1}{\mathsf{k}(x)},\end{aligned} \tag{7.14}$$

this concludes the proof. □

Proposition 7.7. *For all $x, y \in \mathbb{Z}_{>0}$ we have*

$$\begin{aligned}\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\mathsf{K}_{\nu,\xi}(x,y) = \frac{\sqrt{xy}}{x+y}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu+x}\times\\ \times\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu-x}\left(1-w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu+y}\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu-y}\times\\ \times w_1^{-x}w_2^{-y}\frac{w_1w_2+1}{w_1w_2-1}\cdot\frac{dw_1dw_2}{w_1w_2}.\end{aligned}$$

Here the contours $\{w_1\}$ and $\{w_2\}$ are as in the first integral in Proposition 7.6.

Proof. This is proved in the same manner as Proposition 7.6: we start with expression (7.3) for the kernel $\mathsf{K}_{\alpha,\xi}(x,y)$, but now we write $\phi_{m+1}(x)$ and $\phi_{m+1}(y)$ in the form (7.11) with "−" and "+" signs, respectively.

Using (6.21), we can write the kernel $\mathsf{K}_{\alpha,\xi}(x,y)$ as:

$$\begin{aligned}\mathsf{K}_{\alpha,\xi}(x,y) = \frac{2\Xi(x,y)\sqrt{xy}}{x+y}\frac{(-1)^{x+y}(1-\xi)}{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+y-\nu)}\sum_{m=0}^{\infty}\frac{2^{-\delta(m)}}{(2\pi\sqrt{-1})^2}\Bigg\{\\ \oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-\sqrt{\xi}w_1\right)^{-\frac{1}{2}+\nu+x}\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu-x}\left(1-\sqrt{\xi}w_2\right)^{-\frac{1}{2}-\nu+y}\times\\ \times\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu-y}(w_1w_2)^{-m}\frac{dw_1dw_2}{w_1^{x+1}w_2^{y+1}}.\end{aligned}$$

[17] See also the proof of Lemma 3.9 in [BO06a].

We again choose the contours $\{w_1\}$ and $\{w_2\}$ contained in the domain $|w| > 1$. Therefore, $|(w_1w_2)^{-1}| \le r < 1$ for w_1 and w_2 on such contours, and we can interchange the summation and the integration. For the sum we have

$$\sum_{m=0}^{\infty} 2^{-\delta(m)}(w_1w_2)^{-m} = \frac{1}{2} \cdot \frac{w_1w_2+1}{w_1w_2-1}.$$

Now it remains to deal with the factor $\frac{(-1)^{x+y}\Xi(x,y)}{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+y-\nu)}$ in front of the double contour integral. Let us apply transformation (6.21) to all three gamma functions in the expression $\frac{(-1)^x\sqrt{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+x-\nu)}}{\Gamma(\frac{1}{2}+x+\nu)}$. We have

$$\frac{(-1)^x\sqrt{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+x-\nu)}}{\Gamma(\frac{1}{2}+x+\nu)} = \sqrt{\frac{\Gamma(\frac{1}{2}+\nu)^2\Gamma(\frac{1}{2}-\nu)^2}{\Gamma(\frac{1}{2}-x+\nu)\Gamma(\frac{1}{2}-x-\nu)}} \cdot \frac{\Gamma(\frac{1}{2}-x-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}.$$

For our values of ν we have $\Gamma(\frac{1}{2}+\nu+k)\Gamma(\frac{1}{2}-\nu+k) > 0$ for all $k \in \mathbb{Z}$, therefore, using (7.14) we finally get

$$\frac{(-1)^x\sqrt{\Gamma(\frac{1}{2}+x+\nu)\Gamma(\frac{1}{2}+x-\nu)}}{\Gamma(\frac{1}{2}+x-\nu)} = \mathsf{k}(-x).$$

For a similar expression involving y we have

$$\frac{(-1)^y\sqrt{\Gamma(\frac{1}{2}+y+\nu)\Gamma(\frac{1}{2}+y-\nu)}}{\Gamma(\frac{1}{2}+y-\nu)} = \frac{1}{\mathsf{k}(-y)}.$$

Combining the above two identities with the definition of $\Xi(x,y)$ (7.2), we conclude the proof. □

Remark 7.8. The expressions $\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\mathsf{K}_{\alpha,\xi}(x,y)$ and $\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\mathsf{K}_{\alpha,\xi}(x,y)$ of Propositions 7.6 and 7.7, respectively, are so-called gauge transformations of the original correlation kernel $\mathsf{K}_{\alpha,\xi}$, that is, they are related to $\mathsf{K}_{\alpha,\xi}$ by a conjugation by diagonal matrices. This means that each of the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ matrices $\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\mathsf{K}_{\alpha,\xi}(x,y)$ and $\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\mathsf{K}_{\alpha,\xi}(x,y)$ can also serve as a correlation kernel for the point process $\mathsf{M}_{\alpha,\xi}$. Clearly, both these kernels are not symmetric.

7.3 Plancherel degeneration

Here we consider the Plancherel degeneration (2.8) of the hypergeometric-type kernel $\mathsf{K}_{\alpha,\xi}$ studied above in this section. We denote by J_k the Bessel function (of the first kind) of order k and argument $2\sqrt{\theta}$:

$$J_k := J_k(2\sqrt{\theta}) = \sum_{r=0}^{\infty} \frac{(-1)^r\theta^{r+\frac{k}{2}}}{r!\Gamma(r+k+1)}, \qquad k \in \mathbb{Z}. \tag{7.15}$$

Theorem 7.9. *Under the Plancherel degeneration* (2.8)*, the point processes* $\mathsf{M}_{\alpha,\xi}$ *on* $\mathbb{Z}_{>0}$ *converge to the poissonized Plancherel measure* Pl_θ*. This is a determinantal point process on* $\mathbb{Z}_{>0}$ *supported by finite configurations. The correlation kernel* $\mathsf{K}_\theta(x,y)$ *of* Pl_θ *can be expressed through the Bessel functions in two ways: as a series*

$$\mathsf{K}_\theta(x,y)=\frac{2\sqrt{xy}}{x+y}\sum_{m=0}^{\infty}2^{-\delta(m)}J_{m+x}J_{m+y}, \tag{7.16}$$

and in an integrable form (*see also Remark* 7.2):

$$\mathsf{K}_\theta(x,y)=\frac{2\sqrt{xy}}{x^2-y^2}\Big(\sqrt{\theta}J_{x-1}J_y-\sqrt{\theta}J_{y-1}J_x-\tfrac{1}{2}(x-y)J_xJ_y\Big). \tag{7.17}$$

We will discuss three ways to prove this Theorem, but let us first show how one can turn formula (7.16) into (7.17).

Equivalence of (7.16) *and* (7.17). There are three-term relations between the Bessel functions of the same argument (e.g., see [Erd53, 7.2.(56)]) that in our case have the form:

$$J_{k+1}-\frac{k}{\sqrt{\theta}}J_k+J_{k-1}=0,\qquad k\in\mathbb{Z}. \tag{7.18}$$

Using these relations, one can show that (see also [BOO00, Prop. 2.9])

$$\sum_{m=0}^{\infty}2^{-\delta(m)}J_{m+x}J_{m+y}=\sqrt{\theta}\frac{J_{x-1}J_y-J_xJ_{y-1}}{x-y}-\frac{1}{2}J_xJ_y. \tag{7.19}$$

This implies the equivalence of (7.16) and (7.17). □

Proof of Theorem 7.9. I. Formulas (7.16) and (7.17) for K_θ can be obtained from the corresponding formulas (7.3) and (7.4) for $\mathsf{K}_{\alpha,\xi}$ via the Plancherel degeneration (2.8). Namely, under the Plancherel degeneration the functions $\phi_j(x)$ (7.1) have the following limits:

$$\phi_j(x)\longrightarrow\Gamma(j+x)\theta^{\frac{1-j-x}{2}}J_{x+j-1},\qquad x+j\geq 1. \tag{7.20}$$

This is obtained by a termwise limit from the hypergeometric series for $\phi_j(x)$ (this series converges rapidly for fixed x and j). From this one can readily derive (7.16) and (7.17). □

Note that the three-term relations for the Bessel functions (7.18) are obtained via the Plancherel degeneration from the corresponding three-term relations for the functions $\phi_j(x)$ (7.6). After taking the limit in (7.6), one should set $k=x+j-1$.

Proof of Theorem 7.9. II. Another way of proof is to observe that the point process Pl_θ on $\mathbb{Z}_{>0}$ is again an L-ensemble (see Lemma 3.7). Denote by L_θ the following $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ matrix

$$\mathsf{L}_\theta(x,y)=\frac{\theta^{\frac{x+y}{2}}}{x+y}\frac{1}{\sqrt{x!y!(x-1)!(y-1)!}}.$$

This is the Plancherel degeneration of $\mathsf{L}_{\alpha,\xi}$ (7.7), that is, L_θ is the matrix L (3.7) with $w(x) = w_\theta(x) = \frac{\theta^x}{2(x!)^2}$. The probabilities of configurations under the measure Pl_θ on $\mathsf{Conf}_{\text{fin}}(\mathbb{Z}_{>0})$ have the form

$$\mathsf{Pl}_\theta(\lambda) = \frac{\det \mathsf{L}_\theta(\lambda)}{\det(1 + \mathsf{L}_\theta)},$$

where $\lambda = \{\lambda_1, \dots, \lambda_\ell\}$ is any finite subset of $\mathbb{Z}_{>0}$ and $\mathsf{L}_\theta(\lambda)$ denotes the submatrix $[\mathsf{L}_\theta(\lambda_k, \lambda_j)]_{k,j=1}^{\ell}$. Note that this implies that $\det(1 + \mathsf{L}_\theta) = e^{\theta/2}$.

To prove the Theorem it suffices (by Lemma 3.7) to show that the kernel K_θ has the form $\mathsf{K}_\theta = \mathsf{L}_\theta(1 + \mathsf{L}_\theta)^{-1}$. This is equivalent to a certain identity for the Bessel functions that is readily verified using, e.g., Lemma 2.4 in [BOO00]. □

Proof of Theorem 7.9. III. This way of proving the Theorem uses the result of Matsumoto [Mat05, Thm. 3.1] that states that the correlation functions $\rho_\theta^{(n)}(x_1, \dots, x_n)$ of the poissonized Plancherel measure Pl_θ have Pfaffian form similar to (6.3) with the Pfaffian kernel $\mathbf{\Phi}_\theta$ given by

$$\mathbf{\Phi}_\theta(x, y) = \frac{1}{2}(-1)^{x\wedge 0 + y\wedge 0}\,[z^x w^y]\left\{ e^{\sqrt{\theta}(z-1/z)} e^{\sqrt{\theta}(w-1/w)} \frac{z-w}{z+w} \right\}, \qquad x, y \in \mathbb{Z},$$

where $[z^x w^y]\{\dots\}$ denotes the coefficient by $z^x w^y$ in the expression $\{\dots\}$. The function $e^{\sqrt{\theta}(z-1/z)}$ is the generating series for the Bessel functions $J_k(2\sqrt{\theta})$ [Erd53, 7.2.(25)], so one can readily get:

$$\mathbf{\Phi}_\theta(x, y) = (-1)^{x\wedge 0 + y\vee 0} \sum_{m=0}^{\infty} 2^{-\delta(m)} J_{m+x} J_{m-y}. \tag{7.21}$$

For the Pfaffian kernel $\mathbf{\Phi}_\theta$ we have the same reduction formulas as in Corollary 6.10. They can be verified independently using (7.19) and the fact that $J_{-k} = (-1)^k J_k$, or obtained from the reduction formulas for $\mathbf{\Phi}_{\alpha,\xi}$ (because $\mathbf{\Phi}_\theta$ is the Plancherel degeneration of $\mathbf{\Phi}_{\alpha,\xi}$ which can be readily checked). Thus, from Proposition A.2 it follows that the poissonized Plancherel measure is a determinantal point process with the correlation kernel given by

$$\mathsf{K}_\theta(x, y) = \frac{2\sqrt{xy}}{x+y} \mathbf{\Phi}_\theta(x, -y) = \frac{2\sqrt{xy}}{x+y} \sum_{m=0}^{\infty} 2^{-\delta(m)} J_{m+x} J_{m+y},$$

where $x, y \in \mathbb{Z}_{>0}$. Note that for such x and y we have $x \wedge 0 = (-y) \vee 0 = 0$. This is formula (7.16) for K_θ. □

Theorems 7.1 and 7.9 and Propositions 7.6 and 7.7 constitute Theorem 1 from §2.

8 Markov processes

In this section we explain the construction of the dynamical model described in §2.2.

The construction of our Markov processes on the Schur graph is similar to Borodin-Olshanski's construction [BO06a] of the Markov processes on the Young graph that preserve the z-measures. In contrast to [BO06a], we restrict ourselves to the stationary (time homogeneous) case, that is, we assume that the parameter ξ does not vary in time. The introduction of the non-stationary processes in [BO06a] was motivated by the technique of handling the stationary case (in particular, by the method of the computation of the dynamical correlation functions). The technique that we use in the present paper does not require dealing with non-stationary processes.

8.1 Defining Markov processes in terms of jump rates

Let us first recall some basic notions and facts concerning Markov processes. Let E be a finite or countable space. Assume that we have a continuous-time homogeneous Markov process on E with the time parameter $t \in \mathbb{R}_{\geq 0}$. By $(\mathbb{P}(t))_{t\geq 0}$ denote the *transition probabilities* of this Markov process. That is, each $\mathbb{P}(t)$ is a $E \times E$ matrix, and $\mathbb{P}_{ab}(t)$ (where $a, b \in E$) is the probability that the process starting from the state a will be at the state b after time t. The matrices $(\mathbb{P}(t))_{t\geq 0}$ have the following properties:

(P1) $\mathbb{P}_{ab}(t) \geq 0$ for all $t \geq 0$ and $\mathbb{P}_{ab}(0) = \delta_{ab}$ for all $a, b \in E$;

(P2) $\sum_{b\in E} \mathbb{P}_{ab}(t) = 1$ for all $a \in E$;

(P3) $\mathbb{P}(t+s) = \mathbb{P}(t)\,\mathbb{P}(s)$ for $t \geq s \geq 0$, or, in matrix form,

$$\mathbb{P}_{ab}(t+s) = \sum_{c\in E} \mathbb{P}_{ac}(t)\,\mathbb{P}_{cb}(s), \qquad a, b \in E.$$

This is called the *Chapman-Kolmogorov equation*.

Assume that there exists a $E \times E$ matrix $\mathbb{Q}$ such that

$$\mathbb{P}_{ab}(t) = \delta_{ab} + \mathbb{Q}_{ab} \cdot t + o(t), \qquad t \to 0. \tag{8.1}$$

The elements of the matrix $\mathbb{Q}$ are called the *jump rates*. Note that (8.1) implies that each $\mathbb{P}_{ab}(t)$ is continuous at $t = 0$:

(P4) $\lim_{t\downarrow 0} \mathbb{P}_{ab}(t) = \delta_{ab}$, $a, b \in E$.

A family of matrices satisfying (P1)–(P4) is a (*continuous*) *stochasic matrix semigroup*. One can say that $\mathbb{Q}$ is the infinitesimal matrix of this semigroup, that is, $\mathbb{Q} = \frac{d}{dt}\mathbb{P}(t)\big|_{t=0}$. From (8.1) it is clear that

(Q1) $\mathbb{Q}_{ab} \geq 0$ for $a \neq b$ and $\mathbb{Q}_{aa} \leq 0$.

We assume that the jump rates also have the property

(Q2) $\mathbb{Q}_{aa} = -\sum_{b\neq a} \mathbb{Q}_{ab}$ for all $a \in E$.

The property (Q2) implies (e.g., see [KT99, Ch. 14.2]) that the jump rates $\mathbb{Q}$ and the transition probabilities $\mathbb{P}(t)$ are related to each other via the system of *Kolmogorov's backward equations*:

$$\frac{d\,\mathbb{P}_{ab}(t)}{dt} = \sum_{c\in E} \mathbb{Q}_{ac}\mathbb{P}_{cb}(t), \qquad a, b \in E, \tag{8.2}$$

with the initial conditions

$$\mathbb{P}_{ab}(0) = \delta_{ab}, \qquad a, b \in E. \tag{8.3}$$

We would like to start with the jump rates $\mathbb{Q}$ with properties (Q1)–(Q2) and obtain a stochasic matrix semigroup $(\mathbb{P}(t))$ by solving backward equations (8.2)–(8.3). A solution in a wider class of *substochasic matrix semigroups* (when the condition (P2) is replaced by $\sum_{b\in E}\mathbb{P}_{ab}(t) \le 1$) always exists. Among all possible substochasic solutions there is a distinguished *minimal* solution $(\bar{\mathbb{P}}(t))_{t\ge0}$:

$$\mathbb{P}_{ab}(t) \ge \bar{\mathbb{P}}_{ab}(t) \text{ for any } t \ge 0 \text{ and } a, b \in E,$$

where $(\mathbb{P}(t))$ is any substochasic solution. A minimal solution can be constructed using the approximation method (e.g., see [KT99, Ch. 14.3]). If the minimal solution is stochasic, that is,

$$\sum_{b\in E} \bar{\mathbb{P}}_{ab}(t) = 1, \qquad a \in E,$$

then it is the unique solution of (8.2)–(8.3) in the class of substochasic matrices. About solving Kolmogorov's backward equations see also [GSK04, Ch. III.2]

If the system of backward equations (8.2)–(8.3) has a unique solution $(\mathbb{P}(t))_{t\ge0}$ (or, which is equivalent, the minimal solution of the system of backward equations is stochasic), we say that the jump rates $\mathbb{Q}$ define a continuous-time homogeneous Markov process on E (with transition probabilities $\mathbb{P}(t)$) that can start from any point and any probability distribution. A common suffcient condition for this is $\sup_{a\in E}|\mathbb{Q}_{aa}| < +\infty$, which does not hold in our case.

Let us recall another useful suffcient condition for the minimal solution of the system of backward equations (8.2)–(8.3) to be stochasic. We formulate this condition as in [BO06a, Prop. 4.3], it also can be derived from the discussion of [GSK04, Ch. III.2].

Let $X \subset E$ be a finite set and $a \in X$. By $\tau_{a,X}$ denote the time of the first exit from X of the process starting at a. Though we do not know yet if the process itself is uniquely determined by its jump rates $\mathbb{Q}$, the random variable $\tau_{a,X}$ can be constructed from $\mathbb{Q}$ as follows. Contract all the states $b \in E \setminus X$ into one absorbing state $\tilde{b}$ with $\mathbb{Q}_{\tilde{b},c} = 0$ for all $c \in X \cup \{\tilde{b}\}$. On the finite set $X \cup \{\tilde{b}\}$ the backward equations have a unique solution $(\tilde{\mathbb{P}}(t))_{t\ge0}$,[18] where

[18] Indeed, because the state space is finite, one can readily see that the minimal solution is stochasic.

each $\tilde{\mathbb{P}}(t)$ is a matrix with rows and columns indexed by the set $X \cup \{\tilde{b}\}$. The distribution of the random variable $\tau_{a,X}$ is

$$\mathsf{Prob}\,\{\tau_{a,X} \le t\} = \tilde{\mathbb{P}}_{a,\tilde{b}}(t).$$

Proposition 8.1 ([BO06a, Prop. 4.3])**.** *If for any $a \in E$, any $t \ge 0$, and any $\epsilon > 0$ there exists a finite set $X(\epsilon) \subset E$ such that*

$$\mathsf{Prob}\left\{\tau_{a,X(\epsilon)} \le t\right\} \le \epsilon,$$

then the minimal solution of the system of Kolmogorov's backward equations (8.2)–(8.3) *is stochasic.*

8.2 Birth and death processes

Here we discuss the underlying birth and death process on $\mathbb{Z}_{\ge 0}$ that is involved in the construction of the Markov processes on strict partitions (see §2.2).

A general birth and death process on $\mathbb{Z}_{\ge 0}$ is a continuous-time homogeneous Markov process with jump rates $\left\{ \mathfrak{q}_{k,j} \right\}_{k,j \in \mathbb{Z}_{\ge 0}}$ satisfying the conditions (Q1)–(Q2) from the previous subsection, with an additional property that $\mathfrak{q}_{k,j} = 0$ if $|k-j| > 1$. This means that from any point $n \ge 1$ of $\mathbb{Z}_{\ge 0}$ the process can jump only to the neighbor points $n-1$ and $n+1$, and that from 0 it can jump only to 1.

The following necessary and suffcient condition is well-known and can be deduced, e.g., from [GSK04, Ch. III.2, Thm. 4].

Proposition 8.2. *The minimal solution of the system of Kolmogorov's backward equations* (8.2)–(8.3) *for a birth and death process is stochasic iff*

$$\sum_{n=1}^{\infty} \left[\frac{1}{\mathfrak{q}_{n,n+1}} + \frac{\mathfrak{q}_{n,n-1}}{\mathfrak{q}_{n-1,n}\,\mathfrak{q}_{n,n+1}} + \dots + \frac{\mathfrak{q}_{n,n-1} \cdots \mathfrak{q}_{2,1}\,\mathfrak{q}_{1,0}}{\mathfrak{q}_{0,1}\,\mathfrak{q}_{1,2} \cdots \mathfrak{q}_{n-1,n}\,\mathfrak{q}_{n,n+1}} \right] = +\infty. \tag{8.4}$$

Now let us turn to our concrete situation and define the birth and death process that we use in the present paper. Its jump rates depend on our parameters $\alpha > 0$ and $0 < \xi < 1$ and are as follows:

$$\begin{aligned} \mathfrak{q}_{n,n+1} &:= \frac{\xi(n+\alpha/2)}{1-\xi}; \\ \mathfrak{q}_{n,n-1} &:= \frac{n}{1-\xi}; \\ \mathfrak{q}_{n,n} &:= -(\mathfrak{q}_{n,n+1} + \mathfrak{q}_{n,n-1}) = -\frac{\xi(n+\alpha/2)+n}{1-\xi}, \end{aligned} \tag{8.5}$$

where $n = 0, 1, 2, \dots$. All other jump rates are zero.

Proposition 8.3. *The jump rates* (8.5) *satisfy* (8.4).

Proof. We have for $j = 0, 1, \dots, n$:

$$\frac{\mathfrak{q}_{n,n-1} \cdots \mathfrak{q}_{j+1,j}}{\mathfrak{q}_{j,j+1} \cdots \mathfrak{q}_{n-1,n}\, \mathfrak{q}_{n,n+1}} = (1-\xi)\xi^{j-n-1} \frac{n!}{j!} \frac{(\alpha/2)_j}{(\alpha/2)_{n+1}},$$

and

$$\frac{\xi^{-n-1} n!}{(\alpha/2)_{n+1}} \sum_{j=0}^{n} \xi^j \frac{(\alpha/2)_j}{j!}$$
$$= \frac{\xi^{-n-1} n!}{(\alpha/2)_{n+1}} (1-\xi)^{-\alpha/2} - \frac{1}{n+1} {}_2F_1 \left(\begin{array}{c} 1, \ \frac{\alpha}{2}+n+1 \\ n+2 \end{array} \Big| \, \xi \right).$$

The first term grows exponentially fast in n because $\xi < 1$ and

$$\frac{n!}{(\alpha/2)_{n+1}} = \frac{\Gamma(\frac{\alpha}{2})\Gamma(n+1)}{\Gamma(\frac{\alpha}{2}+n+1)} \sim \Gamma\left(\frac{\alpha}{2}\right)(n+1)^{-\alpha/2}, \qquad n \to \infty.$$

On the other hand, the term $\frac{1}{n+1} {}_2F_1(1, \frac{\alpha}{2}+n+1; n+2 \,|\, \xi)$ goes to zero as $n \to +\infty$. Thus, the sum (8.4) diverges. ☐

Propositions 8.2 and 8.3 together with the facts from §8.1 imply that there exists a unique continuous-time Markov process on $\mathbb{Z}_{\ge 0}$ with the jump rates (8.5) that can start from any point and any probability distribution. This process preserves the negative binomial distribution $\pi_{\alpha,\xi}$ (2.4) on $\mathbb{Z}_{\ge 0}$ because $\pi_{\alpha,\xi} \circ \mathfrak{q} = 0$, or, in matrix form,

$$\sum_{k=0}^{\infty} \pi_{\alpha,\xi}(k)\, \mathfrak{q}_{k,j} = 0 \qquad \text{for all } j \in E. \tag{8.6}$$

Denote the equilibrium version of this process (that is, starting from the distribution $\pi_{\alpha,\xi}$) by $(\boldsymbol{n}_{\alpha,\xi}(t))_{t \ge 0}$. Since

$$\pi_{\alpha,\xi}(n)\, \mathfrak{q}_{n,n+1} = \pi_{\alpha,\xi}(n+1)\, \mathfrak{q}_{n+1,n} \qquad \text{for all } n \in \mathbb{Z}_{\ge 0}, \tag{8.7}$$

the process $\boldsymbol{n}_{\alpha,\xi}$ is reversible with respect to $\pi_{\alpha,\xi}$.

The transition probabilities of the process $\boldsymbol{n}_{\alpha,\xi}$ can be expressed through the Meixner orthogonal polynomials, see [BO06a, §4.3], and also [KM57], [KM58] for a much more general formalism.

Remark 8.4. From the discussion of [BO06a, §4] it follows that the process $\boldsymbol{n}_{\alpha,\xi}$ satisfies the hypotheses of Proposition 8.1. This fact will be used in the next subsection in construction of Markov processes on strict partitions that extend the processes $\boldsymbol{n}_{\alpha,\xi}$.

8.3 Markov processes on strict partitions

Here we define continuous-time Markov processes on the set $\mathbb{S}$ of all strict partitions depending on our parameters $\alpha > 0$ and $0 < \xi < 1$. These processes extend the birth and death processes $\boldsymbol{n}_{\alpha,\xi}$ discussed in the previous subsection.

The Markov processes on $\mathbb{S}$ are defined in terms of their jump rates that are as follows (here $\lambda \in \mathbb{S}_n$, $n = 0, 1, \dots$):

$$
\begin{aligned}
\mathbb{Q}_{\lambda,\varkappa} &:= \frac{\xi(n+\alpha/2)}{1-\xi} p_\alpha^\uparrow(n,n+1)_{\lambda,\varkappa}, && \text{where } \varkappa \searrow \lambda; \\
\mathbb{Q}_{\lambda,\mu} &:= \frac{n}{1-\xi} p^\downarrow(n,n-1)_{\lambda,\mu}, && \text{where } \mu \nearrow \lambda; \\
\mathbb{Q}_{\lambda,\lambda} &:= -\sum_{\varkappa\colon \varkappa\searrow\lambda} \mathbb{Q}_{\lambda,\varkappa} - \sum_{\mu\colon \mu\nearrow\lambda} \mathbb{Q}_{\lambda,\mu} \\
&\qquad = -\frac{\xi(\alpha/2+n)+n}{1-\xi}.
\end{aligned}
\tag{8.8}
$$

All other jump rates are zero. Here $p^\downarrow(n,n-1)$ and $p_\alpha^\uparrow(n,n+1)$ are the down and up transition kernels, respectively. Recall that (see §3) if $\lambda \in \mathbb{S}_n$, $\mu \nearrow \lambda$, and $\varkappa \searrow \lambda$, we have

$$
p^\downarrow(n,n-1)_{\lambda,\mu} = \frac{\dim_{\mathbb{S}} \mu}{\dim_{\mathbb{S}} \lambda}, \qquad p_\alpha^\uparrow(n,n+1)_{\lambda,\varkappa} = \frac{\dim_{\mathbb{S}} \lambda}{\dim_{\mathbb{S}} \varkappa} \cdot \frac{\mathsf{M}_{\alpha,n+1}(\varkappa)}{\mathsf{M}_{\alpha,n}(\lambda)},
$$

where $\dim_{\mathbb{S}}(\cdot)$ is given by (3.1) and $\{\mathsf{M}_{\alpha,n}\}$ is the multiplicative coherent system of measures on the Schur graph (§3.3).

Under the projection $\mathbb{S} \to \mathbb{Z}_{\ge 0}$, $\lambda \mapsto |\lambda|$, the $\mathbb{S} \times \mathbb{S}$ matrix $\mathbb{Q}$ (8.8) turns into the $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ matrix of jump rates of the birth and death process $\boldsymbol{n}_{\alpha,\xi}$ from §8.2.

Proposition 8.5. *The minimal solution of the system of Kolmogorov's backward equations* (8.2)–(8.3) *for the matrix* $\mathbb{Q}$ (8.8) *is stochasic.*

Proof. Let $n \le K$ be nonnegative integers. Consider the random variable $\tau_{n,\{0,\dots,K-1\}}$ for the birth and death process $\boldsymbol{n}_{\alpha,\xi}$, that is, the time of the first exit from $\{0, 1, \dots, K-1\}$ of the process with jump rates $\mathfrak{q}$ (8.5) starting at n.

Let $\lambda \in \mathbb{S}_n$. Observe that the time of the first exit from $\mathbb{S}_0 \cup \dots \cup \mathbb{S}_{K-1}$ of the process on strict partitions with jump rates $\mathbb{Q}$ (8.8) starting at λ has the same distribution as $\tau_{n,\{0,\dots,K-1\}}$. Applying Remark 8.4 and Proposition 8.1, we conclude the proof. ∎

Thus, the jump rates $\mathbb{Q}$ (8.8) uniquely define a continuous-time Markov process on $\mathbb{S}$ that can start from any point and any probability distribution.

Recall that the measures $\mathsf{M}_{\alpha,\xi}$ and $\{\mathsf{M}_{\alpha,n}\}_{n\in\mathbb{Z}_{\ge 0}}$ are related as

$$
\mathsf{M}_{\alpha,\xi}(\lambda) = \pi_{\alpha,\xi}(n) \mathsf{M}_{\alpha,n}(\lambda), \qquad \lambda \in \mathbb{S}_n, \tag{8.9}
$$

where $\pi_{\alpha,\xi}$ is the negative binomial distribution (2.4). Moreover, this measure $\pi_{\alpha,\xi}$ is invariant for the birth and death process $\boldsymbol{n}_{\alpha,\xi}$ on $\mathbb{Z}_{\ge 0}$ (see (8.6)), and the measures $\mathsf{M}_{\alpha,n}$ are consistent with the up and down transition kernels (see §3.2):

$$
\mathsf{M}_{\alpha,n} \circ p_\alpha^\uparrow(n,n+1) = \mathsf{M}_{\alpha,n+1} \qquad \text{and} \qquad \mathsf{M}_{\alpha,n+1} \circ p^\downarrow(n+1,n) = \mathsf{M}_{\alpha,n}.
$$

This implies that $\mathsf{M}_{\alpha,\xi} \circ \mathbb{Q} = 0$, and hence the process with jump rates (8.8) preserves the measure $\mathsf{M}_{\alpha,\xi}$ on $\mathbb{S}$. By $(\boldsymbol{\lambda}_{\alpha,\xi}(t))_{t\ge 0}$ we denote the equilibrium version of this process.

The process $\boldsymbol{\lambda}_{\alpha,\xi}$ is reversible with respect to $\mathsf{M}_{\alpha,\xi}$ because $\mathsf{M}_{\alpha,\xi}(\lambda)\,\mathbb{Q}_{\lambda,\mu} = \mathsf{M}_{\alpha,\xi}(\mu)\,\mathbb{Q}_{\mu,\lambda}$ for all $\mu, \lambda \in \mathbb{S}$. Indeed, it suffices to consider $\mu \nearrow \lambda$. Let $|\lambda| = n$. Then

$$\begin{aligned}\mathsf{M}_{\alpha,\xi}(\lambda)\,\mathbb{Q}_{\lambda,\mu} &= \Big(\pi_{\alpha,\xi}(n)\frac{n}{1-\xi}\Big)\left(\mathsf{M}_{\alpha,n}(\lambda)p^{\downarrow}(n,n-1)_{\lambda,\mu}\right)\\ &= \Big(\pi_{\alpha,\xi}(n-1)\frac{\xi(n-1+\frac{\alpha}{2})}{1-\xi}\Big)\Big(\mathsf{M}_{\alpha,n-1}(\mu)p^{\uparrow}_{\alpha}(n-1,n)_{\mu,\lambda}\Big) = \mathsf{M}_{\alpha,\xi}(\mu)\,\mathbb{Q}_{\mu,\lambda}\end{aligned}$$

by (8.7), (8.9), and the definition of the up transition kernel $p^{\uparrow}_{\alpha}(n-1,n)$, see §3.2.

8.4 Pre-generator

Here we discuss the pre-generator of the Markov process $\boldsymbol{\lambda}_{\alpha,\xi}$.

We now regard the $\mathbb{S}\times\mathbb{S}$ matrices $(\mathbb{P}_{\lambda,\mu}(t))_{t\ge 0}$ of transition probabilities of the process $\boldsymbol{\lambda}_{\alpha,\xi}$ as operators acting on functions on $\mathbb{S}$ (from the left):

$$(\mathbb{P}(t)f)(\lambda) := \sum_{\mu\in\mathbb{S}} \mathbb{P}_{\lambda,\mu}(t)f(\mu).$$

The family $(\mathbb{P}(t))_{t\ge 0}$ is a Markov semigroup of self-adjoint contractive operators in the weighted space $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ (see §4.2 for the definition of $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$).

The semigroup $(\mathbb{P}(t))_{t\ge 0}$ in $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ has a generator which is an unbounded operator. By $\mathbb{Q}$ let us denote the restriction of this generator to $\ell^2_{\mathrm{fin}}(\mathbb{S},\mathsf{M}_{\alpha,\xi}) \subset \ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$, the dense subspace of all finitely supported functions in $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$. The operator $\mathbb{Q}$ acts as

$$(\mathbb{Q}f)(\lambda) = \sum_{\mu\in\mathbb{S}} \mathbb{Q}_{\lambda,\mu}f(\mu), \qquad f\in \ell^2_{\mathrm{fin}}(\mathbb{S},\mathsf{M}_{\alpha,\xi}), \tag{8.10}$$

where $\mathbb{Q}_{\lambda,\mu}$ (8.8) are the jump rates of the process $\boldsymbol{\lambda}_{\alpha,\xi}$.

The operator $\mathbb{Q}$ is symmetric with respect to the inner product $(\cdot,\cdot)_{\mathsf{M}_{\alpha,\xi}}$. Moreover, $\mathbb{Q}$ is closable in $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$ and its closure generates the semigroup $(\mathbb{P}(t))_{t\ge 0}$ (see Remark 9.4 below). For this reason we call $\mathbb{Q}$ the *pre-generator* of the process $\boldsymbol{\lambda}_{\alpha,\xi}$.

Remark 8.6. As a wider domain for the operator $\mathbb{Q}$ (8.10) one can take the set of all functions f on $\mathbb{S}$ such that both f and $\mathbb{Q}f$ (defined by (8.10)) belong to $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$. This set clearly includes finitely supported functions.

Recall the isometry $I_{\alpha,\xi}\colon \ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi}) \to \ell^2(\mathbb{S})$ (4.11). Using $I_{\alpha,\xi}$, we get a symmetric operator $\mathbb{B}$ in $\ell^2_{\mathrm{fin}}(\mathbb{S})$ and a Markov semigroup $(\mathbb{V}(t))_{t\ge 0}$ of self-adjoint contractive operators in $\ell^2(\mathbb{S})$ corresponding to $\mathbb{Q}$ and $(\mathbb{P}(t))_{t\ge 0}$, respectively.[19]

[19] We denote, e.g., the operators $\mathbb{B}$ and $\mathbb{Q}$ by different symbols only to indicate in what space they act. Essentially, these operators are the same.

Let us compute the matrix elements of the operator $\mathbb{B}$ in the standard orthonormal basis $\{\underline{\lambda}\}_{\lambda\in\mathbb{S}}$ of $\ell^2(\mathbb{S})$ which are the same as the matrix elements of $\mathbb{Q}$ in the orthonormal basis $\left\{(\mathsf{M}_{\alpha,\xi}(\lambda))^{-\frac12}\underline{\lambda}\right\}_{\lambda\in\mathbb{S}}$ of $\ell^2(\mathbb{S},\mathsf{M}_{\alpha,\xi})$.

Proposition 8.7. *We have*

$$\begin{aligned}\mathbb{B}\underline{\rho} &= \sum_{\sigma\in\mathbb{S}}(\mathbb{B}\underline{\rho},\underline{\sigma})\underline{\sigma}\\ &= -\frac{|\rho|+\xi(|\rho|+\frac{\alpha}{2})}{1-\xi}\underline{\rho}+\frac{\sqrt{\xi}}{1-\xi}\sum_{\mu\colon\mu\nearrow\rho}q_\alpha(\rho/\mu)\underline{\mu}+\frac{\sqrt{\xi}}{1-\xi}\sum_{\varkappa\colon\varkappa\searrow\rho}q_\alpha(\varkappa/\rho)\underline{\varkappa}.\end{aligned}\tag{8.11}$$

Here q_α is the function of a box defined by (4.6).

Proof. We can write

$$\begin{aligned}(\mathbb{B}\underline{\rho},\underline{\sigma}) &= \left((\mathsf{M}_{\alpha,\xi}(\rho))^{-\frac12}\,\mathbb{Q}\underline{\rho},(\mathsf{M}_{\alpha,\xi}(\sigma))^{-\frac12}\underline{\sigma}\right)_{\mathsf{M}_{\alpha,\xi}}\\ &= (\mathsf{M}_{\alpha,\xi}(\rho)\mathsf{M}_{\alpha,\xi}(\sigma))^{-\frac12}(\mathbb{Q}\underline{\rho},\underline{\sigma})_{\mathsf{M}_{\alpha,\xi}} = \left(\frac{\mathsf{M}_{\alpha,\xi}(\sigma)}{\mathsf{M}_{\alpha,\xi}(\rho)}\right)^{\frac12}\mathbb{Q}_{\sigma,\rho}.\end{aligned}\tag{8.12}$$

Recall that for $\lambda\in\mathbb{S}_n$,

$$\mathsf{M}_{\alpha,n}(\lambda)=\frac{(\dim_{\mathbb{S}}\lambda)^2}{n!(\alpha/2)_n}\prod_{\square\in\lambda}q_\alpha(\square)^2,$$
$$\mathsf{M}_{\alpha,\xi}(\lambda)=(1-\xi)^{\alpha/2}\xi^n\left(\frac{\dim_{\mathbb{S}}\lambda}{n!}\right)^2\prod_{\square\in\lambda}q_\alpha(\square)^2,$$

see §3–4. Thus, for $\lambda\in\mathbb{S}_n$ and $\mu\nearrow\lambda$ we can write

$$n\cdot p^{\downarrow}(n,n-1)_{\lambda,\mu}\left(\frac{\mathsf{M}_{\alpha,\xi}(\lambda)}{\mathsf{M}_{\alpha,\xi}(\mu)}\right)^{\frac12}=\sqrt{\xi}q_\alpha(\lambda/\mu),$$

and for $\varkappa\searrow\lambda$ we have

$$\xi\left(n+\frac{\alpha}{2}\right)p^{\uparrow}_\alpha(n,n+1)_{\lambda,\varkappa}\left(\frac{\mathsf{M}_{\alpha,\xi}(\lambda)}{\mathsf{M}_{\alpha,\xi}(\varkappa)}\right)^{\frac12}=\sqrt{\xi}q_\alpha(\varkappa/\lambda).$$

Using (8.12), we conclude the proof. ∎

9 Dynamical correlation functions

In this section we prove a Pfaffian formula for the dynamical correlation functions $\rho^{(n)}_{\alpha,\xi}$ (2.11) of the Markov processes $\boldsymbol{\lambda}_{\alpha,\xi}$ on the set $\mathbb{S}$ of all strict partitions. In the next section we finish the proof of Theorem 2 by giving explicit formulas for the Pfaffian correlation kernel.

9.1 Dynamical correlation functions and Markov semigroups

Let us fix $n \geq 1$ and pairwise distinct space-time points $(t_1, x_1), \dots, (t_n, x_n) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$. We assume that time moments are ordered as $0 \leq t_1 \leq \dots \leq t_n$.

Recall the operators Δ_x (where $x \in \mathbb{Z}_{>0}$) in the Hilbert space $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ defined in §6.1.

Lemma 9.1. *The dynamical correlation functions of the process $\boldsymbol{\lambda}_{\alpha,\xi}$ have the form*

$$\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \dots; t_n, x_n) = \Big(\Delta_{x_1} \mathbb{P}(t_2 - t_1)\Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{P}(t_n - t_{n-1})\Delta_{x_n}\mathbf{1}, \mathbf{1}\Big)_{\mathsf{M}_{\alpha,\xi}},$$

where $(\mathbb{P}(t))_{t \geq 0}$ is the semigroup of the process $\boldsymbol{\lambda}_{\alpha,\xi}$ in the space $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ and $\mathbf{1} \in \ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$ is the constant identity function.

Proof. This is a simple consequence of the Markov property of the process $\boldsymbol{\lambda}_{\alpha,\xi}$. Indeed, let us assume that t_j's are distinct. The n-dimensional distribution of the process $\boldsymbol{\lambda}_{\alpha,\xi}$ at time moments $t_1 < \dots < t_n$ is a probability measure on $\mathbb{S} \times \dots \times \mathbb{S}$ (n copies) that assigns the weight

$$\mathsf{M}_{\alpha,\xi}(\lambda^{(1)})\, \mathbb{P}_{\lambda^{(1)},\lambda^{(2)}}(t_2 - t_1) \dots \mathbb{P}_{\lambda^{(n-1)},\lambda^{(n)}}(t_n - t_{n-1}) \tag{9.1}$$

to every point $(\lambda^{(1)}, \dots, \lambda^{(n)})$, $\lambda^{(i)} \in \mathbb{S}$. On the other hand, $\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \dots; t_n, x_n)$ is exactly the mass of the set $\{\lambda^{(1)} \ni x_1, \dots, \lambda^{(n)} \ni x_n\}$ under the measure (9.1). This proves the claim for distinct t_j's. It can be readily verified that the claim also holds if some of t_j's coincide. This concludes the proof. ∎

Let us consider the following operator in $\mathsf{Fock}(\mathbb{Z}_{>0})$:

$$\Delta_{[\![T,X]\!]} := \Delta_{x_1} \mathbb{V}(t_2 - t_1)\Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{V}(t_n - t_{n-1})\Delta_{x_n}.$$

Here $(\mathbb{V}(t))_{t \geq 0}$ is the semigroup in $\ell^2(\mathbb{S})$ defined in §8.4, and we have identified $\ell^2(\mathbb{S})$ with $\mathsf{Fock}(\mathbb{Z}_{>0})$ as in §5.2. The operators Δ_x, $x \in \mathbb{Z}_{>0}$, are now acting in $\mathsf{Fock}(\mathbb{Z}_{>0})$.

Proposition 9.2. *The correlation functions of $\boldsymbol{\lambda}_{\alpha,\xi}$ have the form*

$$\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \dots; t_n, x_n) = \Big(R(\widetilde{G}_\xi)^{-1}\Delta_{[\![T,X]\!]}R(\widetilde{G}_\xi)\mathsf{vac}, \mathsf{vac}\Big). \tag{9.2}$$

Note that now the expectation is taken in $\mathsf{Fock}(\mathbb{Z}_{>0})$.

Proof. Since $\mathbb{V}(t) = I_{\alpha,\xi}\, \mathbb{P}(t) I^{-1}_{\alpha,\xi}$, the claim is a direct consequence of Lemma 9.1 and formula (5.11) with $A = \Delta_{x_1} \mathbb{P}(t_2 - t_1)\Delta_{x_2} \dots \Delta_{x_{n-1}} \mathbb{P}(t_n - t_{n-1})\Delta_{x_n}$. ∎

Note that in contrast to the static case (6.4), the operator $\Delta_{[\![T,X]\!]}$ is not diagonal. See also Remark 4.9. It is worth noting that for t_j's not ordered as $t_1 \leq \dots \leq t_n$, formula (9.2) does not hold.

9.2 Pre-generator and Kerov's operators

In this subsection we discuss the pre-generator $\mathbb{B}$ of the semigroup $(\mathbb{V}(t))_{t\ge 0}$ and extend the definition of the operators $\mathbb{V}(t)$ to complex t with $\Re t \ge 0$.

Proposition 9.3. *The operator* $\mathbb{B}$ *in* $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ *has the form*

$$\mathbb{B} = -R(H_\xi) + \frac{\alpha}{4}\mathbf{I},$$

where $\mathbf{I}$ *is the identity operator,* $G_\xi \in SU(1,1)$ *is defined in* §4.2*, the representation* R *of* $\mathfrak{sl}(2,\mathbb{C})$ *in* $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ *is defined in* §5.4*, and*

$$H_\xi := \frac{1}{2} G_\xi H G_\xi^{-1} \in \mathfrak{sl}(2,\mathbb{C}).$$

Proof. Identifying $\ell^2(\mathbb{S})$ and $\mathsf{Fock}(\mathbb{Z}_{>0})$ (§5.2), from Proposition 8.7 we see that

$$\mathbb{B} = -\frac{1}{2}\frac{1+\xi}{1-\xi}R(H) + \frac{\sqrt{\xi}}{1-\xi}(R(U)+R(D)) + \frac{\alpha}{4}\mathbf{I}.$$

The matrix computation

$$\frac{1}{2}G_\xi H G_\xi^{-1} = \frac{1}{2(1-\xi)}\begin{bmatrix} 1 & \sqrt{\xi} \\ \sqrt{\xi} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{\xi} \\ -\sqrt{\xi} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2}\frac{1+\xi}{1-\xi} & -\frac{\sqrt{\xi}}{1-\xi} \\ \frac{\sqrt{\xi}}{1-\xi} & -\frac{1}{2}\frac{1+\xi}{1-\xi} \end{bmatrix}$$

concludes the proof. □

Remark 9.4. From the above proposition it follows that the operator $\mathbb{B}$ (with domain $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$) is essentially self-adjoint because all vectors of $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ are analytic for the operator $R(H_\xi)$ by Proposition 4.7. The same also holds for the operator $R(H)$ (corresponding to the case $\xi = 0$). Moreover, the closure of $\mathbb{B}$ looks as

$$\overline{\mathbb{B}} = \frac{\alpha}{4}\mathbf{I} - \overline{R(H_\xi)} = \frac{\alpha}{4}\mathbf{I} - R(\widetilde{G}_\xi)\overline{R(H)}R(\widetilde{G}_\xi)^{-1}.$$

The closure $\overline{\mathbb{B}}$ generates the semigroup $(\mathbb{V}(t))_{t\ge 0}$.

These properties of $\mathbb{B}$ in fact imply (using the isometry $I_{\alpha,\xi}$ (4.11)) that the operator $\mathbb{Q}$ is closable in $\ell^2(\mathbb{S}, \mathsf{M}_{\alpha,\xi})$, and its closure generates the semigroup $(\mathbb{P}(t))_{t\ge 0}$, see §8.4.

Our next aim is to extend the definition of the semigroup $(\mathbb{V}(t))_{t\ge 0}$ from real nonnegative values of t to complex values of t with $\Re t \ge 0$. Observe that the matrix iH_ξ belongs to the real form $\mathfrak{su}(1,1) \subset \mathfrak{sl}(2,\mathbb{C})$ (here $i = \sqrt{-1}$). By $W_\xi(\tau)$ denote the matrix

$$W_\xi(\tau) := e^{-i\tau H_\xi} = G_\xi \begin{bmatrix} e^{-i\tau/2} & 0 \\ 0 & e^{i\tau/2} \end{bmatrix} G_\xi^{-1} \in SU(1,1), \qquad \tau \in \mathbb{R}.$$

The family $\{W_\xi(\tau)\}_{\tau\in\mathbb{R}}$ for any fixed $\xi \in [0,1)$ is a continuous curve in $SU(1,1)$ passing through the unity at $\tau = 0$. Let by $\{\widetilde{W}_\xi(\tau\}_{\tau\in\mathbb{R}}$ denote the lifting of this curve to the universal covering group $SU(1,1)^\sim$.[20]

For real τ one can consider unitary operators

$$R\big(\widetilde{W}_\xi(\tau)\big) = R(\widetilde{G}_\xi)R\big(\widetilde{W}_0(\tau)\big)R(\widetilde{G}_\xi)^{-1}$$

in the Fock space $\mathsf{Fock}(\mathbb{Z}_{>0})$. Here the operator $R(\widetilde{W}_0(\tau))$ corresponds to setting $\xi = 0$. On $\mathsf{Fock}_{\mathrm{fin}}(\mathbb{Z}_{>0})$ it acts as

$$R(\widetilde{W}_0(\tau))\underline{\lambda} = e^{-i\tau R(H)/2}\underline{\lambda} = e^{-i\tau(|\lambda|+\frac{\alpha}{4})}\underline{\lambda}, \qquad \lambda \in \mathbb{S}, \qquad \tau \in \mathbb{R}. \tag{9.3}$$

Informally speaking, for $s \in \mathbb{R}_{\geq 0}$, the operator $\mathbb{V}(s)$ means $e^{s\,\mathbb{B}}$, and for $\tau \in \mathbb{R}$, the operator $R(\widetilde{W}_\xi(\tau))e^{i\tau\frac{\alpha}{4}\mathbf{I}}$ means $e^{i\tau\,\mathbb{B}}$. Thus, it is natural to give the following definition:

Definition 9.5. For $t = s + i\tau \in \mathbb{C}_+ := \{w \in \mathbb{C} \colon \Re w \geq 0\}$ let $\mathbb{V}(t)$ be the operator

$$\mathbb{V}(t) := \mathbb{V}(s)R(\widetilde{W}_\xi(\tau))e^{i\tau\frac{\alpha}{4}\mathbf{I}} \tag{9.4}$$

in $\mathsf{Fock}(\mathbb{Z}_{>0})$. Here $\mathbf{I}$ is the identity operator.

For real nonnegative t the operator $\mathbb{V}(t)$ is self-adjoint and bounded, it was defined in §8.4. For purely imaginary t, the operator $\mathbb{V}(t)$ is unitary. Thus, the operators $\mathbb{V}(t)$ are bounded for all $t \in \mathbb{C}_+$. Moreover, $\mathbb{V}(t_1 + t_2) = \mathbb{V}(t_1)\mathbb{V}(t_2)$ for any $t_1, t_2 \in \mathbb{C}_+$, so $\{\mathbb{V}(t)\}_{t\in\mathbb{C}_+}$ is a semigroup (with complex parameter) that can be viewed as an analytic continuation of the semigroup $\{\mathbb{V}(s)\}_{s\in\mathbb{R}_{\geq 0}}$. In particular, the operators $\mathbb{V}(t)$ commute with each other. Moreover, it is clear that the function $t \mapsto \mathbb{V}(t)h$ is bounded and continuous in $\mathbb{C}_+$ and holomorphic in the interior $\{w \in \mathbb{C}_+ \colon \Re w > 0\}$ of $\mathbb{C}_+$ for any vector $h \in \mathsf{Fock}(\mathbb{Z}_{>0})$ that is analytic for the operator $\overline{\mathbb{B}}$.

9.3 Pfaffian formula for dynamical correlation functions

In this subsection we prove a Pfaffian formula for the dynamical correlation functions $\rho^{(n)}_{\alpha,\xi}$ of the equilibrium Markov process $\boldsymbol{\lambda}_{\alpha,\xi}$. Our aim is to establish the following Theorem:

Theorem 9.6. *There exists a function $\mathbf{\Phi}_{\alpha,\xi}$ on $(\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\neq 0})^2$ such that the dynamical correlation functions of the equilibrium Markov process $\boldsymbol{\lambda}_{\alpha,\xi}$ have the form*

$$\rho^{(n)}_{\alpha,\xi}(t_1, x_1; \ldots; t_n, x_n) = \mathrm{Pf}(\mathbf{\Phi}_{\alpha,\xi}[\![T, X]\!]),$$

where $(t_1, x_1), \ldots, (t_n, x_n) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$ are pairwise distinct space-time points such that $0 \leq t_1 \leq \cdots \leq t_n$, and $\mathbf{\Phi}_{\alpha,\xi}[\![T, X]\!]$ is the $2n \times 2n$ skew-symmetric matrix with rows and columns indexed by the numbers $1, -1, \ldots, n, -n$, such that

[20] Compare this with the definition of $\widetilde{G}_\xi$ in §4.2.

the kj-th entry in $\mathbf{\Phi}_{\alpha,\xi}[\![T,X]\!]$ above the main diagonal is $\mathbf{\Phi}_{\alpha,\xi}(t_{|k|},x_k;t_{|j|},x_j)$, where $k,j=1,-1,\dots,n,-n$ (thus, $|k|\le|j|$).[21]

We will give a formula for the kernel $\mathbf{\Phi}_{\alpha,\xi}$ in (9.11) below. The rest of this subsection is devoted to proving Theorem 9.6. First, we need the following statement:

Lemma 9.7 ([Ols])**.** *Let $F(z)$ be a function on $\mathbb{C}_+$ that is bounded and continuous in $\mathbb{C}_+$ and is holomorphic in the interior of $\mathbb{C}_+$. Then F is uniquely determined by its values on the imaginary axis $\{w\in\mathbb{C}\colon \Re w=0\}$.*

Proof. Transforming the half-plane $\mathbb{C}_+$ to the unit disc $|\zeta|<1$, we get a function G on the disc that is holomorphic in the interior of the disc and bounded and continuous up to the boundary (with possible exception of one point corresponding to $w=\infty\in\mathbb{C}_+$).

For any fixed ζ_0 with $|\zeta_0|<1$, the value $G(\zeta_0)$ is represented by Cauchy's integral over the circle $|\zeta|=r$, for $|\zeta_0|<r<1$. By our hypotheses, this Cauchy's integral has a limit as $r\to1$, which gives an expression of $G(\zeta_0)$ through the boundary values. $\square$

Let us fix pairwise distinct space-time points $(t_1,x_1),\dots,(t_n,x_n)\in\mathbb{R}_{\ge0}\times\mathbb{Z}_{>0}$ such that $0\le t_1\le\dots\le t_n$. For convenience, set $t_{kj}:=t_k-t_j$. Above we have expressed the dynamical correlation functions as (9.2), that is,

$$\begin{aligned}&\rho^{(n)}_{\alpha,\xi}(t_1,x_1;\dots;t_n,x_n)\\&\quad=\Big(R(\widetilde{G}_\xi)^{-1}\Delta_{x_1}\mathbb{V}(t_{2,1})\Delta_{x_2}\dots\Delta_{x_{n-1}}\mathbb{V}(t_{n,n-1})\Delta_{x_n}R(\widetilde{G}_\xi)\mathsf{vac},\mathsf{vac}\Big).\end{aligned}\tag{9.5}$$

Denote the right-hand side of (9.5) by $\mathcal{F}(t_{2,1},\dots,t_{n,n-1};x_1,\dots,x_n)$. As a function in $n-1$ variables $t_{2,1},\dots,t_{n,n-1}$, $\mathcal{F}$ is initially defined for $t_{j,j-1}$ taking real nonnegative values. However, as explained in §9.2, the definition of each operator $\mathbb{V}(t_{j,j-1})$ can be extended to $t_{j,j-1}\in\mathbb{C}_+$. Moreover, $\mathcal{F}$ is continuous and bounded in $(t_{2,1},\dots,t_{n,n-1})$ belonging to the closed domain $(\mathbb{C}_+)^{n-1}\subset\mathbb{C}^{n-1}$ and holomorphic in the interior of this domain. Therefore, by Lemma 9.7, $\mathcal{F}(t_{2,1},\dots,t_{n,n-1};x_1,\dots,x_n)$ is uniquely determined by its values when all the variables $t_{j,j-1}$ are purely imaginary.

From now on in the computation we will assume that the variables $t_j=i\tau_j$ (where $\tau_j\in\mathbb{R}$ and $j=1,\dots,n$) are purely imaginary. This implies that all the differences $t_{k,j}=i\tau_{k,j}$ are also purely imaginary. For such t_j, the operator $\mathbb{V}(t_{j,j-1})$ is unitary and hence invertible (see §9.2). In particular, we can write

$$\mathbb{V}(t_{j,j-1})=\mathbb{V}(t_{j-1,1})^{-1}\mathbb{V}(t_{j,1}),\qquad j=1,\dots,n$$

(here by agreement $t_{1,1}=0$).

Recall that in §6.1 we have defined a representation S of the universal covering group $SU(1,1)^\sim$ in the Hilbert space $V=\ell^2(\mathbb{Z})$ with the standard orthonormal basis $\{v_x\}_{x\in\mathbb{Z}}$. Denote

$$v^{(t)}_{x,\xi}:=S(\widetilde{W}_0(\tau))S(\widetilde{G}_\xi)^{-1}v_x\in V,\qquad x\in\mathbb{Z},\qquad t=i\tau\in i\mathbb{R}.\tag{9.6}$$

[21] Here and below we use convention (6.1).

For $t=0$ this vector becomes $v_{x,\xi}$ defined by (6.14).

We want to rewrite the function $\mathcal{F}(t_{2,1},\dots,t_{n,n-1};x_1,\dots,x_n)$ defined above as a certain Pfaffian.

Proposition 9.8. *For $t_j = i\tau_j \in i\mathbb{R}$ and $x_j \in \mathbb{Z}_{>0}$ $(j=1,\dots,n)$ we have*

$$\mathcal{F}(t_{2,1},\dots,t_{n,n-1};x_1,\dots,x_n) = \mathrm{Pf}(\mathcal{F}[\![T,X]\!]), \tag{9.7}$$

where $\mathcal{F}[\![T,X]\!]$ is the $2n\times 2n$ skew-symmetric matrix with rows and columns indexed by the numbers $1,-1,\dots,n,-n$, such that the kj-th entry in $\mathcal{F}[\![T,X]\!]$ above the main diagonal is $\mathbf{F}_{\mathsf{vac}}\big(v_{x_k,\xi}^{(t_{|k|,1})}, v_{x_j,\xi}^{(t_{|j|,1})}\big)$, where $k,j = 1,-1,\dots,n,-n$ (and thus $|k|\le |j|$).

Here $\mathbf{F}_{\mathsf{vac}}$ is the vacuum average on the Clifford algebra $Cl(V)$ over the space V, see §5. Note that the purely imaginary numbers $t_{r,1}$ (where $r=2,\dots,n$) appearing in $\mathbf{F}_{\mathsf{vac}}\big(v_{x_k,\xi}^{(t_{|k|,1})}, v_{x_j,\xi}^{(t_{|j|,1})}\big)$ can be expressed through the variables $t_{j,j-1}$, $j=2,\dots,n$, that are in the left-hand side of (9.7). Recall that $t_{1,1}=0$.

Proof. The operators Δ_x in the definition of $\mathcal{F}$ (9.5) have the form

$$\Delta_x = \mathcal{T}(v_x)\mathcal{T}(v_{-x}) = \mathcal{T}(v_x v_{-x}), \qquad x\in\mathbb{Z}_{>0}.$$

By (9.6) and Proposition 6.6, we have

$$R(\widetilde{W}_0(\tau))R(\widetilde{G}_\xi)^{-1}\Delta_x R(\widetilde{G}_\xi)R(\widetilde{W}_0(\tau))^{-1} = \mathcal{T}(v_{x,\xi}^{(t)}v_{-x,\xi}^{(t)}), \quad x\in\mathbb{Z},\ t=i\tau\in i\mathbb{R}.$$

A straightforward computation using Definition 9.5 allows us to rewrite the operator in (9.5) as

$$\begin{aligned} R(\widetilde{G}_\xi)^{-1}&\Delta_{x_1}\mathbb{V}(t_{2,1})\Delta_{x_2}\dots\Delta_{x_{n-1}}\mathbb{V}(t_{n,n-1})\Delta_{x_n}R(\widetilde{G}_\xi) \\ &= \mathcal{T}(v_{x_1,\xi}^{(t_{1,1})}v_{-x_1,\xi}^{(t_{1,1})}\dots v_{x_n,\xi}^{(t_{n,1})}v_{-x_n,\xi}^{(t_{n,1})})R(\widetilde{W}_0(\tau_{n,1}))e^{i\tau_{n,1}\frac{\alpha}{4}\mathbf{I}}. \end{aligned}$$

Observe that from (9.3) it follows that $R(\widetilde{W}_0(\tau_{n,1}))e^{i\tau_{n,1}\frac{\alpha}{4}\mathbf{I}}\mathsf{vac} = \mathsf{vac}$, so

$$\begin{aligned} \mathcal{F}(t_{2,1},\dots,t_{n,n-1};x_1,\dots,x_n) &= \big(\mathcal{T}(v_{x_1,\xi}^{(t_{1,1})}v_{-x_1,\xi}^{(t_{1,1})}\dots v_{x_n,\xi}^{(t_{n,1})}v_{-x_n,\xi}^{(t_{n,1})})\mathsf{vac},\mathsf{vac}\big) \\ &= \mathbf{F}_{\mathsf{vac}}\left(v_{x_1,\xi}^{(t_{1,1})}v_{-x_1,\xi}^{(t_{1,1})}\dots v_{x_n,\xi}^{(t_{n,1})}v_{-x_n,\xi}^{(t_{n,1})}\right). \end{aligned}$$

An application of Wick's theorem (Theorem 5.1) concludes the proof. ☐

Now that we have established a Pfaffian formula for purely imaginary time variables $t_{j,j-1}$ $(j=2,\dots,n)$, we want to extend it to the case when all $t_{j,j-1}$'s are real nonnegative. Let us look closer at the function $\mathbf{F}_{\mathsf{vac}}(v_{x,\xi}^{(s)}v_{y,\xi}^{(t)})$, where $x,y\in\mathbb{Z}_{\ne 0}$, and $s=i\sigma$ and $t=i\tau$ are purely imaginary. We have

$$v_{x,\xi}^{(s)} = S(\widetilde{W}_0(\sigma))v_{x,\xi}, \qquad v_{y,\xi}^{(t)} = S(\widetilde{W}_0(\tau))v_{y,\xi},$$

where $v_{x,\xi}$ and $v_{y,\xi}$ are defined by (6.14). Therefore, we get (see also §6.2):

$$\mathbf{F}_{\mathsf{vac}}(v^{(s)}_{x,\xi}v^{(t)}_{y,\xi}) = \sum_{k,l\in\mathbb{Z}} (v_{x,\xi}, v_k)_V (v_{y,\xi}, v_l)_V \mathbf{F}_{\mathsf{vac}}\Big(\big(S(\widetilde{W}_0(\sigma))v_k\big)\big(S(\widetilde{W}_0(\tau))v_l\big)\Big). \tag{9.8}$$

On the space $V_{\text{fin}} \subset V = \ell^2(\mathbb{Z})$ consisting of finite linear combinations of the basis vectors $\{v_x\}_{x\in\mathbb{Z}}$, the operator $S(\widetilde{W}_0(u))$ acts as $e^{-iuS(H)/2}$ (where $u \in \mathbb{R}$), so we have

$$S(\widetilde{W}_0(\sigma))v_k = e^{-ks}v_k, \qquad S(\widetilde{W}_0(\tau))v_l = e^{-lt}v_l.$$

Using (6.17), we see that (9.8) takes the form

$$\mathbf{F}_{\mathsf{vac}}(v^{(s)}_{x,\xi}v^{(t)}_{y,\xi}) = \sum_{m=0}^{\infty} e^{-m(t-s)}(v_{x,\xi}, v_{-m})_V (v_{y,\xi}, v_m)_V. \tag{9.9}$$

Note that here s and t are still purely imaginary.

Using Proposition 6.7, one can estimate for any fixed $r \in \mathbb{Z}$:

$$(v_{r,\xi}, v_{-m})_V = O(1)\cdot m^{-r-\frac{1}{2}}\xi^{\frac{m}{2}}(1-\xi)^m, \qquad m \to +\infty. \tag{9.10}$$

Since $0 < \xi < 1$, this means that the series (9.9) converges rapidly, and so one can view the right-hand side of (9.9) as a function in $(t-s)$ taking values in the right half-plane $\mathbb{C}_+$. This function is bounded and continuous in $\mathbb{C}_+$ and is holomorphic in the interior of $\mathbb{C}_+$. We are interested in the restriction of this function to real nonnegative values of $(t-s)$. Let us denote

$$\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y) := \sum_{m=0}^{\infty} e^{-m(t-s)}(v_{x,\xi}, v_{-m})_V (v_{y,\xi}, v_m)_V, \quad x,y \in \mathbb{Z}_{\neq 0},\ t \ge s \ge 0. \tag{9.11}$$

By application of Lemma 9.7, we see that formula (9.7) holds for real nonnegative $t_{2,1}, \dots, t_{n,n-1}$, that is, for $0 \le t_1 \le \dots \le t_n$. This fact together with Proposition 9.2 implies Theorem 9.6.

9.4 Comments to Theorem 9.6

1. In §10.1 we give an explicit expression for $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y)$ in terms of the Gauss hypergeometric function. Comparing (9.11) and (6.18), we see that the kernel $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y)$ can be viewed as a dynamical extension of the Pfaffian hypergeometric-type kernel $\mathbf{\Phi}_{\alpha,\xi}(x,y)$. Following the common terminology (e.g., see [NF98], [Joh05], [BO06a]), we call the kernel $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y)$ the *extended* (*Pfaffian*) *hypergeometric-type kernel*.

2. The matrix $\mathbf{\Phi}_{\alpha,\xi}[\![T,X]\!]$ of Theorem 9.6 is a dynamical extension of the matrix $\mathbf{\Phi}_{\alpha,\xi}[\![X]\!]$ defined in Remark 6.3. Note that in general the matrix $\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]$ does not have a dynamical extension. This is equivalent to the fact that in the dynamical case the identity

$$v^{(t_{1,1})}_{x_1,\xi}v^{(t_{1,1})}_{-x_1,\xi}\dots v^{(t_{n,1})}_{x_n,\xi}v^{(t_{n,1})}_{-x_n,\xi} = v^{(t_{1,1})}_{x_1,\xi}\dots v^{(t_{n,1})}_{x_n,\xi}v^{(t_{n,1})}_{-x_n,\xi}\dots v^{(t_{1,1})}_{-x_1,\xi}$$

fails (here t_j's are purely imaginary). On the other hand, for $t_{1,1} = \cdots = t_{n,1} = 0$ this identity holds. In the static case this allows to write the correlation functions in two ways:

$$\rho^{(n)}_{\alpha,\xi}(x_1,\dots,x_n) = \mathrm{Pf}\left(\mathbf{\Phi}_{\alpha,\xi}[\![X]\!]\right) = \mathrm{Pf}(\hat{\mathbf{\Phi}}_{\alpha,\xi}[\![X]\!]).$$

10 Extended hypergeometric-type kernel

In §10.1 and §10.2 we study in detail the extended Pfaffian hypergeometric-type kernel $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y)$ introduced in §9.3. In §10.3 we consider the Plancherel degeneration of the kernel $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y)$.

In this section we denote $B := x \vee y = \max(x,y)$ and $b := x \wedge y = \min(x,y)$. For any $a \in \mathbb{R}$ we also set $a_- := \min(a,0)$ and $a_+ := \max(a,0)$. We always assume that $t \geq s \geq 0$.

10.1 Expression through the Gauss hypergeometric function

It is more convenient to consider the function

$$\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y) = \sum_{m=0}^{\infty} e^{-m(t-s)} (v_{x,\xi}, v_{-m})_V (v_{y,\xi}, v_{-m})_V \tag{10.1}$$

instead of $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,y)$ (see (9.11)) because $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ is symmetric in (x,y) by Corollary 6.8.

We write $(v_{x,\xi}, v_{-m})_V$ and $(v_{y,\xi}, v_{-m})_V$ in terms of the Gauss hypergeometric function using Proposition 6.7. There are two different expressions for the matrix elements in this Proposition, so we break the sum on m in (10.1) into three sums as

$$\sum_{m=0}^{\infty} = \sum_{m=0}^{-(x\vee y)-1} + \sum_{m=\max(-(x\vee y),0)}^{-(x\wedge y)-1} + \sum_{m=\max(-(x\wedge y),0)}^{\infty} \tag{10.2}$$

(by convention, the sum over an empty set is zero). By a direct computation we get the following:

Theorem 10.1. *For $x,y \in \mathbb{Z}_{\neq 0}$ and $t \geq s \geq 0$ we have*

$$\begin{aligned}
&\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y) \\
&= \sum_{m=0}^{-B-1} \frac{\xi^{-\frac{x+y}{2}}\,\Xi(-x,-y)\cdot e^{-m(t-s)}\xi^{-m}(1-\xi)^{2m}\phi_{1-m}(-x)\phi_{1-m}(-y)}{2^{\delta(m)}(-x-m)!(-y-m)!\Gamma(\frac12-\nu+m)\Gamma(\frac12+\nu+m)} \\
&\quad + \frac{(-1)^{B\wedge 0}\cos(\pi\nu)}{\pi} \sum_{m=\max(-B,0)}^{-b-1} \frac{\xi^{\frac{|x-y|}{2}}\,\Xi(B,-b)\cdot(-1)^m e^{-m(t-s)}\phi_{m+1}(B)\,\phi_{1-m}(-b)}{2^{\delta(m)}(B+m)!\,(-b-m)!}
\end{aligned}$$

$$+(-1)^{x\wedge 0+y\wedge 0}\sum_{m=\max(-b,0)}^{\infty}\frac{\xi^{\frac{x+y}{2}}\Xi(x,y)\cdot e^{-m(t-s)}\xi^m(1-\xi)^{-2m}\phi_{m+1}(x)\phi_{m+1}(y)}{2^{\delta(m)}(x+m)!(y+m)!\Gamma(\frac12-\nu-m)\Gamma(\frac12+\nu-m)}. \tag{10.3}$$

Here $\phi_j(x)$ and $\Xi(x,y)$ are defined by (7.1) *and* (7.2)*, respectively.*

Remark 10.2. If x and y are positive, in (10.3) only the third sum survives, and it starts from $m=0$. If one of x and y is positive and the other is negative, the first sum in (10.3) vanishes and the other two do not. Finally, for negative x and y we have all three sums in (10.3).

In the static case ($t=s$) and for $x,y\in\mathbb{Z}_{>0}$ we see that $\Phi(s,x;s,-y)$ becomes $\mathsf{K}_{\alpha,\xi}(x,y)$ (7.3) divided by $\frac{2\sqrt{xy}}{x+y}$, as it should be, see (7.5).

10.2 Double contour integral representations

We give two types of double contour integral representations for the kernel $\mathbf{\Phi}_{\alpha,\xi}$ (10.3). In the static case they correspond to Proposition 7.6 and 7.7, respectively. To shorten the formulas, we omit the argument α in $\nu(\alpha)$.

10.2.1 First type of double contour integrals

We consider the three sums in (10.3) separately in the next three Propositions.

Proposition 10.3 (The first sum in (10.3))**.** *Assume that the first sum in* (10.3) (*over* $0\le m\le -B-1$) *does not vanish, that is, both x and y are negative. Then this sum is equal to*

$$\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{\frac12+\nu}$$
$$\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu}\left(1-w_2\sqrt{\xi}\right)^{\frac12-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu}$$
$$\times\frac{1}{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)-e^{s-t}(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}\cdot\frac{dw_1dw_2}{w_1^{1-x}w_2^{1-y}}$$
$$-\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{\frac12+\nu+B}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu-B}$$
$$\times\left(1-w_2\sqrt{\xi}\right)^{\frac12-\nu+B}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu-B}$$
$$\times\frac{e^{B(t-s)}}{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)-e^{s-t}(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}\cdot\frac{dw_1dw_2}{w_1^{1+B-x}w_2^{1+B-y}}$$
$$-\frac12\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu}$$

$$\times \left(1 - w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu} \left(1 - \tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu} \frac{dw_1 dw_2}{w_1^{1-x} w_2^{1-y}}. \quad (10.4)$$

Here $\{w_1\}$ and $\{w_2\}$ are simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside. In the first and second integrals the contours must satisfy an additional condition (which is the same for both integrals): the image of the contour $\{w_1\}$ under the fractional-linear involution

$$w \mapsto \frac{(1-e^{s-t})\sqrt{\xi}w - (1-e^{s-t}\xi)}{(\xi - e^{s-t})w - (1-e^{s-t})\sqrt{\xi}} \quad (10.5)$$

does not intersect the domain bounded by $\{w_2\}$.

Here and below the function $\mathsf{k}(\cdot)$ on $\mathbb{Z}$ is given by (7.13).

Proof. For $0 \le m \le -B-1$, the mth term in the first sum has the form

$$\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)} \frac{1-\xi}{(2\pi\sqrt{-1})^2} \oint_{\{w_1\}} \oint_{\{w_2\}} \left(1 - w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu} \left(1 - \tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu}$$
$$\times \left(1 - w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu} \left(1 - \tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu}$$
$$\times 2^{-\delta(m)} \left(\frac{e^{s-t)}(w_1 - \sqrt{\xi})(w_2 - \sqrt{\xi})}{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)}\right)^m \frac{dw_1 dw_2}{w_1^{1-x} w_2^{1-y}},$$

where $\{w_1\}$ and $\{w_2\}$ are arbitrary positively oriented simple contours that go around 0 and $\sqrt{\xi}$ and leave $1/\sqrt{\xi}$ outside. This is verified by a straightforward computation using expressions (7.10) for the functions $\phi_j(x)$.

Now we interchange the (finite) summation with integration. Under the integral we get the sum of the form

$$\sum_{m=0}^{-B-1} 2^{-\delta(m)} c^m = \frac{1}{1-c} - \frac{c^{-B}}{1-c} - \frac{1}{2}, \quad (10.6)$$

where

$$c = e^{s-t} \frac{(w_1 - \sqrt{\xi})(w_2 - \sqrt{\xi})}{(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)}.$$

When we sum the geometric progression in (10.6), we must impose the additional condition on our contours $\{w_1\}$ and $\{w_2\}$ that the denominator $1-c$ does not vanish. This is done for the first and second integrals in (10.4). Formula (10.5) for the fractional-linear involution is obtained by solving the equation $1-c=0$ with respect to w_1 (or, equivalently, with respect to w_2). □

Remark 10.4. The existence of the contours $\{w_1\}$ and $\{w_2\}$ in Proposition 10.3 is verified as in Comment 2 to Theorem 7.3 in [BO06a] with $\zeta = \eta = \sqrt{\xi}$, because our transformation (10.5) is the same as [BO06a, (7.5)] with $s \le t$. It is worth noting that the additional condition in Proposition 10.3 is symmetric in w_1 and w_2.

Note that when $t = s$, the transformation (10.5) becomes the map $w \mapsto 1/w$. This map has already appeared in the same context in the static case, see Propositions 7.6 and 7.7.

Proposition 10.5 (The second sum in (10.3))**.** *Assume that the second sum in* (10.3) (*over* $\max(-B, 0) \le m \le -b-1$) *does not vanish, that is, at least one of* x *and* y *is negative. Then this sum has the form*

$$
\begin{aligned}
&(-1)^{B\wedge 0}\frac{\mathsf{k}(-b)}{\mathsf{k}(B)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu-B_-}\\
&\quad\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{\frac{1}{2}+\nu+B_-}\left(1-w_2\sqrt{\xi}\right)^{\frac{1}{2}-\nu+B_-}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu-B_-}\\
&\quad\times\frac{e^{B_-(t-s)}}{(w_1-\sqrt{\xi})(1-w_2\sqrt{\xi})-e^{s-t}(w_2-\sqrt{\xi})(1-w_1\sqrt{\xi})}\frac{dw_1dw_2}{w_1^{B_+}w_2^{1+B_--b}}\\
&-(-1)^{B\wedge 0}\frac{\mathsf{k}(-b)}{\mathsf{k}(B)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu-b}\\
&\quad\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{\frac{1}{2}+\nu+b}\left(1-w_2\sqrt{\xi}\right)^{\frac{1}{2}-\nu+b}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu-b}\\
&\quad\times\frac{e^{b(t-s)}}{(w_1-\sqrt{\xi})(1-w_2\sqrt{\xi})-e^{s-t}(w_2-\sqrt{\xi})(1-w_1\sqrt{\xi})}\frac{dw_1dw_2}{w_1^{|x-y|}w_2}\\
&-(-1)^{B\wedge 0}\frac{\mathbb{1}_{B\ge 0}}{2}\frac{\mathsf{k}(-b)}{\mathsf{k}(B)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu}\\
&\quad\times\left(1-w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu}\frac{dw_1dw_2}{w_1^{1+B}w_2^{1-b}}.
\end{aligned}
\tag{10.7}
$$

Here $\{w_1\}$ *and* $\{w_2\}$ *are simple positively oriented contours with points* 0 *and* $\sqrt{\xi}$ *inside and* $1/\sqrt{\xi}$ *outside, and in the first and second integrals the image of the contour* $\{w_1\}$ *under the fractional-linear map*

$$
w \mapsto \frac{(\xi - e^{s-t})w - (1-e^{s-t})\sqrt{\xi}}{(1-e^{s-t})\sqrt{\xi}w - (1-\xi e^{s-t})}
\tag{10.8}
$$

lies inside the contour $\{w_2\}$.

Proof. By (7.10), the mth summand in the second sum in (10.3) has the form

$$
(-1)^{B\wedge 0}\frac{\mathsf{k}(-b)}{\mathsf{k}(B)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu}
$$

$$\times \left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu}$$
$$\times 2^{-\delta(m)}\left(\frac{e^{s-t}(w_2-\sqrt{\xi})(1-\sqrt{\xi}w_1)}{(w_1-\sqrt{\xi})(1-\sqrt{\xi}w_2)}\right)^m\frac{dw_1dw_2}{w_1^{1+B}w_2^{1-b}},$$

where $\max(-B,0)\le m\le -b-1$ and $\{w_1\}$ and $\{w_2\}$ are arbitrary positively oriented simple contours that go around 0 and $\sqrt{\xi}$ and leave $1/\sqrt{\xi}$ outside.

Interchanging (finite) summation and integration, we have the following sum under the integral:

$$\sum_{-B_-}^{-b-1}2^{-\delta(m)}c^m=\frac{c^{-B_-}}{1-c}-\frac{c^{-b}}{1-c}-\frac12\mathbb{1}_{B\ge0},\tag{10.9}$$

where

$$c=\frac{e^{s-t}(w_2-\sqrt{\xi})(1-\sqrt{\xi}w_1)}{(w_1-\sqrt{\xi})(1-\sqrt{\xi}w_2)}.$$

As in the proof of Proposition 10.3, in the first two integrals in (10.4) we impose the additional condition that the denominator $1-c$ does not vanish on the product of the contours $\{w_1\}$ and $\{w_2\}$. Formula (10.8) for the fractional-linear map is obtained by solving the equation $1-c=0$ with respect to w_1. □

Remark 10.6. The map (10.8) is the composition of (10.5) with the inversion $w\mapsto 1/w$. This implies the existence of the contours in Proposition 10.5. In contrast to (10.5), the map (10.8) is not an involution.

Note that the additional condition on the contours $\{w_1\}$ and $\{w_2\}$ in Proposition 10.5 is not symmetric in w_1 and w_2. This is due to the fact that here these contours play different roles, namely, they correspond to $B=x\vee y$ and $b=x\wedge y$, respectively.

Proposition 10.7 (The third sum in (10.3))**.** *The third sum in* (10.3) (*over* $m\ge\max(-b,0)$) *has the form*[22]

$$(-1)^{x\wedge0+y\wedge0}\frac{\mathsf{k}(y)}{\mathsf{k}(x)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu-b_-}$$
$$\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{\frac12+\nu+b_-}\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-b_-}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{\frac12-\nu+b_-}$$
$$\times\frac{e^{b_-(t-s)}}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})-e^{s-t}(1-w_1\sqrt{\xi})(1-w_2\sqrt{\xi})}\frac{dw_1dw_2}{w_1^{x-b_-}w_2^{y-b_-}}$$
$$-(-1)^{x\wedge0+y\wedge0}\frac{\mathbb{1}_{b\ge0}}{2}\frac{\mathsf{k}(y)}{\mathsf{k}(x)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu}$$
$$\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu}\frac{dw_1dw_2}{w_1^{1+x}w_2^{1+y}}.\tag{10.10}$$

[22]Note that this third sum never vanishes.

Here $\{w_1\}$ and $\{w_2\}$ are simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside, and in the first integral the image of the contour $\{w_1\}$ under the fractional-linear involution

$$w \mapsto \frac{(1-e^{t-s})\sqrt{\xi}w - (1-e^{t-s}\xi)}{(\xi - e^{t-s})w - (1-e^{t-s}\sqrt{\xi})} \tag{10.11}$$

must be inside the contour $\{w_2\}$.

Proof. For $m \ge \max(-b,0)$, the mth term in the third sum in (10.3) can be written as

$$\begin{aligned}(-1)^{x\wedge 0+y\wedge 0}\frac{\mathsf{k}(y)}{\mathsf{k}(x)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}&\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu}\\ &\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu}\\ &\times 2^{-\delta(m)}\left(\frac{e^{s-t}(1-\sqrt{\xi}w_1)(1-\sqrt{\xi}w_2)}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}\right)^m\frac{dw_1dw_2}{w_1^{1+x}w_2^{1+y}},\end{aligned}$$

where the contours $\{w_1\}$ and $\{w_2\}$ go around 0 and $\sqrt{\xi}$ and leave $1/\sqrt{\xi}$ outside. This formula is also obtained using (7.10).

We also want to interchange the summation and integration. One can justify that this is possible similarly to the proof of Proposition 7.6. Under the integral we get the sum

$$\sum_{m=-b_-}^{\infty} 2^{-\delta(m)}c^m = \frac{c^{-b_-}}{1-c} - \frac12 \mathbb{1}_{b\ge 0}, \tag{10.12}$$

where

$$c = \frac{e^{s-t}(1-w_1\sqrt{\xi})(1-w_2\sqrt{\xi})}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})}.$$

In the first integral we impose the additional condition that the denominator $1-c$ does not vanish, and formula (10.11) for the fractional-linear involution is obtained by solving the equation $1-c=0$ with respect to w_1 (or, equivalently, with respect to w_2). ☐

Remark 10.8. The fact that the desired contours in Proposition 10.7 exist can also be justified as in Comment 2 to Theorem 7.3 in [BO06a]. Note that the fractional-linear involution (10.11) is the same as (10.5) but with e^{s-t} replaced by e^{t-s}. This explains why in Proposition 10.7 the image of $\{w_1\}$ under (10.11) must be inside $\{w_2\}$, while in Proposition 10.3 the image of $\{w_1\}$ under (10.5) must not intersect the domain bounded by $\{w_2\}$.

Now let us write down double contour integral expression for $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ when both x and y are positive. This is a dynamical counterpart of the static Proposition 7.6. For all other possible choices of signs of x and y one can readily write down double contour integral expressions for $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ using Propositions 10.3, 10.5, and 10.7, but they are very long and do not carry much new information compared to these Propositions.

Proposition 10.9 (Expression for the Pfaffian kernel)**.** *For $x, y \in \mathbb{Z}_{>0}$ and $t \ge s \ge 0$ we have the following expression for the extended hypergeometric-type kernel:*

$$
\begin{aligned}
\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y) &= \frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\\
&\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{\frac12+\nu}\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{\frac12-\nu}\\
&\times\frac{1}{(w_1-\sqrt{\xi})(w_2-\sqrt{\xi})-e^{s-t}(1-w_1\sqrt{\xi})(1-w_2\sqrt{\xi})}\frac{dw_1dw_2}{w_1^xw_2^y}\\
&-\frac12\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu}\\
&\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu}\frac{dw_1dw_2}{w_1^{1+x}w_2^{1+y}}. \qquad (10.13)
\end{aligned}
$$

Here $\{w_1\}$ and $\{w_2\}$ are simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside. In the first integral the image of the contour $\{w_1\}$ under the fractional-linear involution (10.11) *must be inside the contour $\{w_2\}$.*

Remark 10.10. The factors $\frac{\mathsf{k}(x)}{\mathsf{k}(y)}$ in front of the kernel $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ does not mean the same gauge transformation as in the determinantal case. That is, e.g., the kernel $\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ (even for positive x and y) cannot serve as the Pfaffian correlation kernel for our Markov process $\boldsymbol{\lambda}_{\alpha,\xi}$.

10.2.2 Second type of double contour integrals

There is one more type of double contour integral expressions. They are obtained in a similar way as the expressions in §10.2.1 but using formulas (7.11) for the functions $\phi_j(x)$ instead of (7.10). The contours $\{w_1\}$ and $\{w_2\}$ in those integrals are simpler than in §10.2.1 We also consider first the three sums in (10.3), and then give a double contour integral representation for $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ for positive x and y.

Proposition 10.11 (The first sum in (10.3))**.** *If the first sum in* (10.3) (*over* $0 \le m \le -B-1$) *does not vanish* (*that is, both x and y are negative*)*, then it is equal to*

$$
\begin{aligned}
&\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu-x}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu+x}\\
&\qquad\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-y}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu+y}\frac{1}{1-e^{s-t}w_1w_2}\cdot\frac{dw_1dw_2}{w_1^{1-x}w_2^{1-y}}\\
&-\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu-x}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu+x}
\end{aligned}
$$

$$\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-y}\Big(1-\tfrac{\sqrt{\xi}}{w_2}\Big)^{-\frac12-\nu+y}\frac{e^{B(t-s)}}{1-e^{s-t}w_1w_2}\cdot\frac{dw_1dw_2}{w_1^{1+B-x}w_2^{1+B-y}}$$

$$-\frac12\frac{\mathsf{k}(x)}{\mathsf{k}(y)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu-x}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu+x}$$

$$\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-y}\Big(1-\tfrac{\sqrt{\xi}}{w_2}\Big)^{-\frac12-\nu+y}\frac{dw_1dw_2}{w_1^{1-x}w_2^{1-y}}. \tag{10.14}$$

Here $\{w_1\}$ and $\{w_2\}$ are simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside, and in the first and second integrals the contour $\left\{e^{t-s}w_1^{-1}\right\}$ must not intersect the domain bounded by $\{w_2\}$.

Proof. This is proved in the same way as Proposition 10.3. We use integral representation (7.11) for $\phi_j(x)$ to write each mth term in the first sum in (10.3) as a double contour integral. Then we interchange summation with integration and under the integral we get the sum (10.6) with $c=e^{s-t}w_1w_2$. In the first two integrals in (10.14) we must impose the condition that the denominator $1-c$ does not vanish. □

Proposition 10.12 (The second sum in (10.3))**.** *Assume that the second sum in* (10.3) (*over* $\max(-B,0)\le m\le -b-1$) *does not vanish, that is, at least one of x and y is negative. Then this sum has the form*

$$(-1)^{B\wedge 0}\frac{\mathsf{k}(-B)}{\mathsf{k}(b)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu+B}\Big(1-\tfrac{\sqrt{\xi}}{w_1}\Big)^{-\frac12+\nu-B}$$

$$\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-b}\Big(1-\tfrac{\sqrt{\xi}}{w_2}\Big)^{-\frac12-\nu+b}\frac{e^{B_-(t-s)}}{w_1-e^{s-t}w_2}\cdot\frac{dw_1dw_2}{w_1^{B_+}w_2^{1+B_--b}}$$

$$-(-1)^{B\wedge 0}\frac{\mathsf{k}(-B)}{\mathsf{k}(b)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu+B}\Big(1-\tfrac{\sqrt{\xi}}{w_1}\Big)^{-\frac12+\nu-B}$$

$$\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-b}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu+b}\frac{e^{b(t-s)}}{w_1-e^{s-t}w_2}\cdot\frac{dw_1dw_2}{w_1^{|x-y|}w_2}$$

$$-(-1)^{B\wedge 0}\frac{\mathbb{1}_{B\ge 0}}{2}\frac{\mathsf{k}(-B)}{\mathsf{k}(b)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu+B}$$

$$\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu-B}\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu-b}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu+b}\frac{dw_1dw_2}{w_1^{1+B}w_2^{1-b}}. \tag{10.15}$$

Here $\{w_1\}$ and $\{w_2\}$ are simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside, and in the first and second integrals the contour $\{e^{s-t}w_1\}$ lies inside the contour $\{w_2\}$.

Proof. This is proved as Proposition 10.5. We write the mth term in the second sum in (10.3) using (7.11). After that under the integral we get the sum (10.9) with $c = e^{s-t}\frac{w_2}{w_1}$, and in the first two integrals in (10.15) we impose the additional condition that the denominator $1-c$ does not vanish on the product of the contours $\{w_1\}$ and $\{w_2\}$. ☐

Proposition 10.13 (The third sum in (10.3))**.** *The third sum in* (10.3) (*over* $m \ge \max(-b,0)$) *can be written as*

$$
\begin{aligned}
&(-1)^{x\wedge 0+y\wedge 0}\frac{\mathsf{k}(-x)}{\mathsf{k}(-y)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu+x}\\
&\qquad\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu-x}\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu+y}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu-y}\\
&\qquad\times\frac{e^{b_-(t-s)}}{w_1w_2-e^{s-t}}\frac{dw_1dw_2}{w_1^{x-b_-}w_2^{y-b_-}}\\
&-(-1)^{x\wedge 0+y\wedge 0}\frac{\mathbb{1}_{b\ge 0}}{2}\frac{\mathsf{k}(-x)}{\mathsf{k}(-y)}\frac{1-\xi}{(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu+x}\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu-x}\\
&\qquad\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu+y}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu-y}\frac{dw_1dw_2}{w_1^{1+x}w_2^{1+y}}.
\end{aligned}
\tag{10.16}
$$

Here $\{w_1\}$ *and* $\{w_2\}$ *are simple positively oriented contours with points* 0 *and* $\sqrt{\xi}$ *inside and* $1/\sqrt{\xi}$ *outside, and in the first integral the contour* $\left\{e^{s-t}w_1^{-1}\right\}$ *must be inside the contour* $\{w_2\}$.

Proof. The proof is similar to the proof of Proposition 10.7. We write each mth term in the third sum in (10.3) in terms of double contour integrals using formula (7.11) for $\phi_j(x)$, and after that we interchange the summation with integration (this is justified as in the proof of Proposition 7.7). Under the integral we get the sum (10.12) with $c = \frac{e^{s-t}}{w_1w_2}$. In the first integral we must impose the additional condition that the denominator $1-c$ does not vanish. ☐

Now let us give a second type contour integral expression for the kernel $\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y)$ when x and y are positive (see also the end of §10.2.1). For such x and y it is possible to write the difference of two integrals in (10.16) as a single integral.

Proposition 10.14. *For* $x,y \in \mathbb{Z}_{>0}$ *and* $t \ge s \ge 0$ *we have the following expression for the extended hypergeometric-type kernel:*

$$
\begin{aligned}
\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}\mathbf{\Phi}_{\alpha,\xi}(s,x;t,-y) &= \frac{1-\xi}{2(2\pi\sqrt{-1})^2}\oint\limits_{\{w_1\}}\oint\limits_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu+x}\\
&\quad\times\left(1-\tfrac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu-x}\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu+y}\left(1-\tfrac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu-y}
\end{aligned}
$$

$$\times \frac{w_1 w_2 + e^{s-t}}{w_1 w_2 - e^{s-t}} \frac{dw_1 dw_2}{w_1^{1+x} w_2^{1+y}}. \tag{10.17}$$

Here $\{w_1\}$ and $\{w_2\}$ are simple positively oriented contours with points 0 and $\sqrt{\xi}$ inside and $1/\sqrt{\xi}$ outside such that the contour $\{e^{s-t} w_1^{-1}\}$ is inside the contour $\{w_2\}$.

The factor $\frac{\mathsf{k}(-y)}{\mathsf{k}(-x)}$ in front of $\mathbf{\Phi}_{\alpha,\xi}$ also does not mean a gauge transformation, see Remark 10.10.

The results of this subsection together with Theorems 9.6 and 10.1 constitute Theorem 2 from §2.

10.3 Plancherel degeneration

Here we consider the Plancherel degeneration (2.8) of the extended Pfaffian hypergeometric-type kernel $\mathbf{\Phi}_{\alpha,\xi}$ from Theorem 10.1.

Theorem 10.15. *Under the Plancherel degeneration* (2.8)*, the extended hypergeometric-type kernel $\mathbf{\Phi}_{\alpha,\xi}$ has a pointwise limit. The limiting kernel $\mathbf{\Phi}_\theta(s,x;t,-y)$, where $x, y \in \mathbb{Z}_{\ne 0}$ and $t \ge s \ge 0$, is expressed through the Bessel functions of the first kind $J_k = J_k(2\sqrt{\theta})$ defined in* (7.15)*:*

$$\mathbf{\Phi}_\theta(s,x;t,-y) = (-1)^{x\wedge 0 + y\wedge 0} \sum_{m=0}^{\infty} 2^{-\delta(m)} e^{-m(t-s)} J_{m+x} J_{m+y}. \tag{10.18}$$

Proof. Using the limiting behaviour (7.20) of the functions $\phi_j(x)$ (7.1) under the Plancherel degeneration, we can write for the terms in (10.3):

$$\frac{\xi^{-\frac{x+y}{2}} \Xi(-x,-y) \cdot e^{-m(t-s)} \xi^{-m} (1-\xi)^{2m} \phi_{1-m}(-x) \phi_{1-m}(-y)}{2^{\delta(m)} (-x-m)! (-y-m)! \Gamma(\frac{1}{2} - \nu + m) \Gamma(\frac{1}{2} + \nu + m)} \tag{10.19}$$
$$\longrightarrow (-1)^{x\wedge 0 + y\wedge 0} 2^{-\delta(m)} e^{-m(t-s)} J_{m+x} J_{m+y};$$

$$(-1)^{B\wedge 0} \frac{\cos(\pi\nu)}{\pi} \frac{\xi^{\frac{|x-y|}{2}} \Xi(B,-b) \cdot (-1)^m e^{-m(t-s)} \phi_{m+1}(B)\, \phi_{1-m}(-b)}{2^{\delta(m)} (B+m)!\, (-b-m)!} \tag{10.20}$$
$$\longrightarrow (-1)^{x\wedge 0 + y\wedge 0} 2^{-\delta(m)} e^{-m(t-s)} J_{m+x} J_{m+y};$$

$$(-1)^{x\wedge 0 + y\wedge 0} \frac{\xi^{\frac{x+y}{2}} \Xi(x,y) \cdot e^{-m(t-s)} \xi^{m} (1-\xi)^{-2m} \phi_{m+1}(x) \phi_{m+1}(y)}{2^{\delta(m)} (x+m)! (y+m)! \Gamma(\frac{1}{2} - \nu - m) \Gamma(\frac{1}{2} + \nu - m)} \tag{10.21}$$
$$\longrightarrow (-1)^{x\wedge 0 + y\wedge 0} 2^{-\delta(m)} e^{-m(t-s)} J_{m+x} J_{m+y}.$$

Here in (10.19) we assume that both x and y are negative and in (10.20) at least one of x and y is negative. Moreover, in (10.19), (10.20), and (10.21) the bounds on m are as in the three sums in (10.3).

Therefore, taking termwise limits in (10.3)) (this can be justified, e.g., using the asymptotics (9.10), see (9.11)), we obtain (10.18). ☐

Observe that when $t = s$, the kernel $\mathbf{\Phi}_\theta$ (10.18) has the form (7.21) obtained by Matsumoto [Mat05].

A Appendix. Reduction of Pfaffians to determinants

Let us first recall basic definitions and properties related to Pfaffians. We use the following notations for matrices. Let $\mathfrak{X}$ be an abstract finite space of indices and $\mathbf{a} = (a_1, \dots, a_{2n})$ be a sequence of length $2n$ of points of $\mathfrak{X}$. Let $F\colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ be some function. Form a $2n \times 2n$ skew-symmetric matrix

$$\begin{pmatrix} 0 & F(a_1,a_2) & \dots & F(a_1,a_{2n-1}) & F(a_1,a_{2n}) \\ -F(a_1,a_2) & 0 & \dots & F(a_2,a_{2n-1}) & F(a_2,a_{2n}) \\ \dots & \dots & \dots & \dots & \dots \\ -F(a_1,a_{2n-1}) & -F(a_2,a_{2n-1}) & \dots & 0 & F(a_{2n-1},a_{2n}) \\ -F(a_1,a_{2n}) & -F(a_2,a_{2n}) & \dots & -F(a_{2n-1},a_{2n}) & 0 \end{pmatrix}.$$

Denote this matrix by $F[\![\mathbf{a}]\!]$. This matrix has rows and columns indexed by $a_1, \dots, a_{2n}$, such that the ijth element above the main diagonal is equal to $F(a_i, a_j)$ (here $1 \le i < j \le 2n$).

Definition A.1. Let $\mathbf{a} = (a_1, \dots, a_{2n})$ and $F[\![\mathbf{a}]\!]$ be as defined above. The determinant $\det(F[\![\mathbf{a}]\!])$ is a perfect square as a polynomial in $F(a_i, a_j)$ (where $i < j$). The Pfaffian of $F[\![\mathbf{a}]\!]$, denoted by $\mathrm{Pf}\left(F[\![\mathbf{a}]\!]\right)$, is defined to be the square root of $\det F[\![\mathbf{a}]\!]$ having the "+" sign by the monomial $F(a_1,a_2)\dots F(a_{2n-1},a_{2n})$.

The following properties of Pfaffians are well known:

- Let A be a skew-symmetric $2n \times 2n$ matrix and B be any $2n \times 2n$ matrix, then
$$\mathrm{Pf}(BAB^T) = \det B \cdot \mathrm{Pf}(A). \tag{A.1}$$
where B^T means the transposed matrix;
- If M is any $n \times n$ matrix, then
$$\mathrm{Pf}\begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = (-1)^{n(n-1)/2} \det M. \tag{A.2}$$

Now we give a suffcient condition under which a $2n \times 2n$ Pfaffian can be reduced to a certain $n \times n$ determinant. Assume that the set $\mathfrak{X}$ is divided into two parts $\mathfrak{X} = \mathfrak{X}_+ \sqcup \mathfrak{X}_-$, and there exists a bijection between $\mathfrak{X}_+$ and $\mathfrak{X}_-$. By $a \mapsto \hat{a}$ we denote the corresponding involution of the space $\mathfrak{X}$ that interchanges $\mathfrak{X}_+$ and $\mathfrak{X}_-$. Let $\mathbf{a} := (a_1, \dots, a_n, \hat{a}_n, \dots, \hat{a}_1)$, and $a_i \in \mathfrak{X}_+$ (so $\hat{a}_j \in \mathfrak{X}_-$).

Proposition A.2. *Suppose that the function F on $\mathfrak{X} \times \mathfrak{X}$ satisfies the following properties:*[23]

1. $F(a, \hat{b}) = F(b, \hat{a})$ *for any* $a, b \in \mathfrak{X}$.
2. $F(a, b) = -F(b, a)$ *for any* $a, b \in \mathfrak{X}$ *such that* $a \neq \hat{b}$.

[23] cf. Corollary 6.10.

3. There exists a strictly positive function $f\colon \mathfrak{X}_+ \to \mathbb{R}$ with the property $f(a) \neq f(b)$ if $a \neq b$, such that

$$(f(a) - f(b))\, F(a, \hat{b}) = (f(a) + f(b)) F(a, b) \qquad \textit{for any } a, b \in \mathfrak{X}^+.$$

Then

$$\mathrm{Pf}\left(F[\![a_1, \dots, a_n, \hat{a}_n, \dots, \hat{a}_1]\!]\right) = \det[K(a_r, a_s)]_{r,s=1}^n,$$

where K has the form

$$K(u, v) = \frac{2F(u, \hat{v})\sqrt{f(u)f(v)}}{f(u) + f(v)}, \qquad u, v \in \mathfrak{X}_+. \tag{A.3}$$

Note that the third property above implies that $F(a, a) = 0$ for all $a \in \mathfrak{X}_+$.

Proof. In this proof we denote the matrix $F[\![a_1, \dots, a_n, \hat{a}_n, \dots, \hat{a}_1]\!]$ by F.

We act on the matrix F by $SL(2, \mathbb{C})^n$. Each jth copy of $SL(2, \mathbb{C})$ acts by $F \mapsto C_j F C_j^T$, where C_j is the $2n \times 2n$ identity matrix except for the 2×2 submatrix with determinant 1 formed by rows and columns with numbers j and $2n + 1 - j$. By (A.1), this action of $SL(2, \mathbb{C})^n$ does not change the Pfaffian of F. We want to choose $C \in SL(2, \mathbb{C})^n$ such that the matrix CFC^T becomes a block matrix as in (A.2).

Define

$$g(a) := \frac{1}{2} \log f(a), \qquad a \in \mathfrak{X}_+.$$

As the jth element in $C \in SL(2, \mathbb{C})^n$ we take the hyperbolic rotation

$$\begin{pmatrix} \operatorname{ch} g(a_j) & \operatorname{sh} g(a_j) \\ \operatorname{sh} g(a_j) & \operatorname{ch} g(a_j) \end{pmatrix}.$$

It can be readily verified using the properties of F that

$$CFC^T = \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix},$$

where the rows of M are indexed by $i = 1, 2, \dots, n$, and columns are indexed by $j = n + 1, \dots, 2n$, and

$$M_{ij} = \begin{cases} \dfrac{2F(a_i, a_{2n+1-j})\sqrt{f(a_i) f(a_{2n+1-j})}}{f(a_i) - f(a_{2n+1-j})}, & \text{if } i + j \neq 2n, \\[2ex] F(a_i, \hat{a}_i), & \text{otherwise.} \end{cases}$$

Set, for $r, s = 1, \dots, n$,

$$K(a_r, a_s) := M_{r, 2n+s-1},$$

and note that

$$\det\left[K(a_r, a_s)\right]_{r,s=1}^n = (-1)^{n(n-1)/2} \det M.$$

Thus, from (A.1) and (A.2) we get

$$\mathrm{Pf}(F) = \mathrm{Pf}(CFC^T) = (-1)^{n(n-1)/2} \det M = \det \left[K(a_r, a_s)\right]_{r,s=1}^n .$$

It remains to observe that $K(\cdot, \cdot)$ that now has the form

$$K(u,v) = \begin{cases} \dfrac{2F(u,v)\sqrt{f(u)f(v)}}{f(u)-f(v)}, & \text{if } u \neq v, \\ F(u, \hat{u}), & \text{otherwise,} \end{cases}$$

where $u, v \in \mathfrak{X}_+$, can be rewritten as (A.3) using the properties of F. This concludes the proof. □

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