The boundary of the q–Gelfand–Tsetlin graph, interpolation polynomials and q–Toeplitz matrices (Moebius contest version)

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Abstract

The problem of the description of finite factor representations of the infinite-dimensional unitary group, investigated by Voiculescu in 1976, is equivalent to the description of all totally positive Toeplitz matrices. Vershik-Kerov showed that this problem is also equivalent to the description of the simplex of central (i.e. possessing a certain Gibbs property) measures on paths in the Gelfand–Tsetlin graph. We study a quantum version of the latter problem. We introduce a notion of a q–centrality and describe the simplex of all q–central measures on paths in the Gelfand–Tsetlin graph. Conjecturally, q–central measures are related to representations of the quantized universal enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. We also define a class of q–Toeplitz matrices and show that every extreme q–central measure corresponds to a q–Toeplitz matrix with non-negative minors. Finally, there is yet another way to interpret our results: we study the asymptotic behavior of the rational Schur functions, normalized in a certain way, as the number of variables grows to infinity.

We use a class of q–interpolation polynomials. One of the key ingredients of our proofs is the binomial formula for these polynomials proved by Okounkov.

Contents

1 Introduction

1.1 Preface

The infinite dimensional unitary group $U(\infty)$ is the union of the unitary groups $U(N)$ naturally embedded one into another. The study of *characters* of $U(\infty)$, i.e. normalized positive-definite central continuous functions on the group, was initiated by Voiculescu in 1976. He studied *finite factor representations* of $U(\infty)$. These representations are in bijection with *extreme characters* of $U(\infty)$, which are extreme points of the convex set of all characters of $U(\infty)$. In the paper [Vo] Voiculescu gave a list of extreme characters of $U(\infty)$. He also conjectured and partially proved that this list was complete.

There is a correspondence between extreme characters of $U(\infty)$ and *totally positive* Toeplitz matrices. Such matrices were studied much earlier by Shoenberg and his collaborators in the context of classical analysis. A few years after the paper [Vo], Boyer [Bo] and Vershik-Kerov [VK2] independently pointed out that the completeness of the Voiculescu's list of characters follows from the Edrei's result [Ed] on the classification of all totally positive Toeplitz matrices.

In the same paper [VK2] Vershik and Kerov suggested a new approach to the above problem. Their method is based on the approximation of the extreme characters of $U(\infty)$ by the normalized characters of the irreducible representations of the finite-dimensional unitary groups $U(N)$. As a result, classification of extreme characters of $U(\infty)$ is restated in purely combinatorial terms as the problem of the description of the *boundary of the Gelfand–Tsetlin graph*. Okounkov and Olshanski in their paper [OkOl] gave the detailed proof and further generalized the classification theorem for the characters of $U(\infty)$ using the Vershik-Kerov approach.

The aim of the present paper is to introduce and study a q–deformation of the notion of a character of $U(\infty)$. We start from the Vershik-Kerov formulation and define a q–deformed version of the Gelfand–Tsetlin graph. Our main result is the complete description of the boundary of the q–Gelfand–Tsetlin graph. (See Section 1.2 for the details.)

There are several ways to interpret our results. We may go back to the characters and introduce their q–analogues which agree with our deformation of the Gelfand–Tsetlin graph. From this point of view, the problem that we solve in the present paper is the characterization of all possible limits of rational Schur functions normalized in a certain (depending on q) way as the number of variables grows to infinity. (See Section 1.4 for the details.)

Investigating a q–deformation of totally positive Toeplitz matrices, we arrive at a notion of a q–Toeplitz matrix. Every point of the boundary of the q–Gelfand–Tsetlin graph corresponds to a q–Toeplitz matrix with minors satisfying some non-negativity condition. (See Section 1.5 for the details.)

There are strong reasons to believe that the q–Gelfand–Tsetlin graph is related to the representation theory of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. However we do not address this issue in the present paper.

1.2 Statement of the main result

The Gelfand-Tsetlin graph $\mathbb{GT}$ is a graded graph consisting of levels $\mathbb{GT}_N$, $N = 1, 2, \dots$. The vertices of $\mathbb{GT}_N$ are N–tuples $\lambda_1 \geq \dots \geq \lambda_N$ of integers. Following the Weyl's book [W] we call these N–tuples *signatures*. We join two signatures $\lambda \in \mathbb{GT}_N$ and $\mu \in \mathbb{GT}_{N+1}$ by an edge and write $\lambda \prec \mu$ if and only if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \dots \geq \lambda_N \geq \mu_{N+1}.$$

Let $\mathcal{T}$ denote the set of all infinite paths in $\mathbb{GT}$, i.e. $\tau = (\tau(1), \tau(2), \dots)$ belongs to $\mathcal{T}$ if and only if $\tau(N) \in \mathbb{GT}_N$ and $\tau(N)$ is connected with $\tau(N+1)$ by an edge in $\mathbb{GT}$ for every N.

For any probability measure P on $\mathcal{T}$ let $P_N(\cdot)$ denote its projection on $\mathbb{GT}_N$, in other words for any $\lambda \in \mathbb{GT}_N$

$$P_N(\lambda) = P(\tau \in \mathcal{T} \mid \tau(N) = \lambda).$$

For any finite path $\phi = (\phi(1) \prec \phi(2) \prec \dots \prec \phi(N))$, $\phi(k) \in \mathbb{GT}_k$ let C_ϕ denote the corresponding cylinder set in $\mathcal{T}$, i.e.

$$C_\phi = \{\tau \in \mathcal{T} : \tau(1) = \phi(1), \dots, \tau(N) = \phi(N)\}.$$

Let us introduce the *weight* of ϕ:

$$w(\phi) = q^{|\phi(1)|+|\phi(2)|+\cdots+|\phi(N-1)|},$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_k$ for $\lambda \in \mathbb{GT}_k$. In Section 2 we show that these weights have a simple combinatorial interpretation: given a path ϕ one constructs a 3D–body and $w(\phi)$ equals q^{vol}, where *vol* is the volume of this body. For any $\lambda \in \mathbb{GT}_N$ let

$$\mathrm{Dim}_q(\lambda) = \sum_{\phi(1)\prec\cdots\prec\phi(N)|\phi(N)=\lambda} w(\phi).$$

A q–central measure on $\mathcal{T}$ is a probability measure P on $\mathcal{T}$ satisfying

$$P(C_\phi) = P_N(\phi(N)) \frac{w(\phi)}{\mathrm{Dim}_q(\phi(N))}$$

for any finite path $\phi = (\phi(1) \prec \phi(2) \prec \cdots \prec \phi(N))$, $\phi(k) \in \mathbb{GT}_k$. Put it otherwise, if we consider a family of finite paths ϕ ending at the same signature $\phi(N)$, then the probability of ϕ is proportional to the weight $w(\phi)$.

Let Ω_q denote the set of all q–central measures on $\mathcal{T}$. Clearly Ω_q is a convex set. *The minimal boundary of q–Gelfand–Tsetlin graph* is the set $\mathrm{Ex}(\Omega_q)$ of all extreme points of the set Ω_q. The main result of the present paper is the description of the minimal boundary of q–Gelfand–Tsetlin graph. We need to do some preparations before stating the theorem.

We say that signature $\lambda \in \mathbb{GT}_N$ is positive if all coordinates of λ are non-negative, i.e. $\lambda_N \geq 0$. Let $\mathbb{GT}_N^+ \subset \mathbb{GT}_N$ denote the subset of all positive signatures of size N.

Every extreme q–central measure P is uniquely defined by the array of numbers $P_N(\lambda)$. In the case studied by Voiculescu (i.e. $q = 1$) these numbers appear as the coefficients of the expansions of certain functions in the basis of rational Schur functions s_λ. It turns out that in the q–deformed case the formulas have the simplest form if instead of rational Schur functions we use other symmetric polynomials. It is convenient to use *q–interpolation Schur polynomials* $s^*_\lambda(x_1, \ldots, x_N; q)$ defined for every $\lambda \in \mathbb{GT}_N^+$. These polynomials are a particular case of factorial Schur functions (see [Mac2]). They are also a particular case ($q = t$) of interpolation Macdonald polynomials (see [Kn], [S], [Ok2]). In one dimensional case we have

$$s^*_k(x; q) = (x - 1)(x - q)\cdots(x - q^{k-1}).$$

For any probability measure P_N on $\mathbb{GT}_N^+$ we consider the following generating function:

$$\mathcal{S}^*(x_1, \ldots, x_N; P_N) = \sum_{\lambda \in \mathbb{GT}_N^+} P_N(\lambda) \frac{s^*_\lambda(q^{N-1}x_1, \ldots, q^{N-1}u_N; q^{-1})}{s^*_\lambda(0, \ldots, 0; , q^{-1})} \qquad (1)$$

Define $\mathcal{N}$ to be the set of all non-decreasing sequences of integers:

$$\mathcal{N} = \{\nu_1 \leq \nu_2 \leq \nu_3 \leq \ldots\} \subset \mathbb{Z}^\infty.$$

Finally, let A_ℓ be the isomorphism of the Gelfand–Tsetlin graph taking a signature $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{GT}_N$ to $A_\ell(\lambda) = (\lambda_1 + \ell, \dots, \lambda_N + \ell)$.

Now we are ready to state the main theorem of the present paper.

Theorem 1.1. *Let $0 < q < 1$. We have:*

1. $\mathrm{Ex}(\Omega_q)$ *is parameterized by points of $\mathcal{N}$ with $\nu \in \mathcal{N}$ corresponding to the extreme q–central measure $\mathcal{E}^\nu$.*

2. *If $\nu_1 \geq 0$ then*

$$\mathcal{S}^*(x_1, \dots, x_N; \mathcal{E}^\nu_N) = H^\nu(x_1) \cdots H^\nu(x_N),$$

where

$$H^\nu(t) = \frac{\prod_{i=0}^{\infty}(1 - q^i t)}{\prod_{j=1}^{\infty}(1 - q^{\nu_j + j - 1} t)}$$

and $\mathcal{E}^\nu_N$ is the projection of E^ν on $\mathbb{GT}_N$.

3. *For general ν the measure $\mathcal{E}^\nu$ is the image of the measure $\mathcal{E}^{\nu'}$ with $\nu' = (0, \nu_2 - \nu_1, \nu_3 - \nu_1, \dots)$ under isomorphism A_{ν_1} of the graph $\mathbb{GT}$.*

4. *Ω_q is a simplex meaning that every $P \in \Omega_q$ is a unique average of measures E^ν. Put it otherwise, for every $P \in \Omega_q$ there exist a unique probability measure Q on $\mathcal{N}$ such that*

$$P = \int_{\mathcal{N}} E^\nu dQ.$$

There is a natural involution of the graph $\mathbb{GT}$ mapping signature $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ to $-\lambda_N \geq -\lambda_{N-1} \geq \dots \geq -\lambda_1$. Is is easy to see that this map provides a bijection between q–central measures and q^{-1}–central measures. Thus, Theorem 1.1 also provides a description of q–central measures with $q > 1$.

The following proposition describes the support of measure $\mathcal{E}^\nu$ and explains the geometric meaning of the parameter ν.

Proposition 1.2. *Let τ be a random element of $\mathcal{T}$ distributed according to $\mathcal{E}^\nu$, then almost surely the last coordinates of $\tau(N)$ tend to $\{\nu_j\}$, i.e. for every j*

$$\lim_{N \to \infty} \tau(N)_{N+1-j} = \nu_j.$$

The proof of Theorem 1.1 relies on the so-called ergodic method, first proposed by Vershik in [V]. This method was used for the graph $\mathbb{GT}$ in [VK2] and for the generalizations of $\mathbb{GT}$ in [OkOl]. Essentially the same method of the identification of the boundary of a graph was proposed by Diaconis and Freedman [DF] in the context of *partial exchangeability.*

For the q–Gelfand–Tsetlin graph the ergodic method shows that every extreme q–central measure on $\mathcal{T}$ is, in some precise sense, a limit of extreme measures on $\mathbb{GT}_N$. Passing from the measures to their generating functions we arrive at the problem of finding all possible limits of rational Schur functions normalized in a certain way as the number of variables grows to infinity. (In Section 1.4 we discuss this point in more details.) It seems natural to describe extreme q–central measures in terms of the normalized rational Schur functions too. However, if we try to do that, then the formulas turn out to be quite ugly. In the contrast, if we use the polynomials s^* instead of rational Schur functions, then we arrive at simple multiplicative formulas presented in Theorem 1.1.

q–Interpolation Schur polynomials and their properties play a crucial rule in our proofs. One of the key ingredients is the *binomial formula*, which relates q–interpolation Schur polynomials with ordinary Schur polynomials and is a particular case of the binomial formula for interpolation Macdonald polynomials proved by Okounkov in [Ok1].

1.3 Representations of $U(\infty)$ and the origin of the Gelfand–Tsetlin graph

In this section we explain how the study of characters of $U(\infty)$ leads to the Gelfand–Tsetlin graph and compare Theorem 1.1 with the solution of the problem studied by Voiculescu.

There are two different approaches to the representation theory of $U(\infty)$ which lead to a reasonable class of representations (i.e. finite factor representations, see e.g. [Th2], or spherical representations of Gelfand pair $(U(\infty) \times U(\infty), U(\infty))$, see [Olsh2], [Olsh3]). In both approaches representations are in correspondence with *characters*, which are central (i.e. constant on conjugacy classes) positive definite continuous functions on $U(\infty)$ taking value 1 on the unit element of the group.

The Gelfand-Tsetlin graph $\mathbb{GT}$ is a convenient tool for studying characters of $U(\infty)$. Recall that irreducible representations of $U(N)$ are parameterized by their dominant weights, which are N–tuples $\lambda_1 \geq \cdots \geq \lambda_N$ of integers. (See e.g. [Zh].) Thus, the vertices of $\mathbb{GT}_N$ symbolize irreducible representations of $U(N)$. Furthermore, the edges of $\mathbb{GT}$ encode inclusion relations between irreducible representations of $U(N+1)$ and $U(N)$.

Given any character χ of the group $U(\infty)$ we can expand its restriction to the subgroup $U(N)$ into a convex combination of normalized characters (in the conventional sense) of irreducible representations of $U(N)$:

$$\chi\big|_{U(N)} = \sum_{\lambda \in \mathbb{GT}_N} P_N(\lambda) \frac{\chi^\lambda(\cdot)}{\chi^\lambda(e)}. \tag{2}$$

It is known (see e.g. [Zh]) that the characters of irreducible representations of unitary groups are rational Schur functions, i.e.

$$\frac{\chi^\lambda(U)}{\chi^\lambda(e)} = \frac{s_\lambda(u_1, \dots, u_N)}{s_\lambda(1, \dots, 1)},$$

where $u_1, \dots, u_N$ are eigenvalues of U.

The coefficients $P_N(\lambda)$ determine a probability distribution on the set $\mathbb{GT}_N$. In this way we get a bijection between characters χ and sequences $\{P_N\}_{N=1}^{\infty}$ of probability distributions. These sequences are called *coherent systems* because P_N and P_{N+1} satisfy a certain relation.

Any coherent system $\{P_N\}$ defines a measure P on $\mathcal{T}$ in the following way. For a finite path $\phi = (\phi(1) \prec \cdots \prec \phi(N))$ and corresponding cylinder set C_ϕ set

$$P(C_\phi) = \frac{P_N(\phi(N))}{\mathrm{Dim}(\phi(N))},$$

where $\mathrm{Dim}(\lambda)$ is the number of paths $\tau(1) \prec \cdots \prec \tau(N)$ such that $\tau(N) = \lambda$. The coherency relations between P_N and P_{N+1} imply that P is well-defined. Measure P is *a central measure* in the sense that $P(C_\phi)$ depends only on $\phi(N)$. In other words, projection of P on the set of all finite paths $\phi(1), \dots \phi(N)$ ending at a fixed $\phi(N) = \lambda \in \mathbb{GT}_N$ is uniform. Note that P_N is a projection of measure P on $\mathbb{GT}_N$. Also note that a q–central measure becomes a central measure if we set $q = 1$.

Extreme points of the convex set of characters of $U(\infty)$ (in other words, extreme characters) correspond to *irreducible spherical representations* of pair $(U(\infty) \times U(\infty), U(\infty))$ (or, again, finite factor representations). Extreme points of the convex set of all central measures on $\mathcal{T}$ correspond to extreme characters via the above bijections.

Theorem ([Vo],[Bo],[VK2])**.** *Extreme characters of $U(\infty)$ are parameterized by the points ω of the infinite-dimensional domain*

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

The corresponding extreme character is given by the formula

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spectrum}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)}. \tag{3}$$

The remarkable feature shared by the above theorem and Theorem 1.1 is the appearance of the multiplicative functions. There is an independent representation-theoretic argument proving that any extreme character of $U(\infty)$

is multiplicative (i.e. that it should be a product of the values of some function over the eigenvalues of the element of $U(\infty)$). However, the author knows no conceptual reason for the appearance of the multiplicativity in Theorem 1.1 (independent of the classification theorem itself), and it would be interesting to find one.

It is very natural to ask whether extreme central measures on $\mathcal{T}$ can be obtained as $q \to 1$ limits of extreme q–central measures on $\mathcal{T}$. The answer is "Yes" for extreme central measures such that $\alpha_i^+ = \alpha_i^- = 0$. For example, if we chose $\nu(q)$ such that

$$H^{\nu(q)}(t) = (1 - q^{x_1(q)}t)(1 - q^{x_2(q)}t) \cdots (1 - q^{x_k(q)}t)$$

and $q^{x_i(q)} \to \beta_i^+$ as $q \to 1$, then the measures $E^{\nu(q)}$ tend as $q \to 1$ to the extreme central measure on $\mathcal{T}$ parameterized by $\beta_1^+, \dots, \beta_k^+$. Using similar simple arguments one can obtain extreme central measures on $\mathcal{T}$ with arbitrary parameters $\gamma^\pm$, $\{\beta_i^\pm\}$. As for the general case, the author was not able to point out a sequence of extreme q–central measures (with q tending to 1) that converges to an extreme central measure with at least one nonzero coordinate $\alpha_i^\pm$.

1.4 Approximation of characters and limits of symmetric polynomials

Vershik and Kerov proved in [VK2] that every extreme character of $U(\infty)$ is a limit of normalized characters of irreducible representations of $U(N)$, i.e. of the functions

$$\frac{s_\lambda(u_1, \dots, u_N)}{s_\lambda(1, \dots, 1)}. \tag{4}$$

In other words, they reduced the classification problem to the following question: What are the possible limits of symmetric polynomials (4) as the number of variables grows to infinity? The answers for the similar questions for more general polynomials were obtained by Okounkov and Olshanski. See [OkOl, Theorem 1.1] and [OkOl2, Theorem 1.4].

The results of the present paper can be also interpreted as an answer to a certain asymptotic problem for symmetric polynomials as the number of variables grows to infinity.

Let $N(i)$ be an increasing sequence of positive integers and $\lambda(i) \in \mathbb{GT}_{N(i)}$. We call the sequence of signatures $\lambda(i)$ *regular* if for any k the sequence of functions

$$\frac{s_{\lambda(i)}(x_1, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-N(i)})}$$

converges uniformly on the set $\{(x_1, \dots, x_k) \in \mathbb{C}^k \mid |x_i| = q^{1-i}\}$.

The normalization $s_\lambda(1, q^{-1}, \dots, q^{1-k})$, that we use, is well-known. See e.g. [Mac, Examples in Section 3, Chapter 1].

Theorem 1.3. *Let $0 < q < 1$. We have:*

1. *Sequence of signatures* $\lambda(i)$ *is regular if and only if the last coordinates of* $\lambda(i)$ *stabilize, i.e. if there is a nondecreasing sequence of integers* $\nu = \{\nu_j\}$ *such that for any* $j > 0$

$$\lim_{i\to\infty} \lambda(i)_{N(i)+1-j} = \nu_j.$$

2. *The limit function* $Q_k^\nu(x_1, \dots, x_k)$ *is an entire function in* $\mathbb{C}^k$*.*

3. *There is a bijection between limit functions* Q_k^ν *and measures* $\mathcal{E}_k^\nu$*. More precisely*

$$Q_k^\nu = \sum_{\mu\in\mathbb{GT}_k} \mathcal{E}_k^\nu(\mu) \frac{s_{\lambda(i)}(x_1, \dots, x_k)}{s_{\lambda(i)}(1, q^{-1}, \dots, q^{1-k})}$$

4. *If* ν *and* ν' *are such that* $\nu_i = \nu_i' + \ell$ *for every* i*, then*

$$Q_k^\nu = \frac{x_1^\ell \dots, x_k^\ell}{(1 \cdot q^{-1} \cdots q^{1-k})^\ell} Q_k^{\nu'}.$$

5. *If* $\nu_1 \geq 0$*, then*

$$Q_k^\nu = \sum_{\mu\in\mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k; q),$$

where the series converges everywhere in $\mathbb{C}^k$ *and* Spec_ν *is a specialization of algebra of symmetric function* Λ *(put it otherwise, homomorphism from* Λ *to* $\mathbb{C}$*) with* H*–generating function*

$$\sum_{j=0}^{\infty} \mathrm{Spec}_\nu(h_j) t^j = \frac{\prod\limits_{i\geq 0} (1 - q^i t)}{\prod\limits_{j=1}^{\infty} (1 - q^{\nu_j + j - 1} t)}.$$

(Here h_j *stays for the complete symmetric function of degree* j*.)*

Remark. The author knows no simple explicit formula for the functions Q_k^ν. This is the main reason why we do not use these functions as a description of measures $\mathcal{E}^\nu$. Instead we use $\mathcal{S}^*(x_1, \dots, x_k; \mathcal{E}_k^\nu)$, for which a simple multiplicative formula is provided in Theorem 1.1

The functions

$$\frac{s_\lambda(x_1, \dots, x_N)}{s_{\lambda(i)}(1, q, \dots, q^{1-N})}$$

can be viewed as quantum traces of irreducible representations of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_N)$. Furthermore, one can show that the definition of a q–central measure is related to the branching rules for quantum characters of irreducible representations of $U_\epsilon(\mathfrak{gl}_N)$. (This is parallel to the fact that the Gelfand-Tsetlin graph itself is related to the branching rules of irreducible representations of the unitary group.) Thus, it is natural to expect that the functions Q_k^ν and q–central measures are related to certain representations of the quantized enveloping algebra $U_\epsilon(\mathfrak{gl}_\infty)$. The author hopes to address this issue in a later publication.

1.5 Toeplitz and q–Toeplitz matrices

Let us explain the connection between extreme characters of the group $U(\infty)$ and total positivity of Toeplitz matrices. Recall that any extreme character of $U(\infty)$ is a multiplicative function:

$$\chi(U) = \prod_{u_i} \widehat{\chi}(u_i), \quad u_i \in \text{Spectrum}(U).$$

Thus, χ is uniquely defined by $\widehat{\chi}(u)$ which is a continuous function on S^1. $\widehat{\chi}(u)$ can be represented as

$$\widehat{\chi}(u) = \sum_{l \in \mathbb{Z}} c_l u^l.$$

Introduce an infinite Toeplitz matrix

$$c[i,j] \stackrel{def}{=} c_{i-j}.$$

As a corollary of the fact that χ is a positive–definite function, one proves that the matrix $c[i,j]$ is *totally positive*, i.e. all minors of $c[i,j]$ are non-negative. Furthermore, this correspondence is a bijection between extreme characters of $U(\infty)$ and totally positive infinite Toeplitz matrices such that the sum of the matrix elements along the row equals 1. (See [Vo], [Bo], [VK2].)

In order to extend the correspondence between extreme measures and certain matrices to the case of general q we deform the notion of a Toeplitz matrix. We call a semi-infinite matrix $d[i,j]$, $i > 0$, $j > 0$ a *semi-infinite q–Toeplitz matrix* if

$$d[i,j+1] = d[i-1,j] + (q^{1-j} - q^{1-i}) d[i,j] \tag{5}$$

for all $i > 0$, $j > 0$. Here we agree that $d[i,j] = 0$ is either $i < 1$ or $j < 1$. Note that when $q = 1$, the relation (5) turns into

$$d[i,j+1] = d[i-1,j].$$

Hence, a q–Toeplitz matrix becomes a Toeplitz matrix.

Recall that according to Theorem 1.1 every extreme q–central measure corresponds to a multiplicative function $H(x_1)\cdots H(x_N)$. Given a function $H(t)$ we construct a lower-triangular semi-infinite q–Toeplitz matrix $d[i,j]$ in the following way: expand H in series

$$H(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1} (1 - tq^i)$$

and let $d[i,j]$, $i > 0$, $j > 0$ be a unique semi-infinite q–Toeplitz matrix such that

$$d[i,1] = c_{i-1}, \quad i = 1, 2, \ldots.$$

Initial minor of size N of matrix $d[i,j]$ is a minor corresponding to either the first N columns and arbitrary N rows of matrix $d[i,j]$ or the first N rows and arbitrary N columns of $d[i,j]$.

Proposition 1.4. *Let ν be a non-decreasing sequence of non-negative integers $0 \le \nu_1 \le \nu_2 \le \dots$, let $\mathcal{E}^\nu$ be the extreme q–central measure parameterized by ν and corresponding to the function H^ν. If $d^\nu[i,j]$ is a semi-infinite q–Toeplitz matrix constructed by H^ν, then all initial minors of $d^\nu[i,j]$ are non-negative.*

A general theorem (see [FZ]) says that if A is a finite non-degenerate matrix with non-negative initial minors, then A is totally positive, in other words *all* minors of A are non-negative. But, alas, the q–Toeplitz matrices corresponding to q–central measures are usually degenerate. These matrices are triangular and some elements on the main diagonal vanish. Thus, we cannot guarantee that q–Toeplitz matrices corresponding to q–central measures are totally positive, we can only claim that their certain top-left corners are. And, indeed, straightforward computations show that even some matrix elements of these matrices are negative.

However, it might be still interesting to classify all q–Toeplitz matrices with non-negative initial minors. We have the following conjecture here, which is a straightforward analogue of $q = 1$ case.

Conjecture 1.5. *If $d[i,j]$ is a lower-triangular semi-infinite q–Toeplitz matrix such that*

$$\sum_{i=1}^{\infty} d[i,1] = 1,$$

then $d[i,j]$ coincides with one of the matrices from Proposition 1.4, i.e. $d[i,j] = d^\nu[i,j]$ for some ν.

1.6 Other branching graphs and general formalism

The Gelfand–Tsetlin graph and the q–Gelfand–Tsetlin graph are two examples of *branching graphs*. (This term was introduced by Vershik and Kerov.) There is a wide class of problems that can be stated as a problem of identification the boundary of a branching graph.

A bunch of examples comes from the representation theory of "big" groups. Representations of the infinite symmetric group $S(\infty)$ are related to the boundary of the Young graph (see [Th1], [VK1], [Ok3]), projective representations of $S(\infty)$ are related to the Schur graph (see [N], [I]), and (as we already mentioned) representation of the infinite dimensional unitary group $U(\infty)$ are related to the Gelfand–Tsetlin graph.

However, there are other examples of purely probabilistic and combinatorial nature. Perhaps, the most known example is De Finetti's theorem (see. e.g [F, Chapter VII, §4] or [A]) which states that every probability measure on $\{0,1\}^\infty$ invariant with respect to permutations of coordinates is a mixture of Bernoulli measures. Here the underlying branching graph is the Pascal graph.

Motivated by a problem of population genetics Kingman introduced in [Ki] the notion of a partition structure. Kingman's classification of partition structures is equivalent to the description of the boundary of a certain graph, that is now called the Kingman graph, see also [K2].

Other examples can be found in [OkOl3], [GO1], [GO2], [GO3], [GP].

Let us consider a subgraph of $\mathbb{GT}$ consisting of zero–one signatures, i.e. $\lambda_1 \geq \cdots \geq \lambda_N$ such that $1 \geq \lambda_1 \geq \lambda_N \geq 0$. q–central measures on paths in this subgraph can be identified with central measures on paths in the q–Pascal graph studied by Gnedin and Olshanski [GO1]. These measures are related to q–analogues of De Finetti's theorem. (See also earlier paper by Kerov [K1, Chapter 1.4]) The authors of [GO1] proved that for $0 < q < 1$ the boundary of the q–Pascal graph is parameterized by points of the set $\{1, q, q^2, \dots\} \cup \{0\}$. This result agrees with our description of the boundary of the q–Gelfand–Tsetlin graph. Indeed, the only extreme q–central measures concentrated on zero–one signatures are those parameterized by ν with $0 \leq \nu_i \leq 1$ for every i. The extreme central measure on q–Pascal graph parameterized by q^k corresponds to q–central measure on $\mathcal{T}$ parameterized by $\nu = 0 \leq 0 \leq \cdots \leq 0 \leq 1 \leq 1 \dots$ with exactly k zeros.

In another article [GO2] Gnedin and Olshanski studied a multidimensional generalization of the model of [GO1]. The common feature of the results of both these two papers and the present paper is that in the contrast to $q = 1$ case, parameters of the extreme q–central measures (and of the measures studied in [GO1], [GO2]) are discrete.

As a final remark of this section we want to mention the paper [DF] where Diaconis and Freedman introduced a notion of partial exchangeability. Both central and q–central measures of $\mathcal{T}$ are particular cases of *partially exchangeable probabilities.*

1.7 Further connections and developments

Representation theory of $U(\infty)$ shows numerous connections with $S(\infty)$, i.e. inductive limit of symmetric groups S_n. (See [Olsh4] and [KOV] for reviews of the representation theory of $S(\infty)$) For example, while extreme characters of $U(\infty)$ are related to infinite totally positive Toeplitz matrices, extreme character of $S(\infty)$ are similarly related to semi-infinite totally positive Toeplitz matrices. The description of the simplex of q–central measures (in particular, their "right" definition) on the Young graph (substitute of the graph $\mathbb{GT}$ for the group $S(\infty)$) is yet to be done.

The are two important problems in the representation theory of the infinite-dimensional unitary group: identification of all irreducible representations and decomposition of natural representations into irreducible ones (see [Olsh]). Irreducible representations (again, we should speak either about finite factor representations or irreducible spherical representations here) lead to extreme central measures on $\mathcal{T}$ and in the present paper we study a q–analogue of these measures. The natural representations associated with $U(\infty)$, in turn, lead to a remarkable family of central measures on $\mathcal{T}$, i.e. so called (z, w)–measures. At the moment it is unclear whether one can introduce some natural q–central measures that will serve as a q–analogue of (z, w)–measures.

Although, the study of central measures on $\mathcal{T}$ has a representation-theoretic origin, later these measures led to a number of very interesting probability models related to random Young diagrams and random stepped surfaces, see [BK], [BF], [B]. The author hopes that q–central measures introduced in the present paper will also provide a source of new interesting probability models.

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2 Combinatorial setup

$\star$ **Throughout the paper we assume** $0 < q < 1$**.** $\star$

In this section we introduce the basic definitions and give a combinatorial interpretation for the notion of q–centrality.

A *Young diagram* μ is a finite collection of boxes arranged in rows with nonincreasing row lengths μ_i. The total number of boxes in μ is denoted by $|\mu|$. Every box of μ has two coordinates (i, j), the first one is increasing from top to bottom and the second one is increasing from left to right. The top-left corner of a diagram has coordinates $(1, 1)$. If we reflect Young diagram μ with respect to the diagonal $i = j$, then we obtain a *transposed diagram* μ'. Row lengths μ'_i of μ' are column lengths of μ.

The set of all Young diagrams has a natural partial order by inclusion relation, i.e. we write $\lambda \subset \mu$ if for every i we have $\lambda_i \le \mu_i$.

A *signature* λ of size N is an ordered collection of integers

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N,$$

we call λ_i the ith coordinate of λ.

Given a signature λ we construct two Young diagrams λ^+ and λ^- which are called a *positive diagram* and a *negative diagram*, respectively. The row lengths of the former one are all the positive λ_is, while the row lengths of the latter are absolute values of negative λ_is. An example is shown at Figure 1. Let $|\lambda|$ denote the sum of coordinates of λ. Clearly, $|\lambda| = |\lambda^+| - |\lambda^-|$.

Let us write $\lambda \le \mu$ for two signatures of the same size if $\lambda_i \le \mu_i$ for every i. Clearly, $\lambda \le \mu$ is equivalent to $\lambda^+ \subset \mu^+$ and $\mu^- \subset \lambda^-$.

Let $\mathbb{GT}_N$ denote the set of all signatures of size N and let $\mathbb{GT}_N^+$ denote the set of al signatures of size N with nonnegative coordinates. Every element of $\mathbb{GT}_N^+$ can be identified with a Young diagram, however note, that a signature has a certain additional information, i.e. its size N.

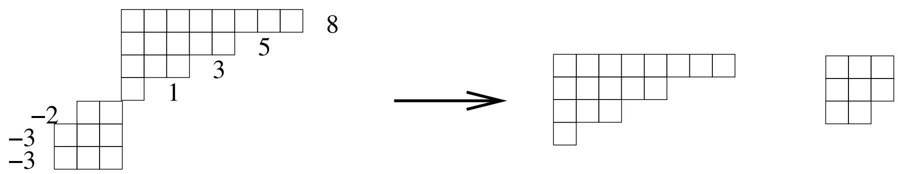

Figure 1: Signature $8 \geq 5 \geq 3 \geq 1 \geq 0 \geq -2 \geq -3 \geq -3$ and two corresponding Young diagrams.

Recall that *a path* $\tau \in \mathcal{T}$ in $\mathbb{GT}$ is a sequence of signatures $\{\lambda(n) \in \mathbb{GT}_n\}_{1,\dots}$ such that for every n, $\lambda(n) \prec \lambda(n+1)$. In other words,

$$\lambda(n+1)_1 \geq \lambda(n)_1 \geq \lambda(n+1)_2 \geq \dots \geq \lambda(n)_n \geq \lambda(n+1)_{n+1}.$$

Paths in $\mathbb{GT}$ are usually called *Gelfand-Tsetlin schemes* in representation-theoretic literature.

Let $\mathcal{T}_N$ denote the set of all paths $\tau(1) \prec \dots \prec \tau(N)$ of length N with $\tau(i) \in \mathbb{GT}_i$. The set $\mathcal{T}$ of all infinite paths is a projective limit of sets $\mathcal{T}_N$. (Here projection is just a removal of the last step of a path.)

We define $\mathbb{GT}^+$ to be the part of $\mathbb{GT}$ consisting of signatures with nonnegative coordinates. Let $\mathcal{T}^+$ and $\mathcal{T}_N^+$ denote the corresponding sets of paths.

A (semistandard Young) *tableau* T of shape $\lambda \in \mathbb{GT}_N$ is an assignment of numbers $1, \dots, N$ to boxes of λ in such a way that the numbers are increasing along the columns and non-decreasing along the rows. Given a path $\tau \in \mathcal{T}_N$ we construct two Young tableaux T_τ^+ and T_τ^- as follows: Shape of T_τ^+ is $\tau(N)^+$, $T_\tau^+(i,j) = k$ if and only if $(i,j) \in \tau(k)^+ \setminus \tau(k-1)^+$, where we agree that $\tau(0)^+ = \emptyset$. Similarly, shape of T_τ^- is $\tau(N)^-$, $T_\tau^-(i,j) = k$ if and only if $(i,j) \in \tau(k)^- \setminus \tau(k-1)^-$.

Every tableau T of shape $\lambda \in \mathbb{GT}_N$ corresponds to a *3D Young diagram* in the following way: For every k put $N-k$ unit cubes on all the boxes of T with number k. The union of all these unit cubes is the desired 3D Young diagram. An example of the above procedure is shown at Figure 2.

Let $V(T)$ denote the volume of 3D Young diagram corresponding to T.

Lemma 2.1. *The following formulas hold for an arbitrary $\tau \in \mathcal{T}_N$:*

$$V(T_\tau^+) - V(T_\tau^-) = \sum_{i=1}^{N-1} |\tau(i)| = \sum_{i=1}^{N} (N-i)(|\tau(i)| - |\tau(i-1)|),$$

where we agree that $|\tau(0)| = 0$

We leave the proof to the reader.

For a finite path $\tau \in \mathcal{T}_N$ let C_τ be a corresponding cylinder set in $\mathcal{T}$, i.e.

$$C_\tau = \{\lambda(1) \prec \lambda(2) \prec \dots \in \mathcal{T} \mid \lambda(1) = \tau(1), \dots, \lambda(N) = \tau(N)\}.$$

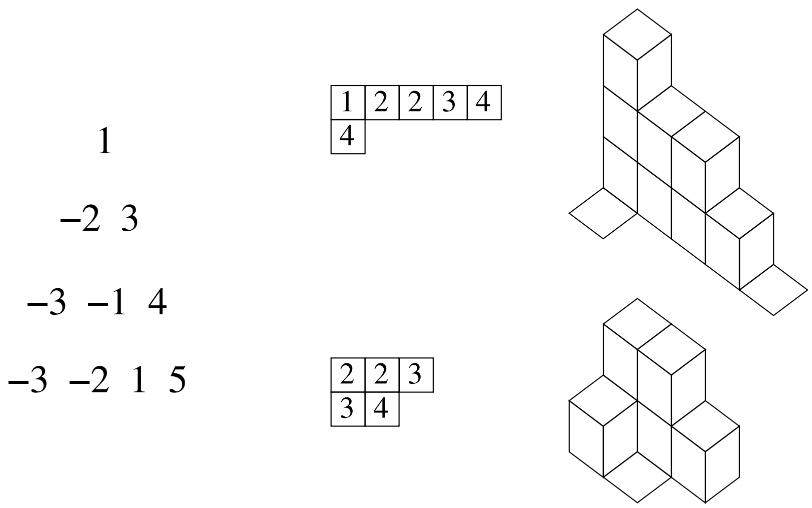

Figure 2: Finite path from $\mathcal{T}_4$, corresponding semistandard Young tableaux and 3D Young diagrams.

We equip $\mathcal{T}$ with a σ-algebra spanned by all cylinder sets. We are going to consider various probability measures on $\mathcal{T}$. For any finite path τ we usually write $P(\tau)$ instead of $P(C_\tau)$ where it leads to no confusion.

For any probability measure P on $\mathcal{T}$ let P_N be its projection on $\mathbb{GT}_N$, i.e.

$$P_N(\{\lambda\}) = P(\{\tau \in \mathcal{T} : \quad \tau(N) = \lambda\}).$$

To simplify the notation we usually write $P_N(\lambda)$ instead of $P_N(\{\lambda\}$

Recall that a probability measure P on $\mathcal{T}$ is called *q–central* if probabilities of paths $\tau \in \mathcal{T}_N$ ending at the same signature λ are proportional to $q^{V(T_\tau^+)-V(T_\tau^-)}$, i.e. for any $\tau \in \mathcal{T}_N$ such that $\tau(N) = \lambda$ we have:

$$P(\tau) = P_N(\lambda) \frac{q^{V(T_\tau^+)-V(T_\tau^-)}}{\sum\limits_{\theta \in \mathcal{T}_N :\, \theta(N)=\lambda} q^{V(T_\theta^+)-V(T_\theta^-)}}.$$

The main goal of the present paper is to describe the convex set of all q–central probability measures on $\mathcal{T}$.

3 Symmetric polynomials

In this section we introduce various symmetric functions and the algebras they belong to. These functions play an important role in our study of q–central measures.

3.1 Schur functions and factorial Schur functions

Recall that a (rational) Schur function $s_\lambda(x_1,\dots,x_N)$ parameterized by an arbitrary signature $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N$, is a symmetric Laurent polynomial given by

$$s_\lambda = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1,\dots,N}}{\prod_{i<j}(x_i-x_j)}.$$

It is known that rational Schur functions form a linear basis in the space of all symmetric Laurent polynomials. If $\lambda \in \mathbb{GT}^+$, then s_λ is a symmetric polynomial. We recommend [Mac] as a general source of information about symmetric polynomials and Schur functions.

Proposition 3.1 (The branching rule for Schur functions)**.** *For any $\lambda \in \mathbb{GT}_N$ we have:*

$$s_\lambda(x_1,\dots,x_N) = \sum_{\mu\prec\lambda} s_\mu(x_1,\dots,x_{N-1})x_N^{|\lambda|-|\mu|},$$

where $|\lambda|$ and $|\mu|$ stand for the sum of the coordinates of signatures λ and μ correspondingly.

See e.g. [Mac, Chapter 1, Section 5] for the proof. Iterating the branching rule for Schur functions we get the following:

Proposition 3.2 (The combinatorial formula)**.** *For any $\lambda \in \mathbb{GT}_k$ we have:*

$$s_\lambda(x_1,\dots,x_N) = \sum_{\tau(0)\prec\dots\prec\tau(N)} x_1^{|\tau(1)|-|\tau(0)|}\dots x_N^{|\tau(N)|-|\tau(N-1)|},$$

where the sum is taken over all paths in $\mathbb{GT}$ $\tau(0) \prec \dots \prec \tau(N)$ such that $\tau(N) = \lambda$ and $|\tau(i)|$ stands for the sum of the coordinates of signature $\tau(i)$.

From now on assume that λ is a positive signature, i.e. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$.

Let $\{a_n\}_{n\in\mathbb{Z}}$ be any sequence of numbers. For any $r \geq 0$ let

$$(x \mid a)^r = (x+a_1)\dots(x+a_r).$$

Factorial Schur function $s_\lambda(x \mid a)(x_1,\dots,x_N)$ is a symmetric polynomial in variables $x_1,\dots x_N$ defined through

$$s_\lambda(x \mid a) = \frac{\det\left[(x_i \mid a)^{\lambda_j+N-j}\right]_{i,j=1,\dots,N}}{\prod_{i<j}(x_i-x_j)}.$$

See [Mac2, 6th Variation] for the properties of these polynomials.

Proposition 3.3 (Combinatorial formula for factorial Schur polynomials)**.** *For any $\lambda \in \mathbb{GT}_N^+$ we have:*

$$s_\lambda(x \mid a) = \sum_T \prod_{(i,j)\in\lambda} (x_{T(i,j)} + a_{T(i,j)+j-i}),$$

where the sum is taken over all semistandard Young tableau $T(i,j)$ of shape λ filled with numbers $1,\dots,N$.

3.2 q–interpolation Schur polynomials

q–interpolation Schur polynomials $s^*_\lambda(x;q)$ are factorial Schur polynomials with $a_i = -q^{i-N}$:

$$s^*_\lambda(x_1,\dots,x_N;q) = s_\lambda(x \mid a), \quad a = \{-q^{i-N}\}.$$

This polynomials are a particular case of Macdonald interpolation polynomials for the case $q=t$. (See [Kn], [S], [Ok2].)

Let us introduce some notations. Suppose $\mu \in \mathbb{GT}_N^+$ and note that μ can be identified with a Young diagram with not more than N rows. Let $\mu_1 \ge \mu_2 \ge \dots$ be the row lengths of μ and let $\mu'_1 \ge \mu'_2 \ge \dots$ be the row lengths of the transposed diagram μ', or, equivalently, the column lengths of μ. For any box $(i,j) \in \mu$ set $c(i,j) = j-i$ and $h(i,j) = \mu_i - i + \mu'_j - j + 1$. Denote $n(\mu) = \sum (i-1)\mu_i$.

Proposition 3.4 (Interpolation property, [Ok2])**.** *The q–interpolation Schur polynomial $s^*_\mu(x_1,\dots,x_N;q)$ is the unique symmetric polynomial in $x_1,\dots,x_N$ such that:*

1. $\deg(s^*_\mu(x_1,\dots,x_N;q)) = |\mu|$
2. $s^*_\mu(q^{\mu-\delta};q) = q^{n(\mu')-2n(\mu)} \prod_{(i,j)\in\mu} (q^{h(i,j)}-1)$
3. $s^*_\mu(q^{\lambda-\delta};q) = 0$ *for all positive signatures (Young diagrams) $\lambda \in \mathbb{GT}_N^+$ such that $\mu \not\subseteq \lambda$,*

where $q^{\lambda-\delta}$ is $(q^{\lambda_1}, q^{\lambda_2-1}, \dots, q^{\lambda_N-N+1})$

Rewriting the combinatorial formula for the factorial Schur polynomials we get:

Proposition 3.5. *We have*

$$s^*_\mu(x_1,\dots,x_N;q) = \sum_T \prod_{(i,j)\in\mu} \left(x_{T(i,j)} - q^{j-i+T(i,j)-N}\right),$$

where the sum is taken over all semistandard Young tableau $T(i,j)$ of shape λ filled with numbers $1,\dots,N$.

In one-dimensional case interpolation Schur polynomials are enumerated by nonnegative integers and

$$s_k^*(x;q) = (x-1)\dots(x-q^{k-1})$$

Polynomials $s_\mu^*(x_1, x_2\cdot, \dots, x_N; q)$ form a linear basis in the space of symmetric polynomials in N variables. The leading term of $s_\mu^*(x_1, x_2\cdot, \dots, x_N\cdot; q)$ coincides with ordinary Schur function $s_\mu(x_1, x_2, \dots, x_N)$.

The basis s_μ^* is connected with the basis s_μ via the following formulas due to Okounkov [Ok1]

$$s_\lambda(x_1, x_2, \dots, x_k) = \sum_\mu \frac{s_\mu^*(q^{\lambda-\delta};q)}{s_\mu^*(q^{\mu-\delta};q)} \frac{s_\lambda(1, q^{-1}, \dots, q^{1-k})}{s_\mu(1, q^{-1}, \dots, q^{1-k})} s_\mu^*(x_1, \dots, x_k; q), \quad (6)$$

$$s_\lambda^*(x_1, x_2, \dots, x_k; q) = \sum_\mu \frac{s_\mu^*(q^{-(\lambda-\delta)};q^{-1})}{s_\mu^*(q^{-(\mu-\delta)});q^{-1})} \frac{s_\lambda^*(0, \dots, 0; q)}{s_\mu^*(0, \dots, 0; q)} s_\mu(x_1, \dots, x_k). \quad (7)$$

Observe that in both formulas only diagrams μ such that $\mu \subset \lambda$ give a nonzero contribution. Thus, both sums are actually finite.

Also note that the formulas (6) and (7) look similar, we are going to further use this fact.

3.3 Algebras of symmetric functions

Let Λ_N be the graded algebra of symmetric polynomials in $x_1, \dots, x_N$. Let Λ be a projective limit of graded algebras Λ_N with respect to projections ρ_N:

$$\rho_N : \Lambda_{N+1} \to \Lambda_N, \quad \rho_N(f(x_1, \dots, x_{N+1})) = f(x_1, \dots, x_N, 0).$$

Λ is usually called *the algebra of symmetric functions.* There are 3 well-known systems of algebraic generators of Λ. They are *Newton power sums* p_k

$$p_k = \sum_{i=1}^{\infty} x_i^k,$$

elementary symmetric functions e_k

$$e_k = \sum_{\ell_1 < \ell_2 < \dots < \ell_k} x_{\ell_1} \cdots x_{\ell_k}$$

and *complete symmetric functions* h_k

$$h_k = \sum_{\ell_1 \le \ell_2 \le \dots \le \ell_k} x_{\ell_1} \cdots x_{\ell_k}.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{GT}_k^+$ and let $\widehat{\lambda} \in \mathbb{GT}_{k+1}^+$ be a signature obtained by adding one zero to λ, i.e. $\widehat{\lambda} = (\lambda_1, \ldots, \lambda_k, 0)$. Schur polynomials corresponding to non-negative signatures are stable in the sense that

$$s_{\widehat{\lambda}}(x_1, \ldots, x_k, 0) = s_\lambda(x_1, \ldots, x_k)$$

for any $\lambda \in \mathbb{GT}_k^+$. Thus, the Schur polynomial corresponding to a non-negative signature λ defines an element of Λ that we denote s_λ. Functions s_λ form a linear basis in algebra Λ.

Let $\widehat{\Lambda}$ be a projective limit of filtered algebras $\widehat{\Lambda}_N$, where $\widehat{\Lambda}_N$ is the ordinary algebra of symmetric polynomials in N variables filtered by the degree, and the projections $\widehat{\rho}_N$ are given by the following formula:

$$\widehat{\rho}_N : \widehat{\Lambda}_{N+1} \to \widehat{\Lambda}_N, \quad \widehat{\rho}_N(f(x_1, \ldots, x_{N+1})) = f(x_1, \ldots, x_N, q^{-N})$$

Polynomials s_μ^* are stable in the sense that

$$s_{\widehat{\mu}}^*(x_1, \ldots, x_N, q^{-N}; q) = s_\mu^*(x_1, \ldots, x_N; q)$$

for any $\mu \in \mathbb{GT}_N^+$. This fact follows from Proposition 3.4. Thus, these polynomials can be viewed as elements of a filtered algebra $\widehat{\Lambda}$. Furthermore, s_μ^* form a linear basis of $\widehat{\Lambda}$.

4 q–Central measures and probability generating functions

In this section we state a number of propositions about q–central measures and probability generating functions related to them. Some of the proofs are quite technical and we omitted them for the convenience of the reader. All the missing proofs are given in Section 6.2.

For any $\mu \in \mathbb{GT}_N$ denote

$$\mathrm{Dim}_q(\mu) = \sum_{\tau \in \mathcal{T}_N,\, \tau(N) = \mu} q^{|\tau(1)| + \cdots + |\tau(N-1)|}$$

Lemma 4.1. *We have*

$$\mathrm{Dim}_q(\lambda) = s_\lambda(1, q, \ldots, q^{N-1}).$$

Proof. Applying the combinatorial formula for Schur functions we obtain

$$\begin{aligned} s_\lambda(1, \ldots, q^{N-1}) &= s_\lambda(q^{N-1}, \ldots, 1) \\ &= \sum_{\tau(0) \prec \cdots \prec \tau(N)} q^{(N-1)\cdot(|\tau(1)| - |\tau(0)|) + \cdots + 0\cdot(|\tau(N)| - |\tau(N-1)|)} \\ &= \sum_{\tau(0) \prec \cdots \prec \tau(N)} q^{|\tau(1)| + |\tau(2)| + \cdots + |\tau(N-1)|} = \mathrm{Dim}_q(\lambda) \end{aligned}$$

□

Let $\lambda \in \mathbb{GT}_{N+1}$, $\mu \in \mathbb{GT}_N$. We define *cotransitional probability* $P(\lambda \to \mu)$:

$$P(\lambda \to \mu) = \begin{cases} q^{|\mu|} \frac{\mathrm{Dim}_q(\mu)}{\mathrm{Dim}_q(\lambda)}, & \mu \prec \lambda, \\ 0, & \textit{otherwise}. \end{cases}$$

Proposition 4.2. *For any $\lambda \in \mathbb{GT}_{N+1}$ we have*

$$\sum_{\mu \in \mathbb{GT}_N} P(\lambda \to \mu) = 1.$$

Proof. Proposition 3.1 implies that

$$s_\lambda(1, \dots, q^N) = \sum_{\mu \prec \lambda} s_\mu(q, q^2, \dots, q^N) = \sum_{\mu \prec \lambda} q^{|\mu|} s_\mu(1, q, \dots, q^{N-1}).$$

Dividing by $s_\lambda(1, \dots, q^N)$ we get the desired equality. □

Suppose that P_N and P_{N+1} are probability distributions on $\mathbb{GT}_N$ and $\mathbb{GT}_{N+1}$, respectively. We call P_N and P_{N+1} *q–coherent* if for any $\mu \in \mathbb{GT}_N$:

$$P_N(\mu) = \sum_{\lambda \in \mathbb{GT}_{N+1}} P_{N+1}(\lambda) P(\lambda \to \mu).$$

We call probability distributions $P_1, P_2, \dots$ on $\mathbb{GT}_1, \mathbb{GT}_2, \dots$ respectively *a q–coherent system*, if P_i and P_{i+1} are q–coherent for every $i = 1, 2, \dots$.

Note that for any probability distribution P_N on $\mathbb{GT}_N$, there exist unique distributions $P_1, \dots, P_{N-1}$ on $\mathbb{GT}_1, \dots, \mathbb{GT}_{N-1}$, respectively, such that $P_1, \dots, P_N$ is a q–coherent system.

Let P be an arbitrary probability measure on $\mathcal{T}_N$. Let P_k denote a projection of P on $\mathbb{GT}_k$. The following two propositions are proved in Section 6.2:

Proposition 4.3. *If measure P is such that*

$$P(\tau(1) \prec \dots \tau(N)) = \frac{q^{|\tau(1)| + \dots + |\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))} P_N(\tau(N)), \tag{8}$$

for any path $\tau \in \mathcal{T}_N$, then $P_1, P_2, \dots, P_N$ is a q–coherent system. In particular, if P is a q–central measure on $\mathcal{T}$, then $P_1, P_2, \dots$ is a q–coherent system.

Proposition 4.4. *Let $P_1, P_2, \dots$ be a q–coherent system. There exists a unique q–central measure P such that P_k is a projection of P on $\mathbb{GT}_k$ for every $k = 1, 2, \dots$.*

Next, we want to introduce a convenient tool for studying q–central measures and q–coherent systems.

Suppose that P is a probability measure on $\mathbb{GT}_N$. *q–Schur generating function* of P is a symmetric function in $x_1, \dots, x_N$ given by:

$$\mathcal{S}(x_1, \dots, x_N; P) = \sum_{\mu \in \mathbb{GT}_N} P(\mu) \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})} \tag{9}$$

Note that when $N = 1$, this definition turns into the usual definition of probability generating function:

$$F(t) = \sum_{\ell} c_\ell t^\ell.$$

For every N we define

$$T_N = \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid |x_i| = q^{1-i}\}$$

and

$$D_N = \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid |x_i| \le q^{1-i}\}.$$

Proposition 4.5. *The series* (9) *converges uniformly on* T_N. *If* $\mathrm{supp}(P) \subset \mathbb{GT}_N^+$, *then the series* (9) *converges uniformly on* D_N.

Proof. The proposition follows from the fact that

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})} \right| \le 1,$$

for all $(x_1, \dots, x_N) \in T_N$, and if $\mu \in \mathbb{GT}_N^+$, then

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})} \right| \le 1,$$

for all $(x_1, \dots, x_N) \in D_N$.

These inequalities, in turn, easily follow from the combinatorial formula for rational Schur functions. □

The following proposition is proved in Section 6.2:

Proposition 4.6. *Suppose that* P_N *and* P_{N+1} *are two probability measures on* $\mathbb{GT}_N$ *and* $\mathbb{GT}_{N+1}$ *respectively.* P_N *and* P_{N+1} *are* q*–coherent if and only if*

$$\mathcal{S}(x_1, \dots, x_N; P_N) = \mathcal{S}(x_1, \dots, x_N, q^{-N}; P_{N+1}).$$

If $\mathrm{supp}(P) \subset \mathbb{GT}_N^+$ (i.e. P is a probability measure on $\mathbb{GT}_N^+$) then we define *q–interpolation Schur generating function* of P through

$$\mathcal{S}^*(x_1, \dots, x_N; P) = \sum_{\mu \in \mathbb{GT}_N^+} P(\mu) \frac{s^*_\mu(q^{N-1}x_1, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})}. \tag{10}$$

Proposition 4.7. *Series* (10) *converges uniformly on compact subsets of* $\mathbb{C}^N$.

Proof. Using the combinatorial formula for q–interpolation polynomials we get:

$$
\begin{aligned}
s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1}) \\
&= q^{(N-1)|\mu|} \sum_T \prod_{(i,j)\in\mu} \left(x_{T(i,j)} - q^{i-j-1+T(i,j)} \right) \\
&= q^{(N-1)|\mu|} (-1)^{|\mu|} \left(\prod_{(i,j)\in\mu} q^{i-j-1} \right) \\
&\times \sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right) \prod_{(i,j)\in\mu} \left(1 - x_{T(i,j)} q^{j-i-T(i,j)+1} \right).
\end{aligned}
$$

Let M be a constant such that $|x_i| < M$ for all i. We have

$$
\begin{aligned}
\left| \frac{s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})} \right| \\
&= \frac{\left| \sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right) \prod_{(i,j)\in\mu} \left(1 - x_{T(i,j)} q^{j-i-T(i,j)+1} \right) \right|}{\sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right)} \\
&\le \frac{\sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right) \prod_{(i,j)\in\mu} \left(1 + M q^{j-i-T(i,j)+1} \right)}{\sum_T \left(\prod_{(i,j)\in\mu} q^{T(i,j)} \right)} \\
&\le \max_T \prod_{(i,j)\in\mu} \left(1 + M q^{j-i-T(i,j)+1} \right) \\
&\le \max_T \exp\left(\sum_{(i,j)\in\mu} M q^{j-i-T(i,j)+1} \right) \le \exp\left(M \sum_{(i,j)\in\mu} q^{j-i-N} \right).
\end{aligned}
$$

Since μ has at most N rows,

$$
\sum_{(i,j)\in\mu} q^{j-i-N} < N(q^{1-2N} + q^{2-2N} + q^{3-2N} + \dots) = \frac{Nq^{1-2N}}{1-q}.
$$

We conclude that if $(x_1, \dots, x_N)$ is such that $|x_i| < M$ for every i, then

$$
\left| \frac{s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})} \right| < A(M).
$$

Consequently,

$$
\sum_{\mu\in\mathbb{GT}^+_N} c_\mu \left| \frac{s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0, \dots, 0; q^{-1})} \right| < \infty,
$$

(10) absolutely converges and this convergence is uniform on compact subsets of $\mathbb{C}^N$. □

The following proposition is proved in Section 6.2:

Proposition 4.8. *Suppose that P_N and P_{N+1} are two probability measures on $\mathbb{GT}_N^+$ and $\mathbb{GT}_{N+1}^+$ respectively. P_N and P_{N+1} are q–coherent if and only if*

$$\mathcal{S}^*(x_1,\ldots,x_N;P_N)=\mathcal{S}^*(x_1,\ldots,x_N,0;P_{N+1}).$$

Next, we want to show that convergence of q–central probability measures is equivalent to uniform convergence of their Schur generating functions.

Let P_N and P_N^i, $i=1,2,\ldots$ be probability measures on $\mathbb{GT}_N$. We say that P_N^i *weakly converges* to P^N as $i\to\infty$ if $\lim_{i\to\infty}P_N^i(\mu)=P_N(\mu)$ for every $\mu\in\mathbb{GT}_N$.

Let P and P^i be probability measures on $\mathcal{T}$. We say that P^i *weakly converges* to P as $i\to\infty$ if $\lim_{i\to\infty}P^i(C_\tau)=P(C_\tau)$ for any cylinder set C_τ.

Suppose that P and P^i are q–central probability measures on $\mathcal{T}$ and let P_N, P_N^i be the corresponding q–coherent systems (i.e. projections of the measures on $\mathbb{GT}_N$).

Proposition 4.9. *Measures P^i weakly converge to P if and only if P_N^i weakly converge to P_N for every N.*

Proof. This proposition follows from the correspondence between q–central measures and q–coherent systems. (See Proposition 4.3 and Proposition 4.4). □

Proposition 4.10. *Let P_N, P_N^i be probability measures on $\mathbb{GT}_N$. If P_N^i weakly converge to P_N, then*

$$\mathcal{S}(x_1,\ldots,x_N;P_N^i)\rightrightarrows\mathcal{S}(x_1,\ldots,x_N;P_N)$$

uniformly on T_N. If P_N, P_N^i are supported on $\mathbb{GT}_N^+$, then the convergence is uniform on D_N.

Proposition 4.11. *Let P_N^i be probability measures on $\mathbb{GT}_N$. If*

$$\mathcal{S}(x_1,\ldots,x_N;P_N^i)$$

converge uniformly on T_N to a function $S(x_1,\ldots,x_N)$, then there exists a probability measure P_N such that

$$S(x_1,\ldots,x_N)=\mathcal{S}(x_1,\ldots,x_N;P_N)$$

and P_N^i weakly converge to P_N.

Proposition 4.12. *Let P_N, P_N^i be probability measures on $\mathbb{GT}_N^+$. P_N^i weakly converge to P_N if and only if*

$$\mathcal{S}^*(x_1,\ldots,x_N;P_N^i)\rightrightarrows\mathcal{S}^*(x_1,\ldots,x_N;P_N)$$

uniformly on compact subsets of $\mathbb{C}^N$.

The proofs are quite involved and we give them in Section 6.2.

5 The boundary of the q–Gelfand–Tsetlin graph

Suppose that $\lambda \in \mathbb{GT}_N$. Let P_i^λ, $i = 1, \dots, N$ be probability measures on $\mathbb{GT}_i$ such that $P_1^\lambda, \dots, P_N^\lambda$ is a q–coherent system and P_N^λ is delta-measure on λ (i.e. $P_N^\lambda(\lambda) = 1$). Clearly, such q–coherent system exists and is unique. We call $\{P_i^\lambda\}$ *a primitive q–coherent system* corresponding to λ.

Let $N(i)$ be an increasing sequence of positive integers and $\lambda(i) \in \mathbb{GT}_{N(i)}$. We call $\lambda(i)$ a *regular* sequence if for every k probability measures $P_k^{\lambda(i)}$ weakly converge to a certain probability measure P_k. When we vary k, we see that the limit measures P_k form a q–coherent system. The set of all possible q–coherent system that can be obtained in such a way is called *the Martin boundary* of the graph. In Sections 5.1-5.4 we describe the Martin boundary of the q–Gelfand–Tsetlin graph. And in Section 5.5 we prove that the minimal boundary of the q–Gelfand–Tsetlin graph coincides with the Martin boundary of the q–Gelfand–Tsetlin graph.

In Section 3.3 we defined the algebra of symmetric function Λ. *A specialization* Spec of Λ is an arbitrary homomorphism from algebra Λ to $\mathbb{C}$. H–generating function of the specialization is the following power series

$$H(t) = \sum_{i=0}^{\infty} \mathrm{Spec}(h_k) t_k.$$

Since complete symmetric functions h_k generate Λ, H–generating function uniquely defines the specialization.

Recall that $\mathcal{N}$ is the set of all non-decreasing sequences of integers $\nu = (\nu_1 \le \nu_2 \le \dots)$.

Theorem 5.1. *The Martin boundary of the q–Gelfand–Tsetlin graph $\{\mathcal{E}_\nu^k\}$ is parameterized by points of $\mathcal{N}$.*

If a sequence of signatures $\lambda(i)$ is regular, then there exists $\nu \in \mathcal{N}$ such that the last coordinates of $\lambda(i)$ stabilize to ν, i.e. for any j

$$\lim_{i \to \infty} \lambda(i)_{N(i)+1-j} = \nu_j.$$

If the last coordinates of $\lambda(i)$ stabilize to ν, then the measures $P_k^{\lambda(i)}$ weakly tend to $\mathcal{E}_k^\nu$.

The q–Schur generating function of the measure $\mathcal{E}_\nu^k$ can be uniquely extended to a function analytic everywhere in $\mathbb{C}^k$ except, perhaps, at the origin.

If $\nu' = \nu + \ell$, i.e. for every j we have $\nu'_j = \nu_j + \ell$, then

$$\mathcal{S}(x_1, \dots, x_k; \mathcal{E}_{\nu'}^k) = \frac{x_1^\ell \cdots x_k^\ell}{1^\ell q^{-\ell} \cdots q^{(1-k)\ell}} \mathcal{S}(x_1, \dots, x_k; \mathcal{E}_{\nu'}^k)$$

If $\nu_1 \ge 0$, then

$$\mathcal{S}(x_1, \dots, x_k; \mathcal{E}_\nu^k) = \sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu) - n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k; q),$$

the series converges everywhere in $\mathbb{C}^k$ *and* Spec_ν *is a specialization of algebra* Λ *with* H*–generating function*

$$H^\nu(t) = \frac{\prod\limits_{i\geq 0}(1-q^i t)}{\prod\limits_{j=1}^{\infty}(1-q^{\nu_j+j-1}t)}$$

Propositions 4.10 and 4.11 imply that $\lambda(i)$ is regular if and only if for any k functions

$$\frac{s_{\lambda(i)}(x_1,x_2,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N(i)})}{s_\lambda(i)(1,q,\dots,q^{1-N(i)})}$$

converge uniformly on T_k. Thus, Theorems 1.3 and 5.1 are equivalent.

5.1 Simple necessary conditions for convergence

Recall that A_ℓ is an automorphisms of graph $\mathbb{GT}$ acting on an arbitrary signature λ by

$$A_\ell(\lambda_1 \geq \dots \geq \lambda_N) = (\lambda_1+\ell) \geq \dots \geq (\lambda_N+\ell).$$

We call A_ℓ an *ℓ–shift.*

Lemma 5.2. *If* $P_1, P_2, \dots$ *is a q–coherent system, then* $A_\ell(P_1), A_\ell(P_2), \dots$ *is a q–coherent system too.*

Proof. This follows form the fact that $P(\lambda \to \mu) = P(A_\ell(\lambda) \to A_\ell(\mu))$, which is straightforward. □

Proposition 5.3. *For any* $N > 0$ *and any* $\lambda = (\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{GT}_N$ *we have*

$$P_1^\lambda(\lambda_N) \geq \prod_{i=1}^{\infty}(1-q^i).$$

Here λ_N *stands for a signature of* $\mathbb{GT}_1$ *with coordinate* λ_N*.*

Proof. Note that

$$P_1^\lambda(\lambda_N) = P_1^{A_\ell(\lambda)}(\lambda_N+\ell).$$

Thus, without loss of generality we may assume that $\lambda_N = 0$.

Since P_N^λ is concentrated on the sinature λ, we have

$$\mathcal{S}(x_1,\dots,x_N;P_N^\lambda) = \frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(1,\dots,q^{1-N})}.$$

Using Proposition 4.6 we conclude that

$$\mathcal{S}(x;P_1^\lambda) = \frac{s_\lambda(x,q^{-1},\dots,q^{1-N})}{s_\lambda(1,\dots,q^{1-N})}.$$

Recall that

$$\mathcal{S}(x; P_1^\lambda) = \sum_\ell P_1^\lambda(\ell) x^\ell.$$

Since $\mathcal{S}(x_1, \dots, x_N; P_N^\lambda)$ is a polynomial, so is $\mathcal{S}(x; P_1^\lambda)$. It follows that

$$P_1^\lambda(\lambda_N) = P_1^\lambda(0) = \mathcal{S}(0; P_1^\lambda) = \frac{s_\lambda(0, q^{-1}, \dots, q^{1-N})}{s_\lambda(1, \dots, q^{1-N})} = \frac{s_\lambda(q^{-1}, \dots, q^{1-N})}{s_\lambda(1, \dots, q^{1-N})}$$

The value $s_\lambda(1, q^{-1}, \dots, q^{1-N})$ can be easily computed using the definition of the rational Schur function (see e.g. [Mac, Example 3.1]). Recall that $n(\lambda) = \sum (i-1)\lambda_i$. We have

$$s_\lambda(1, q, \dots, q^{1-N}) = q^{-n(\lambda)} \prod_{1 \le i < j \le N} \frac{1 - q^{\lambda_j - \lambda_i + i - j}}{1 - q^{i-j}} \tag{11}$$

It follows that

$$\frac{s_\lambda(q^{-1}, \dots, q^{1-N})}{s_\lambda(1, \dots, q^{1-N})} = \frac{q^{-|\lambda|} \prod\limits_{1 \le i < j \le N-1} \frac{1 - q^{-\lambda_i + \lambda_j + i - j}}{1 - q^{i-j}}}{\prod\limits_{1 \le i < j \le N} \frac{1 - q^{-\lambda_i + \lambda_j + i - j}}{1 - q^{i-j}}} = \frac{q^{-|\lambda|}}{\prod\limits_{i=1}^{N-1} \frac{1 - q^{-\lambda_i + i - N}}{1 - q^{i-N}}}$$

$$= \prod_{i=1}^{N-1} \frac{q^{-\lambda_i} - q^{-\lambda_i + i - N}}{1 - q^{-\lambda_i + i - N}} = \prod_{i=1}^{N-1} \frac{1 - q^{N-i}}{1 - q^{\lambda_i - i + N}} \ge \prod_{i=1}^{N-1} (1 - q^{N-i}) \ge \prod_{i=1}^{\infty} (1 - q^i).$$

Thus,

$$P_1^\lambda(\lambda_N) \ge \prod_{i=1}^{\infty} (1 - q^i).$$

□

Corollary 5.4. *If a sequence of measures $P_1^{\lambda(i)}$ with $\lambda(i) \in \mathbb{GT}_{N_i}$, $N_i \to \infty$, weakly converges to a certain probability measure P_1, then $\lambda(i)_{N_i}$ is bounded from below.*

Proof. Let k be an integer such that

$$P_1(\{\lambda \in \mathbb{GT}_1 : -k < \lambda < k\}) > 1 - \frac{1}{2} \prod_{i=1}^{\infty} (1 - q^i).$$

Since $P_1^{\lambda(i)}$ weakly converges to P_1 and the set $\{\lambda \in \mathbb{GT}_1 : -k < \lambda < k\}$ is finite, we conclude that

$$P_1^{\lambda(i)}(\{\lambda \in \mathbb{GT}_1 : -k < \lambda < k\}) > 1 - \prod_{i=1}^{\infty} (1 - q^i)$$

for $i > i_0$. Thus, if $\lambda(i)_{N(i)} < -k$, then $P_1^{\lambda(i)}(\lambda_{N(i)}) < \prod_{i=1}^\infty (1 - q^i)$. This is a contradiction with Proposition 5.3, consequently, $\lambda(i)_{N(i)} \ge (-k)$ for $i \ge i_0$. □

Corollary 5.4 implies that for the full description of the Martin boundary of the q–Gelfand–Tsetlin graph it is enough to study only measures concentrated on positive signatures and their ℓ–shifts.

Proposition 5.5. *Let $\lambda \in \mathbb{GT}_N$ and $k < N$. If $P_k^\lambda(\mu) > 0$, then $\mu \geq (\lambda_{N-k+1}, \lambda_{N-k+2}, \dots, \lambda_N)$.*

Proof. From the definitions of measures P_k^λ and q–coherent systems it follows that if $P_k^\lambda(\mu) > 0$, then there exists a sequence $\tau(k) \prec \tau(k+1) \prec \cdots \prec \tau(N)$ such that $\tau(k) = \mu$ and $\tau(N) = \lambda$. Consequently, for $i = 0, 1, \dots, k-1$ we have

$$\mu_{k-i} = \tau(k)_{k-i} \leq \tau(k+1)_{k+1-i} \leq \cdots \leq \tau(N)_{N-i} = \lambda_{N-i}.$$

□

Corollary 5.6. *If $\lambda(i) \in \mathbb{GT}_{N(i)}$ is a sequence of signatures such that the measures $P_k^{\lambda(i)}$ weakly converge, then the sequence of integers $\lambda(i)_{N(i)-m}$ is bounded from above for any $0 \leq m \leq k-1$.*

5.2 Proof of Theorems 1.3 and 5.1

We start with the following compactness result

Proposition 5.7. *Let $\lambda(i)$ be a sequence of signatures stabilizing to ν. The family of functions*

$$g_i(x_1, \dots, x_k) = \frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_\lambda(i)(1, q, \dots, q^{1-N(i)})}$$

is a relatively compact subset of the set of continuous functions on k–dimensional torus T_k with uniform convergence topology.

The proof is quite technical and we present it in Section 6.3. The main idea is to find a uniform estimate for the derivatives of functions $g_i(x_1, \dots, x_k)$.

In order to identify all the possible limits of the functions g_i, we want to decompose the functions in q–interpolation polynomials series. We need the following technical proposition that is proved in Section 6.1

Proposition 5.8. *Let P_N be a probability measure on $\mathbb{GT}_N$ such that* $\operatorname{supp}(P_N) \subset \mathbb{GT}_N^+$, *in other words the probability of any signature with at least one negative coordinate is zero. The Schur generating function of P_N is well-defined for all $(x_1, \dots, x_N) \in D_N$ and it can be uniquely represented as*

$$\mathcal{S}(x_1, \dots, x_N; P_N) = \sum_{\mu \in \mathbb{GT}_N^+} c_\mu s_\mu^*(x_1, \dots, x_N; q). \tag{12}$$

The series converges uniformly on any ball $B(0, r)$ with radius $0 < r < 1$ and in every point $q^{\lambda - \delta}$.

Furthermore, suppose that P_N^i *and* P_N *are probability measures on* $\mathbb{GT}_N$ *such that* $\mathrm{supp}(P_N^i) \subset \mathbb{GT}_N^+$ *and* $\mathrm{supp}(P_N) \subset \mathbb{GT}_N^+$*. Let* c_μ^i *and* c_μ *be the coefficients of the decomposition* (12) *for the Schur generating functions of* P_N^i *and* P_N*, respectively. If* P_N^i *weakly converge to* P_N *as* $i \to \infty$ *(in other words, if the Schur generating functions of the measures uniformly converge), then for every* μ

$$\lim_{i\to\infty} c_\mu^i = c_\mu.$$

Remark. Note that here we use functions $s_\mu^*(\cdot;q)$ while in the definition of a q–interpolation generating function we use $s_\mu^*(\cdot;q^{-1})$.

Let $\nu = (0 \le \nu_1 \le \nu_2 \le \dots)$ be an arbitrary nondecreasing sequence of non-negative integers.

Recall that $H^\nu(t)$ is the following function:

$$H^\nu(t) = \frac{\prod\limits_{i\ge 0}(1-q^i t)}{\prod\limits_{j=1}^{\infty}(1-q^{\nu_j+j-1}t)}$$

$H^\nu(t)$ is an analytic function in $\mathbb{C}$.

There is a one-to-one correspondence X between νs and subsets of $\mathbb{Z}_{\ge 0}$ with infinite complement given by

$$X(\nu) = \mathbb{Z}_{\ge 0} \setminus \{\nu_i + i - 1\}_{i=1,2,\dots}.$$

Observe that

$$H^\nu(t) = \prod_{x\in X(\nu)} (1-q^x t).$$

The main part of the proof of Theorem 1.3 is contained in the following proposition.

Proposition 5.9. *Let* $N(i)$ *be an increasing sequence of positive integers and* $\lambda(i) \in \mathbb{GT}_{N(i)}^+$*. Suppose that* $P_k^{\lambda(i)}$ *weakly converges as* $i \to \infty$ *to a certain probability measure* P_k *for* $k = 1, 2, \dots$*. Then the last coordinates of* $\lambda(i)$ *stabilize, i.e. there exists a nondecreasing sequence of integers* ν_j *such that for every* $j = 1, 2, \dots$

$$\lambda(i)_{N(i)+1-j} \to \nu_j.$$

The Schur generating function of the limit measure $P_k = \mathcal{E}_k^\nu$ *has the following decomposition:*

$$\mathcal{S}(x_1,\dots,x_k;\mathcal{E}_k^\nu) = \sum_{\mu\in\mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1,\dots,x_k;q),$$

where Spec_ν *is a specialization of algebra* Λ *with* H*–generating function* $H^\nu(t)$

Proof. The probability measure $P_{N(i)}^{\lambda(i)}$ is concentrated on a single signature $\lambda(i)$, thus

$$\mathcal{S}\left(x_1,\dots,x_{N(i)};P_{N(i)}^{\lambda(i)}\right)=\frac{s_{\lambda(i)}(x_1,\dots,x_{N(i)})}{s_{\lambda(i)}(1,\dots,q^{-N(i)})}.$$

Measures $P_1^{\lambda(i)},\dots,P_{N(i)}^{\lambda(i)}$ form a q–coherent system. Consequently, Proposition 4.6 yields

$$\mathcal{S}(x_1,\dots,x_k;P_k^{\lambda(i)})=\frac{s_{\lambda(i)}(x_1,x_2,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N(i)})}{s_{\lambda}(i)(1,q,\dots,q^{1-N(i)})},$$

Let us expand the right side of the last formula into the sum of q-interpolation Schur polynomials. Coefficients of this expansion are given by the formula (6):

$$\frac{s_{\lambda(i)}(x_1,x_2,\dots,x_k,q^{-k},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}=\sum_{\mu\in\mathbb{GT}^+_{N(i)}}\frac{q^{(N(i)-1)|\mu|}s_\mu^*(q^{\lambda(i)-\delta};q)}{s_\mu^*(q^{\mu-\delta};q)}\frac{s_\mu^*(x_1,\dots,x_k,q^{-k},\dots,q^{1-N(i)})}{s_\mu(1,q,\dots,q^{N(i)-1})}\qquad(13)$$

The combinatorial formula for interpolation Schur polynomials (see Proposition 3.5) implies that if $\mu_{k+1}\neq 0$, then

$$s_\mu^*(x_1,\dots,x_k,q^{-k},\dots,q^{1-N(i)})=0.$$

Furthermore, polynomials s_μ^* are stable, i.e. if $\mu_{k+1}=\mu_{k+2}=\dots=0$, then

$$s_\mu^*(x_1,\dots,x_k,q^{-k},\dots,q^{1-N(i)})=s^*_{(\mu_1,\dots,\mu_k)}(x_1,\dots,x_k).$$

It follows that

$$\frac{s_{\lambda(i)}(x_1,x_2,\dots,x_k,q^{-k},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}=\sum_{\mu\in\mathbb{GT}^+_{k}}\frac{q^{(N(i)-1)|\mu|}s_\mu^*(q^{\lambda(i)-\delta};q)}{s_\mu^*(q^{\mu-\delta};q)}\frac{s_\mu^*(x_1,\dots,x_k)}{s_\mu(1,q,\dots,q^{N(i)-1})},\qquad(14)$$

Using Proposition 4.10 and Proposition 5.8 we conclude that weak convergence of measures $P_k^{\lambda(i)}$ implies that for any μ

$$\frac{q^{(N(i)-1)|\mu|}s_\mu^*(q^{\lambda(i)-\delta};q)}{s_\mu^*(q^{\mu-\delta};q)s_\mu(1,q,\dots,q^{N(i)-1})}$$

converges as $i\to\infty$.

Using (11) we obtain

$$\lim_{N\to\infty}s_\mu(1,q,\dots,q^N)=\frac{q^{n(\mu)}}{\prod\limits_{(i,j)\in\mu}(1-q^{h(i,j)})},$$

Proposition 3.4 yields

$$s^*_\mu(q^\mu;q) = q^{n(\mu')-2n(\mu)} \prod_{(i,j)\in\mu} (q^{h(i,j)}-1).$$

Thus,

$$\frac{1}{s^*_\mu(q^\mu;q)s_\mu(1,q,\dots)} \to (-1)^{|\mu|}q^{n(\mu)-n(\mu')}$$

Consequently,

$$q^{(N(i)-1)|\mu|}s^*_\mu(q^{\lambda(i)-\delta};q) \tag{15}$$

should tend to a limit as $i\to\infty$.

Recall that functions s^*_μ form a linear basis of filtered algebra $\widehat{\Lambda}$. Consequently, (15) has a limit for any μ if and only if

$$\lim_{i\to\infty} q^{(N(i)-1)k} f(q^{\lambda(i)-\delta})$$

for any $f\in\widehat{\Lambda}$ of the degree k.

Let us introduce an analogue of Newton power sums in algebra $\widehat{\Lambda}$:

$$p^*_k = \sum_{j\geq 1} x_j^k - (q^{1-j})^k$$

(with $p^*_0 = 1$). Note that functions

$$p^*_{k_1}\cdots p^*_{k_l}$$

also form a linear basis of $\widehat{\Lambda}$,

We conclude that (15) has a limit for every μ if and only if

$$q^{(N(i)-1)(k_1+\dots+k_l)}p^*_{k_1}(q^{\lambda(i)-\delta})\cdots p^*_{k_1}(q^{\lambda(i)-\delta})$$

converges for any positive integers $k_1,\dots,k_l$. The last limits exist if and only if

$$q^{(N(i)-1)k}p^*_k(q^{\lambda(i)-\delta})$$

tends to a finite limit as $i\to\infty$ (for any k).

We have

$$q^{(N(i)-1)k}p^*_k(q^{\lambda(i)-\delta};q) = \sum_{j=1}^{N(i)}(q^{k\lambda(i)_j}-1)q^{k(N(i)-i)} = \sum_{j=0}^{N(i)} q^{kj}(q^{k\lambda(i)_{N(i)-j}}-1).$$

Denote $f(i,j) = \lambda(i)_{N(i)-j}+j$. Note that $\sum_{j=0}^{N(i)} q^{kj}(-1)$ clearly converges for any k. Thus

$$\sum_{j=0}^{N(i)} q^{kj}q^{k\lambda(i)_{N(i)-j}} = \sum_{j=0}^{N(i)} q^{kf_{i,j}}$$

should also converge. But since $f(i,j)$ is increasing in j, we have

$$\left|\sum_{j=0}^{N(i)} q^{kf(i,j)} - q^{kf(i,0)}\right| < q^{k(f(i,0)+1)}(1+q+q^2+\dots) = q^{kf(i,0)}\frac{q^k}{1-q}.$$

Informally speaking, if k is large enough, then

$$\sum_{j=0}^{N(i)} q^{kf(i,j)} \approx q^{kf(i,0)}.$$

Recall that $f(i,0)$ is an integer. Therefore, if $\sum_{j=0}^{N(i)} q^{kf(i,j)}$ converges, then either $f(i,0)\to+\infty$ or $f(i,0)$ stabilize as $i\to\infty$. Repeating the same argument for $\sum_{j=w}^{N(i)} q^{kf_{i,j}}$, $w=1,2,\dots$ we conclude that for every w, either $f(i,w)\to+\infty$ or $f(i,w)$ stabilize as $i\to\infty$. In the former case $\lambda(i)_{N(i)-w}$ is unbounded which is a contradiction with Corollary 5.6. Thus, $f(i,w)$ stabilize, in other words there exists a sequence $0\le\nu_1\le\nu_2\le\dots$ such that

$$\lambda(i)_{N(i)+1-w}\to\nu_w.$$

Then

$$\lim_{i\to\infty} q^{(N(i)-1)k} p_k^*(q^{\lambda(i)-\delta};q) = \sum_{i\geq 1}(q^{k\nu_i}-1)q^{k(i-1)}.$$

For any Young diagram (i.e. positive signature) let s_λ denote the element of Λ corresponding to Schur function $s_\lambda(x_1,\dots,x_N)$. Newton power sums p_k are algebraically independent generators of Λ. Thus, for any λ there exists a unique polynomial R_λ, such that

$$s_\lambda = R_\lambda(p_1,\dots,p_m).$$

(Of course, here m also depends on λ, but we omit this dependence to simplify the notations)

Now consider the following element of $\widehat{\Lambda}$:

$$r_\lambda^* = s_\lambda^* - R(p_1^*,\dots,p_m^*).$$

Note that, if we work with finite sets of variables (i.e. in algebras Λ_N and $\widehat{\Lambda}_N$) then the highest homogenous component of s_λ^* is exactly s_λ, and the same is true for p_k^* and p_k. It follows that the degree of r_λ^* is less than $|\lambda|$. Consequently,

$$\lim_{\imath\to\infty} q^{(N(i)-1)|\lambda|} r_\lambda^*(q^{\lambda(i)-\delta}) = 0.$$

Let Spec_ν be a specialization of Λ defined on generators p_k through

$$\mathrm{Spec}_\nu(p_k) = \sum_{i\geq 1}(q^{k\nu_i}-1)q^{k(i-1)}.$$

The above arguments show that

$$\lim_{\imath\to\infty} q^{(N(i)-1)|\lambda|}s_\lambda^* = \lim_{\imath\to\infty} q^{(N(i)-1)|\lambda|}R(p_1^*,\dots,p_m^*)$$
$$= R_\lambda(\mathrm{Spec}_\nu(p_1),\dots,\mathrm{Spec}_\nu(p_m)) = \mathrm{Spec}_\nu(s_\lambda).$$

Thus, we have proved that if $P_k^{\lambda(i)}$ weakly converges as $i\to\infty$ to a certain probability measure P_k for any $k=1,2,\dots$, then $\lambda(i)_{N(i)+1-j}\to\nu_j$ and

$$\mathcal{S}(x_1,\dots,x_k;P_k) = \lim_{i\to\infty}\mathcal{S}\left(x_1,\dots,x_k;P_k^{\lambda(i)}\right)$$
$$= \sum_{\mu\in\mathbb{GT}_k^+}(-1)^{|\mu|}q^{n(\mu)-n(\mu')}\mathrm{Spec}_\nu(s_\mu)s_\mu^*(x_1,\dots,x_k) \quad (16)$$

It remains to prove that H–generating function of Spec_ν is $H^\nu(t)$.

P–generating function of Spec_ν is the following power series

$$P(t)=\sum_{k=1}^{\infty}\mathrm{Spec}_\nu(p_k)t^{k-1}$$

The following equality relates P–generating function to H–generating function: $P(t)=H'(t)/H(t)$. See e.g. [Mac, 2.10] for the proof. The following computation completes the proof:

$$P(t)=\sum_{k=1}^{\infty}\mathrm{Spec}_\nu(p_k)t^{k-1}=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}(q^{k\nu_i}-1)q^{k(i-1)}t^{k-1}$$
$$=\sum_{i=1}^{\infty}\frac{q^{\nu_i+i-1}}{1-q^{\nu_i+i-1}t}-\sum_{i=1}^{\infty}\frac{q^{i-1}}{1-q^{i-1}t},$$

$$H(t)=\sum_{k=0}^{\infty}\mathrm{Spec}_\nu(h_k)t^k=\exp\left(\int\mathrm{Spec}_\nu(P(t))\right)$$
$$=\exp\left(\sum_{i\geq 1}-\ln(1-q^{\nu_i+i-1}t)+\sum_{i\geq 1}\ln(1-q^{i-1}t)\right)$$
$$=\frac{\prod\limits_{i\geq 0}(1-q^it)}{\prod\limits_{j=1}^{\infty}(1-q^{\nu_j+j-1}t)}=H^\nu(t). \quad (17)$$

□

We also need yet another technical proposition that will be proved in Section 6.4:

Proposition 5.10. *For every* ν *the series*

$$\sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\mu) s^*_\mu(x_1, \dots, x_k)$$

converges for all $x_1, \dots x_k$ *and defines an entire function in* $\mathbb{C}^k$.

Now we are ready to finish the proof of Theorems 1.3 and 5.1.

Proof of Theorem 1.3. Let $\lambda(i) \in \mathbb{GT}_{N(i)}$ be a regular sequence of signatures. It means that for any k probability measures $P_k^{\lambda(i)}$ weakly converge to a certain probability measure P_k. Corollary 5.4 implies that there exists ℓ such that $\lambda(i)_j \geq \ell$ for every i, j. Let $\mu(i) = A_{-\ell}(\lambda(i))$, then $\mu(i) \in \mathbb{GT}^+_{N(i)}$ and for any k probability measures $P_k^{\mu(i)}$ weakly converge to measure $\tilde{P}_k$. Applying Proposition 5.9 we conclude that the last coordinates of $\mu(i)$ stabilize to some ν_j. Thus, the last coordinates of $\lambda(i)$ also stabilize

$$\lim_{i \to \infty} \lambda(i)_{N(i)+1-j} \to \nu_j + \ell.$$

If all the coordinates of $\lambda(i)$ are non-negative starting from some i (equivalently, if the last coordinates stabilize to non-negative numbers), then Proposition 5.9 and Proposition 5.10 imply that q–Schur generating function of P_k is an analytic function with desired interpolation polynomials series decomposition. Again using the correspondence between weak convergence of probability measures and convergence of their q–Schur generating functions (Proposition 4.11 and Proposition 4.10) we conclude that polynomials of Theorem 1.3 converge to the desired analytic function. For general $\lambda(i)$ the limit function is a product of the analytic Schur generating function corresponding to measure $\mathcal{E}_k^\nu$ and polynomial

$$\frac{x_1^\ell \cdots x_k^\ell}{1^\ell q^{-\ell} \cdots q^{-(k-1)\ell}},$$

consequently the limit function is analytic.

Now suppose that a sequence of signatures $\lambda(i) \in \mathbb{GT}_{N(i)}$, $N(i) \to \infty$ is such that

$$\lim_{i \to \infty} \lambda(i)_{N(i)+1-j} \to \nu_j$$

for some sequence $\nu_1 \leq \nu_2, \dots$. Proposition 5.7 yields that the sequence of functions

$$g_i(x_1, \dots, x_k) = \frac{s_{\lambda(i)}(x_1, x_2, \dots, x_k, q^{-k}, q^{-k-1}, \dots, q^{1-N(i)})}{s_\lambda(i)(1, q, \dots, q^{1-N(i)})}$$

has converging subsequences. But in the first part of the theorem we have identified all the possible subsequential limits and their are the same. Thus, $g_i(x_1, \dots, x_k)$ converges uniformly on T_k. $\square$

5.3 Limit measures

The only formula we have for q–Schur generating functions of $\mathcal{E}_k^\nu$ is an infinite series expansion. The situation is different if we turn to q–interpolation Schur generating functions.

Proposition 5.11. *We have*

$$\mathcal{S}^*(x_1, \ldots, x_k; E_k^\nu) = H^\nu(x_1) \cdots H^\nu(x_k).$$

Recall that $\mathcal{S}^*(x_1, \ldots, x_k; P_k)$ is an entire function for any measure P_k with $\mathrm{supp}(P_k) \subset \mathbb{GT}_k^+$. We need the following lemma:

Lemma 5.12. *Let P_k be a probability measure of $\mathbb{GT}_k^+$ Suppose that $\mathcal{S}^*(x_1, \ldots, x_k; P_k)$ has the following Taylor series expansion*

$$\mathcal{S}^*(x_1, \ldots, x_k; P_k) = \sum_{\mu \in \mathbb{GT}_k^+} a_\mu \frac{s_\mu(x_1, \ldots, x_k)}{s_\mu(1, q^{-1}, \ldots, q^{1-k})}$$

Define

$$F = \sum_{\mu \in \mathbb{GT}_k^+} a_\mu \frac{s_\mu^*(x_1, x_2, \ldots, x_k; q)}{s_\mu^*(0, \ldots, 0; q)}. \tag{18}$$

Then the series on the right side of (18) *converges uniformly on any ball $B(0, r)$ with radius $0 < r < 1$ and in every point $q^{\lambda - \delta}$. Furthermore,*

$$F = \mathcal{S}(x_1, \ldots, x_k; P_k).$$

If the support of the measure P_k is finite, then both $\mathcal{S}^*(x_1, \ldots, x_k; P_k)$ and F are polynomials and Lemma 5.12 follows from the symmetry between formulas (7) and (6). The complete proof of the lemma is given in Section 6.1.

Proof of Proposition 5.11. Let us expand $\mathcal{S}^*(x_1, \ldots, x_k; E_k^\nu)$ in Taylor series:

$$\mathcal{S}^*(x_1, \ldots, x_k; E_k^\nu) = \sum_{\mu \in \mathbb{GT}_k^+} a_\mu \frac{s_\mu(x_1, \ldots, x_k)}{s_\mu(1, q^{-1}, \ldots, q^{1-k})}.$$

Using Lemma 5.12 we conclude that

$$\mathcal{S}(x_1, \ldots, x_k; E_k^\nu) = \sum_{\mu \in \mathbb{GT}_k^+} a_\mu \frac{s_\mu^*(x_1, x_2, \ldots, x_k; q)}{s_\mu^*(0, \ldots, 0; q)}.$$

Comparing with Theorem 5.1 we conclude that

$$a_\mu = (-1)^{|\mu|} q^{n(\mu) - n(\mu')} \mathrm{Spec}_\mu(s_\mu) s_\mu^*(0, \ldots, 0; q).$$

The combinatorial formulas for polynomials s_λ and s_λ^* imply that

$$(-1)^{|\mu|}q^{n(\mu)-n(\mu')}\frac{s_\mu^*(0,\dots,0;q)}{s_\mu(1,q^{-1},\dots,q^{1-k})}=1.$$

Thus,

$$\begin{aligned}\mathcal{S}^*(x_1,\dots,x_k;E_k^\nu)\\ =\sum_{\mu\in\mathbb{GT}_k^+}(-1)^{|\mu|}q^{n(\mu)-n(\mu')}\mathrm{Spec}_\mu(s_\mu)s_\mu^*(0,\dots,0;q)\frac{s_\mu(x_1,\dots,x_k)}{s_\mu(1,q^{-1},\dots,q^{1-k})}\\ =\sum_{\mu\in\mathbb{GT}_k^+}\mathrm{Spec}_\mu(s_\mu)s_\mu(x_1,\dots,x_k).\end{aligned}$$

It remains to prove that

$$H^\nu(x_1)\cdots H^\nu(x_k)=\sum_{\mu\in\mathbb{GT}_k^+}\mathrm{Spec}_\nu(s_\mu)s_\mu(x_1,\dots,x_k), \tag{19}$$

where

$$H^\nu(x)=\sum_{j=0}^{\infty}\mathrm{Spec}_\nu(h_j)x^j.$$

The formula (19) is a particular case of the well-known Cauchy identity for symmetric functions (see e.g. [Mac, Chapter 1, Section 4]) and we leave its proof to the reader. □

5.4 Some properties of measures $\mathcal{E}^\nu$

In this section we discuss various properties of the measures $\mathcal{E}_k^\nu$.

For any nondecreasing infinite sequence of integers $\nu=(\nu_1,\nu_2,\dots)$ set

$$A_\ell(\nu)=(\nu_1+\ell,\nu_2+\ell,\dots)$$

Proposition 5.13. *For any nondecreasing sequence of nonnegative integers $\nu=(\nu_1,\nu_2,\dots)$ any any $\ell=1,2,\dots$ we have*

$$A_\ell(\mathcal{E}^\nu)=E^{A_\ell(\nu)}.$$

Proof. There exists a sequence of signatures $\lambda(i)$, such that $\mathcal{E}_k^\nu=\lim P_k^{\lambda(i)}$ for any k. Using Theorem 5.1 we conclude that

$$A_\ell(\mathcal{E}_k^\nu)=\lim A_\ell P_k^{\lambda(i)}=\lim P_k^{A_\ell(\lambda_i)}=\mathcal{E}_k^{A_\ell(\nu)}.$$

□

Let us introduce a partial order on $\mathcal{N}$. We write $\nu\le\nu'$ if $\nu_i\le\nu_i'$ for every i.

Proposition 5.14. *1. For any signature $\mu \in \mathbb{GT}_k$, $\mathcal{E}_k^\nu(\mu) = 0$ unless $\mu \geq (\nu_k, \nu_{k-1}, \ldots, \nu_1)$. Furthermore, $\mathcal{E}_k^\nu((\nu_k, \ldots, \nu_1)) > 0$.*

2. If $\nu' > \nu$, then

$$\mathcal{E}_k^\nu((\nu_k, \ldots, \nu_1)) > \mathcal{E}_k^{\nu'}((\nu_k, \ldots, \nu_1)).$$

Proof. Without loss of generality assume that $\nu_1 \geq 0$. For μ not belonging to $\mathbb{GT}_k^+$, we have $\mathcal{E}_k^\nu(\mu) = 0$. For $\mu \in \mathbb{GT}_k^+$, $\mathcal{E}_k^\nu(\mu)$ is found from the representation:

$$H^\nu(x_1) \cdots H^\nu(x_k) = \sum_{\mu \in \mathbb{GT}_k^+} \mathcal{E}_k^\nu(\mu) \frac{s_\mu^*(q^{k-1}x_1, \ldots, q^{k-1}x_k; q^{-1})}{s_\mu^*(0, \ldots, 0; q^{-1})}. \tag{20}$$

To find $\mathcal{E}_k^\nu(\mu)$ we substitute $x = q^{-(k-1+\lambda-\delta)}$ into (20) and use interpolation property of polynomials s^* (which is stated in Proposition 3.4). We start from the empty diagram $\lambda = \emptyset$ and then add boxes to it and find the numbers $\mathcal{E}_k^\nu(\mu)$ inductively. (We discuss this procedure in more details later. See Proposition 6.1.) It follows from the definition of function $H^\nu(t)$ that

$$H^\nu(q^{-(\lambda_1+k-1)}) \cdots H^\nu(q^{-(\lambda_k)}) = 0,$$

unless $(\lambda_1, \ldots, \lambda_k) \geq (\nu_k, \ldots, \nu_1)$. Thus, $\mathcal{E}_k^\nu(\mu) = 0$ unless $\mu \geq (\nu_k, \nu_{k-1}, \ldots, \nu_1)$.

Next, substitute $x = (q^{-(\nu_k+k-1)}, \ldots, q^{-\nu_1})$ in (20). We obtain

$$H^\nu(q^{-(\nu_k+k-1)}) \cdots H^\nu(q^{-(\nu_1)}) = E_k^{(\nu_k,\ldots,\nu_1)}((\nu_k, \ldots, \nu_1)) \frac{s^*_{(\nu_k,\ldots,\nu_1)}(q^{-\nu_k}, \ldots, q^{-\nu_1+k-1}; q^{-1})}{s^*_{(\nu_k,\ldots,\nu_1)}(0, \ldots, 0; q^{-1})}.$$

Observe that for $i = 1, \ldots, k$ we have $H^\nu(q^{-(\nu_i+i-1)}) > 0$. It follows that

$$\mathcal{E}_k^\nu((\nu_k, \ldots, \nu_1)) > 0.$$

Now, let us prove the second part of Proposition 5.14. If for some $i \in \{1, \ldots, k\}$, $\nu_i' > \nu_i$, then

$$\mathcal{E}_k^{\nu'}((\nu_k, \ldots, \nu_1)) = 0.$$

Otherwise note that for $i = 1, \ldots, k$ we have $H^\nu(q^{-(\nu_i+i-1)}) > H^{\nu'}(q^{-(\nu_i+i-1)})$. In both cases

$$\mathcal{E}_k^\nu((\nu_k, \ldots, \nu_1)) > \mathcal{E}_k^{\nu'}((\nu_k, \ldots, \nu_1)).$$

□

Actually we can say more, i.e. $\mathcal{E}_k^\nu((\nu_k, \ldots, \nu_1)) > c > 0$ for some constant c depending solely on k. Let us prove this fact for $k = 1$.

Lemma 5.15. *We have*

$$\mathcal{E}_1^\nu(\nu_1) \geq \prod_{i=1}^{\infty}(1 - q^i).$$

Proof. Assume without loss of generality that $\nu_1 \geq 0$. Repeating the argument of Proposition 5.14 we conclude that

$$\mathcal{E}_1^{\nu}(\nu_1) = \frac{H^{\nu}(q^{-\nu_1})}{(1-q^{-\nu_1})\cdots(1-q^{-1})} = \frac{\prod\limits_{j\geq 0,\, j\neq \nu_1}(1-q^{-\nu_1}q^j)}{\prod\limits_{j=2}^{\infty}(1-q^{-\nu_1}q^{\nu_j+j-1})}\frac{1}{(1-q^{-\nu_1})\cdots(1-q^{-1})}$$

$$= \frac{\prod\limits_{j=\nu_1+1}^{\infty}(1-q^{-\nu_1}q^j)}{\prod\limits_{j=2}^{\infty}(1-q^{-\nu_1}q^{\nu_j+j-1})} \geq \prod_{j=\nu_1+1}^{\infty}(1-q^{-\nu_1}q^j) = \prod_{i=1}^{\infty}(1-q^i)$$

□

The set $\mathcal{N} \subset \mathbb{Z}^{\infty}$ has a natural topology as a subset of the direct product of discrete spaces $\mathbb{Z}$. A sequence $\theta(i)$ converges to ν in this topology if and only if for every k there exist i_0 such that $\theta(i)_k = \nu_k$ for $i > i_0$.

Proposition 5.16. *Sequence of probability measures $E^{\theta(i)}$ weakly converge to a probability measure P if and only if $P = E^{\nu}$ for some ν, and $\theta(i) \to \nu$.*

Proof. Suppose that $\theta(i) \to \nu$. Without loss of generality assume that $\nu_1 \geq 0$. Then for $i > i_i$, $\theta(i)_1 \geq 0$. Thus, for $i > i_1$ we may use Proposition 5.11.One proves that

$$H^{\theta(i)}(t) \rightrightarrows H^{\nu}(t)$$

uniformly on compact subsets of $\mathbb{C}$. It follows that for any k

$$\mathcal{S}^*(x_1,\ldots,x_k;E^{\theta(i)}) = H^{\theta(i)}(x_1)\cdots H^{\theta(i)}(x_k) \rightrightarrows H^{\nu}(x_1)\cdots H^{\nu}(x_k) = \mathcal{S}^*(x_1,\ldots,x_k;E^{\nu}).$$

Using Proposition 4.12 and Proposition 4.9 we conclude that measures $E^{\theta(i)}$ weakly converge to E^{ν}.

Now suppose that $E^{\theta(i)}$ is a weakly convergent sequence. Lemma 5.15 implies that $E_1^{\theta(i)}(\theta(i)_1) > c$ for some constant $c > 0$. Thus, $\theta(i)_1$ should be bounded from below. (Otherwise, measures $E_1^{\theta(i)}$ "escape to infinity".) Choose $l \geq \max(-\theta(i)_1)$. Applying, if necessary, A_ℓ we may assume without loss of generality that $\theta(i)_1 \geq 0$ for all i. Then Proposition 4.12 yields that q–interpolation Schur generating functions of measures $E_1^{\theta(i)}$ converge as $i \to \infty$ to q–interpolation Schur generating function of measure P uniformly on compact subsets of $\mathbb{C}$. Recall that

$$\mathcal{S}^*(t;E_1^{\theta(i)}) = H^{\theta(i)}(t) = \prod_{t\in X(\theta(i))}(1-q^x t).$$

Due to Rouché's theorem, uniform convergence of analytic functions implies the convergence of positions of their simple zeros. Since zeros of $H^{\theta(i)}(t)$ are precisely q^{-x}, $x \in X(\theta(i))$, we conclude that the sequence $\theta(i)$ convergence to a certain $\nu \in \mathcal{N}$. Thus, $E^{\theta(i)} \to E^{\nu}$. □

Proposition 5.17. *If for some $\theta \in \mathcal{N}$ and for some probability measure π defined on σ–algebra of Borel sets in $\mathcal{N}$ we have*

$$E^{\theta} = \int_{\mathcal{N}} E^{\nu} d\pi,$$

i.e. for any cylinder set C_τ

$$E^{\theta}(C_\tau) = \int_{\mathcal{N}} E^{\nu}(C_\tau) d\pi,$$

then π is a delta measure on θ, in other words $\pi(\theta) = 1$.

Proof. Let us prove that $\pi(\{\nu : \nu \geq \theta\}) = 1$. Assume the opposite. Then there exist k and $\tilde{\nu}_1, \dots, \tilde{\nu}_k$, such that $\pi(\{\nu : \nu_i = \tilde{\nu}_i \text{ for } i = 1, \dots, k\}) > 0$ and $\nu_i < \theta_i$ for some $i \in \{1, \dots, k\}$. But we have

$$E^{\theta}_k = \int_{\mathcal{N}} E^{\nu}_k d\pi,$$

in particular

$$E^{\theta}_k((\tilde{\nu}_1, \dots, \tilde{\nu}_k)) = \int_{\mathcal{N}} E^{\nu}_k((\tilde{\nu}_k, \dots, \tilde{\nu}_1)) d\pi, \tag{21}$$

The first part of Proposition 5.14 implies that the left side of (21) vanishes, while the right side is positive. This contradiction proves that π–almost surely $\nu \geq \theta$.

On the other hand

$$E^{\theta}_k((\theta_k, \dots, \theta_1)) = \int_{\mathcal{N}} E^{\nu}_k((\theta_k, \dots, \theta_1)) d\pi, \tag{22}$$

Using the second part of Proposition 5.14 we conclude that if $\pi(\{\nu : \nu > \theta\}) > 0$ then the right side of (22) should be strictly less than the left side.

Thus, $\pi(\theta) = 1$. □

5.5 Proof of Theorem 1.1

We start with 3 propositions which hold not only for the for the q–Gelfand–Tsetlin graph, but in a much larger generality.

Let Ω_q denote the convex set of q–central probability measures on $\mathcal{T}$. There is a natural topology on Ω_q, i.e. a minimal topology such that for any cylinder set C_τ the map

$$O_\tau : \Omega_q \to \mathbb{R}, \quad O_\tau(P) = P(C_\tau)$$

is continuous. Convergence in this topology coincides with weak convergence of probability measures.

Recall that the minimal boundary of the q–Gelfand-Tsetlin graph $\operatorname{Ex}\Omega_q$ is the set of extremal points of Ω_q. We denote elements of $\operatorname{Ex}\Omega_q$ by ω.

Proposition 5.18. *Ω_q is a simplex, i.e. for any $P \in \Omega_q$ there is a unique measure π on $\operatorname{Ex}\Omega_q$ such that*

$$P = \int_{\operatorname{Ex}\Omega_q} \omega d\pi.$$

Proposition 5.19. *The minimal boundary is a subset of the Martin boundary of the q–Gelfand–Tsetlin graph. More precisely, if $P \in \operatorname{Ex}\Omega_q$ and P_k is a q–coherent system corresponding to P, then P_k belongs to the Martin boundary of the q–Gelfand–Tsetlin graph.*

Remark. For the most non-degenerate examples the Martin boundary of the graph coincides with the minimal boundary. However, there exist graphs for which the minimal boundary is strictly less then the Martin boundary.

Proposition 5.20. *Let $P \in \operatorname{Ex}\Omega_q$ and let $\tau \in \mathcal{T}$. P-almost surely the sequence of signatures $\tau(N)$ is regular and*

$$P_k^{\tau(N)} \to P_k$$

for every k.

For the proofs see [Olsh, Theorem 9.2], [OkOl, Section 6] and [Olsh, Proposition 10.8] respectively. Similar propositions were proved by Diaconis and Freedman in the framework of *partial exchangeability*. See [DF, Theorem 1.1].

Theorem 5.21. *The set Ω_q of all q–central probability measures on $\mathcal{T}$ is a simplex with extreme points $\mathcal{E}^\nu$, i.e. for any q–central probability measure $P \in \Omega_q$ there exists a unique probability measure π on $\mathcal{N}$ such that*

$$P = \int_{\mathcal{N}} E^\nu d\pi.$$

Proof. Proposition 5.18 implies that Ω_q is a simplex. Proposition 5.19 and Theorem 5.1 imply that

$$\operatorname{Ex}\Omega_q \subset \{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}},$$

where $\mathcal{E}^\nu$ is a q–central measure corresponding to q–coherent system $\mathcal{E}_k^\nu$. It follows from Proposition 5.16 that $\{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}$ with topology induced from Ω_q is isomorphic to $\mathcal{N}$. Finally, Proposition 5.17 implies that $\operatorname{Ex}\Omega_q = \{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}$. Indeed, if $Q \in \mathcal{E}^\nu$, then

$$Q = \int_{\operatorname{Ex}\Omega_q} \omega d\pi = \int_{\{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}} \omega d\pi',$$

where π' is a probability measure on $\{\mathcal{E}^\nu\}_{\nu\in\mathcal{N}}$ such that

$$\pi'(\operatorname{Ex}\Omega_q) = 1.$$

But then π' is δ–measure on a single element $Q = E^\nu$. Thus, $E^\nu \in \operatorname{Ex}\Omega_q$. □

The proved theorem is readily seen to be equivalent to Theorem 1.1. Now Proposition 1.2 is a straightforward corollary of Theorem 1.1 and Proposition 5.20

6 Some proofs

6.1 Relations between Schur and interpolation Schur functions

In this section we aim to prove Proposition 5.8 and Lemma 5.12. To do that we need some preparations.

Denote by $\mathcal{F}_N$ a class of Schur generating functions of probability measures supported on $\mathbb{GT}_N^+$. In other words, $F(x_1, \dots, x_N) \in \mathcal{F}_N$ if F is a symmetric analytic functions $F(x_1, \dots, x_N)$ on D_N such that

$$F(x_1, x_2, \dots, x_N) = \sum_{\mu \in \mathbb{GT}_N^+} \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, q^{-1}, \dots, q^{1-N})} c_\mu, \tag{23}$$

for a certain sequence of numbers, c_μ, $\mu \in \mathbb{GT}_N^+$, such that $c_\mu \geq 0$ and $\sum_{\mu \in \mathbb{GT}_N^+} c_\mu = 1$.

Clearly, (23) is essentially just a Taylor series decomposition, thus, if such decomposition exists, then it is unique.

We also want to consider decompositions of symmetric functions into interpolation Schur polynomials:

$$F(x_1, \dots, x_N) = \sum_\mu a_\mu s_\mu^*(x_1, x_2 \cdot, \dots, x_N; q) \tag{24}$$

Proposition 6.1. *There exist coefficients K_μ^λ, ($\mu, \lambda \in \mathbb{GT}_N^+$), such that for every function $F(x_1, \dots, x_N)$ defined in points $q^{\lambda-\delta}$ for every $\lambda \in \mathbb{GT}_N^+$, there is a unique decomposition*

$$F(x_1, \dots, x_N) = \sum_\mu a_\mu s_\mu^*(x_1, x_2 \cdot, \dots, x_N; q), \tag{25}$$

where the series converges in points $x = q^{\lambda-\delta}$ for every $\lambda \in \mathbb{GT}_N^+$. We have

$$a_\mu = \sum_{\lambda \in \mathbb{GT}_N^+} K_\mu^\lambda F(q^{\lambda-\delta}). \tag{26}$$

Furthermore, $K_\mu^\lambda = 0$ unless $\lambda \subset \mu$, thus, all sums in (26) are finite.

Remark. Note that this proposition is valid not only for $0 < q < 1$ but also for $q > 1$.

Proof of Proposition 6.1. Substitute $x = q^{\lambda-\delta}$ for every $\lambda \in \mathbb{GT}_N^+$ in (25). We obtain a system of linear equations

$$F(q^{\lambda-\delta}) = \sum_{\mu \in \mathbb{GT}_N^+} a_\mu L_\lambda^\mu, \tag{27}$$

where

$$L_\lambda^\mu = s_\mu^*(q^{\lambda-\delta}).$$

Proposition 3.4 implies that $L_\lambda^\mu = 0$, unless $\mu \subset \lambda$, and $L_\mu^\mu \neq 0$. Thus, the matrix L_λ^μ has a triangular structure (with respect to the partial order on signatures defined in Section 2) and all the sums in (27) are finite. Consequently, one may solve the system (27) inductively, starting from c_μ with $|\mu| = 0$, then proceeding to c_μ with $|\mu| = 1$ and so on. In other words, there exists a matrix K_μ^λ (which also has a triangular structure), such that

$$\sum_{\lambda \in \mathbb{GT}_N^+} K_{\mu_1}^\lambda L_\lambda^{\mu_2} = \begin{cases} 1, & \mu_1 = \mu_2, \\ 0. & \textit{otherwise}. \end{cases}$$

The coefficients of the matrix K_μ^λ are the desired ones. □

For a general function F we can not claim that the series (25) converges in any points other that $q^{\lambda-\delta}$. However, for functions from $\mathcal{F}_n$ the following lemma holds:

Lemma 6.2. *Suppose that function F belong to $\mathcal{F}_N$. There exists a unique decomposition*

$$F(x_1, \ldots, x_N) = \sum_\mu a_\mu s_\mu^*(x_1, x_2, \ldots, x_N; q).$$

The series converges uniformly on any ball $B(0, r)$ with radius $0 < r < 1$ and in every point $q^{\lambda-\delta}$.

Proof. To simplify the notations let us consider the case $K = 1$. The general case is very similar.

Equality (6) yields

$$x^l = \sum_{m \leq l} \left[\frac{(q^l - 1) \ldots (q^l - q^{m-1})}{(q^m - 1) \ldots (q^m - q^{m-1})} \right] (x - 1) \ldots (x - q^{m-1}) \tag{28}$$

By the definition of $\mathcal{F}_1$ we have

$$F(x) = \sum_{l \geq 0} c_l x^l,$$

where $c_l \geq 0$ and $\sum c_l = 1$.

Substituting (28) we get

$$F(x) = \sum_{l \geq 0} c_l \sum_{m \leq l} \left[\frac{(q^l - 1) \ldots (q^l - q^{m-1})}{(q^m - 1) \ldots (q^m - q^{m-1})} \right] (x - 1) \ldots (x - q^{m-1}) \tag{29}$$

Observe that the coefficients $\frac{(q^l-1)\ldots(q^l-q^{m-1})}{(q^m-1)\ldots(q^m-q^{m-1})}$ are uniformly bounded in l, m. If $x = q^k$, then in (28) only first $k+1$ terms are nonzero and if $|x| < r < 1$, then the series in (28) converges exponentially fast. In both cases (29) absolutely converges and we may change the order of summation.

We obtain

$$F(x) = \sum_{m\geq 0} (x-1)\dots(x-q^{m-1}) \cdot \sum_{l\geq m} c_l \left[\frac{(q^l-1)\dots(q^l-q^{m-1})}{(q^m-1)\dots(q^m-q^{m-1})}\right], \quad (30)$$

which is the required decomposition.

The uniqueness of the decomposition follows from Proposition 6.1. □

Proposition 6.3. *Suppose that functions F^n and F belong to $\mathcal{F}_N$, $F^n \rightrightarrows F$ on T_N. Then the coefficients a^n_μ of the interpolation Schur polynomials expansion (25) for the functions F_n converge to the corresponding coefficients a_μ of the function F.*

Proof. Uniform convergence on T_N implies that $F^n(q^{\lambda-\delta}) \to F(q^{\lambda-\delta})$. Applying Proposition 6.1 we conclude that $a^n_\mu \to a_\mu$. □

Combining Lemma 6.2 with Proposition 6.3 we arrive at Proposition 5.8.

Now let us turn to q–interpolation Schur generating functions. Let $\mathcal{F}^*_N$ denote a class of q–interpolation Schur generating functions of probability measures supported on $\mathbb{GT}^+_N$. In other words, $F(x_1,\dots,x_N) \in \mathcal{F}^*_N$ if F is a symmetric analytic functions $F(x_1,\dots,x_N)$ on $\mathbb{C}^N$ such that

$$F(x_1,\dots,x_N) = \sum_{\mu\in\mathbb{GT}^+_N} c_\mu \frac{s^*_\mu(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s^*_\mu(0,\dots,0;q^{-1})} \quad (31)$$

for a certain sequence of numbers, c_μ, $\mu \in \mathbb{GT}^+_N$, such that $c_\mu \geq 0$ and $\sum_{\mu\in\mathbb{GT}^+_N} c_\mu = 1$.

Note that if decomposition (31) exists, then it is unique. To prove this fact we repeat the argument of Proposition 6.1.

Next, we want to study the relation between q–Schur generating function and q–interpolation Schur generating function of the same probability measure.

Let Sym(N) denote the space of symmetric polynomials in N variables $x_1,\dots,x_N$.

Consider a linear map $G : \mathrm{Sym(N)} \to \mathrm{Sym(N)}$ defined on Schur polynomials' basis through

$$G\left(\frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})}\right) = \frac{s^*_\mu(x_1,x_2,\dots,x_N;q)}{s^*_\mu(0,\dots,0;q)}, \quad (32)$$

or, equivalently,

$$G(s_\mu(x_1,\dots,x_N)) = (-1)^{|\mu|} q^{n(\mu)-n(\mu')} s^*_\mu(x_1,\dots,x_N;q)$$

Note that $s^*_\lambda(x_1,\dots,x_N;1) = s_\lambda(x_1-1,\dots,x_N-1)$. Thus when $q=1$ map G becomes a simple change of variables:

$$G_{q=1}f(x_1,\dots,x_N) = f(1-x_1,\dots,1-x_N).$$

Also consider another map $G' : \mathrm{Sym(N)} \to \mathrm{Sym(N)}$ defined on q–interpolation polynomials through:

$$G'\left(\frac{s_\mu^*(q^{k-1}x_1, q^{k-1}x_2, \dots, q^{k-1}x_k; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})}\right) = \frac{s_\mu(x_1, \dots, x_k)}{s_\mu(1, q^{-1}, \dots, q^{1-k})} \tag{33}$$

Lemma 6.4. *$G = G'$ on all finite degree polynomials.*

Remark. In one-dimensional case we have:

$$G(x^k) = (1-x)(1-xq^{-1})\dots(1-xq^{1-k}),$$

$$G'((1-x)(1-xq)\dots(1-xq^{k-1})) = x^k$$

The fact that $G = G'$ follows from the q–binomial theorem.

Proof of Lemma 6.4. Take the equality (7) with q replaced by q^{-1} and with x replaced by $q^{k-1}x$. We get:

$$s_\lambda^*(q^{k-1}x_1, \dots, q^{k-1}x_N; q^{-1}) \\ = \sum_\mu \frac{s_\mu^*(q^{\lambda-\delta}; q)}{s_\mu^*(q^{\mu-\delta}); q)} \frac{s_\lambda^*(0, \dots, 0; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})} q^{(N-1)|\mu|} s_\mu(x_1, \dots, x_N).$$

Apply G in the sense of formula (32) to the righthand-side and G' in the sense of formula (33) to the lefthand-side. We get the equality

$$s_\lambda(x_1, \dots, x_N) \frac{s_\lambda^*(0, \dots, 0; q^{-1})}{s_\lambda(1, q^{-1}, \dots, q^{1-N})} \\ \stackrel{?}{=} \sum_\mu \frac{s_\mu^*(q^{\lambda-\delta}; q)}{s_\mu^*(q^{\mu-\delta}); q)} \frac{s_\lambda^*(0, \dots, 0; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})} \frac{s_\mu(1, q^{-1}, \dots, q^{1-N})}{s_\mu^*(0, \dots, 0; q)} q^{(N-1)|\mu|} s_\mu^*(x_1, x_2, \dots, x_N k; q). \tag{34}$$

Since

$$\frac{q^{(N-1)|\mu|} s_\mu(1, q^{-1}, \dots, q^{1-N})}{s_\mu^*(0, \dots, 0; q^{-1}) s_\mu^*(0, \dots, 0; q^{-1})} = \frac{1}{s_\mu(1, q^{-1}, \dots, q^{1-N})},$$

(34) is equivalent to

$$s_\lambda(x_1, \dots, x_N) \stackrel{?}{=} \sum_\mu \frac{s_\mu^*(q^{\lambda-\delta}; q)}{s_\mu^*(q^{\mu-\delta}; q)} \frac{s_\lambda(1, q^{-1}, \dots, q^{1-N})}{s_\mu(1, q^{-1}, \dots, q^{1-N})} s_\mu^*(x_1, x_2, \dots, x_N; q),$$

which is exactly (6). Hence, the last equality is true and G coincides with G' on all q–interpolation Schur polynomials. Consequently, they coincide on all polynomials. □

Next, we want to extend the domain of definition of the maps G and G'. For any function $f \in \mathcal{F}_N^*$ we can define $G'(f)$ as follows:

$$f(x_1,\dots,x_N) = \sum_{\mu\in\mathbb{GT}_N^+} c_\mu \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2,\dots,q^{N-1}x_N; q^{-1})}{s_\mu^*(0,\dots,0;q^{-1})},$$

$$G'(f) = \sum_{\mu\in\mathbb{GT}_N^+} c_\mu \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})}.$$

It is clear that G' is a bijection between $\mathcal{F}_N^*$ and $\mathcal{F}_N$. More precisely,

$$G'(\mathcal{S}^*(x_1,\dots,x_N;P)) = \mathcal{S}(x_1,\dots,x_N;P)$$

for any probability measure P on $\mathbb{GT}_N$ with $\mathrm{supp}(P)\subset\mathbb{GT}_N^+$.

Proof of Lemma 5.12. Recall that we want to prove the following statement: Suppose that $F\in\mathcal{F}_N^*$ has the following Taylor series expansion

$$F(x_1,\dots,x_N) = \sum_{\mu\in\mathbb{GT}_N^+} a_\mu \frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,q^{-1},\dots,q^{1-N})}$$

Define

$$G(f) = \sum_{\mu\in\mathbb{GT}_N^+} a_\mu \frac{s_\mu^*(x_1,x_2,\dots,x_N;q)}{s_\mu^*(0,\dots,0;q)}. \tag{35}$$

Then the series on the right side of (35) converges uniformly on any ball $B(0,r)$ with radius $0<r<1$ and in every point $q^{\lambda-\delta}$. Furthermore, $G(f)=G'(f)$.

Let us start the proof. Lemma 6.2 yields that $G'(f)$ can be represented as a linear combination of q–interpolation Schur polynomials:

$$G'(f) = \sum_{\mu\in\mathbb{GT}_N^+} b_\mu \frac{s_\mu^*(x_1,x_2,\dots,x_N;q)}{s_\mu^*(0,\dots,0;q)}.$$

This series converges uniformly on any ball $B(0,r)$ with radius $0<r<1$ and in every point $q^{\lambda-\delta}$. To prove the proposition we should check that $a_\mu=b_\mu$.

Suppose that

$$F(x_1,\dots,x_N) = \sum_{\mu\in\mathbb{GT}_N^+} c_\mu \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2,\dots,q^{N-1}x_N; q^{-1})}{s_\mu^*(0,\dots,0;q^{-1})}.$$

Set

$$f^m(x_1,\dots,x_N) = \sum_{|\mu|\le m} c_\mu \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2,\dots,q^{N-1}x_N; q^{-1})}{s_\mu^*(0,\dots,0;q^{-1})}$$

and $\widehat{f}^m = F - f^m$. Let a_μ^m be the corresponding coefficient of the Taylor series expansion of f^m and let $\widehat{a}_\mu^m$ be the corresponding coefficient of the Taylor series expansion of $\widehat{f}^m$. Clearly, $a_\mu = a_\mu^m + \widehat{a}_\mu^m$. Represent $G'(f^m)$ and $G'(\widehat{f}^m)$ as linear combinations of q–interpolation Schur polynomials and let b_μ^m, $\widehat{b}_\mu^m$ be the corresponding coefficients:

$$G'(f^m) = \sum_{\mu \in \mathbb{GT}_k^+} b_\mu^m \frac{s_\mu^*(x_1, x_2, \ldots, x_k; q)}{s_\mu^*(0, \ldots, 0; q)},$$

$$G'(\widehat{f}^m) = \sum_{\mu \in \mathbb{GT}_k^+} \widehat{b}_\mu^m \frac{s_\mu^*(x_1, x_2, \ldots, x_k; q)}{s_\mu^*(0, \ldots, 0; q)},$$

It is clear that $b_\mu = b_\mu^m + \widehat{b}_\mu^m$.

Lemma 6.4 implies that $b_\mu^m = a_\mu^m$. It remains to prove that both $\widehat{a}_\mu^m$ and $\widehat{b}_\mu^m$ tend to zero as m tends to infinity.

It follows from Proposition 4.7 that $\widehat{f}^m \rightrightarrows 0$ uniformly on compact sets as $m \to \infty$. Uniform convergence of analytical functions implies convergence of their Taylor expansion coefficients. Thus, $\widehat{a}_\mu^m \to 0$ as $m \to \infty$.

$$G'(f^m) = \sum_{|\mu| > m} c_\mu \frac{s_\mu(x_1, \ldots, x_N)}{s_\mu(1, q^{-1}, \ldots, q^{1-N})}.$$

Proposition 4.5 implies that $G'(f^m) \rightrightarrows 0$ uniformly on D_k. Repeating the argument in the proof of Proposition 6.3 we conclude that $\widehat{b}_\mu^m \to 0$ as $m \to \infty$. □

6.2 q–Central measures and probability generating functions

Proof of Proposition 4.3. We want to prove the following statement: if measure P on $\mathcal{T}_N$ is such that

$$P(\tau(1) \prec \ldots \tau(N)) = \frac{q^{|\tau(1)| + \cdots + |\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))} P_N(\tau(N)),$$

for any path $\tau \in \mathcal{T}_N$, then $P_1, P_2, \ldots, P_N$ is a q–coherent system.

We proceed by induction in N. Case $N = 1$ is trivial. For general N we have

$$P_{N-1}(\mu) = \sum_{\tau \in \mathcal{T}_N,\, \tau(N-1)=\mu} P_N(\tau(N)) \frac{q^{|\tau(1)|+\cdots+|\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))}$$
$$= \sum_{\lambda} \frac{P_N(\lambda) q^{|\mu|}}{\mathrm{Dim}_q(\lambda)} \sum_{\tau \in \mathcal{T}_N,\, \tau(N)=\lambda, \tau(N-1)=\mu} q^{|\tau(1)|+\cdots+|\tau(N-2)|}$$
$$= P_N(\lambda) q^{|\mu|} \frac{\mathrm{Dim}_q(\mu)}{\mathrm{Dim}_q(\lambda)} = \sum_{\lambda} P_N(\lambda) P(\lambda \to \mu)$$

Thus, P_{N-1} and P_N are q–coherent.

Next, projection of measure P on $\mathcal{T}_{N-1}$ defines a probability measure that we denote by $\tilde{P}$. If $\tau' \in \mathcal{T}_{N-1}$, then

$$\tilde{P}(\tau') = \sum_{\lambda} P(\tau'(1) \prec \cdots \prec \tau'(N-1) \prec \lambda) = q^{|\tau(1)|+\cdots+|\tau(N-1)|} \sum_{\lambda | \tau'(N-1) \prec \lambda} \frac{P_N(\lambda)}{\mathrm{Dim}_q(\lambda)},$$

It follows that if $\tau^1, \tau^2 \in \mathcal{T}_{N-1}$ and $\tau^1(N-1) = \tau^2(N-1)$, then

$$\frac{\tilde{P}(\tau^1)}{\tilde{P}(\tau^2)} = \frac{q^{|\tau^1(1)|+\cdots+|\tau^1(N-2)|}}{q^{|\tau^2(1)|+\cdots+|\tau^2(N-2)|}},$$

Thus,

$$\tilde{P}(\tau') = \frac{q^{|\tau'(1)|+\cdots+|\tau'(N-2)|}}{\sum\limits_{\tau \in \mathcal{T}_{N-1},\, \tau(N-1)=\tau'(N-1)} q^{|\tau(1)|+\cdots+|\tau(N-2)|}} \sum_{\tau \in \mathcal{T}_{N-1},\, \tau(N-1)=\tau'(N-1)} \tilde{P}(\tau)$$
$$= \frac{q^{|\tau'(1)|+\cdots+|\tau'(N-2)|}}{\mathrm{Dim}_q(\tau'(N-1))} \tilde{P}_{N-1}(\tau'(N-1))$$

Consequently, it follows by induction that $\tilde{P}_1, \ldots, \tilde{P}_{N-1}$ form a q–coherent system. Since $P_i = \tilde{P}_i$ for $i = 1, \ldots, N-1$, $P_1, \ldots P_N$ is a q–coherent system. □

Proof of Proposition 4.4. We want to prove that for any q–coherent system $P_1, P_2, \ldots$, there exists a unique q–central measure P such that P_k is a projection of P on $\mathbb{GT}_k$ for every $k = 1, 2, \ldots$.

For any N let $P^{(N)}$ be a probability measure on $\mathcal{T}_N$ defined through

$$P^{(N)}(\tau) = \frac{q^{|\tau(1)|+\cdots+|\tau(N-1)|}}{\mathrm{Dim}_q(\tau(N))} P_N(\tau(N)).$$

Proposition 4.3 yields that projections $P_k^{(N)}$ of $P^{(N)}$ on $\mathbb{GT}_k$ form a q–coherent system. Since $P_N^{(N)} = P_N$ and $P_1, \ldots, P_N$ also form a q–coherent system, thus, $P_k^{(N)} = P_k$.

Repeating the argument of Proposition 4.3 we conclude that projection of $P^{(N)}$ on $\mathcal{T}_{N-1}$ coincides with $P^{(N-1)}$.

Let P be projective limit of the measures $P^{(N)}$ as $N \to \infty$. Clearly, P is a q–central measure on $\mathcal{T}$ and projections of P on $\mathbb{GT}_k$ are exactly P_k. □

Proof of Proposition 4.6. We want to prove that two probability measures P_N and P_{N+1} on $\mathbb{GT}_N$ and $\mathbb{GT}_{N+1}$, respectively, are q–coherent if and only if

$$\mathcal{S}(x_1,\ldots,x_N;P_N)=\mathcal{S}(x_1,\ldots,x_N,q^{-N};P_{N+1}).$$

Using Proposition (3.1) we obtain

$$\begin{aligned}\frac{s_\lambda(x_1,\ldots,x_N,q^{-N})}{s_\lambda(1,q^{-1},\ldots,q^{-N})}&=\frac{\sum\limits_{\mu\prec\lambda}s_\mu(x_1,\ldots,x_N)q^{-N(|\lambda|-|\mu|)}}{q^{-N|\lambda|}s_\lambda(1,q,\ldots,q^N)}\\ &=\sum_{\mu\prec\lambda}\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}q^{|\mu|}\frac{s_\mu(1,q,\ldots,q^{N-1})}{s_\lambda(1,q,\ldots,q^N)}\\ &=\sum_{\mu\in\mathbb{GT}_N}\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}P(\lambda\to\mu).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{S}(x_1,\ldots,x_N,q^{-N};P_{N+1})&=\sum_{\lambda\in\mathbb{GT}_{N+1}}\frac{s_\lambda(x_1,\ldots,x_N,q^{-N})}{s_\lambda(1,q^{-1},\ldots,q^{-N})}P_{N+1}(\lambda)\\ &=\sum_{\mu\in\mathbb{GT}_N}\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}\sum_{\lambda\in\mathbb{GT}_{N+1}}P(\lambda\to\mu)P_{N+1}(\lambda),\end{aligned}$$

On the other hand,

$$\mathcal{S}(x_1,\ldots,x_N;P_N)=\sum_{\mu\in\mathbb{GT}_N}\frac{s_\mu(x_1,\ldots,x_N)}{s_\mu(1,q^{-1},\ldots,q^{1-N})}P_N(\mu).$$

Comparing the last two expressions we conclude that

$$\mathcal{S}(x_1,\ldots,x_N,q^{-N};P_{N+1})=\mathcal{S}(x_1,\ldots,x_N;P_N)$$

if and only if

$$P_N(\mu)=\sum_{\lambda\in\mathbb{GT}_{N+1}}P(\lambda\to\mu)P_{N+1}(\lambda)$$

for all $\mu\in\mathbb{GT}_N$. □

Proof of Proposition 4.8. We want to prove that two probability measures P_N and P_{N+1} on $\mathbb{GT}_N^+$ and $\mathbb{GT}_{N+1}$, respectively, are q–coherent if and only if

$$\mathcal{S}^*(x_1,\ldots,x_N;P_N)=\mathcal{S}^*(x_1,\ldots,x_N,0;P_{N+1}).$$

To prove this proposition we need a lemma.

Lemma 6.5 (The branching rule for q–interpolation Schur polynomials). *For any $\lambda \in \mathbb{GT}_{N+1}^{+}$ we have*

$$\frac{s_\lambda^*(q^N x_1, q^N x_2, \dots, q^N x_N, 0; q^{-1})}{s_\lambda^*(0,\dots,0;q^{-1})} = \sum_{\mu \in \mathbb{GT}_N} \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0,\dots,0;q^{-1})} P(\lambda \to \mu).$$

Proof. Using the combinatorial formula (see Proposition 3.5) we obtain

$$s_\lambda^*(q^N x_1, \dots, q^N x_N, q^N x_{N+1}; q^{-1}) = \sum_T \prod_{(i,j)\in\lambda} \left(q^N x_{T(i,j)} - q^{i-j-T(i,j)+N+1}\right),$$

where the sum is taken over all semistandard Young tableaux $T(i,j)$ of shape λ filled with numbers $1,\dots,N+1$. Note that the part of T filled with numbers $1,\dots,N$ is a Young diagram of shape μ such that $\mu \prec \lambda$. Consequently, substituting $x_{N+1}=0$ we get

$$s_\lambda^*(q^N x_1, \dots, q^N x_N, 0; q^{-1}) = \sum_{\mu\prec\lambda} \left(\prod_{(i,j)\in\lambda\setminus\mu} \left(0 - q^{i-j}\right)\right) \sum_T \prod_{(i,j)\in\mu} \left(q^N x_{T(i,j)} - q^{i-j-T(i,j)+N+1}\right),$$

where the sums are taken over semistandard Young tableaux $T(i,j)$ of shape μ filled with numbers $1,\dots,N$. Consequently,

$$s_\mu^*(q^N x_1, \dots, q^N x_N, 0; q^{-1}) = \sum_{\mu\prec\lambda} \left(\prod_{(i,j)\in\lambda\setminus\mu} \left(-q^{i-j}\right)\right) q^{|\mu|} s_\mu^*(q^{N-1}x_1, \dots, q^{N-1}x_N; q^{-1}).$$

It remains to prove that

$$\prod_{(i,j)\in\lambda\setminus\mu} \left(-q^{i-j}\right) q^{|\mu|} = \frac{s_\lambda^*(0,\dots,0;q^{-1})}{s_\mu^*(0,\dots,0;q^{-1})} P(\lambda\to\mu). \tag{36}$$

We claim that for any $\lambda \in \mathbb{GT}_N$

$$\begin{aligned} s_\lambda^*(0,\dots,0;q^{-1}) &= s_\lambda(1, q^{-1}, \dots, q^{1-N}) q^{(N-1)|\lambda|} \prod_{(i,j)\in\lambda} \left(-q^{i-j}\right) \\ &= s_\lambda(1, q, \dots, q^{N-1}) \prod_{(i,j)\in\lambda} \left(-q^{i-j}\right) \end{aligned}$$

The claim follows from the combinatorial formulas for Schur polynomials and for q–interpolation Schur polynomials. Using the last relation one immediately proves (36). □

Now we use the last lemma and repeat the argument of Proposition 4.6. $\square$

Proof of Proposition 4.10. We want to prove that if a sequence of probability measure P_N^i on $\mathbb{GT}_N$. weakly converge to P_N, then

$$\mathcal{S}(x_1,\dots,x_N;P_N^i)\rightrightarrows\mathcal{S}(x_1,\dots,x_N;P_N)$$

uniformly on T_N. Furthermore, if P_N, P_N^i are supported on $\mathbb{GT}_N^+$, then the convergence is uniform on D_N.

Since P_N is a probability measure on $\mathbb{GT}_N$, there exists k such that $P_N(\{\mu: -k<\mu_i<k\})>1-\varepsilon$. Observe that $\{\mu: -k<\mu_i<k\}$ is a finite set with less than $(2k)^N$ elements. Thus, weak convergence of measures P_N^i implies that for all $i>i_0$ and all $\lambda\in\mathbb{GT}_N$ such that $-k<\lambda_i<k$ we have

$$\left|P_N^i(\lambda)-P_N(\lambda)\right|<\frac{\varepsilon}{(2k)^N}$$

and

$$P_N^i(\{\mu:\,|\mu|<k\})>1-2\varepsilon.$$

Consequently, since

$$\left|\frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,\dots,q^{1-N})}\right|\le 1$$

on T_N, we have for $i>i_0$ and $(x_1,\dots,x_N)\in T_N$

$$\begin{aligned}&\left|\mathcal{S}(x_1,\dots,x_N;P_N^i)-\mathcal{S}(x_1,\dots,x_N;P_N)\right|\\ &\le\sum_{|\mu|<k}|P_N^i(\mu)-P_N(\mu)|\left|\frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,\dots,q^{1-N})}\right|+\sum_{|\mu|\ge k}(P_N^i(\mu)+P_N(\mu))\left|\frac{s_\mu(x_1,\dots,x_N)}{s_\mu(1,\dots,q^{1-N})}\right|\\ &\qquad\le k^N\frac{\varepsilon}{k^N}+\varepsilon+2\varepsilon=4\varepsilon\end{aligned}$$

It follows, that $\mathcal{S}(x_1,\dots,x_N;P_N^i)$ converges uniformly on T_N.

If P_N, P_N^i are supported on $\mathbb{GT}_N^+$, then the argument is the same with T_N replaced by D_N. $\square$

Proof of Proposition 4.11. We want to prove the following statement: if P_N^i is a sequence of probability measures on $\mathbb{GT}_N$ such that the functions

$$\mathcal{S}(x_1,\dots,x_N;P_N^i)$$

converge uniformly on T_N to a function $S(x_1,\dots,x_N)$, then there exists a probability measure P_N such that

$$S(x_1,\dots,x_N)=\mathcal{S}(x_1,\dots,x_N;P_N)$$

and P_N^i weakly converge to P_N.

Recall that (by its definition) a rational Schur function is the ratio of the alternating sum of monomials and the Vandermonde determinant:

$$s_\lambda(x_1,\dots,x_N)=\frac{\mathrm{Alt}(x^{\lambda_1+N-1}\cdots x^{\lambda_N})}{V(x_1,\dots,x_N)},$$

where

$$V(x_1,\dots,x_N)=\prod_{1\le i<j\le N}(x_i-x_j)$$

and

$$\mathrm{Alt}(x_1^{k(1)}\dots x_N^{k(N)})=\sum_\sigma(-1)^\sigma x_1^{k(\sigma(1))}\cdots x_N^{k(\sigma(N))},$$

sum is taken over all permutations of length N and $(-1)^\sigma$ is the sign of the permutation σ.

The following estimates are useful in the sequel:

Lemma 6.6. *There exist three constants $C_1(N)$, $C_2(N)$, $C_3(N)$ such that for any $(x_1,\dots,x_N)\in T_N$ and any $\lambda\in\mathbb{GT}_N$ we have*

$$C_1(N)<|V(x_1,\dots,x_N)|<C_2(N)$$

and absolute values of the coefficients of the monomials in the alternating sum

$$\frac{\mathrm{Alt}(x^{\lambda_1+N-1}\cdots x^{\lambda_N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})}$$

are bounded by $C_3(N)$.

We leave the proof of this lemma to the reader.

Denote

$$R^i(x_1,\dots,x_N)=V(x_1,\dots,x_N)\mathcal{S}(x_1,\dots,x_N;P_N^i)$$

and

$$R(x_1,\dots,x_N)=V(x_1,\dots,x_N)S(x_1,\dots,x_N).$$

Clearly, $R^i(x_1,\dots,x_N)$ converges to $R(x_1,\dots,x_N)$ uniformly on T_N.

We have

$$R^i(x_1,\dots,x_N)=\sum_{\lambda\in\mathbb{GT}_N}P_N^i(\lambda)\frac{\mathrm{Alt}(x^{\lambda_1+N-1}\cdots x^{\lambda_N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})}. \tag{37}$$

Note that if we expand (37) in a single sum of monomials, then we get a Fourier series expansion of $R^i(x_1,\dots,x_N)$ in the conventional sense. Furthermore, estimates of Lemma 6.6 guarantee that this Fourier series uniformly converges on T_N. It is well known that uniform convergence of the continuous functions implies the convergence of their Fourier coefficients. (This fact follows from the integral formula for the Fourier coefficients.) Consequently, $P_N^i(\lambda)$ converges to a certain number $P_N(\lambda)$ for every λ. Since $P_N^i(\lambda) \geq 0$ and $\sum_\lambda P_N^i(\lambda) = 1$, we have

$$P_N(\lambda) \geq 0, \qquad \sum_{\lambda \in \mathbb{G}T_N} P_N(\lambda) \leq 1. \tag{38}$$

If the last sum equals to 1, then the numbers $P_N(\lambda)$ define a probability measure P_N. Measures P_N^i weakly converge to P_N and we are done. However, the proof of the fact that $\sum_\lambda P_N(\lambda) = 1$ needs an additional argument. Define

$$\widehat{R}(x_1, \dots x_N) = \sum_{\lambda \in \mathbb{GT}_N} P_N(\lambda) \frac{\mathrm{Alt}(x^{\lambda_1 + N - 1} \cdots x^{\lambda_N})}{s_\lambda(1, q^{-1}, \dots, q^{1-N})}.$$

Inequalities (38) and estimates of Lemma 6.6 guarantee that $\widehat{R}(x_1, \dots x_N)$ is well-defined on T_N. Now observe that by the construction all the Fourier coefficients of R and $\widehat{R}$ coincide. Thus, all Fourier coefficients of $R - \widehat{R}$ are equal to zero. Consequently, (since Laurent polynomials are dense in the space of continuous functions of T_N) $R = \widehat{R}$. Therefore

$$\sum_{\lambda \in \mathbb{G}T_N} P_N(\lambda) = \frac{\widehat{R}(1, q^{-1}, \dots, q^{-N})}{V(1, q^{-1}, \dots, q^{-N})} = \frac{R(1, q^{-1}, \dots, q^{-N})}{V(1, q^{-1}, \dots, q^{-N})}$$

$$= \lim_{i \to \infty} \frac{R^i(1, q^{-1}, \dots, q^{-N})}{V(1, q^{-1}, \dots, q^{-N})} = \lim_{i \to \infty} 1 = 1.$$

□

Proof of Proposition 4.12. We want to prove that a sequence of probability measures P_N^i on $\mathbb{GT}_N^+$ weakly converges to P_N if and only if

$$\mathcal{S}^*(x_1, \dots, x_N; P_N^i) \rightrightarrows \mathcal{S}^*(x_1, \dots, x_N; P_N)$$

uniformly on compact subsets of $\mathbb{C}^N$.

If functions $\mathcal{S}^*(x_1, \dots, x_N; P_N^i)$ converge, then using the argument of Proposition 6.1 we conclude that the coefficients of their q–interpolation Schur expansions converge. Thus, measures P_N^i weakly converge.

The second part of the proof repeats the argument of Proposition 4.10 with inequality

$$\left| \frac{s_\mu(x_1, \dots, x_N)}{s_\mu(1, \dots, q^{1-N})} \right| \leq 1$$

replaced by the bound

$$\left| \frac{s_\mu^*(q^{N-1}x_1, q^{N-1}x_2, \dots, q^{N-1}x_N; q^{-1})}{s_\mu^*(0, \dots, 0; q^{-1})} \right| < A(M)$$

valid for $|x_i| \leq M$, as was explained in the proof of Proposition 4.7. □

6.3 Tightness (Proof of Proposition 5.7)

We want to prove that if $\lambda(i)$ is a sequence of signatures stabilizing to ν, then the family of functions

$$g_i(x_1,\dots,x_k) = \frac{s_{\lambda(i)}(x_1,x_2,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N(i)})}{s_\lambda(i)(1,q,\dots,q^{1-N(i)})}$$

is a relatively compact subset of the set of continuous functions on k–dimensional torus T_k with uniform convergence topology.

We need the following lemma.

Lemma 6.7. *Let $\lambda(i)$ be a sequence of positive signatures stabilizing to ν with $\nu_1 = 0$. For any integer $k \geq 0$ set*

$$V_{k,i}(\phi) = \frac{s_{\lambda(i)}(q^{-k}e^{-i\phi},q^{-1},\dots,q^{1-N(i)})}{s_\lambda(i)(1,q^{-1},\dots,q^{1-N(i)})}.$$

There exists a function $c(k,\nu)$ and a number i_0 such that

$$\left|\frac{\partial V_{k,i}(\phi)}{\partial\phi}\right| \leq c(k,\nu)$$

for $i > i_0$.

Proof. Assume without lost of generality that $\lambda(i)_{N(i)} = 0$. Denote $x = q^{-k}e^{-i\phi}$. The branching rule for Schur polynomials yields

$$V_{k,i}(\phi) = \sum_{\mu\prec\lambda(i)} x^{|\lambda|-|\mu|}\frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}$$
$$= \frac{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}\sum_{m\geq 0} x^m \sum_{\mu\prec\lambda(i),\,|\lambda(i)|-|\mu|=m}\frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})}.$$

(Here we used the notation $\lambda(i)$ both for $(\lambda(i)_1,\dots,\lambda(i)_{N(i)})$ and $(\lambda(i)_1,\dots,\lambda(i)_{N(i)-1})$.) Thus,

$$\left|\frac{\partial V_{k,i}(\phi)}{\partial\phi}\right| \leq \frac{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})} \times\sum_{m\geq 0} m|x|^m \sum_{\mu\prec\lambda(i),\,|\lambda(i)|-|\mu|=m}\frac{s_\mu(q^{-1},\dots,q^{1-N(i)})}{s_{\lambda(i)}(q^{-1},\dots,q^{1-N(i)})}. \quad (39)$$

Note that the first fraction in the last formula is not greater than 1. (This follows e.g. from the branching rule for the rational Schur functions applied to $s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N(i)})$. To estimate the sum, we first consider the term with

$m = 1$. Using [Mac, Example 3.1] we obtain

$$\frac{s_\mu(q^{-1},\ldots,q^{1-N(i)})}{s_{\lambda(i)}(q^{-1},\ldots,q^{1-N(i)})} = q^{|\mu|(1-N(i))-|\lambda(i)|(1-N(i))}\frac{s_\mu(1,q,\ldots,q^{N(i)-2})}{s_{\lambda(i)}(1,q,\ldots,q^{N(i)-2})}$$
$$= q^{1-N(i)}q^{n(\mu)-n(\lambda(i))}\prod_{p<q}\frac{1-q^{\mu_p-\mu_q-p+q}}{1-q^{\lambda(i)_p-\lambda(i)_q-p+q}}$$

Suppose that μ differs from $\lambda(i)$ in row $N(i)-j$, then

$$q^{1-N(i)}q^{n(\mu)-n(\lambda(i))} = q^j$$

and

$$\prod_{p<q}\frac{1-q^{\mu_p-\mu_q-p+q}}{1-q^{\lambda(i)_p-\lambda(i)_q-p+q}} \le \prod_{\ell=1}^{\infty}(1-q^\ell)^{-1}.$$

It follows that the terms corresponding to $m = 1$ in the double sum in (39) are bounded from above by

$$\sum_{j=1}^{\infty} q^j \prod_{\ell=1}^{\infty}(1-q^\ell)^{-1}.$$

For general m let $\mu_n = \lambda(i)_n - f_{N(i)-n}$, $n = 1,2,\ldots,N(i)-1$. Similarly to $m = 1$ case we get

$$\frac{s_\mu(q^{-1},\ldots,q^{1-N(i)})}{s_{\lambda(i)}(q^{-1},\ldots,q^{1-N(i)})} \le q^{f_1+2f_2+\cdots+(N(i)-1)f_{N(i)-1}}\prod_{\ell=1}^{\infty}(1-q^\ell)^{-(f_1+f_2+\cdots+f_n)}.$$

Consider the generating function

$$a_i(t) = \sum_{f_1,f_2,\ldots} q^{f_1+2f_2+3f_3\cdots}t^{f_1+f_2+f_3+\cdots},$$

where the sum is taken over all finite collections of integers $\{f_n\}$ satisfying $0 \le f_n \le \lambda_{N(i)-n} - \lambda_{N(i)-n+1}$. The above arguments prove that

$$\left|\frac{\partial V_{k,i}(\phi)}{\partial\phi}\right| \le a_i'\left(q^{-k}\prod_{\ell=1}^{\infty}(1-q^\ell)^{-1}\right).$$

Now choose r such that $q^{-k}\prod_{\ell=1}^{\infty}(1-q^\ell)^{-1} < q^{-r}$. Suppose that i is large enough, so that $\lambda(i)_{N(i)-n} = \nu_n$ for $n = 1,2,\ldots,r+1$. Let

$$b(t) = \prod_{j=1}^{r}\left(\sum_{u=0}^{\nu_{j+1}-\nu_j}(tq^{j+1})^u\right)\cdot\prod_{j=r+1}^{\infty}\left(\sum_{u=0}^{\infty}(tq^{j+1})^u\right).$$

Clearly, $a_i'(t) \le b'(t)$. But

$$b(t) = \prod_{j=1}^{r}\left(\sum_{u=0}^{\nu_{j+1}-\nu_j}(tq^{j+1})^u\right)\cdot\prod_{j=r+1}^{\infty}\left(1-(tq^{i+1})\right)^{-1}.$$

Consequently, $b(t)$ is an analytic function in ball $t \le q^{-r}$. Thus,

$$a_i'\left(q^{-k}\prod_{\ell=1}^{\infty}(1-q^{\ell})^{-1}\right) < b'\left(q^{-k}\prod_{\ell=1}^{\infty}(1-q^{\ell})^{-1}\right) < \infty.$$

□

Proof of Proposition 5.7. First, let $\nu_1 = 0$. Observe that $g_i(x_1, \ldots, x_k)$ is a symmetric polynomial with positive coefficients. Thus, for any $(x_1, \ldots, x_k) \in T_k$ and any $1 \le \ell \le k$ we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial\phi} g_i(\ldots, x_{\ell-1}, e^{i\phi}x_\ell, x_{\ell+1}, \ldots)\right| \\
&\qquad \le \left|\frac{\partial}{\partial\phi} g_i(1, q^{-1}, \ldots, q^{2-\ell}, q^{1-\ell}e^{i\phi}, q^{-\ell}, \ldots, q^{1-k})\right| \\
&\qquad = \left|\frac{\partial}{\partial\phi} g_i(q^{1-\ell}e^{i\phi}, 1, q^{-1}, \ldots, q^{2-\ell}, q^{-\ell}, \ldots, q^{1-k})\right| \\
&\qquad\qquad \le \left|\frac{\partial}{\partial\phi} g_i(q^{1-\ell}e^{i\phi}, q^{-1}, q^{-2}, \ldots, q^{1-k})\right| < const,
\end{aligned}$$

where the last inequality is Lemma 6.7. The above uniform estimate for the derivatives yields that the family of functions g_i is equicontinuous on T_n. We also have $|g_i| \le 1$ on T_n. Thus, by the Arzelà–Ascoli theorem the set of functions $\{g_i\}$ is relatively compact.

For general ν we note that if $\lambda(i)$ stabilizes to ν then $A_{-\nu_1}(\lambda(i))$ stabilizes to $\nu' = A_{-\nu_1}(\nu)$ with $\nu_1' = 0$. Since for any λ and any ℓ

$$s_{A_\ell(\lambda)}(x_1, \ldots, x_N) = (x_1 \cdots x_N)^\ell s_\lambda(x_1, \ldots, x_N),$$

the case of general ν reduces to the case $\nu_1 = 0$. □

6.4 Analyticity (proof of Proposition 5.10)

We want to prove that for every ν the series

$$\sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \ldots, x_k)$$

converges for all $x_1, \ldots x_k$ and defines an entire function.

The combinatorial formula for s_μ^* (Proposition 3.5) implies that for every M there exists a constant $C(M)$ such that for every $x_1, \ldots, x_k$ with $|x_i| < M$, we have

$$\left|s_\mu^*(x_1, \ldots, x_k)\right| < C(M)^{|\mu|} < \left(C(M)^k\right)^{\mu_1}.$$

Let us fix ν and estimate $q^{n(\mu)-n(\mu')}\mathrm{Spec}_\nu(s_\lambda)$.

Lemma 6.8. *There exists a constant W such that*

$$|\mathrm{Spec}_\nu(h_k)| \le W^k q^{k^2/2}$$

for any k. Furthermore, for any $V > 0$ there exists k_0 such that for every $k > k_0$ we have

$$|\mathrm{Spec}_\nu(h_k)| \le V^k q^{k^2/2}.$$

Proof. Recall that

$$\sum_{k=0}^{\infty} \mathrm{Spec}_\nu(h_k) t^k = H^\nu(t) = \prod_{x \in X(\nu)} (1 - q^x t).$$

(See Section 5.2 for the definition of $X(\nu)$) If $X(\nu)$ is finite, then the lemma is obvious. Otherwise,

$$\mathrm{Spec}_\nu(h_k) = (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_k} q^{X(\nu, i_1) + \cdots + X(\nu, i_k)},$$

where $X(\nu, i)$ stays for the ith element in $X(\nu)$. Clearly, $i_\ell \ge \ell$, consequently

$$|\mathrm{Spec}_\nu(h_k)| \le \sum_{i_1 \ge 1,\, i_2 \ge 2, \ldots, i_k \ge k} q^{X(\nu, i_1) + \cdots + X(\nu, i_k)} = q^{X(\nu,1) + \cdots + X(\nu,k)} (1-q)^{-k}.$$

Since $X(\nu, i) \ge i - 1$, the first part of Lemma 6.8 is proved. Now recall that the complement of the set $X(\nu)$ in $\mathbb{Z}_{\ge 0}$ is infinite. Thus, for any r there exists k_0 such that for every $k > k_0$,

$$X(\nu, 1) + \cdots + X(\nu, k) > (0 + 1 + \cdots + k - 1) + rk.$$

Choosing r large enough we prove the second part of Lemma 6.8 $\square$

Jacobi–Trudy formula for Schur polynomials (see e.g. [Mac, 3.4]) implies that for $\lambda \in \mathbb{GT}_k^+$, we have

$$\mathrm{Spec}_\nu(s_\lambda) = \det\{\mathrm{Spec}_\nu(h_{\lambda_i - i + j})\}_{i,j=1,\ldots,k},$$

where we agree that $\mathrm{Spec}_\nu(h_{-1}) = \mathrm{Spec}_\nu(h_{-2}) = \cdots = 0$. Since k is fixed, writing determinant as a signed sum of products and applying Lemma 6.8 we conclude that for any $V > 0$ there exists n_0 such that for every $\lambda \in \mathbb{GT}_k^+$ with $\mu_1 > n_0$ we have

$$|\mathrm{Spec}_\nu(s_\lambda)| \le V^{\lambda_1} \prod_{i=1}^{k} q^{\lambda_i^2/2}.$$

For any $\lambda \in \mathbb{GT}_k^+$ simple computation proves that $n(\lambda') = \sum_{i=1}^k \lambda_i(\lambda_i - 1)/2$. Thus for $x_1, \dots, x_k$ with $|x_i| < M$ we have

$$\left| \sum_{\mu \in \mathbb{GT}_k^+,\, \mu_1 \geq n_0} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k) \right|$$
$$\leq \sum_{\mu \in \mathbb{GT}_k^+,\, \mu_1 \geq n_0} V^{\mu_1} |s_\mu^*(x_1, \dots, x_k)| \leq \sum_{\mu_1 = n+0}^{\infty} V^{\mu_1} ((\mu_1+1)^{k-1}) \left(C(M)^k\right)^{\mu_1}.$$

(Here we used the rough estimate that the number of signatures $\mu \in \mathbb{G}T_k$ with $\mu_1 = n$ is most $(n+1)^{k-1}$.) If V is small enough, then the above series converges. Consequently,

$$\sum_{\mu \in \mathbb{GT}_k^+} (-1)^{|\mu|} q^{n(\mu)-n(\mu')} \mathrm{Spec}_\nu(s_\lambda) s_\mu^*(x_1, \dots, x_k)$$

converges and defines an entire function.

7 q–Toeplitz matrices (proof of Proposition 1.4)

In this section we prove the following theorem:

Theorem 7.1. *Let c_l, $l \geq 0$ be a sequence of non-negative integers and suppose that $\sum_l c_l = 1$. Let $d_{i,j}$, $i > 0$, $j > 0$ be a unique q–Toeplitz matrix such that*

$$d[i,1] = c_{i-1}, \quad i = 1, 2, \dots.$$

Set

$$\phi(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1} (1 - tq^i)$$

an define coefficients c_λ ($\lambda \in \mathbb{GT}_N^+$) as the coefficients of the expansion

$$\phi(x_1) \cdots \phi(x_N) = \sum_{\lambda \in \mathbb{GT}_N^+} c_\lambda (-1)^{|\lambda|} s_\lambda^*(q^{N-1}x_1, \dots, q^{N-1}x_N; q^{-1}). \tag{40}$$

Then

$$c_\lambda = q^{(N-1)|\lambda|} \det\{d[\lambda_{N-i+1} + i, j]\}_{i,j=1,\dots,N}.$$

Remark. If $\phi(t)$ is a polynomial, then the series (40) is finite. For general $\phi(t)$ Proposition 6.1 yields that the series (40) converges at least in points $q^{1-N-\lambda+\delta}$, and such convergence is enough for our purposes.

Before proving Theorem 7.1 let us obtain Proposition 1.4 as its corollary.

Proof of Proposition 1.4. Every initial column minor of the matrix $d^\nu[i,j]$ is

$$\det\{d[\lambda_{N-i+1}+i,j]\}_{i,j=1,\dots,N}$$

for a certain N and $\lambda \in \mathbb{GT}_N^+$. Theorem 7.1 and Theorem 1.1 imply that

$$\det\{d_{\lambda_{N-i+1}+i,j}\}_{i,j=1,\dots,N} = q^{-(N-1)|\lambda|}c_\lambda = q^{-(N-1)|\lambda|}(-1)^{|\lambda}\frac{\mathcal{E}_N^\nu(\lambda)}{s_\lambda^*(0,\dots,0;q^{-1})}.$$

Observe that

$$\frac{(-1)^{|\lambda}}{s_\lambda^*(0,\dots,0;q^{-1})} > 0,$$

thus, all initial column minors of $d^\nu[i,j]$ are non-negative.

To finish the proof note that since $d^\nu[i,j]$ is a triangular matrix all its initial row minors (i.e. minors corresponding to the first N rows and arbitrary N columns) are either zero or equal to certain initial column minors. □

To prove Theorem 7.1 we need two general formulas involving factorial Schur polynomials.

Proposition 7.2. *We have*

$$\prod_{i=1}^N\prod_{j=1}^m(y_j-x_i) = \sum_{\lambda\subset m^N}(-1)^{|\lambda|}s_\lambda(x_1,\dots,x_N\mid a)s_{\hat{\lambda}'}(y_1,\dots,y_n\mid a),$$

where λ' is the transpose diagram and $\widehat{\lambda}'$ is the complement of λ' in diagram N^m

Proof. See [Mac2, Proof of (6.17)]. □

Denote

$$e_k(y_1,\dots,y_m\mid a) = \begin{cases} s_{1^k}(y\mid a), & 0<k\le m,\\ 1, & k=0,\\ 0, & k<0 \text{ or } k>m.\end{cases}$$

Proposition 7.3. *Factorial Schur polynomials admit a determinantal formula:*

$$s_\lambda(y_1,\dots,y_m\mid a) = \det[e_{\lambda_i'-i+j}(y_1,\dots,y_m\mid \tau^{j-1}a)]_{i,j=1,\dots,m},$$

where m is an arbitrary integer greater or equal than λ_1, λ_i' are row lengths of the transposed diagram (equivalently, they are column lengths of λ), and $\tau^{j-1}a$ stands for the sequence $(\tau^{j-1}a)_i = a_{j-1+i}$.

Proof. See [Mac2, (6.7)]. □

Now let $\widehat{a}$ be the following sequence:

$$\widehat{a}_j = -q^{1-j}, \quad j\in\mathbb{Z}.$$

Now fix m arbitrary numbers $y_1,\dots,y_m$ and let $d[i,j]_{i,j=1,2,\dots}$ be an infinite matrix given by

$$d[i,j] = e_{m-i+j}(y_1,\dots,y_m\mid \tau^{j-1}\widehat{a}).$$

Lemma 7.4. *$d[i,j]$ is a q–Toeplitz matrix, i.e.*

$$d[i,j+1] = d[i-1,j] + (q^{1-j} - q^{1-i})d[i,j].$$

(Here we agree that $d[i,j]=0$ is either $i<1$ or $j<1$.)

Proof. Setting $N=1$ in Proposition 7.2, we get

$$\prod_{i=1}^{m}(y_i - t) = \sum_{k=0}^{m}(-1)^k h_k(t \mid a) e_{m-k}(y_1,\dots,y_m \mid a),$$

where

$$h_k(t \mid a) = s_k(t \mid a) = (t+a_1)(t+a_2)\dots(t+a_k).$$

We see that the numbers $e_{m-k}(y \mid a)$ are the coefficients of the decomposition of the function $\prod_{i=1}^{m}(y_i - t)$ into the sum of polynomials $(-1)^k h_k(t \mid a)$. We can use any sequence a. In particular, if we set $a_i = -q^{2-i-\ell}$ then we get

$$\prod_{i=1}^{m}(y_i - t) = \sum_{k=0}^{m} d[k+\ell,\ell](q^{1-\ell}-t)(q^{1-\ell-1}-t)\dots(q^{2-k-\ell}-t). \qquad (41)$$

Observe that the left side of (41) does not depend on ℓ. Comparing (41) for ℓ and $\ell+1$ and using the fact that

$$\begin{aligned}(q^{1-\ell}-t)&(q^{1-\ell-1}-t)\dots(q^{2-k-\ell}-t)\\ &= (q^{1-(\ell+1)}-t)(q^{1-(\ell+1)-1}-t)\dots(q^{2-k-(\ell+1)}-t)\\ &+ (q^{1-\ell}-q^{2-k-(\ell+1)})(q^{1-(\ell+1)}-t)(q^{1-(\ell+1)-1}-t)\dots(q^{2-k-\ell}-t),\end{aligned}$$

we get

$$d[k+\ell+1,\ell+1] = d[k+\ell,\ell] + (q^{1-\ell} - q^{-k-\ell})d[k+\ell+1,\ell].$$

To finish the proof substitute $j=\ell$, $i=k+\ell+1$. □

Proposition 7.5. *Let $H(t)$ be a polynomial of degree m with $H(0)=1$. Let the coefficients c_ℓ, $i=0,\dots,m$ be defined from the expansion*

$$H(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1}(1-tq^i).$$

More generally, define coefficients c_λ ($\lambda \in \mathbb{GT}_N^+$) as the coefficients of the expansion

$$H(x_1)\cdots H(x_N) = \sum_{\lambda\in\mathbb{GT}^+} c_\lambda (-1)^{|\lambda} s^*_\lambda(q^{N-1}x_1,\dots,q^{N-1}x_N;q^{-1}). \qquad (42)$$

Let $\widetilde{d}[i,j]$ denote a unique q–Toeplitz matrix such that

$$\widetilde{d}[i,1] = c_{i-1}.$$

Then we have

$$c_\lambda = q^{(N-1)|\lambda|}\det\{d[\lambda_{N-i+1}+i,j]\}_{i,j=1,\dots,N}.$$

Proof. Let y_i be the roots of $H(t)$, i.e.

$$H(t) = \prod_{i=1}^{m} (1 - y_i^{-1} t).$$

Let, as above, define

$$\widehat{a}_j = -q^{1-j}, \quad j \in \mathbb{Z}$$

and

$$d[i,j] = e_{m-i+j}(y_1, \dots, y_m \mid \tau^{j-1} \widehat{a}).$$

Setting $N = 1$ in Proposition 7.2, we conclude that

$$d[i,1] = c_{i-1} \prod_{\ell=1}^{m} y_\ell^{-1}.$$

Lemma 7.4 yields that $d[i,j]$ is a q–Toeplitz matrix, thus,

$$d[i,j] = \widetilde{d}[i,j] \prod_{\ell=1}^{m} y_\ell^{-1}.$$

Recall that

$$s_\lambda^*(x_1, \dots, x_N; q^{-1}) = s_\lambda(x_1, \dots, x_N \mid a)$$

with

$$a_j = -q^{N-j}.$$

Definition of factorial Schur functions $s_\lambda(x \mid a)$ implies that

$$s_\lambda(qy \mid qa) = q^{|\lambda|} s_\lambda(y \mid a).$$

(Here qa stands for the sequence with $(qa)_j = qa_j$.)

It follows that

$$s_\lambda^*(q^{N-1} x_1, \dots, q^{N-1} x_N; q^{-1}) = q^{(N-1)|\lambda|} s_\lambda(x_1, \dots, x_N \mid \widehat{a}).$$

Now let b_λ be the coefficient of the expansion

$$\prod_{i=1}^{N} \prod_{j=1}^{m} (y_j - x_i) = \sum_{\lambda \in m^N} (-1)^{|\lambda|} s_\lambda(x_1, \dots, x_N \mid \widehat{a}) b_\lambda.$$

Comparing the last formula with the definition (42) of c_λ we see that

$$c_\lambda = q^{(N-1)|\lambda|} \prod_{\ell=1}^{m} y_\ell^{-N} b_\lambda.$$

On the other hand, Proposition 7.2 yields that

$$b_\lambda = s_{\widehat{\lambda}'}(y_1, \dots, y_m \mid \widehat{a}).$$

Applying Proposition 7.3 we conclude that

$$b_\lambda = \det[e_{\widehat{\lambda}_i - i + j}(y_1, \ldots, y_m \mid \tau^{j-1}\widehat{a})]_{i,j=1,\ldots,N}.$$

It is clear that

$$\widehat{\lambda}_i - i = m - (\lambda_{N-i} + i).$$

Thus,

$$e_{\widehat{\lambda}_i - i + j}(y_1, \ldots, y_m \mid \tau^{j-1}\widehat{a}) = d[\lambda_{N-i} + i, j].$$

Consequently,

$$c_\lambda = q^{(N-1)|\lambda|} \prod_{\ell=1}^{m} y_\ell^{-N} \det\{d[\lambda_{N-i}+i, j]\}_{i,j=1,\ldots,N} = q^{(N-1)|\lambda|} \det\{\widetilde{d}[\lambda_{N-i}+i, j]\}_{i,j=1,\ldots,N}.$$

□

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. Recall that c_l, $l \geq 0$ is a sequence of non-negative integers and $\sum_l c_l = 1$.

Let us denote

$$c_l^{(m)} = \begin{cases} \dfrac{c_l}{\sum_{i=0}^m c_i}, & 0 \leq l \leq k, \\ 0, \text{ otherwise.} \end{cases}$$

And let $d^{(m)}[i,j]$, $i > 0$, $j > 0$ be a unique q–Toeplitz matrix such that

$$d^{(m)}[i, 1] = c_{i-1}^{(m)}, \quad i = 1, 2, \ldots.$$

Observe that

$$\lim_{m\to\infty} d^{(m)}[i,j] = d[i,j].$$

Consequently, all minors of $d[i,j]$ are limits of the corresponding minors of $d^{(m)}[i,j]$.

Next, let $H^{(m)}(t)$ be a degree m polynomial such that

$$H^{(m)}(t) = \sum_{\ell=0}^{\infty} c_\ell \prod_{i=0}^{\ell-1} (1 - tq^i).$$

Let $c_\lambda^{(k)}$ be the coefficients of the expansion

$$H^{(m)}(x_1) \cdots H^{(m)}(x_N) = \sum_{\lambda \in \mathbb{GT}^+} c_\lambda^{(k)} (-1)^{|\lambda} s_\lambda^*(q^{N-1}x_1, \ldots, q^{N-1}x_N; q^{-1}).$$

Observe that

$$H^{(m)}(t) \rightrightarrows \phi(t)$$

uniformly on compact subsets of $\mathbb{C}$. Hence,

$$H^{(m)}(x_1) \cdots H^{(m)}(x_N) \rightrightarrows \phi(x_1) \cdots \phi(x_N).$$

Then Proposition 6.1 implies that

$$c_\lambda^{(m)} \to c_\lambda.$$

Applying Proposition 7.5 we conclude that

$$c_\lambda = \lim_{m\to\infty} c_\lambda^{(m)} = \lim_{m\to\infty} \det\{d^{(m)}[\lambda_{N-i}+i, j]\}_{i,j=1,\dots,N} \\ = \det\{d[\lambda_{N-i}+i, j]\}_{i,j=1,\dots,N}.$$

□

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