FUKAYA CATEGORIES, LANDAU-GINZBURG MODELS AND HOMOLOGICAL MIRROR SYMMETRY

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ABSTRACT. In this paper we prove two conjectures related to homological mirror symmetry.

The first one concerns with rigorous mathematical definition of Fukaya A_∞ -category associated to a symplectic variety. The main problen with general definition is that the morphism spaces are defined only for transversal pairs of Lagrangians, and higher products are defined only for transversal sequences of Lagrangians. Thus "Fukaya category" is actually an A_∞ -pre-category in the sense of Kontsevich and Soibelman. They conjectured that quasi-equivalence classes of A_∞- pre-categories are in bijection with quasi-equivalence classes of A_∞- categories. We prove this conjecture for essentially small A_∞- (pre-)categories (working over a field). We also present natural construction of pre-triangulated envelope in the framework of A_∞- pre-categories.

The second main result is homological mirror symmetry for curves of genus ≥ 3. We treat such curves as symplectic varieties, and associate to them Fukaya A_∞ -category. The mirror is Landau-Ginzburg model, and we associate to it the category of singularities of the (unique) singular fiber. We prove equivalence between perfect complexes over the Fukaya A_∞ -category and the Karoubian completion of the category of singularities. Our proof bases on the ideas of Seidel in his proof of the genus 2 case.

We also prove a kind of reconstruction theorem for hypersurface singularities. Namely, formal type of hypersurface singularity (i.e. a formal power series up to a formal change of variables) can be reconstructed, with some technical assumptions, from its D $(\mathbb{Z}/2)$- G category of Landau-Ginzburg branes.

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1. Introduction

A remarkable construction of K. Fukaya [F] associates to a symplectic manifold a ($\mathbb{Z}$- or $\mathbb{Z}/2$-)graded A_∞- pre-category in sense of Kontsevich and Soibelman [KS]. Its objects are Lagrangian submanifolds with some additional structures. This is not an actual A_∞- category since in general the morphism spaces are defined only for transversal pairs of Lagrangians, and higher products are defined only for transversal sequences of Lagrangians. Fukaya's construction is used in the categorical interpretation of mirror symmetry [Ko1] for Calabi-Yau varieties, and further generalizations to Fano and general cases, the so-called homological mirror symmetry conjecture. For the systematic exposition of different versions of Fukaya A_∞- pre-categories, see [Se2].

However, in order to prove HMS conjecture at least in some special cases, one should first replace Fukaya A_∞- pre-category with a (quasi-equivalent) actual A_∞- category. Clearly, each A_∞- category (with weak identity morphisms) can be considered also as an A_∞- pre-category. Kontsevich and Soibelman [KS] formulated the following natural conjecture.

Conjecture 1.1. *(*[KS]*)Let k be a graded commutative ring. Then quasi-equivalence classes of A_∞- pre-categories over k are in bijection with quasi-equivalence classes of A_∞- categories over k with strict (or weak) identity morphisms.*

We prove this conjecture for essentially small A_∞ -pre-categories in the case when k is a field.

Theorem 1.2. *Let k be a field. Then quasi-equivalence classes of essentially small A_∞- pre-categories over k are in bijection with quasi-equivalence classes of essentially small A_∞- categories over k with strict (or weak) identity morphisms.*

Another subject of the paper is the natural construction of twisted complexes in the framework of A_∞- pre-categories. Here the main statement is the invariance of twisted complexes under quasi-equivalences (Proposition 5.6). For ordinary A_∞- categories we obtain standard pre-triangulated envelopes.

Our second main result is homological mirror symmetry for curves of genus ≥ 3. As it is said above, originally, HMS conjecture was proposed by Kontsevich [Ko1] for Calabi-Yau varieties. It was proved in some special cases [AS, PZ, Se3].

An analogue of the conjecture for Fano varieties has been proposed soon after. In this case the mirror is a Landau-Ginzburg model — a smooth algebraic variety together with a regular function. More generally, it is wxpected that HMS should take place for varieties with effective anti-canonical divisor, see [Au].

Katzarkov [Ka, KKP] has proposed a generalization of homological mirror symmetry, which includes some varieties of general type. The mirror to such variety is a Landau-Ginzburg model. One direction of Katzarkov's conjecture was proved by Seidel in the case of the genus 2 curve [Se1]. We prove an analogous statement for curves of genus ≥ 3. We treat such curves as symplectic varieties, and associate to them Fukaya A_∞ -category $\mathcal{F}(M)$. The mirror is three-dimensional Landau-Ginzburg model (X, W) , and we associate to it the category of singularities $D_{sg}(X_0)$ of the (unique) singular fiber X_0. We prove equivalence between perfect complexes over the Fukaya A_∞ -category and the Karoubian completion of the category of singularities. Our proof bases on the ideas of Seidel in his proof of the genus 2 case.

Theorem 1.3. *The triangulated categories $D^\pi(F(M))$ and $\overline{D_{sg}}(X_0)$ are equivalent.*

Here $D^{\pi}(\mathcal{F}(M))$ is the triangulated category of perfect complexes over $\mathcal{F}(M)$, and $\overline{D_{sg}}(X_0)$ is a Karoubian completion of the triangulated category of singularities $D_{sg}(X_0) = D^b_{coh}(X_0)/Perf(X_0)$.

The paper is organized as follows.

In Section 2 we outline the proof of Kontsevich-Soibelman conjecture (Theorem 2.1).

In Section 3 we define A_∞- categories, strict and weak identity morphisms, quasi-equivalences, and A_∞- pre-categories, following [KS].

Section 4 is devoted to the proof of Theorem 2.1.

In Section 5 we present the construction of pre-triangulated envelope for A_∞- pre-categories over arbitrary graded commutative ring. We verify that it is well-defined and is invariant under quasi-equivalences.

In Section 6 we outline the proof of homological mirror symmetry for curves of higher genus.

Section 7 contains an introduction to Maurer-Cartan theory for pro-nilpotent DG Lie algebras, and Kontsevich formality theorem. They are used to relate A_∞ -structures on the exterior super-algebra and polyvector fields.

Sections 8 and 9 are rather technical, and are devoted to some quasi-equivalence class of A_∞ -structures on the super-algebra $\Lambda(\mathbb{C}^3)$. It is proved, roughly speaking, that it is uniquely determined by μ^3 and μ^{2g+1} (Theorem 9.2).

Sections 10 and 11 give a description of the category of Landau-Ginzburg branes (Corollary 11.2). It turns out that it is described by a certain A_∞ -structure on $\Lambda(\mathbb{C}^3)$, which was constructed in Section 8.

In Section 12 we prove a kind of reconstruction theorem for hypersurface singularities (Theorem 12.1). Namely, formal type of hypersurface singularity (i.e. a formal power series up to a formal change of variables) can be reconstructed, with some technical assumptions, from its D $(\mathbb{Z}/2)$- G category of Landau-Ginzburg branes.

In Section 13 we describe two different LG models, both mirror to genus g curve, such that the resulting categories are equivalent (Theorem 13.1). This is actually a famous Mckay correspondence.

Section 14 contains the definition of Fukaya categories for curves of higher genus, and various technical results. Here we follow [Se1].

In Section 15 we construct a generator in Fukaya category of a genus $g \geq 3$ curve, and prove that its endomorphism A_∞ -algebra is quasi-isomorphic to that in the Landau-Ginzburg model. This completes the proof of HMS for higher genus curves.

In Appendix we prove one necessary technical result from Maurer-Cartan theory for pro-nilpotent DG Lie algebras.

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2. Kontsevich-Soibelman conjecture

Our first main result in this paper is the following theorem.

Theorem 2.1. *Let k be a field. Then quasi-equivalence classes of essentially small A_∞- pre-categories over k are in bijection with quasi-equivalence classes of essentially small A_∞- categories over k with strict (or weak) identity morphisms.*

We deal with A_∞- (pre-)categories over a field because we need to pass to minimal A_∞- (pre-)categories (i.e. with $m_1 = 0$). Further, we deal with essentially small A_∞- (pre-)categories for purely set-theoretical reason: we need to consider Hochshild cohomology of graded (pre-)categories.

In Section 4 we prove Theorem 2.1 (Theorem 4.2). The proof goes as follows. In Subsection 4.1 we pass from essentially small A_∞- (pre-)categories to small ones. Further, in Subsection 4.2 we pass from small to small minimal A_∞- (pre-)categories. In Subsection 4.3 we define Hochshild cohomology of graded pre-categories. Roughly speaking, obstructions to constructing, step by step, of A_∞- structures and A_∞- morphisms live in these cohomology spaces.

In Subsection 4.4 we formulate and prove important Lemma (Lemma 4.4) about invariance of Hochshild cohomology under equivalences of graded pre-categories. In contrast to ordinary DG and A_∞- categories, this is non-trivial, and this is in fact the crucial point in the proof of Theorem 2.1. Here we use the language of simplicial local systems.

In Subsection 4.5 we introduce the sets of equivalence classes of minimal A_∞- structures on graded pre-categories, and develop basic obstruction theory for lifting A_∞- structures and A_∞- homotopies. In Subsection 4.6 we use invariance of Hochshild cohomology to prove the invariance of the set of equivalence classes of minimal A_∞- structures on graded pre-categories. Finally, in Subsection 4.7 we prove Theorem 2.1 using the invariance result.

3. Preliminaries on A_∞ -(pre-)categories

Fix some basic field k of arbitrary characteristic.

3.1. Non-unital A_∞-algebras and A_∞-categories.

Let $A = \bigoplus_{i\in\mathbb{Z}} A^i$ be a $\mathbb{Z}$ (resp. $\mathbb{Z}/2$)-graded vector space. Denote by $A[n]$ its shift by n: $A[n]^i := A^{i+n}$.

Definition 3.1. *A structure of a non-unital A_∞- algebra on A is given by degree $+1$ coderivation $b : T_+(A[1]) \to T_+(A[1])$, such that $b^2 = 0$. Here $T_+(A[1]) = \bigoplus_{n\geq 1} A[1]^{\otimes n}$ is a cofree tensor coalgebra.*

The coderivation b is uniquely determined by its "Taylor coefficients" $m_n : A^{\otimes n} \to A[2-n],\ n\geq 1.$ The condition $b^2=0$ is equivalent to the series of quadratic relations

$$\sum_{i+j=n+2}\sum_{0\leq l\leq i-1}(-1)^{\epsilon}m_i(a_0,\dots,a_{l-1},m_j(a_l,\dots,a_{l+j-1}),a_{l+j},\dots,a_n)=0, \tag{3.1}$$

where $a_m\in A,$ and $\epsilon=j\sum\limits_{0\leq s\leq l-1}\deg(a_s)+l(j-1)+j(i-1).$ In particular, for $n=0,$ we have $m_1^2=0.$

Definition 3.2. *An A_{∞}- morphism of non-unital A_{∞}- algebras $A\to B$ is a morphism of the corresponding non-counital DG coalgebras $T_+(A[1])\to T_+(B[1])$.*

Such an A_{∞}- morphism is uniquely determined by its "Taylor coefficients" $f_n : A^{\otimes n}\to B[1-n],$ satisfying the following system of equations:

$$\sum_{1\leq l_1<l_2<\dots<l_i=n}(-1)^{\gamma_i}m_i^B(f_{l_1}(a_1,\dots,a_{l_1}),\dots,f_{n-l_{i-1}}(a_{l_{i-1}+1},\dots,a_n))= \tag{3.2}$$
$$\sum_{s+r=n+1}\sum_{1\leq j\leq s}(-1)^{\epsilon_j}f_s(a_1,\dots,a_{j-1},m_r^A(a_j,\dots,a_{j+r-1}),a_{j+r},\dots,a_n),$$

where $a_m\in A,$ $\epsilon_j=r\sum\limits_{1\leq p\leq j-1}\deg(a_p)+j-1+r(s-j),$ $\gamma_i=\sum\limits_{1\leq p\leq i-1}(i-p)(l_p-l_{p-1}-1)+\sum\limits_{1\leq p\leq i-1}\nu(l_p)\sum\limits_{l_{p-1}+1\leq q\leq l_p}\deg(a_q),$ where we use the notation $\nu(l_p)=\sum\limits_{p+1\leq m\leq i}(l_m-l_{m-1}-1),$ and put $l_0=0.$

Definition 3.3. *A non-unital $\mathbb{Z}$ (resp. $\mathbb{Z}/2$)-graded A_{∞}- category $\mathcal{C}$ is given by the following data:*

1) A class of objects $Ob(\mathcal{C})$.

2) For any two objects X_1 and X_2, a $\mathbb{Z}$ (resp. $\mathbb{Z}/2$)-graded vector space of morphisms $\mathrm{Hom}(X_1,X_2)$.

3) For any sequence of objects $X_0,\dots,X_n$, a map of graded vector spaces m_n : $\bigotimes\limits_{0\leq i\leq n-1}\mathrm{Hom}(X_i,X_{i+1})\to\mathrm{Hom}(X_0,X_n)[2-n].$

This data us required to satisfy the following property: for each finite collection of objects $X_0,\dots,X_N,$ $N\geq 0,$ the graded vector space $\bigoplus\limits_{0\leq i,j\leq N}\mathrm{Hom}(X_i,X_j)$ *equipped with operations m_n, becomes an A_{∞}- algebra.*

Remark 3.4. *A non-unital A_{∞}- algebra can be considered as a non-unital A_{∞}- category with one object X such that* $\mathrm{Hom}(X,X)=A.$

Remark 3.5. *If $\mathcal{C}$ is a non-unital A_∞-category, then we have a "non-unital" graded category $H(\mathcal{C})$, which is defined by replacing the spaces of morphisms by their cohomologies with respect to m_1. Here "non-unital" means that we may not have identity morphisms $id_X \in \mathrm{Hom}^0_{H(\mathcal{C})}(X,X)$.*

Definition 3.6. *An A_∞-functor between non-unital A_∞-categories $F:\mathcal{C}_1\to\mathcal{C}_2$ is given by the following data:*

1) A map of classes of objects $F: Ob(\mathcal{C}_1)\to Ob(\mathcal{C}_2)$.

2) For any finite sequence of objects $X_0,\dots,X_n$ in $\mathcal{C}_1$, a morphism of graded vector spaces $f_n:\bigotimes\limits_{0\le i\le n-1}\mathrm{Hom}(X_i,X_{i+1})\to \mathrm{Hom}(F(X_0),F(X_n))[1-n]$.

It is required that for any finite collection of objects $X_0,\dots,X_N$, $N\geq 0$, the sequence f_n, $n\geq 1$, defines an A_∞-morphism

$$\bigoplus_{0\le i,j\le N}\mathrm{Hom}(X_i,X_j)\to\bigoplus_{0\le i,j\le N}\mathrm{Hom}(F(X_i),F(X_j)). \tag{3.3}$$

3.2. **Identity morphisms.** Now we define strict and weak identity morphisms.

Definition 3.7. *Let $\mathcal{C}$ be a non-unital A_∞-category, and $X\in Ob(\mathcal{C})$. A morphism $e\in \mathrm{Hom}^0(X,X)$ is called*

a) a strict identity if $m_2(f,e)=f$, $m_2(e,g)=g$, for $n\neq 2$ $m_n(f_1,\dots,f_{i-1},e,f_{i+1},\dots,f_n)=0$ for any morphisms f,g,f_j such that the equalities make sense. In this case we put $1_X:=e$.

b) a weak identity if $m_1(e)=0$, and for any closed morphisms $f:X\to Y$, $g:Z\to X$, we have that in $H(\mathcal{C})$ $\bar{e}\cdot\bar{g}=\bar{g}$, $\bar{f}\cdot\bar{e}=\bar{f}$.

Clearly, a strict identity is also a weak identity.

Remark 3.8. *If a non-unital A_∞-category $\mathcal{C}$ has at least weak identity morphisms, then $H(\mathcal{C})$ is an actual graded category, and vice versa.*

Definition 3.9. *An A_∞-functor $F:\mathcal{C}\to\mathcal{D}$ between A_∞-categories with strict (resp. weak) identity morphisms is called strictly (resp. weakly) unital if it preserves strict units and $f_n(g_1,\dots,1_X,\dots,g_n)=0$ whenever $n>1$ (resp. if it preserves weak identity morphisms).*

Definition 3.10. *A strictly (resp. weakly) unital A_∞-functor $F:\mathcal{C}\to\mathcal{D}$ between strictly (resp. weakly) unital A_∞-categories is called a quasi-equivalence if the induced functor $H(F):H(\mathcal{C})\to H(\mathcal{D})$ is an equivalence of graded categories.*

Two A_∞-categories with weak identity morphisms $\mathcal{C}$ and $\mathcal{D}$ are called quasi-equivalent if there exists a finite sequence of A_∞-categories with weak identity morphisms $\mathcal{C}_0 =$

$\mathcal{C}, \mathcal{C}_1, \dots, \mathcal{C}_n = \mathcal{D}$ such that for $0 \leq i \leq n-1$ there exists a quasi-equivalence $\mathcal{C}_i \to \mathcal{C}_{i+1}$ or vice versa.

The following statement is well-known, see [L-H].

Proposition 3.11. *If we consider only A_∞- categories with strict identity morphisms and strictly unital quasi-equivalences, then the resulting quasi-equivalence classes are in bijection with quasi-equivalence classes of A_∞- categories with weak identity morphisms.*

3.3. **A_∞ -pre-categories.** Now we recall the definition of A_∞- pre-categories which were originally defined in [KS]. We start with the notion of a non-unital A_∞- pre-category.

Definition 3.12. *A non-unital $\mathbb{Z}$ (resp. $\mathbb{Z}/2$)-graded A_∞- pre-category $\mathcal{C}$ is the following data:*

a) A class of objects $Ob(\mathcal{C})$,

b) For any $n \geq 1$ a subclass $\mathcal{C}_{tr}^n \subset Ob(\mathcal{C})^n$ of transversal sequences. It is required that $\mathcal{C}_{tr}^1 = Ob(\mathcal{C})$.

c) For each pair $(X_1, X_2) \in \mathcal{C}_{tr}^2$, a graded vector space $\mathrm{Hom}(X_1, X_2)$.

d) For a transversal sequence of objects $X_0, \dots, X_n$, a map of graded vector spaces $m_n : \bigotimes_{0 \leq i \leq n-1} \mathrm{Hom}(X_i, X_{i+1}) \to \mathrm{Hom}(X_0, X_n)[2-n]$.

It is required that each subsequence $(X_{i_1}, X_{i_2}, \dots, X_{i_l})$, $0 \leq i_1 < i_2 < \cdots < i_l \leq n$ of a transversal sequence $(X_0, \dots, X_N)$ is transversal, and that the graded vector space $\bigoplus_{0 \leq i < j \leq N} \mathrm{Hom}(X_i, X_j)$ with operations m_n becomes a non-unital A_∞- algebra.

Note that the property of transversality for a pair of objects is not required to be symmetric. Clearly, a non-unital A_∞- category is the same as a non-unital A_∞- pre-category $\mathcal{C}$ with $\mathcal{C}_{tr}^n = Ob(\mathcal{C})^n$.

Definition 3.13. *An A_∞- functor between non-unital A_∞- pre-categories $F : \mathcal{C} \to \mathcal{D}$ is given by the following data:*

1) A map of classes of objects $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$, such that $F(\mathcal{C}_{tr}^n) \subset \mathcal{D}_{tr}^n$.

2) For any finite transversal sequence of objects $X_0, \dots, X_n$ in $\mathcal{C}$, a morphism of graded vector spaces $f_n : \bigotimes_{0 \leq i \leq n-1} \mathrm{Hom}(X_i, X_{i+1}) \to \mathrm{Hom}(F(X_0), F(X_n))[1-n]$.

It is required that for any transversal sequence of objects $X_0, \dots, X_N$, $N \geq 0$, the sequence f_n, $n \geq 1$, defines an A_∞- morphism

$$\bigoplus_{0 \leq i < j \leq N} \mathrm{Hom}(X_i, X_j) \to \bigoplus_{0 \leq i < j \leq N} \mathrm{Hom}(F(X_i), F(X_j)). \tag{3.4}$$

Roughly speaking, an A_∞- pre-category is a non-unital A_∞- pre-category with sufficiently many transversal sequences. To define this notion rigorously, we first define the notion of a quasi-isomorphism.

Definition 3.14. *Let $\mathcal{C}$ be a non-unital A_∞- pre-category, and $(X_1, X_2) \in \mathcal{C}^2_{tr}$. A closed morphism $f \in \mathrm{Hom}^0(X_1, X_2)$ is called a quasi-isomorphism if for any objects X_0 and X_3 such that $(X_0, X_1, X_2) \in \mathcal{C}^3_{tr}$, $(X_1, X_2, X_3) \in \mathcal{C}^3_{tr}$, one has that the maps*

(3.5)
$$m_2(f, \cdot) : \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(X_0, X_2) \quad \text{and} \quad m_2(\cdot, f) : \mathrm{Hom}(X_2, X_3) \to \mathrm{Hom}(X_1, X_3)$$

are quasi-isomorphisms.

Definition 3.15. *An A_∞- pre-category is a non-unital A_∞- pre-category $\mathcal{C}$ which satisfies the following extension property:*

For any finite collection of transversal sequences $(S_i)_{i \in I}$ in $\mathcal{C}$ and an object X there exist objects X_-, X_+ and quasi-isomorphisms $f_- : X_- \to X$, $f_+ : X \to X_+$ such that the sequences (X_-, S_i, X_+), $i \in I$, are transversal.

Definition 3.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be A_∞- pre-categories. An A_∞- functor $F : \mathcal{C} \to \mathcal{D}$ is an A_∞- functor between the corresponding non-unital A_∞- pre-categories which takes quasi-isomorphisms in $\mathcal{C}$ to quasi-isomorphisms in $\mathcal{D}$.*

Remark 3.17. *An A_∞ -category with weak identity morphisms is the same as an A_∞- pre-category $\mathcal{C}$ with $\mathcal{C}^n_{tr} = Ob(\mathcal{C})^n$.*

Now we define the notion of quasi-equivalence for A_∞- pre-categories.

Definition 3.18. *An A_∞- functor $F : \mathcal{C} \to \mathcal{D}$ between A_∞- pre-categories is called quasi-equivalence if:*

a) For any pair $(X_1, X_2) \in \mathcal{C}^2_{tr}$, the map $f_1 : \mathrm{Hom}(X_1, X_2) \to \mathrm{Hom}(F(X_1), F(X_2))$ is a quasi-isomorphism.

b) Each object Y of $\mathcal{D}$ is quasi-isomorphic to some $F(X)$, $X \in Ob(\mathcal{C})$.

Definition 3.19. *Two A_∞- pre-categories $\mathcal{C}$ and $\mathcal{D}$ are called quasi-equivalent if there exists a sequence $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_n$ of A_∞- pre-categories with $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_n = \mathcal{D}$, such that for each $o \leq i \leq n-1$ there exists a quasi-equivalence from $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ or vice versa.*

4. Proof of Theorem 2.1

Definition 4.1. *An A_∞- (pre-)category $\mathcal{C}$ is called essentially small (resp. small) if the quasi-isomorphism classes in $\mathcal{C}$ form a set (resp. $Ob(\mathcal{C})$ is a set).*

In the rest of the paper "A_∞- category" stands for "A_∞- category with weak identity morphisms".

Theorem 4.2. *Quasi-equivalence classes of essentially small A_∞- pre-categories are in bijection with quasi-equivalence classes of essentially small A_∞- categories.*

The proof of this theorem will occupy this section.

4.1. **From essentially small to small.** Define the quasi-equivalence classes of small A_∞- (pre-)categories by requiring all the intermediate A_∞- (pre-)categories $\mathcal{C}_i$ in the chain to be small.

Lemma 4.3. *Quasi-equivalence classes of essentially small A_∞- (pre-)categories are in bijection with quasi-equivalence classes of small A_∞- (pre-)categories.*

Proof. We will just show how to construct a small A_∞- (pre-)category starting from essentially small one. All the rest checkings are straightforward.

The case of A_∞- categories is obvious. Namely, given essentially small A_∞- category $\mathcal{C}$, we can choose one object from each quasi-isomorphism class, and take the corresponding full A_∞- subcategory. By construction, it is small and the inclusion A_∞- functor is a quasi-equivalence.

To treat the case of A_∞- pre-categories, we need the following non-canonical operation on the small subclasses. Let $\mathcal{C}$ be an A_∞- pre-category, and $\mathcal{D} \subset Ob(\mathcal{C})$ be a small subclass (i.e. which is a set). For each finite collection $S_1, \dots, S_n$ of transversal sequences in $\mathcal{D}$, choose the objects $X_\pm = X_\pm(S.) \in Ob(\mathcal{C})$ such that all the sequences (X_-, S_i, X_+) are transversal. Let $T(\mathcal{D})$ be the union of $\mathcal{D}$ and all pairs $(X_-(S.), X_+(S.))$. Then $T(\mathcal{D})$ is again a small subclass of $Ob(\mathcal{C})$. Put

$$T^\infty(\mathcal{D}) := \bigcup_{n=0}^{\infty} T^n(\mathcal{D}). \tag{4.1}$$

Now, let $\mathcal{D}$ be a small subclass of $Ob(\mathcal{C})$ which contains precisely one object from each quasi-isomorphism class. Take the full A_∞- sub-pre-category $\mathcal{E} = T^\infty(\mathcal{D}) \subset \mathcal{C}$. Clearly, $\mathcal{E}$ is a small A_∞- pre-category by construction, and the inclusion A_∞- functor $\mathcal{E} \hookrightarrow \mathcal{C}$ is a quasi-equivalence. □

In the rest of this section, we deal only with small A_∞- (pre-)categories.

4.2. **Minimal models.**

Definition 4.4. *An A_∞ -(pre-)category is called minimal if $m_1 = 0$.*

Define quasi-equivalence classes of minimal A_∞- (pre-)categories by requiring all the intermediate A_∞- (pre-)categories $\mathcal{C}_i$ in the chain to be minimal.

Lemma 4.5. *Quasi-equivalence classes of A_∞- (pre-)categories are in bijection with quasi-equivalence classes of minimal A_∞- (pre-)categories.*

Proof. Let $\mathcal{C}$ be an A_∞- pre-category. For each pair of objects (X_1, X_2) (resp. for $(X_1, X_2) \in \mathcal{C}_{tr}^2$) choose a direct sum decomposition $\mathrm{Hom}(X_1, X_2) = K(X_1, X_2) \oplus Ac(X_1, X_2)$, where $K(X_1, X_2)$ is a subcomplex with zero differential, and $Ac(X_1, X_2)$ is an acyclic subcomplex. Denote by $i(X_1, X_2) : K(X_1, X_2) \to \mathrm{Hom}(X_1, X_2)$ the natural inclusion, $p(X_1, X_2) : \mathrm{Hom}(X_1, X_2) \to Ac(X_1, X_2)$ the natural projection (both i and p are quasi-isomorphisms). Choose some contracting homotopy $h = h(X_1, X_2) : Ac(X_1, X_2) \to Ac(X_1, X_2)$, such that $h^2 = 0$. Finally, denote by $H(X_1, X_2) : \mathrm{Hom}(X_1, X_2) \to \mathrm{Hom}(X_1, X_2)$ the extension of h by zero.

Starting from the data $i(X_1, X_2)$, $p(X_1, X_2)$ and $H(X_1, X_2)$, and applying the standard formulae for transferring A_∞- structures (see [KS]), one obtains:

1) An A_∞ -(pre-)category $\mathcal{C}_{min}$ with $Ob(\mathcal{C}_{min}) = Ob(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}_{min}}(X_1, X_2) = K(X_1, X_2)$. Also, in the case of A_∞- pre-categories, we have $(\mathcal{C}_{min})_{tr}^n = \mathcal{C}_{tr}^n$.

2) A quasi-equivalence $F : \mathcal{C}_{min} \to \mathcal{C}$, such that $F(X) = X$ and $f_1 = i$.

3) A quasi-equivalence $G : \mathcal{C} \to \mathcal{C}_{min}$, such that $G(X) = X$ and $g_1 = p$.

Lemma follows easily. □

Hence, it suffices to deal only with minimal A_∞- (pre-)categories.

4.3. **Hochshild cohomology of graded pre-categories.**

Definition 4.6. *Define a ($\mathbb{Z}$- or $\mathbb{Z}/2$-)graded pre-category as an A_∞- pre-category with $m_n = 0$ for $n \neq 2$.*

Define a functor between graded pre-categories to be an A_∞- functor with $f_n = 0$ for $n \neq 1$. We say that such a functor is an equivalence if it is a quasi-equivalence of A_∞- pre-categories.

Proposition 4.7. *Let $\mathcal{C}$ be a graded pre-category. Then it can be canonically extended to an actual graded category $\mathcal{C}_{full}$ with $Ob(\mathcal{C}_{full}) = Ob(\mathcal{C})$, together with a natural equivalence of graded pre-categories $\iota : \mathcal{C} \to \mathcal{C}_{full}$, $\iota(X) = X$ for any $X \in Ob(\mathcal{C})$.*

Proof. Category $\mathcal{C}_{full}$ can be obtained by formal inverting of quasi-isomorphisms in $\mathcal{C}$. We leave the straightforward checking to the reader. □

Let $\mathcal{C}$ be a graded pre-category. Define the bigraded Hochshild complex of $\mathcal{C}$ by the following formula:

$$CC^{i,j}(\mathcal{C}) = \prod_{(X_0,\dots,X_i)\in\mathcal{C}^i_{tr}} \mathrm{Hom}^j(\bigotimes_{1\le l\le i} \mathrm{Hom}_{\mathcal{C}}(X_{l-1},X_l), \mathrm{Hom}_{\mathcal{C}_{full}}(X_0,X_i)). \tag{4.2}$$

Here $i \in \mathbb{Z}_{\geq 0}$, and $j \in \mathbb{Z}$ (resp. $j \in \mathbb{Z}/2$). We write $\mathcal{C}_{full}$ in the above equation, because for $i = 0$ the sequence (X_0, X_0) is not transversal in general.

The differential is the standard Hochshild one. It maps $CC^{i,j}(\mathcal{C})$ to $CC^{i+1,j}(\mathcal{C})$. Namely, for $\phi \in CC^{n,p}(\mathcal{C})$, we have

$$\begin{aligned}\partial(\phi)(a_{n+1},\dots,a_1) = \sum_{i=1}^{n}(-1)^{n-i}\phi(a_{n+1},\dots,a_{i+1}a_i,\dots,a_1)+ \\ (-1)^n\phi(a_{n+1},\dots,a_2)a_1 + (-1)^{p\deg(a_{n+1})+1}a_{n+1}\phi(a_n,\dots,a_1).\end{aligned} \tag{4.3}$$

Therefore, we have bigraded Hochshild cohomology $HH^{i,j}(\mathcal{C})$.

Now suppose that we have a fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ between graded pre-categories. Clearly, it induces a morphism of complexes (preserving the gradings)

$$F^* : CC^{\cdot,\cdot}(\mathcal{D}) \to CC^{\cdot,\cdot}(\mathcal{C}). \tag{4.4}$$

4.4. **Important Lemma.** The key statement in our proof of Theorem 4.2 is the following.

Lemma 4.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of graded pre-categories. Then the induced morphism of complexes*

$$F^* : CC^{\cdot,\cdot}(\mathcal{D}) \to CC^{\cdot,\cdot}(\mathcal{C}) \tag{4.5}$$

induces isomorphisms $F^* : HH^{i,j}(\mathcal{D}) \to HH^{i,j}(\mathcal{C})$.

We will prove this lemma in the framework of local systems on simplicial sets. Recall the category Δ, which is a (non-full) subcategory of the category of sets. Its objects are denoted by $[n] = \{0, 1 \dots, n\}$, where $n \in \mathbb{Z}_{\geq 0}$. Further,

$$\mathrm{Hom}_{\Delta}([m],[n]) = \{\text{non-decreasing maps of sets } f : [m] \to [n]\}. \tag{4.6}$$

The category Δ admits the well-known system of generating morphisms, which consists of the face maps $\sigma^n_i : [n] \to [n+1]$, $i = 0, 1, \dots, n+1$, and degeneration maps $s^n_i : [n] \to [n-1]$, $i = 0, 1 \dots, n-1$. Here

$$\sigma^n_i(j) = \begin{cases} j & \text{for } j \leq i-1 \\ j+1 & \text{for } j \geq i, \end{cases} \qquad s^n_i(j) = \begin{cases} j & \text{for } j \leq i \\ j-1 & \text{for } j \geq i+1. \end{cases} \tag{4.7}$$

A simplicial set $\mathcal{X}.$ is a functor $\mathcal{X} : \Delta^{op} \to \mathrm{Set}$, $\mathcal{X}_n = \mathcal{X}([n])$. We treat $\mathcal{X}$ also as a category (which we denote by the same letter). Its objects are elements $X_n \in \mathcal{X}_n$. Further,

$$\mathrm{Hom}_{\mathcal{X}}(X_m, X_n) = \{f : [m] \to [n] | f^*(X_n) = X_m\}, \tag{4.8}$$

and the composition is obvious.

Given a simplicial set $\mathcal{X}.$, a (cohomological) local system of k- vector spaces on $\mathcal{X}.$ is a functor $\mathcal{A}:\mathcal{X}\to k\text{-}Vect$. In particular, we have a constant local system $\underline{k}$ with fiber k. Namely, $\underline{k}(X_n)=k$, and $\underline{k}(f)=\mathrm{id}$ for $f:X_m\to X_n$.

Local systems obviously form an abelian category. It is also easy to see that it has enough projectives. Indeed, for each object $X_n\in\mathcal{X}$, we have the corresponding local system P_{X_n}, $P_{X_n}(Y_m)=k[\mathrm{Hom}_{\mathcal{X}}(X_n,Y_m)]$, where $k[\cdot]$ means the formal linear envelope. By Ioneda Lemma, we have that

$$\mathrm{Hom}(P_{X_n},\mathcal{A})=\mathcal{A}(X_n). \tag{4.9}$$

Hence, P_{X_n} is projective. Each local system has a resolution by direct sums of such projectives.

Cohomology of a local system $\mathcal{A}$ is defined as $Ext^{\cdot}(\underline{k},\mathcal{A})$.

Now recall the well-known projective resolution of the local system $\underline{k}$. Put

$$\mathcal{P}_n=\bigoplus_{X_n\in\mathcal{X}_n}P_{X_n}. \tag{4.10}$$

The differential $\partial:\mathcal{P}_n\to\mathcal{P}_{n-1}$ is the sum of maps $(-1)^i\sigma_i^{n-1}:P_{X_n}\to P_{(\sigma_i^{n-1})^*(X_n)}$. The standard complex computing cohomology of $\mathcal{A}$ is just a complex of morphisms from this projective resolution to $\mathcal{A}$.

Proof of lemma 4.8. Define graded category $\mathcal{E}$ to be a full subcategory of $\mathcal{D}_{full}$, which contains exactly one object from each isomorphism class in $\mathcal{D}$. Choose some functor $G:\mathcal{D}_{full}\to\mathcal{E}$, which equals to identity on $\mathcal{E}$, and denote by the same letter its restriction to $\mathcal{D}$. To proof the desired quasi-isomorphism, it suffices to proof that maps $G^*:CC^{\cdot,\cdot}(\mathcal{E})\to CC^{\cdot,\cdot}(\mathcal{D})$ and $(GF)^*:CC^{\cdot,\cdot}(\mathcal{E})\to CC^{\cdot,\cdot}(\mathcal{C})$ are quasi-isomorphisms.

Therefore, we may assume that $\mathcal{D}$ is itself an actual graded category, and the functor F induces a bijection between the set of quasi-isomorphism classes of objects in $\mathcal{C}$ and the set $Ob(\mathcal{D})$.

Now define the simplcial set $\mathcal{X}.$ by the formula $\mathcal{X}_n=Ob(\mathcal{D})^{n+1}$. Further, for $(Y_0,\dots,Y_n)\in\mathcal{X}_n$ and $f:[m]\to[n]$, put $f^*(Y_0,\dots,Y_n)=(Y_{f(0)},Y_{f(1)},\dots,Y_{f(m)})$.

Define the local system $\mathcal{A}$ on $\mathcal{X}$ by the formula

$$\mathcal{A}(Y_0,\dots,Y_n)=\bigoplus_j\mathrm{Hom}^j(\bigotimes_{1\leq l\leq n}\mathrm{Hom}_{\mathcal{D}}(Y_{l-1},Y_l),\mathrm{Hom}_{\mathcal{D}}(Y_0,Y_n)). \tag{4.11}$$

Further, for $f : [m] \to [n]$, and homogeneous $\phi^m \in \mathcal{A}(f^*(Y_0,\dots,Y_n))$, $a_i \in \mathrm{Hom}_{\mathcal{D}}(Y_{i-1},Y_n)$ we put

$$(4.12)\quad f(\phi^m)(a_n,\dots,a_1) = \\ (-1)^{\epsilon(f)} a_n \dots a_{f(m)+1}\phi^m(a_{f(m)}\dots a_{f(m-1)+1},\dots,a_{f(1)}\dots,a_{f(0)+1})a_{f0)}\dots a_1,$$

where $\epsilon(f) = \deg(\phi^m)\sum_{i=f(m)+1}^{n}\deg(a_i) + \frac{(n-m)(n+m+1)}{2}$, and the product over the empty set equals to the corresponding identity morphism.

Take the standard projective resolution $\mathcal{P}.$ of the local system $\underline{k}$, as above. Then the complex $\mathrm{Hom}(\mathcal{P}.,\underline{k})$ is naturally isomorphic to the complex $\bigoplus_j CC^{\cdot,j}$, hence

$$(4.13)\qquad H^{\cdot}(\mathcal{A}) \cong \bigoplus_j HH^{\cdot,j}(\mathcal{D}).$$

Now we would like to express $HH^{\cdot,\cdot}(\mathcal{C})$ in similar terms. For $S \in \mathcal{C}_{tr}^{n+1}$, put $Q_S := P_{F(S)}$ — a projective local system on $\mathcal{X}$. If $S = (Y_0,\dots,Y_n)$, then put $(\sigma_i^{n-1})^*(S) := (Y_0,\dots,\widehat{Y_i},\dots,Y_n)$. Clearly, $F((\sigma_i^{n-1})^*(S)) = (\sigma_i^{n-1})^*(F(S))$.

Define a chain complex $\mathcal{Q}.$ as follows. Put $\mathcal{Q}_n = \bigoplus_{S\in\mathcal{C}_{tr}^{n+1}} Q_S$. Further, the differential $\partial : \mathcal{Q}_n \to \mathcal{Q}_{n-1}$ is the sum of maps $(-1)^i\sigma_i^{n-1} : \mathcal{Q}_S \to \mathcal{Q}_{(\sigma_i^{n-1})^*(S)}$. We have obvious projection $\mathcal{Q}_0/Im(\partial) \to \underline{k}$.

It is clear that the complex $\mathrm{Hom}(\mathcal{Q}.,\mathcal{A})$ is naturally quasi-isomorphic to the complex $\bigoplus_j CC^{\cdot,j}(\mathcal{C})$. We have an obvious morphism of complexes $\Phi : \mathcal{Q}. \to \mathcal{P}.$, with non-zero components being identity maps $Q_S \to P_{F(S)}$. The morphism Φ is compatible with projections $\mathcal{Q}_0 \to \underline{k}$, $\mathcal{P}_0 \to \underline{k}$. Also, we have the commutative diagram

$$(4.14)\qquad \begin{array}{ccc} \bigoplus_j CC^{i,j}(\mathcal{D}) & \xrightarrow{F^*} & \bigoplus_j CC^{i,j}(\mathcal{D}) \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Hom}(\mathcal{P}_i,\mathcal{A}) & \xrightarrow{\cdot\circ\Phi} & \mathrm{Hom}(\mathcal{Q}_i,\mathcal{A}). \end{array}$$

Therefore, we are left to prove that $\mathcal{Q}.$ is a resolution of $\underline{k}$.

Sublemma. *The complex $\mathcal{Q}.$ is a resolution of $\underline{k}$.*

Proof. For convenience put $\mathcal{Q}_{-1} = \underline{k}$, and treat $k = \underline{k}(Y_0,\dots,Y_m)$ as $k[\mathrm{Hom}([-1],[n])]$, where $[-1] := \varnothing$. So we need to prove that the complex $\mathcal{Q}.$ is acyclic.

Fix some sequence $T = (Y_0,\dots,Y_n) \in \mathcal{X}_n$, $n \geq 0$. and consider the complex $\mathcal{Q}.(T)$ of vector spaces. We have that $\mathcal{Q}_m(T)$ consists of linear combinations of pairs (f,S), where $f : [m] \to [n]$, $S \in \mathcal{C}_{tr}^{m+1}$, such that $F(S) = f^*(T)$. Take some closed element

$a = \sum_{j=1}^{p} \lambda_j(f_j, S_j) \in \mathcal{Q}_m(T)$, $m \geq -1$. We would like to construct its (non-canonically defined) pre-image $H(a) \in \mathcal{Q}_{m+1}(T)$ under the differential ∂.

Define the maps $g_j : [m+1] \to [n]$, $j = 1, \ldots, p$, by the formula

$$g_j(l) = \begin{cases} f_j(l-1) & \text{for } l > 0 \\ 0 & \text{for } l = 0. \end{cases} \tag{4.15}$$

Further, by the definition of an A_∞- pre-category, there exists an object $\widetilde{Y} \in Ob(\mathcal{C})$ and a quasi-isomorphism $\widetilde{Y} \to Y_0$ such that all sequences $(\widetilde{Y}, S_j)$ are transversal. Since F induces a bijection between quasi-isomorphism classes in $\mathcal{C}$ and objects in $\mathcal{D}$, we have that $F(\widetilde{Y}) = F(Y_0)$. Therefore, $F(\widetilde{Y}, S_j) = (F(Y_0), F(S_j)) = g_j^*(T)$. Put

$$H(a) = \sum_{j=1}^{p} \lambda_j(g_j, (\widetilde{Y}, S_j)). \tag{4.16}$$

Then from the closedness of a we immediately obtain that $\partial(H(a)) = a$. This proves Sublemma. □

Lemma is proved. □

4.5. A_∞**- structures on a graded pre-category.**

Definition 4.9. *An A_∞- structure on a graded pre-category $\mathcal{C}$ is a collection of maps m_n, $n \geq 3$, $\deg(m_n) = 2 - n$ for all transversal sequences, such that together with $m_2(a, b) = ab$ and $m_1 = 0$ they give a structure of A_∞- pre-category on $\mathcal{C}$.*

Two A_∞- structures m and m' on $\mathcal{C}$ are called strongly homotopic if there exists an A_∞- morphism $F : (\mathcal{C}, m) \to (\mathcal{C}, m')$ with $F(X) = X$ for $X \in Ob(\mathcal{C})$, and $f_1 = \mathrm{id}$. In this case F is called a strong homotopy between m and m'.

Formal collections of maps $f_n : \bigotimes_{1 \leq i \leq n} \mathrm{Hom}(X_{i-1}, X_i) \to \mathrm{Hom}(X_0, X_n)$ of degree $1 - n$ for all transversal sequences $(X_0, \ldots, X_n) \in \mathcal{C}_{tr}^{n+1}$, $n \geq 1$, with $f_1 = \mathrm{id}$, form a group $G_{\mathcal{C}}$. The product of f and g is given by the same formula as the composition of A_∞- functors. This group acts on the set $A_\infty(\mathcal{C})$ of A_∞- structures on $\mathcal{C}$. Namely, $f(m) = m'$ iff f is an A_∞- morphism from $(\mathcal{C}, m)$ to $(\mathcal{C}, m')$. Tautologically, two A_∞- structures are strongly homotopic iff they lie in the same orbit of this action.

The following Lemmas are well-known for A_∞- algebras, and the proof is in fact straightforward. We omit the proof.

Lemma 4.10. *Let $(m_3, \ldots, m_{n-1})$ be partially defined A_∞- structure on a graded pre-category $\mathcal{C}$, i.e. the maps $m_{\leq n-1}$ satisfy all the required equations which do not contain*

$m_{\geq n}$. *Write the first* A_∞*- constraint containing* m_n *in the form*

$$\partial(m_n) = \Phi, \tag{4.17}$$

where ∂ *is the Hochshild differential and* $\Phi = \Phi(m_3, \dots, m_{n-1})$ *is quadratic expression. Then we always have* $\partial(\Phi) = 0$.

Lemma 4.11. *Let* m *and* m' *be two* A_∞ *-structures on a graded pre-category* $\mathcal{C}$. *Let* $f : (\mathcal{C}, m) \to (\mathcal{C}, m')$ *be an* A_∞*- morphism with* $f_1 = \mathrm{id}$, *and* $f_i = 0$ *for* $2 \leq i \leq n-2$. *Then* $m'_i = m_i$ *for* $i \leq n-1$, *and* $m'_n = m_n + \partial(f_{n-1})$.

Lemma 4.12. *Let* m *and* m' *be two* A_∞ *-structures on a graded pre-category* $\mathcal{C}$. *Suppose that* $(f_1 = \mathrm{id}, f_2, \dots, f_{n-1})$ *is a partially defined strong homotopy between* m *and* m'. *i.e. the maps* $f_{\leq n-1}$ *satisfy all the required equations which do not contain* $f_{\geq n}$. *Write the first* A_∞*- constraint containing* f_n *in the form*

$$\partial(f_n) = \Psi, \tag{4.18}$$

where ∂ *is the Hochshild differential and* $\Psi = \Psi(f_2, \dots, f_{n-1}; m, m')$ *is a polynomial expression. Then we always have* $\partial(\Psi) = 0$.

We will also need the notion of homotopy between two A_∞- functors. First, let $f, f' : A \to B$ be two A_∞- morphisms of (possibly non-unital) A_∞- algebras. We have the associated morphisms of DG coalgebras $f, f' : T_+(A[1]) \to T_+(B[1])$. A homotopy between f and f' is a map $H : T_+(A[1]) \to T_+(B[1])$ satisfying the identities

$$\Delta \circ H = (f \otimes H + H \otimes f') \circ \Delta, \tag{4.19}$$

and

$$f - f' = b_B \circ H + H \circ b_A. \tag{4.20}$$

Any map H satisfying (4.19) is uniquely determined by its components $h_n : A^{\otimes n} \to B$, $\deg(h_n) = -n$.

Definition 4.13. *Let* $F, F' : \mathcal{C} \to \mathcal{D}$ *be* A_∞*-functors between* A_∞*- pre-categories, such that* $F(X) = F'(X)$ *for each* $X \in Ob(\mathcal{C})$. *A homotopy* H *between* f *and* f' *is a collection of maps* $h_n : \bigotimes\limits_{1 \leq i \leq n} \mathrm{Hom}_{\mathcal{C}}(X_{i-1}, X_i) \to \mathrm{Hom}_{\mathcal{D}}(F(X_0), F(X_n))$ *of degree* $-n$, *for all transversal sequences* $(X_0, \dots, X_n)$, *satisfying the following property. For each transversal sequence* $(X_0, \dots, X_N) \in \mathcal{C}_{tr}^{n+1}$, *we have that the maps* h_n *define a homotopy between the restricted* A_∞*-functors*

$$F, F' : \bigoplus_{i<j} \mathrm{Hom}_{\mathcal{C}}(X_i, X_j) \to F, F' : \bigoplus_{i<j} \mathrm{Hom}_{\mathcal{D}}(F(X_i), F(X_j)). \tag{4.21}$$

We will need the following Lemma.

Lemma 4.14. *Let* $F:\mathcal{C}\to\mathcal{D}$ *be an* A_∞*-functor. Suppose that we are given with a collection of maps* $h_n:\bigotimes\limits_{1\leq i\leq n}\mathrm{Hom}_{\mathcal{C}}(X_{i-1},X_i)\to\mathrm{Hom}_{\mathcal{D}}(F(X_0),F(X_n))$ *for all transversal sequences. Then there exists a unique* A_∞*-functor* $F':\mathcal{C}\to\mathcal{D}$, $F'(X)=F(X)$, *such that the sequence* h_n *defines a homotopy between* F *and* F'.

Moreover, in the case when $\mathcal{C}=(\mathcal{E},m)$, $\mathcal{D}=(\mathcal{E},m')$ *are* A_∞*-structures on the same graded pre-category* $\mathcal{E}$, F *belongs to* $G_{\mathcal{E}}$, *and* $h_i=0$ *for* $1\leq i\leq n-2$, *we have that*

$$\begin{cases} f_i'=f_i & \text{for } 1\leq i\leq n-1,\\ f_n'=f_n+\partial(h_{n-1}) \end{cases} \tag{4.22}$$

Proof. The first statement is a direct consequence of [P], Lemma 2.1. The second one is checked straightforwardly. ☐

4.6. **Invariance Theorem.** Let $\phi:\mathcal{C}\to\mathcal{D}$ be an equivalence of graded pre-categories. We have a natural map $\phi^*:A_\infty(\mathcal{D})\to A_\infty(\mathcal{C})$, and a homomorphism $\phi^*:G_{\mathcal{D}}\to G_{\mathcal{C}}$, compatible with our group actions. Therefore we have a map of strong homotopy equivalence classes of A_∞- structures:

$$\phi^*:A_\infty(\mathcal{D})/G_{\mathcal{D}}\to A_\infty(\mathcal{C})/G_{\mathcal{C}}. \tag{4.23}$$

Theorem 4.15. *The map* (4.23) *is a bijection.*

Proof. **Surjectivity.** First we prove that our map is surjective. Take some A_∞- structure m on $\mathcal{C}$. We want to prove that it is strongly homotopic to some A_∞- structure of the form $\phi^*(\widetilde{m})$, where $\widetilde{m}$ is an A_∞- structure on $\mathcal{D}$. Clearly, it suffices to prove the following Lemma.

Lemma 4.16. *Let* m *be an* A_∞*- structure on* $\mathcal{C}$ *such that* $m_i=\phi^*(\widetilde{m_i})$ *for* $3\leq i\leq n-1$, *where* $(\widetilde{m_3},\dots,\widetilde{m_{n-1}})$ *is a partially defined* A_∞*- structure on* $\mathcal{D}$*. Then* m *is strongly homotopic to some* A_∞*- structure* m' *such that* $m_i'=m_i$ *for* $i\leq n-1$, *and* $m_n'=\phi^*(\widetilde{m_n})$, *so that* $(\widetilde{m_3},\dots,\widetilde{m_n})$ *is a partially defined* A_∞*- structure on* $\mathcal{D}$*. Moreover, strong homotopy* $(f_1,f_2,\dots)$ *between* m *and* m' *can be taken to be such that* $f_2=\dots=f_{n-2}=0$.

Proof. Write the first A_∞- constraint containing m_n in the form

$$\partial(m_n)=\Phi, \tag{4.24}$$

as in Lemma 4.10. We have that $\Phi(m_3,\dots,m_{n-1})$ is a Hochshild coboundary. By Lemma 4.10, we have that $\Phi(\widetilde{m_3},\dots,\widetilde{m_{n-1}})$ is a Hochchild cocycle. Further, we have that

$$\Phi(m_3,\dots,m_{n-1})=\phi^*\Phi(\widetilde{m_3},\dots,\widetilde{m_{n-1}}). \tag{4.25}$$

Therefore, Lemma 4.8 implies that $\Phi(\widetilde{m_3},\dots,\widetilde{m_{n-1}})$ is a Hochshild coboundary. Take some $\widetilde{m_n}\in CC^{n,2-n}(\mathcal{D})$ such that $\partial(\widetilde{m_n})=\Phi(\widetilde{m_3},\dots,\widetilde{m_{n-1}})$.

We have that $\phi^*(\widetilde{m_n})-m_n$ is a Hochshild cocycle. Again by Lemma 4.8, we can choose $\widetilde{m_n}$ in such a way that this difference is a Hochshild coboundary. Take some element $f\in G_{\mathcal{C}}$ such that $f_2=\dots=f_{n-2}=0$, and $\partial(f_{n-1})=\phi^*(\widetilde{m_n})-m_n$. Put $m'=f(m)$. By Lemma 4.11, we have that $m'_i=\phi^*(\widetilde{m_i})$ for $i\leq n$. This proves Lemma. □

Surjectivity is proved.

Injectivity. We are left to prove that our map is injective. Let m,m' be A_∞-structures on $\mathcal{D}$, and let $F:(\mathcal{C},\phi^*(m))\to(\mathcal{C},\phi^*(m'))$ be a strong homotopy. We need to prove the existence of a strong homotopy between m and m'. Clearly, it suffices to prove the following Lemma.

Lemma 4.17. *Let m,m' be A_∞-structures on $\mathcal{C}$. Let f be a strong homotopy between $\phi^*(m)$ and $\phi^*(m')$. Suppose that $f_i=\phi^*(\widetilde{f_i})$ for $2\leq i\leq n-1$, where $(\widetilde{f_2},\dots,\widetilde{f_{n-1}})$ is a partially defined strong homotopy between m and m'. Then there exists some strong homotopy f' between $\phi^*(m)$ and $\phi^*(m')$ such that $f'_i=f_i$ for $i\leq n-1$, and $f'_n=\phi^*(\widetilde{f_n})$, so that $(\widetilde{f_2},\dots,\widetilde{f_n})$ is a partially defined strong homotopy between m and m'.*

Proof. Write the first A_∞-constraint containing f_n in the form

$$\partial(f_n)=\Psi, \tag{4.26}$$

as in Lemma 4.12. We have that $\Psi(f_2,\dots,f_{n-1};\phi^*(m),\phi^*(m'))$ is a Hochshild coboundary. By Lemma 4.12, we have that $\Psi(\widetilde{f_2},\dots,\widetilde{f_{n-1}};m,m')$ is a Hochchild cocycle. Further, we have that

$$\Psi(f_2,\dots,f_{n-1},\phi^*(m),\phi^*(m'))=\phi^*\Psi(\widetilde{f_2},\dots,\widetilde{f_{n-1}};m,m'). \tag{4.27}$$

Therefore, Lemma 4.8 implies that $\Psi(\widetilde{f_2},\dots,\widetilde{f_{n-1}},m,m')$ is a Hochshild coboundary. Take some $\widetilde{f_n}\in CC^{n,1-n}(\mathcal{D})$ such that $\partial(\widetilde{f_n})=\Psi(\widetilde{f_2},\dots,\widetilde{f_{n-1}};m,m')$.

We have that $\phi^*(\widetilde{f_n})-f_n$ is a Hochshild cocycle. Again by Lemma 4.8, we can choose $\widetilde{f_n}$ in such a way that this difference is a Hochshild coboundary. Take some sequence of elements $h_n\in CC^{n,-n}$, $n\geq 1$, such that $h_2=\dots=h_{n-2}=0$, and $\partial(h_{n-1})=\phi^*(\widetilde{f_n})-f_n$. By Lemma 4.14, there exists a unique strong homotopy f', such that the sequence h_n defines a homotopy between f and f'. Again by Lemma 4.14, we have that $f'_i=\phi^*(\widetilde{f_i})$ for $i\leq n$. This proves Lemma. □

Injectivity is proved. □

4.7. **Completion of the proof.** Before we prove Theorem 4.2, we need one more lemma.

Lemma 4.18. *Let $\phi : \mathcal{C} \to \mathcal{D}$ be an equivalence of graded pre-categories. Let m, m' be some A_∞- structures on $\mathcal{C}$ and $\mathcal{D}$ respectively. The following are equivalent:*

(i) The functor ϕ can be extended to an A_∞- functor $\Phi : (\mathcal{C}, m) \to (\mathcal{D}, m')$;

(ii) The A_∞- structures m and $\phi^*(m')$, are strongly homotopic.

Proof. Evident. □

Proof of Theorem 4.2. By Lemmas 4.3 and 4.5, it suffices to prove that quasi-equivalence classes of small minimal A_∞- categories are in bijection with quasi-equivalence classes of small minimal A_∞- pre-categories.

Given a minimal A_∞- category $\mathcal{C}$, it can also be considered as an A_∞- pre-category with $\mathcal{C}^n_{tr} = Ob(\mathcal{C})^n$. Clearly, if $\mathcal{C}$ and $\mathcal{D}$ are quasi-equivalent minimal A_∞- categories, then the associated minimal A_∞- pre-categories are quasi-equivalent.

Now, let $\mathcal{C}$ be a minimal A_∞- pre-category. Denote by m the A_∞- structure on $\mathcal{C}^{gr}$ corresponding to $\mathcal{C}$.

We have an equivalence of graded pre-categories $\iota_{\mathcal{C}^{gr}} : \mathcal{C}^{gr} \to \mathcal{C}^{gr}_{full}$. By Theorem 4.15, there exists an A_∞- structure $\widetilde{m}$ on $\mathcal{C}^{gr}_{full}$, such that the A_∞- structure $\iota^*_{\mathcal{C}^{gr}}(\widetilde{m})$ is strongly homotopic to m. By lemma 4.18, the functor $\iota_{\mathcal{C}^{gr}}$ can be extended to the quasi-equivalence $\mathcal{C} \to (\mathcal{C}^{gr}_{full}, m)$. Hence, starting from a minimal A_∞- pre-category, we constructed some minimal A_∞- category $\widetilde{\mathcal{C}}$, together with a quasi-equivalence $\mathcal{C} \to \widetilde{\mathcal{C}}$. We are left to prove that, starting from quasi-equivalent A_∞- pre-categories, we obtain quasi-equivalent A_∞- categories.

Let $F : \mathcal{C} \to \mathcal{D}$ be a quasi-equivalence of minimal A_∞- pre-categories. The functor of graded categories $F_1 : \mathcal{C}^{gr} \to \mathcal{D}^{gr}$ can be obviously extended to a functor $\Phi_1 : \mathcal{C}^{gr}_{full} \to \mathcal{D}^{gr}_{full}$, so that we have a commutative square of functors:

$$\begin{array}{ccc} \mathcal{C}^{gr} & \xrightarrow{F_1} & \mathcal{D}^{gr} \\ \iota_{\mathcal{C}^{gr}} \downarrow & & \iota_{\mathcal{D}^{gr}} \downarrow \\ \mathcal{C}^{gr}_{full} & \xrightarrow{\Phi_1} & \mathcal{D}^{gr}_{full}. \end{array} \tag{4.28}$$

Denote by m (resp. m') the A_∞- structure on $\mathcal{C}^{gr}$ (resp. on $\mathcal{D}^{gr}$) corresponding to $\mathcal{C}$ (resp. to $\mathcal{D}$). Further, denote by $\widetilde{m}$ (resp. $\widetilde{m'}$) the A_∞- structure on $\mathcal{C}^{gr}_{full}$ (resp. on $\mathcal{D}^{gr}_{full}$) such that $\iota^*_{\mathcal{C}^{gr}}(\widetilde{m})$ is strongly homotopic to m (resp. $\iota^*_{\mathcal{D}^{gr}}(\widetilde{m'})$ is strongly homotopic to m'). By Lemma 4.18, A_∞- structures $F_1^*(m')$ and m are also strongly homotopic. Hence, from the commutative square 4.28 and Theorem 4.15, we obtain that A_∞- structures $\Phi_1^*(\widetilde{m'})$ and $\widetilde{m}$ are strongly homotopic. Therefore, by Lemma 4.18, the functor Φ_1 can be extended to a quasi-equivalence

$$\Phi : (\mathcal{C}^{gr}_{full}, \widetilde{m}) \to \mathcal{D}^{gr}_{full}, \widetilde{m'}). \tag{4.29}$$

Thus, starting from quasi-equivalent minimal A_∞- pre-categories, we obtain quasi-equivalent minimal A_∞- categories. Theorem is proved. □

5. Twisted complexes over A_∞- pre-categories

It is clear that Theorem 4.2 implies that we can take pre-triangulated envelope and perfect derived category of any essentially small A_∞- pre-category, by replacing it with some quasi-equivalent A_∞- category. However, it is useful in practice to have a construction of pre-triangulated envelope in the framework of A_∞- pre-categories. We present such a construction in this section. It is in fact straightforward generalization of twisted complexes over ordinary A_∞- categories [Ko1].

We work here over arbitrary graded commutative ring k.

Let $\mathcal{C}$ be an A_∞- pre-category. Define the A_∞- pre-category $\widetilde{\mathcal{C}}$ as follows:

1) $Ob(\widetilde{\mathcal{C}}) = \{X[n] \mid X \in Ob(\mathcal{C}), n \in \mathbb{Z}\}$;

2) $(X_1[n_1], \dots, X_k[n_k]) \in \widetilde{\mathcal{C}}^k_{tr} \Leftrightarrow (X_1, \dots, X_k) \in \mathcal{C}^k_{tr}$.

3) For a transversal pair $(X_1[n_1], X_2[n_2])$, we put

$$\mathrm{Hom}(X_1[n_1], X_2[n_2]) := \mathrm{Hom}(X_1, X_2)[n_2 - n_1]. \tag{5.1}$$

The higher products equal to that in $\mathcal{C}$.

Now, we define the twisted complexes. Given a transversal sequence $S = (X_1, \dots, X_n) \in \mathcal{C}^n_{tr}$, we put

$$\mathrm{End}_+(S) := \bigoplus_{1 \le i < j \le n} \mathrm{Hom}(X_i, X_j). \tag{5.2}$$

Definition 5.1. *A non-unital A_∞- algebra A is called nilpotent if it is equipped with finite decreasing filtration $A = F_1 A \supset F_2 A \supset \dots F_n A = 0$, such that*

$$m_k(F_{r_1} A \otimes \dots \otimes F_{r_k} A) \subset F_{r_1 + \dots + r_k} A. \tag{5.3}$$

Clearly, $\mathrm{End}_+(S)$ is a nilpotent A_∞- algebra, with filtration $F_\cdot \mathrm{End}_+(S)$, where

$$F_r \mathrm{End}_+(S) = \bigoplus_{1 \le i \le j - r \le n - r} \mathrm{Hom}(X_i, X_j), \quad r \ge 1. \tag{5.4}$$

Just as in [ELO2], for each nilpotent A_∞- algebra A, we have a groupoid $\mathcal{MC}(A)$ of Maurer-Cartan solutions in A. The following Lemma is straightforward generalization of [ELO2], Theorem 7.2.

Lemma 5.2. *Let $f : A_1 \to A_2$ be a filtered A_∞- quasi-isomorphism of nilpotent A_∞- algebras. Then the induced functor*

$$f^* : \mathcal{MC}(A_1) \to \mathcal{MC}(A_2) \tag{5.5}$$

is an equivalence.

Definition 5.3. *Let $\mathcal{C}$ be an A_∞- pre-category. Define the A_∞- pre-category $\mathcal{C}^{pre-tr}$ of twisted complexes over $\mathcal{C}$ as follows:*

1) Objects of $\mathcal{C}^{pre-tr}$ are pairs (S,α), where S is some transversal sequence in $\widetilde{\mathcal{C}}$, and $\alpha \in \mathrm{End}_+(S)^1$ is a Maurer-Cartan solution.

2) The sequence $((S_1,\alpha_1),\dots,(S_n,\alpha_n)) \in (\mathcal{C}^{pre-tr})^n$ is transversal iff the sequence $(S_1,\dots,S_n)$ is transversal in $\widetilde{\mathcal{C}}$.

3) For a transversal pair $((S_1,\alpha_1),(S_2,\alpha_2))$ in $\mathcal{C}^{pre-tr}$, we put

$$\mathrm{Hom}((S_1,\alpha_1),(S_2,\alpha_2)) = \bigoplus_{X\in S_1, Y\in S_2} \mathrm{Hom}(X,Y). \tag{5.6}$$

4) For a transversal sequence, $((S_0,\alpha_0),\dots,(S_n,\alpha_n)) \in (\mathcal{C}^{pre-tr})^{n+1}_{tr}$, and homogeneous morphisms $x_i \in \mathrm{Hom}((S_{i-1},\alpha_{i-1}),(S_i,\alpha_i))$ we put

$$m_n(x_n,\dots,x_1) = \sum_{k_0,\dots,k_n\geq 0} (-1)^{\epsilon} m_{n+k_0+\dots+k_n}(\alpha_n^{k_n}, x_n, \alpha_{n-1}^{k_{n-1}},\dots,x_1,\alpha_0^{k_0}), \tag{5.7}$$

where $\epsilon = \sum\limits_{n\geq i>j\geq 0} (\deg(x_i)+k_i)k_j + \sum\limits_{i=0}^{n} \frac{k_i(k_i+1)}{2} + \sum\limits_{i=1}^{n} ik_i.$

Proposition 5.4. *The A_∞- pre-category $\mathcal{C}^{pre-tr}$ is well-defined.*

Proof. The only non-obvious thing to check is the extension property. To prove it, we need the following lemma.

Lemma 5.5. *Let $\mathcal{D}$ be an A_∞- pre-category, and $(X_1,\dots,X_n,Y_1,\dots,Y_n)$ a transversal sequence in $\mathcal{D}$. Suppose that we are given with quasi-isomorphisms $F_i : X_i \to Y_i$, $1 \leq i \leq n$. Then for each Maurer-Cartan solution $\beta \in \mathrm{End}_+(Y_1,\dots,Y_n)$ (resp. $\alpha \in \mathrm{End}_+(X_1,\dots,X_n)$) there exists a Maurer-Cartan solution $\alpha \in \mathrm{End}_+(X_1,\dots,X_n)$ (resp. $\beta \in \mathrm{End}_+(Y_1,\dots,Y_n)$), together with a quasi-isomorphism $G : ((X_1,\dots,X_n),\alpha) \to ((Y_1,\dots,Y_n),\beta)$ in $\mathcal{D}^{pre-tr}$.*

Proof. Consider the following A_∞- algebras: $\mathcal{A}_1 = \mathrm{End}_+(X_1,\dots,X_n)$, $\mathcal{A}_2 = \mathrm{End}_+(Y_1,\dots,Y_n)$ and $\mathcal{B}$ described as follows. As a k- module,

$$\mathcal{B} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \bigoplus_{1\leq i<j\leq n} \mathrm{Hom}(X_i,Y_j)[-1], \tag{5.8}$$

and the higher products are direct sums of that in $\mathcal{D}$, and also

$$m_{k+l+1}(y_l,\dots,y_1,F_p,x_k,\dots,x_1) \quad \text{for } y_i\in \mathrm{Hom}(Y_{j_{i-1}},Y_{j_i}), x_i\in \mathrm{Hom}(X_{q_{i-1}},X_{q_i}), \\ 1\leq q_0<\dots<q_k=p=j_0<\dots<j_l\leq n. \tag{5.9}$$

We have obvious projections $\pi_i:\mathcal{B}\to\mathcal{A}_i$, $i=1,2$, which are quasi-isomorphisms.

Now suppose that $\beta\in\mathcal{A}_2^1$ is an MC solution. We will show how to construct the required $\alpha\in\mathcal{A}_1^1$. The construction in the other direction is analogous.

By Lemma 7.6, there exists an MC solution $\tilde{\beta}\in\mathcal{B}^1$ such that $\pi_2(\tilde{\beta})$ is homotopic to β. The components of $\tilde{\beta}$, together with $F_i:X_i\to Y_i$ give objects $E_1=((X_1,\dots,X_n),\pi_1(\tilde{\beta})), E_2=((Y_1,\dots,Y_n),\pi_2(\tilde{\beta}))\in Ob(\mathcal{D}^{pre-tr})$, and a quasi-isomorphism $F:E_1\to E_2$ (this is straightforward).

Further, if $h\in\mathcal{A}_2^0$ is a morphism $\pi_2(\tilde{\beta})\to\beta$ in the groupoid $\mathcal{MC}(\mathcal{A}_2)$, then $G:E_1\to((Y_1,\dots,Y_n),\beta)$,

$$G=F+\sum_{k_0,k_1,k_2\geq 0} m^{\mathcal{D}}_{2+k_0+k_2}(\beta^{k_2},h,\pi_2(\tilde{\beta})^{k_1},F,\pi_1(\tilde{\beta})^{k_0}) \tag{5.10}$$

is a quasi-isomorphism in $\mathcal{D}^{pre-tr}$. This proves Lemma. □

The above Lemma immediately implies the extension property for $\mathcal{D}^{pre-tr}$. □

Proposition 5.6. *1) Each A_∞-functor $F:\mathcal{D}_1\to\mathcal{D}_2$ induces in the natural way an A_∞-functor $F^*:\mathcal{D}_1^{pre-tr}\to\mathcal{D}_2^{pre-tr}$.*

2) In the case when F is a quasi-equivalence, F^* is also such.

Proof. 1) The A_∞-functor F^* is given by the same formulas as in [ELO2], Section 7. We should prove that it preserves quasi-isomorphisms. We note that this is evident for the following class of "good" quasi-isomorphisms.

We call a quasi-isomorphism $f:((X_1,\dots,X_n),\alpha)\to((Y_1,\dots,Y_m),\beta)$ good if $n=m$, the components $f_{ij}:X_i\to Y_j$ vanish for $i>j$ and the components $f_{ii}:X_i\to Y_i$ are quasi-isomorphisms.

Clearly, good quasi-isomorphisms are preserved by F^*. It follows from Lemma 5.5 that for each quasi-isomorphism $f_1:E_1\to E_2$ in $\mathcal{D}_1^{pre-tr}$, there exist quasi-isomorphisms $f_0:E_0\to E_1$, $f_2:E_2\to E_3$ such that the sequence (E_0,E_1,E_2,E_3) is transversal and $m_2(f_1,f_0)$, $m_2(f_2,f_1)$ are homotopic to good quasi-isomorphisms. This proves part 1) of Proposition.

2) If F is a quasi-isomorphism, then all morphisms $f_1:\mathrm{Hom}_{\mathcal{D}_1^{pre-tr}}(E_1,E_2)\to \mathrm{Hom}_{\mathcal{D}_2}(F(E_1),F(E_2))$ induce quasi-isomorphisms on the subquotients with respect to the natural filtrations

$$F_r \operatorname{Hom}(((X_1,\ldots,X_n),\alpha),((Y_1,\ldots,Y_m),\beta)) = \bigoplus_{j-i\geq r} \operatorname{Hom}(X_i,Y_j). \tag{5.11}$$

Essential surjectivity is implied by Lemma 5.5, together with Lemma 7.6. □

For the ordinary A_∞- categories, our construction gives standard A_∞- categories of twisted complexes introduced in [BK] for DG categories and generalized in [Ko1] to A_∞- categories.

Now suppose that k is again a field. By Proposition 5.6 2), we have that passing from an essentially small A_∞- pre-category to quasi-equivalent A_∞- category commutes (up to quasi-equivalence) with taking of twisted complexes.

6. Homological mirror symmetry for curves of higher genus

The second main result of this paper is the prove of a version of HMS conjecture for curves genus $g \geq 3$. Actually, we follow the steps of Seidel's proof in the genus 2 case, and generalize it to genus $g \geq 3$ case.

We treat genus ≥ 3 curves as symplectic varieties, and associate to them Fukaya categories. Further, Landau-Ginzburg models are considered algebro-geometrically. The associated categories are that of matrix factorizations or, equivalently, triangulated categories of singularities [Or1].

Let M be a symplectic compact oriented surface of genus $g \geq 3$. The mirror Landau-Ginzburg (LG for short) model $W : X \to \mathbb{C}$ is three-dimensional. The only singular fibre $H := X_0 \subset X$ is a union of $(g+1)$ surfaces. This LG model will be constructed explicitly in Section 13.

We denote by $\mathcal{F}(M)$ the Fukaya A_∞-category of M, and by $D^\pi(\mathcal{F}(M))$ the category of perfect complexes over $\mathcal{F}(M)$. Further, let $D_{sg}(H)$ be the category of singularities oof the surface H, and denote by $\overline{D_{sg}}(H)$ its Karoubian completion. We will prove the following Theorem.

Theorem 6.1. *The triangulated categories $D^\pi(F(M))$ and $\overline{D_{sg}}(H)$ are equivalent.*

The main ideas in the proof are the same as in [Se1]. We sketch the steps of the proof.

Take $V = \mathbb{C}^3$. We denote by $\xi_k \in V$, $k = 1,2,3$ the standard basis vectors of V, and by $z_k \in V^*$, $k = 1,2,3$ the dual basis. Take the K-invariant polynomial

$$W = -z_1z_2z_3 + z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1} \in \mathbb{C}[V^\vee]^K, \tag{6.1}$$

where $K \cong \mathbb{Z}/(2g+1) \subset SL(V)$ is the cyclic subgroup generated by the diagonal matrix $diag(\zeta,\zeta,\zeta^{2g-1})$, with $\zeta = \exp(\frac{2\pi i}{2g+1})$.

A generator of Fukaya category. The generator of $D^{\pi}(\mathcal{F}(M))$ is constructed as follows. We consider a cyclic covering $\pi : M \to \bar{M}$, where $\bar{M}$ is $\mathbb{P}^1$ with three orbifold points. The Galois group of this covering is $\Sigma = \mathrm{Hom}(K, \mathbb{C}^*) \cong \mathbb{Z}/(2g+1)$. There is a nice Galois-invariant collection of curves $L_1, \dots, L_{2g+1} \subset M$, such that

1) the object $L_1 \oplus \dots \oplus L_{2g+1} \in D^{\pi}(\mathcal{F}(M))$ is a generator;

2) the projection $\pi(L_i)$ of each of these curves is the immersed curve $\bar{L} \subset \bar{M}$.

Here to prove generation we use the criterions of Seidel ([Se1], Lemmas 6.4 and 6.5). The endomorphism A_∞-algebra $\mathrm{End}(\bigoplus_{1\leq i\leq 2g+1} L_i)$ is a smash product $\mathrm{End}(\bar{L})\#\mathbb{C}[K]$. The Floer cohomology super-algebra $HF^{\cdot}(\bar{L}, \bar{L})$ is isomorphic to the exterior super-algebra $\Lambda(V)$. We compute some higher A_∞-operations which uniquely determine the whole A_∞-structure (up to homotopy). This computation is analogous to that of [Se1], Section 10, and is in fact combinatorial, as in the approach of Abouzaid [Ab].

Classification of A_∞-structures. The super-algebra $\Lambda(V)$ has a lot of (homotopy classes of) $\mathbb{Z}/2$-graded A_∞-structures. These A_∞-structures are actually Maurer-Cartan solutions in the differential graded Lie algebra of Hochschild cochains. We use Kontsevich's formality theorem [Ko2] (in the suitable version) to reduce classification of A_∞-structures to some questions on formal polyvector fields on V. It turns out that the A_∞-algebra $\mathrm{End}(\bar{L})$ above (which gives an A_∞-structure on $\Lambda(V) \cong HF^{\cdot}(\bar{L}, \bar{L})$), corresponds to the (gauge equivalence class of) the superpotential W (considered as a polyvector field). This part of the paper generalizes [Se1], Sections 4 and 5. Technical details here are more complicated than in [Se1].

Matrix factorizations. It is well known that the triangulated categories of singularities of a fiber $W^{-1}(0)$ is equivalent to the homotopy category of matrix factorizations of W [Or1]. In our case, the structure sheaf of the origin $\mathcal{O}_0$ is a split-generator in the category of singularities. We take the matrix factorization corresponding to this skyscraper sheaf $\mathcal{O}_0$. The endomorphism DGA of this matrix factorization turns out to be quasi-isomorphic to the A_∞-algebra computed on the Fukaya side. Namely, the cohomology super-algebra of this DGA is isomorphic to the exterior algebra $\Lambda(V)$ and again the resulted A_∞-structure corresponds to the superpotential W in polyvector fields. This part generalizes [Se1], Sections 11, 12.

Here we also prove the following general reconstruction theorem (more precise formulation is Theorem 12.1):

Theorem 6.2. *Let* k *be a field of characteristic zero,* $n \geq 1$, *and* $V = \mathrm{k}^n$. *Let* $W = \sum_{i=3}^{d} W_i \in \mathrm{k}[V^\vee]$ *be a non-zero polynomial, where* $W_i \in \mathrm{Sym}^i(V^\vee)$. *Then* W *can be reconstructed, up to a formal change of variables, from the quasi-isomorphism class of* $D(\mathbb{Z}/2)$*-* G *algebra* $\mathcal{B}_W \cong \mathbf{R}\,\mathrm{Hom}_{D_{sg}(W^{-1}(0))}(\mathcal{O}_0, \mathcal{O}_0)$, *the endomorphism DG* $(\mathbb{Z}/2)$*-graded algebra of* *the structure sheaf* $\mathcal{O}_0$ *in* $D_{sg}(W^{-1}(0))$, *together with identification* $H^{\cdot}(\mathcal{B}_W)\cong\Lambda(V)$. *Moreover, formal change of variables is of the form*

$$z_i\to z_i+O(z^2). \tag{6.2}$$

Equivalence between two LG models. We have two natural LG models both mirror to the curve M. The first one is a stack $V//K$ together with a function W. Another one is a crepant resolution $\psi: X\to\bar{X}=V/K$ given by the K-Hilbert scheme [CR], together with pullback of W. In both cases the only singular fiber is over zero. Denote by $H\subset X$ be the preimage of $\bar{H}=W^{-1}(0)/K\subset\bar{X}$. We can describe the surface H very explicitly (Section 13). By the famous Mckay correspondence for derived categories [BKR], we have an equivalence $D^b_K(V)\cong D^b(X)$. We use an analogous result for categories of singularities [BP, QV]: $D_{sg,K}(W^{-1}(0))\cong D_{sg}(H)$. This is a generalization of [Se1], Section 13.

The sign convention. From this moment We will treat an A_∞-algebra as a $\mathbb{Z}$-(or $(\mathbb{Z}/2)$-)graded vector space equipped with a sequence of maps $\mu^d: A^{\otimes d}\to A$ of degree $2-d$ (resp. of parity d) such that the maps $m_d: A^{\otimes d}\to A$, where

$$m_d(a_d,\dots,a_1)=(-1)^{|a_1|+2|a_2|+\dots+d|a_d|}\mu^d(a_d,\dots,a_1), \tag{6.3}$$

define an A_∞-structure in standard sign convention.

7. A_∞-structures and formal polyvector fields

Let $\mathfrak{g}$ be some DG Lie algebra over $\mathbb{C}$. Recall Maurer-Cartan (MC) equation for $\mathfrak{g}$:

$$\partial\alpha+\frac{1}{2}[\alpha,\alpha]=0,\quad \alpha\in\mathfrak{g}^1. \tag{7.1}$$

An element $\alpha\in\mathfrak{g}^1$ is called Maurer-Cartan (MC) element if it satisfies MC equation. For each $\gamma\in\mathfrak{g}^0$ we have affine vector field on $\mathfrak{g}^1$, $\alpha\mapsto-\partial\gamma+[\gamma,\alpha]$. This defines a morphism o Lie algebras from $\mathfrak{g}^0$ to the Lie algebra of affine vector fields on $\mathfrak{g}^1$. It is easy to check that all vector fields in the image are tangent to the subscheme of solutions of (7.1). Under some natural assumptions on $\mathfrak{g}$ (see below), there is a group G^0 (which is exponent of $\mathfrak{g}^0$) acting on the set of Maurer-Cartan elements.

We will need to deal with L_∞-morphisms between DG Lie algebras. An L_∞-morphisn $\Phi:\mathfrak{g}\to\mathfrak{h}$ is given by a sequence of maps $\Phi^k:\mathfrak{g}^{\otimes k}\to\mathfrak{h}$. These maps must be anti-symmetric (in super sense) and satisfy natural compatibility equations [LM]. In particular, Φ^1 is a morphism of complexes, and $H^{\cdot}(\Phi^1): H^{\cdot}(\mathfrak{g})\to H^{\cdot}(\mathfrak{h})$ is a morphism of graded Lie algebras.

Such Φ is called a quasi-isomorphism if Φ^1 is a quasi-isomorphism. We will need the following statement.

Lemma 7.1. *Let $\mathfrak{g}$ be a graded Lie algebra considered as a DG Lie algebra with zero differential. Let $\mathfrak{h}$ be a DG Lie algebra, and $\Psi : \mathfrak{g} \to \mathfrak{h}$ an L_∞-quasi-isomorphism. Take some morphism of complexes $\Phi^1 : \mathfrak{h} \to \mathfrak{g}$ together with a homogeneous map $H : \mathfrak{h} \to \mathfrak{h}$ of degree -1, such that*

$$\Phi^1\Psi^1 = \mathrm{id}, \quad \Psi^1\Phi^1 - \mathrm{id} = \partial H + H\partial. \tag{7.2}$$

Then Φ^1 can be extended to an L_∞-morphism $\Phi : \mathfrak{h} \to \mathfrak{g}$, so that the higher order terms Φ^k are given by a universal formulae, depending only on Ψ, Φ^1 and H.

Moreover, one can choose Φ in such a way that the composition $\Phi \circ \Psi$ equals to the identity L_∞-morphism.

Proof. For the proof of the first statement, see [Se1], Lemma 3.1. Further, for the constructed Φ, we have that the composition $\Phi \circ \Psi$ is an L_∞-automorphism of $\mathfrak{h}$. Define $\Phi' = (\Phi \circ \Psi)^{-1}\Phi$. Then Φ' satisfies the required property, and the higher order terms Φ'^k are again given by a universal formulae, depending only on Ψ, Φ^1 and H. □

In order to be able to exponentiate the gauge vector fields on $\mathfrak{g}^1$, we will deal with *pro-nilpotent* DG Lie algebras.

Definition 7.2. *A DG Lie algebra $\mathfrak{g}$ is called pro-nilpotent if it is equipped with a complete decreasing filtration $\mathfrak{g} = L_1\mathfrak{g} \supset L_2\mathfrak{g} \supset \dots$, such that*

$$\partial(L_r\mathfrak{g}) \subset L_r\mathfrak{g}, \quad [L_r\mathfrak{g}, L_s\mathfrak{g}] \subset L_{r+s}\mathfrak{g}. \tag{7.3}$$

If $\mathfrak{g}$ is pro-nilpotent, then Lie algebra $\mathfrak{g}^0$ is also such, and hence we get a pro-nilpotent group G^0. As a set, it equals to $\mathfrak{g}^0$, and the product is given by the Baker-Campbell-Hausdorff formula. The group G^0 then acts on MC elements $\alpha \in \mathfrak{g}^1$. Two MC elements are called equivalent if they lie in the same G^0-orbit.

Definition 7.3. *Let $\mathfrak{g}$, $\mathfrak{h}$ be pro-nilpotent DG Lie algebras. An L_∞-morphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ is called filtered if*

$$\Phi^k(L_{r_1}\mathfrak{g} \otimes \dots \otimes L_{r_k}\mathfrak{g}) \subset L_{r_1+\dots+r_k}\mathfrak{h}. \tag{7.4}$$

Definition 7.4. *A filtered L_∞-morphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ of pro-nilpotent DG Lie algebras is called a filtered L_∞-quasi-isomorphism if the induced morphisms of complexes $L_r\mathfrak{g}/L_{r+1}\mathfrak{g} \to L_r\mathfrak{h}/L_{r+1}\mathfrak{h}$ are quasi-isomorphisms.*

Remark 7.5. *In Lemma 7.1 we can require $\mathfrak{g}$, $\mathfrak{h}$ to be pro-nilpotent, Ψ to be filtered L_∞-quasi-isomorphims, and Φ^1, H to be compatible with filtrations. Then the constructed L_∞-morphism Φ is also filtered.*

If $\Phi : \mathfrak{g} \to \mathfrak{h}$ is a filtered L_∞- morphism of pro-nilpotent DG Lie algebras, then we have an induced map on Maurer-Cartan elements

$$\alpha \mapsto \Phi_*(\alpha) := \sum_{k\geq 1} \frac{1}{k!}\Phi^k(\alpha,\dots,\alpha). \tag{7.5}$$

This map preserves equivalence relation (see Appendix). The following statement is a generalization of the corresponding result in [Ko2].

Lemma 7.6. *Let $\Phi : \mathfrak{g} \to \mathfrak{h}$ be a filtered L_∞- quasi-isomorphism of filtered DG Lie algebras. Then the induced map on equivalence classes of MC elements is a bijection.*

This lemma is proved in Appendix by using obstruction theory, similar to [GM] (or [ELO2] for A_∞- algebras).

Now we define some necessary notions to formulate a version of Kontsevich formality theorem [Ko2]. Let V be a finite-dimensional $\mathbb{C}$- vector space. The graded Lie algebra of formal polyvector fields on V is the following:

$$\mathbb{C}[[V^\vee]] \otimes \Lambda(V) = \prod_{i,j} \mathrm{Sym}^i(V^\vee) \otimes \Lambda^j(V). \tag{7.6}$$

We assign to the summand $\mathbb{C}[[V^\vee]] \otimes \Lambda^j(V)$ the grading $j-1$. The Lie bracket is the Schouten one:

$$\begin{aligned}(7.7)\quad & [f\xi_{i_1}\wedge\dots\wedge\xi_{i_k}, g\xi_{j_1}\wedge\dots\wedge\xi_{j_l}] = \\ & \sum_{q=1}^{k}(-1)^{k-q}(f\partial_{i_q}g)\xi_{i_1}\wedge\dots\wedge\widehat{\xi_{i_q}}\wedge\dots\wedge\xi_{i_k}\wedge\xi_{j_1}\wedge\dots\wedge\xi_{j_l}+ \\ & \sum_{p=1}^{l}(-1)^{l-p-1+(k-1)(l-1)}(g\partial_{j_p}f)\xi_{j_1}\wedge\dots\wedge\widehat{\xi_{j_p}}\wedge\dots\wedge\xi_{j_l}\wedge\xi_{i_1}\wedge\dots\wedge\xi_{i_k}.\end{aligned}$$

A formal bivector field $\alpha \in \mathbb{C}[[V^\vee]]\otimes\Lambda^2(V)$ is MC element iff α defines a formal Poisson structure. The elements $\gamma \in \mathbb{C}[[V^\vee]] \otimes V,$ which are formal vector fields, act on Poisson brackets by their Lie derivatives. If the value of γ at the origin vanishes, then it can be exponentiated to a formal diffeomorphism of $V.$ The corresponding action on Poisson brackets is just the pushforward action by formal diffeomorphisms.

Now let A be a graded algebra over $\mathbb{C}$. We would like to make the The Hochshild cochain complex of A into a DG Lie algebra $CC^\cdot(A,A)$. As a graded vector space, it consists of graded multilinear maps:

$$CC^d(A,A)=\prod_{i+j-1=d}\mathrm{Hom}^j(A^{\otimes i},A). \tag{7.8}$$

The differential on the Hochshild complex is given by the formula

$$(\partial\phi)^j(a_j,\dots,a_1)=\sum_k(-1)^{|\phi|+|a_1|+\dots+|a_k|+k}\phi^{j-1}(a_j,\dots,a_{k+1}a_k,\dots,a_1)+ \\ (-1)^{|\phi|+|a_1|+\dots+|a_{j-1}|+j}a_j\phi^{j-1}(a_{j-1},\dots,a_1)+ \\ (-1)^{(|\phi|-1)(|a_1|-1)+1}\phi^{j-1}(a_j,\dots,a_2)a_1. \tag{7.9}$$

There is a naturaL Gerstenhaber bracket on the Hochshild complex which makes it into a DG Lie algebra:

(7.10)

$$[\phi,\psi]^j(a_j,\dots,a_1)=\sum_{k,l}(-1)^{|\psi|(|a_1|+\dots+|a_k|-k)}\phi^{j-l+1}(a_j,\dots,a_{k+l+1},\psi^l(a_{k+l},\dots,a_{k+1}),a_k,\dots,a_1)- \\ \sum_{k,l}(-1)^{|\phi||\psi|+|\phi|(|a_1|+\dots+|a_k|-k)}\psi^{j-l+1}(a_j,\dots,a_{k+l+1},\phi^l(a_{k+l},\dots,a_{k+1}),a_k,\dots,a_1).$$

Our grading on the Hochshild complex is shifted by 1 from the usual one (otherwise the Gerstenhaber bracket would have degree -1).

We would like to illustrate the Maurer-Cartan theory for pro-nilpotent DG Lie algebras by describing A_∞-structures on A up to a strict homotopy. Consider the DG Lie subalgebra $\mathfrak{g}_A\subset CC^{\cdot}(A,A)$ with

$$\mathfrak{g}_A^d=\prod_{\substack{i+j-1=d,\\ i\geq d+2}}\mathrm{Hom}^j(A^{\otimes i},A). \tag{7.11}$$

We have that $\mathfrak{g}_A$ is pro-nilpotent, with filtration

$$L_r\mathfrak{g}_A^d=\prod_{\substack{i+j-1=d,\\ i\geq d+1+r}}\mathrm{Hom}^j(A^{\otimes i},A),\quad r\geq 1. \tag{7.12}$$

It is well known (and is easy to see) that A_∞-structures on the graded algebra A correspond to MC elements $\alpha\in CC^1(A,A)$. Namely, each $\alpha\in CC^1(A,A)$ is given by maps s $\alpha^j:A^{\otimes j}\to A$ of degree $2-j$, for each $j\geq 3$. Put

$$\begin{cases}\mu^j=\alpha^j \text{ for } j\geq 3;\\ \mu^2(a_2,a_1)=\alpha^2(a_2,a_1)+(-1)^{|a_1|}a_2a_1;\\ \mu_1=0.\end{cases} \tag{7.13}$$

Then μ^j define an A_∞- structure if and only if α is Maurer-Cartan element.

Remark 7.7. *As we have already mentioned in Introduction, our sign convention differs from the standard one. To obtain an A_∞- structure in standard sign convention, one should put*

$$m_j(a_j,\dots,a_1)=(-1)^{|a_1|+2|a_2|+\dots+j|a_j|}\mu^j(a_j,\dots,a_1). \tag{7.14}$$

The exponentiated action of $\exp(\mathfrak{g}_A^0)$ on MC elements (A_∞ -structures) is the following. Take some $\gamma\in\mathfrak{g}_A^0$. Take homogeneous maps $\phi^r: A^{\otimes r}\to A$, $\deg(\phi^r)=1-r$, $r\geq 1$, given by the formulas:

(7.15)

$$\begin{cases} \phi^1=\mathrm{id};\\ \phi^2=\gamma^2;\\ \phi^3=\gamma^3+\frac{1}{2}\gamma^2(\gamma^2\otimes\mathrm{id})+\frac{1}{2}\gamma^2(\mathrm{id}\otimes\gamma^2);\\ \phi^4=\gamma^4+\frac{1}{2}\gamma^2(\gamma^3\otimes\mathrm{id})+\frac{1}{2}\gamma^2(\mathrm{id}\otimes\gamma^3)+\frac{1}{2}\gamma^3(\gamma^2\otimes\mathrm{id}\otimes\mathrm{id})+\frac{1}{2}\gamma^3(\mathrm{id}\otimes\gamma^2\otimes\mathrm{id})+\\ \frac{1}{2}\gamma^3(\mathrm{id}\otimes\mathrm{id}\otimes\gamma^2)+\frac{1}{3}\gamma^2(\gamma^2\otimes\gamma^2);\\ \dots \end{cases}$$

In general, ϕ^j is the sum over all ways of concatenating the components of γ to get a j- linear map. The associated term is taken with the coefficient $\frac{s}{r!}$, where r is the number of components of γ, and s is the number of ways of ordering the components, compatibly with their appearance in concatenation. If two MC elements α and $\tilde{\alpha}$ lie in the same orbit of the action of $\mathfrak{g}_A^0$, so that $\tilde{\alpha}=\exp(\gamma)(\alpha)$, then the corresponding A_∞- structures are strictly homotopic, and ϕ is an A_∞- isomorphism.

Now let again V be a finite-dimensional vector space, and take $A=\Lambda(V)$. By Hochshild-Kostant-Rosenberg Theorem (see [HKR]), we have $HH^{\cdot}(A,A)\cong\mathbb{C}[[V^\vee]]\otimes\Lambda(V)$. This isomorphism is induced by Hochshild-Kostant-Rosenberg map

$$\Phi^1: CC^{\cdot}(A,A)\to\mathbb{C}[[V^\vee]]\otimes\Lambda(V), \tag{7.16}$$

given by the formula

$$\Phi^1(\beta)(\xi)=\sum_{j\geq 1}\beta^j(\xi,\dots\xi). \tag{7.17}$$

Here we consider polyvector fields as formal power series with values in $\Lambda(V)$.

Theorem 7.8. *(*[Ko3]*) The map Φ^1 is the first term of some L_∞- morphism Φ, which can be taken to be $GL(V)$- equivariant.*

Theorem 7.8 is implied by Kontsevich formality Theorem [Ko3] using Lemma 7.1 and reductiveness of $GL(V)$, see [Se1] and Remark 7.9.

Remark 7.9. *In contrast to our situation, Kontsevich deals with the algebra of smooth functions on smooth manifolds. He proves that for each smooth manifold X the graded Lie algebra of polyvector fields $T_{poly}(X)$ is quasi-isomorphic to the DG Lie algebra of polydifferential operators $D_{poly}(X)$. In the case when X is an open domain U in affine space $\mathbb{R}^d$, he constructs an explicit L_∞- quasi-isomorphism. One can replace the smooth functions by polynomials (or formal power series) over $\mathbb{C}$, and his construction works as well. Then one exchanges even an odd variables, and obtains an L_∞- quasi-isomorphism*

$$\Psi : \mathbb{C}[[V^\vee]] \otimes \Lambda(V) \to CC^{\cdot}(A, A). \tag{7.18}$$

This Ψ is $GL(V)$- equivariant, and using Lemma 7.1 and reductiveness of $GL(V)$, one obtains the required $\Phi : CC^{\cdot}(A, A) \to \mathbb{C}[[V^\vee]] \otimes \Lambda(V)$ which can be taken to be left inverse to Ψ.

8. Classification lemma for polyvector fields

Put $V = \mathbb{C}^3$. Take the subgroup $G \subset SL(V)$ which consists of diagonal matrices with $(2g+1)$- th roots of unity on the diagonal. Clearly, $G \cong (\mathbb{Z}/(2g+1))^2$. Define the pro-nilpotent graded Lie algebra $\mathfrak{g}$ as follows:

$$\mathfrak{g}^d = \prod_{\substack{2i+j-(4g-4)k=3d+3\\ k\geq 0,\, i\geq d+2}} (\mathrm{Sym}^i V^\vee \otimes \Lambda^j V)^G \hbar^k. \tag{8.1}$$

The Lie bracket comes from Schouten bracket on polyvector fields, and $L_r\mathfrak{g}^d$ is the part of the product which consists of terms with $i \geq d+1+r$.

We can omit $\hbar^k$ but remember that

$$2i+j-3d-3 \geq 0, \quad \text{and} \quad 2i+j-3d-3 \equiv 0 \bmod 4g-4. \tag{8.2}$$

We would like to describe explicitly elements of $\mathfrak{g}^1$ and $\mathfrak{g}^0$, and Maurer-Cartan equation. Any element $\alpha \in \mathfrak{g}^1$ can be written as (α^0, α^2), where $\alpha^0 \in \mathbb{C}[[V^\vee]]$, and $\alpha^2 \in \mathbb{C}[[V^\vee]] \otimes \Lambda^2 V$. Both α^0 and α^2 must be G-invariant, and the degrees of non-zero homogeneous components of α^0 and α^2 must fullfill the conditions (8.2). In particular, $\alpha^0 \in F_3\mathbb{C}[[V^\vee]]$, and $\alpha^2 \in F_{2g}\mathbb{C}[[V^\vee]] \otimes \Lambda^2 V$. Here $F_\bullet\mathbb{C}[[V^\vee]]$ is the complete decreasing filtration, s.t.

$$F_r\mathbb{C}[[V^\vee]] = \prod_{i\geq r} \mathrm{Sym}^i(V^\vee). \tag{8.3}$$

Similarly, any element $\gamma \in \mathfrak{g}^0$ can be written as (γ^1, γ^3), where $\gamma^1 \in F_{2g-1}\mathbb{C}[[V^\vee]] \otimes V$, and $\gamma^3 \in F_{2g-2}\mathbb{C}[[V^\vee]] \otimes \Lambda^3 V$. Again, both γ^1 and γ^3 must be G-invariant, and non-zero homogeneous components of γ^1 and γ^3 must satisfy (8.2).

Maurer-Cartan equation for $\alpha = (\alpha^0, \alpha^2)$ splits into the components:

$$\frac{1}{2}[\alpha^2, \alpha^2] = 0, \quad [\alpha^0, \alpha^2] = 0. \tag{8.4}$$

This means that

1) The bivector field α^2 is Poisson (the first equation);

2) The Poisson vector field associated to the function α^0 is identically zero. It will be convenient to reformulate this. Consider the complex $\mathbb{C}[[V^\vee]] \otimes \Lambda^\cdot(V)$ with differential being contraction with $d\alpha^0$ (Koszul complex). Then the second equation means that α^2 is a cocycle in this complex.

The exponentiated adjoint action of $\gamma = (\gamma^1, 0) \in \mathfrak{g}^0$ on the solutions of MC equation is the usual action by formal diffeomorphisms. For $\gamma = (0, \gamma^3)$, this action is given by the formula

$$(\alpha^0, \alpha^2) \mapsto (\alpha^0, \alpha^2 + \iota_{d\alpha^0}\gamma^3). \tag{8.5}$$

Take the polynomial

$$W = -z_1 z_2 z_3 + z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1} \in \mathbb{C}[V^\vee]^G, \tag{8.6}$$

which we have already mentioned in Introduction as a superpotential. Then $(W, 0) \in \mathfrak{g}^1$ is a solution of MC equation (as any other $\alpha \in \mathfrak{g}^1$ of type $(\alpha^0, 0)$). Our main technical result in this section is the following.

Lemma 8.1. *Let $\alpha = (\alpha^0, \alpha^2) \in \mathfrak{g}^1$ be an MC element. Suppose that*

$$\alpha^0 \equiv \begin{cases} W \ \mathit{mod}\ F_{2g+2}\mathbb{C}[[V^\vee]] & \mathit{if}\ g \not\equiv 1\ \mathit{mod}\ 3 \\ W + \lambda(z_1 z_2 z_3)^{\frac{2g+1}{3}},\ \mathit{where}\ \lambda \in \mathbb{C} & \mathit{if}\ g \equiv 1\ \mathit{mod}\ 3. \end{cases} \tag{8.7}$$

Then α is equivalent to $(W, 0)$.

Proof. First we note that in the case ($g \equiv 1$ mod 3) one may assume that $\lambda = 0$. Indeed, in this case we have

$$\exp(\lambda z_1^{\frac{2g+1}{3}} z_2^{\frac{2g-2}{3}} z_3^{\frac{2g-2}{3}} \otimes \xi_1)^* \alpha^0 \equiv \\ \alpha^0 + \lambda z_1^{\frac{2g+1}{3}} z_2^{\frac{2g-2}{3}} z_3^{\frac{2g-2}{3}} \frac{\partial \alpha^0}{\partial z_1} \equiv W \text{ mod } F_{2g+2}\mathbb{C}[[V^\vee]]. \tag{8.8}$$

Thus, we may and will assume that $\alpha^0 \equiv W \text{ mod } F_{2g+2}\mathbb{C}[[V^\vee]]$.

Let $I \subset \mathbb{C}[V^\vee]$ be an ideal generated by $\frac{\partial W}{\partial z_i}$, $i = 1, 2, 3$. It is easy to see that

$$z_i z_j \in I + F_{2g}\mathbb{C}[[V^\vee]] \text{ for } i < j, \quad z_i^{2g+2} \in I \cdot F_2\mathbb{C}[[V^\vee]] + F_{4g}\mathbb{C}[[V^\vee]]. \tag{8.9}$$

Indeed, for example $z_1z_2 \equiv -\frac{\partial W}{\partial z_3} \mod F_{2g}\mathbb{C}[[V^\vee]]$, and

$$z_1^{2g+2} \equiv \frac{1}{2g+1}z_1^2\frac{\partial W}{\partial z_1} - \frac{1}{2g+1}z_1z_2\frac{\partial W}{\partial z_2} - z_2^{2g}\frac{\partial W}{\partial z_3} \mod F_{4g}\mathbb{C}[[V^\vee]]. \tag{8.10}$$

Put $W_{4g-1} = \alpha^0$. It follows from (8.2) that α^0 contains only monomials of degree $3+(2g-2)k$, where $k \geq 0$. The difference $W - W_{4g-1}$ does not contain monomials z_i^{4g-1}, since they are not G-invariant. It follows from (8.9) that $W - W_{4g-1} \in I \cdot F_{4g-3}\mathbb{C}[[V^\vee]] + F_{6g-3}\mathbb{C}[[V^\vee]]$. Therefore, there exist homogeneous polynomials $f_{4g-3,1}, f_{4g-3,2}, f_{4g-3,3}$ of degree $(4g-3)$, such that

$$\begin{aligned} W_{6g-3} &= \exp(f_{4g-3,1}\otimes\xi_1 + f_{4g-3,2}\otimes\xi_2 + f_{4g-3,3}\otimes\xi_3)^* W_{4g-3} \\ &\equiv W_{2g+1} + f_{4g-3,1}\frac{\partial W}{\partial z_1} + f_{4g-3,2}\frac{\partial W}{\partial z_2} + f_{4g-3,3}\frac{\partial W}{\partial z_3} \mod F_{6g-3}\mathbb{C}[[V^\vee]] \\ &\equiv W \mod F_{6g-3}\mathbb{C}[[V^\vee]]. \end{aligned} \tag{8.11}$$

Moreover, we can take $f_{4g-3,i}$ such that $(f_{4g-3,1}\otimes\xi_1 + f_{4g-3,2}\otimes\xi_2 + f_{4g-3,3}\otimes\xi_3, 0) \in \mathfrak{g}^0$. We obtain a new formal function $W_{6g-3} \equiv W \mod F_{6g-3}\mathbb{C}[[V^\vee]]$.

Now suppose that we are given with some formal function $W_{3+(2g-2)k}$, where $k \geq 3$, such that $(W_{3+(2g-2)k}, 0) \in \mathfrak{g}^1$ and $W_{3+(2g-2)k} \equiv W \mod F_{3+(2g-2)k}\mathbb{C}[[V^\vee]]$. It follows from (8.9) that $W - W_{3+(2g-2)k} \in I \cdot F_{1+(2g-2)(k-1)}\mathbb{C}[[V^\vee]] + F_{3+(2g-2)(k+1)}$. Thus, there exist homogeneous polynomials $f_{1+(2g-2)(k-1),1}, f_{1+(2g-2)(k-1),2}, f_{1+(2g-2)(k-1),3}$ of degree $1+(2g-2)(k-1)$ such that

$$\begin{aligned} &\exp(f_{1+(2g-2)(k-1),1}\otimes\xi_1 + f_{1+(2g-2)(k-1),2}\otimes\xi_2 + f_{1+(2g-2)(k-1),3}\otimes\xi_3)^* W_{3+(2g-2)k} \equiv \\ &\qquad W \mod F_{3+(2g-2)(k+1)}. \end{aligned} \tag{8.12}$$

Again, the exponentiated formal vector field can be taken to belong to $\mathfrak{g}^0$. We obtain a new formal function $W_{3+(2g-2)(k+1)}$, such that $(W_{3+(2g-2)(k+1)}, 0) \in \mathfrak{g}^1$ and $W_{3+(2g-2)(k+1)} \equiv W \mod F_{3+(2g-2)(k+1)}\mathbb{C}[[V^\vee]]$.

Iterating, we obtain infinite sequence of formal diffeomorphisms, and their product obviously converges. As a result, our MC solution α is equivalent to (W, α'^2) for some $\alpha'^2 \in F_{2g}\mathbb{C}[[V^\vee]] \otimes \Lambda^2 V$. Since the quotient $\mathbb{C}[[V^\vee]]/I$ is finite-dimensional, it follows that the sequence $(\frac{\partial W}{\partial z_1}, \frac{\partial W}{\partial z_2}, \frac{\partial W}{\partial z_3})$ is regular in $\mathbb{C}[[V^\vee]]$, and hence the Koszul complex $\mathbb{C}[[V^\vee]] \otimes \Lambda(V)$ with differential ι_{dW} is a resolution of $\mathbb{C}[[V^\vee]]/I$. Since α'^2 is a cocycle in the Koszul complex, it is also a coboundary. Hence there exists $\gamma^3 \in \mathbb{C}[[V^\vee]] \otimes \Lambda^3 V$ such that $\iota_{dW}(\gamma^3) = -\alpha'^2$. Again, γ^3 can be choosen to belong to $\mathfrak{g}^0$. By the explicit formula (8.5), the exponential of $(0, \gamma^3)$ maps (W, α'^2) to $(W, 0)$, and we are done. □

9. Classification theorem on A_∞-structures

Take the algebra $A=\Lambda(V)$ with standard grading ($\deg(V)=1$). Consider the following DG Lie algebra $\mathfrak{h}$:

$$\mathfrak{h}^d \quad = \prod_{\substack{3i+j-(4g-4)k=3d+3\\ k\geq 0,\, i\geq d+2}} \mathrm{Hom}^j(A^{\otimes i},A)^G\,\hbar^k. \tag{9.1}$$

The differential is Hochshild differential and the bracket is Gerstenhaber bracket. Again, $\mathfrak{h}$ is pro-nilpotent with respect to the filtration $L_\bullet\mathfrak{h}$, where $L_r\mathfrak{h}^d$ is the part of the product which consists of terms with $i\geq d+1+r$.

Theorem 7.8 implies the following lemma (see [Se1] for detailed explanation).

Lemma 9.1. *There exists a filtered L_∞- quasi-isomorphism $\Phi:\mathfrak{h}\to\mathfrak{g}$, with Φ^1 being the obvious $\hbar$- linear extension of Hochshild-Kostant-Rosenberg map.*

Note that, analogously to the discussion in Section 7, each MC element $\alpha\in\mathfrak{h}^1$ defines a $\mathbb{Z}/2$-graded A_∞-structure on A. Moreover, equivalent MC elements yield strictly homotopic A_∞-structures. Below, on the A side and on the B side, we will encounter two different A_∞-structures on A, which come from the same equivalence class in $MC(\mathfrak{h})$.

We are going to describe this equivalence class below.

Consider arbitrary $\alpha\in\mathfrak{h}^1$. Its components are G-equivariant i- linear maps $\alpha^i:A^{\otimes i}\to A$, for $i\geq 3$. Further, each α^i has (finite) decomposition $\alpha^i=\alpha^i_0+\alpha^i_1\hbar+\alpha^i_2\hbar^2+\dots$, where

$$\alpha^i_k\in\mathrm{Hom}^{6-3i+(4g-4)k}(A^{\otimes i},A)^G. \tag{9.2}$$

Note that if $\alpha^i_k\neq 0$, then $(6-3i+(4g-4)k)\leq 3$. It follows that $\alpha^i_1=0$ for $3\leq i<\frac{4g-1}{3}$. We will also need the following elementary observations:

$$L_{2g}\mathfrak{g}^1=(\hbar^2\mathfrak{g})^1; \tag{9.3}$$

$$\Phi^1(\mathrm{Hom}^{2-2g}(A^{\otimes 2g},A)^G)=(\mathrm{Sym}^{2g}(V^\vee)\otimes\Lambda^2(V))^G= \tag{9.4}$$

$$\begin{cases}\mathbb{C}\cdot z_1^{2g}\otimes(\xi_2\wedge\xi_3)+\mathbb{C}\cdot z_2^{2g}\otimes(\xi_3\wedge\xi_1)+\mathbb{C}\cdot z_3^{2g}\otimes(\xi_1\wedge\xi_2) & \text{if } g\not\equiv 1 \bmod 3\\ \mathbb{C}\cdot z_1^{2g}\otimes(\xi_2\wedge\xi_3)+\mathbb{C}\cdot z_2^{2g}\otimes(\xi_3\wedge\xi_1)+\mathbb{C}\cdot z_3^{2g}\otimes(\xi_1\wedge\xi_2)+ & \\ \mathbb{C}\cdot z_1^{\frac{2g+1}{3}}z_2^{\frac{2g+1}{3}}z_3^{\frac{2g-2}{3}}\otimes(\xi_1\wedge\xi_2)+\mathbb{C}\cdot z_1^{\frac{2g+1}{3}}z_2^{\frac{2g-2}{3}}z_3^{\frac{2g+1}{3}}\otimes(\xi_3\wedge\xi_1)+ & \\ \mathbb{C}\cdot z_1^{\frac{2g-2}{3}}z_2^{\frac{2g+1}{3}}z_3^{\frac{2g+1}{3}}\otimes(\xi_2\wedge\xi_3) & \text{if } g\equiv 1 \bmod 3;\end{cases}$$

$$\Phi^1(\mathrm{Hom}^{-2g-1}(A^{\otimes(2g+1)},A)^G)=(\mathrm{Sym}^{2g+1}(V^\vee)^G= \tag{9.5}$$
$$\begin{cases}\mathbb{C}\cdot z_1^{2g+1}+\mathbb{C}\cdot z_2^{2g+1}+\mathbb{C}\cdot z_3^{2g+1} & \text{if } g\not\equiv 1 \bmod 3\\ \mathbb{C}\cdot z_1^{2g+1}+\mathbb{C}\cdot z_2^{2g+1}+\mathbb{C}\cdot z_3^{2g+1}+\mathbb{C}\cdot(z_1z_2z_3)^{\frac{2g+1}{3}} & \text{if } g\equiv 1 \bmod 3.\end{cases}$$

Theorem 9.2. *Let* $\alpha\in\mathfrak{h}^1$ *be an MC element such that* $\Phi^1(\alpha_0^3)=-z_1z_2z_3$ *and*
(9.6)
$$\Phi^1(\alpha_1^{2g+1})=\begin{cases}z_1^{2g+1}+z_2^{2g+1}+z_3^{2g+1} & \text{if } g\not\equiv 1 \bmod 3\\ z_1^{2g+1}+z_2^{2g+1}+z_3^{2g+1}+\lambda(z_1z_2z_3)^{\frac{2g+1}{3}},\ \text{where } \lambda\in\mathbb{C} & \text{if } g\equiv 1 \bmod 3.\end{cases}$$
Then we have that MC element $\Phi_*(\alpha)\in MC(\mathfrak{g})$ *is equivalent to* $(W,0)\in MC(\mathfrak{g})$, *in the notation of the previous section. In particular, all such* α *are equivalent to each other.*

Proof. First we will replace α with another α' satisfying the assumptions of the theorem, and such that $\alpha_1'^i=0$ for $3\le i<2g$. We will need the following

Lemma 9.3. *1) Take some* $\gamma_1^i\in\mathfrak{h}^0$ *lying in the component* $\mathrm{Hom}^{3-3i+(4g-4)}(A^{\otimes i},A)$. *Then for each MC element* $\alpha\in\mathfrak{h}^1$ *we have*
$$\alpha'=\exp(\gamma_1^i)\cdot\alpha\equiv\alpha-\partial\gamma+[\gamma,\alpha]\ mod\ (\hbar^2\mathfrak{h})^1. \tag{9.7}$$

2) If. moreover, $i\le 2g-2$, *then we have that* α' *satisfies the assumptions of the theorem.*

Proof. 1) This is evident.

2) According to 1) and (9.3), we only need to check that the polynomial $\Phi^1([\gamma_1^i,\alpha_0^{2g+2-i}])$ does not contain monomials z_i^{2g+1}. But for degree reasons, for $2\le i\le 2g-2$ we have that α_0^{2g+2-i} vanishes when restricted to $V^{\otimes(2g+2-i)}$. Further, for $2\le i\le 2g-3$, we have that γ_1^i vanishes when restricted to $V^{\otimes i}$. Therefore, in the case $2\le$ $[\gamma_1^i,\alpha_0^{2g+2-i}]$ vanishes on $V^{\otimes(2g+1)}$, hence the assertion.

Further, in the case $i=2g-2$, it suffices to notice that $\gamma_1^{2g-2}(\xi_i^{\otimes 2g-2})=0$ from the G-equivariance condition. □

Take the smallest i_0 such that $\alpha_1^{i_0}\ne 0$. Suppose that $i_0<2g$. Since α is MC solution, we have that $\partial\alpha_1^{i_0}=0$. Denote by $\bar{A}=\sum\limits_{k\ge 1}\Lambda^k(V)$ the augmentation ideal of A. Simple degree counting shows that $\mathrm{Hom}^{6-3i_0+4g-4}(\bar{A}^{\otimes i_0},A)=0$. Since the reduced Hochshild complex embeds quasi-isomorphically to the standard one, we have that there exists $\gamma_1^{i_0-1}\in\mathfrak{h}^0$ such that $\partial\gamma_1^{i_0-1}=\alpha_1^{i_0}$. Then, it follows from Lemma 9.3 that $\alpha'=\exp(\gamma_1^{i_0-1})\alpha$ satisfies the assumptions of the theorem. Moreover, $\alpha_1'^i=0$ for $3\le i\le i_0$.

Iterating, we obtain some equivalent MC solution $\alpha' \in \mathfrak{h}^1$ satisfying the assumptions of the theorem and such that $\alpha_1'^i = 0$ for $3 \leq i < 2g$. Assume from this moment that α itself satisfies this property.

Since α is MC solution, we have

$$\partial\alpha_0^3 = 0, \quad \partial\alpha_1^{2g} = 0, \quad \partial\alpha_1^{2g+1} + [\alpha_0^3, \alpha_1^{2g}] = 0. \tag{9.8}$$

Therefore, α_1^{2g} satisfies the identity

$$[z_1z_2z_3, \Phi^1(\alpha_1^{2g})] = -[\Phi^1(\alpha_0^3), \Phi^1(\alpha_1^{2g})] = -\Phi^1([\alpha_0^3, \alpha_1^{2g}]) = \Phi^1(\partial\alpha_1^{2g+1}) = 0. \tag{9.9}$$

From (9.9) and from (9.4) we conclude that

(9.10)

$$\Phi^1(\alpha_1^{2g}) = \begin{cases} 0 & \text{if } g \not\equiv 1 \bmod 3 \\ \lambda'(z_1^{\frac{2g+1}{3}} z_2^{\frac{2g+1}{3}} z_3^{\frac{2g-2}{3}} \otimes (\xi_1 \wedge \xi_2) + z_1^{\frac{2g+1}{3}} z_2^{\frac{2g-2}{3}} z_3^{\frac{2g+1}{3}} \otimes (\xi_3 \wedge \xi_1) + & \\ z_1^{\frac{2g-2}{3}} z_2^{\frac{2g+1}{3}} z_3^{\frac{2g+1}{3}} \otimes (\xi_2 \wedge \xi_3)),\ \lambda' \in \mathbb{C} & \text{if } g \equiv 1 \bmod 3. \end{cases}$$

Simple degree counting shows that

(9.11)

$$\tilde{\alpha} := \sum_{n \geq 1} \frac{1}{n!}\Phi^n(\alpha, \dots, \alpha) \equiv \Phi^1(\alpha_0^3) + \hbar\Phi^1(\alpha_1^{2g+1}) + \hbar\Phi^2(\alpha_0^3, \alpha_1^{2g}) \bmod L_{2g}\mathfrak{g}^1 = (\hbar^2\mathfrak{g})^1.$$

Lemma 9.4. *The polynomial* $\Phi^2(\alpha_0^3, \alpha_1^{2g}) \in \mathrm{Sym}^{2g+1}(V^\vee)$ *does not contain terms* z_i^{2g+1}.

Proof. If $\alpha_1'^{2g} \in \mathrm{Hom}^{2-2g}(A^{\otimes 2g}, A)$ is a Hochshild cocycle homologous to α_1^{2g} and $\gamma_0^2 \in \mathrm{Hom}^{-3}(A^{\otimes 2}, A)$, then

(9.12)

$$\Phi^2(\partial\gamma_0^2, \alpha_1'^{2g}) = \pm\Phi^2(\gamma_0^2, \partial\alpha_1'^{2g}) \pm \Phi^1([\gamma_0^2, \alpha_1'^{2g}]) \pm \partial\Phi^2(\gamma_0^2, \alpha_1'^{2g}) \pm [\Phi^1(\gamma_0^2), \Phi^1(\alpha_1'^{2g})] = \pm\Phi^1([\gamma_0^2, \alpha_1'^{2g}]).$$

It follows from (9.10) that the RHS of the above chain of identities does not contain monomials z_i^{2g+1}. Analogously, if $\alpha_0'^3 \in \mathrm{Hom}^{-3}(A^{\otimes 3}, A)$ is a Hochshild cocycle homologous to α_0^3 and $\gamma_1^{2g-1} \in \mathrm{Hom}^{2-2g}(A^{\otimes(2g-1)}, A)$, then we have that $\Phi^2(\alpha_0'^3, \partial\gamma_1^{2g-1})$ does not contain terms z_i^{2g+1}. Therefore, we may assume that

$$\alpha_0^3 = \Psi^1\Phi^1(\alpha_0^3), \quad \alpha_1^{2g} = \Psi^1\Phi^1(\alpha_1^{2g}), \tag{9.13}$$

where $\Psi : \mathfrak{g} \to \mathfrak{h}$ is (the obvious $\hbar$-linear extension of) Kontsevich's L_∞-quasi-isomorphism. Further, L_∞-morphism Φ can be taken to be strictly left inverse to Ψ, that is $\Phi\Psi = \mathrm{Id}$ (Remark 7.9). Under this assumptions, the coefficients of $\Phi^2(\alpha_0^3, \alpha^{2g_1})$ in the monomials z_i^{2g+1} equal to

$$\pm\Psi^2(\Phi^1(\alpha_0^3), \Phi^1(\alpha_1^{2g}))(\xi_i^{\otimes(2g+1)}), \quad i = 1, 2, 3. \tag{9.14}$$

From the precise formulas for $\Phi^1(\alpha_0^3)\,(=-z_1z_2z_3\,)$ and $\Phi^1(\alpha_1^{2g})$ (formula (9.10)), as well as for the component Ψ^2 ([Ko2], subsection 6.4, with suitable changes) one obtains that (9.14) equals to zero, as follows. In the notation of [Ko2], subsection 6.4, for each relevant admissible graph Γ we have $\mathcal{U}_\Gamma(\Phi^1(\alpha_0^3),\Phi^1(\alpha_1^{2g}))(\xi_i^{\otimes(2g+1)})=0$. Since Ψ^2 is a linear combination of $\mathcal{U}_\Gamma$, we obtain that (9.14) equals to zero. □

Further, $L_{2g}\mathfrak{g}^1=(\hbar^2\mathfrak{g})^1$ consists of pairs $(\tilde{\alpha}^0,\tilde{\alpha}^2)$ such that $\tilde{\alpha}^0\in F_{4g-1}\mathbb{C}[[V^\vee]]$, and $\tilde{\alpha}^2\in F_{4g-2}\mathbb{C}[[V^\vee]]\otimes\Lambda^2V$. From (9.11) and Lemma 9.4 it follows that $\tilde{\alpha}$ satisfies the assumptions of Lemma 8.1. Therefore, $\tilde{\alpha}$ is equivalent to $(W,0)$. By Lemma 7.6, Φ induces a bijection on the equivalence classes of Maurer-Cartan solutions. It follows that α with required properties is unique up to equivalence. □

We are interested in the following reformulation of the above Theorem. Suppose that we are given with a $(\mathbb{Z}/2)$-graded A_∞-structure $(\mu^1,\mu^2,\dots)$ on $A=\Lambda(V)$. Moreover, assume that all μ^i are G-equivariant, $\mu^1=0$, μ^2 is the usual wedge product (twisted by sign), and for $i\geq 3$ we have (finite) decomposition $\mu^i=\mu_0^i+\mu_1^i+\dots$, where μ_k^i is homogeneous of degree $6-3i+(4g-4)k$ with respect to $\mathbb{Z}$-gradings. Suppose that for $z\in V\subset A$ we have

$$\mu_0^3(z,z,z)=-z_1z_2z_3, \tag{9.15}$$

and

(9.16)
$$\mu_1^{2g+1}(z,\dots,z)=\begin{cases} z_1^{2g+1}+z_2^{2g+1}+z_3^{2g+1} & \text{if } g\not\equiv 1 \bmod 3\\ z_1^{2g+1}+z_2^{2g+1}+z_3^{2g+1}+\lambda(z_1z_2z_3)^{\frac{2g+1}{3}},\ \lambda\in\mathbb{C} & \text{if } g\equiv 1 \bmod 3.\end{cases}$$

Then by Theorem 9.2 such a structure is determined uniquely up to G-equivariant A_∞-quasi-isomorphisms. We denote this class of G-equivariant A_∞-structures by $\mathcal{A}'$.

10. Categories of singularities and matrix factorizations

Let $V=\mathbb{C}^n$ and take some non-zero polynomial $W\in\mathbb{C}[V^\vee]$ such that the hypersurface $W^{-1}(0)$ has (not necessarily isolated) singularity at the origin. Following Orlov [Or1], associate to it the triangulated category of singularities:

$$D_{sg}(W^{-1}(0))=D^b_{coh}(W^{-1}(0))/Perf(W^{-1}(0)). \tag{10.1}$$

Denote by $\overline{D_{sg}}(W^{-1}(0))$ the idempotent completion of $D_{sg}(W^{-1}(0))$. The following Lemma easily follows from the results in [Or2] (see [Se1], proof of Lemma 12.1):

Lemma 10.1. *If W has the only singular point at the origin, then the triangulated category $\overline{D_{sg}}(W^{-1}(0))$ is split-generated by the image of the structure sheaf $\mathcal{O}_0$.*

It turns out that the triangulated category $D_{sg}(W^{-1}(0))$ is $(\mathbb{Z}/2)$- graded, i.e. the shift by 2 in $D_{sg}(W^{-1}(0))$ is canonically isomorphic to the identity (this follows from Theorem 10.2 below).

Now we define the D $(\mathbb{Z}/2)$- G category $MF(W)$ of matrix factorizations of W. Matrix factorizations give a $(\mathbb{Z}/2)$- graded enhancement of this category. A matrix factorization for W is a pair of projective (hence free) finitely generated $\mathbb{C}[V^\vee]$- modules (E^0, E^1), together with a pair of morphisms $\delta_E^1 : E^1 \to E^0$, $\delta_E^0 : E^0 \to E^1$, such that

$$\delta_E^1 \delta_E^0 = W \cdot \mathrm{id}_{E^0}, \quad \delta_E^0 \delta_E^1 = W \cdot \mathrm{id}_{E^1}\,. \tag{10.2}$$

In particular, E^0 and E^1 have the same rank. Denote by $E = E^0 \oplus E^1$ the $\mathbb{Z}/2$-graded $\mathbb{C}[V^\vee]$-module, and $\delta_E = \delta_E^0 \oplus \delta_E^1 : E \to E$ the corresponding odd map. We call the map δ_E "differential" , although its square does not equal to zero.

If (E, δ_E) and (F, δ_F) are matrix factorizations, then we have 2-periodic complex of morphisms $\mathrm{Hom}(E, F)$. Namely, As a $\mathbb{Z}/2$-graded vector space, it consists of all even and odd maps of $\mathbb{Z}/2$-graded modules. The differential is a super-commutator with δ. It is easy to see that $MF(W)$ is a strongly pre-triangulated D $(\mathbb{Z}/2)$- G category.

Theorem 10.2. *(*[Or1]*, Theorem 3.9)There is a natural exact equivalence of triangulated categories* $\mathrm{Ho}(MF(W)) \sim D_{sg}(W^{-1}(0))$.

This equivalence associates to a matrix factorization (E, δ_E) a projection of $\mathrm{Coker}(\delta^1 : E^1 \to E^0)$ (clearly, W annihilates this $\mathbb{C}[V^\vee]$- module, hence it can be considered as an object of $D^b_{coh}(W^{-1}(0))$).

We would like to write down explicitly the matrix factorization which corresponds to the structure sheaf of origin under the equivalence of Theorem 10.2. Decompose the polynomial W into the sum of its graded components:

$$W = \sum_{i=2}^{k} W_i, \quad W_i \in \mathrm{Sym}^i(V^\vee). \tag{10.3}$$

Take the one-form

$$\gamma = \sum_{i=2}^{k} \frac{dW_i}{i}. \tag{10.4}$$

Denote by $\eta = \sum z_k \xi_k$ the Euler vector field on V.

Now take the matrix factorization (E, δ_E) with $E = \Omega(V) = \mathbb{C}[V^\vee] \otimes \Lambda(V^\vee)$, and $\delta_E = \iota_\eta + \gamma \wedge \cdot$. It is easy to see that $\delta_E^2 = \gamma(\eta) \cdot \mathrm{id} = W \cdot \mathrm{id}$.

Lemma 10.3. *(*[Se1]*, Lemma 12.3) The object* $\mathrm{Coker}(\delta_E^1)$ *is isomorphic to* $\mathcal{O}_0$ *in* $D_{sg}(W^{-1}(0))$.

Remark 10.4. *In a similar way, one can write down matrix factorization, corresponding to $\mathcal{O}_Z$, where $Z \subset W^{-1}(0)$ is any closed subscheme, which is complete intersection in V.*

Take the D $(\mathbb{Z}/2)$- G algebra

$$\mathcal{B}_W := \mathrm{End}_{MF(W)}(E). \tag{10.5}$$

By Lemma 10.3, it is quasi-isomorphic to the D $(\mathbb{Z}/2)$- G algebra $\mathbf{R}\,\mathrm{Hom}_{D_{sg}(W^{-1}(0))}(\mathcal{O}_0, \mathcal{O}_0)$. We have the following

Corollary 10.5. *Suppose that W has the only singular point at the origin. Then there is an equivalence $\overline{D_{sg}}(W^{-1}(0)) \cong \mathrm{Perf}(\mathcal{B}_W)$.*

11. Minimal A_∞-model for $\mathcal{B}_W$

In this section we describe more explicitly the DG algebra $\mathcal{B}_W$ introduced in 10.5. We also prove that in the special case of our LG model, it is (equivariantly) quasi-isomorphic to the A_∞- algebra $\mathcal{A}'$ from the end of Section 9 (Proposition 11.1)

Let $V = \mathbb{C}^n$. Consider $\Omega(V) = \mathbb{C}[V^\vee] \otimes \Lambda(V^\vee)$ as a complex of $\mathbb{C}[V^\vee]$- modules with $\deg(\mathbb{C}[V^\vee] \otimes \Lambda^k V^\vee) = -k$ and differential ι_η, where $\eta = \sum\limits_{k=1}^{n} z_k \xi_k$ is the Euler vector field. This complex is just a Koszul resolution of the structure sheaf of the origin $\mathcal{O}_0$.

Consider the DG algebra $B = \mathrm{End}_{\mathbb{C}[V^\vee]}(\Omega(V))$. We have $H^{\cdot}(B) \cong \mathrm{Ext}^{\cdot}_{\mathbb{C}[V^\vee]}(\mathcal{O}_0, \mathcal{O}_0) \cong \Lambda(V)$. Further, we can identify

$$B \cong \Omega(V) \otimes \Lambda(V), \tag{11.1}$$

where for $f \in \mathrm{Sym}(V^\vee)$, $\beta \in \Lambda(V^\vee)$, $\theta \in \Lambda(V)$ the element $f\beta \otimes \theta \in \Omega(V) \otimes \Lambda(V)$ corresponds to the endomorphism $f\beta \wedge \iota_\theta(\cdot) \in B = \mathrm{End}_{\mathbb{C}[V^\vee]}(\Omega(V))$.

Explicitly, the differential $\partial : \Omega(V) \otimes \Lambda(V) \to \Omega(V) \otimes \Lambda(V)$ is given by the formula

$$\partial(f\beta \otimes \theta) = \iota_\eta(f\beta) \otimes \theta. \tag{11.2}$$

It is well known that DG algebra B is formal. Moreover, we can write down explicitly the quasi-isomorphism of DG algebras $i : \Lambda(V) \to B$,

$$i(\theta) = 1 \otimes \theta. \tag{11.3}$$

Also, consider the natural projection $p : B \to \Lambda(V)$,

$$\begin{cases} p(1 \otimes \theta) = \theta & \text{for } \theta \in \Lambda(V); \\ p(f\beta \otimes \theta) = 0 & \text{for } f \in \mathrm{Sym}^r(V^\vee), \beta \in \Lambda^s(V^\vee), \theta \in \Lambda(V), r+s > 0. \end{cases} \tag{11.4}$$

Clearly, $pi = id_{\Lambda(V)}$. Further, ip differs from id_B by homotopy given by the fromula

$$h(f\beta\otimes\theta) = \begin{cases} 0 & \text{if } f\beta=\lambda,\ \lambda\in\mathbb{C} \\ \frac{1}{w}(df\wedge\beta)\otimes\theta & \text{otherwise,} \end{cases} \tag{11.5}$$

where $w = r+s$, $f\in \mathrm{Sym}^r(V^\vee)$, $\beta\in\Lambda^s(V^\vee)$. Moreover, the maps h, p, i satisfy the following identities:

$$h^2 = 0, \quad ph = 0, \quad hi = 0. \tag{11.6}$$

Now take the polynomial $W\in\mathbb{C}[V^\vee]$ with singularity at the origin. In the previous section we have written down the one-form $\gamma\in\Omega^1(V)$, such that $\iota_\eta(\gamma)=W$. Such γ defines a matriz factorization $E = (\Omega(V),\iota_\eta+\gamma\wedge\cdot)$. We defined the D$(\mathbb{Z}/2)$-G algebra $\mathcal{B}_W := \mathrm{End}(E)$. It is clear that $\mathcal{B}_W^{gr}\cong B^{gr}$, where $\mathcal{B}_W^{gr}$ (resp. B^{gr}) is the underlying $(\mathbb{Z}/2)$-graded algebra of $\mathcal{B}_W$ (resp. B). Denote the differential on $\mathcal{B}_W$ by $\tilde{\partial}$. We have the following explicit formula for the difference of differentials:

$$(\tilde{\partial}-\partial)(f\beta\otimes\theta) = (-1)^{|\beta|-1}\sum_{k=1}^n g_k f\beta\otimes\iota_{dz_k}\theta, \text{ where } \gamma=\sum_{k=1}^n g_k dz_k. \tag{11.7}$$

We are going to describe the minimal A_∞-model for $\mathcal{B}_W$. It is obtained from the maps h,p,i above using standard formula of summing up over trees. We obtain a $(\mathbb{Z}/2)$-graded A_∞-structure $\mathcal{A}$ on the $(\mathbb{Z}/2)$-graded vector space $A=\Lambda(V)$ together with A_∞-quasi-isomorphism $\mathcal{A}\to\mathcal{B}$. Explicit computation of $\mu^k:\mathcal{A}^{\otimes k}\to\mathcal{A}$ goes as follows. Consider a ribbon tree with $(k+1)$ semi-infinite edges, k incoming and one outgoing, which has only bivalent and trivalent vertices. Associate with each vertex and each edge an operation by the following formulas:

$$\begin{cases} \text{for a bivalent vertex} & b\mapsto(-1)^{|b|}(\tilde{\partial}-\partial)(b),\ \mathcal{B}\to\mathcal{B}; \\ \text{for a trivalent vertex} & (b_2,b_1)\mapsto(-1)^{|b_1|}b_2b_1,\ \mathcal{B}^{\otimes 2}\to\mathcal{B}; \\ \text{for a finite edge} & b\mapsto(-1)^{|b|-1}h(b),\ \mathcal{B}\to\mathcal{B}; \\ \text{for an incoming edge} & a\to i(a),\ A\to\mathcal{B}; \\ \text{for an outgoing edge} & b\to p(b),\ \mathcal{B}\to A. \end{cases} \tag{11.8}$$

Then each such tree gives a map $A^{\otimes k}\to A$ in an obvious way. The explicit expression is just the sum of contributions of all possible trees. The sum is actually finite because

$$(\tilde{\partial}-\partial)(C[[V^\vee]]\otimes\Lambda^k(V^\vee)\otimes\Lambda(V))\subset C[[V^\vee]]\otimes\Lambda^k(V^\vee)\otimes\Lambda(V), \text{ and} \tag{11.9}$$

$$h(C[[V^\vee]]\otimes\Lambda^k(V^\vee)\otimes\Lambda(V))\subset C[[V^\vee]]\otimes\Lambda^{k+1}(V^\vee)\otimes\Lambda(V). \tag{11.10}$$

The components $f_k : \mathcal{A}^{\otimes k} \to \mathcal{B}$ of the A_∞- quasi-isomorphism are defined in the same way with the only difference: to the outgoing edge one attaches the operation $b \to h(b)$. Again, the sum over trees is actually finite.

To see that f_1 is quasi-isomorphism, take the increasing filtrations by subcomplexes:

$$F_r\mathcal{B}_W = \Omega(V) \otimes \Lambda^{\leq r}(V), \quad F_r\Lambda(V) = \Lambda^{\leq r}(V). \tag{11.11}$$

Then the map $f_1 : \Lambda(V) \to \mathcal{B}_W$ is compatible with these filtrations, and it induces quasi-isomorphisms on the subquotients.

Return to the special case $V = \mathbb{C}^3$, $W = -z_1z_2z_3 + z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1}$. Then we have

$$g_1 = -\frac{z_2z_3}{3} + z_1^{2g}, \quad g_2 = -\frac{z_1z_3}{3} + z_2^{2g}, \quad g_3 = -\frac{z_1z_2}{3} + z_3^{2g}. \tag{11.12}$$

Proposition 11.1. *In the above notation, the resulting A_∞- algebra $\mathcal{A}$ is G- equivariantly equivalent to $\Lambda(V)$ with the A_∞- structure $\mathcal{A}'$ from the end of section 9.*

Proof. It is useful to take the following $\mathbb{Z}$- grading on $B = \Omega(V) \otimes \Lambda(V)$.

$$\deg(\mathrm{Sym}^i(V^\vee) \otimes \Lambda^j(V^\vee) \otimes \Lambda^k(V)) = 2i - j + k. \tag{11.13}$$

Then ∂ has degree 3, h has degree -3. If we want $\tilde{\partial}$ to have degree 3, we should introduce a formal parameter $\hbar$ with degree $(4-4g)$. Further, we should write $g_1 = -\frac{z_2z_3}{3} + \hbar z_1^{2g}$ and analogously for other g_i. The operations μ^d are then decomposed as follows: $\mu^d = \mu_0^d + \mu_1^d\hbar + \mu_2^d\hbar^2 + \dots$, with μ_k^d being of degree $(6 - 3d + (4g-4)k)$. Also, it is easy to see that all μ^d are G- equivariant. It is straightforward to check that $\mu_{\mathcal{A}}^1 = 0$, and $\mu_{\mathcal{A}}^2$ the usual wedge product (this follows from vanishing of the degree 2 component of W). Further, the only tree (see Figure 1) contributes to $\Phi^1(\mu_0^3)$, and it equals to $-z_1z_2z_3$. Analogously, the only tree (see Figure 2) contributes to $\Phi^1(\mu_1^{2g+1})$, and it equals to $z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1}$, as prescribed. This proves Proposition. □

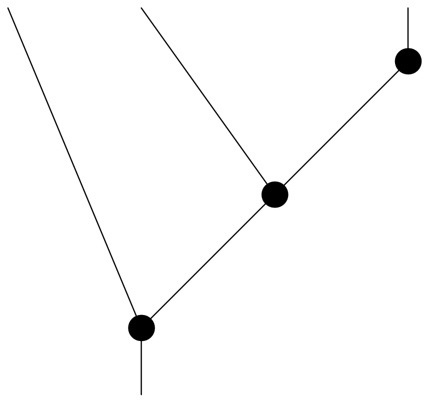

Figure 1.

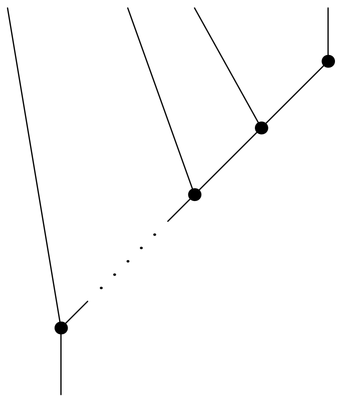

Figure 2.

From Corollary 10.5 and Proposition 11.1 we obtain the equivalence

$$\overline{D_{sg}}(W^{-1}(0)) \cong \mathrm{Perf}(\mathcal{A}'). \tag{11.14}$$

Further, Orlov's theorem can be extended to the equivariant setting. Let $K \subset G$ be the cyclic subgroup of order $2g+1$, generated by the diagonal matrix with diagonal entries $(\zeta, \zeta, \zeta^{2g-1})$, where ζ is the primitive $(2g+1)$-th root of unity. Then $D_{sg,K}(W^{-1}(0))$ is equivalent to $MF_K(W)$. The projection of $\mathcal{O}_0 \otimes \mathbb{C}[K]$ split-generates $D_{sg,K}(W^{-1}(0))$. In K-equivariant matrix factorizations it corresponds to $(\Omega(V) \otimes \mathbb{C}[K], \iota_\eta + \gamma \wedge \cdot)$. Its endomorphism DG algebra is naturally isomorphic to the smash product $\mathbb{C}[K]\#\mathcal{B}_W$, which is further A_∞-quasi-isomorphic to $\mathbb{C}[K]\#\mathcal{A}'$. The result is

Corollary 11.2. *The category* $\overline{D_{sg,K}}(W^{-1}(0))$ *is equivalent to* $\mathrm{Perf}(\mathbb{C}[K]\#\mathcal{A}')$.

12. Reconstruction theorem

The results of this section will not be used in the proof of Theorem 6.1.

Here we show that one can recover the polynomial W (up to formal change of variables) from the A_∞-structure on $\Lambda(V)$ transferred from $\mathrm{D}(\mathbb{Z}/2)$-G algebra $\mathcal{B}_W$, as in the previous section, for general W. Our proof is based on Kontsevich formality theorem, and on Keller's paper [Ke1].

More precisely, our setting is the following. Let k be any field of characteristic zero and $V = \mathrm{k}^n$, $n \geq 1$. Consider a polynomial $W = \sum\limits_{i=3}^{r} W_r \in \mathrm{k}[V^\vee]$, with $W_i \in \mathrm{Sym}^i(V^\vee)$. Take the $\mathrm{D}(\mathbb{Z}/2)$-G algebra $\mathcal{B}_W$. We have the canonical isomorphism of super-algebras

$$\Lambda(V) \cong H^{\cdot}(\mathcal{B}_W). \tag{12.1}$$

Theorem 12.1. *Let* W, W' *be non-zero polynomials as above. Suppose that DG algebras* $\mathcal{B}_W$ *and* $\mathcal{B}_{W'}$ *are quasi-isomorphic, and the chain of quasi-isomorphisms connecting* $\mathcal{B}_W$ *with* $\mathcal{B}_{W'}$ *induces the identity in cohomology via identifications* (12.1). *Then* W' *can be obtained from* W *by a formal change of variables of the form*

$$z_i \to z_i + O(z^2). \tag{12.2}$$

Proof. We introduce four pro-nilpotent DG algebras. First define the DGLA $\widetilde{\mathfrak{g}}$ by the formula

$$\widetilde{\mathfrak{g}}^d = \prod_{\substack{j-2k=d+1\\ k\in\mathbb{Z},\, i\geq d+2}} (\mathrm{Sym}^i(V^\vee) \otimes \Lambda^j(V)) \cdot \hbar^k, \tag{12.3}$$

and $L_r\widetilde{\mathfrak{g}}^d$ is the part of the product with $i\geq d+1+r,\ \ r\geq 1$ (the differential is zero, and the bracket is Schouten one). Further, put

$$\widetilde{\mathfrak{h}_1}^d=\prod_{\substack{i+j-2k=d+1\\ k\in\mathbb{Z},\, i\geq d+2}}(\mathrm{Hom}^j(\Lambda(V)^{\otimes i},\Lambda(V))\cdot\hbar^k, \tag{12.4}$$

and $L_r\widetilde{\mathfrak{h}_1}^d$ is the part with $i\geq d+1+r$ (the differential is Hochshild one and the bracket is Gerstenhaber one). Now, take the "lower" grading on $\mathrm{k}[[V^\vee]]$, with $\mathrm{k}[[V^\vee]]_d=\mathrm{Sym}^d(V^\vee)$. Of course, $\mathrm{k}[[V^\vee]]$ is the *direct product* of its graded components, but *not direct sum.* For the rest of this section we will denote the standard grading by upper indices, and the "lower" grading by the lower indices. Define the DGLA $\widetilde{\mathfrak{h}_2}$ by the formula

$$\widetilde{\mathfrak{h}_2}^d=\prod_{\substack{i-2k=d+1\\ k\in\mathbb{Z},\, i\geq 0,\, j'+2k\geq 1.}}(\mathrm{Hom}_{j'}(\mathrm{k}[[V^\vee]]^{\otimes i},\mathrm{k}[[V^\vee]])\cdot\hbar^k, \tag{12.5}$$

with $L_r\widetilde{\mathfrak{h}_2}^d$ being the part of the product with $j'+2k\geq r.$

Now take the Koszul DG $\mathrm{k}[[V^\vee]]$-$\Lambda(V)$- bimodule $X=\Lambda(V^\vee)\otimes\mathrm{k}[[V^\vee]]$ with the "upper" and "lower" gradings $X^j_{j'}=\Lambda^{-j}(V^\vee)\otimes\mathrm{Sym}^{j'}(V^\vee),$ and with differential ι_η of bidegree $(1,1)$. Define the DGLA Q by the formula

$$Q^d=\widetilde{\mathfrak{h}_1}^d\oplus\widetilde{\mathfrak{h}_2}^d\oplus\prod_{\substack{i_1+i_2+j-2k=d\\ 2k+j'-j\geq 1}}\mathrm{Hom}^j_{j'}(\Lambda(V)^{\otimes i_1}\otimes X\otimes\mathrm{k}[[V^\vee]]^{\otimes i_2},X)\cdot\hbar^k, \tag{12.6}$$

where the differential and the bracket are induced by those in the Hochshild complex of the DG category $\mathcal{C}$, where

- $Ob(\mathcal{C})=\{Y_1,Y_2\}$;
- $\mathrm{Hom}_{\mathcal{C}}(Y_1,Y_1)=\mathrm{k}[[V^\vee]]$;
- $\mathrm{Hom}_{\mathcal{C}}(Y_2,Y_2)=\Lambda(V)$;
- $\mathrm{Hom}_{\mathcal{C}}(Y_1,Y_2)=X$;
- $\mathrm{Hom}_{\mathcal{C}}(Y_2,Y_1)=0,$

Composition law in $\mathcal{C}$ comes from the bimodule structure on X (and from algebra structures on $\mathrm{k}[[V^\vee]],\ \Lambda(V)$). Thus, the DGLA structure on Q is defined. Further, define

$$L_rQ^d=L_r\widetilde{\mathfrak{h}_1}^d\oplus L_r\widetilde{\mathfrak{h}_2}^d\oplus(\text{part of the product with } 2k+j'-j\geq r),\quad r\geq 1. \tag{12.7}$$

It follows from [Ke1], Lemma in subsection 4.5, that natural projections $p_i:Q\to\widetilde{h}_i$, $i=1,2,$ are quasi-isomorphisms of DGLA's. Moreover, both p_1,p_2 are filtered quasi-isomorphisms, as it is straightforward to check.

According to [Ko3], one can attach to all Kontsevich admissible graphs (relevant for the formality theorem) *rational* weights, in such a way that they give formality L_∞- quasi-isomorphism (i.e. satisfy the relevant system of quadratic equations). In this way we obtain filtered L_∞- quasi-isomorphism $\mathcal{U}:\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{h}_2}$.

Since $p_1, p_2, \mathcal{U}$ are filtered L_∞- quasi-isomorphisms, we have by Lemma 7.6 that the composition $p_{1*}\circ(p_{2*})^{-1}\circ\mathcal{U}_*\,:\,MC(\widetilde{\mathfrak{g}})\to MC(\widetilde{\mathfrak{h}_1})$ is a bijection between the sets of equivalence classes of MC solutions in $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}_1}$.

To prove the theorem, we need to prove that, under the assumptions of the theorem, MC equations $W, W'\in\widetilde{\mathfrak{g}}^1$ are equivalent. Indeed, this means that W' is the pullback of W under the formal diffeomorphism of V with zero differential at the origin. Therefore, it suffices to prove the following

Lemma 12.2. *Under the above bijection between equivalence classes of MC solutions, the class of* $W\in\widetilde{\mathfrak{g}}^1$ *corresponds to the class* $\alpha\in\widetilde{\mathfrak{h}_1}^1$ *of the* $(\mathbb{Z}/2)$*- graded)* A_∞*- structure on* $\Lambda(V)$ *transferred from* $\mathcal{B}_W$ *to* $H^\cdot(\mathcal{B}_W)\cong\Lambda(V)$.

Proof. First note that $\mathcal{U}^k(W,\dots,W)=0$ for $k>1$, and $\mathcal{U}^1(W)$ has the only constant component which is equal to W.

Denote by $\mu=(\mu^3,\mu^4,\dots)$ the A_∞- structure on $\Lambda(V)\cong H^\cdot(\mathcal{B}_W)$ transferred from $\mathcal{B}_W$, as in the previous section. Let $\mathcal{A}$ be the resulting A_∞- algebra. Denote by $f=(f_1,f_2\dots)$ the A_∞- quasi-isomorphism $\mathcal{A}\to\mathcal{B}_W$. Also denote by $f_0\in\mathcal{B}_W^1$ the multiplication by the 1- form γ. We can consider f_i as maps $f_i\,:\,A^{\otimes i}\otimes X\to X$. Now define $\tilde{\alpha}\in Q^1$ with components μ^i, $i\geq 3$, f_j, $j\geq 0$, and $W\in\widetilde{\mathfrak{h}_2}^1$. Then $\tilde{\alpha}$ is MC solution,

$$p_1(\tilde{\alpha})=\alpha,\ \text{and}\ p_2(\tilde{\alpha})=\mathcal{U}^1(W)=\sum_{k\geq 1}\frac{1}{k!}\mathcal{U}^k(W,\dots,W). \tag{12.8}$$

Thus, classes of MC solutions $W\in\widetilde{\mathfrak{g}}^1$ and $\alpha\in\widetilde{\mathfrak{h}}^1$ correspond to each other. Lemma is proved. □

Theorem is proved. □

It follows from the proof of the above Theorem that there exists filtered L_∞- morphism $\tilde{\Phi}:\widetilde{\mathfrak{h}_1}\to\widetilde{\mathfrak{g}}$ such that the polynomial W can be reconstructed from $\mathcal{B}_W$ as follows. Take $\alpha\in\widetilde{\mathfrak{h}_1}^1$ to be MC solution corresponding to the A_∞- structure on $\Lambda(V)$ transferred from $\mathcal{B}_W$. Put

$$\beta=\sum_{k\geq 1}\frac{1}{k!}\tilde{\Phi}^k(\alpha,\dots,\alpha). \tag{12.9}$$

Decompose β into the sum $\beta^0+\beta^2+\dots+\beta^{2[\frac{n}{2}]}$, with $\beta^{2j}\in \mathrm{k}[[V^\vee]]\otimes\Lambda^{2j}(V)$. Then W can be obtained from β^0 by a formal change of variables of type (12.2).

Remark 12.3. *Note that in Theorem 12.1 we required our polynomials* W, W' *not to have terms of order* 2, *and also required the induced isomorphism* $H^{\cdot}(\mathcal{B}_W) \to H^{\cdot}(\mathcal{B}_{W'})$ *to be compatible with identifications* (12.1). *The reason is that in general Maurer-Cartan theory for DGLA's works well only in the pro-nilpotent case. However, it should be plausible that in the case* $\mathrm{k} = \mathbb{C}$ *one can drop these assumptions. Of course, in this case one also should drop the requirement on the change of variables to be of type* (12.2).

13. Equivalence of two LG models

Take $V = \mathbb{C}^3$ and $K \subset G \subset SL(V)$ be as before. In this section we describe two different LG models, such that the resulting categories are equivalent.

The first one is stacky: $(V//K, W)$, where W is our superpotential. The associated category $\overline{D_{sg,K}}(W^{-1}(0))$ has already been described (Corollary 11.2).

Now we describe another LG model, which is taken in Theorem 6.1. There is a well-known crepant resolution of the quotient V/K :

$$X = \mathrm{Hilb_K(V)} \to \mathrm{V/K}. \tag{13.1}$$

More explicitly, X can be given by a fan describing it, due to the paper [CR]. Take $N \subset \mathbb{R}^3$, $N = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{2g+1}(1, 1, 2g-1)$. Now, if we take a fan Σ consisting of a positive octant and its faces, then we have $X_\Sigma \cong V/K$. To describe X, we should subdivide the fan Σ. Namely, take the fan Σ' consisting of the cones generated by

$$\left(\frac{1}{2g+1}(k, k, 2g+1-2k), \frac{1}{2g+1}(k+1, k+1, 2g-1-2k), (1, 0, 0)\right), \quad 0 \le k \le g-1; \tag{13.2}$$

$$\left(\frac{1}{2g+1}(k, k, 2g+1-2k), \frac{1}{2g+1}(k+1, k+1, 2g-1-2k), (0, 1, 0)\right), \quad 0 \le k \le g-1; \tag{13.3}$$

$$\left(\frac{1}{2g+1}(g, g, 1), (1, 0, 0), (0, 1, 0)\right), \tag{13.4}$$

and all their faces (see Figure 3 for the case $g = 3$). Then $X \cong X_{\Sigma'}$.

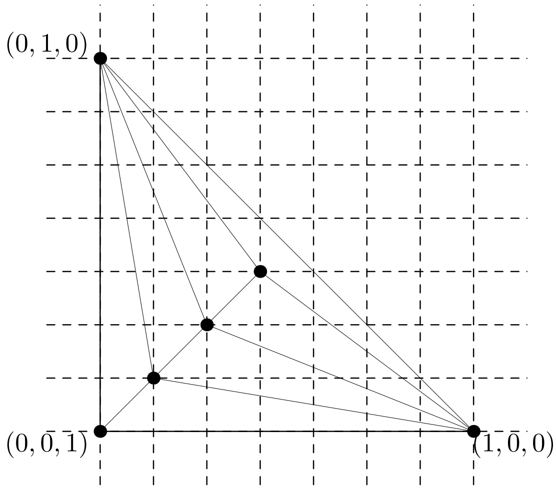

Figure 3.

The function $W \in \mathbb{C}[V^\vee]$ is K- invariant, hence gives a function on V/K, and on X. The LG model (X, W) is a mirror to the genus g curve. The only singular fiber of W on X is $X_0 =: H$. The surface H has $(g+1)$ irreducible components $H_1, \ldots, H_{g+1}$, where H_i is defined below for $1 \leq i \leq g$, and H_{g+1} is the proper pre-image of $W^{-1}(0) \subset V/K$.

The exceptional surface $H_k \subset X$, $q \leq k \leq g$, corresponding to the vector $\frac{1}{2g+1}(k, k, 2g+1-2k) \in N$ is

(13.5)

$$\begin{cases} \text{the rational ruled surface } F_{2g+1-2k} \cong \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O} \oplus \mathcal{O}(2g+1-2k)) & \text{for } 1 \leq k \leq g-1 \\ \mathbb{CP}^2 & \text{for } k = g. \end{cases}$$

The surfaces H_i and H_j have empty intersection if $|i - j| \geq 2$. Further, the surfaces H_i and H_{i+1} intersect transversally along the curve $C_i \subset X$, where $1 \leq i \leq g-1$. The curve C_i is

(13.6)

$$\begin{cases} \text{the "}\infty\text{-section" } \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O}(2g+1-2i)) \subset \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O} \oplus \mathcal{O}(2g+1-2i)) \cong H_i \text{ on } H_i & \text{for } 1 \leq i \leq g-1 \\ \text{the "zero-section" } \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O}) \subset \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O} \oplus \mathcal{O}(2g-1-2i)) \cong H_{i+1} \text{ on } H_{i+1} & \text{for } 1 \leq i \leq g-2 \\ \text{the line on } \mathbb{CP}^2 \cong H_g & \text{for } k = g. \end{cases}$$

The divisor H has simple normal crossings. We have already described the intersections between H_i for $1 \leq i \leq g$. Further, the intersection $H_i \cap H_{g+1}$ is:

(13.7)

$$\begin{cases} \text{the section } \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O}\times(y_0y_1, y_0^{2g+1}+y_1^{2g+1}))\subset \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O}(2)\oplus\mathcal{O}(2g+1))\cong H_1 & \text{for } i=1\\ \text{the union of two fibers } \{y_0y_1=0\}\subset \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O}\oplus\mathcal{O}(2g+1-2i)) & \text{for } 2\leq i\leq g-1\\ \text{a non-degenerate conic in } \mathbb{CP}^2\cong H_g & \text{for } i=g.\end{cases}$$

Here $(y_0 : y_1)$ are homogeneous coordinates on $\mathbb{CP}^1$.

The triple intersection $H_i\cap H_{i+1}\cap H_{g+1}$ consists of two points for each $1\leq i\leq g-1$. The corresponding dual CW complex of this configuration is homeomorphic to S^2.

Theorem 13.1. *The triangulated category $D_{sg}(H)$ is equivalent to $D_{sg,K}(W^{-1}(0))$.*

Proof. This follows from [BKR] and [BP], Theorem 1.1. Alternatively, Theorem is implied by [QV], Theorem 8.6. □

Denote by $\overline{D_{sg}}(H)$ the split-closure of the triangulated category of singularities $D_{sg}(H)$.

Corollary 13.2. *There is an equivalence $\overline{D_{sg}}(H)\cong \mathrm{Perf}(\mathbb{C}[K]\#\mathcal{A}')$.*

Proof. This follows from Theorem 13.1 and Corollary 11.2.

□

14. Generalities on Fukaya categories

This section is devoted to generalities on Fukaya A_∞-categories of compact symplectic surfaces of genus ≥ 2. We follow [Se1], Sections 6-10.

14.1. **The definition.** Let M be such a surface, with a symplectic form ω. Let $\pi : S(TM)\to M$ be a bundle of unit circles in the tangent bundle (it does not depend on Riemann metric). Fix a 1-form θ on $S(TM)$, such that $d\theta=\pi^*\omega$. In the definition of the Fukaya category $\mathcal{F}(M)$, we will need to fix equivalence class of θ modulo exact 1-forms.

Consider some connected Lagrangian submanifold of M, i.e. just simple closed curve $L\subset M$. Let $\sigma : L\to S(TM)_{|L}$ be the natural section for some choice of orientation on L. The curve L is called balanced if $\int_L \sigma^*\theta=0$. This property does not depend on the orientation on L. Contractible curves can not be balanced. Further, if L is non-contractible, then it is isotopic to some balanced L'. Moreover, such L' is unic up to Hamiltonian isotopy.

Objects of the Fukaya category $\mathcal{F}(M)$ are oriented balanced curves $L\subset M$, with a chice of a spin structure (there are two Spin structures: trivial and non- trivial). For each two objects $L_0, L_1\in\mathcal{F}(M)$ we choose a Hamiltonian isotopy $(\phi_t)_{t\in[0,1]}$, such that $\phi_1(L_0)$

is transverse to L_1. Then we define the $\mathbb{Z}/2$-graded space $\mathrm{Hom}(L_0, L_1)$ as the space of Floer cochains:

$$\mathrm{Hom}_{\mathcal{F}(M)}(L_0, L_1) = CF^{\cdot}(L_0, L_1) = \bigoplus_{x \in L_0 \cap \phi_1^{-1}(L_1)} \mathbb{C}x. \tag{14.1}$$

The $\mathbb{Z}/2$-grading of x is even (resp.) odd if the local intersection number $L_0 \cdot \phi_1^{-1}(L_1)$ at x is -1 (resp. 1).

Now we are going to define higher products in the A_∞-category $\mathcal{F}(M)$. Take some objects $L_0, \ldots, L_d$ for some $d \geq 1$. Let $x_0 \in M$ be a point, which corresponds to some basis element of $CF^{\cdot}(L_0, L_d)$. Similarly, choose some $x_k \in M$, giving some basis elements of $CF^{\cdot}(L_{k-1}, L_k)$, $1 \leq k \leq d$. Making some choices of additional data, one obtains a moduli space of perturbed pseudo-holomorphic polygons. We denote it by $\mathcal{M}(x_0, x_1, \ldots, x_d)$. Points in this space are represented by maps $u : S \to M$, where

1) S is a closed 2-dimensional disk with standard complex structure, minus $(d+1)$ boundary points. Denote this boundary points by $\zeta_0, \ldots, \zeta_d$. Their ordering is fixed, so that they go in the clockwise direction. For convenience, put $\zeta_{i+d} := \zeta_i$. Further, for $0 \leq k \leq d$, denote by $\partial_k S$ the boundary component between ζ_k and ζ_{k+1}. With this notation, $u : S \to M$ is a smooth map such that $u(\partial_k S) \subset L_k$. When $z \in S$ tends to ζ_k, $u(z)$ must converge to x_k. Also, u is required to satisfy the inhomogeneous Cauchey-Riemann equation

$$\frac{1}{2}(du + J_z(u(z)) \circ du \circ i) = \nu_z(u(z)), \tag{14.2}$$

where i is the complex structure on S, J is a family of almost complex structures on M parameterized by S, and ν is the inhomogeneous term.

2) J and ν are the auxiliary data, which should be chosen carefully.

For details of the definition, see [Se2].

For each pseudo-holomorphic polygon u, we have its virtual dimension $\mathrm{vdim}(u)$. It is defined using index theory. Suppose that the necessary choice of auxiliary data is sufficiently general. Then, the moduli space will be smooth, and near any u its dimension will equal to $\mathrm{vdim}(u)$.

We are interested in points of virtual dimension zero. Such points form a finite set. To each such u, one can associate a sign $\epsilon(u) = \pm 1$ (at this point, Spin structures are used). We put

$$m(x_0, \ldots, x_d) = \sum_{\substack{u \in \mathcal{M}(x_0, \ldots, x_d), \\ \mathrm{vdim}(u) = 0}} \epsilon(u). \tag{14.3}$$

Define higher products in $\mathcal{F}(M)$ by the formula

$$\mu^d(x_d,\dots,x_1)=\sum_{x_0\in\mathcal{M}(x_0,\dots,x_d)} m(x_0,\dots,x_d)x_0. \tag{14.4}$$

We will use the following Seidel's version ([Se1], Section 7) of the definition of morphisms and higher products μ^k in the Fukaya category $\mathcal{F}(M)$, which works under some assumptions, and is sufficient for our purposes.

From now on, we fix a complex structure on M. Further, we will deal only with objects of $\mathcal{F}(M)$ from some chosen finite collection $\mathcal{L}\subset Ob(\mathcal{F}(M).)$ We assume the collection $\mathcal{L}$ is in general position. For $L_0\neq L_1$, let $\mathrm{Hom}(L_0,L_1)=CF^{\cdot}(L_0,L_1)$ be the direct sum of 1-dimensional spaces $\mathbb{C}x$, where $x\in L_0\cap L_1$. $\mathbb{Z}/2$-grading is as above (in other words, we just take ϕ_t to be trivial Hamiltonian isotopy). Further, for $L_0=L_1=L$ fix some generic point $*\in L$ and define $\mathrm{Hom}(L,L)=CF^{\cdot}(L,L):=\mathbb{C}e\oplus\mathbb{C}q$, where e and q are even and odd respectively.

Let $(L_0,L_1,\dots,L_d)$ be a collection of objects, and let $x_0,x_1,\dots,x_d$ be generators of Floer complexes $CF^{\cdot}(L_0,L_d),CF^{\cdot}(L_0,L_1),\dots,CF^{\cdot}(L_{d-1},L_d)$ respectively. Define $\mathcal{M}(x_0,\dots,x_d)'$ to be the moduli space of maps $u':S\to M$, where S is again a $(d+1)$-pointed disk, $u'(\partial_k S)\subset L_k$, and u' is holomorphic. If the generator x_k is transversal intersection point, then the convergence $\lim\limits_{z\to\zeta_k}u'(z)=x_k$ must hold. If x_k is of the type e or q, then u' is required to extend over ζ_k. Further, in the cases ($k=0$ and $x_0=e$), ($k>0$ and $x_k=q$) the condition $u'(\zeta_k)=*$ is required. Otherwise $u'(\zeta_k)$ can be any point of L_k.

Lemma 14.1. *(*[Se1] *Sections 6,7) All $u'\in\mathcal{M}(x_0,\dots,x_d)'$ which are non-constant, are regular points. The virtual dimension at such a point is at least the number of k such that no point constraint are imposed on $u'(\zeta_k)$. Equality holds iff u' is an immersion, extends to a map with no branching at x_k which are of type e and q, and is locally embedded as a convex corner at those x_k which are transverse intersection points.*

We assume that all points u' of virtual dimension zero are regular, and they form a finite set. The number $m(x_0,\dots,x_d)'$ is defined to be the signed count of such points $u'\in\mathcal{M}(x_0,\dots,x_d)'$.

Proposition 14.2. *(*[Se1]*, Sections 6,7) Take $(L_0,\dots,L_d)\in\mathcal{L}$. Then, by making appropriate choices, one can achieve that $m(x_0,\dots,x_d)=m(x_0,\dots,x_d)'$ in the following two situations: if the L_k are pairwise different; or only two of them coincide and these two are either (L_{k-1},L_k), or (L_d,L_0).*

Now we consider some examples.

Constant triangles. Let $L_0 \neq L_1$. Then the constant triangle at any point $x \in L_0 \cap L_1$ contributes to the products

$$\mu^2(e,x), \mu^2(x,e) : CF^{\cdot}(L_0, L_1) \to CF^{\cdot}(L_0, L_1); \tag{14.5}$$

$$\mu^2(x,x) : CF^{\cdot}(L_1, L_0) \otimes CF^{\cdot}(L_0, L_1) \to CF^{\cdot}(L_0, L_0). \tag{14.6}$$

No non-constant triangles can contribute to these products, and taking signs into account one obtains

$$\mu^2(x,e) = x, \quad \mu^2(e,x) = (-1)^{|x|}x, \quad \mu^2(x,x) = (-1)^{|x|}q. \tag{14.7}$$

Similarly,

$$\mu^2(e,e,) = e, \quad \mu^2(e,q) = -q, \quad \mu^2(q,e) = q, \quad \mu^2(q,q) = 0. \tag{14.8}$$

Non-constant polygons. Here we should take Spin structures into account. If there is a non-trivial Spin structure on a Lagrangian L, then we mark a generic point $\circ \neq *$. Otherwise, we do not mark any other point.

Consider the case when the curves $L_0, \dots, L_d$ are pairwise distinct. Let $x_0, \dots, x_d$ be some generators of the corresponding Floer complexes. Further, let $u' \in \mathcal{M}(x_0, \dots, x_d)'$ be some holomorphic polygon of virtual dimension zero. Suppose that $L_1, \dots, L_d$ are oriented compatibly with the anti-clockwise orientation on ∂S, and none of the points $\circ$ lie on the boundary of u. Reversing the orientation of L_k, $0 < k < d$, changes the sign by $(-1)^{|x_k|}$. Reversing the orientation of L_d changes the sign by $(-1)^{|x_0|+|x_d|}$. Finally, each time when the boundary of u' passes through one of the points $\circ \in L_k$, the sign changes by (-1).

The other case we are interested in is when $L_0 = L_d$, and $L_0, \dots, L_{d-1}$ are pairwise distinct. Take $u' \in \mathcal{M}(e, x_1, \dots, x_d)$ with $\mathrm{vdim}(u') = 0$. If $L_1, \dots, L_d$ are oriented compatibly with the anti-clockwise orientation on ∂S, and the boundary of u' does not meet $\circ$, then u' contributes with the sign $+1$. Otherwise reversing the orientations and meeting points $\circ$ has the same effect on sign as in the previous case.

14.2. **Split-generators in Fukaya categories.** Suppose that $\mathcal{A}$ is some $(\mathbb{Z}/2)$- graded A_∞- category with weak units, and $E \in \mathrm{Perf}(\mathcal{A})$ is an object which split-generates $\mathrm{Perf}(\mathcal{A})$. Then it is well-known that the natural A_∞- functor $\mathrm{Hom}(-, E) : \mathrm{Perf}(\mathcal{A}) \to \mathrm{Perf}(\mathrm{End}(E))$ is a quasi-equivaelence, see [**?**].

Let L_0, L_1 be two objects of the Fukaya category $\mathcal{F}(M)$, and the Spin structure on L_1 is non-trivial. The Dehn twist τ_{L_1} is a balanced symplectic automorphism of M, hence

$\tau_{L_1}(L_0)$ is again balanced. According to [Se1] and [Se2], we then have the following exact triangle in $D^\pi\mathcal{F}(M)$:

$$HF^{\cdot}(L_1, L_0) \otimes L_1 \to L_0 \to \tau_{L_1}(L_0). \tag{14.9}$$

We will need the following two Lemmas from [Se1], which we will use to prove that a given objects is a generator of $D^\pi\mathcal{F}(M)$..

Lemma 14.3. *(*[Se1]*, Lemma 6.4) Let $L_1, \dots, L_r$ be objects of $\mathcal{F}(M)$ whose Spin structures are non-trivial. Suppose that L_0 is another object, and $\tau_{L_r} \dots \tau_{L_1}(L_0) \cong L_0[1]$. Then L_0 is split-generated by $L_1, \dots, L_r$.*

Lemma 14.4. *(*[Se1]*, Lemma 6.5) Let $L_1, \dots, L_r$ be objects of $\mathcal{F}(M)$ whose Spin structures are non-trivial and which are such that $\tau_{L_r} \dots \tau_{L_1}$ is isotopic to the identity. Then they split-generate $D^\pi(\mathcal{F}(M))$.*

14.3. **Additional $\mathbb{Z}$-gradings.** Since M is not Calabi-Yau, the $(\mathbb{Z}/2)$-grading on M cannot be improved to $\mathbb{Z}$-gradings. However, it turns out that one can still put some $\mathbb{Z}$-grading for some Lagrangians, but and then control the $\mathbb{Z}$-homogeneous components of higher products.

Fix a complex structure on M. Take a meromorphic section η^r of the line bundle $\omega^{\otimes r}$ $T^*M^{\otimes r}$. Let D be its divisor. For any oriented $L \subset M \setminus \mathrm{Supp}(D)$, our section η^r gives a map

$$L \to S^1, \quad x \to \frac{\eta^r(X^{\otimes r})}{|\eta^r(X^{\otimes r})|}, \tag{14.10}$$

where X is a tangent vector to L at x, which points in the positive direction.

We define an $\frac{1}{r}$-grading on L as a lift $L \to \mathbb{R}$ of the map (14.10). Let $\mathcal{F}(M, D)$ be a version of Fukaya category, with the only difference that Lagrangian submanifolds L should lie in $M \setminus \mathrm{Supp}(D)$, and to be equipped with $\frac{1}{r}$-grading. In particular, we have full and faithful A_∞-functor $\mathcal{F}(M, D) \to \mathcal{F}(M)$.

Suppose that two objects L_0, L_1 of $\mathcal{F}(M, D)$ have only transversal intersection. Then each $x \in L_0 \cap L_1$, is equipped with an integer $i^r(x)$. Namely, let $\alpha \in (0, \pi)$ be the angle counted clockwise from $TL_{0,x}$ to $TL_{1,x}$. Let $\alpha_0(x)$, $\alpha_1(x)$ be the values of $\frac{1}{r}$-gradings of L_0 and L_1 at x respectively. Then

$$i^r(x) := \frac{r\alpha + \alpha_1(x) - \alpha_0(x)}{\pi}. \tag{14.11}$$

If r is odd, then $i^r(x) \bmod 2$ coincides with the value of $(\mathbb{Z}/2)$-grading on x. Further, if $L_0 = L_1$, then $i^r(e) = 0$, $i^r(q) = r$.

Let $u \in \mathcal{M}(x_0, \ldots, x_d)$ be a perturbed pseudo-holomorphic polygon of virtual dimension zero, hence contributing to the higher product. For each $z \in Supp(D)$, denote by $\deg(u, z)$ the multiplicity with which u hints z. Then it follows from the index formula that

$$i^r(x_0) - i^r(x_1) - \cdots - i^r(x_d) = r(2-d) + 2 \sum_{z \in Supp(D)} \mathrm{ord}(\eta^r, z) \deg(u, z). \tag{14.12}$$

Now suppose that for all points $z \in \mathrm{Supp}(D)$ the order $\mathrm{ord}(\eta^r, z)$ is the same positive integer $m > 0.$. With respect to our $\mathbb{Z}$-gradings $i^r(x)$, the higher operations μ^i will decompose into the sum

$$\mu^i = \mu_0^i + \mu_1^i + \ldots, \tag{14.13}$$

where μ_k^i, $k \geq 0$, are homogeneous maps of degree $r(2-d) + 2mk$. Note that in section 9 we considered precisely these conditions on gradings, with $r = 3$ and $m = 2g - 2$.

14.4. **Fukaya categories of orbifolds.** Let $\bar{M}$ be a Riemann surface with a finite subset $\bar{D} \subset \bar{M}$ of orbifold points. Let *omēga* be an orbifold symplectic form on $\bar{M}$. Near each $z \in \bar{D}$ we have a chart in which the neighborhood of z is represented as a quotient of disc by a cyclic group $\mathbb{Z}/iso(z)$, where $iso(z) \geq 2$. We assume that the orbifold Euler characteristic

$$\chi_{orb}(\bar{M}) = \chi_{top}(\bar{M}) - \sum_{z \in \bar{D}} \frac{iso(z) - 1}{iso(z)} \tag{14.14}$$

is negative. Then there exists a 1-form $\bar{\theta}$ on $S(T\bar{M})$ such that $d\bar{\theta}$ equals to the pullback of $\bar{\omega}$.

Consider some immersion $\bar{l} : L \to \bar{M}$, where $L = S^1$. is a circle Denote the image of $\bar{l}$ by $\bar{L}$. Again, we call $\bar{L}$ balanced if the integral of the pullback of $\bar{\theta}$ with respect to the natural map $L \to S(T\bar{M})$ equals to zero. Further, the self-intersections of $\bar{L}$ are required to be generic, and we also require absence of teardrops:

Definition 14.5. *Let $x_-, x_+ \in L$, $x_- \neq x_+$, and $\bar{l}(x_-) = \bar{l}(x_+) = x$. Denote by H the closed upper half-plane. A tear-drop is a pair $(\bar{u}, w)$, where $\bar{u} : H \to \bar{M} \setminus \bar{D}$ is a smooth immersion, and $w : \partial H \to L$ is a smooth map, such that $\bar{l} \circ w = \bar{u}_{|\partial H}$, and $\lim\limits_{x \to \pm\infty} = x_{\pm}$, $\lim\limits_{z \to +i\infty} \bar{u}(z) = x$.*

We also put a Spin structure on L. One can work out Floer theory for such L, together with Fukaya higher products, see [Se1], Section 9.

From this moment we assume that $\bar{M} = M/\Gamma$, where M is a (compact) Riemann surface, and the group Γ is finite. Further, assume that and $\bar{L}$ is an image of some embedded $L \subset M$. This implies the absence of teardrops. For each generator x of $CF^{\cdot}(\bar{L}, \bar{L})$, we have the associated element $\gamma \in \Gamma$. Namely, let $\bar{\phi}_t$ be the Hamiltonian isotopy used to define $CF^{\cdot}(\bar{L},\bar{L})$, and ϕ_t its lift on M. If x corresponds to a pair $x_-, x_+ \in L$ with $\bar{l}(\phi_1 x_+) = \bar{l}(x_-)$, then $\gamma(x_+) = x_-$. Write the corresponding decomposition of the Floer complex as

$$CF^{\cdot}(\bar{L},\bar{L}) = \bigoplus_{\gamma\in\Gamma} CF^{\cdot}(\bar{L},\bar{L})^{\gamma} \tag{14.15}$$

Then it is clear that

$$\mu^d(CF^{\cdot}(\bar{L},\bar{L})^{\gamma_d} \otimes \cdots \otimes CF^{\cdot}(\bar{L},\bar{L})^{\gamma_1}) \subset CF^{\cdot}(\bar{L},\bar{L})^{\gamma_d\cdots\gamma_1}. \tag{14.16}$$

Suppose that Γ is abelian. Then we have the action of the group of characters $G = \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ on $CF^{\cdot}(\bar{L},\bar{L})$: $g \in G$ acts on $CF^{\cdot}(\bar{L},\bar{L})^{\gamma}$ with multiplication by $g(\gamma)$.

Now, take the collection of curves $\gamma(L)$, $\gamma \in \Gamma$. These are all non-trivial lifts of $\bar{L}$ onto M. Then it is elementary that we have G- equivariant A_∞- isomorphism

$$\bigoplus_{\gamma_0,\gamma_1\in\Gamma} CF^{\cdot}(\gamma_0(L),\gamma_1(L)) \cong \mathbb{C}[G]\# CF^{\cdot}(\bar{L},\bar{L}). \tag{14.17}$$

15. Fukaya category of a genus $g \geq 3$ curve

It is convenient to represent the genus $g \geq 3$ curve M as a 2- fold covering of $\mathbb{CP}^1$, branched in $(2g+2)$ points: $(2g+1)$- th roots of unity and 0. Take the curves $L_1,\dots,L_{2g+1}$, which are preimages of intervals $[\zeta^0,\zeta^2]$, $[\zeta^1,\zeta^3],\dots,[\zeta^{2g-1},\zeta^0]$, $[\zeta^{2g},\zeta^1]$ respectively, where $\zeta = \exp(\frac{2\pi i}{2g+1})$. The special case $g = 3$ is shown in Figure 4.

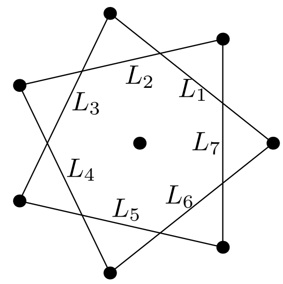

Figure 4.

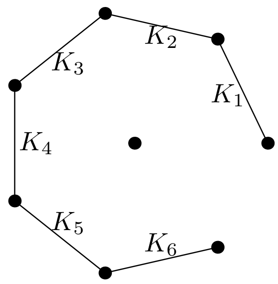

Figure 5.

Lemma 15.1. *The curves $L_1,\dots,L_{2g+1}$, equipped with non-trivial spin structures, split-generate $D^\pi\mathcal{F}(M)$.*

Proof. Take the curves $K_1,\dots,K_{2g}$, which are preimages of intervals $[\zeta^0,\zeta^1],[\zeta^1,\zeta^2],\dots,[\zeta^{2g-1},\zeta^{2g}]$ respectively (the special case $g = 3$ is illustrated in Figure 5). Then by [Ma] we have $(\tau_{K_{2g}}\dots\tau_{K_1})^{4g+2} \sim \mathrm{id}$. From Lemma 14.4, it follows that the curves $K_1,\dots,K_{2g}$, equipped with non-trivial spin structures, split-generate $D^\pi\mathcal{F}(M)$. Further, it is straightforward to check that $\tau_{L_{2g+1}}\dots\tau_{L_1}(K_1)$ is isotopic to

$K_1[1]$. Thus, it follows from Lemma 14.3 that K_1 is split-generated by $L_1,\dots,L_{2g+1}$. Analogously, all the other K_i are split-generated by $L_1,\dots,L_{2g+1}$.

Hence, $L_1,\dots,L_{2g+1}$ split-generate $D^{\pi}\mathcal{F}(M)$. □

We now compute partially the A_∞- algebra $\bigoplus\limits_{1\leq i,j\leq 2g+1} CF^{\cdot}(L_i,L_j)$. Our computation is in fact analogous to the computations in [Se1], Section 10.

Take a natural $\Sigma=\mathbb{Z}/(2g+1)$- action on M which lifts the rotational action on $\mathbb{CP}^1$. The quotient M/Σ is a sphere $\bar{M}$ with 3 orbifold points. Denote the set of orbifold points by $\bar{D}$.

Explicitly, the hyperelliptic curve M is given (in affine chart) by the equation

$$y^2=z(z^{2g+1}-1). \tag{15.1}$$

The generator of Σ acts by the formula

$$(y,z)\to(\zeta^{g+1}y,\zeta z). \tag{15.2}$$

We have that $\mathbb{C}(M)^{\Sigma}\cong\mathbb{C}(\frac{y}{z^{g+1}})$, hence $t=\frac{y}{z^{g+1}}$ is a coordinate on an affine chart $\mathbb{C}\subset\mathbb{CP}^1\cong\overline{M}$. The set $\bar{D}$ consists of the points $t=1$, $t=-1$, and $t=\infty$.

Each of the curves L_i projects to the same curve $\bar{L}\subset\bar{M}$. It lies in $\mathbb{C}\setminus\{-1,1\}\subset\bar{M}$ and has the same isotopy type for all $g\geq 3$. The case $g=3$ is shown in Figure 6. We have natural A_∞- isomorphism, as in (14.17):

$$\bigoplus_{1\leq i,j\leq 2g+1} CF^{\cdot}(L_i,L_j)\cong\mathbb{C}[K]\# CF^{\cdot}(\bar{L},\bar{L}), \tag{15.3}$$

where $K=\mathrm{Hom}(\Sigma,\mathbb{C}^*.)$ This is actually the same K as in the end of Section 11.

The super vector space $CF^{\cdot}(\bar{L},\bar{L})$ has 8 generators: two standard e (even) and q (odd), together with three pairs ($\bar{x}_i$ (even), x_i (odd)), $1\leq i\leq 3$, coming from each self-intersection point of $\bar{L}$ (see Figure 6). Take $\widetilde{\Gamma}=\pi_1^{orb}(\bar{M})$, and put $\Gamma=[\widetilde{\Gamma},\widetilde{\Gamma}]$. Then Γ is naturally the quotient of $(\mathbb{Z}/(2g+1))^3$ by the diagonal subgroup $\mathbb{Z}/(2g+1)$. The class of our immersed curve $\bar{L}$ in Γ is trivial, hence the generators of $CF^{\cdot}(L,L)$ are labelled by the weights which are elements of Γ.

Further, take a meromorphic section η^3 of $(T^*\bar{M})^{\otimes 3}$, having double pole at each point of $\bar{D}$. Explicitly,

$$\eta^3=\frac{(dt)^{\otimes 3}}{(t-1)^2(t+1)^2}. \tag{15.4}$$

Each generator of $CF^{\cdot}(\bar{L},\bar{L})$ is equipped with additional integer grading, together with weight in Γ :

(15.5)

generator	e	x_1	x_2	x_3
weight	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
index	0	1	1	1

generator	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	q
weight	$(0,1,1)$ $=(-1,0,0)$	$(1,0,1)$ $=(0,-1,0)$	$(1,1,0)$ $=(0,0,-1)$	$(1,1,1)$ $=(0,0,0)$
index	2	2	2	3

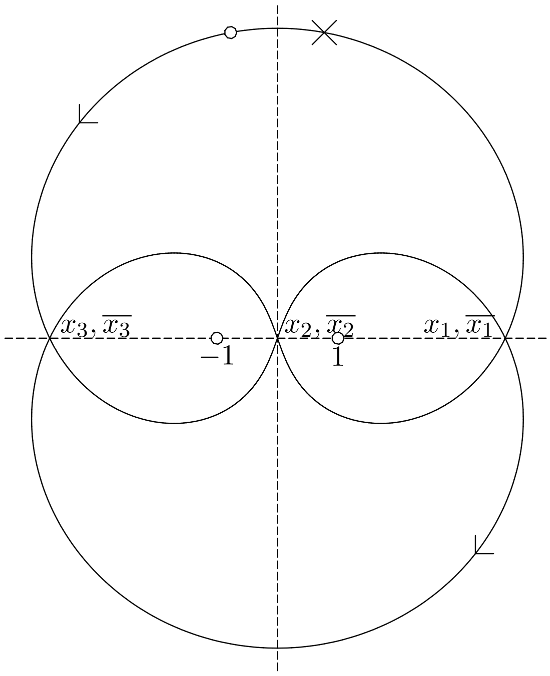

Figure 6.

Since the A_∞-structure is homogeneous with respect to Γ by (14.16) we have that $\mu^1 = 0$.

Further, the inverse image of η^3 on M has three poles of order $(2g-2)$. Therefore, according to (14.13), we have a decomposition $\mu^i = \mu^i_0 + \mu^i_1 + \dots$, where μ^i_k has degree $6 - 3i + (4g-4)k$.

For degree reasons, μ^2_k vanishes for $k > 0$. Further, according to (14.7), (14.8), we have

$$\mu^2(x_i, e) = x_i = -\mu^2(e, x_i), \quad \mu^2(\bar{x}_i, e) = \bar{x}_i = \mu^2(e, \bar{x}_i), \quad \mu^2(q, e) = q = -\mu^2(e, q),$$
$$\mu^2(q, q) = 0, \quad \mu^2(x_i, \bar{x}_i) = q = -\mu^2(\bar{x}_i, x_i). \tag{15.6}$$

Further, there are only six (taking into account the ordering of the vertices) non-constant triangles which avoid $\bar{D}$. To determine the sign of their contributions, choose generic points $\circ \neq *$ on $\bar{L}$, as in Figure 6. Then we have

$$\mu^2(x_1,x_2) = \bar{x_3} = -\mu^2(x_2,x_1); \qquad (15.7)$$

$$\mu^2(x_2,x_3) = \bar{x_1} = -\mu^2(x_3,x_2);$$

$$\mu^2(x_3,x_1) = \bar{x_2} = -\mu^2(x_1,x_3).$$

Further, one of the triangles (passing through $*$) can be thought as a four-pointed disc with one of the vertex being $*$. It gives contribution to

$$\mu_0^3(x_3,x_2,x_1) = -e. \qquad (15.8)$$

Further, $\mu_0^3(x_{i_1},x_{i_2},x_{i_1}) = 0$ for $(i_1,i_2,i_3) \neq (3,2,1)$, since such an expression is a multiple of e (for degree reasons), and all the relevant spaces $\bar{\mathcal{M}}(e,x_{i_1},x_{i_2},x_{i_3})$ are empty.

There are six holomorphic $(2g+1)$-gons in our picture. Namely, each point $x_i \in \bar{L}$ breaks the curve $\bar{L}$ into two loops γ',γ''. Choose the orientations on them in such a way that they go anti-clockwise around the corresponding orbifold point $t_{\gamma'} = t_{\gamma''}$.. Then for each such loop γ_j we have a bi-holomorphic map $v_j : S \to \bar{M}$, where S is a 1-pointed disk. The image of v_j is the area bounded by γ_j and containing the orbifold point t_{γ_j}. Also require v_j to map the center of S to t_{γ_j} and the marked point to the corresponding x_i. Further, define u_j to be the composition of v_j with the map $z \to z^{2g+1}$. Then u_j maps the $(2g+1)$-th roots of unity to x_i.

Further, each u_j hits exactly one of the points of $\bar{D}$ and has $(2g+1)$-fold ramification there, and no ramification elsewhere, which means that it lifts to a genuine immersed $(2g+1)$-gon in M. We take the three $(2g+1)$-gons that go through $*$, and determine their contributions to μ_1^{2g+1}, namely:

$$\mu_1^{2g+1}(x_i,\dots,x_i) = e. \qquad (15.9)$$

Now identify $CF^{\cdot}(\bar{L},\bar{L})$ with $\Lambda(V)$, $V = \mathbb{C}^3$, mapping e to 1, x_i to ξ_i, $\bar{x_1}$ to $\xi_2 \wedge \xi_3$ and analogously for other $\bar{x_i}$, and q to $-\xi_1 \wedge \xi_2 \wedge \xi_3$. Then, it follows from the above computations and Theorem 9.2 that the resulting A_∞-structure on $\Lambda(V)$ is $G \cong \mathrm{Hom}(\Gamma,\mathbb{C}^*)$-equivariantly A_∞-isomorphic to $\mathcal{A}'$ from the end of the Section 9. The covering $M \to \bar{M}$ is classified by the surjective homomorphism $\Gamma \to \Sigma$, which is dual to the inclusion $K \subset G$. Combining this with Lemma 15.1 and (14.17), we obtain the following

Corollary 15.2. *We have an equivalence* $D^{\pi}\mathcal{F}(M)\cong \mathrm{Perf}(\mathbb{C}[K]\#\mathcal{A}')$.

Proof of Theorem 6.1. By Corollary 13.2, we have an equivalence

$$\overline{D_{sg}}(H)\cong \mathrm{Perf}(\mathbb{C}[K]\#\mathcal{A}'). \tag{15.10}$$

By Corollary 15.2, we have an equivalence

$$D^{\pi}\mathcal{F}(M)\cong \mathrm{Perf}(\mathbb{C}[K]\#\mathcal{A}'). \tag{15.11}$$

Therefore, we have the desired equivalence

$$D^{\pi}\mathcal{F}(M)\cong \overline{D_{sg}}(H). \tag{15.12}$$

Theorem is proved. □

16. Appendix

Here we prove the statement of Lemma 7.6. It is in fact standard, but we could not find a reference. Fix some basic field k of characteristic zero.

In fact one can define, following Getzler [Ge] a simplicial set $\mathcal{MC}_{\bullet}(\mathfrak{g})$ for any pro-nilpotent L_{∞}- algebra, such that $\pi_0(|\mathcal{MC}_{\bullet}(\mathfrak{g})|)$ is the set of equivalence classes of MC solutions in $\mathfrak{g}$. Further, one can prove that filtered L_{∞}- quasi-isomorphism $\Phi:\mathfrak{g}\to\mathfrak{h}$ induces a homotopy equivalence of these simplicial sets. However, we prove in this Appendix precisely what we need.

Let $\mathfrak{g}$ be a nilpotent (in the standard sense) DG Lie algebra. Denote by $MC(\mathfrak{g})$ the set of MC solutions. We have the nilpotent group $\exp(\mathfrak{g}^0)$, which acts on $MC(\mathfrak{g})$ as it is described in Section 7.

Now let $\mathfrak{h}\subset\mathfrak{g}$ be a DG ideal such that $[\mathfrak{g},\mathfrak{h}]=0$. We have natural maps $\pi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ and $\pi_*:MC(\mathfrak{g})\to MC(\mathfrak{g}/\mathfrak{h})$. Then one has the following obstruction theory.

Proposition 16.1. *1) There is a natural map* $o_2: MC(\mathfrak{g}/\mathfrak{h})\to H^2(\mathfrak{h})$ *satisfying the following property: if* $\alpha\in MC(\mathfrak{g}/\mathfrak{h})$, *then the following are equivalent:*

(i) The set $\pi_*^{-1}(\alpha)$ *is non-empty.*

(ii) $o_2(\alpha)=0$.

Moreover, if $\alpha,\beta\in MC(\mathfrak{g}/\mathfrak{h})$ *are equivalent then* $o_2(\alpha)=0$ *iff* $o_2(\beta)=0$.

2) Suppose that $\alpha\in MC(\mathfrak{g}/\mathfrak{h})$ *is such that the set* $(\pi_*)^{-1}(\alpha)$ *is not empty. Then there is a natural simply transitive* $Z^1(\mathfrak{h})$ *-action on the set* $(\pi_*)^{-1}(\alpha)$.

3) Let $\alpha,\beta\in MC(\mathfrak{g}/\mathfrak{h})$ *and* $X\in(\mathfrak{g}/\mathfrak{h})^0$ *be such that* $\exp(X)(\alpha)=\beta$. *Suppose that the set* $(\pi_*)^{-1}(\alpha)$ *(and hence also* $(\pi_*)^{-1}(\beta)$ *) is non-empty. Take a* $Z^1(\mathfrak{h})$*- action on*

$(\pi_*)^{-1}(\beta)$ as in 2) and on $(\pi_*)^{-1}(\alpha)$ inverse to the action in 2). Then there exists a natural $Z^1(\mathfrak{h})$- equivariant map

$$o_1^X : (\pi_*)^{-1}(\alpha) \times (\pi_*)^{-1}(\beta) \to H^1(\mathfrak{h}) \tag{16.1}$$

satisfying the following property: if $\tilde{\alpha} \in (\pi_*)^{-1}(\alpha)$, $\tilde{\beta} \in (\pi_*)^{-1}(\beta)$ then the following are equivalent:

(iii) there exists an element $\tilde{X} \in \mathfrak{g}^0$ such that $\pi(\tilde{X}) = X$ and $\exp(\tilde{X})(\tilde{\alpha}) = \tilde{\beta}$.

(iv) $o_1^X(\alpha, \beta) = 0$.

4) Let $\alpha, \beta \in MC(\mathfrak{g})$, and let $X \in (\mathfrak{g}/\mathfrak{h})^0$ be such that $\exp(X)(\pi_*(\alpha)) = \pi_*(\beta)$. Suppose that the set $(\pi_*)^{-1}(X) = \{\tilde{X} \in \mathfrak{g}^0 \mid \exp(\tilde{X})(\alpha) = \beta\}$ is non-empty. Then there is a natural simply transitive action of $Z^0(\mathfrak{h})$ on the set $(\pi_*)^{-1}(X)$.

Proof. 1) Let $\alpha \in MC(\mathfrak{g}/\mathfrak{h})$. Take some $\tilde{\alpha} \in \mathfrak{g}^1$ such that $\pi(\tilde{\alpha}) = \alpha$. Then it is easy to check that. $\mathcal{F}(\tilde{\alpha}) := \partial\tilde{\alpha} + \frac{1}{2}[\tilde{\alpha}, \tilde{\alpha}] \in Z^2(\mathfrak{g})$. Define $o_2(\alpha)$ to be the class of $\mathcal{F}(\tilde{\alpha})$.

Check that this is well defined. Take some other lift $\tilde{\alpha}' \in \mathfrak{g}^1$ of α. Since $\alpha - \alpha' \in \mathfrak{h}$ is central, we have that $\mathcal{F}(\tilde{\alpha}) - \mathcal{F}(\tilde{\alpha}') = \partial(\tilde{\alpha} - \tilde{\alpha}')$. Therefore, $o_2(\alpha)$ is well defined. Now we prove that (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii). Let $\tilde{\alpha} \in MC(\mathfrak{g})$ be such that $\pi_*(\tilde{\alpha}) = \alpha$. Then $\mathcal{F}(\tilde{\alpha}) = 0$, hence $o_2(\alpha) = 0$.

(ii) $\Rightarrow$ (i). Let $\tilde{\alpha} \in \mathfrak{g}^1$ be such that $\pi(\tilde{\alpha}) = \alpha$. Since $o_2(\alpha) = 0$, there exists $u \in \mathfrak{h}^1$ such that $\mathcal{F}(\tilde{\alpha}) = \partial(u)$. Then $\tilde{\alpha} - u \in MC(\mathfrak{g})$ and $\pi_*(\tilde{\alpha} - u) = \alpha$.

Now, suppose that $\alpha, \beta \in MC(\mathfrak{g}/\mathfrak{h})$ are equivalent, and $X \in (\mathfrak{g}/\mathfrak{h})^0$ is such that $\exp(X)(\alpha) = \beta$. Suppose that $o_2(\alpha) = 0$. Take some lift $\tilde{\alpha} \in MC(\mathfrak{g})$ of α, and a lift $\tilde{X} \in \mathfrak{g}^0$ of X. Then $\exp(\tilde{X})(\tilde{\alpha}) \in MC(\mathfrak{g})$ is a lift of β, hence $o_2(\beta) = 0$. Analogously, vanishing of $o_2(\beta)$ implies vanishing of $o_2(\alpha)$.

2) The desired action is just the translation one. It is obviously simply transitive.

3) Let $\tilde{\alpha} \in (\pi_*)^{-1}(\alpha)$, $\tilde{\beta} \in (\pi_*)^{-1}(\beta)$. Take some lift $\tilde{X} \in \mathfrak{g}^0$ of X. Define $o_1^X(\tilde{\alpha}, \tilde{\beta})$ to be the class of $\tilde{\beta} - \exp(\tilde{X})(\alpha)$ in $H^1(\mathfrak{h})$.

First check that this is well defined. Let $\tilde{X}' \in \mathfrak{g}^0$ be another lift of $\tilde{X}$. Then we have that

$$(\tilde{\beta} - \exp(\tilde{X})(\alpha)) - (\tilde{\beta} - \exp(\tilde{X}')(\alpha)) = \partial(\tilde{X} - \tilde{X}'). \tag{16.2}$$

Therefore, the map o_1^X is well defined. It is clear that it is $Z^1(\mathfrak{h})$- equivariant. Now prove that (iii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (iv). It suffices to choose $\tilde{X} \in \mathfrak{g}^0$ such that $\exp(\tilde{X})(\tilde{\alpha}) = \tilde{\beta}$.

(iv) $\Rightarrow$ (iii). Choose some lift $\tilde{X} \in \mathfrak{g}^0$ of X. Since $o_1^X(\tilde{\alpha}, \tilde{\beta}) = 0$, there exists $u \in \mathfrak{h}^0$ such that $\tilde{\beta} - \exp(\tilde{X})(\alpha) = \partial u$. Then $\exp(\tilde{X} - u)(\alpha) = \beta$ and $\pi(\tilde{X} - u) = X$.

4) The desired action is just the translation one. It is obviously simply transitive. $\square$

Let $\mathfrak{g}$ be some pro-nilpotent DG Lie algebra.

Now we recall the notion of homotopy between two exponents in $\exp(\mathfrak{g}^0)$. If $\alpha \in \mathfrak{g}^1$ is an MC solution, then we have the deformed differential $\partial_\alpha(u) = \partial u + [\alpha, u]$.

Definition 16.2. *Let $\alpha, \alpha' \in \mathfrak{g}^1$ be MC solutions, and let $X, Y \in \mathfrak{g}^0$ be such that $\exp(X) \cdot \alpha = \exp(Y) \cdot \alpha = \alpha'$. Then an element $H \in \mathfrak{g}^{-1}$ is called a homotopy between X and Y if*

$$\exp(Y) = \exp(X) \exp(\partial_\alpha H). \tag{16.3}$$

It is clear that for each X and u as in definition there exists precisely one $Y \in \mathfrak{g}^0$ such that u is a homotopy between X and Y.

Now we prove the special case of Lemma 7.6.

Proposition 16.3. *Let $\Phi : \mathfrak{g} \to \mathfrak{h}$ be a DG filtered quasi-isomorphism of pro-nilpotent DG Lie algebras. Then the induced map $\Phi_* : MC(\mathfrak{g})/\exp(\mathfrak{g}^0) \to MC(\mathfrak{h})/\exp(\mathfrak{h}^0)$ is a bijection.*

Proof. 1) First we prove that the induced map on the equivalence classes of MC solutions is *surjective.* Take some $r \geq 1$. Denote by $\pi_1 : \mathfrak{g}/L_{r+1}\mathfrak{g} \to \mathfrak{g}/L_r\mathfrak{g}$, $\pi_2 : \mathfrak{h}/L_{r+1}\mathfrak{h} \to \mathfrak{g}/L_r\mathfrak{h}$ the natural projections. Clearly, it suffices to prove the following

Lemma 16.4. *Take some $\alpha \in MC(\mathfrak{g}/L_r\mathfrak{g})$. Suppose that there exists $\beta \in MC(\mathfrak{h}/L_{r+1}\mathfrak{h})$ such that $\pi_{2*}(\beta) = \Phi_*(\alpha)$. Then there exists $\tilde{\alpha} \in MC(\mathfrak{g}/L_{r+1}\mathfrak{g})$ and $X \in L_r\mathfrak{h}^0/L_{r+1}\mathfrak{h}^0$, such that $\pi_{1*}(\tilde{\alpha}) = \alpha$, and*

$$\Phi_*(\tilde{\alpha}) = \exp(X)(\beta) = \beta - \partial X. \tag{16.4}$$

Proof. First, we have that $o_2(\Phi_*(\alpha)) = \Phi(o_2(\alpha))$. Since $\pi_{2*}(\beta) = \Phi_*(\alpha)$, we have by Proposition 16.1 that $o_2(\Phi_*(\alpha)) = 0$. Since Φ is filtered quasi-isomorphism, we have that $o_2(\alpha) = 0$. Therefore, by Proposition 16.1, there exists some $\tilde{\alpha} \in MC(\mathfrak{g}/L_{r+1}\mathfrak{g})$, such that $\pi_{1*}(\tilde{\alpha}) = \alpha$.

Let $u \in Z^1(L_r\mathfrak{g}/L_{r+1}\mathfrak{g})$. Then we have that $o_1^0(\Phi_*(\tilde{\alpha} + u), \beta) = o_1(\Phi_*(\tilde{\alpha}), \beta) - \Phi^1(u)$. Again, since Φ is filtered quasi-isomorphism, we can choose u in such a way that $o_1^0(\Phi_*(\tilde{\alpha}+u), \beta) = 0$. In this case, by Proposition 16.1 3), we have that there exists $X \in L_r\mathfrak{h}^0/L_{r+1}\mathfrak{h}^0$, such that $\Phi_*(\tilde{\alpha}) = \exp(X)(\beta)$. Lemma is proved. □

Surjectivity is proved.

2) Now, prove that our map is injective. Take some $r \geq 1$. Denote by $\pi_1 : \mathfrak{g}/L_{r+1}\mathfrak{g} \to \mathfrak{g}/L_r\mathfrak{g}$, $\pi_2 : \mathfrak{h}/L_{r+1}\mathfrak{h} \to \mathfrak{g}/L_r\mathfrak{h}$ the natural projections. Clearly, it suffices to prove the following

Lemma 16.5. *Let* $\alpha, \beta \in MC(\mathfrak{g}/L_{r+1}\mathfrak{g})$, $X \in (\mathfrak{g}/L_r\mathfrak{g})^0$, *and* $Y \in MC(\mathfrak{h}/L_{r+1}\mathfrak{h})$ *be such that* $\exp(Y)(\Phi_*(\alpha)) = \Phi_*(\beta)$, $\exp(X)(\pi_{1*}(\alpha)) = \pi_{1*}(\beta)$, *and* $\Phi(X) = \pi_2(Y)$. *Then there exists some* $\tilde{X} \in (\mathfrak{g}/L_{r+1}\mathfrak{g})^0$ *such that*

$$\pi_1(\tilde{X}) = X, \quad \exp(\tilde{X})(\alpha) = \beta, \tag{16.5}$$

and $\Phi(\tilde{X})$ *is homotopic to* Y *(as a homotopy between* $\Phi_*(\alpha)$ *and* $\Phi_*(\beta)$ *).*

Proof. First, we have that $\Phi(o_1^X(\alpha, \beta)) = o_1^{\Phi(X)}(\Phi_*(\alpha), \Phi_*(\beta))$. By Proposition 16.1 3), we have that $o_1^{\Phi(X)}(\Phi_*(\alpha), \Phi_*(\beta)) = 0$. Since Φ is filtered quasi-isomorphism, we have that $o_1^X(\alpha, \beta) = 0$. Therefore, by Proposition 16.1 3), there exists some $\tilde{X} \in (\mathfrak{g}/L_{r+1}\mathfrak{g})^0$ such that (16.5) holds.It follows from Proposition 16.1 4) and surjectivity of the map $H^0(L_r\mathfrak{g}/L_{r+1}\mathfrak{g}) \to H^0(L_r\mathfrak{h}/L_{r+1}\mathfrak{h})$, that $\tilde{X}$ can be choosen in such a way that $Y - \Phi(\tilde{X}) = \partial u$ for some $u \in (L_r\mathfrak{h}/L_{r+1}\mathfrak{h})^{-1}$. Then u is a homotopy between $\Phi(\tilde{X})$ and Y. Lemma is proved. □

Injectivity is proved. □

To prove Lemma 7.6, we need first to modify the notion of homotopy between MC solutions (so that it generalizes naturally to pro-nilpotent L_∞- algebras). Denote by Ω_1 the commutative DG algebra of polynomial differential form on the affine line. Denote by t the parameter on the line. If $\mathfrak{g}$ is a DG Lie algebra, then $\mathfrak{g} \otimes \Omega_1$ is also a DG Lie algebra.

In the case when $\mathfrak{g}$ is pro-nilpotent, we may and will consider the completed tensor product:

$$\mathfrak{g}\hat{\otimes}\Omega_1 := \lim_{\leftarrow}(\mathfrak{g}/L_r\mathfrak{g}) \otimes \Omega_1. \tag{16.6}$$

This is also naturally filtered pro-nilpotent DG Lie algebra. We have natural inclusion $\iota : \mathfrak{g} \to \mathfrak{g}\hat{\otimes}\Omega_1$ which is a filtered quasi-isomorphism. Further, for each $t_0 \in \mathrm{k}$ we have the evaluation morphism $\mathrm{ev}_{t_0} : \mathfrak{g}\hat{\otimes}\Omega_1 \to \mathfrak{g}$, which is left inverse to ι, and hence is also filtered quasi-isomorphism.

Proposition 16.6. *Let* $\mathfrak{g}$ *be a pro-nilpotent DG Lie algebra. Take some* $\alpha, \beta \in MC(\mathfrak{g})$. *Then the following are equivalent:*

(i) α *and* β *are homotopic.*

(ii)There exists some $A \in MC(\mathfrak{g}\hat{\otimes}\Omega_1)$ *such that* $\mathrm{ev}_{0*}(A) = \alpha$ *and* $\mathrm{ev}_{1*}(A) = \beta$.

Proof. (ii) $\Rightarrow$ (i). From Proposition 16.3 we deduce that A is homotopic both to $\iota_*(\alpha)$ and $\iota_*(\beta)$. Again by Proposition 16.3, we have that α and β are homotopic.

(i) $\Rightarrow$ (ii). Take $X \in \mathfrak{g}^0$ such that $\exp(X)(\alpha) = \beta$. Then it suffices to put

$$A = \exp(tX)(\alpha) + X \otimes dt. \tag{16.7}$$

□

From this moment, by a homotopy between MC solutions α,β in the pro-nilpotent DGLA $\mathfrak{g}$ we mean an MC solution $A\in\mathfrak{g}\hat{\otimes}\Omega_1$ such that $\mathrm{ev}_{0*}(A)=\alpha$ and $\mathrm{ev}_{1*}(A)=\beta$.

We also modify the notion of homotopy between homotopies.

Definition 16.7. *Let $A,B\in MC(\mathfrak{g}\hat{\otimes}\Omega_1)$ be homotopies between $\alpha,\beta\in MC(\mathfrak{g})$. We call A and B homotopic if*

$$B=\exp(\partial_\beta(u)t)\exp(t(1-t)X)(A), \tag{16.8}$$

where $X\in\mathfrak{g}^0\hat{\otimes}\Omega_1^0$, and $u\in\mathfrak{g}^{-1}$.

We need to adapt the obstruction theory for our modified homotopies.

Proposition 16.8. *Let $\mathfrak{g}$ be a nilpotent DGLA, and $\mathfrak{h}\subset\mathfrak{g}$ be its central DG ideal, and $\pi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ the natural projection. Let $\alpha,\beta\in MC(\mathfrak{g})$.*

1) There is a natural map $A\mapsto o_1^A(\alpha,\beta)$ which assigns to each homotopy $A\in MC((\mathfrak{g}/\mathfrak{h})\otimes\Omega_1)$ between $\pi_*(\alpha)$ and $\pi_*(\beta)$, an element $o_1^A(\alpha,\beta)\in H^1(\mathfrak{h})$, such that the following are equivalent:

(i) There exists a homotopy $\tilde{A}\in MC(\mathfrak{g}\otimes\Omega_1)$ between α and β such that $\pi_*(\tilde{A})=A$.

(ii) $o_1^A(\alpha,\beta)=0$.

2) Suppose that A is the homotopy between $\pi_*(\alpha)$ and $\pi_*(\beta)$. Then there is a natural transitive action of $H^0(\mathfrak{g})$ on homotopy classes of elements in the set $(\pi_*)^{-1}(A)$. Here $(\pi_*)^{-1}(A)$ is the set of homotopies $\tilde{A}$ between α and β such that $\pi_*(\tilde{A})=A$.

Proof. 1) Take some element $\tilde{A}\in(\mathfrak{g}\otimes\Omega_1)^1$ such that $\pi(\tilde{A})=A$, $\mathrm{ev}_0(\tilde{A})=\alpha$, $\mathrm{ev}_1(\tilde{A})=\beta$. Put $\mathcal{F}(\tilde{A})=\partial\tilde{A}+\frac{1}{2}[\tilde{A},\tilde{A}]$. Then $\mathcal{F}(\tilde{A})$ is a cocycle in the complex $\mathfrak{h}\otimes L^{\cdot}\subset\mathfrak{g}\otimes\Omega_1$, where

$$L^{\cdot}\subset\Omega_1,\quad L^0=t(1-t)\Omega_1^0,\quad L^1=\Omega_1^1. \tag{16.9}$$

Clearly, we have that $H^0(L^{\cdot})=0$, $H^1(L^{\cdot})=\mathrm{k}$, and the natural projection $L^1\to\mathrm{k}$ is given by the formula

$$\sum_{i=0}^N a_i t^i dt\mapsto\sum_{i=0}^N\frac{a_i}{i+1}. \tag{16.10}$$

We define $o_1^A(\alpha,\beta)$ to be the class of $\mathcal{F}(\tilde{A})$ in $H^1(\mathfrak{h})\cong H^2(\mathfrak{h}\otimes L^{\cdot})$. The checking of correctness and equivalence (i)$\Leftrightarrow$(ii) is analogous to that of Proposition 16.1 1).

2) Suppose that the set $(\pi_*)^{-1}(A)$ is non-empty (otherwise there is nothing to prove). It is clear from the proof of 1) that there is a simply transitive translation action of the group

$Z^1(\mathfrak{h}\otimes L^\cdot)$ on the set $(\pi_*)^{-1}(A)$. Further, any coboundary b in $\mathfrak{h}\otimes L^\cdot$ can be represented as $\partial(\iota(\partial u)t+X)$, where $X\in\mathfrak{h}\otimes L^0$ and $u\in\mathfrak{h}$. Thus, we have that

$$\tilde{A}+b=\exp(\partial_\beta(u)t)\exp(X)(\tilde{A}) \tag{16.11}$$

— homotopic to $\tilde{A}$. Therefore, we have the desired transitive action of $H^0(\mathfrak{h})$. □

Now we are able to prove Lemma 7.6.

Proof of Lemma 7.6. Proof of surjectivity is the same as in Proposition 16.3.

Now we prove the injectivity. Take some $r\geq 1$. Denote by $\pi_1:\mathfrak{g}/L_{r+1}\mathfrak{g}\to\mathfrak{g}/L_r\mathfrak{g}$, $\pi_2:\mathfrak{h}/L_{r+1}\mathfrak{h}\to\mathfrak{g}/L_r\mathfrak{h}$ the natural projections. It suffices to prove the following

Lemma 16.9. *Let $\alpha,\beta\in MC(\mathfrak{g}/L_{r+1}\mathfrak{g})$, $A\in MC((\mathfrak{g}/L_r\mathfrak{g})\otimes\Omega_1)$, and $B\in MC((\mathfrak{h}/L_{r+1}\mathfrak{h})\otimes\Omega_1)$ be such that B is the homotopy between $\Phi_*(\alpha)$ and $\Phi_*(\beta)$, A is the homotopy between $\pi_{1*}(\alpha)$ and $\pi_{1*}(\beta)$, and $\Phi_*(A)=\pi_2(B)$. Then there exists some homotopy $\tilde{A}\in MC((\mathfrak{g}/L_r\mathfrak{g})\otimes\Omega_1)$ between α and β such that $\pi_{1*}(\tilde{A})=A$, and $\Phi_*(\tilde{A})$ is homotopic to B.*

Proof. This follows from Proposition 16.8, analogously to Lemma 16.5. □

Lemma is proved □

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