THE CLASSES OF THE QUASIHOMOGENEOUS HILBERT SCHEMES OF POINTS ON THE PLANE

A. BURYAK

Abstract. In this paper we give a formula for the classes (in the Grothendieck ring of complex quasi-projective varieties) of irreducible components of $(1, k)$-quasi-homogeneous Hilbert schemes of points on the plane. We formulate a conjecture about the generating function of the classes of (a, b)-quasi-homogeneous Hilbert schemes. Finally, we investigate connections between $(1, k)$-quasi-homogeneous Hilbert schemes and homogeneous nested Hilbert schemes.

1. Introduction

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in the plane $\mathbb{C}^2$ parametrizes the ideals $I \subset \mathbb{C}[x, y]$ of colength n, $dim_{\mathbb{C}}\mathbb{C}[x, y]/I = n$. There is an open dense subset of $(\mathbb{C}^2)^{[n]}$ that parametrizes the ideals associated with configurations of n distinct points. The Hilbert scheme of n points in the plane is a nonsingular, irreducible, quasiprojective algebraic variety of dimension $2n$ with a rich and much studied geometry, see [9, 15] for an introduction.

The cohomology groups of $(\mathbb{C}^2)^{[n]}$ were calculated in [6] and we refer the reader to the papers [5, 12, 13, 14, 16] for the description of the ring structure in the cohomology $H^*((\mathbb{C}^2)^{[n]})$. Let $\overline{n} = (n_1, \ldots, n_k)$. The nested Hilbert scheme $(\mathbb{C}^2)^{[\overline{n}]}$ parametrizes k-tuples $(I_1, I_2, \ldots, I_k)$ of ideals $I_j \subset \mathbb{C}[x, y]$ such that $I_j \subset I_h$ for $j < h$ and $dim_{\mathbb{C}}\mathbb{C}[x, y]/I_j = n_j$. In [4] J. Cheah studied smoothness and the homology groups of the nested Hilbert scheme $(\mathbb{C}^2)^{[\overline{n}]}$.

There is a $(\mathbb{C}^*)^2$-action on $(\mathbb{C}^2)^{[n]}$ that plays the central role in this subject. The algebraic torus $T = (\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by scaling the coordinates, $(t_1, t_2)(x, y) = (t_1x, t_2y)$. This action lifts to the T-action on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$.

Let $T_{a,b} = \{(t^a, t^b) \in T | t \in \mathbb{C}^*\}$, where $a, b \geq 1$ and $gcd(a, b) = 1$, be a one dimensional subtorus. Let $(\mathbb{C}^2)^{[n]}_{a,b}$ be the set of fixed points of the $T_{a,b}$-action on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$. Obviously, the variety $(\mathbb{C}^2)^{[n]}_{a,b}$ is smooth and parameterizes quasi-homogeneous ideals of colength n in the ring $\mathbb{C}[x, y]$. Irreducible components of $(\mathbb{C}^2)^{[n]}_{1,1}$ were described

The author is partially supported by the grant RFBR-10-01-00678, the Vidi grant of NWO and by the Moebius Contest Foundation for Young Scientists.

in [10] and the description of the irreducible components of $(\mathbb{C}^2)^{[n]}_{a,b}$ for arbitrary a and b was obtained in [7].

We denote by $K_0(\nu_{\mathbb{C}})$ the Grothendieck ring of complex quasiprojective varieties. The classes of irreducible components of the Hilbert scheme $(\mathbb{C}^2)^{[n]}_{1,1}$ in $K_0(\nu_{\mathbb{C}})$ were computed in [11].

Let $(\mathbb{C}^2)^{[\overline{n}]}_{a,b}$ be the set of fixed points of the $T_{a,b}$-action on the nested Hilbert scheme $(\mathbb{C}^2)^{[\overline{n}]}$. The dimensions of irreducible components of $(\mathbb{C}^2)^{[(n,n+1)]}_{1,1}$ were computed in [4].

This paper has two parts. In the first part we generalize the result of [11] and give a formula for the classes in $K_0(\nu_{\mathbb{C}})$ of the irreducible components of the variety $(\mathbb{C}^2)^{[n]}_{1,k}$ for an arbitrary positive k. We discuss the problem of existence of a similar formula for the components of $(\mathbb{C}^2)^{[n]}_{a,b}$ for arbitrary a and b. We formulate the conjectural formula for the generating function of the classes $\left[(\mathbb{C}^2)^{[n]}_{a,b}\right]$.

In the second part of the paper we construct a natural map $\pi\colon (\mathbb{C}^2)^{[n]}_{1,k} \to (\mathbb{C}^2)^{[\overline{n}]}_{1,1}$. We find the sufficient condition for the restriction of this map to an irreducible component to be an isomorphism. In particular, this condition is satisfied when $\overline{n} = (n+1, n)$. Hence, we generalize the result from [4], where the dimensions of the irreducible components in this case were computed.

1.1. **Grothendieck ring.**

Here we recall a definition of the Grothendieck ring $K_0(\nu_{\mathbb{C}})$ of complex quasi-projective varieties. It is an abelian group generated by the classes $[X]$ of all complex quasi-projective varieties X modulo the relations:

1. if varieties X and Y are isomorphic, then $[X] = [Y]$;
2. if Y is a Zariski closed subvariety of X, then $[X] = [Y] + [X\backslash Y]$.

The multiplication in $K_0(\nu_{\mathbb{C}})$ is defined by the product of varieties: $[X_1]\cdot[X_2] = [X_1 \times X_2]$. The class $\mathbb{A}^1_{\mathbb{C}} \in K_0(\nu_{\mathbb{C}})$ of the complex affine line is denoted by $\mathbb{L}$.

1.2. **Description of the irreducible components of $(\mathbb{C}^2)^{[n]}_{a,b}$.**

Let us recall a description of the irreducible components of the variety $(\mathbb{C}^2)^{[n]}_{a,b}$. Let $\mathbb{C}[x,y]^d_{a,b} \subset \mathbb{C}[x,y]$ be the subspace of quasihomogeneous polynomials of degree d with respect to the action of $T_{a,b}$. For an ideal $I \subset \mathbb{C}[x,y]$ let $I^d_{a,b} = I \cap \mathbb{C}[x,y]^d_{a,b}$. Let $H = (d_0, d_1, \ldots)$ be a sequence of non-negative integers such that $\sum_{i\geq 0} d_i = n$. Let $(\mathbb{C}^2)^{[n]}_{a,b}(H) \subset (\mathbb{C}^2)^{[n]}_{a,b}$ be the set of points corresponding to quasihomogeneous ideals $I \subset \mathbb{C}[x,y]$ such that $dim(\mathbb{C}[x,y]^i_{a,b}/I^i_{a,b}) = d_i$.

Proposition 1.1 ([7]). *If $(\mathbb{C}^2)^{[n]}_{a,b}(H) \neq \emptyset$ then $(\mathbb{C}^2)^{[n]}_{a,b}(H)$ is an irreducible component of $(\mathbb{C}^2)^{[n]}_{a,b}$.*

1.3. **Classes of the irreducible components of $(\mathbb{C}^2)^{[n]}_{1,k}$.** In this section we fix $k \geq 1$. For numbers $M, N \geq 0$ let $G(M,N)_q = \frac{\prod_{i=1}^{M+N}(1-q^i)}{\prod_{i=1}^{M}(1-q^i)\prod_{i=1}^{N}(1-q^i)}$. Let $\eta(H)$ be the biggest l, such that $d_l = |\{(i,j) \in \mathbb{Z}^2_{\geq 0} | i + kj = l\}|$. We adopt the following conventions, $\eta(H) = -1$, if $H = 0$; $d_i = 0$, if $-k \leq i \leq -1$ and $d_{-k-1} = -1$. We introduce an auxiliary function τ defined by the following rule, $\tau(i) = 1$, if $k \mid i+1$ and $\tau(i) = 0$, if $k \nmid i+1$. The main result of the paper is the following theorem.

Theorem 1.2. *Let $H = (d_0, d_1, \ldots), n = \sum\limits_{i \geq 0} d_i$. If $(\mathbb{C}^2)^{[n]}_{1,k}(H) \neq \emptyset$, then*

$$\left[(\mathbb{C}^2)^{[n]}_{1,k}(H)\right] = \frac{1-\mathbb{L}}{1-\mathbb{L}^{d_{\eta-k+1}+1-d_{\eta+1}}} \prod_{i \geq \eta+1} G(d_i - d_{i+1} + \tau(i), d_{i-k} - d_i)_{\mathbb{L}}.$$

Remark 1.3. *We see that the classes of the irreducible components of $(\mathbb{C}^2)^{[n]}_{1,k}$ are polynomials in $\mathbb{L}$. Moreover, all roots of these polynomials are the roots of unity. In the case of an arbitrary (a,b), it is not true. For example, it is easy to compute that*

$$(\mathbb{C}^2)^{[12]}_{2,3}(1,0,1,1,1,1,2,1,1,1,1,0,1) = 1 + 3\mathbb{L} + \mathbb{L}^2.$$

1.4. **Conjecture.** The following conjectural formula for the generating function of the classes $\left[(\mathbb{C}^2)^{[n]}_{a,b}\right]$ is based on computer calculations.

Conjecture 1.4.

$$\sum_{n \geq 0} \left[(\mathbb{C}^2)^{[n]}_{a,b}\right] t^n = \prod_{\substack{i \geq 1 \\ (a+b) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1-\mathbb{L}t^{(a+b)i}}.$$

Similar conjectural formulas for the generating functions of the classes of some equivariant Hilbert schemes can be found in [8].

1.5. **Homogeneous nested Hilbert schemes $(\mathbb{C}^2)^{[\overline{n}]}_{1,1}$.** Let $\overline{n} = (n_1, n_2, \ldots, n_k)$, where $n_1, \ldots, n_k$ are non-negative integers such that $n_1 \geq n_2 \geq \ldots \geq n_k$. Let $\overline{H} = (H_1, H_2, \ldots, H_k)$, where $H_i = (d_{i,0}, d_{i,1}, \ldots)$ and $\sum_{j \geq 0} d_{i,j} = n_i$. Let $(\mathbb{C}^2)^{[\overline{n}]}_{a,b}(\overline{H}) = \{(Z_1, \ldots, Z_k) \in (\mathbb{C}^2)^{[\overline{n}]} | Z_i \in (\mathbb{C}^2)^{[n_i]}_{a,b}(H_i)\}$. Let $E(\overline{H}) = \{i \in \mathbb{Z}_{\geq 0} | d_{1,i} = d_{2,i} = \ldots = d_{k,i}\}$. Let $n = \sum_{i=1}^k n_i$ and $H = (d_0, d_1, \ldots)$, where $d_{i+kj} = d_{i+1,j}, 0 \leq i < k, j \geq 0$. We will prove the following theorem.

Theorem 1.5. *Suppose that for any two number $i, j \in \mathbb{Z}_{\geq 0} \backslash E(\overline{H}), i < j$, we have $j - i \geq 2$. Then the variety $(\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H})$ is isomorphic to $(\mathbb{C}^2)^{[n]}_{1,k}(H)$.*

1.6. **Organization of the paper.** In section 2 we construct a cellular decomposition of the quasihomogeneous Hilbert scheme and reduce Theorem 1.2 to a combinatorial identity. In section 3 we construct a bijection that is a generalization of the hook code from [11]. The main result of this section is Proposition 3.8. Finally, we apply it in section 4 to conclude the proof of Theorem 1.2. Section 5 contains the proof of Theorem 1.5.

1.7. **Acknowledgments.** The author is grateful to S. M. Gusein-Zade for suggesting the area of research. The author is grateful to B. L. Feigin and A. N. Kirillov who noticed that the main result from [11] can be generalized to $(1,k)$-case and suggested the author to work on this problem. The author is grateful to S. Shadrin, M. Kazarian, S. Lando and A. Oblomkov for useful discussions.

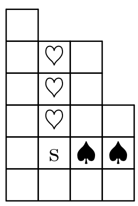
2. Cellular decomposition of $(\mathbb{C}^2)^{[n]}_{1,k}$

In this section we reduce Theorem 1.2 to the combinatorial identity (1) using the cellular decomposition of $(\mathbb{C}^2)^{[n]}_{1,k}$.

Consider the T-action on $(\mathbb{C}^2)^{[n]}$. The fixed points of this action correspond to the monomial ideals in $\mathbb{C}[x,y]$. Let $I \subset \mathbb{C}[x,y]$ be a monomial ideal of length n. Let $D_I = \{(i,j) \in \mathbb{Z}^2_{\geq 0} | x^i y^j \notin I\}$ be the corresponding Young diagram. We will use the following notations. For a Young diagram D let

$$r_l(D) = |\{(i,j) \in D | j = l\}|,$$
$$c_l(D) = |\{(i,j) \in D | i = l\}|,$$
$$diag^{a,b}_l(D) = |\{(i,j) \in D | ai + bj = l\}|,$$
$$diag^{a,b}(D) = (diag^{a,b}_0(D), diag^{a,b}_1(D), diag^{a,b}_2(D), \ldots).$$

For a box $s = (i,j) \in D$ let $a(s) = c_i(D) - j - 1$ and $l(s) = r_j(D) - i - 1$ (see Figure 1).



$l(s)$ = number of ♠
$a(s)$ = number of ♡

Figure 1

Let $p \in (\mathbb{C}^2)^{[n]}$ be the fixed point corresponding to a Young diagram D. Let $R(T) = \mathbb{Z}[t_1, t_2]$ be the representation ring of T. Then the weight decomposition of $T_p(\mathbb{C}^2)^{[n]}$ is given by (see [6])

$$T_p(\mathbb{C}^2)^{[n]} = \sum_{s \in D} \left(t_1^{l(s)+1} t_2^{-a(s)} + t_1^{-l(s)} t_2^{a(s)+1} \right).$$

Obviously, the variety $(\mathbb{C}^2)^{[n]}_{1,k}$ is invariant under the T-action and contains all fixed points of the T-action on $(\mathbb{C}^2)^{[n]}$. Hence, the weight decomposition of $T_p(\mathbb{C}^2)^{[n]}_{1,k}$ is given by

$$T_p(\mathbb{C}^2)^{[n]}_{1,k} = \sum_{\substack{s\in D\\ l(s)+1=ka(s)}} t_1^{l(s)+1}t_2^{-a(s)} + \sum_{\substack{s\in D\\ l(s)=k(a(s)+1)}} t_1^{-l(s)}t_2^{a(s)+1}.$$

Consider the $T_{\alpha,\beta}$-action on $(\mathbb{C}^2)^{[n]}_{1,k}$, where α,β are positive integers such that $\alpha << \beta$. If α and β are general enough then the set of fixed points of the $T_{\alpha,\beta}$-action coincides with the set of fixed points of the T-action. For a fixed point $p\in(\mathbb{C}^2)^{[n]}_{1,k}$ let $C_p=\{z\in(\mathbb{C}^2)^{[n]}_{1,k}|\lim_{t\to 0}tz=p\}$. The set C_p is isomorphic to $\mathbb{C}^N$ and $(\mathbb{C}^2)^{[n]}_{1,k}$ has a cellular decomposition with cells C_p (see [2, 3]). If a point p corresponds to a Young diagram D then $C_p\cong\mathbb{C}^{|\{s\in D|l(s)=k(a(s)+1)\}|}$. Moreover, $p\in(\mathbb{C}^2)^{[n]}_{1,k}(H)\Leftrightarrow diag^{1,k}(D)=H$, where $H=(d_0,d_1,\ldots)$ is an arbitrary sequence of non-negative integers.

Let $\mathcal{D}$ be the set of Young diagrams. We see that

$$\left[(\mathbb{C}^2)^{[n]}_{1,k}(H)\right] = \sum_{\substack{D\in\mathcal{D}\\ diag^{1,k}(D)=H}} \mathbb{L}^{|\{s\in D|l(s)=k(a(s)+1)\}|}.$$

Therefore, Theorem 1.2 follows from the combinatorial identity:

$$\sum_{\substack{D\in\mathcal{D}\\ diag^{1,k}(D)=H}} q^{|\{s\in D|l(s)=k(a(s)+1)\}|} = \tag{1}$$

$$= \frac{1-q}{1-q^{d_{\eta-k+1}+1-d_{\eta+1}}}\prod_{i\ge\eta+1}G(d_i-d_{i+1}+\tau(i),d_{i-k}-d_i)_q.$$

3. Bijection

This section shows how to encode an element of the set $\{D\in\mathcal{D}|diag^{1,k}(D)=H\}$ as a sequence of partitions. The main result of this section is Proposition 3.8. In section 3.1 we define a map F from the set $\{D\in\mathcal{D}|diag^{1,k}(D)=H\}$ to the set of sequences $(P_0,P_1,\ldots)$, where P_i are Young diagrams. In section 3.2 we prove the main properties of the map F. In section 3.3 we prove an injectivity of the map F and in section 3.4 we describe the image of F.

In this section we fix an arbitrary sequence $H=(d_0,d_1,\ldots)$ of non-negative integers.

3.1. **The definition of the map F.** For a Young diagram D let

$$B_m(D)=\{j\in\mathbb{Z}_{\ge 0}|r_j(D)\neq 0, kj+r_j(D)-1=m\},$$
$$h_m(D)=|\{s=(i,j)\in D|j=m, l(s)=k(a(s)+1)\}|.$$

Let $\lambda(D,m)$ be the partition with numbers $h_j(D)$, where $j\in B_m(D)$.

For a partition $\lambda = \lambda_0, \ldots, \lambda_r, \lambda_0 \geq \ldots \geq \lambda_r$ let $D_\lambda = \{(i,j) \in \mathbb{Z}^2_{\geq 0} | i \leq r, j \leq \lambda_i - 1\}$ be the corresponding Young diagram. Let $\theta(H)$ be the maximal $i \leq \eta(H)$ such that $i \equiv k-1 \mod k$.

Definition 3.1. *Let D be a Young diagram such that $diag^{1,k}(D) = H$. We denote by $F(D)$ a sequence of Young diagrams $(F(D)_0, F(D)_1, \ldots)$ such that $F(D)_i = D_{\lambda(D,i+\theta)}$.*

3.2. **The main properties of F.** We use the following notations.

$$w_i(H) = d_{i-k+\theta} - d_{i+\theta} + 1,$$

$$f_i(H) = \begin{cases} d_{i+\theta} - d_{i+1+\theta}, & \text{if } k \nmid i, \\ d_{i+\theta} - d_{i+1+\theta} + 1, & \text{if } k \mid i. \end{cases}$$

We denote by $R(M,N)$ the square in the integral lattice defined by $R(M,N) = \{(i,j) \in \mathbb{Z}^2_{\geq 0} | i \leq M-1, j \leq N-1\}$. We denote by $\mathcal{D}(M,N)$ the set $\{D \in \mathcal{D} | D \subset R(M,N)\}$.

Lemma 3.2. *The Young diagram $F(D)_i$ lies in the square $R(f_i, w_i)$.*

Proof. Consider a point $(i,j) \in D$. Let $i + kj = l$. Suppose $k \nmid l$, then $(i-1,j) \in D$ and $j \notin B_{l-1}(D)$. Hence, $|B_{l-1}(D)| = d_{l-1} - d_l$. Suppose $k \mid l$. If $i \neq 0$, then $(i-1,j) \in D$ and $j \notin B_{l-1}(D)$. Hence, $|B_{l-1}(D)| \leq d_{l-1} - d_l + 1$. Thus, we prove that $r_0(F(D)_i) \leq f_i$.

Consider a number $a \in B_l(D)$. Let $d'_m = |\{(i,j) \in D | j \geq a, i + kj = m\}|$. Clearly, $h_a = d'_{l-k} - d'_l + 1 \leq d_{l-k} - d_l + 1$. Thus, we prove that $c_0(F(D)_i) \leq w_i$. □

In the following lemma we state the important properties of the numbers $w_i(H)$ and $f_i(H)$.

Lemma 3.3. *The set $\{D \in \mathcal{D} | diag^{1,k}(D) = H\}$ is not empty if and only if for any $i > \eta - \theta$ the following condition holds: $f_i \geq 0, w_i \geq 1$.*

Proof. It is easy to check that the set $\{D \in \mathcal{D} | diag^{1,k}(D) = H\}$ is not empty if and only if for any $i > \eta$ the following three conditions hold: 1) $d_i \leq d_{i-k}$; 2) if $k \nmid i$, then $d_i \leq d_{i-1}$; 3) if $k \mid i$, then $d_i \leq d_{i-1} + 1$. These conditions are equivalent to the condition from the lemma. □

Consider a sequence of Young diagrams $P = (P_0, P_1, \ldots)$ such that $P_i \in \mathcal{D}(f_i, w_i)$ (a short notation for that will be $P \in \prod_{i \geq 0} \mathcal{D}(f_i, w_i)$). Let $\nu(P)$ be the maximal i such that $c_0(P_i) = w_i$. The number $\nu(P)$ is well-defined since $w_0 = 0$, but it can be equal to ∞. It is easy to see that if $P = F(D)$, then $\nu(P) < \infty$.

Lemma 3.4. *Let D be a Young diagram such that $diag^{1,k}(D) = H$. Then $r_0(D) = \theta(H) + \nu(F(D)) + 1$.*

Proof. Consider a number $a \in B_l(D)$. Suppose $h_a(D) = d_{l-k} - d_l + 1$, then for any $0 \leq j \leq a$ we have $(r_a(D) - 1 + kj, a - j) \in D$. In particular, $(0,l) \in D$, hence $r_0(D) \geq l+1$. On the other hand, $h_0(D) = d_{r_0(D)-1-k} - d_{r_0(D)-1} + 1$. This completes the proof of the lemma. □

For a Young diagram D let $D(a,b)=\{(i,j)\in\mathbb{Z}^2_{\geq 0}|(i+a,j+b)\in D\}$. Consider an arbitrary Young diagram D such that $diag^{1,k}(D)=H$. Let $D'=D(0,1), H'=diag^{1,k}(D'), F(D)=(P_0,P_1,\ldots), F(D')=(P'_0,P'_1,\ldots),\ f'_i=f_i(H'), w'_i=w_i(H'), \theta'=\theta(H'), \nu=\nu(P), \nu'=\nu(P')$.

Lemma 3.5. $d'_i=\begin{cases} d_{i+k}-1, & \text{if } i+k\leq \nu+\theta,\\ d_{i+k}, & \text{if } i+k>\nu+\theta.\end{cases}$

1) If $\nu\geq k$ or $w_k\geq 2$, then

$$\theta'=\theta-k; \qquad P'_i=\begin{cases} P_i, & \text{if } i\neq\nu,\\ P_i(1,0), & \text{if } i=\nu;\end{cases}$$

$$f'_i=\begin{cases} f_i, & \text{if } i\neq\nu,\\ f_i-1, & \text{if } i=\nu;\end{cases} \qquad w'_i=\begin{cases} w_i, & \text{if } i\notin[\nu+1,\nu+k],\\ w_i-1, & \text{if } i\in[\nu+1,\nu+k];\end{cases}$$

2) If $\nu\leq k-1$ and $w_k=1$, then

$$\theta'=\theta; \qquad P'_i=P_i+k;$$

$$f'_i=f_{i+k}; \qquad w'_i=\begin{cases} w_i, & \text{if } i>\nu,\\ w_i-1, & \text{if } i\leq\nu;\end{cases}$$

Proof. The proof is clear from Lemma 3.4 and the definition of the map F. □

3.3. **An injectivity of F.**

Lemma 3.6. *The map $F\colon\{D\in\mathcal{D}|diag^{1,k}(D)=H\}\to\prod_{i\geq 0}\mathcal{D}(f_i,w_i)$ is an injection.*

Proof. The proof is by induction on $|D|$. For $|D|=0$, there is nothing to prove. Assume that $|D|>0$. Using Lemma 3.5, we can reconstruct $F(D')$. By the inductive assumption, we can reconstruct D'. From Lemma 3.4 it follows that $F(D)$ determines $r_0(D)$. The diagram D' and the number $r_0(D)$ determines D. This completes the proof of the lemma. □

3.4. **The image of F.** Consider a sequence $P\in\prod_{i\geq 0}\mathcal{D}(f_i,w_i)$. For a number $i\geq 0$ let $\Phi_P(i)$ be the minimal $j>i$ such that $r_0(P_j)<f_j$. If for any $j>i$ we have $r_0(P_j)=f_j$, then we put $\Phi_P(i)=\infty$.

Lemma 3.7. *Let D be a Young diagram such that $diag^{1,k}(D)=H$, then for any $i\geq 0$ we have $\Phi_{F(D)}(i)-i\leq k$.*

Proof. The proof is by induction on $|D|$. For $|D|=0$, there is nothing to prove. Assume that $|D|>0$. We use the notations from Lemma 3.5. Suppose $\nu>\eta-\theta$ or $\nu=\eta-\theta$ and $f_{\eta-\theta}\geq 2$, then from Lemma 3.5 it follows that for any $i\geq 0$ we have $r_0(P_i)<f_i\Leftrightarrow r_0(P'_i)<f'_i$. Thus, Lemma 3.7 follows from the inductive assumption. Assume that

$\nu = \eta - \theta$ and $f_{\eta-\theta} = 1$. From Lemma 3.5 it follows that we must only prove that $\Phi_P(\eta - \theta) - (\eta - \theta) \le k$. Assume the converse. Clearly, $w_{\eta-\theta+1} = 1$. From the definition of the number ν and the assumption $\Phi_P(\eta - \theta) - (\eta - \theta) > k$ it follows that $f_{\eta-\theta+1} = 0$. Continuing in the same way, we see that $w_{\eta-\theta+1} = w_{\eta-\theta+2} = \ldots = w_{\eta-\theta+k} = 1$ and $f_{\eta-\theta+1} = f_{\eta-\theta+2} = \ldots = f_{\eta-\theta+k} = 0$. Clearly, $w_{\eta-\theta+k+1} = 0$, but this contradicts Lemma 3.3. The lemma is proved. □

Proposition 3.8. *Suppose* $\{D \in \mathcal{D} | diag^{1,k}(D) = H\} \neq \emptyset$*, then the map*

$$F\colon \{D \in \mathcal{D} | diag^{1,k}(D) = H\} \to \left\{ P \in \prod_{i\ge 0} \mathcal{D}(f_i, w_i) \,\middle|\, \begin{smallmatrix} \forall i \ge 0: \\ \Phi_P(i) - i \le k \end{smallmatrix} \right\}.$$

is a bijection such that $|\{s \in D | l(s) = k(a(s)+1)\}| = \sum_{i\ge 0} |F(D)_i|$.

Proof. The second statement of the proposition is clear from the definition of the map F. Let us prove that F is a bijection. We have already proved an injectivity. Let us prove a surjectivity of the map F. The proof is by induction on $n = \sum_{i\ge 0} d_i$. For $n = 0$, there is nothing to prove. Assume that $n \ge 1$. Consider a sequence $P \in \prod_{i\ge 0} \mathcal{D}(f_i, w_i)$ such that for any $i \ge 0$ we have $\Phi_P(i) - i \le k$. Define H' and P' by formulas from Lemma 3.5.

We want to apply the inductive assumption to the sequence H', so we need to check that the set $\{D \in \mathcal{D} | diag^{1,k}(D) = H'\}$ is not empty. If $\nu = \eta - \theta$, then it easily follows from Lemma 3.3. Assume that $\nu > \eta - \theta$. By Lemmas 3.5 and 3.3, we must only prove that for any $\nu < i \le \nu + k$ we have $w_i \ge 2$. Assume the converse. Hence, there exists a number $\nu < i \le \nu + k$ such that $w_i = 1$. Therefore, $\sum_{j=1}^{k} f_{i-j} = 1$. Hence, $\Phi_P(i-k-1) = i$. This contradicts the condition $\Phi_P(i-k-1) - (i-k-1) \le k$. Thus, we prove that $\{D \in \mathcal{D} | diag^{1,k} = H'\} \neq \emptyset$.

By the inductive assumption, there exists a Young diagram D' such that $diag^{1,k}(D') = H'$ and $F(D') = P'$. Let us prove that $r_0(D') \le \nu + \theta + 1$. By Lemma 3.4, it is equivalent to $\nu' + \theta' \le \nu + \theta$ and it follows from Lemma 3.5.

If we add the row of length $\nu + \theta + 1$ to the diagram D', we get the diagram D. Clearly, $F(D) = P$. This completes the proof of the proposition. □

4. Proof of Theorem 1.2

In this section we prove (1) using Proposition 3.8.

We fix a sequence $H = (d_0, d_1, \ldots)$ such that the set $\{D \in \mathcal{D} | diag^{1,k}(D) = H\}$ is not empty. We will use the following well known fact (see e.g. [1])

$$\sum_{D \in \mathcal{D}(M,N)} q^{|D|} = G(M, N).$$

Let $S(H) = \{P \in \prod_{i\geq 0}\mathcal{D}(f_i, w_i) | \forall i \geq 0 : \Phi_P(i) - i \leq k\}$. Using Proposition 3.8 and our notations we see that (1) is equivalent to the following formula

$$\sum_{P\in S(H)} q^{|P|} = \frac{1-q}{1-q^{w_{\eta-\theta+1}}} \prod_{i\geq \eta-\theta+1} G(f_i, w_i - 1), \tag{2}$$

where $|P| = \sum_{i\geq 0} |P_i|$. Let $\sigma(H)$ be the minimal $i \geq 0$ such that for any $j > \theta + i$ we have $d_j = 0$. Let $\psi(H)$ be the maximal $i \leq \sigma(H)$ such that $k \mid i$. For a sequence $P \in S(H)$ let $\phi_P(i)$ be the maximal $j < i$ such that $r_0(P_j) < f_j$. We claim that

$$\sum_{\substack{P\in S(H)\\ \phi_P(\psi+k)=p}} q^{|P|} = q^{\sum_{i=p+1}^{\psi+k-1} f_i} \frac{1-q^{f_p}}{1-q^{w_{\psi+k}}} \left(\sum_{P\in S(H)} q^{|P|}\right), \tag{3}$$

where $\psi \leq p < \psi + k$.

Let us prove (2) and (3) by induction on σ. Suppose $\sigma < k$, then

$$\sum_{P\in S(H)} q^{|P|} = \prod_{i=\eta-\theta+1}^{k-1} G(f_i, w_i) = \frac{1-q}{1-q^{w_{\eta-\theta+1}}} \prod_{i\geq\eta-\theta+1} G(f_i, w_i - 1).$$

Hence, (2) is proved. It is clear that

$$\sum_{\substack{P\in S(H)\\ \phi_P(k)=p}} q^{|P|} = \prod_{i=\eta-\theta+1}^{p} G(f_i - \delta_i^p, w_i) \prod_{i=p+1}^{k-1} q^{f_i} G(f_i, w_i - 1) =$$

$$= q^{\sum_{i=p+1}^{k-1} f_i} \frac{1-q^{f_p}}{1-q^{w_k}} \left(\frac{1-q}{1-q^{w_{\eta-\theta+1}}} \prod_{i\geq\eta-\theta+1} G(f_i, w_i - 1)\right).$$

Therefore, (3) is proved.

Suppose $\sigma \geq k$. For $p > \eta(H)$ let

$$H(p) = (d_0(p), d_1(p), d_2(p), \ldots), \text{where}$$
$$d_i(p) = \begin{cases} d_{kd_{p+1}+i} - d_{p+1}, & \text{if } kd_{p+1} + i \leq p, \\ 0, & \text{if } kd_{p+1} + i > p. \end{cases}$$

If $d_p \geq d_{p+1}$, then $\{D \in \mathcal{D} | diag^{1,k}(D) = H(p)\} \neq \emptyset$. We adopt the following convention, $S(H(p)) = \emptyset$, if $d_p < d_{p+1}$. Note that if $d_p < d_{p+1}$, then $k \mid p + 1$. Let $H' = H(\theta + \sigma - 1)$ and $H'' = H(\theta + \psi - 1)$.

Suppose $\psi = \sigma$, then obviously

$$\sum_{P\in S(H)} q^{|P|} = \left(\sum_{P'\in S(H')} q^{|P'|}\right) G(f_\psi - 1, w_\psi).$$

By the inductive assumption, it is equal to $\frac{1-q}{1-q^{w_{\eta-\theta+1}}}\prod\limits_{i>\eta-\theta}G(f_i,w_i-1)$.
Suppose $\psi<\sigma$, then

$$\sum_{P\in S(H)}q^{|P|}=\left(\sum_{P'\in S(H')}q^{|P'|}\right)G(f_\sigma,w_\sigma)+$$
$$+\sum_{p=\sigma-k}^{\psi-1}\left(\sum_{\substack{P''\in S(H'')\\ \phi_{P''}(\psi)=p}}q^{|P''|}\right)\left(\prod_{i=\psi}^{\sigma-1}q^{f_i}G(f_i,w_i-1)\right)G(f_\sigma-1,w_\sigma).$$

By the inductive assumption, it is equal to

$$\frac{1-q}{1-q^{w_{\eta-\theta+1}}}\left[\prod_{i=\eta-\theta+1}^{\sigma-1}G(f_i,w_i-1)\right]\times$$
$$\times\left(\frac{1-q^{\sum_{i=\psi}^{\sigma-1}f_i}}{1-q}G(f_\sigma,w_\sigma)+\frac{1-q^{\sum_{i=\sigma-k}^{\psi-1}f_i}}{1-q}q^{\sum_{i=\psi}^{\sigma-1}f_i}G(f_\sigma-1,w_\sigma)\right).$$

It is easy to check that it is equal to $\frac{1-q}{1-q^{w_{\eta-\theta+1}}}\prod\limits_{i>\eta-\theta}G(f_i,w_i-1)$. Hence, (2) is proved.

Let us prove (3). Suppose $p>\sigma$, then (3) is trivial because both sides are equal to zero. Suppose $p<\sigma$, then we have

$$\sum_{\substack{P\in S(H)\\ \phi_P(\psi+k)=p}}q^{|P|}=\left(\sum_{\substack{P'\in S(H')\\ \phi_{P'}(\psi+k)=p}}q^{|P'|}\right)q^{f_\sigma}G(f_\sigma,w_\sigma-1).$$

By the inductive assumption, it is equal to $q^{\sum_{i=p+1}^{\psi+k-1}f_i}\frac{1-q^{f_p}}{1-q^{w_{\psi+k}}}\left(\sum_{P\in S(H)}q^{|P|}\right)$.

Suppose $p=\sigma$, then we have

$$\sum_{\substack{P\in S(H)\\ \phi(\psi+k)=\sigma}}q^{|P|}=\left(\sum_{P'\in S(H')}q^{|P'|}\right)G(f_\sigma-1,w_\sigma)+$$
$$+\sum_{u=\sigma-k}^{\psi-1}\left(\sum_{\substack{P''\in S(H'')\\ \phi_{P''}(\psi)=u}}q^{|P''|}\right)\left(\prod_{i=\psi}^{\sigma-1}q^{f_i}G(f_i,w_i-1)\right)G(f_\sigma-1,w_\sigma).$$

By the inductive assumption, it is equal to

$$\frac{1-q}{1-q^{w_{\eta-\theta+1}}}\left[\prod_{i=\eta-\theta+1}^{\sigma-1} G(f_i,w_i-1)\right] G(f_\sigma-1,w_\sigma)\times$$
$$\times\left(\frac{1-q^{\sum_{i=\psi}^{\sigma-1} f_i}}{1-q}+\frac{1-q^{\sum_{i=\sigma-k}^{\psi-1} f_i}}{1-q}q^{\sum_{i=\psi}^{\sigma-1} f_i}\right).$$

It is easy to check that it is equal to $\frac{1-q^{f_\sigma}}{1-q^{w_{\psi+k}}}\left(\sum_{P\in S(H)} q^{|P|}\right)$. Thus, (3) is proved. This completes the proof of the theorem.

5. Homogeneous nested Hilbert schemes

In this section we prove Theorem 1.5. In section 5.1 we recall the quiver descriptions of the varieties $(\mathbb{C}^2)^{[n]}_{1,k}(H)$ and $(\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H})$. In section 5.2 we apply this description to conclude the proof of the theorem.

5.1. **A quiver description.** The variety $(\mathbb{C}^2)^{[n]}$ has the following description (see e.g.[15]).

$$(\mathbb{C}^2)^{[n]}\cong\left\{(B_1,B_2,i)\;\middle|\;\begin{array}{c}1)[B_1,B_2]=0\\ 2)\text{(stability) There is no subspace}\\ S\subsetneq\mathbb{C}^n \text{ such that } B_\alpha(S)\subset S\ (\alpha=1,2)\\ \text{and } im(i)\subset S\end{array}\right\}\Big/ GL_n(\mathbb{C}),$$

where $B_\alpha\in End(\mathbb{C}^n)$ and $i\in Hom(\mathbb{C},\mathbb{C}^n)$ with the action given by $g\cdot(B_1,B_2,i)=(gB_1g^{-1},gB_2g^{-1},gi)$, for $g\in GL_n(\mathbb{C})$.

Let $H=(d_0,d_1,\ldots)$. Let $V_i=\mathbb{C}^{d_i}$. It is easy to see that the variety $(\mathbb{C}^2)^{[n]}_{1,k}(H)$ has the following description (see Figure 2).

$$(\mathbb{C}^2)^{[n]}_{1,k}(H)\cong$$
$$\cong\left\{((B_{1,j},B_{2,j})_{j\ge0},i)\;\middle|\;\begin{array}{c}1)B_{1,j+k}B_{2,j}-B_{2,j+1}B_{1,j}=0\\ 2)\text{There is no graded subspace}\\ S\subsetneq\bigoplus_{j\ge0}V_j \text{ such that } B_\alpha(S)\subset S\\ (\alpha=1,2) \text{ and } im(i)\subset S\end{array}\right\}\Big/\prod_{j\ge0} GL_{d_j}(\mathbb{C}),$$

where $B_{1,j}\in Hom(V_j,V_{j+1})$, $B_{2,j}\in Hom(V_j,V_{j+k})$ and $i\in Hom(\mathbb{C},V_0)$.

Let $\overline{H}=(H_1,\ldots,H_k)$, where $H_i=(d_{i,0},d_{i,1},\ldots)$. Let $V_{i,j}=\mathbb{C}^{d_{i,j}}$. It is easy to see that the variety $(\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H})$ has the following description (see Figure 3).

$$(\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H})\cong$$
$$\cong\left\{\left((C_{1,j,h},C_{2,j,h})_{\substack{1\le j\le k\\0\le h}},(p_{j,h})_{\substack{1\le j\le k-1\\0\le h}},i\right)\;\middle|\;\begin{array}{c}1)C_{1,j,h+1}C_{2,j,h}-C_{2,j,h+1}C_{1,j,h}=0\\ 2)C_{\alpha,j+1,h}p_{j,h}-p_{j,h+1}C_{\alpha,j,h}=0\\ 3)\text{There is no graded subspace } S\subsetneq\bigoplus_{j,h}V_{j,h}\\ \text{such that } B_\alpha(S)\subset S,\ p(S)\subset S \text{ and } im(i)\subset S\end{array}\right\}\Big/\prod_{j,h} GL_{d_{j,h}}(\mathbb{C}),$$

where $C_{\alpha,j,h}\in Hom(V_{j,h},V_{j,h+1})$, $p_{j,h}\in Hom(V_{j,h},V_{j+1,h})$ and $i\in Hom(\mathbb{C},V_{1,0})$.

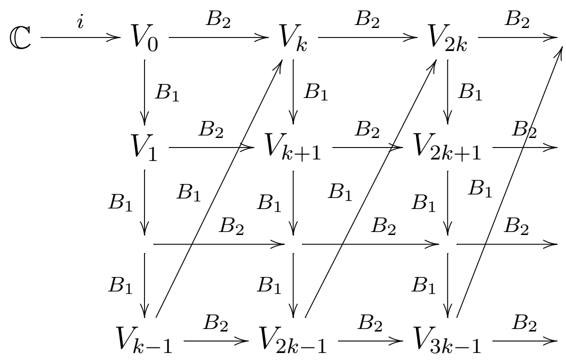


FIGURE 2. The quiver description of $(\mathbb{C}^2)^{[n]}_{1,k}(H)$

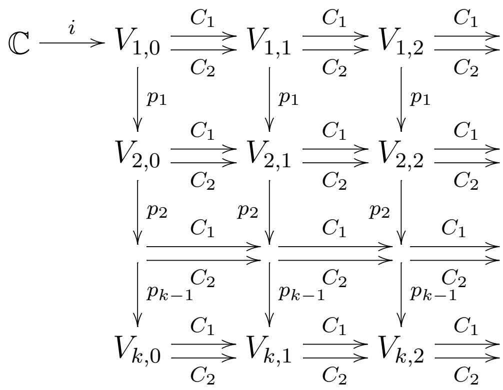


FIGURE 3. The quiver description of $(\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H})$

5.2. **Proof of Theorem 1.5.** We use the notations from section 1.5.

Proposition 5.1. *There is a natural map* $\pi\colon (\mathbb{C}^2)^{[n]}_{1,k}(H) \to (\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H})$.

Proof. Clearly, we have $V_{j,h} = V_{j-1+kh}$, for $1 \le j \le k, 0 \le h$. We define a map π by the following formula $\pi\colon (B_1, B_2, i) \mapsto (C_1, C_2, p, i)$, where $C_1 = B_1^k, C_2 = B_2, p = B_1$. This completes the proof of the proposition. □

Proposition 5.2. *Under the conditions of Theorem 1.5, the map* π *is an isomorphism.*

Proof. From the stability condition and the commutation relations it follows that the map $p_{j,h}$ is an isomorphism if $d_{j,h} = d_{j+1,h}$. Let us define a map $\phi\colon (\mathbb{C}^2)^{[\overline{n}]}_{1,1}(\overline{H}) \to (\mathbb{C}^2)^{[n]}_{1,k}(H)$ by the following formula

$\phi\colon (C_1, C_2, p, i) \mapsto (B_1, B_2, i)$, where $B_2 = C_2$ and

$$B_{1,j-1+kh} = \begin{cases} p_{j,h}, & \text{if } 1 \leq j \leq k-1, \\ C_{1,1,h} p_{1,h}^{-1} \cdots p_{k-2,h}^{-1} p_{k-1,h}^{-1}, & \text{if } j = k \text{ and } h \in E(\overline{H}), \\ p_{1,h+1}^{-1} \cdots p_{k-2,h+1}^{-1} p_{k-1,h+1}^{-1} C_{1,j,h}, & \text{if } j = k \text{ and } h+1 \in E(\overline{H}). \end{cases}$$

Clearly, the map ϕ is inverse to π. □

Theorem 1.5 follows from these two propositions.

References

[1] G. E. Andrews. The theory of partitions. Encyclopedia of Mathematics and its Applications, Vol. 2. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, 255 pp.

[2] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. Ann. Math. 98, 480-497 (1973).

[3] A. Bialynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. Bull. Acad. Pol. Sci. S6r. Sci. Math. astron. Phys. 24, (No. 9) 667-674 (1976).

[4] J. Cheah. Cellular decompositions for nested Hilbert schemes of points. Pacific J. Math. 183 (1998), no. 1, 39-90.

[5] K. Costello and I. Grojnowski. Hilbert schemes, Hecke algebras and the Calogero-Sutherland system. math.AG/0310189.

[6] G. Ellingsrud, S. A. Stromme. On the homology of the Hilbert scheme of points in the plane. Invent. math. 87, 343-352 (1987).

[7] L. Evain. Irreducible components of the equivariant punctual Hilbert schemes. Adv. Math. 185 (2004), no. 2, 328-346.

[8] S. M. Gusein-Zade, I. Luengo, A. Melle Hernandez. On generating series of classes of equivariant Hilbert schemes of fat points. arXiv:0905.1779.

[9] L. Gottsche. Hilbert schemes of points on surfaces. ICM Proceedings, Vol. II (Beijing, 2002), 483-494.

[10] A. Iarrobino. Punctual Hilbert schemes. Mem. Am. Math. Soc., (188), 1977.

[11] A. Iarrobino, J. Yameogo. The family G_T of graded artinian quotients of $k[x, y]$ of given Hilbert function. Special issue in honor of Steven L. Kleiman. Comm. Algebra 31 (2003), no. 8, 3863-3916.

[12] M. Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. Invent. Math. 136 (1999), no. 1, 157-207.

[13] M. Lehn and C. Sorger. Symmetric groups and the cup product on the cohomology of Hilbert schemes. Duke Math. J. 110 (2001), no. 2, 345-357.

[14] W.-P. Li, Z. Qin, W. Wang, Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces. Math. Ann. 324 (2002), no. 1, 105133.

[15] H. Nakajima. Lectures on Hilbert schemes of points on surfaces. AMS, Providence, RI, 1999.

[16] E. Vasserot, Sur lanneau de cohomologie du schma de Hilbert de $\mathbb{C}^2$, C. R. Acad. Sci. Paris Ser. I Math. 332 (2001), no. 1, 712.