Shuffling algorithm for boxed plane partitions

Moebius contest version*

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Abstract

We introduce discrete time Markov chains that preserve uniform measures on boxed plane partitions. Elementary Markov steps change the size of the box from $a \times b \times c$ to $(a-1) \times (b+1) \times c$ or $(a+1) \times (b-1) \times c$. Algorithmic realization of each step involves $O((a+b)c)$ operations. One application is an efficient perfect random sampling algorithm for uniformly distributed boxed plane partitions.

Trajectories of our Markov chains can be viewed as random point configurations in the three-dimensional lattice. We compute the bulk limits of the correlation functions of the resulting random point process on suitable two-dimensional sections. The limiting correlation functions define a two-dimensional determinantal point processes with certain Gibbs properties.

Intoduction

For any integers $a, b, c \geq 1$ consider a hexagon with sides a, b, c, a, b, c drawn on a regular triangular lattice. Denote by $\Omega_{a\times b\times c}$ the set of all tilings of this hexagon by rhombi obtained by gluing two of the neighboring elementary triangles together (such rhombi are called *lozenges*). Equivalently, $\Omega_{a\times b\times c}$ is the set of all dimers on the part of the dual hexagonal lattice cut out by our (a, b, c)-hexagon. An element of $\Omega_{4\times 5\times 5}$ is shown on Figure 1.

Elements of $\Omega_{a\times b\times c}$ have a number of different interpretations, see e.g. Section 1 below. In particular, they can be viewed as plane partitions or stepped surfaces inside a three-dimensional box of size $a \times b \times c$. A uniformly distributed

*This is a joint paper (arXiv:0804.3071) by Alexei Borodin (California Institute of Technology and Institute for Information Transmission Problems, Moscow) and Vadim Gorin. A.B. originally suggested that the general construction introduced in [BF] can be used in the case of random tilings of a hexagon. V.G. found a family of commuting stochastic matrices (see Section 2) that is necessary for this construction. The algorithmical description of Markov chains (see Section 4) was developed during a discussion between authors. Correlation functions (see Section 5) and their limits (see Section 6) were mainly computed by V.G. The final version of the text was written by both authors together.

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element of $\Omega_{a\times b\times c}$ then provides a basic model of a *random surface*. This model and its generalizations have been thoroughly studied, see e.g. [CLP], [CKP], [DMB], [Des], [J1] [J2], [JN], [Ke1], [KO], [Kr], [P1], [P2], [W1], [W2].

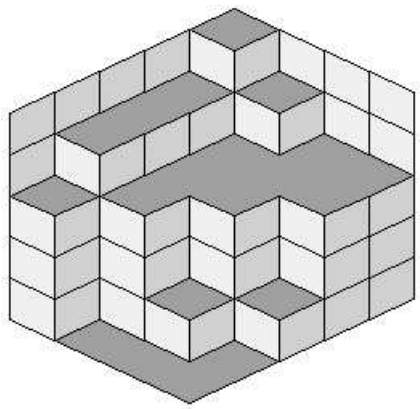

Figure 1. Tiling, stepped surface or boxed plane partition. Separating lines between horizontal lozenges are removed.

The main goal of this paper is to introduce and study certain discrete time Markov chains on boxed plane partitions that preserve the uniform measures.

Denote by $\mu_{a\times b\times c}$ the uniform probability measure on $\Omega_{a\times b\times c}$. We construct two families of stochastic matrices

$$P^{+}_{a\times b\times c} : \Omega_{a\times b\times c} \times \Omega_{(a-1)\times(b+1)\times c} \to [0,1],$$
$$P^{-}_{a\times b\times c} : \Omega_{a\times b\times c} \times \Omega_{(a+1)\times(b-1)\times c} \to [0,1],$$

such that

$$\sum_{\omega\in\Omega_{a\times b\times c}} \mu_{a\times b\times c}(\omega) P^{\pm}_{a\times b\times c}(\omega,\omega') = \mu_{(a\mp 1)\times(b\pm 1)\times c}(\omega')$$

for all a, b, c such that the participating sets Ω are nonempty.

Although it is a little awkward to write matrices $P^{\pm}$ in one formula, application of Markov operators corresponding to these matrices is fairly easy to describe algorithmically. The exact algorithm can be found in Section 4. Roughly speaking, the algorithm does the following: Given $\omega \in \Omega_{a\times b\times c}$, in order to construct ω' distributed according to $P^{\pm}_{a\times b\times c}(\omega,\omega')$ it needs to consider all horizontal lozenges of ω sequentially from left to right. For each such lozenge the algorithm decides on its new position using a simple one-dimensional probability distribution. (Here we ignored occasional appearance/diappearance of horizontal lozenges on top and at the bottom, that sometimes take place.) In a way, this means that $P^{\pm}$ decompose into products of one-dimensional Markov steps.

The algorithm has some similarity to the shuffling algorithm for domino tilings of the Aztec diamonds introduced in [EKLP]. Indeed, that algorithm

also maps uniform measures to uniform measures (actually, it works for a one-dimensional family of measures that includes the uniform one), and it also decomposes into simple (Bernoulli) Markov steps. For that reason we call our algorithm the *shuffling algorithm for boxed plane partitions*.

We further consider Markov chains obtained by successive application of arbitrary sequences of matrices P^+ and P^-. The initial condition and the one-time distributions are all uniform measures on the appropriate spaces Ω. One example is the application of the sequence of b matrices P^+ to the unique probability measure on singleton $\Omega_{(a+b)\times 0\times c}$. This gives a perfect sampling algorithm for $\mu_{a\times b\times c}$ that takes $O((a+b)bc)$ one-dimensional steps. When a, b, and c are comparable, the algorithm is roughly as efficient as that from [Kr], and it is more efficient than other known algorithms, cf. [BM], [LRS], [P1], [P2], [W1], [W2]. Another example is an alternating sequence $(P^+P^-)(P^+P^-)\cdots$ that provides an equilibrium dynamics on $\Omega_{a\times b\times c}$ with $\mu_{a\times b\times c}$ as the equilibrium measure.

These Markov chains can be viewed as two-dimensional random growth/decay models. One important feature of these models is the fact that on certain two-dimensional sections of the three-dimensional space-time, their suitably defined correlation functions are computable in a closed determinantal form. For the static correlation functions (those of the measures $\mu_{a\times b\times c}$) this fact is well known, see [Ka], [Ke2], [J2], [Gor].

We then focus on the *bulk asymptotics* of the correlation functions. Namely, when $a, b, c \to \infty$ with $a : b : c$ fixed, we look at the three-dimensional lattice process near a fixed point of the global limit shape of our random surface. On the same two-dimensional sections we compute the limiting correlation functions that also have the determinantal form. In fact, for every such section, they define a two-dimensional determinantal random point process with a certain Gibbs property, see [BS] and references therein. Our results extend similar results for the static case obtained in [Gor].

The construction of $P^\pm$ is an application of the general algebraic formalism developed in [BF]. The continuous time Markov chain considered in [BF] in detail can be viewed as the degeneration of $P^\pm$ near a corner of the hexagon as a, b, c become large, and either a and b is substantially larger than the other two. It is worth noting that the shuffling algorithm for domino tilings of the Aztec diamonds also fits into the formalism of [BF], see Section 2.6 of [BF] and [N].

The proof of the bulk asymptotics involves spectral decomposition of the matrices $P^\pm$ in terms of Hahn classical orthogonal polynomials. However, the limiting argument does not require the asymptotics of Hahn polynomials themselves — it only requires the much simpler asymptotics of the difference operators related to Hahn polynomials. This approach to bulk asymptotics of determinantal point processes is due to G. Olshanski; it was first used in [BO2] for Charlier and Krawtchouk orthogonal polynomials, and in [Gor] it was further developed in the more complex case of Hahn polynomials.

The paper is organized as follows. In Section 1 we discuss different combinatorial definitions of our model and introduce notations. In Section 2, we

introduce four stochastic matrices on one-dimensional point configurations and prove certain commutativity relations between them. In Section 3, we use these four matrices to define $P^{\pm}$; the construction is based on an idea from [DF]. Section 4 contains the algorithmic description of $P^{\pm}$ and images obtained from their computer realizations. In Section 5 we compute the correlation functions on suitable two-dimensional sections of the space-time, and in Section 6 we obtain the bulk limits.

1 Basic model

The main object of our study has many different combinatorial interpretations. In this section we discuss some of them.

Consider a tiling of an equi-angular hexagon of side lengths a, b, c, a, b, c by rhombi with angles $\pi/3$ and $2\pi/3$ and side lengths 1. Such rhombi are called lozenges.

Lozenge tilings of a hexagon can be identified with 3d Young diagrams (equivalently, boxed plane partitions) or with stepped surfaces. The bijection is best described pictorially. Examine Figure 1, where a tiling of the $(4 \times 5 \times 5)$ hexagon is shown, and view this picture as a 3d shape.

Given a tiling we construct a family of non-intersecting paths on the surface of the corresponding 3d Young diagram as in [LRS], [J2]. Figure 2 provides an example.

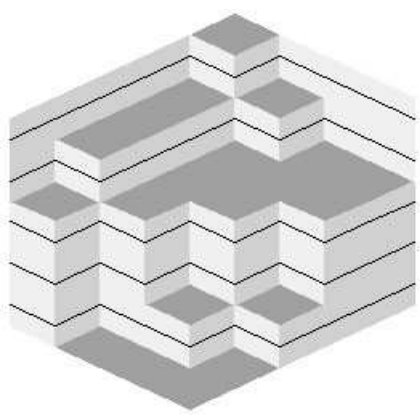

Figure 2. Non-intersecting paths on the surface of 3d Young diagram.

We view this family as a family of paths on the plane. It is convenient for us to do one more modification. We replace the downgoing segments of paths by horizontal ones and we replace upgoing segments by segments of slope 1. Consequently, our family is interpreted as a family of non-intersecting paths on $\mathbb{Z}^2$. Figure 3 shows the family corresponding to the tiling on Figures 1 and 2.

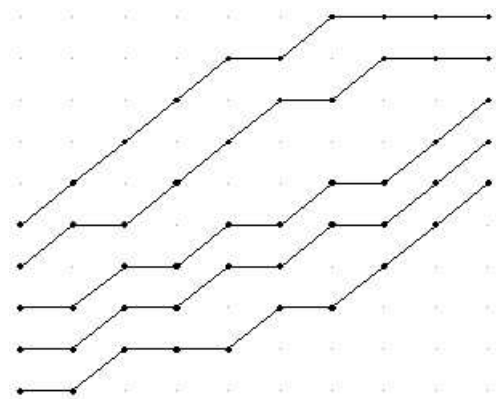

Figure 3. Non-intersecting paths on $\mathbb{Z}^2$.

Below we are going to use the last interpretation.

Let us introduce some notations. Denote by N the number of paths (in our example $N = 5$). Introduce coordinates on $\mathbb{Z}^2$ so that the first path starts at the point $(0, 0)$ and ends at the point (T, S). The second path starts at the point $(0, 1)$ and ends at the point $(T, S + 1)$, and so on. Finally, the Nth path starts at the point $(0, N - 1)$ and ends at the point $(T, S + N - 1)$. In our example $T = 9$ and $S = 5$.

Denote by $\Omega(N, T, S)$ the set of families of N non-intersecting paths made of segments of slopes 0,1, starting at $(0, 0), \dots, (0, N - 1)$ and ending at $(T, S), \dots, (T, S+N-1)$. Note that $\Omega(N, T, S) \neq \emptyset$, is equivalent to $0 \leq S \leq T$.

Denote by $\mu(N, T, S)$ the uniform measure on $\Omega(N, T, S)$.

Note that $\Omega(N, T, S)$ was called $\Omega_{a\times b\times c}$ in Introduction, while measure $\mu(N, T, S)$ was called $\mu_{a\times b\times c}$. Here $a = T - S$, $b = S$, $c = N$.

Set

$$\mathfrak{X}^{S,t}_{N,T} = \{x \in \mathbb{Z} : \max(0, t + S - T) \leq x \leq \min(t + N - 1, S + N - 1)\}$$

and

$$\mathcal{X}^{S,t}_{N,T} = \{(x_1, x_2, \dots, x_N) \in (\mathfrak{X}^{S,t}_{N,T})^N : x_1 < x_2 < \dots < x_N\}.$$

$\mathfrak{X}^{S,t}_{N,T}$ is the section of our hexagon by the vertical line with coordinate t, and $\mathcal{X}^{S,t}_{N,T}$ is the set of all N–tuples in this section.

For any $\mathcal{T} \in \Omega(N, T, S)$ denote by $\tau_1, \tau_2, \dots, \tau_N$ the corresponding paths, numbering starts from the bottom one. Thus,

$$\tau_i = \{i - 1 = \tau_i(0), \tau_i(1), \dots, \tau_i(T - 1), \tau_i(T) = S + i - 1\},$$

so that $\tau_i(t + 1) - \tau_i(t) \in \{0, 1\}$. Note that

$$\tau_i(t) \in \mathfrak{X}^{S,t}_{N,T}, \quad (\tau_1(t), \dots, \tau_N(t)) \in \mathcal{X}^{S,t}_{N,T}.$$

Consequently, any family of paths $\mathcal{T}$ can be identified with a sequence

$$\{X(1), \dots, X(T)\}, \quad X(t) \in \mathcal{X}^{S,t}_{N,T},$$

where

$$X(t) = (\tau_1(t), \dots, \tau_N(t)).$$

In fact, $X(t)$ is a Markov chain with time t, see [J1], [J2], [JN], [Gor]. Its transition probabilities are given in Proposition 2.2 below.

Through the Sections 2-5 the parameters N and T remain fixed and we omit them in different notations. We write $\mathfrak{X}^{S,t}$ instead of $\mathfrak{X}^{S,t}_{N,T}$, $\mathcal{X}^{S,t}$ instead of $\mathcal{X}^{S,t}_{N,T}$, and so on.

In the present paper we introduce a discrete time Markov chain $M(r)$, where r is a time variable. $M(r)$ takes values in $\Omega(N,T,r)$ and one-dimensional distributions of $M(r)$ coincide with $\mu(N,T,r)$.

2 Four families of stochastic matrices

2.1 Properties of sections $X(t)$

Denote by $\rho_{S,t}$ the projection of $\mu(N,T,S)$ to $\mathcal{X}^{S,t}$, i.e.

$$\rho_{S,t}(Y) = \mathrm{Prob}\{X(t) = Y\}, \quad Y = (y_1 < \dots < y_N) \in \mathcal{X}^{S,t}.$$

The following two propositions were proved in [J1, Theorem 4.1] and [Gor, Lemma 4]. Below we use the Pochhammer symbol $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$.

Proposition 2.1. *We have*

$$\rho_{S,t}(Y) = Z_{S,t} \prod_{1\le i<j\le N} (y_i - y_j)^2 \prod_{i=1}^N w^{S,t}(y_i),$$

where

$$w^{S,t}(x) = \frac{1}{x!(t+N-1-x)!(S+N-1-x)!(T-t-S+x)!}$$

and

$$Z_{S,t} = \prod_{i=1}^N \frac{(t+1)_{i-1}(T-t+1)_{i-1}(S-i+N)!(T-S+i-1)!}{(T+1)_{i-1}(i-1)!} \cdot \left(\frac{t!(T-t)!}{T!}\right)^N.$$

Proposition 2.2. *For $t = 0, 1, \dots, T-1$,*

$$\begin{aligned}&\mathrm{Prob}\{X(t+1) = (y_1,\dots,y_N) \mid X(t) = (x_1,\dots,x_N)\}\\ &\quad= \frac{\prod_{i<j}(y_j - y_i) \cdot \prod_{i:\, y_i = x_i+1} (N+S-x_i-1) \cdot \prod_{i:\, y_i = x_i} (x_i+T-t-S)}{(T-t)_N \cdot \prod_{i<j}(x_j - x_i)},\end{aligned}$$

if $y_i - x_i \in \{0,1\}$ for every i, and the conditional probability is equal to zero otherwise.

We also need "co-transition probabilities" of the Markov chain $X(t)$.

Proposition 2.3. *For $t = 1, 2, \dots, T$*

$$\begin{aligned}&\mathrm{Prob}\{X(t-1) = (y_1, \dots, y_N) \mid X(t) = (x_1, \dots, x_N)\}\\ &\qquad = \frac{\prod\limits_{i<j}(y_j - y_i) \cdot \prod\limits_{i:\, y_i = x_i - 1} x_i \cdot \prod\limits_{i:\, y_i = x_i}(t + N - 1 - x_i)}{(t)_N \cdot \prod\limits_{i<j}(x_j - x_i)},\end{aligned}$$

if $y_i - x_i \in \{-1, 0\}$ for every i, and the conditional probability is equal to zero otherwise.

Proof. Straightforward computation using

$$\begin{aligned}&\mathrm{Prob}\{X(t-1) = (y_1, \dots, y_N) \mid X(t) = (x_1, \dots, x_N)\}\\ &= \mathrm{Prob}\{X(t) = (x_1, \dots, x_N) \mid X(t-1) = (y_1, \dots, y_N)\} \cdot \frac{\rho_{S,t-1}(y_1, \dots, y_N)}{\rho_{S,t}(x_1, \dots, x_N)}\end{aligned}$$

□

2.2 Stochastic matrices

We need four families of stochastic matrices $P^{S,t}_{t+}$, $P^{S,t}_{t-}$, $P^{S,t}_{S+}$, $P^{S,t}_{S-}$.

$P^{S,t}_{t+}(X, Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S,t+1}|$ matrix, $X = (x_1 < \dots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \dots < y_n) \in \mathcal{X}^{S,t+1}$;

$$P^{S,t}_{t+}(X, Y) = \frac{\prod\limits_{i<j}(y_j - y_i) \prod\limits_{i:\, y_i = x_i + 1}(N + S - x_i - 1) \prod\limits_{i:\, y_i = x_i}(x_i + T - t - S)}{(T - t)_N \cdot \prod\limits_{i<j}(x_j - x_i)},$$

if $y_i - x_i \in \{0, 1\}$ for every i, and $P^{S,t}_{t+}(X, Y) = 0$ otherwise.

$P^{S,t}_{S+}(X, Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S+1,t}|$ matrix, $X = (x_1 < \dots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \dots < y_n) \in \mathcal{X}^{S+1,t}$;

$$P^{S,t}_{S+}(X, Y) = \frac{\prod\limits_{i<j}(y_j - y_i) \prod\limits_{i:\, y_i = x_i + 1}(N + t - x_i - 1) \prod\limits_{i:\, y_i = x_i}(x_i + T - t - S)}{(T - S)_N \cdot \prod\limits_{i<j}(x_j - x_i)},$$

if $y_i - x_i \in \{0, 1\}$ for every i, and $P^{S,t}_{S+}(X, Y) = 0$ otherwise.

$P^{S,t}_{t-}(X, Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S,t-1}|$ matrix, $X = (x_1 < \dots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \dots < y_n) \in \mathcal{X}^{S,t-1}$;

$$P^{S,t}_{t-}(X, Y) = \frac{\prod\limits_{i<j}(y_j - y_i) \prod\limits_{i:\, y_i = x_i - 1} x_i \prod\limits_{i:\, y_i = x_i}(t + N - 1 - x_i)}{(t)_N \cdot \prod\limits_{i<j}(x_j - x_i)},$$

if $y_i - x_i \in \{-1, 0\}$ for every i, and $P^{S,t}_{t-}(X,Y) = 0$ otherwise.

$P^{S,t}_{S-}(X,Y)$ is an $|\mathcal{X}^{S,t}| \times |\mathcal{X}^{S-1,t}|$ matrix, $X = (x_1 < \cdots < x_N) \in \mathcal{X}^{S,t}$, $Y = (y_1 < \cdots < y_n) \in \mathcal{X}^{S-1,t}$;

$$P^{S,t}_{S-}(X,Y) = \frac{\prod\limits_{i<j}(y_j - y_i) \prod\limits_{i:\, y_i = x_i - 1} x_i \prod\limits_{i:\, y_i = x_i} (S + N - 1 - x_i)}{(S)_N \cdot \prod\limits_{i<j}(x_j - x_i)},$$

if $y_i - x_i \in \{-1, 0\}$ for every i, and $P^{S,t}_{S-}(X,Y) = 0$ otherwise.

Looking at spaces that parameterize rows and columns of these matrices one can say that $P^{S,t}_{t+}$ increases t, $P^{S,t}_{t-}$ decreases t, while $P^{S,t}_{S+}$ increases S and $P^{S,t}_{S-}$ decreases S.

Theorem 2.4. *All four types of matrices defined above are stochastic, and they preserve the family of measures $\rho_{S,t}$. In other words*

$$\sum_{Y \in \mathcal{X}^{S,t\pm1}} P^{S,t}_{t\pm}(X,Y) = 1, \qquad \sum_{Y \in \mathcal{X}^{S,S\pm1}} P^{S,t}_{t\pm}(X,Y) = 1, \tag{1}$$

$$\rho_{S,t\pm1}(Y) = \sum_{X \in \mathcal{X}^{S,t}} P^{S,t}_{t\pm}(X,Y) \cdot \rho_{S,t}(X), \; \rho_{S\pm1,t}(Y) = \sum_{X \in \mathcal{X}^{S,t}} P^{S,t}_{S\pm}(X,Y) \cdot \rho_{S,t}(X).$$

Proof. Propositions 2.2 and 2.3 imply the claim for $P^{S,t}_{t+}(X,Y)$ and $P^{S,t}_{t-}(X,Y)$.

Now observe that the space $\mathcal{X}^{S,t}$ is unaffected when we interchange parameters t and S, i.e.

$$\mathcal{X}^{S,t} = \mathcal{X}^{t,S}.$$

Moreover, the measures $\rho_{S,t}$ are also invariant under $S \leftrightarrow t$, i.e.

$$\rho_{S,t} = \rho_{t,S}.$$

Finally, note that $P^{S,t}_{t+}(X,Y)$ becomes $P^{S,t}_{S+}(X,Y)$ under $S \leftrightarrow t$ and $P^{S,t}_{t-}(X,Y)$ becomes $P^{S,t}_{S-}(X,Y)$.

Consequently, applying $S \leftrightarrow t$ to the relations for $P_{t\pm}$ we obtain needed relations for $P_{S\pm}$. □

2.3 Determinantal representation

In this section we write our stochastic matrices in a determinantal form. This representation is very convenient for various computations.

First, we introduce 4 new two-diagonal matrices:

$$U^{S,t}_{t+}(x,y) = \begin{cases} N + S - 1 - x, & \text{if } y = x + 1, \\ T - t - S + x, & \text{if } y = x, \\ 0, & \text{otherwise}, \end{cases} \qquad x \in \mathfrak{X}^{S,t}, \quad y \in \mathfrak{X}^{S,t+1};$$

$$U^{S,t}_{S+}(x,y)=\begin{cases} N+t-1-x, & \text{if } y=x+1,\\ T-t-S+x, & \text{if } y=x,\\ 0, & \text{otherwise,}\end{cases} \qquad x\in\mathfrak{X}^{S,t},\quad y\in\mathfrak{X}^{S+1,t};$$

$$U^{S,t}_{t-}(x,y)=\begin{cases} x, & \text{if } y=x-1,\\ t+N-1-x, & \text{if } y=x,\\ 0, & \text{otherwise,}\end{cases} \qquad x\in\mathfrak{X}^{S,t},\quad y\in\mathfrak{X}^{S,t-1};$$

$$U^{S,t}_{S-}(x,y)=\begin{cases} x, & \text{if } y=x-1,\\ S+N-1-x, & \text{if } y=x,\\ 0, & \text{otherwise,}\end{cases} \qquad x\in\mathfrak{X}^{S,t},\quad y\in\mathfrak{X}^{S-1,t}.$$

It is possible to express stochastic matrices $P^{S,t}_{t\pm}$, $P^{S,t}_{S\pm}$ through certain minors of the matrices defined above.

Proposition 2.5. *We have*

$$P^{S,t}_{t+}(X,Y)=\frac{\prod\limits_{i<j}(y_j-y_i)\det[U^{S,t}_{t+}(x_i,y_j)]_{i,j=1,\dots,N}}{(T-t)_N\cdot\prod\limits_{i<j}(x_j-x_i)}$$

$$P^{S,t}_{S+}(X,Y)=\frac{\prod\limits_{i<j}(y_j-y_i)\det[U^{S,t}_{S+}(x_i,y_j)]_{i,j=1,\dots,N}}{(T-S)_N\cdot\prod\limits_{i<j}(x_j-x_i)}$$

$$P^{S,t}_{t-}(X,Y)=\frac{\prod\limits_{i<j}(y_j-y_i)\det[U^{S,t}_{t-}(x_i,y_j)]_{i,j=1,\dots,N}}{(t)_N\cdot\prod\limits_{i<j}(x_j-x_i)}$$

$$P^{S,t}_{S-}(X,Y)=\frac{\prod\limits_{i<j}(y_j-y_i)\det[U^{S,t}_{S-}(x_i,y_j)]_{i,j=1,\dots,N}}{(S)_N\cdot\prod\limits_{i<j}(x_j-x_i)}$$

Proof. Straightforward computation using the definitions of stochastic matrices $P^{S,t}_{t\pm}$, $P^{S,t}_{S\pm}$ and matrices $U^{S,t}_{t\pm}$, $U^{S,t}_{S\pm}$.

Any submatrix of a two-diagonal matrix, which has a nonzero determinant, is block-diagonal, where each block is either upper or lower triangular matrix. Thus, any nonzero minor is a product of suitable matrix elements. □

2.4 Spectral decomposition of stochastic matrices

In this section we modify the determinantal representation of the stochastic matrices and introduce new representation, which we call *spectral decomposition*. Spectral decomposition is of crucial importance for computing correlation functions of the processes that will be constructed later on, and for finding bulk limits of the processes. Results of this section will be used in Sections 5 and 6, while Sections 3 and 4 are independent of these results.

Let us introduce some notations.

Denote by $H_k^{S,t}(x)$ the Hahn polynomial of the degree k corresponding to the parameters N, T, t, S. Domain of definition of these polynomials coincides with $\mathfrak{X}^{S,t}$, and the polynomials are orthogonal with respect to the weight function $w^{S,t}(x)$ defined in Proposition 2.1. For definition and explicit formulas for Hahn polynomials see [KS], more information about the usage of Hahn polynomials in our case can be found in [Gor].

Let us denote by $\Psi_k^{S,t}(x)$ the normalized Hahn polynomials

$$\Psi_k^{S,t}(x) = \frac{H_k^{S,t}(x)\sqrt{w^{S,t}(x)}}{\sqrt{(H_k^{S,t}, H_k^{S,t})}}.$$

Here $(H_k^{S,t}, H_k^{S,t})$ is the squared norm of the polynomial $H_k^{S,t}$ in $L_2(\mathfrak{X}^{S,t}, w^{S,t}(x))$. The functions $\Psi_k^{S,t}(x)$ form an orthonormal basis in the space $L_2(\mathfrak{X}^{S,t})$ (this L_2 is with respect to the uniform measure on $\mathfrak{X}^{S,t}$).

Let

$$c_{t+}^{S,t}(i) = \sqrt{\left(1 - \frac{i}{t+N}\right)\left(1 - \frac{i}{T+N-t-1}\right)}.$$

Note that $c_{t+}^{S,t}(i)$ does not actually depend on S, but it is convenient to use it for the notation.

Finally, denote by $v_{t+}^{S,t}$ the $|\mathfrak{X}^{S,t}| \times |\mathfrak{X}^{t+1,S}|$ matrix given by

$$v_{t+}^{S,t}(x,y) = \sum_{i\geq 0} c_{t+}^{S,t}(i)\Psi_i^{S,t}(x)\Psi_i^{S,t+1}(y), \quad x \in \mathfrak{X}^{S,t}, \quad y \in \mathfrak{X}^{S,t+1}.$$

The following proposition was proved in [Gor, proof of Proposition 5]

Proposition 2.6. *Let $X = (x_1 < \cdots < x_N)$ and $Y = (y_1 < \cdots < y_N)$ be elements of $\mathcal{X}^{S,t}$ and $\mathcal{X}^{S,t+1}$, respectively. Then*

$$P_{t+}^{S,t}(X,Y) = \frac{\sqrt{\rho_{S,t+1}(Y)}}{\sqrt{\rho_{S,t}(X)}} \frac{\det[v_{t+}^{S,t}(x_i,y_j)]_{i,j=1,\ldots,N}}{\prod_{i=0}^{N-1} c_{t+}^{S,t}(i)}$$

Stochastic matrices $P_{t-}^{S,t}$ admit similar spectral decomposition with transposed matrices.

Denote

$$c_{t-}^{S,t} := c_{t+}^{S,t-1}$$

and

$$v_{t-}^{S,t} := (v_{t+}^{S,t-1})^T$$

(here $(\quad)^T$ means matrix transposition).

Proposition 2.7. *Let $X = (x_1 < \cdots < x_N)$ and $Y = (y_1 < \cdots < y_N)$ be elements of $\mathcal{X}^{S,t}$ and $\mathcal{X}^{S,t-1}$, respectively. Then*

$$P_{t-}^{S,t}(X,Y) = \frac{\sqrt{\rho_{S,t-1}(Y)}}{\sqrt{\rho_{S,t}(X)}} \frac{\det[v_{t-}^{S,t}(x_i,y_j)]_{i,j=1,\dots,N}}{\prod\limits_{i=0}^{N-1} c_{t-}^{S,t}(i)}$$

Proof. Recall that while $P_{t+}^{S,t}$ coincides with transition matrix of the Markov process $X(t)$, matrix elements of $P_{t-}^{S,t}$ are co-transition probabilities of Proposition 2.3. Consequently,

$$\begin{aligned} P_{t-}^{S,t}(X,Y) &= \frac{\rho_{S,t-1}(Y)}{\rho_{S,t}(X)} \cdot P_{t+}^{S,t-1}(Y,X) \\ &= \frac{\rho_{S,t-1}(Y)}{\rho_{S,t}(X)} \cdot \frac{\sqrt{\rho_{S,t}(X)}}{\sqrt{\rho_{S,t-1}(Y)}} \frac{\det[v_{t+}^{S,t-1}(y_i,x_j)]_{i,j=1,\dots,N}}{\prod\limits_{i=0}^{N-1} c_{t+}^{S,t-1}(i)} \\ &= \frac{\sqrt{\rho_{S,t-1}(Y)}}{\sqrt{\rho_{S,t}(X)}} \frac{\det[v_{t-}^{S,t}(y_i,x_j)]_{i,j=1,\dots,N}}{\prod\limits_{i=0}^{N-1} c_{t-}^{S,t}(i)} \end{aligned}$$

□

Define

$$c_{S+}^{S,t}(i) = \sqrt{\left(1 - \frac{i}{S+N}\right)\left(1 - \frac{i}{T+N-S-1}\right)}$$

(these constants do not acually depend on t). Set

$$v_{S+}^{S,t}(x,y) = \sum_{i\geq 0} c_{S+}^{S,t}(i)\Psi_i^{S,t}(x)\Psi_i^{S+1,t}(y), \quad x \in \mathfrak{X}^{S,t}, \quad y \in \mathfrak{X}^{S+1,t}.$$

Recall that matrices $P_{t+}^{S,t}$ and $P_{S+}^{S,t}$ are connected by the involution $t \leftrightarrow S$, thus, Propositions 2.6 and 2.7 imply the following statements.

Proposition 2.8. *Let $X = (x_1 < \cdots < x_N)$ and $Y = (y_1 < \cdots < y_N)$ be elements of $\mathcal{X}^{S,t}$ and $\mathcal{X}^{S+1,t}$, respectively. Then*

$$P_{S+}^{S,t}(X,Y) = \frac{\sqrt{\rho_{S+1,t}(Y)}}{\sqrt{\rho_{S,t}(X)}} \frac{\det[v_{S+}^{S,t}(x_i,y_j)]_{i,j=1,\dots,N}}{\prod\limits_{i=0}^{N-1} c_{S+}^{S,t}(i)}.$$

Proposition 2.9. *Let $X = (x_1 < \cdots < x_N)$ and $Y = (y_1 < \cdots < y_N)$ be elements of $\mathcal{X}^{S,t}$ and $\mathcal{X}^{S-1,t}$, respectively. Then*

$$P^{S,t}_{S-}(X,Y) = \frac{\sqrt{\rho_{S-1,t}(Y)}}{\sqrt{\rho_{S,t}(X)}} \frac{\det[v^{S-1,t}_{S-}(y_i,x_j)]_{i,j=1,\dots,N}}{\prod\limits_{i=0}^{N-1} c^{S,t}_{S-}(i)}.$$

2.5 Commutativity

Theorem 2.10. *The families of stochastic matrices $P^{S,t}_{t\pm}$ and $P^{S,t}_{S\pm}$ commute, that is*

$$\begin{aligned}
P^{S,t}_{t+} \cdot P^{S,t+1}_{S-} &= P^{S,t}_{S-} \cdot P^{S-1,t}_{t+},\\
P^{S,t}_{t-} \cdot P^{S,t-1}_{S-} &= P^{S,t}_{S-} \cdot P^{S-1,t}_{t-},\\
P^{S,t}_{t+} \cdot P^{S,t+1}_{S+} &= P^{S,t}_{S+} \cdot P^{S+1,t}_{t+},\\
P^{S,t}_{t-} \cdot P^{S,t-1}_{S+} &= P^{S,t}_{S+} \cdot P^{S+1,t}_{t-},
\end{aligned}$$

for any meaningful values of S and t.

Proof. Proofs of all four cases are very similar and we consider only the first one.

$$\begin{aligned}
(P^{S,t}_{t+} \cdot P^{S,t+1}_{S-})(X,Y) &= \sum_{Z\in\mathcal{X}^{S,t+1}} P^{S,t}_{t+}(X,Z) \cdot P^{S,t+1}_{S-}(Z,Y)\\
= \frac{\prod\limits_{i>j}(y_i-y_j) \sum\limits_{Z\in\mathcal{X}^{S,t+1}} \det[U^{S,t}_{t+}(x_i,z_j)]_{i,j=1,\dots,N} \det[U^{S,t+1}_{S-}(z_i,y_j)]_{i,j=1,\dots,N}}{(T-t)_N \cdot (S)_N \cdot \prod\limits_{i>j}(x_i-x_j)}
\end{aligned}$$

Applying Cauchy-Binet identity we obtain

$$\sum_{Z\in\mathcal{X}^{S,t+1}} \det[U^{S,t}_{t+}(x_i,z_j)]_{i,j=1,\dots,N} \det[U^{S,t+1}_{S-}(z_i,y_j)]_{i,j=1,\dots,N}$$
$$= \det[(U^{S,t}_{t+} \cdot U^{S,t+1}_{S-})(x_i,y_j)]_{i,j=1,\dots,N}.$$

Thus,

$$(P^{S,t}_{t+} \cdot P^{S,t+1}_{S-})(X,Y) = \frac{\prod\limits_{i>j}(y_i-y_j) \det[(U^{S,t}_{t+} \cdot U^{S,t+1}_{S-})(x_i,y_j)]_{i,j=1,\dots,N}}{(T-t)_N \cdot (S)_N \cdot \prod\limits_{i>j}(x_i-x_j)}.$$

Similarly

$$(P^{S,t}_{S-} \cdot P^{S-1,t}_{t+})(X,Y) = \frac{\prod\limits_{i>j}(y_i-y_j) \det[(U^{S,t}_{S-} \cdot U^{S-1,t}_{t+})(x_i,y_j)]_{i,j=1,\dots,N}}{(T-t)_N \cdot (S)_N \cdot \prod\limits_{i>j}(x_i-x_j)}.$$

Our problem reduces to verifying the equality

$$U_{t+}^{S,t} \cdot U_{S-}^{S,t+1} = U_{S-}^{S,t} \cdot U_{t+}^{S-1,t}.$$

By straightforward computation one proves that

$$U_{t+S-}^{S,t} = U_{t+}^{S,t} \cdot U_{S-}^{S,t+1} = U_{S-}^{S,t} \cdot U_{t+}^{S-1,t},$$

where

$$U_{t+S-}^{S,t}(x,y) = \begin{cases} (N+S-1-x)(N+S-2-x), & \text{if } y = x+1, \\ (N+S-1-x)(T-t-S+2x+1), & \text{if } y = x, \\ x(T-t-S+x), & \text{if } y = x-1, \\ 0, & \text{otherwise.} \end{cases}$$

□

Remark. Another way to prove the commutativity is to use the spectral decomposition introduced in Section 2.4 and to observe that coefficients $c_{t\pm}^{S,t}(i)$ do not depend on S, while coefficients $c_{S\pm}^{S,t}(i)$ do not depend on t. One computes

$$\sqrt{\frac{w^{S,t}(x)/w^{S-1,t+1}(y)}{(t+N)(T+N-t-1)(S+N-1)(T+N-S)}} \cdot U_{t+S-}^{S,t}(x,y)$$
$$= \sum_{i\geq 0} c_{t+}^{S,t}(i) c_{S-}^{S,t+1}(i) \Psi_i^{S,t}(x) \Psi_i^{S-1,t+1}(y)$$
$$= \sum_{i\geq 0} c_{S-}^{S,t}(i) c_{t+}^{S-1,t}(i) \Psi_i^{S,t}(x) \Psi_i^{S-1,t+1}(y).$$

3 Markov step $S \mapsto S \pm 1$

In this section we aim to define two new stochastic matrices

$$P_{S\mapsto S+1}^{S}(X,Y), \quad X \in \Omega(N,T,S), \quad Y \in \Omega(N,T,S+1)$$

and

$$P_{S\mapsto S-1}^{S}(X,Y), \quad X \in \Omega(N,T,S), \quad Y \in \Omega(N,T,S-1)$$

that preserve the measures $\mu(N,T,S)$. Both $P_{S\mapsto S+1}^{S}$ and $P_{S\mapsto S-1}^{S}$ depend on parameters N and T but we again omit these indices. In Introduction we called these matrices $P_{a\times b\times c}^{\pm}$ with $a = T-S$, $b = S$, $c = N$.

Suppose we are given a sequence $X = (X(0), X(1), \ldots, X(T)) \in \Omega(N,T,S)$ (recall that $X(t) \in \mathcal{X}^{S,t}$). Below we construct a random sequence $Y = (Y(0), \ldots, Y(T)) \in \Omega(N,T,S+1)$ and therefore define the transition probability (or, equivalently, stochastic matrix) $P_{S\mapsto S+1}^{S}(X,Y)$.

First note that $Y(0) \in \mathcal{X}^{S+1,0}$ and $|\mathcal{X}^{S+1,0}| = 1$. Thus, $Y(0)$ is uniquely defined. We will perform a *sequential update*. Suppose $Y(0), Y(1), \ldots, Y(t)$

have been already defined. Define conditional distribution of $Y(t+1)$ given X, $Y(0), Y(1), \dots, Y(t)$ by

$$\begin{aligned}\operatorname{Prob}\{Y(t+1)=Z\} &= \frac{P_{t+}^{S+1,t}(Y(t),Z)\cdot P_{S-}^{S+1,t+1}(Z,X(t+1))}{(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))}\\ &= \frac{P_{S+}^{S,t+1}(X(t+1),Z)\cdot P_{t-}^{S+1,t+1}(Z,Y(t))}{(P_{S+}^{S,t+1}P_{t-}^{S+1,t+1})(X(t+1),Y(t))}. \qquad (2)\end{aligned}$$

(The second equality follows from $\rho_{S+1,t+1}(X)P_{t-}^{S+1,t+1}(X,Y) = \rho_{S+1,t}(Y)P_{t+}^{S+1,t}(Y,X)$.)

This definitions follows the idea of [DF, Section 2.3], see also [BF].

Observe that $(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1)) > 0$. Indeed

$$\begin{aligned}(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1)) &= (P_{S-}^{S+1,t}P_{t+}^{S,t})(Y(t),X(t+1))\\ &\geq P_{S-}^{S+1,t}(Y(t),X(t))\cdot P_{t+}^{S,t}(X(t),X(t+1)) > 0, \qquad (3)\end{aligned}$$

because $Y(t)$ was chosen on the previous step so that $P_{S-}^{S+1,t}(Y(t),X(t)) > 0$.

One could say that we choose $Y(t+1)$ using conditional distribution of the middle point in the successive application of $P_{t+}^{S+1,t}$ and $P_{S-}^{S+1,t+1}$ (or $P_{S+}^{S,t+1}$ and $P_{t-}^{S+1,t+1}$), provided that we start at $Y(t)$ and finish at $X(t+1)$ (or start at $X(t+1)$ and finish at $Y(t)$).

After performing T updates we obtain the sequence Y.

Equivalently, define $P_{S\mapsto S+1}^{S}$ by (cf. [BF, Section 2.2])

$$P_{S\mapsto S+1}^{S}(X,Y) = \begin{cases}\displaystyle\prod_{t=0}^{T-1}\frac{P_{t+}^{S+1,t}(Y(t),Y(t+1))\cdot P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))},\\ \qquad \text{if } \displaystyle\prod_{t=0}^{T-1}(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))>0,\\ 0, \text{ otherwise.}\end{cases}$$

Theorem 3.1. *The matrix $P_{S\mapsto S+1}^{S}$ on $\Omega(N,T,S)\times\Omega(N,T,S+1)$ is stochastic. The transition probabilities $P_{S\mapsto S+1}^{S}(X,Y)$ preserve the uniform measures $\mu(N,T,S)$:*

$$\mu(N,T,S+1)(Y) = \sum_{X\in\Omega(N,T,S)} P_{S\mapsto S+1}^{S}(X,Y)\mu(N,T,S)(X).$$

Proof. First, let us prove that the matrix $P_{S\mapsto S+1}^{S}$ is stochastic, equivalently:

$$\begin{aligned}1 &= \sum_{Y\in\Omega(N,T,S+1)} P_{S\mapsto S+1}^{S}(X,Y)\\ &= \sum_{Y(0),\dots,Y(T)}\prod_{t=0}^{T-1}\frac{P_{t+}^{S+1,t}(Y(t),Y(t+1))\cdot P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t}P_{S-}^{S+1,t+1})(Y(t),X(t+1))}, \qquad (4)\end{aligned}$$

where the summation goes over all $(Y(0), Y(1), \dots, Y(T)) \in \Omega(N, T, S+1)$ such that

$$\prod_{t=0}^{T-1} (P_{t+}^{S+1,t} P_{S-}^{S+1,t+1})(Y(t), X(t+1)) > 0. \tag{5}$$

We write (4) in the form

$$\sum_{Y(0),\dots,Y(T-1)} \prod_{t=0}^{T-2} \frac{P_{t+}^{S+1,t}(Y(t), Y(t+1)) \cdot P_{S-}^{S+1,t+1}(Y(t+1), X(t+1))}{(P_{t+}^{S+1,t} P_{S-}^{S+1,t+1})(Y(t), X(t+1))}$$
$$\times \sum_{Y(T)} \frac{P_{t+}^{S+1,T-1}(Y(T-1), Y(T)) \cdot P_{S-}^{S+1,T}(Y(T), X(T))}{(P_{t+}^{S+1,T-1} P_{S-}^{S+1,T})(Y(T-1), X(T))}.$$

Summing over $Y(T)$ we obtain

$$\sum_{Y(0),\dots,Y(T-2)} \prod_{t=0}^{T-3} \frac{P_{t+}^{S+1,t}(Y(t), Y(t+1)) \cdot P_{S-}^{S+1,t+1}(Y(t+1), X(t+1))}{(P_{t+}^{S+1,t} P_{S-}^{S+1,t+1})(Y(t), X(t+1))}$$
$$\times \sum_{Y(T-1)} \frac{P_{t+}^{S+1,T-2}(Y(T-2), Y(T-1)) \cdot P_{S-}^{S+1,T-1}(Y(T-1), X(T-1))}{(P_{t+}^{S+1,T-2} P_{S-}^{S+1,T-1})(Y(T-2), X(T-1))}.$$

Next, we want to sum over $Y(T-1)$. Inequality (5) implies that the summation goes over $Y(T-1)$ such that $(P_{t+}^{S+1,T-1} P_{S-}^{S+1,T})(Y(T-1), X(T)) > 0$. Note that

$$(P_{t+}^{S+1,T-1} P_{S-}^{S+1,T})(Y(T-1), X(T)) = (P_{S-}^{S+1,T-1} P_{t+}^{S,T-1})(Y(T-1), X(T))$$
$$\geq P_{S-}^{S+1,T-1}(Y(T-1), X(T-1)) P_{t+}^{S,T-1}(X(t-1), X(T)).$$

Consequently, if $(P_{t+}^{S+1,T-1} P_{S-}^{S+1,T})(Y(T-1), X(T))$ vanishes, then $P_{S-}^{S+1,T-1}(Y(T-1), X(T-1))$ vanishes too. Thus, we may drop the inequality that restricts the summation and sum over all possible $Y(T-1)$. After that we sum over $Y(T-2)$, and so on. After summing over all $Y(t)$, $t = T, T-1, \dots, 1$ and noticing that $Y(0)$ has just one possible value we arrive at (4).

Now we prove that the transition probabilities $P_{S \mapsto S+1}^{S}(X, Y)$ preserve the uniform measures $\mu(N, T, S)$. It is equivalent to

$$\mu(N, T, S+1)(Y) = \sum_{X=(X(0),X(1),\dots,X(T))} P_{S \mapsto S+1}^{S}(X, Y) \mu(N, T, S)(X). \tag{6}$$

Since $X(t)$ can be viewed as a Markov chain with time t,

$$\mu(N, T, S)(X) = m_S^0(X(0)) \cdot P_{t+}^{S,0}(X(0), X(1)) \dots P_{t+}^{S,T-1}(X(T-1), X(T)),$$

where $m_S^0(X(0))$ is the unique probability measure on singleton $\mathcal{X}^{S,0}$.

Thus, the right-hand side of (6) is equal to

$$\sum_{X(0),\dots,X(T)} m_S^0(X(0)) \prod_{t=0}^{T-1} P_{t+}^{S,t}(X(t),X(t+1)) \times \prod_{t=0}^{T-1} \frac{P_{t+}^{S+1,t}(Y(t),Y(t+1)) \cdot P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t} P_{S-}^{S+1,t+1})(Y(t),X(t+1))} \quad (7)$$

Note that

$$m_S^0(X(0)) = m_{S+1}^0(Y(0)) = P_{S-1}^{S+1,0}(Y(0),X(0)) = P_{S-}^{S+1,T}(Y(T),X(T)) = 1$$

and write (7) in the form

$$m_{S+1}^0(Y(0)) \prod_{t=0}^{T-1} P_{t+}^{S+1,t}(Y(t),Y(t+1)) \times \sum_{X(0),\dots,X(T)} \prod_{t=0}^{T-1} \frac{P_{S-}^{S+1,t}(Y(t),X(t)) P_{t+}^{S,t}(X(t),X(t+1))}{(P_{S-}^{S+1,t} P_{t+}^{S,t})(Y(t),X(t+1))}.$$

(We used the equality $P_{S-}^{S+1,t} P_{t+}^{S,t} = P_{t+}^{S+1,t} P_{S-}^{S+1,t+1}$.) Summing first over $X(0)$, then over $X(1)$, and so on, we get

$$m_{S+1}^0(Y(0)) \prod_{t=0}^{T-1} P_{t+}^{S+1,t}(Y(t),Y(t+1)),$$

which is exactly the distribution $\mu(N,T,S+1)(Y)$.

□

Similarly to $P_{S\mapsto S+1}$, one defines a transition matrix

$$P_{S\mapsto S-1}^{S}(X,Y), \quad X \in \Omega(N,T,S), \quad Y \in \Omega(N,T,S-1),$$

by

$$P_{S\mapsto S-1}^{S}(X,Y) = \begin{cases} \prod_{t=0}^{T-1} \frac{P_{t+}^{S-1,t}(Y(t),Y(t+1)) \cdot P_{S+}^{S-1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S-1,t} P_{S+}^{S-1,t+1})(Y(t),X(t+1))}, \\ \qquad \text{if } \prod_{t=0}^{T-1} (P_{t+}^{S-1,t} P_{S+}^{S-1,t+1})(Y(t),X(t+1)) > 0, \\ 0, \text{ otherwise.} \end{cases}$$

Similarly to (2) there is another way to write $P_{S\mapsto S-1}^{S}$ because of the equality

$$\frac{P_{t+}^{S-1,t}(Y(t),Y(t+1)) \cdot P_{S+}^{S-1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S-1,t} P_{S+}^{S-1,t+1})(Y(t),X(t+1))} = \frac{P_{S-}^{S,t+1}(X(t+1),Y(t+1)) \cdot P_{t-}^{S-1,t+1}(Y(t+1),Y(t))}{(P_{S-}^{S,t+1} P_{t-}^{S-1,t+1})(X(t+1),Y(t))}$$

Similarly to Theorem 3.1 one proves the following claim.

Theorem 3.2. *The matrix $P^S_{S\mapsto S-1}$ on $\Omega(N,T,S)\times\Omega(N,T,S-1)$ is stochastic. The transition probabilities $P^S_{S\mapsto S-1}(X,Y)$ preserve the uniform measures $\mu(N,T,S)$:*

$$\mu(N,T,S-1)(Y)=\sum_{X\in\Omega(N,T,S)}P^S_{S\mapsto S-1}(X,Y)\mu(N,T,S)(X).$$

Remark. The above construction performs sequential update from $t=0$ to $t=T$. One can equally well update from $t=T$ to $t=0$ by suitably modifying the definitions. The resulting Markov chains also preserve the uniform measure $\mu(N,T,S)$, and they are different from the Markov chains defined above.

4 Algorithmic description

In this section we suggest an algorithmic description of the Markov chain from the previous section.

Denote by $D(a,b,n)$ the probability distribution on $\{0,1,\dots,n\}$ given by

$$\begin{aligned}\mathrm{Prob}(\{k\})=D(a,b,n)\{k\}&=\frac{\frac{(a)_k}{(b)_k}}{\sum\limits_{j=0,\dots,n}\frac{(a)_j}{(b)_j}}\\&=\frac{a(a+1)\dots(a+k-1)(b+k)\dots(b+n-1)}{\sum_{j=0}^n a(a+1)\dots(a+j-1)(b+j)\dots(b+n-1)}.\end{aligned}\tag{8}$$

4.1 Algorithm for $S\mapsto S+1$ step.

Suppose we are given $X=(X(0),X(1),\dots,X(T))\in\Omega(N,T,S)$. We want to construct $Y=(Y(0),Y(1),\dots,Y(T))\in\Omega(N,T,S+1)$.

In the first place we note that $Y(0)$ is uniquely defined,

$$Y(0)=(0,1,\dots,N-1).$$

Then we perform T sequential updates, i.e. for $t=0,1,\dots T-1$ we construct $Y(t+1)$ using $Y(t)$ and $X(t+1)$. Let us describe each step.

Let $Y(t)=(y_1<y_2<\dots<y_N)$ and $X(t+1)=(x_1<x_2<\dots<x_N)$. We are going to construct $Y(t+1)=(z_1<z_2<\dots<z_N)$.

Recall that

$$z_i\in\mathfrak{X}^{S+1,t+1}=\{x\in\mathbb{Z}\mid\max(0,t+S-T+2)\le x\le\min(t+N,S+N)\}.$$

$Y(t)$ and $X(t+1)$ satisfy (3). This implies that x_i-y_i is equal to either -1, 0 or 1 for every i.

• First, consider all indices i such that $x_i-y_i=1$. For every such i we set $z_i=x_i$.

• Second, consider all indices i such that $x_i-y_i=-1$ and set $z_i=y_i$.

• Finally, consider all remaining indices, i.e. all i such that $x_i = y_i$. Divide the corresponding x_i's into blocks of neighboring integers of distance at least one from each other. Call such a block a (k,l)–block, where k is the smallest number in the block and l is its size. Thus, we have

$$x_i = y_i = k, \quad x_{i+1} = y_{i+1} = k+1, \quad \ldots, \quad x_{i+l-1} = y_{i+l-1} = k+l-1$$

and

$$y_{i-1} < k-1, \quad y_{i+l} > k+l.$$

For each (k,l)–block we perform the following procedure: consider random variable ξ distributed according to $D(k+T-t-S-1, k+1, l)$ (ξ's corresponding to different (k,l)–blocks are independent). Set $z_i = x_i$ for the first ξ integers of the block (their coordinates are $k, k+1, \ldots, k+\xi-1$) and set $z_i = x_i + 1$ for the rest of the block.

At Figure 4 we provide an example of constructing $Y(t+1)$ using $X(t+1)$ and $Y(t)$: there is only one (k,l)–block and it splits into two groups, here $\xi = 2$.

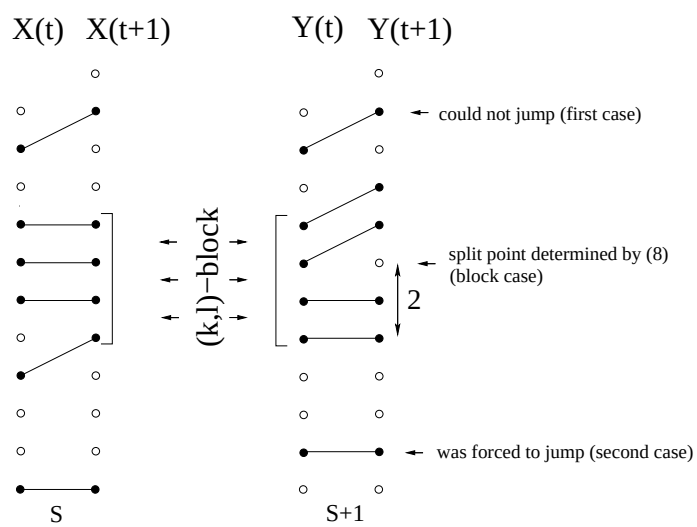

Figure 4. Example of (k,l)–block split, $l = 4$, $\xi = 2$.

Theorem 4.1. *The algorithm described above is precisely $S \mapsto S+1$ Markov step given by $P^S_{S\mapsto S+1}$.*

Proof. As was shown in the previous section, the transition $S \to S+1$ consists of T updates. Namely, given $Y(t)$ and $X(t+1)$ we define $Y(t+1)$ by

$$\mathrm{Prob}\{Y(t+1) = Z\} = \frac{P^{S+1,t}_{t+}(Y(t), Z) \cdot P^{S+1,t+1}_{S-}(Z, X(t+1))}{(P^{S+1,t}_{t+} P^{S+1,t+1}_{S-})(Y(t), X(t+1))}.$$

Let $Y(t) = (y_1 < y_2 < \cdots < y_N)$, $X(t+1) = (x_1 < x_2 < \cdots < x_N)$ and we are defining $Y(t+1) = (z_1 < z_2 < \cdots < z_N)$. Inequality (3) implies that for every i the difference $x_i - y_i$ is equal to either -1, 0 or 1. Thus, we have 3 cases:

1. $x_i - y_i = 1$: Definitions of $P_{t+}^{S+1,t}$ and $P_{S-}^{S+1,t+1}$ imply that both $z_i - y_i$ and $z_i - x_i$ must be equal to either 0 or 1. Consequently, z_i is uniquely defined and $z_i = x_i$.

2. $x_i - y_i = -1$: Again z_i is uniquely defined, and $z_i = y_i$.

3. $x_i - y_i = 0$: This is the only nontrivial case. Here one has two possibilities, either $z_i = x_i$ or $z_i = x_i + 1$.

   When we pass from x_i to z_i, every (k, l)–block is split into at most two groups. Namely, we have $l + 1$ possibilities for the split point $j \in \{0, 1, \ldots, l\}$ $0, 1 \ldots, l$: $z_i = x_i$ for the lowest j points of the block and $z_i = x_i + 1$ for the other points of this block.

   Now we want to compute the probabilities of different splits of the blocks. We have

$$\begin{aligned}\mathrm{Prob}\{Y(t+1) = Z\} &= \frac{P_{t+}^{S+1,t}(Y(t), Z) \cdot P_{S-}^{S+1,t+1}(Z, X(t+1))}{(P_{t+}^{S+1,t} P_{S-}^{S+1,t+1})(Y(t), X(t+1))} \\ &= \prod_{i: z_i = y_i} (y_i + T - t - S - 1) \cdot \prod_{i: z_i = y_i + 1} (N + S - y_i) \\ &\times \prod_{i: z_i = x_i} (N + S - x_i) \cdot \prod_{i: z_i = x_i + 1} (x_i + 1) \cdot (\text{factors independent of } Z)\end{aligned}$$

   This formula implies that blocks split independently. For each (k, l)–block the probability of split position j is equal to

$$\prod_{a=k}^{k+j-1} (a + T - t - S - 1)(N + S - a) \cdot \prod_{a=k+j}^{k+l-1} (a + 1)(N + S - a) \times (\text{factors independent of } j)$$

   Since $(N + S - a)$ is present in both products, this probability can be written as

$$\prod_{a=k}^{k+j-1} (a + T - t - S - 1) \cdot \prod_{a=k+j}^{k+l-1} (a + 1) \cdot (\text{factors independent of } j)$$

   which is exactly the distribution $D(k + T - t - S - 1, k + 1, l)$.

□

4.2 Algorithm for $S \mapsto S - 1$ step

The $S \mapsto S - 1$ step algorithm is very similar to the $S \mapsto S + 1$ one.

Suppose we are given $X = (X(0), X(1), \ldots, X(T)) \in \Omega(N, T, S)$. We want to construct $Y = (Y(0), Y(1), \ldots, Y(T)) \in \Omega(N, T, S - 1)$.

As above, note that $Y(0)$ is uniquely defined,

$$Y(0) = (0, 1, \dots, N-1).$$

Then we again perform T sequential updates, i.e. for $t = 0, 1, \dots T-1$ we construct $Y(t+1)$ using $Y(t)$ and $X(t+1)$. Let us describe each step.

Let $Y(t) = (y_1 < y_2 < \dots < y_N)$ and $X(t+1) = (x_1 < x_2 < \dots < x_N)$. We are going to construct $Y(t+1) = (z_1 < z_2 < \dots < z_N)$.

Recall that

$$z_i \in \mathfrak{X}^{S-1,t+1} = \{x \in \mathbb{Z} \mid \max(0, t+S-T) \le x \le \min(t+N, S+N-2)\}.$$

$Y(t)$ and $X(t+1)$ satisfy $(P_{t+}^{S-1,t} P_{S+}^{S-1,t+1})(Y(t), X(t+1)) > 0$. This implies that $x_i - y_i$ is equal to either 0, 1 or 2 for every i.

• First, consider all indices i such that $x_i - y_i = 0$. For every such i we set $z_i = x_i$.

• Second, consider all indices i such that $x_i - y_i = 2$ and set $z_i = y_i + 1$.

• Finally, consider all remaining indices, i.e. all i such that $x_i = y_i + 1$. Divide the corresponding x_i's into blocks of neighboring integers of distance at least one from each other. Call such a block a $(k,l)'$–block, where k is the smallest number in the block and l is its size. Thus, we have

$$x_i = y_i + 1 = k, \quad x_{i+1} = y_{i+1} + 1 = k+1, \quad \dots, \quad x_{i+l-1} = y_{i+l-1} = k+l-1.$$

For each $(k,l)'$–block we perform the following procedure: consider random variable ξ distributed according to $D(N+t-k+1, N+S-k-1, l)$ (ξ's corresponding to different $(k,l)'$–blocks are independent). Set $z_i = y_i$ for the first ξ integers of the block (their coordinates are $k-1, k, \dots, k+\xi-2$) and set $z_i = y_i + 1$ for the rest of the block.

Theorem 4.2. *The algorithm described above is precisely $S \mapsto S-1$ Markov step defined by $P^S_{S \mapsto S-1}$.*

The proof is similar to Theorem 4.1 and we omit it.

4.3 Numeric experiments

The $S \mapsto S \pm 1$ steps can be used in different ways.

1. Suppose that our aim is to sample a random tiling (equivalently, random family of paths) $\mathcal{T} \in \Omega(N,T,S)$ from the uniform measure $\mu(N,T,S)$.

We start from the unique family of paths $\mathcal{T}_0 \in \Omega(N,T,0)$. Indeed, $|\Omega(N,T,0)| = 1$.

Next, we perform S steps. During rth step we construct $\mathcal{T}_r \in \Omega(N,T,r)$ distributed as $\mu(N,T,r)$, using already constructed family of paths $\mathcal{T}_{r-1} \in \Omega(N,T,r-1)$. Theorem 3.1 implies that $\mathcal{T}_S$ is the desired random element of $\Omega(N,T,S)$.

Let us estimate the number of operations. Every update takes $O(N)$ operations. For every $S \to S+1$ step we have to perform T updates. Consequently to sample from $\Omega(N,T,S)$ we need $O(NTS)$ operations.

On Figure 5 we show a random surface generated by our algorithm. Here $N = 1000$, $T = 2000$, $S = 1000$. It took less than 4 minutes on our laptop (Intel Core2 Duo 2.2GHz, 2Gb Ram) to generate this tiling. Theoretically predicted "arctic ellipse", see [CLP], is clearly visible on our picture.

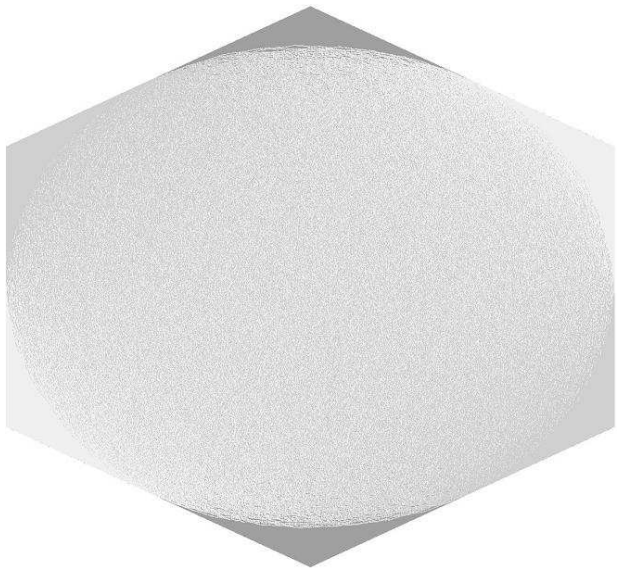

Figure 5. Random surface corresponding to the big tiling.

2. Using our steps one can also construct equilibrium dynamics $S \mapsto S+1 \mapsto S$ or $S \mapsto S-1 \mapsto S$.

On Figure 6 we show the evolution of the "filled box" tiling under $S \mapsto S+1 \mapsto S$ dynamics. Here $N = 50$, $T = 50$, $S = 20$.

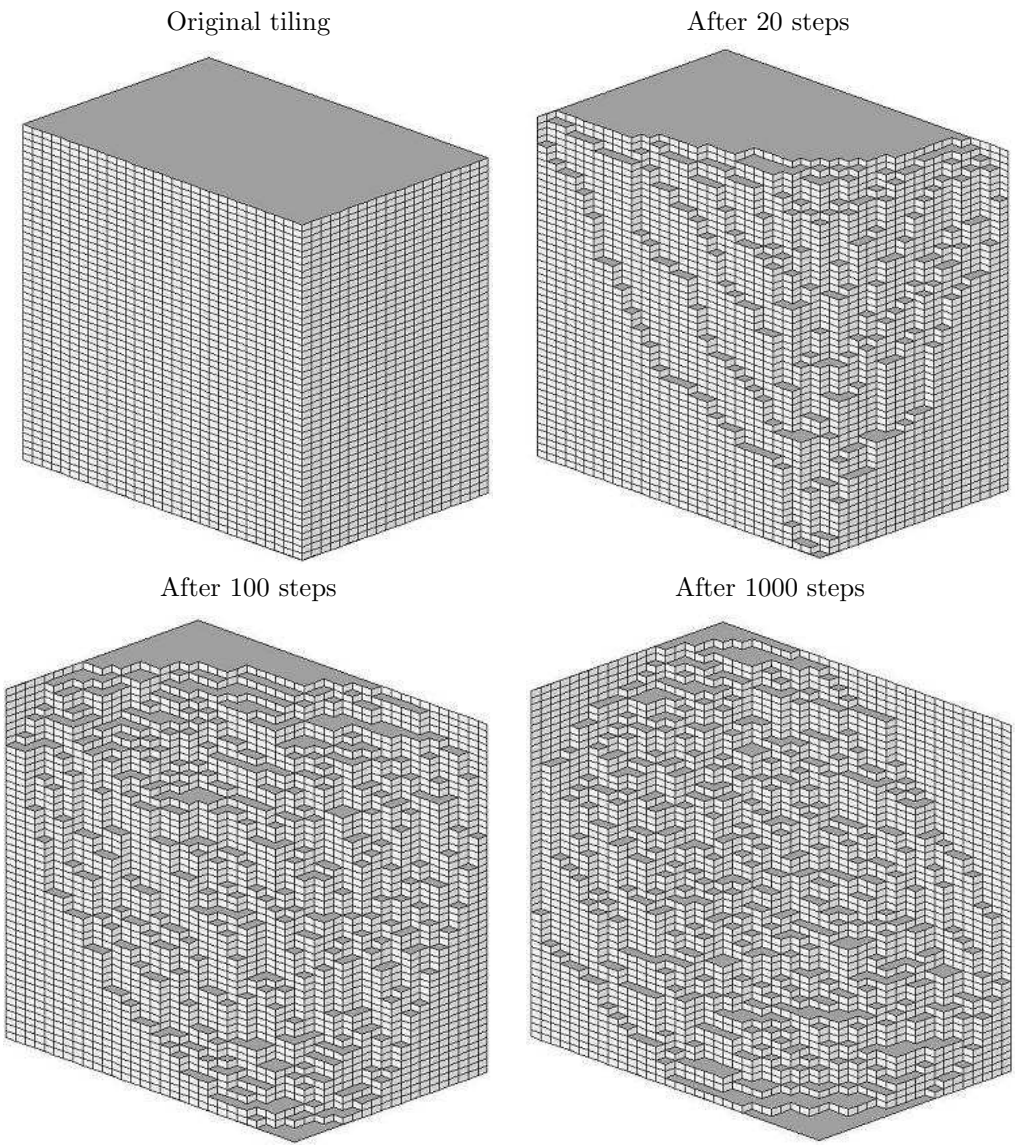

Figure 6. Evolution of tiling under $S \mapsto S+1 \mapsto S$ dynamics.

5 General 2-dimensional dynamics, its sections and correlation functions

5.1 Construction of dynamics and its correlation functions

In this section we construct a family of Markov chains on the spaces $\Omega(N,T,S)$ using the transition probabilities $P^S_{S\mapsto S+1}(X,Y)$ and $P^S_{S\mapsto S-1}(X,Y)$ introduced

in Section 3. Namely, we want to combine $S \mapsto S+1$ and $S \mapsto S-1$ steps. To fix the order of "+1" and "−1" steps we introduce an auxiliary sequence $\{\epsilon_i\}$ of +1's and −1's.

Formally, let $0 \le S_0 \le T$ and let $\epsilon = \{\epsilon_i\}_{i=1,2,\dots}$ be an arbitrary finite or infinite sequence of +1's and −1's such that

$$0 \le S_0 + \sum_{n=1}^{m} \epsilon_i \le T$$

for every m. The last condition is necessary to ensure that the state spaces $\Omega(N,T,S)$ of the process are not empty.

For any integer r we denote

$$S(r) = \begin{cases} S_0, & r = 0, \\ S_0 + \sum_{i=1}^{r} \epsilon_i, & r > 0. \end{cases}$$

Given S_0 and ϵ_i let us define a Markov chain

$$\mathfrak{M}_{S_0,\epsilon}(r), \quad r = 0, 1, \dots$$

$\mathfrak{M}_{S_0,\epsilon}(r)$ takes values in $\Omega(N,T,S(r))$, its initial distribution is $\mu(N,T,S_0)$:

$$\mathrm{Prob}\{M_{S_0,\epsilon}(0) = X\} = \mu(N,T,S_0)(\{X\}).$$

Transition probabilities of our process are given by

$$\mathrm{Prob}\{\mathfrak{M}_{S_0,\epsilon}(r+1) = Y \mid \mathfrak{M}_{S_0,\epsilon}(r) = X\} = \begin{cases} P^{S(r)}_{S\mapsto S+1}(X,Y), & \text{if } \epsilon_{r+1} = 1 \\ P^{S(r)}_{S\mapsto S-1}(X,Y), & \text{if } \epsilon_{r+1} = -1. \end{cases}$$

Theorems 3.1 and 3.2 imply that one-time distributions of $\mathfrak{M}_{S_0,\epsilon}(r)$ are exactly $\mu(N,T,S(r))$.

Example 1. If $S_0 = 0$ and $\{\epsilon_i\} = \{1,1,1,\dots\}$, then $\mathfrak{M}_{S_0,\epsilon}(r)$ is precisely the chain used for the random tiling sampling in Section 4.3.

Example 2. If we set $\epsilon_i = (-1)^i$ and then restrict $\mathfrak{M}_{S_0,\epsilon}(r)$ on even r we get a stationary Markov chain from Section 4.3.

Recall that $X \in \Omega(N,T,S)$ is a sequence $X = (X(0), X(1), \dots, X(T))$. Given a trajectory of the Markov chain $\mathfrak{M}_{S_0,\epsilon}(r)$ we can construct a point configuration in $\mathbb{Z}^3$ with coordinates (r,t,x) such that the point (r_0,t_0,x_0) is occupied if and only if $x_0 \in (\mathfrak{M}_{S_0,\epsilon}(r_0))(t_0)$. Thus, our Markov chain defines a measure on such point configuration or, equivalently, a random point process in $\mathbb{Z}^3$. Denote it by $\mathbb{M}$.

Define the nth *correlation function* of $\mathbb{M}$ by

$$\begin{aligned} &R_n(r_1,t_1,x_1;r_2,t_2,x_2;\dots;r_n,t_n,x_n) \\ &= \mathbb{M}\{\mathcal{M} \in \mathrm{Conf}(\mathbb{Z}^3) \mid (r_1,t_1,x_1) \in \mathcal{M}, (r_2,t_2,x_2) \in \mathcal{M}, \dots, (r_n,t_n,x_n) \in \mathcal{M}\}. \end{aligned}$$

These correlation functions uniquely define the process $\mathbb{M}$.

Through the rest of the paper we concentrate on computation of correlation functions R_n. Unfortunately, we are not able to fully describe R_n for all possible arguments. But we can compute R_n on certain two-dimensional sections of $\mathbb{Z}^3$.

The main result of this section is the following statement.

Theorem 5.1. *Let $r_1 \le r_2 \le \dots \le r_n$, $t_1 \ge t_2 \ge \dots \ge t_n$. Then*

$$R_n(r_1, t_1, x_1; r_2, t_2, x_2; \dots; r_n, t_n, x_n) = \det \left[K(r_i, t_i, x_i; r_j, t_j, x_j) \right]_{i,j=1,\dots,n},$$

where

$$K(r, t, x; r', t', x') = \sum_{i=0}^{N-1} \frac{1}{c_i^{r',t';r,t}} \Psi_i^{S(r),t}(x) \Psi_i^{S(r'),t'}(y),$$

$$\text{if } r \ge r', t \le t';$$

$$K(r, t, x; r', t', x') = -\sum_{i \ge N} c_i^{r,t;r',t'} \Psi_i^{S(r),t}(x) \Psi_i^{S(r'),t'}(y),$$

$$\text{if } r < r' \text{ or } r = r', t > t';$$

$$c_i^{r,t;r,t} = 1,$$

$$c_i^{r,t;r',t'} = \prod_{k=t'+1}^{t} c_{t-}^{S(r),k}(i) \prod_{k=r}^{r'-1} c_{S\epsilon_{k+1}}^{S(k),t}(i),$$

where $S\epsilon_{k+1}$ stands for $S+$ if $\epsilon_{k+1} = +1$ and $S-$ otherwise.

In Section 6 we will study the bulk asymptotics of these correlation functions.

For $r_1 = r_2 = \dots = r_n$, Theorem 5.1 was obtained in [J2], [JN], [Gor].

5.2 Admissible sections

We call a sequence $\mathcal{A} = ((r_0, t_0), (r_1, t_1), \dots (r_n, t_n))$ an *admissible section* of $\mathbb{Z}^2$ provided that

1. $(r_i, t_i) \in \{0, 1 \dots\} \times \{0, 1, \dots, T\}$
2. $r_0 \le r_1 \le \dots \le r_n$
3. $t_0 \ge t_1 \ge \dots \ge t_n$
4. For every $i = 0, 1, \dots, n-1$ either $r_{i+1} = r_i + 1$, $t_{i+1} = t_i$ or $r_{i+1} = r_i$, $t_{i+1} = t_i - 1$

Figure 6 gives an example of an admissible section.

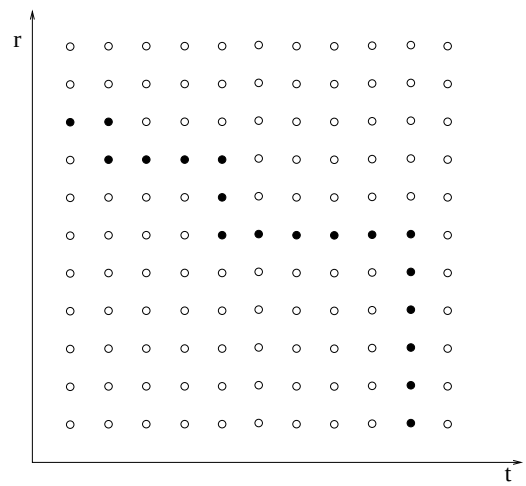

Figure 6. Admissible section.

Given an admissible section $\mathcal{A}$ we introduce a *sectional process* $\mathcal{M}_{\mathcal{A}}(h)$, $\quad h = 0, 1 \ldots, n$, by

$$\mathcal{M}_{\mathcal{A}}(h) = \left(\mathfrak{M}_{S_0,\epsilon}(r_h)\right)(t_h)$$

Theorem 5.2. *$\mathcal{M}_{\mathcal{A}}(h)$ is a Markov chain with initial distribution*

$$\mathrm{Prob}\{\mathcal{M}_{\mathcal{A}}(0) = X\} = \rho_{N,T}^{S(r_0),t_0}(X)$$

and transition probabilities

$$\begin{aligned}\mathrm{Prob}\{\mathcal{M}_{\mathcal{A}}(h+1) = Y \mid \mathcal{M}_{\mathcal{A}}(h) = X\}& \\ = \begin{cases} P_{t-}^{S(r_h),t_h}(X,Y), & \text{if } t_{h+1} = t_h - 1, \\ P_{S+}^{S(r_h),t_h}(X,Y), & \text{if } S(r_{h+1}) = S(r_h) + 1, \\ P_{S-}^{S(r_h),t_h}(X,Y), & \text{if } S(r_{h+1}) = S(r_h) - 1. \end{cases}\end{aligned}$$

Proof. This theorem follows from [BF, Proposition 2.7]. Let us explain the correspondence between our notations and notations of [BF]. To avoid confusions we denote by τ the time variable t from [BF]. Then τ corresponds to time r of Markov chain $\mathfrak{M}_{S_0,\epsilon}(r)$; k of [BF] corresponds to t; state space $\mathcal{S}_k(\tau)$ is $\mathcal{X}^{S(\tau),k}$; matrices $\Lambda_{k-1}^k(\,\cdot\,,\,\cdot\mid\tau)$ of [BF] are $P_{t-}^{S(\tau),k}(\,\cdot\,,\,\cdot\,)$; matrices $P_k(\,\cdot\,,\,\cdot\mid\tau)$ are $P_{S\epsilon_{\tau+1}}^{S(\tau),k}(\,\cdot\,,\,\cdot\,)$; the commutation relations $\Lambda_{k-1}^k P_{k-1} = P_k \Lambda_{k-1}^k$ are exactly $P_{S\epsilon}^{S,t} P_{t-}^{S+\epsilon,t} = P_{t-}^{S,t} P_{S\epsilon}^{S,t-1}$ above; τ_0 is r_0 and $m_n(\,\cdot\,)$ of [BF] corresponds to $\rho_{S(r_0),t_0}$. $\square$

5.3 Correlation functions

To compute the correlation functions R_n we are going to use a variant of the Eynard-Metha theorem (see [EM] and [BO, Section 7.4]). Let us state it first.

Proposition 5.3. *Assume that for every time moment h we are given an orthonormal system $\{g_n^h\}_{n\geq 0}$ in linear space $l_2(\{0,1,\dots,L\})$ and a set of numbers $c_0^h, c_1^h, \dots$. Denote*

$$v_{h,h+1}(x,y) = \sum_{n\geq 0} c_n^h g_n^h(x) g_n^{h+1}(y).$$

Assume also that we are given a discrete time Markov process $\mathcal{P}_h$ taking values in N–tuples of elements of the set $\{0,1,\dots,L\}$, with one–dimensional distributions

$$\left(\det\left[g_{i-1}^h(x_j)\right]_{i,j=1,\dots,N}\right)^2$$

and transition probabilities

$$\frac{\det\left[v_{h,h+1}(x_i,y_j)\right]_{i,j=1,\dots,N} \det\left[g_{i-1}^{h+1}(y_j)\right]_{i,j=1,\dots,N}}{\det\left[g_{i-1}^h(x_j)\right]_{i,j=1,\dots,N} \prod_{n=0}^{N-1} c_n^h}.$$

Then

$$\mathrm{Prob}\{x_1 \in \mathcal{P}_{k_1}, \dots, x_n \in \mathcal{P}_{k_n}\} = \det\left[K(k_i,x_i;k_j,x_j)\right]_{i,j=1,\dots,n},$$

where

$$K(k,x;l,y) = \sum_{i=0}^{N-1} \frac{1}{c_i^{l,k}} g_i^k(x) g_i^l(y),\ k \geq l;$$
$$K(k,x;l,y) = -\sum_{i\geq N} c_i^{k,l} g_i^k(x) g_i^l(y),\ k < l;$$
$$c_i^{k,k} = 1,\ c_i^{k,l} = c_i^k \cdot c_i^{k+1} \cdot \dots \cdot c_i^{l-1}.$$

Now set $\mathcal{P}_h := \mathcal{M}_{\mathcal{A}}(h)$. Then we can take orthonormal functions g_n^h to be the functions $\Psi_n^{S(r_h),t_h}(x)$ defined in Section 2.4, and

$$c_i^h := \begin{cases} c_{t-}^{S(r_h),t_h}(i), & \text{if } t_{h+1} = t_h - 1, \\ c_{S+}^{S(r_h),t_h}(i), & \text{if } S(r_{h+1}) = S(r_h) + 1, \\ c_{S-}^{S(r_h),t_h}(i), & \text{if } S(r_{h+1}) = S(r_h) - 1, \end{cases} \tag{9}$$

$$v_{h,h+1}(x,y) := \begin{cases} v_{t-}^{S(r_h),t_h}(x,y), & \text{if } t_{h+1} = t_h - 1, \\ v_{S+}^{S(r_h),t_h}(x,y), & \text{if } S(r_{h+1}) = S(r_h) + 1, \\ v_{S-}^{S(r_h),t_h}(y,x), & \text{if } S(r_{h+1}) = S(r_h) - 1. \end{cases}$$

Proposition 5.4. *Markov chain $\mathcal{M}_{\mathcal{A}}(h)$ satisfies the assumptions of Proposition 5.3.*

Proof. By Theorem 2.4 one-dimensional distributions of $\mathcal{M}_{\mathcal{A}}(h)$ are given by the measures $\rho_{S(r_h),t_h}$. These are Hahn *orthogonal polynomial ensembles* (see e.g. [J2],[Gor]). It is well known (see for example [Kön, Section 2.7]) that this distribution can be written in the form

$$\rho_{S(r_h),t_h}(X) = \left(\det\left[\Psi_{i-1}^{S(r_h),t_h}(x_j)\right]_{i,j=1,\dots,N}\right)^2.$$

Propositions 2.7–2.9 imply that the transition probabilities can be expressed in the required form too. □

Applying Proposition 5.3 we obtain the following

Proposition 5.5.

$$\mathrm{Prob}\{x_1 \in \mathcal{M}_{\mathcal{A}}(h_1), \dots, x_n \in \mathcal{M}_{\mathcal{A}}(h_n)\} = \det\left[K(k_i, x_i; k_j, x_j)\right]_{i,j=1,\dots,n},$$

$$K(k,x;l,y) = \sum_{i=0}^{N-1} \frac{1}{c_i^{l,k}} \Psi_i^{S(r_k),t_k}(x)\Psi_i^{S(r_l),t_l}(y),\ k \geq l;$$
$$K(k,x;l,y) = -\sum_{i \geq N} c_i^{k,l} \Psi_i^{S(r_k),t_k}(x)\Psi_i^{S(r_l),t_l}(y),\ k < l;$$
$$c_i^{k,k} = 1,\ c_i^{k,l} = c_i^k \cdot c_i^{k+1} \cdot \dots \cdot c_i^{l-1},$$

and coefficients c_i^j *are given by* (9).

Proof of Theorem 5.1. If $r_1 \leq r_2 \leq \dots \leq r_n$, $t_1 \geq t_2 \geq \dots \geq t_n$, then the sequence $\{(r_1,t_1),\dots,(r_n,t_n)\}$ can be included into some admissible section $\mathcal{A}$. Applying Proposition 5.5 and substituting the values of all parameters c_i^j we obtain the desired formula for correlation functions R_n. □

6 Bulk limits

In this section we aim to compute so-called "bulk limits" of the correlation functions introduced in Section 5.

Note that while in the previous sections parameters N, T were always the same, through this section they will change.

We are interested in the following limit regime: Let us fix positive numbers $\tilde{S}_0$, $\tilde{T}$, $\tilde{N}$, $\tilde{t}$, $\tilde{x}$. Introduce a small parameter $\varepsilon \ll 1$, and set

$$S_0 = \tilde{S}_0\varepsilon^{-1} + o(\varepsilon^{-1}), \quad T = \tilde{T}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad N = \tilde{N}\varepsilon^{-1} + o(\varepsilon^{-1}).$$

Consider also integer valued functions $t_i = t_i(\varepsilon)$ and $x_i = x_i(\varepsilon)$, $i = 1,\dots,n$, such that

$$\lim_{\varepsilon\to 0} \varepsilon t_i(\varepsilon) = \tilde{t}, \quad \lim_{\varepsilon\to 0} \varepsilon x_i(\varepsilon) = \tilde{x}, \quad i = 1,\dots,n,$$

and pairwise differences $t_i - t_j$, and $x_i - x_j$ do not depend on ε.

Then correlation functions R_n defined in Theorem 5.1 tend to a limit $\hat{R}_n$ which depends on the parameters of the limit regime $\tilde{S}_0$, $\tilde{T}$, $\tilde{N}$, $\tilde{t}$, $\tilde{x}$.

We consider the region where the limit correlation functions are nontrivial. This region is commonly referred to as "bulk" and it is characterized (see e.g. [Gor]), by the fact that the expression

$$\frac{-\tilde{N}(\tilde{N}+\tilde{T})+(-\tilde{x}+\tilde{S}_0+\tilde{N})(\tilde{t}+\tilde{N}-\tilde{x})+\tilde{x}(\tilde{T}+\tilde{x}-\tilde{S}_0-\tilde{t})}{2\sqrt{\tilde{x}(-\tilde{x}+\tilde{S}_0+\tilde{N})(\tilde{t}+\tilde{N}-\tilde{x})(\tilde{x}+\tilde{T}-\tilde{S}_0-\tilde{t})}}.$$

is strictly between -1 and 1. Denote by $\phi=\phi(\tilde{S}_0,\tilde{T},\tilde{N},\tilde{t},\tilde{x})$ the arccosine of this expression.

Let us denote by Q_i the triplet (r_i,t_i,x_i).

Theorem 6.1. *Let $r_1\le r_2\le\cdots\le r_n$, $t_1\ge t_2\ge\cdots\ge t_n$ and $R_n(Q_1,\dots,Q_n)$ be defined as in Theorem 5.1. Then*

$$\lim_{\varepsilon\to 0} R_n(Q_1,\dots,Q_n)=\det[K_{ij}^{\mathrm{bulk}}]_{i,j=1,\dots,n},$$

where for (i,j) such that $r_i<r_j$, or $r_i=r_j$, $t_i>t_j$

$$K_{ij}^{\mathrm{bulk}}=\frac{1}{2\pi i}\int_{\gamma_-}(1+c_1z)^{t_i-t_j}\cdot\prod_{k=r_i+1}^{r_j}\left(1+c_2z^{-\epsilon_k}\right)\cdot\frac{dz}{z^{x_i-x_j+1}}$$

and for (i,j) such that $r_i\ge r_j$, $t_i\le t_j$

$$K_{ij}^{\mathrm{bulk}}=\frac{1}{2\pi i}\int_{\gamma_+}(1+c_1z)^{t_i-t_j}\cdot\prod_{k=r_j+1}^{r_i}\left(1+c_2z^{-\epsilon_k}\right)^{-1}\cdot\frac{dz}{z^{x_i-x_j+1}},$$

$$c_1=\sqrt{\frac{\tilde{x}(\tilde{S}_0+\tilde{N}-\tilde{x})}{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{t}+\tilde{N}-\tilde{x})}},$$

$$c_2=\sqrt{\frac{\tilde{x}(\tilde{t}+\tilde{N}-\tilde{x})}{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{S}_0+\tilde{N}-\tilde{x})}}.$$

Here $\gamma_\pm$ are contours in $\mathbb{C}$ joining $e^{-i\phi(\tilde{S}_0,\tilde{T},\tilde{N},\tilde{t},\tilde{x})}$ and $e^{i\phi(\tilde{S}_0,\tilde{T},\tilde{N},\tilde{t},\tilde{x})}$ and crossing $\mathbb{R}_\pm$, respectively.

Comments. The limiting correlation functions

$$\hat{R}_n(Q_1,\dots,Q_n)=\det[K_{ij}^{\mathrm{bulk}}]_{i,j=1,\dots,n}$$

are correlation functions of the limit process defined on a fixed admissible section. The proof of the existence of the limit process can be found for instance in [Bor, Lemma 4.1]. The limit process satisfies certain Gibbs property, see [BS].

The case $r_1=r_2=\cdots=r_n=0$ was thoroughly studied in [Gor]. The limit process for this case first appeared in [OR]. Our argument is based on the facts proved in [Gor].

Proof. Our first goal is to find the limit of the correlation kernels

$$\lim_{\varepsilon\to 0} K(r_i, t_i(\varepsilon), x_i(\varepsilon); r_j, t_j(\varepsilon), x_j(\varepsilon)).$$

Let us introduce six auxiliary families of $\mathbb{Z}\times\mathbb{Z}$ matrices or, equivalently, operators acting in $l_2(\mathbb{Z})$. Set

$$\mathbf{P}^{N,T,S,t}(x,y) = \begin{cases} \sum\limits_{k=0}^{N-1} \Psi_k^{S,t}(x)\Psi_k^{S,t}(y), & x,y\in\mathfrak{X}^{S,t},\\ 0 & \text{for other } x,y,\end{cases}$$

(recall that $\Psi_k^{S,t}(x)$ and $\mathfrak{X}^{S,t}$ depend on N and T, although these indices are omitted)

$$\mathbf{P}'^{N,T,S,t}(x,y) = \begin{cases} -\sum\limits_{k\geq N} \Psi_k^{S,t}(x)\Psi_k^{S,t}(y), & x,y\in\mathfrak{X}^{S,t},\\ 0 & \text{for other } x,y.\end{cases}$$

Observe that $\mathbf{P}-\mathbf{P}' = \mathrm{Id}_{\mathfrak{X}^{S,t}}$. Define also

$$V_{t+}^{N,T,S,t}(x,y) = \begin{cases} \sum\limits_{k\geq 0} c_{t+}^{S,t}(k)\Psi_k^{S,t}(x)\Psi_k^{S,t+1}(y) & x\in\mathfrak{X}^{S,t}, y\in\mathfrak{X}^{S,t+1},\\ 0 & \text{for other } x,y,\end{cases}$$

$$V_{t+}^{N,T,S,t}(x,y) = \begin{cases} \sum\limits_{k\geq 0} c_{t-}^{S,t}(k)\Psi_k^{S,t}(x)\Psi_k^{S,t-1}(y) & x\in\mathfrak{X}^{S,t}, y\in\mathfrak{X}^{S,t-1},\\ 0 & \text{for other } x,y,\end{cases}$$

$$V_{S+}^{N,T,S,t}(x,y) = \begin{cases} \sum\limits_{k\geq 0} c_{S+}^{S,t}(k)\Psi_k^{S,t}(x)\Psi_k^{S+1,t}(y) & x\in\mathfrak{X}^{S,t}, y\in\mathfrak{X}^{S+1,t},\\ 0 & \text{for other } x,y,\end{cases}$$

$$V_{S-}^{N,T,S,t}(x,y) = \begin{cases} \sum\limits_{k\geq 0} c_{S-}^{S,t}(k)\Psi_k^{S,t}(x)\Psi_k^{S-1,t}(y) & x\in\mathfrak{X}^{S,t}, y\in\mathfrak{X}^{S-1,t},\\ 0 & \text{for other } x,y.\end{cases}$$

These are (trivial) extensions of finite matrices $v_{t\pm}^{S,t}$, $v_{S\pm}^{S,t}$ of Section 2.4.

Now let

$$\begin{array}{lll} S=\tilde{S}_0\varepsilon^{-1}+o(\varepsilon^{-1}), & T=\tilde{T}\varepsilon^{-1}+o(\varepsilon^{-1}), & N=\tilde{N}\varepsilon^{-1}+o(\varepsilon^{-1}),\\ t=\tilde{t}\varepsilon^{-1}+o(\varepsilon^{-1}), & x=[\tilde{x}\varepsilon^{-1}]+\varkappa, & y=[\tilde{x}\varepsilon^{-1}]+\nu,\end{array}$$

and send ε to 0. All 6 families tend to some limits. Let us denote these limit operators by $\hat{\mathbf{P}}(\varkappa,\nu)$, $\hat{\mathbf{P}}'(\varkappa,\nu)$, $\hat{V}_{t+}(\varkappa,\nu)$, $\hat{V}_{t-}(\varkappa,\nu)$, $\hat{V}_{S+}(\varkappa,\nu)$ and $\hat{V}_{S-}(\varkappa,\nu)$, respectively. These limit operators depend on the parameters of limit regime $\tilde{N}$,$\tilde{T}$,$\tilde{S}_0$, $\tilde{t}$, $\tilde{x}$.

As we are dealing with linear operators in $l_2(\mathbb{Z})$, it is convenient to employ the Fourier transform

$$F: l_2(\mathbb{Z}) \mapsto L_2(S^1),$$

where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We denote the images of our operators under Fourier transform by $F\hat{\mathbf{P}}$, $F\hat{\mathbf{P}'}$ and so on.

Proposition 6.2. *When $\varepsilon \to 0$ operators $\mathbf{P}^{N,T,S,t}$ and $\mathbf{P'}^{N,T,S,t}$ strongly converge to limits $\hat{\mathbf{P}}$ and $\hat{\mathbf{P}'}$, respectively.*

$F\hat{\mathbf{P}}$ is the operator of multiplication by the characteristic function of the right arc of the unit circle contained between the angles $-\phi$ and ϕ.

$F\hat{\mathbf{P}'}$ is the operator of multiplication by the characteristic function of the left arc of the unit circle contained between the angles $-\phi$ and ϕ.

Proof. See Section 3.2 in [Gor]. □

Proposition 6.3. *When $\varepsilon \to 0$ operators $V_{t+}^{N,T,S,t}$ strongly converge to a limit $\hat{V}_{t+}$. $F\hat{V}_{t+}$ is the operator of multiplication by the function*

$$\sqrt{\frac{(\tilde{T} - \tilde{t} - \tilde{S}_0 + \tilde{x})(\tilde{t} + \tilde{N} - \tilde{x})}{(\tilde{t} + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{t})}} + z^{-1} \cdot \sqrt{\frac{(\tilde{S}_0 + \tilde{N} - \tilde{x})\tilde{x}}{(\tilde{t} + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{t})}}.$$

Proof. See Section 3.3 in [Gor] □

Proposition 6.4. *When $\varepsilon \to 0$ operators $V_{t-}^{N,T,S,t}$ strongly converge to a limit $\hat{V}_{t-}$. $F\hat{V}_{t-}$ is the operator of multiplication by the function*

$$\sqrt{\frac{(\tilde{T} - \tilde{t} - \tilde{S}_0 + \tilde{x})(\tilde{t} + \tilde{N} - \tilde{x})}{(\tilde{t} + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{t})}} + z \cdot \sqrt{\frac{(\tilde{S}_0 + \tilde{N} - \tilde{x})\tilde{x}}{(\tilde{t} + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{t})}}.$$

Proof. Recall that $c_{t-}^{S,t}(k) = c_{t+}^{S,t-1}(k)$ and $c_{t-}^{S,t}(k)$ is a real number. Consequently, $V_{t-}^{N,T,S,t} = (V_{t+}^{N,T,S,t-1})^*$. Therefore, operators $V_{t-}^{N,T,S,t}$ tend to a limit $\hat{V}_{t-}$ and $F\hat{V}_{t-} = (F\hat{V}_{t+})^*$. Finally, observe that if A is the operator of multiplication by z^{-1} acting in $L_2(S^1)$, then A^* is the operation of multiplication by z. Thus, $F\hat{V}_{t-}$ is given by the desired formula. □

Proposition 6.5. *When $\varepsilon \to 0$ operators $V_{S+}^{N,T,S,t}$ strongly converge to a limit $\hat{V}_{S+}$. $F\hat{V}_{S+}$ is the operator of multiplication by the function*

$$\sqrt{\frac{(\tilde{T} - \tilde{t} - \tilde{S}_0 + \tilde{x})(\tilde{S}_0 + \tilde{N} - \tilde{x})}{(\tilde{S}_0 + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{S}_0)}} + z^{-1} \cdot \sqrt{\frac{(\tilde{t} + \tilde{N} - \tilde{x})\tilde{x}}{(\tilde{S}_0 + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{S}_0)}},$$

Proof. Results of Section 2 imply that $V_{S+}^{N,T,S,t} \leftrightarrow V_{t+}^{N,T,t,S}$ under the $S \leftrightarrow t$. Perform this involution, then send ε to zero and then switch $S \leftrightarrow t$ again. Proposition 6.3 implies the result. □

Proposition 6.6. *When $\varepsilon \to 0$ operators $V_{S-}^{N,T,S,t}$ strongly converge to a limit $\hat{V}_{S-}$. $F\hat{V}_{S-}$ is the operator of multiplication by the function*

$$\sqrt{\frac{(\tilde{T} - \tilde{t} - \tilde{S}_0 + \tilde{x})(\tilde{S}_0 + \tilde{N} - \tilde{x})}{(\tilde{S}_0 + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{S}_0)}} + z \cdot \sqrt{\frac{(\tilde{t} + \tilde{N} - \tilde{x})\tilde{x}}{(\tilde{S}_0 + \tilde{N})(\tilde{T} + \tilde{N} - \tilde{S}_0)}},$$

Proof. Same argument as in Proposition 6.5. □

Now we proceed to the proof of Theorem 6.1. The correlation kernel $K(r_i, t_i, x_i; r_j, t_j, x_j)$ defines a $\mathbb{Z}\times\mathbb{Z}$ matrix or, equivalently, an operator acting in $l_2(\mathbb{Z})$ by

$$\mathcal{K}(x,y) = \begin{cases} K(r_i, t_i, x; r_j, t_j, y) & x \in \mathfrak{X}_{N,T}^{S(r_i),t_i}, y \in \mathfrak{X}_{N,T}^{S(r_j),t_j}, \\ 0 & \text{for other } x, y. \end{cases}$$

First, suppose that $r_i < r_j$ or $r_i = r_j$, $t_i > t_j$. In this case $\mathcal{K}(x,y)$ can be decomposed in the following way:

$$\begin{aligned} \mathcal{K}(x,y) = \mathbf{P}'^{N,T,S(r_j),t_j} \cdot V_{t-}^{N,T,S(r_j),t_j+1} \cdot V_{t-}^{N,T,S(r_j),t_j+2} \dots V_{t-}^{N,T,S(r_j),t_i} \\ \times V_{S\epsilon_{r_j}}^{N,T,S(r_j-1),t_i} \cdot V_{S\epsilon_{r_j-1}}^{N,T,S(r_j-2),t_i} \dots V_{S\epsilon_{r_i+1}}^{N,T,S(r_i),t_i} \end{aligned} \tag{10}$$

This relation readily follows from the definition of $\mathcal{K}(x,y)$ and orthogonality relations on functions $\Psi_l^{S,t}(x)$.

Observe that norms of all factors in (10) are bounded by 1. ($\mathbf{P}'^{N,T,S(r_2),t_2}$ are ortoprojection operators; norms of V's are bounded since constants c used in their definition are bounded by 1). Consequently, convergence of each factor in (10) as $\varepsilon \to 0$ implies strong convergence of $\mathcal{K}(x,y)$ to the limit operator

$$\hat{\mathcal{K}} = \hat{\mathbf{P}}'(\hat{V}_{t-})^{t_i-t_j} \cdot \hat{V}_{S\epsilon_{r_j}} \cdot \hat{V}_{S\epsilon_{r_j-1}} \dots \hat{V}_{S\epsilon_{r_i+1}}.$$

(Indeed, multiplication of operators is jointly continuous on bounded sets in strong operator topology.) The image of $\hat{\mathcal{K}}$ under the Fourier transform is given by

$$F\hat{\mathcal{K}} = F\hat{\mathbf{P}}'(F\hat{V}_{t-})^{t_i-t_j} \cdot \prod_{k=r_i}^{r_j-1} F\hat{V}_{S\epsilon_{k+1}}.$$

Performing the inverse Fourier transform we obtain the formula

$$\begin{aligned} &\lim_{\varepsilon\to 0} K(r_i, t_i(\varepsilon), x_i(\varepsilon); r_j, t_j(\varepsilon), x_j(\varepsilon)) = \hat{\mathcal{K}}(\varkappa_i, \varkappa_j) \\ &= \frac{1}{2\pi i}\int_{e^{-i\phi}}^{e^{i\phi}} \left(\sqrt{\frac{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{t}+\tilde{N}-\tilde{x})}{(\tilde{t}+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{t})}} + z\cdot\sqrt{\frac{(\tilde{S}_0+\tilde{N}-\tilde{x})\tilde{x}}{(\tilde{t}+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{t})}} \right)^{t_i-t_j} \\ &\times \prod_{k=r_i+1}^{r_j} \left(\sqrt{\frac{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{S}_0+\tilde{N}-\tilde{x})}{(\tilde{S}_0+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{S}_0)}} + z^{-\epsilon_k}\sqrt{\frac{(\tilde{t}+\tilde{N}-\tilde{x})\tilde{x}}{(\tilde{S}_0+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{S}_0)}} \right) \\ &\qquad\qquad \times \frac{dz}{z^{x_i-x_j+1}} = const_1^{t_i-t_j} \cdot const_2^{r_j-r_i} \cdot K_{ij}^{\text{bulk}} \end{aligned}$$

with

$$const_1 = \sqrt{\frac{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{t}+\tilde{N}-\tilde{x})}{(\tilde{t}+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{t})}},$$

$$const_2 = \sqrt{\frac{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{S}_0+\tilde{N}-\tilde{x})}{(\tilde{S}_0+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{S}_0)}}.$$

(The integration is performed over the left side of the unit circle.)

The case $r_i \geq r_j$, $t_i \leq t_j$ is quite similar. The only difference is that instead of the operators $V_{t-}^{N,T,S,t}$ and $V_{S\pm}^{N,T,S,t}$ we have to use in some sense inverse operators. Actually, these operators are not invertible and limit operators $\hat{V}_{S\pm}$ and $\hat{V}_{t-}$ might be non-invertible too, but the difficulties can be avoided if we restrict all operators on the images of $\mathbf{P}^{N,T,S,t}$ and $\hat{\mathbf{P}}$. Details of this trick can be found in Section 3.3 of [Gor].

The answer for $r_i \geq r_j$, $t_i \leq t_j$ is

$$\begin{aligned}
&\lim_{\varepsilon\to 0} K(r_i, t_i(\varepsilon), x_i(\varepsilon); r_j, t_j(\varepsilon), x_j(\varepsilon)) \\
&= \frac{1}{2\pi i}\int_{e^{-i\phi}}^{e^{i\phi}} \left(\sqrt{\frac{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{t}+\tilde{N}-\tilde{x})}{(\tilde{t}+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{t})}} + z\cdot\sqrt{\frac{(\tilde{S}_0+\tilde{N}-\tilde{x})\tilde{x}}{(\tilde{t}+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{t})}}\right)^{t_i-t_j} \\
&\times \prod_{k=r_j+1}^{r_i}\left(\sqrt{\frac{(\tilde{T}-\tilde{t}-\tilde{S}_0+\tilde{x})(\tilde{S}_0+\tilde{N}-\tilde{x})}{(\tilde{S}_0+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{S}_0)}} + z^{-\epsilon_k}\sqrt{\frac{(\tilde{t}+\tilde{N}-\tilde{x})\tilde{x}}{(\tilde{S}_0+\tilde{N})(\tilde{T}+\tilde{N}-\tilde{S}_0)}}\right)^{-1} \\
&\qquad\qquad \times \frac{dz}{z^{x_i-x_j+1}} = const_1^{t_i-t_j}\cdot const_2^{r_j-r_i}\cdot K_{ij}^{\mathrm{bulk}},
\end{aligned}$$

where $const_1$ and $const_2$ are as above. (The integration is performed over the right side of the unit circle.)

Since in the determinant for correlation functions the prefactors $const_1^{t_i-t_j} const_2^{r_j-r_i}$ cancel out, the proof is complete.

□

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