DEFORMATIONS OF OBJECTS IN DERIVED CATEGORIES AND NONCOMMUTATIVE GRASSMANIANS

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Abstract. This paper is the part of our joint papers with D.O. Orlov and V.A. Lunts [ELOI], [ELOII], [ELOIII]. All results in this paper are obtained by the author.

In [ELOI] we developed deformation theory of objects in homotopy and derived categories of DG categories. In [ELOII] we extended the (derived) deformation functors of an object $E$ to an appropriate bicategory of artinian DG algebras. In this paper we prove (under some assumptions) that these extended functors are pro-representable in a strong sense by complete pro-artinian DG algebra. It is defined in terms of the minimal $A_\infty$-structure on $\text{Ext}(E,E)$. One of the main ingredients in the proof is Maurer-Cartan pseudo-functor for $A_\infty$-algebras. Moreover, using $A_\infty$-methods, we obtain some results on comparison of the homotopy and derived deformation functors.

We also construct noncommutative Grassmanians as examples of noncommutative moduli spaces of objects in derived categories. The derived categories of quasi-coherent sheaves on these noncommutative Grassmanians are described. Further, we make an attempt to relate two different approaches to noncommutative algebraic geometry. After that, we describe the $k$-points of noncommutative Grassmanians.

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1. Introduction

In the paper [ELOI] we developed a general deformation theory of objects in homotopy and derived categories of DG categories. The corresponding deformation pseudo-functors are defined on the category of artinian DG algebras $\text{dgart}$ and take values in the 2-category $\text{Gpd}$ of groupoids. More precisely if $A$ is a DG category and $E$ is a right DG module over $A$ we defined four pseudo-functors $\text{Def}^h(E)$, $\text{coDef}^h(E)$, $\text{Def}(E)$, $\text{coDef}(E) : \text{dgart} \to \text{Gpd}$.

The first two are the homotopy deformation and co-deformation pseudo-functors, i.e. they describe deformations (and co-deformations) of $E$ in the homotopy category of DG $A^{op}$-modules; and the last two are their derived analogues. The pseudo-functors $\text{Def}^h(E)$, $\text{coDef}^h(E)$ are equivalent and depend only on the quasi-isomorphism class of the DG algebra $\text{End}(E)$. The derived pseudo-functors $\text{Def}(E)$, $\text{coDef}(E)$ need some boundedness conditions to give the "right" answer and in that case they are equivalent to $\text{Def}^h(F)$ and $\text{coDef}^h(F)$ respectively for an appropriately chosen h-projective or h-injective DG module $F$ which is quasi-isomorphic to $E$ (one also needs to restrict the pseudo-functors to the category $\text{dgart}_{\geq}$ of negative artinian DG algebras).

In this paper we prove pro-representability of these pseudo-functors. Recall that "classically" one defines representability only for functors with values in the category of sets (since the collection of morphisms between two objects in a category is a set). For example, given a moduli problem in the form of a pseudo-functor with values in the 2-category of groupoids one then composes it with the functor $\pi_0$ to get a set valued functor,
which one then tries to (pro-) represent. This is certainly a loss of information. But in order to represent the original pseudo-functor one needs the source category to be a bicategory.

It turns out that there is a natural bicategory 2-adgalg of augmented DG algebras. It is defined in [ELOII]. (Actually we consider two versions of this bicategory, 2-adgalg and 2'-adgalg , but then show that they are equivalent). The derived deformation functors can be naturally extended to pseudo-functors

\[ \text{coDEF}(E) : 2\text{-dgart} \to \text{Gpd}, \quad \text{DEF}(E) : 2'\text{-dgart} \to \text{Gpd}. \]

Then (under some finiteness conditions on the graded algebra \( \text{Ext}(E, E) = H(C) \), where \( C = \mathbb{R}\text{Hom}(E, E) \)), we prove pro-representability of these pseudo-functors restricted to negative artinian DG algebras. The pro-representing DG algebra \( \hat{S} = (B\hat{A})^* \) is the linear dual of the bar construction \( B\hat{A} \) of the minimal \( A_\infty \)-model of \( C \) (Theorems 14.1, 14.2, 15.1, 15.2).

This pro-representability appears to be more "natural" for the pseudo-functor \( \text{coDEF}_{-} \), because the bar complex \( B\hat{A} \otimes_{\tau A} A \) is the "universal co-deformation" of \( A \) considered as an \( A_\infty \)-module over \( A^{\text{op}} \). The pro-representability of the pseudo-functor \( \text{DEF}_- \) may then be formally deduced from that of \( \text{coDEF}_- \), but we can find the corresponding "universal deformation" of \( A \) only under an additional assumption on \( A \) (Theorem 15.9). We also make the equivalence \( \text{DEF}_-(E) \cong 1\text{-Hom}(\hat{S}, -) \) explicit in this case (Corollary 15.12).

We define and investigate the noncommutative Grassmanians \( \text{NGr}(m, V) \). The noncommutative Grassmannian \( \text{NGr}(m, V) \) is the true global noncommutative moduli space of the sheaves \( \mathcal{O}_F(W) \in D^b_{\text{coh}}(\mathbb{P}(V)) \), where \( W \subset V \) are vector subspaces of dimension \( \dim W = m \).

The paper is organized as follows.

The first part of the paper is devoted to preliminaries on \( A_\infty \)-algebras, \( A_\infty \)-modules and \( A_\infty \)-categories. The only non-standard point here is the DG category of \( A_\infty \) \( A_\infty \)-modules for an \( A_\infty \)-algebra \( A \) and a DG algebra \( C \), and the corresponding derived category \( D_\infty(A_C) \). We also recollect some facts on admissible \( A_\infty \)-algebras from [ELOII].

Second part is devoted to the Maurer-Cartan pseudo-functor \( \mathcal{MC}(A) : \text{dgart} \to \text{Gpd} \) for a strictly unital \( A_\infty \)-algebra \( A \). The Maurer-Cartan groupoid \( \mathcal{MC}_R(A) \) can be defined by means of some \( A_\infty \)-category with the same objects, which are solutions of the generalized Maurer-Cartan equation. We develop the obstruction theory for this Maurer-Cartan pseudo-functor (Proposition 6.1). Further, we show the invariance of (quasi-) equivalence classes of the constructed \( A_\infty \)-categories and Maurer-Cartan pseudo-functors under the quasi-isomorphisms of \( A_\infty \)-algebras (Theorems 7.1, 7.2).

In the third part we recall the bicategories 2-adgalg and 2'-adgalg from [ELOII] and the pseudo-functors \( \text{coDEF}_- \) and \( \text{DEF}_- \) and prove the theorem on their relation (Theorem 13.4). We also obtain here some results on the equivalences between the homotopy and derived (co-)deformation functors (Lemma 9.7, Theorem 11.7).

In the fourth part we prove the pro-representability theorems (Theorems 14.1, 14.2, 15.1, 15.2).

The fifth part is devoted to noncommutative Grassmanians. First we recall some facts and notions concerning \( \mathbb{Z} \)-algebras (Section 16). Then we define noncommutative Grassmanians \( \text{NGr}(m, V), 1 \leq m \leq \dim V - 1 \) as \( \text{Proj}(A^{m,V}) \), where \( A^{m,V} \) is a certain (periodic) \( \mathbb{Z} \)-algebra. We explain what means that \( \text{NGr}(m, V) \) is a true noncommutative moduli space of the sheaves \( \mathcal{O}_F(W) \in D^b_{\text{coh}}(\mathbb{P}(V)) \), where \( W \subset V \) are subspaces of dimension \( \dim W = m \) (Section 17). Then we describe the derived categories of quasi-coherent sheaves on \( \text{NGr}(m, V) \) (Theorem 18.19, Corollary 18.22). In Section 19 we describe the \( k \)-points of noncommutative Grassmanians. To do that, we first associate to each (positively oriented) \( \mathbb{Z} \)-algebra \( A \) a presheaf of groupoids \( X_A \) on the category \( \text{Alg}_{k}^{\text{op}} \) dual to the category of unital associative algebras. This relates two
approaches to noncommutative geometry. One approach is to think of noncommutative stacks as of Proj of
some noncommutative graded ring, see [M], [V1], [V2], [AZ]. We deal with its improvement using \( \mathbb{Z} \)-algebras,
see [BP], [P]. The other approach is to think of noncommutative stacks as of presheaves of groupoids on the
category of noncommutative affine schemes, see [Or]. In particular, we obtain the groupoid of \( k \)-points of
Proj(\( \mathcal{A} \)). It turns out that in the case of noncommutative Grassmanians all groupoids are trivial. There is
a bijection between the set \( \text{NGr}(m, V)(k) = X_{\mathcal{A}^m, \mathcal{V}}(k) \) of \( k \)-points of \( \text{NGr}(m, V) \) and the set of all vector
subspaces \( W \subset V \) such that \( 1 \leq \dim W \leq m \) (Theorem 19.10).

We freely use the notation and results of [ELOI].

Part 1. \( A_\infty \)-structures and the bar complex

2. Coalgebras

2.1. Coalgebras and comodules. We will consider DG coalgebras. For a DG coalgebra \( \mathcal{G} \) we denote by
\( \mathcal{G}^{gr} \) the corresponding graded coalgebra obtained from \( \mathcal{G} \) by forgetting the differential. Recall that if
\( \mathcal{G} \) is a DG coalgebra, then its graded dual \( \mathcal{G}^* \) is naturally a DG algebra. Also given a finite dimensional DG algebra
\( \mathcal{B} \) its dual \( \mathcal{B}^* \) is a DG coalgebra.

A morphism of DG coalgebras \( k \to \mathcal{G} \) (resp. \( \mathcal{G} \to k \)) is called a co-augmentation (resp. a co-unit) of
\( \mathcal{G} \) if it satisfies some obvious compatibility condition. We denote by \( \mathcal{G} \) the cokernel of the co-augmentation map.

Denote by \( \mathcal{G}^{[n]} \) the kernel of the \( n \)-th iterate of the co-multiplication map \( \Delta^n : \mathcal{G} \to \mathcal{G}^{\otimes n} \). The DG
coalgebra \( \mathcal{G} \) is called co-complete if
\[ \mathcal{G} = \bigcup_{n \geq 2} \mathcal{G}^{[n]} . \]

A \( \mathcal{G} \)-comodule means a left DG comodule over \( \mathcal{G} \).

A \( \mathcal{G}^{gr} \)-comodule is cofree if it is isomorphic to \( \mathcal{G} \otimes V \) with the obvious comodule structure for some graded
vector space \( V \).

Denote by \( \mathcal{G}^{op} \) the DG coalgebra with the opposite co-multiplication.

Let \( g : \mathcal{H} \to \mathcal{G} \) be a homomorphism of DG coalgebras. Then \( \mathcal{H} \) is a DG \( \mathcal{G} \)-comodule with the co-action
\( g \otimes 1 : \Delta \mathcal{H} : \mathcal{H} \to \mathcal{G} \otimes \mathcal{H} \) and a DG \( \mathcal{G}^{op} \)-comodule with the co-action \( 1 \otimes g : \Delta \mathcal{H} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G} \).

Let \( M \) and \( N \) be a right and left DG \( \mathcal{G} \)-comodules respectively. Their cotensor product \( M \Box_{\mathcal{G}} N \) is defined
as the kernel of the map
\[ \Delta_M \otimes 1 - 1 \otimes \Delta_N : M \otimes N \to M \otimes \mathcal{G} \otimes N , \]
where \( \Delta_M : M \to M \otimes \mathcal{G} \) and \( \Delta_N : N \to \mathcal{G} \otimes N \) are the co-action maps.

A DG coalgebra \( \mathcal{G} \) is a left and right DG comodule over itself. Given a DG \( \mathcal{G} \)-comodule \( M \) the co-action
morphism \( M \to \mathcal{G} \otimes M \) induces an isomorphism \( M = \mathcal{G} \Box_{\mathcal{G}} M \). Similarly for DG \( \mathcal{G}^{op} \)-modules.

Definition 2.1. The dual \( \mathcal{R}^* \) of an artinian DG algebra \( \mathcal{R} \) is called an artinian DG coalgebra.

Given an artinian DG algebra \( \mathcal{R} \), its augmentation \( \mathcal{R} \to k \) induces the co-augmentation \( k \to \mathcal{R}^* \) and its
unit \( k \to \mathcal{R} \) induces the co-unit \( \mathcal{R}^* \to k \).

2.2. From comodules to modules. If \( P \) is a DG comodule over a DG coalgebra \( \mathcal{G} \), then \( P \) is naturally a
DG module over the DG algebra \( (\mathcal{G}^*)^{op} \). Namely, the \( (\mathcal{G}^*)^{op} \)-module structure is defined as the composition
\[ P \otimes \mathcal{G}^* \xrightarrow{\Delta \otimes 1} \mathcal{G} \otimes P \otimes \mathcal{G}^* \xrightarrow{T \otimes 1} P \otimes \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{1 \otimes \text{ev}} P , \]
where \( T : \mathcal{G} \otimes P \to P \otimes \mathcal{G} \) is the transposition map.
Similarly, if $Q$ is a DG $G^\text{op}$-comodule, then $Q$ is a DG module over $G^\ast$.

Let $P$ and $Q$ be a left and right DG $G$-comodules respectively. Then $P \otimes Q$ is a DG $G^\ast$-bimodule, i.e. a DG $G^\ast \otimes G^{\ast 0}$-module by the above construction. Note that its center is naturally a DG algebra.

$$Z(P \otimes Q) := \{ x \in P \otimes Q \mid ax = (-1)^{\bar{a}\bar{x}} xa \text{ for all } a \in G^\ast \}$$

is isomorphic to the cotensor product $Q \Box G P$.

3. Preliminaries on $A_\infty$-algebras, $A_\infty$-categories and $A_\infty$-modules

3.1. $A_\infty$-algebras and $A_\infty$-modules. Our basic reference for $A_\infty$-algebras, $A_\infty$-modules and $A_\infty$-categories is [Le-Ha].

Let $A = \bigoplus_{n \in \mathbb{Z}} A^n$ be a $\mathbb{Z}$-graded $k$-vector space. Put $BA = T(A[1]) = \bigoplus_{n \geq 0} A[1]^{\otimes n}$. Then the graded vector space $BA$ has natural structure of a graded coalgebra with counit:

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{m=0}^{n} (a_1 \otimes \cdots \otimes a_m) \otimes (a_{m+1} \otimes \cdots \otimes a_n),$$

$$\varepsilon(a_1 \otimes \cdots \otimes a_n) = \begin{cases} 0 & \text{for } n \geq 1; \\ 1 & \text{for } n = 0. \end{cases}$$

Here we put $a_1 \otimes \cdots \otimes a_n = 1$ for $n = 0$. Put also $\overline{BA} = BA/k$. Then $\overline{BA}$ is also a graded coalgebra, but it is non-counital. The most effective way to define the notion of a $\mathbb{Z}$-graded (non-unital) $A_\infty$-algebra is the following:

**Definition 3.1.** A structure of a (non-unital) $A_\infty$-algebra on $\mathbb{Z}$-graded vector space $A$ is a coderivation $b : \overline{BA} \to \overline{BA}$ of degree 1 such that $b^2 = 0$, i.e. a structure of a DG coalgebra on the graded coalgebra $\overline{BA}$.

Such a coderivation is equivalent to a sequence of maps $b_n = b_n^A : A[1]^{\otimes n} \to A[1]$ $n \geq 1$, of degree 1 satisfying for each $n \geq 1$ the following identity:

$$\sum_{r+s+t=n} b_{r+1+t}(1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) = 0. \quad (3.1)$$

Note that the coderivation $b : \overline{BA} \to \overline{BA}$ naturally extends to a coderivation $b : BA \to BA$ (which we denote by the same letter), thus $BA$ also becomes a DG coalgebra, and $\varepsilon \cdot b = 0$. Thus, its dual $\hat{S} = (BA)^\ast$ is naturally a DG algebra.

Let $s : A \to A[1]$ be the translation map. Identify $A^{\otimes n}$ with $A[1]^{\otimes n}$ via the map $s^{\otimes n}$, and $A$ with $A[1]$ via the map $s$. Let $m_n = m_n^A : A^{\otimes n} \to A$ be the maps corresponding to $b_n$. Then $m_n$ has degree $(2 - n)$ and this sequence of maps satisfies for each $n \geq 1$ the following identity:

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0. \quad (3.2)$$

In particular, $m_1$ is a differential on $A$, hence $A$ is a complex. Further, $m_2$ is a morphisms of complexes and it is associative up to homotopy given by $m_3$. Thus, the cohomology $H(A)$ is naturally a (possibly non-unital) graded algebra. Further, if $m_n = 0$ for $n \geq 3$, then $A$ is a (possibly non-unital) DG algebra.

Let $A_1$, $A_2$ be (non-unital) $A_\infty$-algebras. The most effective way to define the notion of an $A_\infty$-morphism between them is the following:
Definition 3.2. An $A_\infty$-morphism $f : A_1 \to A_2$ is a (counital) homomorphism of DG coalgebras $f : BA_1 \to BA_2$ (which we denote by the same letter).

Thus, the assignment $A \mapsto BA$ is the full embedding of the category of (non-unital) $A_\infty$-algebras and $A_\infty$-morphisms to the category of counital DG coalgebras.

An $A_\infty$-morphism $f : A_1 \to A_2$ is equivalent to a sequence of maps $\bar{f}_n : A[1]^{\otimes n} \to A[1], \ n \geq 1$, of degree zero satisfying for each $n \geq 1$ the following identity:

\begin{equation}
\sum_{i_1 + \cdots + i_s = n} b^{A_2}_s (f_{i_1} \otimes \cdots \otimes f_{i_s}) = \sum_{r+s+t} \bar{f}_{r+1+t} (1^{\otimes r} \otimes b^{A_1}_s \otimes 1^{\otimes t}).
\end{equation}

Let $f_n : A^{\otimes n}_1 \to A_2$ be the maps corresponding to $\bar{f}_n$ with respect to our identifications. Then $f_n$ has degree $(1-n)$ and this sequence of maps satisfies for each $n \geq 1$ the following identity:

\begin{equation}
\sum_{i_1 + \cdots + i_s = n} (-1)^{e(i_1, \ldots, i_{s-1}, s)} m^{A_2}_s (f_{i_1} \otimes \cdots \otimes f_{i_s}) = \sum_{r+s+t} (-1)^{r+s+t} \bar{f}_{r+1+t} (1^{\otimes r} \otimes m^{A_1}_s \otimes 1^{\otimes t}),
\end{equation}

where $e(i_1, \ldots, i_{s-1}, s) = (s-1)i_1 + \cdots + i_{s-1} + \frac{s(s+1)}{2}$.

In particular, $f_1$ is a morphism of complexes, and $H(f) : H(A_1) \to H(A_2)$ is a morphism of (non-unital) graded associative algebras. An $A_\infty$-morphism $f$ is called quasi-isomorphism if $f_1$ is a quasi-isomorphism of complexes.

Further, we define the DG category $A_{-mod_{\infty}}$ of $A_\infty$ $A$-modules for an $A_\infty$-algebra $A$.

Definition 3.3. A structure of an $A_\infty$-module over $A$ on the graded vector space $M$ is the differential $b^M : BA \otimes M[1] \to BA \otimes M[1]$ of degree 1, which defines a structure of a DG $BA$-comodule on the graded cofree $(BA)^{gr}$-comodule $BA \otimes M[1]$.

Such a structure is equivalent to a sequence of maps $b_n = b^M_n : A[1]^{\otimes (n-1)} \otimes M[1] \to M[1], \ n \geq 1$, of degree 1, satisfying for each $n \geq 1$ the identity (3.1), where $b_i$ is interpreted as $b^A_i$ or $b^M_i$, according to the type of its arguments. It is also equivalent to the sequence of maps $m_n = m^M_n : A^{\otimes (n-1)} \otimes M \to M, \ n \geq 1$, of degree $(2-n)$ satisfying for each $n \geq 1$ the identity (3.2), where $m_i$ is interpreted as $m^A_i$ or $m^M_i$, according to the type of its arguments. In particular, $(m^M_i)^2 = 0$, hence $M$ is a complex. Again, $m^M_2$ is a morphism of complexes and $m_2$ is associative up to a homotopy given by $m^M_3$. Thus, $H(M)$ is naturally a graded $H(A)$-module.

If $M$ and $N$ are $A_\infty$ $A$-modules, then we put

$$\text{Hom}_{A_{-mod_{\infty}}}(M, N) := \text{Hom}_{BA_{-comod}}(BA \otimes M[1], BA \otimes N[1]).$$

More explicitly,

$$\text{Hom}_{A_{-mod_{\infty}}}(M, N) = \prod_{m \geq 1} \text{Hom}^n_n(A[1]^{\otimes (m-1)} \otimes M[1], N[1]),$$

and for $\phi = (\phi_m) \in \text{Hom}_{A_{-mod_{\infty}}}(M, N)$ one has

\begin{equation}
(d\phi)_m = \sum_{1 \leq i \leq m} b^N_{m-i+1} (1^{\otimes (l-i)} \otimes \phi_i) - (-1)^n \sum_{r+s+t=m} \phi_{r+1+t} (1^{\otimes r} \otimes b^M_s \otimes 1^{\otimes t}),
\end{equation}

where $b_s$ in the RHS is interpreted as $b^A_s$ or $b^M_s$, according to the type of its arguments. If $\phi = (\phi_m) \in \text{Hom}_{A_{-mod_{\infty}}}(M, N)$ and $\psi = (\psi_m) \in \text{Hom}_{A_{-mod_{\infty}}}(N, L)$, then
We will write \( \text{Hom}_A(M, N) \) instead of \( \text{Hom}_{A\text{-mod}_{\infty}}(M, N) \).

The closed morphism \( \phi \in \text{Hom}^0(\text{A-mod}_{\infty}) \) is called quasi-isomorphism if \( \phi_1 \) is a quasi-isomorphism of complexes.

The homotopy category \( K_{\infty}(A) \) is defined as \( \text{Ho}(\text{A-mod}_{\infty}) \). It is always triangulated. It turns out that all acyclic \( A_{\infty} \) \( \text{A-modules} \) in \( K_{\infty}(A) \) are already null-homotopic. Hence the corresponding derived category \( D_{\infty}(A) \) is the same as \( K_{\infty}(A) \). However, we will write \( D_{\infty}(A) \) instead of \( K_{\infty}(A) \).

Let \( f : A_1 \to A_2 \) be an \( A_{\infty} \)-morphism. Then we have the DG functor \( f_* : A_{2\text{-mod}_{\infty}} \to A_{1\text{-mod}_{\infty}} \), which we call the "restriction of scalars". Namely, if \( M \in A_{2\text{-mod}_{\infty}} \), then \( f_*(M) \) coincides with \( M \) as a graded vector space, and the differential on \( BA_1 \otimes f_*(M)[1] \) coincides with the differential on \( BA_1 \bigcirc_{BA_2}(BA_2 \otimes M[1]) \) after the natural identification

\[
BA_1 \otimes f_*(M)[1] \cong BA_1 \bigcirc_{BA_2}(BA_2 \otimes M[1]).
\]

We also have the resulting exact functor \( f_* : A_{2\text{-mod}_{\infty}} \to A_{1\text{-mod}_{\infty}} \). If \( f \) is a quasi-isomorphism, then the DG functor \( f_* : A_{2\text{-mod}_{\infty}} \to A_{1\text{-mod}_{\infty}} \) is quasi-equivalence, and hence the functor \( f_* : D_{\infty}(A_2) \to D_{\infty}(A_1) \) is an equivalence.

We would like also to define the \( A_{\infty} \)-bimodules.

**Definition 3.4.** Let \( A_1 \) and \( A_2 \) be \( A_{\infty} \)-algebras. A structure of an \( A_{\infty} \) \( A_1\text{-A}_2 \)-bimodule on the graded vector space \( M \) is a differential \( b^M : BA_1 \otimes M[1] \otimes BA_2 \to BA_1 \otimes M[1] \otimes BA_2 \) which defines the structure of a DG comodule over \( BA_1 \otimes (BA_2)^{op} \) on the \( (BA_1 \otimes (BA_2)^{op})^{gr} \) -bicomodule \( BA_1 \otimes M[1] \otimes BA_2 \).

Such a differential is given by a sequence of maps

\[
b_{i,j} : A_1[1]^{\otimes i} \otimes M[1] \otimes A_2[1]^{\otimes j} \to M[1]
\]

satisfying analogous equations. In particular, we have a regular \( A_1\text{-A}_2 \)-bimodule \( A_1 \otimes A_2 \). In the case when \( A_1 = A_2 \), we have a diagonal bimodule \( A \). The DG category \( A_{1\text{-mod}}A_2 \) of \( A_{\infty} \) \( A_1\text{-A}_2 \)-bimodules is defined analogously (see also [KoSo]). Again, we define \( K_{\infty}(A_1\text{-A}_2) \) as the homotopy category \( \text{Ho}(A_{1\text{-mod}}A_2) \).

All acyclic \( A_{\infty} \)-bimodules in \( K_{\infty}(A_1\text{-A}_2) \) are null-homotopic and hence the corresponding derived category \( D_{\infty}(A_1\text{-A}_2) \) coincides with \( K_{\infty}(A_1\text{-A}_2) \).

3.2. **Strictly unital \( A_{\infty} \)-algebras.**

**Definition 3.5.** An \( A_{\infty} \)-algebra is called strictly unital if there exists an element \( 1_A \in A \) of degree zero satisfying the following properties:

- \((U1)\) \( m_1(1_A) = 0 \);
- \((U2)\) \( m_2(a,1_A) = m_2(1_A,a) = a \) for each \( a \in A \);
- \((U3)\) for \( n \geq 3 \), \( m_n(a_1,\ldots,a_n) \) vanishes if at least one of \( a_i \) equals to \( 1_A \).

Such an element \( 1_A \) is called a strict unit.

Clearly, if a strict unit exists then it is unique. An \( A_{\infty} \)-morphism \( f : A_1 \to A_2 \) of strictly unital \( A_{\infty} \)-algebras is called strictly unital if \( f_1(1_{A_1}) = 1_{A_2} \), and for \( n \geq 2 \) \( f_n(a_1,\ldots,a_n) \) vanishes if at least one of \( a_i \) equals to \( 1_{A_1} \). Further, an \( A_{\infty} \)-module \( M \in A_{\text{mod}_{\infty}} \) is called strictly unital if \( m_2^M(1_A,m) = m \) for each
m ∈ M and for n ≥ 3 \( m_n^M(a_1, \ldots, a_{n-1}, m) = 0 \) if at least one of \( a_i \) equals to \( 1_A \). If \( A \) is strictly unital then we denote by \( D^u_\infty(A) \subset D_\infty(A) \) the full subcategory which consists of strictly unital \( A_\infty \) \( A \)-modules.

Analogously, if \( A_1 \) and \( A_2 \) are strictly unital \( A_\infty \)-algebras, then we have a notion of strictly unital \( A_\infty \) \( A_1 \)-\( A_2 \)-bimodules, and we define \( D^u_\infty(A_1 \cdot A_2) \subset D_\infty(A_1 \cdot A_2) \) as the full subcategory which consists of strictly unital \( A_\infty \)-bimodules.

If \( C \) is a DG algebra then it is also a strictly unital \( A_\infty \)-algebra with \( m_n = 0 \) for \( n ≥ 3 \). We have an obvious DG functor \( C \)-mod \( \rightarrow \) \( C \)-mod∞. It induces an equivalence

\[
D(C) \overset{\sim}{\longrightarrow} D^u_\infty(C).
\]

Let \( A \) be an arbitrary \( A_\infty \)-algebra. Then its unitization \( A_+ := k \cdot 1_+ \oplus A \), which is a strictly unital \( A_\infty \)-algebra, is defined as follows:

\[
m_n^{A+}(a_1, \ldots, a_n) = m_n^A(a_1, \ldots, a_n) \text{ for any } a_1, \ldots, a_n ∈ A,
m_1(1_+) = 0,
m_2^{A+}(1_+, a) = m_2^A(a, 1_+) = a \text{ for each } a ∈ A_+,
m_n^{A+}(a_1, \ldots, a_n) = 0 \text{ if at least one of } a_i \text{ equals to } 1_+.
\]

Clearly, the assignment \( A \mapsto A_+ \) defines faithful functor from the category of \( A_\infty \)-algebras and \( A_\infty \)-morphisms to the category of strictly unital \( A_\infty \)-algebras and strictly unital \( A_\infty \)-morphisms. Further, we have an obvious faithful DG functor \( A \)-mod∞ \( \rightarrow \) \( A_+ \)-mod∞. Its image consists of strictly unital \( A_\infty \)-modules. The induced functor \( D_\infty(A) \rightarrow D^u_\infty(A_+) \) is an equivalence.

We call \( A_\infty \)-algebras of the form \( A_+ \) augmented \( A_\infty \)-algebras. We also use the notation \( A = \overline{A_+} \).

**Definition 3.6.** Let \( A \) be an augmented \( A_\infty \)-algebra. Its bar-cobar contraction \( U(A) \), which is a DG algebra, together with a strictly unital \( A_\infty \) quasi-isomorphism \( f_A : A \rightarrow U(A) \) are defined by the following universal property. If \( B \) is a DG algebra, and \( f : A \rightarrow B \) is a strictly unital \( A_\infty \)-morphism then there exists a unique morphism of DG algebras \( \varphi : U(A) \rightarrow B \) such that \( f = \varphi \cdot f_A \).

More explicitly, \( U(A) \) equals to \( T(BA[-1]) \) as a graded algebra, and the differential comes from the differential and comultiplication on \( BA \). The \( A_\infty \)-morphism \( f_A \) is the obvious one.

3.3. **Minimal models of \( A_\infty \)-algebras.** An \( A_\infty \)-algebra \( A \) is called minimal if \( m_1^A = 0 \). Each (strictly unital) \( A_\infty \)-algebra is quasi-isomorphic to the minimal (strictly unital) \( A_\infty \)-algebra.

**Proposition 3.7.** ([Le-Ha], Corollaire 1.4.1.4, Proposition 3.2.4.1) Let \( A \) be an \( A_\infty \)-algebra. There exists an \( A_\infty \)-algebra structure on \( H(A) \) such that

a) \( m_1 = 0 \) and \( m_2 \) is induced by \( m_2^A \);

b) there exists an \( A_\infty \)-quasi-isomorphism of \( A_\infty \)-algebras \( f : H(A) \rightarrow A \) such that \( f_1 \) induces the identity in cohomology.

Moreover, if \( A \) is strictly unital then this \( A_\infty \)-structure on \( H(A) \) and the quasi-isomorphism can be chosen to be strictly unital.
3.4. Perfect $A_\infty$-modules and $A_\infty$-bimodules. Let $A$ be a strictly unital $A_\infty$-algebra. The category $\text{Perf}(A)$ of perfect $A_\infty$-modules is the minimal full thick triangulated subcategory of $D_{\infty}^w(A)$ which contains $A$.

Further, if $A_1$ and $A_2$ are strictly unital $A_\infty$-algebras then the category $\text{Perf}(A_1-A_2)$ of perfect $A_\infty$-$A_1$-$A_2$-bimodules is the minimal full thick triangulated subcategory of $D_{\infty}^w(A_1-A_2)$ which contains $A_1 \otimes A_2$.

3.5. $A_\infty$-categories. The notion of an $A_\infty$-category is a straightforward generalization of the notion of an $A_\infty$-algebra. Namely, a non-unital $A_\infty$-category $\mathcal{A}$ is the following data:
- the class of objects of $\mathcal{A}$;
- for each two objects $X_1, X_2$ the graded vector space $\text{Hom}(X_1, X_2)$;
- for each finite sequence of objects $X_0, X_1, \ldots, X_n \in \mathcal{A}$, $n \geq 1$, the map

$$m_n^{\mathcal{A}}(x_0, \ldots, x_n) : \text{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \text{Hom}(X_0, X_1) \rightarrow \text{Hom}(X_0, X_n)$$

of degree $(2-n)$, such that for any $Y_1, \ldots, Y_m \in \mathcal{A}$ the graded vector space $\bigoplus_{1 \leq i, j \leq m} \text{Hom}(Y_i, Y_j)$ becomes an $A_\infty$-algebra.

If $\mathcal{A}$ is an $A_\infty$-category then $\text{Ho}(\mathcal{A})$ is a pre-category, i.e. a "category" which may not have identity morphisms.

An element $1_X \in \text{Hom}(X, X)$ of degree zero is called a strict identity morphism if it satisfies the conditions $(U1), (U2), (U3)$ from Definition 3.5, where $a$ and $a_i$ are arbitrary morphisms such that the equalities make sense. An $A_\infty$-category is called strictly unital if each object has a strict identity morphism. If $\mathcal{A}$ is a strictly unital $A_\infty$-category then $\text{Ho}(\mathcal{A})$ is a true category.

A (strictly unital) $A_\infty$-algebra can be thought of as a (strictly unital) $A_\infty$-category with one object.

Let $\mathcal{A}_1, \mathcal{A}_2$ be $A_\infty$-categories. An $A_\infty$-functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is the following data:
- an object $F(X) \in \mathcal{A}_2$ for each object $X \in \mathcal{A}_1$;
- for each finite sequence of objects $X_0, X_1, \ldots, X_n \in \mathcal{A}_1$, $n \geq 1$, the map

$$F(X_0, \ldots, X_n) : \text{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \text{Hom}(X_0, X_1) \rightarrow \text{Hom}(F(X_0), F(X_n))$$

of degree $(1-n)$, such that for any $Y_1, \ldots, Y_m \in \mathcal{A}_1$ we obtain an $A_\infty$-morphism between $A_\infty$-algebras

$$\bigoplus_{1 \leq i, j \leq m} \text{Hom}(Y_i, Y_j) \rightarrow \bigoplus_{1 \leq i, j \leq m} \text{Hom}(F(Y_i), F(Y_j)).$$

The definition of a strictly unital $A_\infty$-functor between strictly unital $A_\infty$-categories is analogous to the definition of a strictly unital $A_\infty$-morphism between strictly unital $A_\infty$-algebras.

A strictly unital $A_\infty$-functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ between strictly unital $A_\infty$-categories is called quasi-equivalence if the following conditions hold:
- the map $F(X, Y) : \text{Hom}(X, Y) \rightarrow \text{Hom}(F(X), F(Y))$ is a quasi-isomorphism of complexes for any $X, Y \in \mathcal{A}_1$;
- the induced functor $\text{Ho}(F) : \text{Ho}(\mathcal{A}_1) \rightarrow \text{Ho}(\mathcal{A}_2)$ is an equivalence.

3.6. The tensor product of an $A_\infty$-algebra and a DG algebra. Let $A$ be an $A_\infty$-algebra and $C$ be a DG algebra. Then their tensor $A \otimes C$ is naturally an $A_\infty$-algebra with the following multiplications:

$$m_1^{A \otimes C} = m_1^A \otimes 1_C + 1_A \otimes d_C;$$
$$m_n^{A \otimes C}(a_1 \otimes c_1, \ldots, a_n \otimes c_n) = (-1)^{n-1}m_n^A(a_1, \ldots, a_n) \otimes (c_1 \cdots c_n)$$
for $n \geq 2$, where $m_n^A : \bigotimes_{i=1}^n A \rightarrow A$ is the $n$-fold product.
where \( \epsilon = \sum_{i<j} a_j c_i \) (all \( a_i \) and \( c_i \) are homogeneous). If \( A \) is strictly unital, then \( A \otimes C \) is also strictly unital and \( 1_{A \otimes C} = 1_A \otimes 1_C \).

**Remark 3.8.** The constructed tensor product is a specialization of the complicated construction of the tensor product of \( A_\infty \)-algebras which was first proposed in [SU]. We also remark that in the case when \( A_1 \) and \( A_2 \) are strictly unital \( A_\infty \)-algebras, there is a canonical DG model for \( A_1 \otimes A_2 : \)

\[
A_1^n \otimes \prime A_2 = \text{End}_{A_1\text{-mod},A_2^{gr}}(A_1 \otimes A_2),
\]

see [KoSo].

3.7. The category of \( A_C \)-modules for an \( A_\infty \)-algebra \( A \) and a DG algebra \( C \). Let \( A \) be an \( A_\infty \)-algebra and let \( C \) be a DG algebra. We want to define the DG category of \( A_\infty \) \( A_C \)-modules which is analogue of the category of \( (A \otimes C) \)-modules in the case when \( A \) is a DG algebra.

**Definition 3.9.** A structure of an \( A_\infty \) \( A_C \)-module on the graded vector space \( M \) is the following data:

1) A structure of a \( C^{gr} \)-module on \( M \);

2) A differential \( b^M : BA \otimes M[1] \rightarrow BA \otimes M[1] \) of degree 1 which makes \( BA \otimes M[1] \) into a DG comodule over \( BA \) and into a DG module over \( C \).

If we are already given with the structure of a \( C^{gr} \)-module on \( M \), then such a differential \( b^M \) is equivalent to the sequence of maps \( b_n = b_n^M : A[1] \otimes (n-1) \otimes M[1] \rightarrow M[1], \ n \geq 1 \), satisfying the following properties:

1) The maps \( b_n^M \) satisfy the identities (3.1) (in the same sense as for \( A_\infty \) \( A \)-modules);

2) The differential \( b_1^M \) makes \( M[1] \) into a DG module over \( C \);

3) The maps \( b_n^M \) are \( C^{gr} \)-linear for \( n \geq 2 \).

Further, the corresponding maps \( m_n = m_n^M : A \otimes (n-1) \otimes M \rightarrow M \) have to satisfy the following properties:

1) The maps \( m_n^M \) satisfy the identities (3.2) (in the same sense as for \( A_\infty \) \( A \)-modules);

2) The differential \( m_1^M \) makes \( M \) into a DG module over \( C \);

3) The maps \( m_n^M \) are \( C^{gr} \)-linear for \( n \geq 2 \).

If \( M, N \) are \( A_\infty \) \( A_C \)-modules, then we put

\[
\text{Hom}_{A_C\text{-mod}_{\infty}}(M, N) := \text{Hom}_{BA\text{-comod}} \cap \text{Hom}_{C\text{-mod}}(BA \otimes M[1], BA \otimes N[1]).
\]

More explicitly,

\[
\text{Hom}_{A_C\text{-mod}_{\infty}}^n(M, N) = \prod_{m \geq 1} \text{Hom}_{C^{gr}}^n(A[1] \otimes (m-1) \otimes M[1], N[1]),
\]

the differential and the compositions are defined by the formulas (3.5) and (3.6) respectively.

We will write \( \text{Hom}_{A_C}(M, N) \) instead of \( \text{Hom}_{A_C\text{-mod}_{\infty}}(M, N) \).

Again, the homotopy category \( K_{\infty}(A_C) \) is defined as \( \text{Ho}(A_C\text{-mod}_{\infty}) \). The acyclic \( A_\infty \) \( A_C \)-module in \( K_{\infty}(A_C) \) are not null-homotopic in general, hence we define the derived category \( D_{\infty}(A_C) \) as the Verdier quotient of \( K_{\infty}(A_C) \) by the subcategory of acyclic \( A_\infty \) \( A_C \)-modules.

**Remark 3.10.** Notice that the structure of an \( A_\infty \) \( A_C \)-module is not equivalent to the structure of an \( A_\infty \) \( A \otimes C \)-module. Moreover, there is a natural DG functor \( A_C\text{-mod}_{\infty} \rightarrow A_+ \otimes C\text{-mod}_{\infty} \) which induces an equivalence \( D_{\infty}(A_C) \sim D_{\infty}^+ (A_+ \otimes C) \). Also, in the case when \( A \) is strictly unital, the DG functor \( A_C\text{-mod}_{\infty} \rightarrow A \otimes C\text{-mod}_{\infty} \) induces an equivalence \( D_{\infty}^+(A_C) \sim D_{\infty}^+(A \otimes C) \).
Definition 3.11. An $A_{\infty}$ $A_{\mathcal{C}}$-module $M$ is called $h$-projective (resp. $h$-injective) if for each acyclic $N \in A_{\mathcal{C}}$-$mod_{\infty}$ the complex $\text{Hom}_{A_{\mathcal{C}}}(M, N)$ (resp. $\text{Hom}_{A_{\mathcal{C}}}(N, M)$) is acyclic.

It turns out that an $A_{\infty}$ $A_{\mathcal{C}}$-module is $h$-projective (resp. $h$-injective) iff it is such as a DG $\mathcal{C}$-module.

Proposition 3.12. Let $M$ be an $A_{\infty}$ $A_{\mathcal{C}}$-module. Suppose that $M$ is $h$-projective (resp. $h$-injective) as a DG $\mathcal{C}$-module. Then $M$ is also $h$-projective (resp. $h$-injective) as an $A_{\infty}$ $A_{\mathcal{C}}$-module.

Proof. We will prove Proposition for $h$-projectives. The proof for $h$-injectives is analogous.

So let $M \in A_{\mathcal{C}}$-$mod_{\infty}$ and suppose that $M$ is $h$-projective as a DG $\mathcal{C}$-module. Let $N$ be an acyclic $A_{\infty}$ $A_{\mathcal{C}}$-module. The complex $K = \text{Hom}_{A_{\mathcal{C}}}(M, N)$ admits a decreasing filtration by subcomplexes

$$F^pK = \prod_{n \geq p} \text{Hom}_{\mathcal{C}}(A^{\otimes n} \otimes M, N).$$

The subquotients

$$F^pK/F^{p+1}K = \text{Hom}_{\mathcal{C}}(A^{\otimes p} \otimes M, N)$$

are acyclic since the DG modules $A^{\otimes p} \otimes M$ are $h$-projective. Since

$$K = \lim_{\longleftarrow} F^pK,$$

the complex $K$ is also acyclic. Therefore, $M$ is $h$-projective as an $A_{\infty}$ $A_{\mathcal{C}}$-module.

We denote by $K_{\infty}^P(A_{\mathcal{C}}) \subset K_{\infty}(A_{\mathcal{C}})$ (resp. by $K_{\infty}^L(A_{\mathcal{C}}) \subset K_{\infty}(A_{\mathcal{C}})$) the full subcategory which consists of $h$-projective (resp. $h$-injective) $A_{\infty}$ $A_{\mathcal{C}}$-modules.

Theorem 3.13. For each $M \in A_{\mathcal{C}}$-$mod_{\infty}$, there exist quasi-isomorphisms $M \rightarrow I$, $P \rightarrow M$, where $I \in A_{\mathcal{C}}$-$mod_{\infty}$ is $h$-injective and $P \in A_{\mathcal{C}}$-$mod_{\infty}$ is $h$-projective. The natural functor $K_{\infty}^P(A_{\mathcal{C}}) \rightarrow D_{\infty}(A_{\mathcal{C}})$ (resp. $K_{\infty}^L(A_{\mathcal{C}}) \rightarrow D_{\infty}(A_{\mathcal{C}})$) is an equivalence.

Proof. First we construct a quasi-isomorphism $pM \rightarrow M$ with $h$-projective $P$. Namely, let $pM$ be the total complex of the bicomplex

$$\cdots \rightarrow \mathcal{C}^{\otimes n} \otimes M \xrightarrow{d^n} \mathcal{C}^{\otimes n-1} \otimes M \rightarrow \cdots \rightarrow \mathcal{C} \otimes M,$$

where $d^n$ is the bar differential. Then $pM$ is naturally an $A_{\infty}$ $A_{\mathcal{C}}$-module. A quasi-isomorphism of complexes $pM \rightarrow M$ is a quasi-isomorphism in $A_{\mathcal{C}}$-$mod_{\infty}$ (with zero components $f_n : A^{\otimes n-1} \otimes M \rightarrow M$ for $n \geq 2$).

Further, $pM$ satisfies property (P) ([ELOI], Definition 3.2) as a DG $\mathcal{C}$-module. Hence, $pM$ is an $h$-projective $A_{\mathcal{C}}$-module.

The construction $M \rightarrow pM$ extends to the functor $p : K_{\infty}(A_{\mathcal{C}}) \rightarrow K_{\infty}^P(A_{\mathcal{C}})$ which is right adjoint to the inclusion $K_{\infty}^P(A_{\mathcal{C}}) \rightarrow K_{\infty}(A_{\mathcal{C}})$. The kernel of $p$ consists of acyclic $A_{\mathcal{C}}$-modules. Thus, the functor $K_{\infty}^P(A_{\mathcal{C}}) \rightarrow D_{\infty}(A_{\mathcal{C}})$ is an equivalence.

Analogously, one can construct a functor $i : K_{\infty}(A_{\mathcal{C}}) \rightarrow K_{\infty}^L(A_{\mathcal{C}})$ which is left adjoint to the inclusion $K_{\infty}^L(A_{\mathcal{C}}) \rightarrow K_{\infty}(A_{\mathcal{C}})$, and the kernel of $i$ also consists of acyclic complexes. Thus, the functor $K_{\infty}^L(A_{\mathcal{C}}) \rightarrow D_{\infty}(A_{\mathcal{C}})$ is an equivalence. Theorem is proved.

Notice that if $G : K_{\infty}(A_{\mathcal{C}}) \rightarrow T$ is an exact functor between triangulated categories then we can define its left and right derived functors

$$LG : D_{\infty}(A_{\mathcal{C}}) \rightarrow T, \quad RG : D_{\infty}(A_{\mathcal{C}}) \rightarrow T.$$
Namely, for each $M \in A_{\mathcal{C}}\text{-mod}_{\infty}$ choose quasi-isomorphisms $P \to M$, $M \to I$ with $h$-projective $P$ and $h$-injective $I$, and put 

\[ \text{LG}(M) = G(P), \quad \text{RG}(M) = G(I). \]

**Proposition 3.14.** The derived categories $D_{\infty}(A_{\mathcal{C}})$ and $D(U(A_{+}) \otimes \mathcal{C})$ are naturally equivalent.

**Proof.** Indeed, the "restriction of scalars" DG functor

\[ f_{A*} : (U(A_{+}) \otimes \mathcal{C})\text{-mod} \to A_{\mathcal{C}}\text{-mod}_{\infty} \]

admits a right adjoint DG functor

\[ f^{A}_{*} : A_{\mathcal{C}}\text{-mod}_{\infty} \to (U(A_{+}) \otimes \mathcal{C})\text{-mod}, \]

given by the formula

\[ f^{A}_{*}(M) = \text{Hom}_{A}(U(A_{+}), M). \]

For any $M \in (U(A_{+}) \otimes \mathcal{C})\text{-mod}$, $N \in A_{\mathcal{C}}\text{-mod}_{\infty}$, the adjunction morphisms $M \to f^{A}_{*}f_{A*}M$, $f_{A*}f^{A}_{*}N \to N$ are quasi-isomorphisms. Moreover, both $f_{A*}$ and $f^{A}_{*}$ preserve acyclic modules. Thus, the induced functors

\[ f_{A*} : D(U(A_{+}) \otimes \mathcal{C}) \to D_{\infty}(A_{\mathcal{C}}), \quad f^{A}_{*} : D_{\infty}(A_{\mathcal{C}}) \to D(U(A) \otimes \mathcal{C}) \]

are mutually inverse equivalences. \[ \square \]

### 3.8. The bar complex.

Let $A$ be an $A_{\infty}$-algebra. The graded vector space $BA \otimes A[1] \otimes BA$ carries a natural differential which makes it into a DG bicomodule over $BA$. Namely, such a differential is determined by its components


and we put $b_{i,j} = b^{A}_{i+j+1}$. In particular, $BA \otimes A$ is an $A_{\infty}$-module over $A^{op}$. It is called the bar complex and is denoted by $BA \otimes A_{\tau_{A}}$. Now let $A$ be an augmented $A_{\infty}$-algebra, and put $\bar{S} = (BA)^{s}$. The graded vector space $B\bar{A} \otimes A[1] \otimes B\bar{A}$ also carries a natural differential which makes it into a DG bicomodule over $B\bar{A}$. In particular, $B\bar{A} \otimes A$ is an $A_{\infty}$-module over $A^{op}$. It is also called the bar complex and is denoted by $B\bar{A} \otimes A_{\tau_{A}}$. Note that $B\bar{A} \otimes A_{\tau_{A}}$ is a $B\bar{A}$-comodule, and hence is a $\bar{S}^{op}$-module. This makes it into an object of $\bar{S}^{op}_{\tau_{A}}\text{-mod}_{\infty}$.

Analogously, we have an $A_{\infty}$ $\hat{A}_{S}$-module $A_{\otimes_{\tau_{A}}}B\bar{A}$. 

### 4. Admissible $A_{\infty}$-algebras.

Here we recollect some facts about admissible $A_{\infty}$-algebras from [ELOII].

**Definition 4.1.** ([ELOII]) An augmented $A_{\infty}$-algebra $\mathcal{C}$ is called

a) nonnegative if $C^{i} = 0$ for $i < 0$;

b) connected if $C^{0} = \mathbb{k}$;

c) locally finite if $\dim_{\mathbb{k}}C^{i} < \infty$ for all $i$.

We say that $\mathcal{C}$ is admissible if it satisfies a), b), c).

Fix an augmented $A_{\infty}$-algebra $\mathcal{C}$. Consider the bar construction $B\bar{C}$, the corresponding DG algebra $\bar{S} = (B\bar{C})^{s}$ and the $A_{\infty}$ $\bar{C}^{op}_{\bar{S}^{op}}$-module $B\bar{C} \otimes_{\tau_{\bar{C}}} \mathcal{C}$ (the bar complex). If $\mathcal{C}$ is connected and nonnegative, then $B\bar{C}$ is concentrated in nonnegative degrees and consequently $\bar{S}$ is concentrated in nonpositive degrees.

Let $B$ be a DG algebra. Denote by $D_{f}(B^{op}) \subset D(B^{op})$ the full triangulated subcategory consisting of DG modules with finite dimensional cohomology.
Lemma 4.2. ([ELOII]) Assume that DG algebra $B$ is augmented and local and complete. Also assume that $B^i = 0$ for $i > 0$. Then the category $D_f(B^{op})$ is the triangulated envelope of the DG $B^{op}$-module $k$.

Choose a quasi-isomorphism of $A_\infty \overset{\text{gr}}{\Hom(S)}$-modules $B \overset{\text{gr}}{\Hom(S)} C \rightarrow J$, where $J$ is h-injective.

Proposition 4.3. ([ELOII]) Under the assumptions of the above theorem the following holds.

a) The complex $R \overset{\text{gr}}{\Hom(S)}(k,k)$ is quasi-isomorphic to $C$.

b) The natural morphism of complexes $\overset{\text{gr}}{\Hom(S)}(k,B \overset{\text{gr}}{\Hom(S)} C) \rightarrow \overset{\text{gr}}{\Hom(S)}(k,J)$ is a quasi-isomorphism.

Remark 4.4. Notice that for any augmented $A_\infty$-algebra $C$ we have $\overset{\text{gr}}{\Hom(S)}(k,B \overset{\text{gr}}{\Hom(S)} C) = C$. Thus the $A_\infty \overset{\text{gr}}{\Hom(S)}$-module $B \overset{\text{gr}}{\Hom(S)} C$ is a “homotopy $\hat{S}$-co-deformation” of $C$. The Proposition 4.3 implies that for an admissible $C$ this $A_\infty \overset{\text{gr}}{\Hom(S)}$-module is a “derived $\hat{S}$-co-deformation” of $C$. (Of course we have only defined co-deformations along artinian DG algebras.)

Lemma 4.5. ([ELOII]) Let $K$ be a DG $\overset{\text{gr}}{\Hom(S)}$-module such that $\dim_k K < \infty$. Then the natural morphism of complexes

$$\overset{\text{gr}}{\Hom(S)}(K,B \overset{\text{gr}}{\Hom(S)} C) \rightarrow \overset{\text{gr}}{\Hom(S)}(K,J)$$

is a quasi-isomorphism.

Part 2. Maurer-Cartan pseudo-functor for $A_\infty$-algebras

5. THE DEFINITION

Let $A$ be a strictly unital $A_\infty$-algebra, and $R$ be an artinian DG algebra with the maximal ideal $m$. Recall that $A \otimes R$ is naturally a strictly unital $A_\infty$-algebra (see subsection 3.6). We define the set $MC(A \otimes m)$ as the set of $\alpha \in (A \otimes m)^1$ such that the generalized Maurer-Cartan equation holds:

$$\sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}} m_n(\alpha, \ldots, \alpha) = 0. \tag{5.1}$$

This equation is well defined since $m \subset R$ is nilpotent ideal. Below for convenience we will write $\alpha^n$ instead of $\alpha, \ldots, \alpha$.

There is a natural $A_\infty$-category $\overset{\text{gr}}{MC}(A)$ with the set of objects $MC(A \otimes m)$. Namely, for $\alpha_1, \alpha_2 \in MC(A \otimes m)$ we define

$$\overset{\text{gr}}{\Hom}_{MC}(A)(\alpha_1, \alpha_2) := (A \otimes R)^{gr}$$

as a graded vector space. Further, for $\alpha_0, \alpha_1, \ldots, \alpha_m \in MC(A \otimes m)$ and for homogeneous $x_1 \in \overset{\text{gr}}{\Hom}(\alpha_0, \alpha_1), \ldots, x_n \in \overset{\text{gr}}{\Hom}(\alpha_{n-1}, \alpha_n)$ we define

$$m_n^{\overset{\text{gr}}{MC}(A)}(\alpha_0, \ldots, \alpha_n)(x_1, \ldots, x_1) = \sum_{i_0, \ldots, i_n \geq 0} (-1)^{\epsilon} m_{n+i_0+\ldots+i_n}^A(\alpha_{i_0}, x_n, \alpha_{n-1}, \ldots, \alpha_{i_1}, x_1, \alpha_0^{i_0}),$$

where

$$\epsilon = \sum_{n \geq k \geq 0} (\bar{x}_k + i_k)j + \sum_{k=0}^n \frac{i_k(i_k + 1)}{2} + \sum_{k=1}^n ki_k.$$

One checks without difficulties that this indeed defines an $A_\infty$-category and that $1 \in (A \otimes R)^{gr} = \overset{\text{gr}}{\Hom}(\alpha, \alpha)$ is a strict identity for each $\alpha \in \overset{\text{gr}}{MC}(A)$. Below we will write $m_n^{\overset{\text{gr}}{MC}(A)(\alpha_0, \ldots, \alpha_n)}$ instead of $m_n^{\overset{\text{gr}}{MC}(A)}(\alpha_0, \ldots, \alpha_n)$. 

Remark 5.1. The Maurer-Cartan equation and the formulas for higher multiplications are the same as in the definition of the $A_{\infty}$-category of one-sided twisted complexes, see [Ko]. Note that in the case of one-sided twisted complexes all the solutions of Maurer-Cartan equation are automatically "nilpotent".

Now we define the Maurer-Cartan pseudo-functor $MC(A) : \text{dgart} \to \text{Gpd}$ as follows. Let $R$ and $m$ be as above. The objects of the groupoid $MC_R(A)$ are the same as the objects of $MC_R^\infty(A)$. For $\alpha, \beta \in MC_R^\infty(A)$, let $G(\alpha, \beta)$ be the set of elements $g \in 1 + (A \otimes m)^0$ such that

$$m_1^{\alpha, \beta}(g) = \sum_{i_0,i_1 \geq 0} (-1)^{i_0i_1 + \frac{i_0(i_0-1)}{2} + \frac{i_1(i_1-1)}{2}} m_{1+i_0+i_1}^A(\partial^{i_1}, g, \alpha^{i_0}) = 0.$$ 

Then we have an obvious action of the group $(A \otimes m)^{-1}$ on the set $G(\alpha, \beta):

$$h \cdot g = g + m_1^{\alpha, \beta}(h) = g + \sum_{i_0,i_1 \geq 0} (-1)^{i_0i_1 + \frac{i_0(i_0-1)}{2} + \frac{i_1(i_1-1)}{2}} m_{1+i_0+i_1}^A(\partial^{i_1}, g, \alpha^{i_0}).$$

We define $\text{Hom}_{MC_R(A)}(\alpha, \beta)$ as the set of orbits $G(\alpha, \beta)/(A \otimes m)^{-1}$. The composition of morphisms in $MC_R(A)$ is induced by $m_2^{MC_R^\infty(A)}$. It follows from the axioms of $A_{\infty}$-structures that we obtain a well defined category.

**Proposition 5.2.** The category $MC_R(A)$ is a groupoid.

**Proof.** Let $g \in \text{Hom}_{MC_R(A)}(\alpha, \beta)$. Prove that it has a left inverse $g' \in \text{Hom}_{MC_R(A)}(\beta, \alpha)$.

Let $\tilde{g} \in G(\alpha, \beta)$ be a lift of $g$. First prove that there exists $\tilde{g}' \in 1 + (A \otimes m)^0$ such that

$$m_2^{\alpha, \alpha}(\tilde{g}', \tilde{g}) = 1. \tag{5.2}$$

Let $n$ be the minimal positive integer such that $m^n = 0$. The proof is by induction over $n$.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m \geq 1$. Prove it for $n = m + 1$. From the induction hypothesis it follows that there exists $\tilde{g}' \in \text{Hom}_{MC_R(A)}(\beta, \alpha)$. Suppose $m_2^{\alpha, \alpha}(\tilde{g}', \tilde{g}) = 1 + x$, where $x \in (A \otimes m^{n})^0$. Then we can obviously have

$$m_2^{\alpha, \alpha}(\tilde{g}' - x, \tilde{g}) = 1.$$

Thus, the induction hypothesis is proved for $n = m + 1$. The statement is proved.

Further, take $\tilde{g}' \in 1 + (A \otimes m)^0$ such that (5.2) holds. To prove that $g$ has a left inverse it suffices to prove that

$$m_1^{\beta, \alpha}(\tilde{g}') = 0.$$

From the equality (5.2), and since $m_1^{\alpha, \beta}(g) = 0$, we obtain that

$$m_2^{\alpha, \alpha}(m_1^{\beta, \alpha}(\tilde{g}'), \tilde{g}) = 0.$$

Suppose that $m_1^{\beta, \alpha}(\tilde{g}') \neq 0$. Take the maximal positive integer $m$ such that $m_1^{\beta, \alpha}(\tilde{g}') \in (A \otimes m^n)^0$. Then we can obviously obtain that $m_2^{\alpha, \alpha}(m_1^{\beta, \alpha}(\tilde{g}'), \tilde{g}) \in (A \otimes m^m)^0 \setminus (A \otimes m^{m+1})^0$, which leads to contradiction.

Thus, $g$ has a left inverse. Analogously, it has a right inverse, hence $g$ is invertible. Therefore, the category $MC_R(A)$ is a groupoid. \hfill $\square$

Clearly, the assignment $R \mapsto MC_R(A)$ defines a pseudo-functor from $\text{dgart}$ to $\text{Gpd}$. We denote this pseudo-functor by $MC(A)$ and call it Maurer-Cartan pseudo-functor.

Notice that if $A$ is a DG algebra, i.e. $m_n^A = 0$ for $n \geq 3$, then $MC_R^\infty(A)$ is a DG category. Further, for $\phi \in \text{Hom}(\alpha, \beta)$ we have
and the composition in $\mathcal{MC}_R(A)$ is just the product in $A \otimes R$. It follows that the constructed Maurer-Cartan pseudo-functor coincides in this case with that constructed in [ELOI], Section 5.

**Remark 5.3.** The Maurer-Cartan groupoid $\mathcal{MC}_R(A)$ can be extended to a $\infty$-groupoid $\mathcal{MC}_R^\infty(A)$ so that $\mathcal{MC}_R(A) = \pi_0(\mathcal{MC}_R^\infty(A))$. Further, the assignment $R \to \mathcal{MC}_R^\infty(A)$ defines a pseudo-functor $\mathcal{MC}^\infty(A) : \text{dgart} \to \text{Gpd}^\infty$, where $\text{Gpd}^\infty$ is an $\infty$-category of $\infty$-groupoids.

6. Obstruction theory

Fix a strictly unital $A_\infty$-algebra $A$.

Let $R$ be an artinian DG algebra with the maximal ideal $m$. Further, let $n$ be the minimal positive integer such that $m^{n+1} = 0$. Put $I = m^n$, $R = R/I$, and $\pi : R \to \widehat{R}$ — the projection morphism. The next Proposition describes the obstruction theory for lifting of objects and morphisms along the functor $\pi^* : \mathcal{MC}_R(A) \to \mathcal{MC}_R(A)$.

**Proposition 6.1.** 1) There exists a map $o_2 : \text{Ob}(\mathcal{MC}_R(A)) \to H^2(A \otimes I)$ such that $\alpha \in \mathcal{MC}_R(A)$ is in the image of $\pi^*$ if and only if $o_2(\alpha) = 0$. Furthermore, if $\alpha, \beta \in \mathcal{MC}_R(A)$ are isomorphic then $o_2(\alpha) = 0$ iff $o_2(\beta) = 0$.

2) Let $\xi \in \text{Ob}(\mathcal{MC}_R(A))$ be such that the fiber $(\pi^*)^{-1}(\xi)$ is non-empty. Then there exists a simply transitive action of the group $Z^1(A \otimes I)$ on the set $\text{Ob}((\pi^*)^{-1}(\xi))$. Let $\xi_1, \xi_2 \in \text{Ob}(\mathcal{MC}_R(A))$ be isomorphic objects such that both fibers $(\pi^*)^{-1}(\xi_1)$, $(\pi^*)^{-1}(\xi_2)$ are non-empty, and let $f : \xi_1 \to \xi_2$ be a morphism. Take the action of $Z^1(A \otimes I)$ on $\text{Ob}((\pi^*)^{-1}(\xi_2))$ as above and the action on $\text{Ob}((\pi^*)^{-1}(\xi_1))$ which is inverse to the above action. Then there is a (non-canonical) $Z^1(A \otimes I)$-equivariant map

$$\tilde{o}_1 : \text{Ob}((\pi^*)^{-1}(\xi_1)) \times \text{Ob}((\pi^*)^{-1}(\xi_2)) \to Z^1(A \otimes I),$$

such that the composition of it with the projection

$$Z^1(A \otimes I) \to H^1(A \otimes I),$$

which we denote by $o^1_1$, is canonically defined and satisfies the following property: for $\alpha_1 \in \text{Ob}((\pi^*)^{-1}(\xi_1))$, $\alpha_2 \in \text{Ob}((\pi^*)^{-1}(\xi_2))$ there exists a morphism $\gamma : \alpha_1 \to \alpha_2$ such that $\pi^*(\gamma) = f$ iff $o^1_1(\alpha_1, \alpha_2) = 0$.

3) Let $\tilde{\alpha}, \tilde{\beta} \in \mathcal{MC}_R(A)$ be objects and let $f : \alpha \to \beta$ be a morphism from $\alpha = \pi^*(\tilde{\alpha})$ to $\beta = \pi^*(\tilde{\beta})$. Suppose that the set $(\pi^*)^{-1}(f)$ of morphisms $\tilde{f} : \tilde{\alpha} \to \tilde{\beta}$ such that $\pi^*(\tilde{f}) = f$ is non-empty. Then there is a simple transitive action of the group $\text{Im}(H^0(A \otimes I) \to H^0(A \otimes m, m^1_{\alpha, \beta}))$ on the set $(\pi^*)^{-1}(f)$. In particular, the difference map

$$o_0 : (\pi^*)^{-1}(f) \times (\pi^*)^{-1}(f) \to \text{Im}(H^0(A \otimes I) \to H^0(A \otimes m, m^1_{\alpha, \beta}))$$

satisfies the following property: if $\tilde{f}, \tilde{f}' \in (\pi^*)^{-1}(f)$ then $\tilde{f} = \tilde{f}'$ iff $o_0(\tilde{f}, \tilde{f}') = 0$.

**Proof.** 1) Let $\alpha \in \mathcal{MC}_R(A)$. Take some $\tilde{\alpha} \in (A \otimes m)^1$ such that $\pi(\tilde{\alpha}) = \alpha$. Then we have

$$\sum_{n \geq 1} (-1)^{n(n+1)/2} m^A \otimes R(\tilde{\alpha}, \ldots, \tilde{\alpha}) \in (A \otimes I)^2.$$
A straightforward applying of (3.2) shows that
\[ \sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A \otimes R}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \in Z^2(A \otimes I). \]

Further, if \( \tilde{\alpha}' \in A \otimes m \) is another lift of \( \alpha \) then
\[ \sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A \otimes R}(\tilde{\alpha}', \ldots, \tilde{\alpha}') - \sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A \otimes R}(\tilde{\alpha}, \ldots, \tilde{\alpha}) = m_1^{A \otimes R}(\tilde{\alpha}' - \tilde{\alpha}). \]

Hence, we obtain the well defined element \( o_2(\alpha) \in H^2(A \otimes I) \) and therefore the map \( o_2 : Ob(MC_R(A)) \to H^2(A \otimes I) \). The first property of \( o_2 \) is obviously satisfied.

Further, let \( \alpha, \beta \in MC_R(A) \), and \( f : \alpha \to \beta \) be a morphism. Suppose that \( o_2(\alpha) = 0 \). Take some \( \tilde{\alpha} \in (\pi^*)^{-1}(\alpha) \). Further, take some \( \tilde{\alpha} \in (\pi^*)^{-1}(\alpha) \). Hence, we have that
\[ \sum_{i_0, i_1 \geq 0} (-1)^{i_0 i_1 + \frac{i_0(i_0+1)}{2} + \frac{i_1(i_1+1)}{2}} m_{i_0+i_1+1}^{A \otimes R}(\tilde{\beta}, \tilde{\alpha}) = \sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A \otimes R}(\tilde{\beta}, \tilde{\alpha}). \]

A straightforward applying of (3.2) shows that
\[ m_1^{A \otimes R}(\sum_{i_0, i_1 \geq 0} (-1)^{i_0 i_1 + \frac{i_0(i_0+1)}{2} + \frac{i_1(i_1+1)}{2}} m_{i_0+i_1+1}^{A \otimes R}(\tilde{\beta}, \tilde{\alpha})) = \sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A \otimes R}(\tilde{\beta}, \tilde{\alpha}). \]

Therefore, \( o_2(\beta) = 0 \). This proves 1).

2). Let \( \eta \in Z^1(A \otimes I) \). It follows from (6.1) that the formula
\[ \eta : \alpha \mapsto \alpha + \eta \]
defines a simply transitive action of the group \( Z^1(A \otimes I) \) on the set \( Ob((\pi^*)^{-1}(\xi)) \). Let \( \xi_1, \xi_2, \gamma \) be as in Proposition. Take some \( \tilde{f} \in 1 + (A \otimes m)^0 \) such that \( \pi(\tilde{f}) = f \). Define \( \tilde{o}_1 \) by the formula
\[ \tilde{o}_1(\alpha, \beta) = \tilde{o}_1^{\tilde{f}}(\alpha, \beta) = m_1^{A \otimes R}(\tilde{f}). \]

It is easy to see that the image of \( \tilde{o}_1^{\tilde{f}} \) lies in \( Z^1(A \otimes I) \) and that \( \tilde{o}_1^{\tilde{f}} \) is \( Z^1(A \otimes I) \)-equivariant.

If \( \tilde{f}' \) is another lift of \( f \), then there exists \( h \in (A \otimes m)^{-1} \) such that
\[ \tilde{v} = \tilde{f}' - \tilde{f} - m_1^{A \otimes R}(h) \in (A \otimes I)^0. \]

Further,
\[ \tilde{o}_1^{\tilde{f}'}(\alpha, \beta) - \tilde{o}_1^{\tilde{f}}(\alpha, \beta) = m_1^{A \otimes R}(v), \]
hence the map \( \tilde{o}_1^{\tilde{f}'} \) is canonically defined.

Suppose that \( \tilde{o}_1^{\tilde{f}'}(\alpha, \beta) = 0 \) for some \( \alpha \in (\pi^*)^{-1}(\xi_1), \beta \in (\pi^*)^{-1}(\xi_2) \). Let \( \tilde{f} \) be as above. Then there exists \( x \in (A \otimes I)^0 \) such that
\[ m_1^{A \otimes R}(\tilde{f}) = m_1^{A \otimes R}(x). \]
We have \( \tilde{f} - x \in G(\alpha, \beta) \), and \( \pi^*(\tilde{f} - x) = f \).

Conversely, suppose that there exists a morphism \( \gamma \in \text{Hom}_{MC_R(A)}(\alpha, \beta) \) for some \( \alpha \in (\pi^*)^{-1}(\xi_1), \beta \in (\pi^*)^{-1}(\xi_2) \), such that \( \pi^*(\gamma) = f \). Let \( \tilde{\gamma} \in G(\alpha, \beta) \) be a representative of \( \gamma \). Then we have \( \tilde{o}_1^{\tilde{\gamma}}(\alpha, \beta) = 0 \), hence \( \tilde{o}_1^{\tilde{f}'}(\alpha, \beta) = 0 \). This proves 2).
3). First we define the action of the group $Z^0(A \otimes I)$ on the set $(\pi^*)^{-1}(f)$ by the formula
\[
\eta : \tilde{f} \to \tilde{f} + \eta,
\]
where $\eta \in Z^0(A \otimes I)$, and $\tilde{f} \in G(\alpha, \beta)$ is such that $\pi^*(\tilde{f}) = f$. Clearly, this is correct. Further, if $\eta = m_1^{A \otimes R}(\zeta)$ for some $\zeta \in (A \otimes m)^{-1}$, then
\[
\eta(\tilde{f}) = \tilde{f} + m_1^{\alpha, \beta}(\zeta) = \tilde{f}.
\]
Hence, we have an action of $\text{Im}(H^0(A \otimes I) \to H^0(A \otimes m, m_1^{\alpha, \beta}))$ on the set $(\pi^*)^{-1}(f)$.

Tautologically, this action is simple.

Prove that it is transitive. Let $\tilde{f}, \tilde{f}' \in G(\alpha, \beta)$ be such that $\pi^*(\tilde{f}) = \pi^*(\tilde{f}') = f$. Then, by definition, there exists $h \in (A \otimes m)^{-1}$ such that
\[
\tilde{f}' - \tilde{f} = m_1^{\alpha, \beta}(h) \in (A \otimes I)^0.
\]
Replacing $\tilde{f}$ by $\tilde{f} + m_1^{\alpha, \beta}(h)$, we obtain $\tilde{f}' = \tilde{f} + \eta$, where $\eta \in (A \otimes I)^0$. Since $\tilde{f}, \tilde{f}' \in G(\alpha, \beta)$, we have that $\eta \in Z^0(A \otimes I)$. This shows transitivity and proves 3).

Proposition is proved. \[ \Box \]

**Remark 6.2.** One can also construct the obstruction theory for lifting of objects and all $k$-morphisms along the $\infty$-functor
\[
\pi^* : MC^\infty_R(A) \to MC^\infty_R(A).
\]

7. Invariance Theorems

Let $A_1, A_2$ be strictly unital $A_\infty$-algebras and $f : A_1 \to A_2$ be a strictly unital $A_\infty$-morphism between them given by a sequence of maps
\[
f_n : A_1^\otimes n \to A_2.
\]
Further, let $R$ be an artinian DG algebra with the maximal ideal $m$.

Then we have a (strictly unital) $A_\infty$-functor
\[
f_R^* : MC^R_R(A_1) \to MC^R_R(A_2)
\]
defined by the formulas
\[
f_R^*(\alpha) = \sum_{n \geq 1} (-1)^{n(n-1)/2} f_n(\alpha, \ldots, \alpha);
\]
\[
f_R^*(\alpha_0, \ldots, \alpha_n)(x_1, \ldots, x_n) = \sum_{i_0, \ldots, i_n \geq 0} (-1)^{\epsilon} f_{n+i_0+\cdots+i_n}^{\alpha_0, \ldots, \alpha_n}(\alpha_0^{i_0}, \ldots, \alpha_n^{i_n}, x_1, \ldots, x_n),
\]
where
\[
\epsilon = \sum_{n \geq k \geq j \geq 0} (x_k + i_k)j + \sum_{k=0}^n \frac{i_k(i_k - 1)}{2} + \sum_{k=1}^n k i_k.
\]
One checks without difficulties that these formulas indeed define a strictly unital $A_\infty$-functor.

It induces a functor $f_R^* : MC_R(A_1) \to MC_R(A_2)$ and we obtain a morphism of pseudo-functors
\[
f^* : MC(A_1) \to MC(A_2).
\]

The following theorems show that for quasi-isomorphic strictly unital $A_\infty$-algebras the corresponding Maurer-Cartan $A_\infty$-categories (resp. Maurer-Cartan pseudo-functors) are quasi-equivalent (resp. equivalent).
Theorem 7.1. Let $f : A_1 \to A_2$ be a strictly unital quasi-isomorphism of strictly unital $A_\infty$-algebras and let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Then the $A_\infty$-functor

$$f^*_R : \mathcal{MC}_R^\infty(A_1) \to \mathcal{MC}_R^\infty(A_2)$$

is a quasi-equivalence.

Proof. 1). Prove that for any $\alpha, \beta \in \mathcal{MC}_R^\infty(A_1)$ the morphism of complexes

$$f^*_R(\alpha, \beta) : \text{Hom}_{\mathcal{MC}_R^\infty(A_1)}(\alpha, \beta) \to \text{Hom}_{\mathcal{MC}_R^\infty(A_2)}(f^*(\alpha), f^*(\beta))$$

is quasi-isomorphism. Note that both complexes have finite filtrations by subcomplexes $A_1 \otimes \mathfrak{m}^i$ and $A_2 \otimes \mathfrak{m}^i$. The morphism $f^*_R(\alpha, \beta)$ is compatible with these filtrations and induces quasi-isomorphisms on the subquotients. Hence, it is quasi-isomorphism.

2). Now we prove that the functor

$$\text{Ho}(f^*) : \text{Ho}(\mathcal{MC}_R^\infty(A_1)) \to \text{Ho}(\mathcal{MC}_R^\infty(A_2))$$

is an equivalence. We have already proved that it is full faithful, hence it remains to prove that it is essentially surjective. We will prove the stronger statement: the functor

$$f^*_R : \mathcal{MC}_R(A_1) \to \mathcal{MC}_R(A_2)$$

is essentially surjective.

Let $n$ be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over $n$.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m$. Prove it for $n = m + 1$. Let $\mathcal{I}$, $\mathcal{R}$, $\mathfrak{m}$, $\pi$ be as above. A straightforward checking shows that the following diagram commutes:

\[\begin{array}{ccc}
\text{Ob}(\mathcal{MC}_R(A_1)) & \xrightarrow{f^*_R} & \text{Ob}(\mathcal{MC}_R(A_2)) \\
o_2 \downarrow & & \downarrow no_2 \\
H^2(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^2(A_2 \otimes \mathcal{I}),
\end{array}\]  

(7.1)

where the map $o_2$ is defined in Proposition 6.1.

Let $\alpha \in \mathcal{MC}_R(A_2)$. By the induction hypothesis, there exists $\beta \in \mathcal{MC}_R(A_1)$ such that $f^*_R(\beta)$ is isomorphic to $\pi^*(\alpha)$ in $\mathcal{MC}_R(A_2)$. Since the diagram (7.1) commutes, we have that $o_2(\beta) = 0$. Thus, by Proposition 6.1, the fiber $(\pi^*)^{-1}(\beta)$ is nonempty. Fix some $\tilde{\beta} \in (\pi^*)^{-1}(\beta)$. Let $\gamma : f^*_R(\tilde{\beta}) \to \pi^*(\alpha)$ be a morphism. A straightforward checking shows that the following diagram commutes:

\[\begin{array}{ccc}
\text{Ob}((\pi^*)^{-1}(\beta)) & \xrightarrow{f^*_R} & \text{Ob}((\pi^*)^{-1}(f^*_R(\tilde{\beta}))) \\
o_1^{id_{\beta}}(\tilde{\beta}) \downarrow & & \downarrow no_1^{f^*_R(\tilde{\beta})} \circ no_1^{\gamma} \circ f^*_R(\tilde{\beta}) \\
H^1(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^1(A_2 \otimes \mathcal{I}),
\end{array}\]  

(7.2)

where the vertical arrows are defined in Proposition 6.1. Since the map $o_1^{id_{\beta}}(\tilde{\beta})$ is surjective and the diagram (7.2) commutes, there exists an object $\tilde{\beta}' \in \text{Ob}((\pi^*)^{-1}(\beta))$ such that $o_1^{f^*_R(\tilde{\beta})}(\tilde{\beta}', \alpha) = 0$. Then, by Proposition 6.1, there exists a morphism $\tilde{\gamma} : f^*_R(\tilde{\beta}') \to \alpha$ (such that $\pi^*(\tilde{\gamma}) = \gamma$). Therefore, the functor $f^*_R$ is essentially surjective, and the induction hypothesis is proved for $n = m + 1$. The statement is proved.

Theorem is proved. \qed
**Theorem 7.2.** Let \( f : A_1 \rightarrow A_2 \) be a strictly unital quasi-isomorphism of strictly unital \( A_\infty \)-algebras. Then the morphism of pseudo-functors

\[
 f^* : \mathcal{MC}(A_1) \rightarrow \mathcal{MC}(A_2)
\]

is an equivalence.

**Proof.** Fix an artinian DG algebra \( R \) with the maximal ideal \( m \). We must prove that the functor

\[
 f_R^* : \mathcal{MC}_R(A_1) \rightarrow \mathcal{MC}_R(A_2)
\]

is an equivalence.

In the proof of the previous Theorem we have already shown that it is essentially surjective. So it remains to prove that it is full and faithful.

Let \( n \) be the minimal positive integer such that \( m^n = 0 \). The proof is by induction over \( n \).

For \( n = 1 \), there is nothing to prove.

Suppose that the induction hypothesis holds for \( n = m \geq 1 \). Prove it for \( n = m + 1 \).

**Full.** Let \( \alpha, \beta \in \mathcal{MC}_R(A_1) \) and let \( \gamma : f_R^*(\alpha) \rightarrow f_R^*(\beta) \) be a morphism. By induction hypothesis, there exists a morphism \( g : \pi^*(\alpha) \rightarrow \pi^*(\beta) \) such that

\[
 f_R^*(g) = \pi^*(\gamma).
\]

A straightforward checking shows that the following diagram commutes

\[
\begin{array}{ccc}
(\pi^*)^{-1}(\pi^*(\alpha)) \times (\pi^*)^{-1}(\pi^*(\beta)) & \longrightarrow & (\pi^*)^{-1}(\pi^*(f_R^*(\alpha))) \times (\pi^*)^{-1}(\pi^*(f_R^*(\beta))) \\
\downarrow o_1^* & & \downarrow o_1^* \\
H^1(A_1 \otimes I) & \sim & H^1(A_2 \otimes I).
\end{array}
\]

(7.3)

By Proposition 6.1 and since the diagram (7.3) commutes there exists a morphism \( \tilde{g} : \alpha \rightarrow \beta \) such that \( \pi^*(\tilde{g}) = g \). Further, a straightforward checking shows that the following diagram commutes:

\[
\begin{array}{ccc}
\pi^*(\gamma) & \longrightarrow & \text{Im}(H^0(A_1 \otimes I) \rightarrow H^0(A_1 \otimes m, m_1^{\alpha, \beta})) \\
\downarrow f_R^* & & \downarrow ~ \\
(\pi^*)^{-1}(\pi^*(\tilde{g})) & \longrightarrow & \text{Im}(H^0(A_2 \otimes I) \rightarrow H^0(A_2 \otimes m, m_1^{f_R^*(\alpha), f_R^*(\beta)}))
\end{array}
\]

(7.4)

Since the upper arrow is surjective, there exists a morphism \( \tilde{g}' \in (\pi^*)^{-1}(g) \) such that

\[
o_0(f_R^*(\tilde{g}'), \gamma) = 0,
\]

i.e. \( f_R^*(\tilde{g}') = \gamma \). Hence, the functor \( f_R^* \) is full.

**Faithful.** Let \( \gamma_1, \gamma_2 : \alpha \rightarrow \beta \) be two morphisms in \( \mathcal{MC}_R(A_1) \). Suppose that \( f_R^*(\gamma_1) = f_R^*(\gamma_2) \). Then we have also \( f_R^*(\pi^*(\gamma_1)) = f_R^*(\pi^*(\gamma_2)) \), hence by induction hypothesis \( \pi^*(\gamma_1) = \pi^*(\gamma_2) \). A straightforward checking shows that the following diagram commutes:

\[
\begin{array}{ccc}
(\pi^*)^{-1}(\pi^*(\gamma_1)) \times (\pi^*)^{-1}(\pi^*(\gamma_1)) & \longrightarrow & \text{Im}(H^0(A_1 \otimes I) \rightarrow H^0(A_1 \otimes m, m_1^{\alpha, \beta})) \\
\downarrow f_R^* & & \downarrow ~ \\
(\pi^*)^{-1}(\pi^*(f_R^*(\gamma_1))) \times (\pi^*)^{-1}(\pi^*(f_R^*(\gamma_1))) & \longrightarrow & \text{Im}(H^0(A_2 \otimes I) \rightarrow H^0(A_2 \otimes m, m_1^{f_R^*(\alpha), f_R^*(\beta)})).
\end{array}
\]

(7.5)
By Proposition 6.1 and since the diagram (7.5) commutes we have that
\[ o_0(\gamma_1, \gamma_2) = 0, \]
hence \( \gamma_1 = \gamma_2 \). Thus, the functor \( f^*_R \) is full.

The induction hypothesis is proved for \( n = m + 1 \). The statement is proved.

Theorem is proved. \( \square \)

**Remark 7.3.** It can be proved that an \( A_\infty \)-quasi-isomorphism \( f : A_1 \rightarrow A_2 \) induces an equivalence of \( \infty \)-groupoids \( f^*_R : MC^\infty_R(A_1) \rightarrow MC^\infty_R(A_2) \).

8. Twisting cochains

Let \( G \) be a co-augmented DG coalgebra. Let \( A \) be an arbitrary \( A_\infty \)-algebra. Then the graded vector space \( Hom_k(G, A) \) has natural structure of an \( A_\infty \)-algebra. If \( \dim G < \infty \) or \( \dim A < \infty \), then \( Hom_k(G, A) \) is canonically identified with \( A \otimes G^* \) as an \( A_\infty \)-algebra.

Suppose that the DG coalgebra \( G \) is co-complete. The map \( \tau : G \rightarrow A \) of degree 1 is called a twisting cochain if it passes through \( \bar{G} \) and satisfies the generalized Maurer-Cartan equation (5.1) as an element of \( A_\infty \)-algebra \( Hom_k(G, A) \). This is well defined since \( G \) is co-complete. If \( R \) is an artinian DG algebra and \( A \) is strictly unital, then we have natural bijection between the set of twisting cochains \( \tau : R^* \rightarrow A \) and the set \( MC(A \otimes m) \). In the case when \( A \) is augmented, the twisting cochain is called admissible if it passes through \( \bar{A} \).

Tautologically, admissible twisting cochains \( G \rightarrow A \) are in one-to-one correspondence with twisting cochains \( G \rightarrow \bar{A} \).

**Proposition 8.1.** Let \( A \) be an \( A_\infty \)-algebra. The composition \( \tau_A : BA \rightarrow A \) of the natural projection \( BA \rightarrow A[1] \) with the shift map \( A[1] \rightarrow A \) is the universal twisting cochain. That is, if \( G \) is a co-augmented co-complete DG coalgebra and \( \tau : G \rightarrow A \) is a twisting cochain then there exists a unique homomorphism \( g_\tau : G \rightarrow BA \) of co-augmented DG coalgebras, such that \( \tau_A \cdot g_\tau = \tau \).

It follows that if \( A \) is augmented then the composition of \( \tau_{\bar{A}} \) with the embedding \( \bar{A} \hookrightarrow A \), which we also denote by \( \tau_A \), is the universal admissible twisting cochain in the same sense.

**Proof.** A straightforward checking. \( \square \)

Further, if \( G \) is a co-augmented co-complete DG coalgebra, and \( \tau : G \rightarrow A \) is a twisting cochain then
\[ G \otimes A := G \square_B A \]
is an object of \( A_{op}^{op} \text{-mod}_{\infty} \).

**Proposition 8.2.** Let \( f : A_1 \rightarrow A_2 \) be an \( A_\infty \)-quasi-isomorphism of \( A_\infty \)-algebras, \( G \) be a co-augmented co-complete DG coalgebra, and \( \tau : G \rightarrow A_1 \) be a twisting cochain. Then there is a natural homotopy equivalence in \( A_{op}^{op} \text{-mod}_{\infty} \):
\[ G \otimes A_1 \rightarrow f_*(G \otimes f_{\tau} A_2). \]

**Proof.** We have a natural homotopy equivalence of DG bicomodules over \( BA_1 \):
\[ BA_1 \otimes A_1[1] \otimes BA_1 \rightarrow BA_1 \square_{BA_2} (BA_2 \otimes A_2[1] \otimes BA_2) \square_{BA_1} BA_1. \]

Co-tensoring it on the left by \( G \), we obtain the required homotopy equivalence. \( \square \)
Now let $A$ be an augmented $A_\infty$-algebra. If $G$ is a co-augmented co-complete DG coalgebra and $\tau : G \rightarrow A$ is an admissible twisting cochain then

$$G \otimes_{\tau} A := G \square_{\bar{B}}(B\bar{A} \otimes_{\tau_A} A)$$

is an object of $\bar{A}_{(G^*)^{op}\text{-mod}}$. 

**Proposition 8.3.** Let $f : A_1 \rightarrow A_2$ be an $A_\infty$-quasi-isomorphism of augmented $A_\infty$-algebras, $G$ be a co-augmented co-complete DG coalgebra and $\tau : G \rightarrow A_1$ be an admissible twisting cochain. Then there is a natural homotopy equivalence in $\bar{A}_{(G^*)^{op}\text{-mod}}$:

$$G \otimes_{\tau} A_1 \rightarrow f_*(G \otimes_{f,\tau} A_2).$$

**Proof.** We have a natural homotopy equivalence of DG bicomodules over $B\bar{A}_1$:

$$B\bar{A}_1 \otimes A_1[1] \otimes B\bar{A}_1 \rightarrow B\bar{A}_1 \square_{\bar{B}\bar{A}_2}(B\bar{A}_2 \otimes A_2[1] \otimes B\bar{A}_2) \square_{\bar{B}\bar{A}_2} B\bar{A}_1.$$

Co-tensoring it on the left by $G$, we obtain the required homotopy equivalence. \qed

Let $R$ be an artinian DG algebra, and $\tau : R^* \rightarrow A$ be an admissible twisting cochain. Then by Proposition 8.1 we have a natural morphism of DG coalgebras $g_\tau : R^* \rightarrow B\bar{A}$. Further, we have the dual morphism of DG algebras $g_\tau^* : \hat{S} \rightarrow R$. In particular, $\hat{R}$ becomes a DG $\hat{S}^{op}$-module.

**Lemma 8.4.** In the above notation $A_\infty$ $\bar{A}^{op}_{\hat{S}^{op}}$-modules $\Hom_{\hat{S}^{op}}(R^*, B\bar{A} \otimes_{\tau_A} A)$ and $R^* \otimes_{\tau} A$ are isomorphic.

**Proof.** Evident. \qed

If $\tau : R^* \rightarrow A$ is an admissible twisting cochain and $\alpha \in \mathcal{MC}_R(A)$ is the corresponding object, then we will write also $A \otimes_\alpha R^*$ instead of $R^* \otimes_{\tau} A$.

Further, for $\alpha \in \mathcal{MC}_R(A)$ corresponding to an admissible twisting cochain we put

$$A \otimes_\alpha R := \Hom_R(R^*, A \otimes_\alpha R^*).$$

This is an object of $\bar{A}^{op}_{R^{op}\text{-mod}}$. Its $(R^{op})^{op}$-module structure is obvious and $A_\infty$-module structure can also be given by the explicit formulas:

$$m_n^{A \otimes_\alpha R}(m, a_1, \ldots, a_{n-1}) = m_n^{0, \ldots, 0, \alpha}(m, a_1 \otimes 1_R, \ldots, a_{n-1} \otimes 1_R). \quad (8.1)$$

**Proposition 8.5.** Let $f : A_1 \rightarrow A_2$ be an $A_\infty$-quasi-isomorphism of augmented $A_\infty$-algebras, $R$ be an artinian DG algebra and let $\alpha \in \mathcal{MC}_R(A_1)$. Then there is a natural homotopy equivalence in $\bar{A}^{op}_{(R^{op})^{op}\text{-mod}}$:

$$A_1 \otimes_\alpha R \rightarrow f_*(A_2 \otimes f^*_R(\alpha) R).$$

**Proof.** The required homotopy equivalence is obtained by applying the functor $\Hom_R(R^*, -)$ to the homotopy equivalence

$$A_1 \otimes_\alpha R^* \rightarrow f_*(A_2 \otimes f^*_R(\alpha) R^*)$$

from Proposition 8.3. \qed
Note that if $A$ is a DG algebra, then $A \otimes_\alpha R^*$ and $A \otimes_\alpha R$ are the DG modules from $\text{coDef}^h_R(A)$ and $\text{Def}^h_R(A)$ respectively, which correspond to $\alpha$.

Finally, if $A$ is a strictly unital but not necessarily augmented $A_\infty$-algebra, $R$ is an artinian DG algebra, $\alpha$ is an object of $\mathcal{MC}_R(A)$ and $\tau : R^* \to A$ is the corresponding twisting cochain then we also write $A \otimes_\alpha R^*$ instead of $R^* \otimes_\tau A$. Further, we put

$$A \otimes_\alpha R = \text{Hom}_R(R^*, A \otimes_\alpha R^*).$$

This is the object of $A^{op}_R R^*\text{-mod}_\infty$. Again, its $(R^{op})^{gr}\text{-module}$ structure is obvious and the $A_\infty$-module structure is given by the formulas 8.1. The following Proposition is absolutely analogous to the previous one and we omit the proof.

**Proposition 8.6.** Let $f : A_1 \to A_2$ be a strictly unital $A_\infty$-morphism of strictly unital $A_\infty$-algebras, $R$ be an artinian DG algebra and let $\alpha \in MC_R(A_1)$. Then there is a natural homotopy equivalence in $A^{op}_1 R^{op}\text{-mod}_\infty$:

$$A_1 \otimes_\alpha R \to f_*(A_2 \otimes f_R^*(\alpha) R).$$

**Part 3. The pseudo-functors $\text{DEF}$ and $\text{coDEF}$**

9. **The bicategory $2\text{-adgalg}$ and deformation pseudo-functor $\text{coDEF}$**

Let $\mathcal{E}$ be a bicategory and $F,G : \mathcal{E} \to \text{Gpd}$ two pseudo-functors. A morphism $\epsilon : F \to G$ is called an equivalence if for each $X \in \text{Ob}\mathcal{E}$ the functor $\epsilon_X : F(X) \to G(X)$ is an equivalence of categories.

**Definition 9.1.** ([ELOII]) We define the bicategory $2\text{-adgalg}$ of augmented DG algebras as follows. The objects are augmented DG algebras. For DG algebras $B,C$ the collection of 1-morphisms $1\text{-Hom}(B,C)$ consists of pairs $(M,\theta)$, where

- $M \in D(B^{op} \otimes C)$ is such that there exists an isomorphism (in $D(C)$) $C \to \nu_* M$ (where $\nu_* : D(B^{op} \otimes C) \to D(C)$ is the functor of restriction of scalars corresponding to the natural homomorphism $\nu : C \to B^{op} \otimes C$);
- and $\theta : k \otimes C M \to k$ is an isomorphism in $D(B^{op})$.

The composition of 1-morphisms

$$1\text{-Hom}(B,C) \times 1\text{-Hom}(C,D) \to 1\text{-Hom}(B,D)$$

is defined by the tensor product $\cdot \otimes \cdot$. Given 1-morphisms $(M_1,\theta_1), (M_2,\theta_2) \in 1\text{-Hom}(B,C)$ a 2-morphism $f : (M_1,\theta_1) \to (M_2,\theta_2)$ is an isomorphism (in $D(B^{op} \otimes C)$) $f : M_2 \to M_1$ (not from $M_1$ to $M_2$!) such that $\theta_1 \cdot k \otimes C f = \theta_2$. So in particular the category $1\text{-Hom}(B,C)$ is a groupoid. Denote by $2\text{-dgart}$ the full subbicategory of $2\text{-adgalg}$ consisting of artinian DG algebras. Similarly we define the full subcategories $2\text{-dgart}_+, 2\text{-dgart}_-, 2\text{-art}, 2\text{-cart}$ ([ELOI], Definition 2.3).

**Remark 9.2.** Assume that augmented DG algebras $B$ and $C$ are such that $B^i = C^i = 0$ for $i > 0$, $\dim B^i, \dim C^i < \infty$ for all $i$ and $\dim H(C) < \infty$. Denote by $\langle k \rangle \subset D(B^{op} \otimes C)$ the triangulated envelope of the DG $B^{op} \otimes C$-module $k$. Let $(M,\theta) \in 1\text{-Hom}(B,C)$. Then by [ELOI], Corollary 3.22 $M \in \langle k \rangle$.

For any augmented DG algebra $B$ we obtain a pseudo-functor $h_B$ between the bicategories $2\text{-adgalg}$ and $\text{Gpd}$ defined by $h_B(C) = 1\text{-Hom}(B,C)$. 

Note that a usual homomorphism of augmented DG algebras $\gamma : B \to C$ defines the structure of a DG $B^\text{op}$-module on $C$ with the canonical isomorphism of DG $B^\text{op}$-modules $\text{id} : k \otimes_k C \to k$. Hence, we have a natural pseudo-functor $F : \text{adgalg} \to \text{2-adgalg}$.

The next proposition from [ELOII] asserts that the deformation functor $\text{coDef}$ has a natural “lift” to the bicategory $\text{2-dgart}$. We present its proof for completeness.

**Proposition 9.3.** ([ELOII]) There exist a pseudo-functor $\text{coDEF}(E)$ from $\text{2-dgart}$ to $\text{Gpd}$ which is an extension to $\text{2-dgart}$ of the pseudo-functor $\text{coDef}$ i.e. there is an equivalence of pseudo-functors $\text{coDef}(E) \simeq \text{coDEF}(E) \cdot F$.

**Proof.** Given artinian DG algebras $R, Q$ and $M = (M, \theta) \in 1\text{-Hom}(R, Q)$ we need to define the corresponding functor

$$M^! : \text{coDef}_R(E) \to \text{coDef}_Q(E).$$

Let $S = (S, \sigma) \in \text{coDef}_R(E)$. Put

$$M^!(S) := R\text{Hom}_{R^\text{op}}(M, S) \in D(A^\text{op}_Q).$$

We claim that $M^!(S)$ defines an object in $\text{coDef}_Q(E)$, i.e. $R\text{Hom}_{Q^\text{op}}(k, M^!(S))$ is naturally isomorphic to $E$ (by the isomorphisms $\theta$ and $\sigma$).

Indeed, choose quasi-isomorphisms $P \to k$ and $S \to I$ for $P \in \mathcal{P}(A^\text{op}_Q)$ and $I \in \mathcal{I}(A^\text{op}_R)$. Then

$$R\text{Hom}_{Q^\text{op}}(k, M^!(S)) = \text{Hom}_{Q^\text{op}}(P, R\text{Hom}_{R^\text{op}}(M, I)).$$

By [ELOI], Lemma 3.17 the last term is equal to $\text{Hom}_{R^\text{op}}(P \otimes Q M, I)$. Now the isomorphism $\theta$ defines an isomorphism between $P \otimes Q M = k \otimes Q M$ and $k$, and we compose it with the isomorphism $\sigma : E \to R\text{Hom}_{R^\text{op}}(k, I) = i^! S$.

So $M^!$ is a functor from $\text{coDef}_R(E)$ to $\text{coDef}_Q(E)$.

Given another artinian DG algebra $Q'$ and $M' \in 1\text{-Hom}(Q, Q')$ there is a natural isomorphism of functors

$$(M' \otimes Q M)^!(-) \simeq M'^! \cdot M^!(-) .$$

(This follows again from [ELOI], Lemma 3.17).

Also a 2-morphism $f \in 2\text{-Hom}(M, M_1)$ between objects $M, M_1 \in 1\text{-Hom}(R, Q)$ induces an isomorphism of the corresponding functors $M^! \sim M_1^!$.

Thus we obtain a pseudo-functor $\text{coDEF}(E) : 2\text{-dgart} \to \text{Gpd}$, such that $\text{coDEF}(E) \cdot F = \text{coDef}(E)$. □

We denote by $\text{coDEF}_+(E), \text{coDEF}_-(E), \text{coDEF}_0(E), \text{coDEF}_d(E)$ the restriction of the pseudo-functor $\text{coDEF}(E)$ to subcategories $2\text{-dgart}_+, 2\text{-dgart}_-, 2\text{-art}$ and $2\text{-cart}$ respectively.

**Proposition 9.4.** ([ELOII]) A quasi-isomorphism $\delta : E_1 \to E_2$ of DG $A^\text{op}$-modules induces an equivalence of pseudo-functors

$$\delta^* : \text{coDEF}(E_2) \to \text{coDEF}(E_1)$$

defined by $\delta^*(S, \sigma) = (S, \sigma \cdot \delta)$.

**Proof.** This is clear. □
Proposition 9.5. ([ELOII]) Let $F : \mathcal{A} \to \mathcal{A}'$ be a DG functor which induces a quasi-equivalence $F_{\text{pre-tr}}^{\text{pre-tr}} : \mathcal{A}_{\text{pre-tr}} \to \mathcal{A}'_{\text{pre-tr}}$ (this happens for example if $F$ is a quasi-equivalence). Then for any $E \in D(\mathcal{A}^{\text{op}})$ the pseudo-functors $\text{coDef}_-(E)$ and $\text{coDef}_-(RF^{\text{w}}(E))$ are equivalent (hence also $\text{coDef}(F_*(E'))$ and $\text{coDef}_-(E')$ are equivalent for any $E' \in D(\mathcal{A}^{\text{op}})$).

Corollary 9.6. ([ELOII]) Assume that DG algebras $\mathcal{B}$ and $\mathcal{C}$ are quasi-isomorphic. Then the pseudo-functors $\text{coDef}_-(\mathcal{B})$ and $\text{coDef}_-(\mathcal{C})$ are equivalent.

Proof. We may assume that there exists a homomorphism of DG algebras $\mathcal{B} \to \mathcal{C}$ which is a quasi-isomorphism. Then put $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}' = \mathcal{C}$ in the last proposition.

The following Lemma is stronger than [ELOI], Corollary 11.15 for the pseudo-functors $\text{coDef}_-$ and $\text{coDef}^h_-$.

It is proved by $A_{\infty}$-methods.

Lemma 9.7. Let $\mathcal{B}$ be a DG algebra. Suppose that the following conditions hold:

a) $H^{-1}(\mathcal{B}) = 0$;

b) the graded algebra $H(\mathcal{B})$ is bounded below.

Then the pseudo-functors $\text{coDef}_-(\mathcal{B})$ and $\text{coDef}^h_-(\mathcal{B})$ are equivalent.

Proof. Fix some negative artinian DG algebra $\mathcal{R} \in \text{dgar}_-$. Take some $(T, \text{id}) \in \text{coDef}^h_-(\mathcal{B})$. Due to [ELOI], Corollary 11.4 b) it suffices to prove that $i^jT = R_{i^j}T$. Let $A$ be a strictly unital minimal model of $\mathcal{B}$, and let $f : A \to \mathcal{B}$ be a strictly unital $A_{\infty}$ quasi-isomorphism. By our assumption on $H(\mathcal{B})$, $A$ is bounded below.

By Theorem 7.2 there exists an object $\alpha \in MC(\mathcal{A})$ such that $S \cong \mathcal{B} \otimes f_{\mathcal{R}}(\alpha) \mathcal{R}^*$. The DG $\mathcal{R}^{\text{op}}$-modules $\mathcal{B} \otimes f_{\mathcal{R}}(\alpha) \mathcal{R}^*$ and $f_*(\mathcal{B} \otimes f_{\mathcal{R}}(\alpha) \mathcal{R}^*)$ are naturally identified. Further, by Proposition 8.6 we have natural homotopy equivalence (in $A_{\infty}^{\text{op}}$-mod$_{\infty}$)

$$\gamma : A \otimes_{\alpha} \mathcal{R}^* \to f_*(\mathcal{B} \otimes f_{\mathcal{R}}(\alpha) \mathcal{R}^*)$$

Thus, it remains to prove that

$$i^j(A \otimes_{\alpha} \mathcal{R}^*) = R_{i^j}(A \otimes_{\alpha} \mathcal{R}^*).$$

We claim that $A \otimes_{\alpha} \mathcal{R}^*$ is h-injective. Indeed, since $A$ is bounded below and $\mathcal{R} \in \text{dgar}_-$, this DG $\mathcal{R}^{\text{op}}$-module has a decreasing filtration by DG $\mathcal{R}^{\text{op}}$-submodules $A^{\geq i} \otimes \mathcal{R}^*$ with subquotients being cofree DG $\mathcal{R}^{\text{op}}$-modules $A^i \otimes \mathcal{R}^*$. Thus $A \otimes_{\alpha} \mathcal{R}^*$ satisfies property (I) as DG $\mathcal{R}^{\text{op}}$-module ([ELOI], Definition 3.3) and hence is h-injective. Lemma is proved.

The next result implies stronger statement for pseudo-functor $\text{coDef}_-$ then [ELOI], Proposition 11.16.

Proposition 9.8. Let $E \in \mathcal{A}^{\text{op}}$-mod. Assume that

a) $\text{Ext}^{-1}(E, E) = 0$;

b) the graded algebra $\text{Ext}(E, E)$ is bounded below;

b) there exists a bounded below h-projective or h-injective DG $\mathcal{A}^{\text{op}}$-module $F$ which is quasi-isomorphic to $E$.

Put $\mathcal{B} = \text{End}(F)$. Then the pseudo-functors $\text{coDef}_-(\mathcal{B})$ and $\text{coDef}_-(E)$ are equivalent.

Proof. Consider the DG functor

$$L := \Sigma^F \cdot \psi^* : \mathcal{B}^{\text{op}}$-mod \to \mathcal{A}^{\text{op}}$-mod, \quad L(N) = N \otimes_{\mathcal{C}} F$$
as in [ELOI], Remark 11.17. It induces the equivalence of pseudo-functors
\[ \text{coDef}_h^\text{h}(\mathcal{L}) : \text{coDef}_h^\text{h}(\mathcal{B}) \sim \text{coDef}_h^\text{h}(\mathcal{F}), \]
i.e. for every artinian DG algebra \( R \in \text{dgar}_- \) the corresponding DG functor
\[ \mathcal{L}_R : (\mathcal{B} \otimes \mathcal{R})^{\text{op}-\text{mod}} \rightarrow A_R^{\text{op}-\text{mod}} \]
induces the equivalence of groupoids \( \text{coDef}_h^\text{h}(\mathcal{B}) \sim \text{coDef}_h^\text{h}(\mathcal{F}) \) ([ELOI], Propositions 9.2, 9.4). By [ELOI], Theorem 11.6 b) there is a natural equivalence of pseudo-functors
\[ \text{coDef}_h^\text{h}(\mathcal{F}) \simeq \text{coDef}_-^\text{h}(\mathcal{E}). \]
By Lemma 9.7 there is an equivalence of pseudo-functors
\[ \text{coDef}_-^\text{h}(\mathcal{B}) \simeq \text{coDef}_-^\text{h}(\mathcal{F}). \]
Hence the functor \( \mathcal{L} \mathcal{L} \) induces the equivalence
\[ \mathcal{L} \mathcal{L} : \text{coDef}_-^\text{h}(\mathcal{B}) \sim \text{coDef}_-^\text{h}(\mathcal{F}). \]

Fix \( \mathcal{R}, \mathcal{Q} \in \text{2-dgar}_- \) and \( M \in \text{1-Hom}(\mathcal{R}, \mathcal{Q}) \). We need to show that there exists a natural isomorphism
\[ \mathcal{L} \mathcal{L}_Q \cdot M^! \simeq M^! \cdot \mathcal{L} \mathcal{L}_R \]
between functors from \( \text{coDef}_R^h(\mathcal{B}) \) to \( \text{coDef}_Q^h(\mathcal{E}) \).

Since the cohomology of \( M \) is finite dimensional, and the DG algebra \( \mathcal{R} \otimes \mathcal{Q} \) has no components in positive degrees, by [ELOI], Corollary 3.21 we may assume that \( M \) is finite dimensional.

**Lemma 9.9.** Let \( (S, \text{id}) \) be an object in \( \text{coDef}_R^h(\mathcal{B}) \) or in \( \text{coDef}_R^h(\mathcal{F}) \). Then \( S \) is acyclic for the functor \( \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \), i.e. \( M^!(S) = \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \).

**Proof.** In the proof of Lemma 9.7 (resp. in [ELOI], Lemma 11.8) we showed that \( S \) is h-injective when considered as a DG \( \mathcal{R}^{\text{op}} \)-module.

Choose \( (S, \text{id}) \in \text{coDef}_R^h(\mathcal{B}) \). By the above lemma \( M^!(S) = \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \).

We claim that the DG \( \mathcal{B}^{\text{op}} \)-module \( \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \) is h-projective. Indeed, first notice that the graded \( \mathcal{R}^{\text{op}} \)-module \( S \) is injective being isomorphic to a direct sum of copies of shifted graded \( \mathcal{R}^{\text{op}} \)-module \( \mathcal{R}^* \) (the abelian category of graded \( \mathcal{R}^{\text{op}} \)-modules is locally notherian, hence a direct sum of injectives is injective). Second, the DG \( \mathcal{R}^{\text{op}} \)-module \( M \) has a (finite) filtration with quotients isomorphic to \( k \). Thus the DG \( \mathcal{B}^{\text{op}} \)-module \( \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \) has a filtration with quotients isomorphic to \( \text{Hom}_{\mathcal{R}^{\text{op}}}(k, S) = i^! S \simeq \mathcal{B} \). So it satisfies property (P).

Hence \( \mathcal{L} \mathcal{L} \cdot M^!(S) = \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \otimes_{\mathcal{B}} F \). For the same reasons \( M^! \cdot \mathcal{L} \mathcal{L}_R(S) = \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S \otimes_{\mathcal{B}} F) \). The isomorphism
\[ \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S) \otimes_{\mathcal{B}} F = \text{Hom}_{\mathcal{R}^{\text{op}}}(M, S \otimes_{\mathcal{B}} F) \]
follows from the fact that \( S \) as a graded module is a tensor product of graded \( \mathcal{C}^{\text{op}} \) and \( \mathcal{R}^{\text{op}} \) modules and also because \( \dim_k M < \infty \). \( \square \)
10. Deformation pseudo-functor \( \text{coDEF} \) for an augmented \( A_\infty \)-algebra

Let \( A \) be an augmented \( A_\infty \)-algebra. We are going to define the pseudo-functor \( \text{coDEF}(A) : 2\text{-dgart} \rightarrow \text{Gpd} \).

Let \( \mathcal{R} \) be an artinian DG algebra. An object of the groupoid \( \text{coDEF}_\mathcal{R}(A) \) is a pair \((S, \sigma)\), where \( S \in D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}) \), and \( \sigma \) is an isomorphism (in \( D_\infty(\bar{A}^{op}) \))

\[
\sigma : A \rightarrow R^i(S).
\]

A morphism \( f : (S, \sigma) \rightarrow (T, \tau) \) in \( \text{coDEF}_\mathcal{R}(A) \) is an isomorphism (in \( D(\bar{A}^{op}_{\mathcal{R}^{op}}) \)) \( f : S \rightarrow T \) such that

\[
R^i(f) \circ \sigma = \tau.
\]

This defines the pseudo-functor \( \text{coDEF}(A) \) on objects. Further, let \((M, \theta) \in 1\text{-Hom}(\mathcal{R}, \mathcal{Q}) \). Define the corresponding functor

\[
M^i : \text{coDEF}_\mathcal{R}(A) \rightarrow \text{coDEF}_{\mathcal{Q}}(A)
\]

as follows. For an object \((S, \sigma) \in \text{coDEF}_\mathcal{R}(A)\) put

\[
M^i(S) = R\text{Hom}_{\mathcal{R}^{op}}(M, S) \in D_\infty(\bar{A}^{op}_{\mathcal{Q}^{op}}).
\]

Then we have natural isomorphisms in \( D_\infty(\bar{A}) \):

\[
R\text{Hom}_{\mathcal{Q}^{op}}(k, M^i(S)) \cong R\text{Hom}_{\mathcal{R}^{op}}(k \otimes_{\mathcal{R}^{op}} M, S) \cong R\text{Hom}_{\mathcal{R}^{op}}(k, S) = R^i(S)
\]

(the second isomorphism is induced by \( \theta \)). Thus, \( M^i \) is a functor from \( \text{coDEF}_\mathcal{R}(A) \) to \( \text{coDEF}_{\mathcal{Q}}(A) \).

If \( \mathcal{Q}' \) is another artinian DG algebra and \((M', \theta') \in 1\text{-Hom}(\mathcal{Q}, \mathcal{Q}')\) then there is a natural isomorphism of functors

\[
(M' \otimes_{\mathcal{Q}} M)^i \cong M'^i \cdot M^i.
\]

Further, if \( f \in 2\text{-Hom}((M, \theta), (M, \theta_1)) \) is a 2-morphism between objects \((M, \theta), (M, \theta_1) \in 1\text{-Hom}(\mathcal{R}, \mathcal{Q})\) then it induces an isomorphism between the corresponding functors \( M^i \simeq M'^i \).

Thus we obtain a pseudo-functor \( \text{coDEF}(A) : 2\text{-dgart} \rightarrow \text{Gpd} \). We denote by \( \text{coDEF}_- (A) \) its restriction to the sub-2-category \( 2\text{-dgart}_- \).

**Proposition 10.1.** Let \( A \) be an augmented \( A_\infty \)-algebra and \( U(A) \) its bar-cobar construction. Then there is a natural equivalence of pseudo-functors \( \text{coDEF}(U(A)) \cong \text{coDEF}(A) \).

**Proof.** Let \( f_A : A \rightarrow U(A) \) be the universal strictly unital \( A_\infty \)-morphism. Let \( \mathcal{R} \) be an artinian DG algebra. Recall that by Proposition 3.14 we have an equivalence

\[
f_A^* : D((U(A) \otimes \mathcal{R})^{op}) \rightarrow D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}).
\]

Moreover, the following diagram of functors commutes up to an isomorphism:

\[
\begin{array}{ccc}
D((U(A) \otimes \mathcal{R})^{op}) & \xrightarrow{f_A^*} & D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}) \\
\downarrow R^i & & \downarrow R^i \\
D(U(A)^{op}) & \xrightarrow{f_A^*} & D_\infty(A^{op}).
\end{array}
\]

Hence, the functor \( f_A^* \) induces an equivalence of groupoids \( \text{coDEF}_\mathcal{R}(U(A)) \rightarrow \text{coDEF}_\mathcal{R}(A) \) and we obtain the required equivalence of pseudo-functors. \( \square \)
Corollary 10.2. Let $A$ be an augmented $A_{\infty}$-algebra and let $\mathcal{B}$ be a DG algebra quasi-isomorphic to $A$. Then the pseudo-functor $\text{coDEF}(A)$ and $\text{coDEF}(\mathcal{B})$ are equivalent.

Proof. Indeed, by Proposition 10.1 the pseudo-functors $\text{coDEF}(A)$ and $\text{DEF}(U(A))$ are equivalent, and by Corollary 9.6 the pseudo-functors $\text{coDEF}(U(A))$ and $\text{coDEF}(\mathcal{B})$ are equivalent. □

Corollary 10.3. Let $A$ be an admissible $A_{\infty}$-algebra, and $\mathcal{R}$ be an artinian negative DG algebra. Then for any $(S, \sigma) \in \text{coDEF}_{\mathcal{R}}(A)$ there exists a morphism of DG algebras $\hat{S} \to \mathcal{R}$ such that the pair $(T, \text{id}), \text{ where } T = \text{Hom}_{\mathcal{R}}(\mathcal{R}, B\hat{A} \otimes_{T, A} A)$, defines an object of $\text{coDEF}_{\mathcal{R}}(A)$ which is isomorphic to $(S, \sigma)$.

Proof. This follows easily from Proposition 10.1, the proof of Lemma 9.7 in the case $\mathcal{B} = U(A)$, and Lemma 8.4. □

11. The bicategory $2^\prime$-adgalg and deformation pseudo-functor DEF

It turns out that the deformation pseudo-functor Def lifts naturally to a different version of a bicategory of augmented DG algebras. We denote this bicategory $2^\prime$-adgalg. It differs from $2$-adgalg in two respects: the 1-morphisms are objects in $D(\mathcal{B} \otimes \mathcal{C}^{\text{op}})$ (instead of $D(\mathcal{B}^{\text{op}} \otimes \mathcal{C})$) and 2-morphisms go in the opposite direction.

We will relate the bicategories $2$-adgalg and $2^\prime$-adgalg (and the pseudo-functors $\text{coDEF}$ and $\text{DEF}$) in section 13 below.

Definition 11.1. ([ELOII]) We define the bicategory $2^\prime$-adgalg of augmented DG algebras as follows. The objects are augmented DG algebras. For DG algebras $\mathcal{B}, \mathcal{C}$ the collection of 1-morphisms $1\text{-Hom}(\mathcal{B}, \mathcal{C})$ consists of pairs $(M, \theta)$, where

- $M \in D(\mathcal{B} \otimes \mathcal{C}^{\text{op}})$ and there exists an isomorphism (in $D(\mathcal{C}^{\text{op}})$) $\nu_\ast M$ (where $\nu_\ast : D(\mathcal{B} \otimes \mathcal{C}^{\text{op}}) \to D(\mathcal{C}^{\text{op}})$ is the functor of restriction of scalars corresponding to the natural homomorphism $\nu : \mathcal{C}^{\text{op}} \to \mathcal{B} \otimes \mathcal{C}^{\text{op}}$);
- and $\theta : M \otimes_{\mathcal{C}k} k$ is an isomorphism in $D(\mathcal{B})$.

The composition of 1-morphisms

$1\text{-Hom}(\mathcal{B}, \mathcal{C}) \times 1\text{-Hom}(\mathcal{C}, \mathcal{D}) \to 1\text{-Hom}(\mathcal{B}, \mathcal{D})$

is defined by the tensor product $\cdot \otimes_{\mathcal{C}} \cdot$. Given 1-morphisms $(M_1, \theta_1), (M_2, \theta_2) \in 1\text{-Hom}(\mathcal{B}, \mathcal{C})$ a 2-morphism $f : (M_1, \theta_1) \to (M_2, \theta_2)$ is an isomorphism (in $D(\mathcal{B} \otimes \mathcal{C}^{\text{op}})$) $f : M_1 \to M_2$ such that $\theta_1 = \theta_2 \cdot (f)_{\otimes_{\mathcal{C}k}}$. So in particular the category $1\text{-Hom}(\mathcal{B}, \mathcal{C})$ is a groupoid. Denote by $2^\prime$-dgart the full subbicategory of $2^\prime$-adgalg consisting of artinian DG algebras. Similarly we define the full subbicategories $2^\prime$-dgart$_{+}$, $2^\prime$-dgart$_{-}$, $2^\prime$-art, $2^\prime$-cart ([ELOI], Definition 2.3).

Remark 11.2. The exact analogue of Remark 9.2 holds for the bicategory $2^\prime$-adgalg.

For any augmented DG algebra $\mathcal{B}$ we obtain a pseudo-functor $h_B^\prime$ between the bicategories $2^\prime$-adgalg and $\text{Gpd}$ defined by $h_B^\prime(\mathcal{C}) = 1\text{-Hom}(\mathcal{B}, \mathcal{C})$.

Note that a usual homomorphism of DG algebras $\gamma : \mathcal{B} \to \mathcal{C}$ defines the structure of a $\mathcal{B}$-module on $\mathcal{C}$ with the canonical isomorphism of DG $\mathcal{B}$-modules $\mathcal{L} \cdot \mathcal{C} \otimes_{\mathcal{C}k} \cdot$. Hence, we have a natural pseudo-functor $\mathcal{F}^\prime : \text{adgalg} \to 2^\prime$-adgalg.

The next Proposition from [ELOII] asserts that the pseudo-functor Def has a natural lift on the bicategory $2^\prime$-adgalg. We present its proof for completeness.
Proposition 11.3. There exist a pseudo-functor $\text{DEF}(E)$ from $2'$-dgart to $\text{Gpd}$ and which is an extension to $2'$-dgart of the pseudo-functor $\text{Def}(E)$, i.e. there is an equivalence of pseudo-functors $\text{Def}(E) \simeq \text{DEF}(E) \cdot \mathcal{F}'$.

Proof. Let $\mathcal{R}, \mathcal{Q}$ be artinian DG algebras. Given $(M, \theta) \in 1\cdot \text{Hom}(\mathcal{R}, \mathcal{Q})$ we define the corresponding functor

$$M^* : \text{Def}_\mathcal{R}(E) \rightarrow \text{Def}_\mathcal{Q}(E)$$

as follows

$$M^*(S) := S \otimes_{\mathcal{R}} M$$

for $(S, \sigma) \in \text{Def}_\mathcal{R}(E)$. Then we have the canonical isomorphism

$$M^*(S) \overset{L}{\otimes}_Q k = S \overset{L}{\otimes}_R (M \overset{L}{\otimes}_Q k) \overset{\theta}{\rightarrow} S \overset{L}{\otimes}_R k \overset{\sim}{\rightarrow} E.$$ 

So that $M^*(S) \in \text{Def}_\mathcal{Q}(E)$ indeed.

Given another artinian DG algebra $\mathcal{Q}'$ and $M' \in 1\cdot \text{Hom}(\mathcal{Q}, \mathcal{Q}')$ there is a natural isomorphism of functors

$$M'^* \cdot M^* = (M \otimes Q M')^*.$$ 

Also a 2-morphism $f \in 2\cdot \text{Hom}(M, M_1)$ between $M, M_1 \in 1\cdot \text{Hom}(\mathcal{R}, \mathcal{Q})$ induces an isomorphism of corresponding functors $M^* \overset{\sim}{\rightarrow} M_1^*$.

Thus we obtain a pseudo-functor $\text{DEF}(E) : 2'$-dgart $\rightarrow \text{Gpd}$, such that $\text{DEF}(E) \cdot \mathcal{F}' = \text{Def}(E)$. \hfill \Box

We denote by $\text{DEF}_+(E)$, $\text{DEF}_-(E)$, $\text{DEF}_0(E)$, $\text{DEF}_{cl}(E)$ the restriction of the pseudo-functor $\text{DEF}(E)$ to subcategories $2'$-dgart, $2'$-dgart, $2'$-art and $2'$-cart respectively.

Proposition 11.4. ([ELOII]) A quasi-isomorphism $\delta : E_1 \rightarrow E_2$ of DG $A^{op}$-modules induces an equivalence of pseudo-functors

$$\delta_* : \text{DEF}(E_1) \rightarrow \text{DEF}(E_2)$$

defined by $\delta_*(S, \sigma) = (S, \delta \cdot \sigma)$.

Proof. This is clear. \hfill \Box

Proposition 11.5. ([ELOII]) Let $F : A \rightarrow A'$ be a DG functor which induces a quasi-equivalence $F_{pre-tr} : A_{pre-tr} \rightarrow A'_{pre-tr}$ (this happens for example if $F$ is a quasi-equivalence). Then for any $E \in D(A^{op})$ the pseudo-functors $\text{DEF}_-(E)$ and $\text{DEF}_-(LF^*(E))$ are equivalent (hence also $\text{DEF}_-(F_*(E'))$ and $\text{DEF}_-(E')$ are equivalent for any $E' \in D(A^{op})$).

Corollary 11.6. ([ELOII]) Assume that DG algebras $\mathcal{B}$ and $\mathcal{C}$ are quasi-isomorphic. Then the pseudo-functors $\text{DEF}_-(\mathcal{B})$ and $\text{DEF}_-(\mathcal{C})$ are equivalent.

Proof. We may assume that there exists a morphism of DG algebras $\mathcal{B} \rightarrow \mathcal{C}$ which is a quasi-isomorphism. Then put $A = \mathcal{B}$ and $A' = \mathcal{C}$ in the last proposition. \hfill \Box

The following Theorem is stronger then [ELOI], Corollary 11.15 for the pseudo-functors Def and $\text{Def}_h$.

Theorem 11.7. Let $E \in A^{op}$-mod be a DG module. Suppose that the following conditions hold:

a) $\text{Ext}^{-1}(E, E) = 0$;

b) the graded algebra $\text{Ext}(E, E)$ is bounded above.

Let $F : E$ be a quasi-isomorphism with h-projective $F$. Then the pseudo-functors $\text{Def}_-(E)$ and $\text{Def}_h^h(F)$ are equivalent.
Proof. Replace the pseudo-functor Def(E) by the equivalent pseudo-functor Def(F). Fix some negative artinian DG algebra \( R \in \text{dgart}_- \).

Due to [ELOI], Corollary 11.4 a) it suffices to prove that for each \((S, id) \in \text{Def}_R^h(F)\) one has \(i^*(S) = L_i^*(S)\). Consider the DG algebra \( B = \text{End}(F) \). First we will prove the following special case:

**Lemma 11.8.** The pseudo-functors \( \text{Def}_-(B) \) and \( \text{Def}_R^h(B) \) are equivalent.

**Proof.** Take some \((S, \sigma) \in \text{Def}_R^h(B)\). Let \( A \) be a strictly unital minimal model of \( B \), and let \( f : A \to B \) be a strictly unital \( A_{\infty} \) quasi-isomorphism. By our assumption on \( \text{Ext}(E, E) \cong H(B), A \) is bounded above.

By Theorem 7.2 there exists an object \( \alpha \in MC_R(A) \) such that \( S \cong B \otimes f^*_R(\alpha) R \). The DG \( R^{op} \)-modules \( B \otimes f^*_R(\alpha) R \) and \( f_* (B \otimes f^*_R(\alpha) R) \) are naturally identified. Further, by Proposition 8.6 we have natural homotopy equivalence (in \( A_R^{op}_{\text{mod}} \))

\[
\gamma : R \otimes_\alpha A \to f_* (B \otimes f^*_R(\alpha) R).
\]

Thus, it remains to prove that

\[
i^*(A \otimes_\alpha R) = L_i^*(A \otimes_\alpha R).
\]

We claim that \( R \otimes_\alpha A \) is h-projective. Indeed, since \( A \) is bounded above and \( R \in \text{dgart}_- \), this DG \( R^{op} \)-module has an increasing filtration by DG \( R^{op} \)-submodules \( A^{\geq i} \otimes R \) with subquotients being free DG \( R^{op} \)-modules \( A^i \otimes R \). Thus \( A \otimes_\alpha R \) satisfies property (P) as DG \( R^{op} \)-module and hence is h-projective. Lemma is proved.

Now take some \((S, id) \in \text{Def}_R^h(F)\). We claim that \( S \) is h-projective. Recall the DG functor

\[
\Sigma_R : (B \otimes R)^{op}_{\text{mod}} \to A_R^{op}_{\text{mod}}, \quad \Sigma(M) = M \otimes_B F.
\]

From [ELOI], Proposition 9.2 e) we know that \( S \cong \Sigma_R(S') \) for some \((S', id) \in \text{Def}_R^h(B)\). By the above Lemma and [ELOI], Proposition 11.2, DG \( (B \otimes R)^{op}_{\text{mod}} \)-module \( S' \) is h-projective. Since the DG functor \( \Sigma_R \) preserves h-projectives, it follows that \( S \) is also h-projective. Theorem is proved.

The next proposition is the analogue of Proposition 9.8 for the pseudo-functor \( \text{DEF}_- \). Note that here we do not need boundedness assumptions on the h-projective DG module.

**Proposition 11.9.** Let \( E \in A^{op}_{\text{mod}} \) be a DG module. Suppose that the following conditions hold:

a) \( \text{Ext}^{-1}(E, E) = 0 \);

b) the graded algebra \( \text{Ext}(E, E) \) is bounded above.

Put \( B = R^{\text{Hom}}(E, E) \). Then pseudo-functors \( \text{DEF}_-(B) \) and \( \text{DEF}_-(E) \) are equivalent.

**Proof.** Take some h-projective \( F \) quasi-isomorphic to \( E \) and replace \( \text{DEF}_-(E) \) by the equivalent pseudo-functor \( \text{DEF}_-(F) \). We may assume that \( B = \text{End}(F) \).

By [ELOI], Proposition 9.2 e) the DG functor \( \Sigma = \Sigma^F : B^{op}_{\text{mod}} \to A^{op}_{\text{mod}}, \Sigma(N) = N \otimes_B F \) induces an equivalence of pseudo-functors

\[
\text{Def}_R^h(\Sigma) : \text{Def}_R^h(B) \to \text{Def}_R^h(F).
\]

By Lemma 11.7 we have that the pseudo-functors \( \text{Def}_-(F) \) and \( \text{Def}_R^h(F) \) (resp. \( \text{Def}_-(B) \) and \( \text{Def}_R^h(B) \)) are equivalent. We conclude that \( \Sigma \) also induces an equivalence of pseudo-functors

\[
\text{Def}_-(\Sigma) : \text{Def}_-(B) \to \text{Def}_-(F).
\]
Let us prove that it extends to an equivalence

$$\text{DEF}_-(\Sigma) : \text{DEF}_-(\mathcal{B}) \to \text{DEF}_-(\mathcal{F}).$$

Let $\mathcal{R}, \mathcal{Q} \in \text{dgart}_-$, $M \in 1\text{-Hom}(\mathcal{R}, \mathcal{Q})$. We need to show that the functorial diagram

$$\begin{array}{ccc}
\text{DEF}_\mathcal{R}(\mathcal{B}) & \xrightarrow{\text{DEF}_\mathcal{R}(\Sigma)} & \text{DEF}_\mathcal{R}(\mathcal{F}) \\
M^* \downarrow & & \downarrow M^* \\
\text{DEF}_\mathcal{Q}(\mathcal{B}) & \xrightarrow{\text{DEF}_\mathcal{Q}(\Sigma)} & \text{DEF}_\mathcal{Q}(\mathcal{F}).
\end{array}$$

commutes. This follows from the natural isomorphism

$$N \otimes_{\mathcal{B}} F \otimes_{\mathcal{R}} M \cong N \otimes_{\mathcal{R}} M \otimes_{\mathcal{B}} F.$$

\[\square\]

12. **Deformation pseudo-functor DEF for an augmented $A_\infty$-algebra**

Let $A$ be an augmented $A_\infty$-algebra. We are going to define the pseudo-functor $\text{DEF}(A) : 2'\text{-dgart} \to \text{Gpd}$.

Let $\mathcal{R}$ be an artinian DG algebra. An object of the groupoid $\text{DEF}_\mathcal{R}(A)$ is a pair $(S, \sigma)$, where $S \in D_\infty(A^{\text{op}}_{\mathcal{R}^{\text{op}}})$, and $\sigma$ is an isomorphism (in $D_\infty(A^{\text{op}})$)

$$\sigma : \text{Li}^*(S) \to A.$$

A morphism $f : (S, \sigma) \to (T, \tau)$ in $\text{DEF}_\mathcal{R}(A)$ is an isomorphism (in $D(A^{\text{op}}_{\mathcal{R}^{\text{op}}})$) $f : S \to T$ such that

$$\tau \circ \text{Li}^*(f) = \sigma.$$

This defines the pseudo-functor $\text{DEF}(A)$ on objects. Further, let $(M, \theta) \in 1\text{-Hom}(\mathcal{R}, \mathcal{Q})$. Define the corresponding functor

$$M^* : \text{DEF}_\mathcal{R}(A) \to \text{DEF}_\mathcal{Q}(A)$$

as follows. For an object $(S, \sigma) \in \text{DEF}_\mathcal{R}(A)$ put

$$M^*(S) = S \otimes^L_{\mathcal{R}} M \in D_\infty(A^{\text{op}}_{\mathcal{Q}^{\text{op}}}).$$

Then we have natural isomorphisms in $D_\infty(A)$:

$$M^*(S) \otimes^L_{\mathcal{Q}} k \cong S \otimes^L_{\mathcal{R}} (M \otimes^L_{\mathcal{Q}} k) \cong S \otimes^L_{\mathcal{R}} k \cong A$$

(the second isomorphism is induced by $\theta$). Thus, $M^*$ is a functor form $\text{DEF}_\mathcal{R}(A)$ to $\text{DEF}_\mathcal{Q}(A)$.

If $\mathcal{Q}'$ is another artinian DG algebra and $(M', \theta') \in 1\text{-Hom}(\mathcal{Q}, \mathcal{Q}')$ then there is a natural isomorphism of functors

$$(M' \otimes^L_{\mathcal{Q}} M)^* \cong M'^* \cdot M^*.$$

Further, if $f \in 2\text{-Hom}((M, \theta), (M, \theta_1))$ is a 2-morphism between objects $(M, \theta), (M, \theta_1) \in 1\text{-Hom}(\mathcal{R}, \mathcal{Q})$ then it induces an isomorphism between corresponding functors $M^* \to M_1^*.$

Thus we obtain a pseudo-functor $\text{DEF}(A) : 2\text{-dgart} \to \text{Gpd}$. We denote by $\text{DEF}_-(A)$ its restriction to the sub-2-category $2\text{-dgart}_-$.

**Proposition 12.1.** Let $A$ be an augmented $A_\infty$-algebra and $U(A)$ its universal DG algebra. Then there is a natural equivalence of pseudo-functors $\text{DEF}(U(A)) \cong \text{DEF}(A)$. 
Proof. The proof is the same as of Proposition 10.1 and we omit it. □

**Corollary 12.2.** Let \( A \) be an augmented \( A_\infty \)-algebra and let \( B \) be a DG algebra quasi-isomorphic to \( A \). Then the pseudo-functor \( \text{DEF}(A) \) and \( \text{DEF}(B) \) are equivalent.

Proof. Indeed, by Proposition 10.1 the pseudo-functors \( \text{DEF}(A) \) and \( \text{DEF}(U(A)) \) are equivalent, and by Corollary 11.6 the pseudo-functors \( \text{DEF}(U(A)) \) and \( \text{DEF}(B) \) are equivalent. □

**Corollary 12.3.** Let \( A \) be an admissible \( A_\infty \)-algebra, and \( R \) be an artinian negative DG algebra. Then for any \( (S, \sigma) \in \text{DEF}_R(A) \) there exists an \( \alpha \in \mathcal{MC}_R(A) \) such that the pair \( (T, \text{id}) \), where \( T = A \otimes_\alpha R \), defines an object of \( \text{DEF}_R(A) \) which is isomorphic to \( (S, \sigma) \).

Proof. This follows easily from Proposition 12.1 and the proof of Lemma 11.7 in the case \( B = U(A) \). □

### 13. Comparison of pseudo-functors \( \text{coDEF}_- \) and \( \text{DEF}_- \)

We have proved in [ELOI], Corollary 11.9 that under some conditions on \( E \) the pseudo-functors \( \text{coDef}_-(E) \) and \( \text{Def}_-(E) \) from \( \text{dgart} \) to \( \text{Gpd} \) are equivalent. Note that we cannot speak about an equivalence of pseudo-functors \( \text{coDEF}_-(E) \) and \( \text{DEF}_-(E) \) since they are defined on different bicategories. So our first goal is to establish an equivalence of the bicategories \( 2\text{-adgalg} \) and \( 2'\text{-adgalg} \) in the following sense: we will construct pseudo-functors

\[
\mathcal{D} : 2\text{-adgalg} \to 2'\text{-adgalg}, \\
\mathcal{D}' : 2'\text{-adgalg} \to 2\text{-adgalg},
\]

which have the following properties

1) \( \mathcal{D} \) (resp. \( \mathcal{D}' \)) is the identity on objects;

2) for each \( B, C \in \text{Ob}(2\text{-adgalg}) \) they define mutually inverse equivalences of groupoids

\[
\mathcal{D} : \text{Hom}_{2\text{-adgalg}}(B, C) \to \text{Hom}_{2'\text{-adgalg}}(B, C), \\
\mathcal{D}' : \text{Hom}_{2'\text{-adgalg}}(B, C) \to \text{Hom}_{2\text{-adgalg}}(B, C).
\]

Fix augmented DG algebras \( B, C \) and let \( M \) be a DG \( C \otimes B^{op} \)-module. Define the DG \( B \otimes C^{op} \)-module \( \mathcal{D}(M) \) as

\[
\mathcal{D}(M) := R \text{Hom}_C(M, C).
\]

Further, let \( N \) be a DG \( B^{op} \otimes C \)-module. Define the DG \( B^{op} \otimes C^{op} \)-module \( \mathcal{D}'(N) \) as

\[
\mathcal{D}'(N) = R \text{Hom}_{C^{op}}(N, C).
\]

**Proposition 13.1.** ([ELOII]) The operations \( \mathcal{D} \), \( \mathcal{D}' \) as above induces the pseudo-functors

\[
\mathcal{D} : 2\text{-adgalg} \to 2'\text{-adgalg}, \\
\mathcal{D}' : 2'\text{-adgalg} \to 2\text{-adgalg},
\]

so that the properties 1) and 2) hold.
Corollary 13.2. ([ELOII]) For any augmented DG algebra \( B \) the pseudo-functor \( D : 2\text{-adgalg} \to 2'\text{-adgalg} \) induces a morphism of pseudo-functors

\[
h_B \to h'_B \cdot D,
\]

which is an equivalence.

Similarly, the pseudo-functor \( D' : 2'\text{-adgalg} \to 2\text{-adgalg} \) induces an equivalence of pseudo-functors

\[
h'_B \to h_B \cdot D'.
\]

Proof. This is clear. \( \square \)

Lemma 13.3. ([ELOII]) Let \( B_1, B_2, B_3 \in \text{Ob}(2\text{-adgalg}) \), \( M_1 \in 1\text{-Hom}(B_1, B_2) \), \( M_2 \in 1\text{-Hom}(B_2, B_3) \). Assume that \( M_1 \) and \( M_2 \) are h-projective as DG \( B_2 \otimes B_1^{op} \)- and \( B_3 \otimes B_2^{op} \)-modules respectively. Then

a) The DG \( B_2^{op} \)-module \( \text{Hom}_{B_2}(M_1, B_2) \) is h-projective.

b) The DG \( B_3 \)-module \( M_2 \otimes B_3 M_1 \) is h-projective.

The next theorem is related to [ELOII], Theorem 13.4. It asserts the stronger statement in the case when \( E \) is a DG algebra considered as a DG module over itself.

Theorem 13.4. Let \( B \) be a DG algebra. Suppose that the following conditions hold:

a) \( H^{-1}(B) = 0 \);

b) the cohomology algebra \( H(B) \) is bounded above and bounded below. Then the pseudo-functors \( \text{coDEF}_-(B) \) and \( \text{DEF}_-(B) \cdot D \) from \( 2\text{-gart}_- \) to \( \text{Gpd} \) are equivalent.

Proof. Let \( R \) be a negative artinian DG algebra. Recall the DG functors

\[
\epsilon_R : (B \otimes R)^{op}\text{-mod} \to (B \otimes R)^{op}\text{-mod}, \quad \epsilon_R(M) = M \otimes_R R^*,
\]

\[
\eta_R : (B \otimes R)^{op}\text{-mod} \to (B \otimes R)^{op}\text{-mod}, \quad \eta_R(M) = \text{Hom}_R(R^*, M).
\]

By [ELOI], Proposition 4.7 they induce quasi-inverse equivalences

\[
\epsilon_R : \text{Def}^h_R(B) \to \text{coDef}^h_R(B),
\]

\[
\eta_R : \text{coDef}^h_R(B) \to \text{Def}^h_R(B).
\]

By Theorem 11.7 the pseudo-functors \( \text{Def}_-(B) \) and \( \text{Def}^h_R(B) \) are equivalent. By Lemma 9.7 the pseudo-functors \( \text{coDef}_-(B) \) and \( \text{coDef}^h_R(B) \). It follows that the derived functors \( L\epsilon_R, R\eta_R \) induce mutually inverse equivalences

\[
L\epsilon_R : \text{Def}_R(B) \to \text{coDef}_R(B),
\]

\[
R\eta_R : \text{coDef}_R(B) \to \text{Def}_R(B).
\]

Let now \( Q \in \text{gart}_- \) and \( M \in 1\text{-Hom}(R, Q) \). It suffices to prove that the functorial diagram

\[
\begin{array}{ccc}
\text{Def}_R(B) & \xrightarrow{L\epsilon_R} & \text{coDef}_R(B) \\
D(M)^* \downarrow & & M' \downarrow \\
\text{coDef}_Q(B) & \xrightarrow{L\epsilon_Q} & \text{coDef}_Q(B)
\end{array}
\]

naturally commutes.
Choose a bounded above h-projective or h-injective $P$ quasi-isomorphic to $E$. By [ELOI], Theorem 11.6 a) the groupoids $\text{Def}_R(E)$ and $\text{Def}^h_R(P)$ are equivalent. Hence given $(S, \text{id}) \in \text{Def}^h_R(P)$ it suffices to prove that there exists a natural isomorphism of objects in $D(A^p_Q)$

$$M^! \cdot \mathbb{L}c_R(S) \simeq \mathbb{L}c_Q \cdot D(M)^*(S),$$

i.e.

$$\mathbb{R} \text{Hom}_{\mathcal{Q} \otimes R^{op}}(M, S) \simeq S \otimes_{R} \mathbb{R} \text{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathbb{Q}^*.$$

We may and will assume that the DG $\mathcal{Q} \otimes R^{op}$-module $M$ is h-projective. In the proof of [ELOI], Lemma 11.7 we showed that the DG $\mathcal{A}^{op}_R$-module $S$ is h-projective as a DG $\mathcal{R}^{op}$-module. Therefore by Lemma 13.3 a) it suffices to prove that the morphism of DG $\mathcal{A}^{op}_Q$-modules

$$\eta : S \otimes_{\mathcal{R}} \text{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathbb{Q}^* \to \text{Hom}_{\mathcal{R}^{op}}(M, S \otimes_{\mathcal{R}} \mathcal{R}^*)$$

defined by

$$\eta(s \otimes f \otimes g)(m)(r) = sg(f(m)r)$$

is a quasi-isomorphism.

It suffices to prove that $\eta$ is a quasi-isomorphism of DG $\mathcal{Q}^{op}$-modules. Notice that just the $\mathcal{R}^{op}$-module structure on $S$ is important for us. Furthermore we may assume that $S$ satisfies property (P) as DG $\mathcal{R}^{op}$-module. Thus it suffices to prove that $\eta$ is a quasi-isomorphism if $S = R$. Then

$$\eta : \text{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathbb{Q}^* \to \text{Hom}_{\mathcal{R}^{op}}(M, \mathcal{R}^*).$$

We have the canonical isomorphisms

$$\text{Hom}_{\mathcal{R}^{op}}(M, \text{Hom}_{\mathcal{R}}(\mathcal{R}, k)) = \text{Hom}_{\mathcal{R}}(M \otimes_{\mathcal{R}} \mathcal{R}, k) = M^*.$$

Also, since the DG $\mathcal{Q}^{op}$-module $M$ is homotopy equivalent to $\mathcal{Q}$, we have the homotopy equivalences

$$\text{Hom}_{\mathcal{Q}}(M, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathbb{Q}^* \simeq \text{Hom}_{\mathcal{Q}}(\mathcal{Q}, \mathcal{Q}) \otimes_{\mathcal{Q}} \mathbb{Q}^* \simeq \mathbb{Q}^* \simeq M^*.$$

Part 4. Pro-representability theorems

14. PRO-REPRESENTABILITY OF THE PSEUDO-FUNCTOR $\text{coDEF}_-$

The next theorem claims that under some conditions on the DG algebra $\mathcal{C}$ that the functor $\text{coDEF}_-(\mathcal{C})$ is pro-representable.

**Theorem 14.1.** Let $\mathcal{C}$ be a DG algebra such that the cohomology algebra $H(\mathcal{C})$ is admissible finite-dimensional. Let $A$ be a strictly unital minimal model of $\mathcal{C}$. Then the pseudo-functor $\text{coDEF}_-(\mathcal{C})$ is pro-representable by the DG algebra $\hat{S} = (BA)^*$. That is, there exists an equivalence of pseudo-functors $\text{coDEF}_-(\mathcal{C}) \simeq h_{\hat{S}}$ from 2-dgart$_-$ to Gpd.

As a corollary, we obtain the following

**Theorem 14.2.** Let $E \in \mathcal{A}^{op}$-mod. Assume that the following conditions hold:

a) the graded algebra $\text{Ext}(E, E)$ is admissible finite-dimensional;

b) $E$ is quasi-isomorphic to a bounded below $F$ which is h-projective or h-injective.

Then the pseudo-functor $\text{coDEF}_-(E)$ is pro-representable by the DG algebra $\hat{S} = (BA)^*$, where $A$ is a strictly unital minimal model of $\mathbb{R} \text{Hom}(E, E)$.
Define the contravariant DG functor $\Phi : \hat{\mathcal{D}}$.
Denote by $\langle \cdot \rangle$ Full and Faithful. Surjective on isomorphism classes.

Consider the $A_\infty$ $\hat{\mathcal{D}}$-module $B \hat{A} \otimes A \to J$, where $J$ is an $h$-injective $A_\infty$ $\hat{\mathcal{D}}$-module. Note that $J$ is also $h$-injective as a DG $\hat{\mathcal{D}}$-module.

Given an artinian DG algebra $R$ and a 1-morphism $(M, \theta) \in 1\cdot \mathcal{H}(\hat{S}, R)$ we define

$$\Theta(M) := \text{Hom}_{\hat{\mathcal{D}}}(M, J).$$

We have $R \text{Hom}_{\hat{\mathcal{D}}}(k, \text{Hom}_{\hat{\mathcal{D}}}(M, J)) = R \text{Hom}_{\hat{\mathcal{D}}}(k \otimes \hat{R} M, J)$. Hence the quasi-isomorphism $\theta : k \otimes \hat{R} M \to k$ induces a quasi-isomorphism

$$R \text{Hom}_{\hat{\mathcal{D}}}(k, \Theta(M)) \simeq R \text{Hom}_{\hat{\mathcal{D}}}(k, J) = \text{Hom}_{\hat{\mathcal{D}}}(k, J),$$

and by Proposition 4.3 the last term is canonically quasi-isomorphic to $A$ as an $A_\infty$ $A$-module.

If we are given with another artinian DG algebra $Q$ and a 1-morphism $(N, \delta) \in 1\cdot \mathcal{H}(R, Q)$, then the object $\Theta(N \otimes \hat{R} M)$ is canonically quasi-isomorphic to the object $R \text{Hom}(N, \Theta(M))$. Thus, $\Theta$ is a morphism of pseudo-functors.

It remains to prove that for each $R \in 2\cdot \text{dcart}_-$ the induced functor $\Theta_R : 1\cdot \mathcal{H}(\hat{S}, R) \to \text{coDEF}(R)$ is an equivalence of groupoids. So fix a DG algebra $R \in 2\cdot \text{dcart}_-$.

**Surjective on isomorphism classes.** Let $(S, \sigma)$ be an object of $\text{coDEF}_R(A)$. By Corollary 10.3, there exists a morphism of DG algebras $\phi : \hat{S} \to R$ such that the pair $(T, \text{id})$, where $T = \text{Hom}_{\hat{\mathcal{D}}}(R, B \hat{A} \otimes A)$, defines an object of $\text{coDEF}_R(A)$ which is isomorphic to $(S, \sigma)$. Further, by Proposition 4.5 the morphism $\text{Hom}_{\hat{\mathcal{D}}}(R, B \hat{A} \otimes A) \to \text{Hom}_{\hat{\mathcal{D}}}(R, J)$ is quasi-isomorphism. Therefore, the object $(T, \text{id})$ is isomorphic to $\Theta(M)$, where $M = R$ is DG $\hat{\mathcal{D}}$-$R$-module via the homomorphism $\phi$.

**Full and Faithful.** Consider the above $\Theta$ as a contravariant DG functor from $R \otimes \hat{\mathcal{D}}$-mod to $\hat{\mathcal{D}}_{\hat{R} \otimes \hat{\mathcal{D}}}$-mod. Define the contravariant DG functor $\Phi : \hat{\mathcal{D}}_{R \otimes \hat{\mathcal{D}}}$-mod $\to R \otimes \hat{\mathcal{D}}$-mod defined by the similar formula:

$$\Phi(N) = \text{Hom}_{\hat{\mathcal{D}}}(N, J).$$

These DG functors induce the corresponding DG functors between derived categories

$$\Theta : D(R \otimes \hat{\mathcal{D}}) \to D_{\infty}(\hat{\mathcal{D}}_{\hat{R} \otimes \hat{\mathcal{D}}}), \quad \Phi : D_{\infty}(\hat{\mathcal{D}}_{\hat{R} \otimes \hat{\mathcal{D}}}) \to D(R \otimes \hat{\mathcal{D}}).$$

Denote by $\langle k \rangle \subset D(R \otimes \hat{\mathcal{D}})$ and $\langle A \rangle \subset D_{\infty}(\hat{\mathcal{D}}_{\hat{R} \otimes \hat{\mathcal{D}}})$ the triangulated envelopes of the DG $R \otimes \hat{\mathcal{D}}$-module $k$ and $A_{\infty}$ $\hat{\mathcal{D}}_{\hat{R} \otimes \hat{\mathcal{D}}}$-module $A$ respectively.

**Lemma 14.3.** The functors $\Theta$ and $\Phi$ induce mutually inverse anti-equivalences of the triangulated categories $\langle k \rangle$ and $\langle A \rangle$.

**Proof.** For $M \in R \otimes \hat{\mathcal{D}}$, and $N \in \hat{\mathcal{D}}_{\hat{R} \otimes \hat{\mathcal{D}}}$ we have the functorial closed morphisms

$$\beta_M : M \to \Phi(\Theta(M)), \quad \beta_M(x)(f) = (-1)^{|f||x|} f(x), \quad \beta_M(x)_n = 0 \text{ for } n \geq 2;$$

$$\gamma_N : N \to \Theta(\Phi(N)), \quad (\gamma_N)(a_1, \ldots, a_{n-1}, y)(f) = (-1)^{n(|a_1|+\cdots+|a_{n-1}|+|y|)} f_n(a_1, \ldots, a_{n-1}, y).$$
By Proposition 4.3 the $A_{\infty}$ $\overline{A}_{\text{op}}$-module $\Theta(k)$ is quasi-isomorphic to $A$. Further, $\Phi(A)$ is quasi-isomorphic to $J$ and hence to $k$. Therefore, $\beta_k$ and $\gamma_A$ are quasi-isomorphisms, and Lemma is proved. □

Note that for $(M, \theta) \in 1\text{-Hom}(\hat{S}, R)$ (resp. for $(S, \sigma) \in \text{coDEF}_R(A)$) $M \in \langle k \rangle$ (resp. $S \in \langle A \rangle$). Hence the functor $\Theta_R : 1\text{-Hom}(\hat{S}, R) \to \text{coDEF}_R(A)$ is fully faithful. This proves the theorem. □

15. Pro-representability of the pseudo-functor $\text{DEF}_-$

Pro-representability Theorems 14.1 and 14.2 imply analogous results for the pseudo-functor $\text{DEF}_-$. Namely, we have the following Theorems.

**Theorem 15.1.** Let $C$ be a DG algebra such that the cohomology algebra $H(C)$ is admissible finite-dimensional. Let $A$ be a strictly unital minimal model of $C$. Then the pseudo-functor $\text{DEF}_-(C)$ is pro-representable by the DG algebra $\hat{S} = (B\overline{A})^*$. That is, there exists an equivalence of pseudo-functors $\text{DEF}_-(C) \simeq h'_\hat{S}$ from 2'-dgart to $\text{Gpd}$.

**Proof.** By Theorem 14.1 we have the equivalence $\text{coDEF}_-(C) \simeq h_\hat{S}$ of pseudo-functors from 2-dgart to $\text{Gpd}$.

By Theorem 13.4 we have the equivalence $\text{coDEF}_-(C) \simeq \text{DEF}_-(C) \cdot D$ of pseudo-functors from 2-dgart to $\text{Gpd}$.

Further, by Corollary 13.2 $h_\hat{S} \simeq h'_\hat{S} \cdot D$. Hence $\text{DEF}_-(C) \cdot D \simeq h'_\hat{S} \cdot D$ and therefore $\text{DEF}_-(C) \simeq h'_\hat{S}$. □

We get the following corollary.

**Theorem 15.2.** Let $E \in A^{\text{op}}\text{-mod}$. Assume that the graded algebra $\text{Ext}(E, E)$ is admissible finite-dimensional. Then the pseudo-functor $\text{DEF}_-(E)$ is pro-representable by the DG algebra $\hat{S} = (B\overline{A})^*$, where $A$ is a strictly unital minimal model of the DG algebra $R\text{Hom}(E, E)$.

**Proof.** Indeed, by Proposition 11.9 the pseudo-functors $\text{DEF}_-(E)$ and $\text{DEF}_-(R\text{Hom}(E, E))$ are equivalent. And by Theorem 15.1 the pseudo-functors $\text{DEF}_-(R\text{Hom}(E, E))$ and $h'_\hat{S}$ are equivalent. □

In the proof of Theorem 14.1 we showed that the bar complex $B\overline{A} \otimes_{\gamma_A} A$ is the ”universal co-deformation” of the $A_{\infty}$ $\overline{A}^{\text{op}}$-module $A$. However, Theorem 15.1 is deduced from Theorem 14.1 without finding the analogous ”universal deformation” of the $A_{\infty}$ $\overline{A}^{\text{op}}$-module $A$. We do not know if this ”universal deformation” exists in general (under the assumptions of Theorem 15.1). But we can find it and hence give a direct proof of Theorem 15.1 if the minimal model $A$ of $C$ satisfies an extra assumption (*) below.

For the rest of this section we assume that $A$ is an augmented $A_{\infty}$-algebra.
Definition 15.3. Let $A$ be an augmented $A_\infty$-algebra. Consider $k$ as a left $A_\infty$-module. We say that $A$ satisfies the condition (*) if the canonical morphism

$$k \to \text{Hom}_{A^{op}}(\text{Hom}_A(k, A), A)$$

of left $A_\infty$-modules is a quasi-isomorphism.

Example 15.4. Let $A$ be an augmented $A_\infty$-algebra. If $k$ lies in $\text{Perf}(A)$ then $A$ satisfies the condition (*).

In particular, suppose that $A$ is homologically smooth and compact. That is, the diagonal $A_\infty$ $A-A$-bimodule $A$ lies in $\text{Perf}(A-A)$ (smoothness), and $\dim H(A) < \infty$ (compactness). Then the $A_\infty$ $A$-module is perfect if it has finite-dimensional total cohomology. Thus, $k \in \text{Perf}(A)$ and $A$ satisfies the condition (*).

Example 15.5. Let $A$ be an augmented $A_\infty$-algebra which is left and right Gorenstein of dimension $d$. This means that $\text{Ext}_{A_\infty}^p(A, A) = \begin{cases} k, & \text{if } p=d \\ 0, & \text{otherwise,} \end{cases}$ and $\text{Ext}_{A^{op}_\infty}^p(k, A) = \begin{cases} k, & \text{if } p=d \\ 0, & \text{otherwise.} \end{cases}$ Then $A$ satisfies the condition (*).

For the rest of this section assume that $A$ is admissible, finite-dimensional and satisfies the condition (*).

Denote by $E$ the $A_\infty$ $A^{op}_{\hat{S}^{op}}$-module

$$E := \text{Hom}_A(k, A).$$

This $A_\infty$-module is isomorphic to $A \otimes \hat{S}$ as a graded $(\hat{S}^{op})^{gr}$-module and can be given explicitly by the formula

$$(15.1) \quad m^E_n(m, a_1, \ldots, a_{n-1}) = m^\text{MC}^{\hat{S}}_{\infty}(A)(0, \ldots, 0, \tau_A)(m, a_1 \otimes 1, \ldots, a_{n-1} \otimes 1).$$

Remark 15.6. The definition of the $A_\infty$-category $\text{MC}^{\hat{S}}_{\infty}(A)$ is the same as if $\hat{S}$ would be artinian. It is correct because $\hat{S}$ is complete in $m$-adic topology and $A$ is finite-dimensional. In the above formula $\tau_A$ is considered as an element of $A \otimes \hat{S} = \text{Hom}_k(B \hat{A}, A)$. We denote the $A_\infty$ $A^{op}_{\hat{S}^{op}}$-module $E$ by $A \otimes_{\tau_A} \hat{S}$.

We claim that $E$ is the "universal deformation" of $A$. This is justified by Theorem 15.9 below. Let us start with a few lemmas.

Lemma 15.7. The object $E$ considered as a DG $\hat{S}^{op}$-module is $h$-projective.

Proof. Notice that the stupid filtration $A^\geq i$ of the complex $A$ is finite. Since $A$ is admissible it follows that the differential $m^E_1$ preserves the $(\hat{S}^{op})^{gr}$-submodule $A^\geq i \otimes \hat{S}$. Hence the DG $\hat{S}^{op}$-module $E = A \otimes_{\tau_A} \hat{S}$ has a finite filtration by DG $\hat{S}^{op}$-submodules $A^\geq i \otimes \hat{S}$ with subquotients being free $\hat{S}^{op}$-modules $A^i \otimes \hat{S}$. Thus the DG $\hat{S}^{op}$-module $E$ is $h$-projective. □

Lemma 15.8. The $A_\infty$ $A^{op}$-module $E \otimes_{\hat{S}} k$ is canonically quasi-isomorphic to $A$. 

Proof. By Lemma 15.7 and Remark 15.6 we have
\[ E \otimes \hat{S} k = E \otimes \hat{S} k = A \otimes_{\tau_A} \hat{S} \otimes \hat{S} k, \]
and the last \( A_{\infty} \) \( A^{op} \)-module is isomorphic to \( A \) since \( \tau_A \in A \otimes m \), where \( m \subset \hat{S} \) is the augmentation ideal.

Now we are ready to define a morphism of pseudo-functors
\[ \Psi : h_{\hat{S}}' \to \text{DEF}_-(A). \]
Let \( R \in \text{dgart}_- \) and \( M = (M, \theta) \in 1\text{-Hom}(\hat{S}, R) \). We put
\[ \Psi(M) := E \otimes \hat{S} M \in D_{\infty}(\hat{A}_{R^{op}}). \]
Notice that the structure isomorphism \( \theta : M \otimes_R k \to k \) defines an isomorphism
\[ L_i^* \Psi(M) = \Psi(M) \otimes_R k = E \otimes \hat{S} M \otimes_R k \iso E \otimes \hat{S} k, \]
and the last term is canonically quasi-isomorphic to \( A \) as \( A_{\infty} \) \( \hat{A}^{op} \)-module by Lemma 15.8. Hence \( \Psi(M) \) is indeed an object in the groupoid \( \text{DEF}_R(A) \).

If \( \delta : M \to N \) is a 2-morphism, where \( M, N \in 1\text{-Hom}(\hat{S}, R) \), then \( \Psi(\delta) : \Psi(M) \to \Psi(N) \) is a morphism of objects in the groupoid \( \text{DEF}_R(A) \). Thus \( \Psi \) is indeed a morphism of pseudo-functors.

Theorem 15.9. The morphism \( \Psi : h_{\hat{S}}' \to \text{DEF}_-(A) \) is an equivalence.

Proof. It remains to show that for each \( R \in \text{dgart}_- \) the induced functor
\[ \Psi : 1\text{-Hom}(\hat{S}, R) \to \text{DEF}_R(A) \]
is an equivalence of groupoids.

We fix \( R \).

Surjective on isomorphism classes.

Let \( (S, \sigma) \in \text{DEF}_R(A) \). By Corollary 12.3 there exists an element \( \alpha \in MC_R(A) \) such that the pair \( (T, \text{id}) \), where \( T = A \otimes_{\alpha} R \), defines an object of \( \text{DEF}_R(A) \) and \( (T, \text{id}) \) is isomorphic to \( (S, \sigma) \) in \( \text{DEF}_R(A) \). The element \( \alpha \) corresponds to a (unique) admissible twisting cochain \( \tau : R^* \to A \), which in turn corresponds to a homomorphism of DG coalgebras \( g_{\tau} : R^* \to BA \) (Proposition 8.1). By dualizing we obtain a homomorphism of DG algebras \( g_{\tau}^* : \hat{S} \to R \) and hence the corresponding object \( M_{\alpha} = (\hat{S} R R, \text{id}) \in 1\text{-Hom}(\hat{S}, R) \).

Lemma 15.10. The object \( \Psi(M_{\alpha}) \in \text{Def}_R(A) \) is isomorphic to \( (T, \text{id}) \).

Proof. By Remark 15.6
\[ E = A \otimes_{\tau_A} \hat{S}, \]
and hence by Lemma 15.7
\[ \Psi(M_{\alpha}) = (A \otimes_{\tau_A} \hat{S}) \otimes g_{\tau}^* R. \]
Notice that the image of \( \tau_A \) under the map
\[ 1_A \otimes g_{\tau}^* : A \otimes \hat{S} \to A \otimes R \]
coincides with \( \tau \). Thus \( \Psi(M_{\alpha}) = T \).
Full and faithful.

Let us define a functor $\Pi : \text{DEF}_R(A) \to 1\text{-}\text{Hom}(\hat{S}, R)$ as follows: for $S = (S, \sigma) \in \text{DEF}_R(A)$ we put
\[
\Pi(S) := \text{Hom}_{\text{A}^{\text{op}}}(E, S) \in D(\hat{S}^{\text{op}} \otimes R).
\]
We claim that $\Pi(S)$ is an object in $1\text{-}\text{Hom}(\hat{S}, R)$, i.e. it is quasi-isomorphic to $R$ as a DG $R^{\text{op}}$-module and the isomorphism $\sigma$ defines an isomorphism $\Pi(S) \xrightarrow{\sim} k$.

Indeed, again by Corollary 12.3 we may and will assume that $(S, \sigma) = (T, \text{id})$, where $T = A \otimes_{\alpha} R$, $\alpha \in \mathcal{M}_C(A)$. We have
\[
\Pi(T) = \text{Hom}_{\text{A}^{\text{op}}}(E, A \otimes_{\alpha} R) = \text{Hom}_{\text{A}^{\text{op}}}(\text{Hom}(k, A), A) \otimes_{\alpha} R.
\]
Since the $A_\infty$-algebra $A$ satisfies the condition (*) the last term as a DG $R^{\text{op}}$-module is canonically quasi-isomorphic to $k \otimes R = R$. Thus we have a canonical isomorphism $\Pi(S) \otimes_R k \xrightarrow{\sim} k$.

Note that the functors $\Psi$ and $\Pi$ are adjoint:
\[
\text{Hom}_{\text{DEF}_R(A)}(\Psi(M), S) = \text{Hom}_{\text{Hom}(\hat{S}, R)}(M, \Pi(S)).
\]
Now let us consider $\Psi$ and $\Pi$ as functors simply between the derived categories $D(\hat{S} \otimes R^{\text{op}})$ and $D_\infty(\hat{A}^{\text{op}}_{R^{\text{op}}})$. (They remain adjoint). Denote by $\langle k \rangle \subset D(\hat{S} \otimes R^{\text{op}})$ and $\langle A \rangle \subset D_\infty(\hat{A}^{\text{op}}_{R^{\text{op}}})$ the triangulated envelopes of the DG module $k$ and the $A_\infty \hat{A}^{\text{op}}_{R^{\text{op}}}$-module $A$ respectively. Let $(S, \sigma) \in \text{DEF}_R(A)$. By Corollary 12.3 we may and will assume that $(S, \sigma) = (T, \text{id})$, where $T = A \otimes_{\alpha} R$, $\alpha \in \mathcal{M}_C(A)$. Hence $S \in \langle A \rangle$. Choose $(M, \theta) \in 1\text{-}\text{Hom}(\hat{S}, R)$. Since the DG algebra $\hat{S} \otimes R^{\text{op}}$ is local and complete by Lemma 4.2 we have $M \in \langle k \rangle$.

Therefore it suffices to prove the following lemma.

**Lemma 15.11.** The functors $\Psi$ and $\Pi$ induce mutually inverse equivalences of triangulated categories $\langle k \rangle$ and $\langle A \rangle$.

**Proof.** It suffices to prove that the adjunction maps $k \to \Pi \Psi(k)$ and $\Psi \Pi(A) \to A$ are isomorphisms.

We have $\Pi \Psi(k) = \text{Hom}_{\text{A}^{\text{op}}}(E, E \otimes_{\hat{S}} k) = \text{Hom}_{\text{A}^{\text{op}}}(E, A)$ (Lemma 15.8). Hence $k \to \Pi \Psi(k)$ is a quasi-isomorphism because $A$ satisfies property (*).

Vice versa, $\Psi \Pi(A) = E \otimes_{\hat{S}} (\text{Hom}_{\text{A}^{\text{op}}}(E, A)) = E \otimes_{\hat{S}} k$, since $A$ satisfies property (*). But $E \otimes_{\hat{S}} k = A$ by Lemma 15.8.

This proves the lemma. □

Theorem is proved. □

15.1. **Explicit equivalence** $\text{DEF}_-(E) \cong h'_S$. Let $E \in A^{\text{op}}\text{-mod}$. Suppose that the graded algebra $\text{Ext}(E, E)$ is admissible and finite-dimensional. Let $A$ be a strictly unital minimal model of the DG algebra $R \text{Hom}(E, E)$. Suppose that $A$ satisfies the condition (*) above. Further, let $F \to E$ be a quasi-isomorphism with h-projective $F$, $C = \text{End}(F)$ and let $f : A \to C$ be a strictly unital $A_\infty$-quasi-isomorphism. By Theorem 15.9, the $A_\infty \hat{A}^{\text{op}}_{S^{\text{op}}}$-module $\text{Hom}(k, A)$ is the "universal deformation" of the $A_\infty \hat{A}^{\text{op}}_{R^{\text{op}}}$-module $A$. It follows from the equivalence $\text{DEF}_-(A) \cong \text{DEF}_-(C)$ (Corollary 12.2) that that the $(C \otimes \hat{S})^{\text{op}}$-module
\[
\text{Hom}(k, C) = C \otimes f^*(\tau_A) \hat{S}
\]
is the "universal deformation" of the DG $C^{\text{op}}$-module $C$.

Put
\[
\mathcal{F} = \text{Hom}(k, C) \otimes C F = (C \otimes f^*(\tau_A) \hat{S}) \otimes C F.
\]
Then $\mathcal{F}$ is a DG $A^\text{op}_S$-module. We claim that it is a "universal deformation" of the DG module $E$. More precisely, we get the following

**Corollary 15.12.** Let $E$ and $\mathcal{F}$ be as above. Then the functors $\Phi_R : D(\hat{S} \otimes R^\text{op}) \to D(A^\text{op}_R)$,

$$\Phi_R(M) = \mathcal{F} \otimes_{\hat{S}} M,$$

induce the equivalence of pseudo-functors

$$\Phi : h'_S \to \text{DEF}_-(E)$$

from $\text{dgart}_-$ to $\text{Gpd}$.

**Proof.** Indeed, the morphism $\Phi : h'_S \to \text{DEF}_-(E)$ is isomorphic to the composition of the equivalence

$$\Psi : h'_S \to \text{DEF}_-(A)$$

from Theorem 15.9, the equivalence

$$\text{DEF}_-(A) \cong \text{DEF}_-(C)$$

from Corollary 12.2, and the equivalence

$$\text{DEF}_-(\Sigma) : \text{DEF}_-(C) \to \text{DEF}_-(E)$$

from the proof of Proposition 11.9. \hfill \square

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**Part 5. Noncommutative Grassmanians**

16. Preliminaries on $\mathbb{Z}$-algebras

In this section we define the notion of a $\mathbb{Z}$-algebra and associate to it an abelian category which should be thought of as a category of quasi-coherent sheaves on the corresponding noncommutative stack. We also define Koszul $\mathbb{Z}$-algebras.

**Definition 16.1.** A $\mathbb{Z}$-algebra $\mathcal{A}$ over the field $k$ is a $k$-linear category with the set of objects $\mathbb{Z}$. For $i, j \in \mathbb{Z}$, we write $\mathcal{A}_{ij}$ instead of $\text{Hom}_\mathcal{A}(i, j)$. Sometimes we will identify a $\mathbb{Z}$-algebra $\mathcal{A}$ with the corresponding ordinary non-unital algebra $\text{Alg}_A$.

Further, if $\mathcal{A}$ is a $\mathbb{Z}$-algebra, then we define the abelian category $\text{Mod-} \mathcal{A}$ as the category $\text{Fun}(\mathcal{A}^\text{op}, k\text{-Vect})$ of contravariant functors from $\mathcal{A}$ to $k$-vector spaces. Equivalently $\text{Mod-} \mathcal{A}$ is the full subcategory of $\text{Mod-} \text{Alg}_\mathcal{A}$ which consists of right $\text{Alg}_\mathcal{A}$-modules $M$ such that

$$M = \bigoplus_{i \in \mathbb{Z}} M \cdot 1_i$$

(quasi-unital modules). We call the objects of $\text{Mod-} \mathcal{A} \ \mathcal{A}^\text{op}$-modules. For each $i \in \mathbb{Z}$ put

$$P_i := \text{Hom}(-, i) = 1_i \text{Alg}_\mathcal{A} \in \text{Mod-} \mathcal{A}.$$

By Yoneda Lemma, for each $M \in \text{Mod-} \mathcal{A}$ we have

$$\text{Hom}_{\mathcal{A}^\text{op}}(P_i, M) = M(i),$$

hence $P_i$ are projectives. Clearly, each $M \in \text{Mod-} \mathcal{A}$ can be covered by a direct sum of $P_i$'s, hence the abelian category $\text{Mod-} \mathcal{A}$ has enough projectives.
Definition 16.2. Let $M \in \text{Mod-}\mathcal{A}$ be an $\mathcal{A}^{\text{op}}$-module. An element $x \in M(i)$ is called torsion if we have $x \cdot A_{ji} = 0$ for $j < i$. Torsion elements form a submodule of $M$ which we denote by $\tau(M)$. An $\mathcal{A}^{\text{op}}$-module $M$ is called torsion if we have $M = \tau(M)$. We denote by $\text{Tors}(\mathcal{A})$ the full subcategory of $\text{Mod-}\mathcal{A}$ which consists of torsion $\mathcal{A}^{\text{op}}$-modules.

The category $\text{QMod}(\mathcal{A})$ is defined as the quotient category $\text{Mod-}\mathcal{A}/\text{Tors}(\mathcal{A})$. We denote by $\pi : \text{Mod-}\mathcal{A} \to \text{QMod}(\mathcal{A})$ the projection functor.

If $M, N$ are $\mathcal{A}^{\text{op}}$-modules then

$$\text{Hom}_{\text{QMod}(\mathcal{A})}(\pi(M), \pi(N)) = \lim_{\to} \text{Hom}_{\mathcal{A}^{\text{op}}}(M', N/\tau(N)),$$

where $M'$ runs over the quasi-directed category of submodules $M' \subset M$ such that $M/M'$ is torsion.

The category $\text{QMod}(\mathcal{A})$ should be thought of as the category $\text{QCoh}(\text{Proj}(\mathcal{A}))$ of quasi-coherent sheaves on the noncommutative projective stack $\text{Proj}(\mathcal{A})$. Furthermore, the object $\pi(P_0) \in \text{QMod}(\mathcal{A})$ should be thought of as the structure sheaf $\mathcal{O}_{\text{Proj}(\mathcal{A})}$.

Remark 16.3. Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded (unital) algebra. Then one can associate to it a $\mathbb{Z}$-algebra $\mathcal{A}$ with $A_{ij} = A^{i-j}$ so that the composition in $\mathcal{A}$ comes from the multiplication in $A$. Recall that in $[AZ]$ there defined the category $\text{QGr}(\mathcal{A})$ as the quotient category $\text{Gr} A/\text{Tors}$ of the category $\text{Gr} A$ of graded $A$-modules by the subcategory $\text{Tors}$ of torsion modules. It is clear that the categories $\text{Gr} A$, $\text{Tors}$, $\text{QGr} A$ are equivalent to $\text{Mod-}\mathcal{A}$, $\text{Tors}(\mathcal{A})$, $\text{QMod}(\mathcal{A})$ respectively.

Notice that it can happen that the graded algebras $A_1$ and $A_2$ are not isomorphic but the associated $\mathbb{Z}$-algebras are equivalent. Thus, it is more reasonable to consider $\mathbb{Z}$-algebras but not graded algebras.

The projection functor $\pi : \text{Mod-}\mathcal{A} \to \text{QMod}(\mathcal{A})$ admits a right adjoint functor $\omega : \text{QMod}(\mathcal{A}) \to \text{Mod-}\mathcal{A}$ defined by the formula

$$\omega(X)(i) = \text{Hom}(\pi(P_i), X).$$

The adjunction morphism $\pi \omega \to \text{id}$ is an isomorphism.

Definition 16.4. A $\mathbb{Z}$-algebra $\mathcal{A}$ is called

a) positively (resp. negatively) oriented if $A_{ij} = 0$ for $i > j$ (resp. for $i < j$);

b) connected if $A_{ii} \cong k$ for each $i \in \mathbb{Z}$;

c) locally finite if $\dim A_{ij} < \infty$ for any $i, j \in \mathbb{Z}$.

Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. We denote by $\mathcal{A}_{\leq i}$ the full subcategory of $\mathcal{A}$ such that $\text{Ob}(\mathcal{A}_{\leq i}) = \{ j : j \leq i \}$. Clearly, we also have the categories $\text{Mod-}\mathcal{A}_{\leq i}$ and $\text{Tors}(\mathcal{A}_{\leq i})$. It is easy to see that the quotient category $\text{Mod-}\mathcal{A}_{\leq i}/\text{Tors}(\mathcal{A}_{\leq i})$ is equivalent to $\text{QMod}(\mathcal{A})$. We denote by $\pi_{\leq i} : \text{Mod-}\mathcal{A}_{\leq i} \to \text{QMod}(\mathcal{A})$ and $\omega_{\leq i} : \text{QMod}(\mathcal{A}) \to \text{Mod-}\mathcal{A}_{\leq i}$ the projection functor and its right adjoint respectively.

If $\mathcal{A}$ is a positively oriented $\mathbb{Z}$-algebra then we put

$$T_{ij} = P_j/(P_j)_{\leq i},$$

where

$$(P_j)_{\leq i} = \bigoplus_{k \leq i} A_{kj}.$$

Clearly, the $\mathcal{A}^{\text{op}}$-modules $T_{ij}$ are torsion.
If $\mathcal{A}$ is a positively or negatively oriented connected $\mathbb{Z}$-algebra then we denote by $S_n$ the simple $\mathcal{A}^{op}$-modules defined by the formula

$$S_n(i) = \begin{cases} k & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $\mathcal{A}$ is positively oriented then $S_n = T_{n,n}$.

**Definition 16.5.** A connected positively (resp. negatively) oriented $\mathbb{Z}$-algebra is called quadratic if the algebra $Alg_{\mathcal{A}}$ is generated by the subspaces $\mathcal{A}_0 = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$ and $\mathcal{A}_i = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{i,i+1}$ (resp. $\mathcal{A}_{-1} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{i+1,i}$) and is determined by the quadratic relations $I_{i,i+2} \in \mathcal{A}_{i+1,i+2} \otimes \mathcal{A}_{i,i+1}$ (resp. $I_{i+2,i} \in \mathcal{A}_{i+1,i} \otimes \mathcal{A}_{i+2,i+1}$).

For a locally finite positively (resp. negatively) oriented quadratic $\mathbb{Z}$-algebra $\mathcal{A}$ one can define the dual quadratic $\mathbb{Z}$-algebra $\mathcal{A}^!$ with the opposite orientation. It is defined by the dual generators $\mathcal{A}_{i+1,i}^! = \mathcal{A}_{i,i+1}^*$ (resp. $\mathcal{A}_{i,i+1}^! = \mathcal{A}_{i,i+1}^*$) and the dual quadratic relations $S(I_{i,i+2}^!) \subset \mathcal{A}_{i+1,i}^* \otimes \mathcal{A}_{i,i+1}^*$ (resp. $S(I_{i+2,i}^!) \subset \mathcal{A}_{i+1,i}^* \otimes \mathcal{A}_{i+2,i+1}^*$), where $I_{i,i+2}^! \subset \mathcal{A}_{i+1,i+2}^* \otimes \mathcal{A}_{i,i+1}^*$ (resp. $I_{i+2,i}^! \subset \mathcal{A}_{i+1,i}^* \otimes \mathcal{A}_{i+2,i+1}^*$) is the dual subspace and $S$ is the transposition of factors.

Further, one can define a Koszul complex

$$K := \mathcal{A}^{op} \otimes_{\mathcal{A}_0} \mathcal{A} = \bigoplus \mathcal{A}_{k,j}^{op} \otimes_k \mathcal{A}_{i,j}.$$

Here $\mathcal{A}^{op} = \bigoplus \mathcal{A}_{i,j}$ is a bounded dual of $\mathcal{A}^!$. It is an $\mathcal{A}^!$-bimodule.

The differential $d : K \to K$ is defined as follows. Suppose that $\mathcal{A}$ is positively oriented. We have the natural maps

$$\mathcal{A}_{j,j+1} \otimes \mathcal{A}_{i,j} \to \mathcal{A}_{j,j+1}, \quad \text{and} \quad \mathcal{A}_{j,j+1}^* \otimes \mathcal{A}_{k,j}^* \to \mathcal{A}_{k,j+1}^*.$$

In particular, we have the maps

$$\psi_{ijk} : \mathcal{A}_{j,j+1}^* \otimes \mathcal{A}_{j,j+1} \to \text{Hom}_k(\mathcal{A}_{i,j}^{op} \otimes_k \mathcal{A}_{i,j}, \mathcal{A}_{k,j+1}^* \otimes_k \mathcal{A}_{k,j+1}).$$

The non-zero components of $d$ are the maps $d_{ijk} = \psi_{ijk}(1_{A_{j+1,j+1}})$. Note that $d$ is $\mathcal{A}_0$-linear and $\mathcal{A}^! \otimes \mathcal{A}^{op}$-linear. Thus, $K_n = 1_n K$ and $K_n^m = K_n 1_m$ are $d$-invariant. The complex $K_n$ is of the form

$$\cdots \to \mathcal{A}_{n,n-2}^{op} \otimes P_{n-2} \to \mathcal{A}_{n,n-1}^{op} \otimes P_{n-1} \to P_n \to 0,$$

and the complex $K_n^m$ is of the form

$$\cdots \to \mathcal{A}_{n,n-2}^{op} \otimes A_{m,n-2} \to \mathcal{A}_{n,n-1}^{op} \otimes A_{m,n-1} \to A_{m,n} \to 0.$$

In particular, $K_n^m \cong k$.

Analogously for negatively oriented $\mathbb{Z}$-algebras.

For the rest of this section we assume that $\mathcal{A}$ is positively oriented.

**Definition 16.6.** A quadratic locally finite $\mathbb{Z}$-algebra is called Koszul if the complex $K_n^m$ is acyclic for $n \neq m$, or, equivalently, $K_n$ is a resolution of $S_n$.

We refer to [BP] for the definition of co-Koszul and Gorenstein $\mathbb{Z}$-algebras. We will not need these definitions but we will need the following proposition:
Proposition 16.7. ([BP]) Let $A$ be a Koszul (positively oriented) $\mathbb{Z}$-algebra of finite homological dimension $n$. Then the following conditions are equivalent:

(i) $A$ is co-Koszul;

(ii) $A$ is Gorenstein;

(iii) $A^!$ is Frobenious, i.e. $A^!_{i+n,i} = k$ for all $i$, and the multiplication $A_{i+n,j} \otimes A_{i+n,j} \to k$ is a perfect pairing for all $i$ and $j$.

Now we define the notion of a coherent $\mathbb{Z}$-algebra and the category $\text{qmod}(A)$ for a coherent $\mathbb{Z}$-algebra $A$.

Definition 16.8. Let $A$ be a $\mathbb{Z}$-algebra. An $A^{\text{op}}$-module $M$ is called finitely generated if there exists a surjective morphism (in $\text{Mod-} A$)

$$\bigoplus_{j=1}^{m} P_{i_j} \to M,$$

where $i_1, \ldots, i_m \in \mathbb{Z}$. Further, a finitely generated $A^{\text{op}}$-module $M$ is called coherent if for each (not necessarily surjective) morphism (in $\text{Mod-} A$)

$$\phi : \bigoplus_{j=1}^{m} P_{i_j} \to M$$

the $A^{\text{op}}$-module $\ker(\phi)$ is finitely generated. A $\mathbb{Z}$-algebra $A$ is called coherent if for each $i \in \mathbb{Z}$ the module $P_i$ is coherent.

If $A$ is coherent then we denote by $\text{qmod}(A)$ the full (abelian) subcategory of $\text{QMod}(A)$ which consists of the images of coherent $A^{\text{op}}$-modules.

The category $\text{qmod}(A)$ should be thought of as the category $\text{Coh}(\text{Proj}(A))$. By definition, we have that the structure sheaf $\mathcal{O}_{\text{Proj}(A)}$ is coherent.

17. The definition of noncommutative Grassmannians

Let $V$ be a finite-dimensional $k$-vector space of dimension $n > 0$. Let $m$ be an integer such that $1 \leq m \leq n - 1$. We define the noncommutative Grassmanians by the formula

$$\text{NGr}(m, V) := \text{Proj}(A^{m,V}),$$

where $A^{m,V}$ is the following quadratic $\mathbb{Z}$-algebra:

$$A^{m,V}_{i,i+1} = \begin{cases} V^* & \text{for } (n - m + 1) \nmid i, \\ \Lambda^{n-m}V & \text{otherwise}, \end{cases}$$

and the quadratic relations are defined by the natural exact sequences

$$\begin{cases} \Lambda^2 V^* \to A^{m,V}_{i+1,i+1+2} \otimes A^{m,V}_{i+1,i+1} \to A^{m,V}_{i,i+2} \to 0 & \text{for } (n - m + 1) \nmid i, i+1, \\ \Lambda^{n-m-1}V \to A^{m,V}_{i+1,i+1+2} \otimes A^{m,V}_{i+1,i+1} \to A^{m,V}_{i,i+2} \to 0 & \text{otherwise}. \end{cases}$$

Notice that if we fix a volume form $\omega \in \Lambda^n V$, then the $A^{1,V}$ is naturally equivalent to the $\mathbb{Z}$-algebra associated to the symmetric algebra $\bigoplus_{l \geq 0} s^l V^*$, where $\deg(V^*) = 1$. Hence, the stack $\text{NGr}(1, V)$ is isomorphic to the commutative projective space $\mathbb{P}(V)$.

We claim that $\text{NGr}(m, V)$ is a true noncommutative moduli space of structure sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{\text{coh}}(\mathbb{P}(V))$, where $W \subset V$ are vector subspaces of dimension $\dim W = m$. Namely, it satisfies the following properties:
(i) there is a natural fully faithful functor $\Phi$ from the perfect derived category $\text{Perf}(\text{NGr}(m, V)) = D_{\text{perf}}(\text{QMod}(A^{m,V}))$ (Definition 18.20) to $D^b_{\text{cohom}}(\mathbb{P}(V))$. Its image is the double orthogonal to the family of objects $\mathcal{O}_{\mathbb{P}(V)}$, i.e., the full subcategory generated by objects $\mathcal{O}_{\mathbb{P}(V)}(m-n), \ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}$. This is Corollary 18.22 below:

(ii) There is a $k$-point $x_W \in \text{NGr}(m, V)(k) = X_{A^{m,V}}(k)$ (see Section 19 below) for each subspace $W \subset V$ of dimension $\dim W = m$. Further, $(x_W)_*(k)$ lies in $\text{Perf}(\text{NGr}(m, V))$ and $\Phi((x_W)_*(k)) \cong \mathcal{O}_{\mathbb{P}(W)}$. This is a part of Theorem 19.10 below.

Further, according to our pro-representability Theorems (namely, Theorem 15.2), the completion of the local ring of the $k$-point $x_W$ must be isomorphic to $H^0(\hat{S})$, where $\hat{S}$ is dual to the bar construction of the minimal $A_\infty$-structure on $\text{Ext}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$. Indeed, it can be shown that the DG algebra $R\text{Hom}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is formal and the graded algebra $\text{Ext}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is quadratic Koszul, and hence the projection $\hat{S} \rightarrow H^0(\hat{S})$ is a quasi-isomorphism. However, we do not have a suitably defined notion of a (completion of) the local ring at the $k$-point of noncommutative stack. Thus, we can not establish and prove the corresponding result at the moment.

Furthermore, we do not have a moduli functor of our family of objects $\mathcal{O}_{\mathbb{P}(W)}$, which should be defined on the category of noncommutative affine schemes. However, the properties (i) and (ii) suggest that $\text{NGr}(m, V)$ is a true moduli space of this family of objects, in our context of deformations of objects in derived categories.

It is remarkable that there is a natural morphism from the commutative Grassmanian $\text{Gr}(m, V)$ to noncommutative one $\text{NGr}(m, V)$. Moreover, the functor $\Phi : \text{Perf}(\text{NGr}(m, V)) \rightarrow D^b_{\text{cohom}}(\mathbb{P}(V))$ above coincides with $L_{f_{1,m,V}}^\ast$, where $f_{1,m,V} : \mathbb{P}(V) \rightarrow \text{NGr}(m, V)$ is a natural morphism. Both these statements are parts of Proposition 19.12 below.

18. THE DERIVED CATEGORIES OF NONCOMMUTATIVE GRASSMANIANS

Before we formulate and prove results on the derived categories of quasi-coherent sheaves on noncommutative Grassmanians $\text{NGr}(m, V)$ we need to remind some notions and results from [BP].

Let $D$ be a $k$-linear enhanced triangulated category.

**Definition 18.1.** An object $E \in \text{Ob}(D)$ is called exceptional if $\text{Hom}^i(E, E) = 0$ for $i \neq 0$, and $\text{Hom}^0(E, E) = k$.

**Definition 18.2.** A collection $(E_1, \ldots, E_m)$ of exceptional objects in $D$ is called exceptional if $\text{Hom}^i(E_i, E_j) = 0$ for $i > j$.

**Definition 18.3.** A full exceptional collection of objects in the category $D$ is a collection which generates $D$ as triangulated category.

**Definition 18.4.** An exceptional collection $(E_1, \ldots, E_n)$ is called strong exceptional if it satisfies the additional assumption $\text{Hom}^i(E_k, E_l) = 0$ for $i \neq 0$ and all $k$ and $l$.

Let $(E, F)$ be an exceptional pair. Define the objects $L_E F$ and $R_F E$ by the exact triangles

$$L_E F \rightarrow \text{Hom}^1(E, F) \otimes E \rightarrow F; \quad E \rightarrow \text{Hom}^1(E, F)^\vee \otimes F \rightarrow R_F E.$$

Let $\sigma = (E_1, \ldots, E_n)$ be an exceptional collection. If $1 \leq i \leq n - 1$ (resp. $2 \leq i \leq n$), then the right (resp. left) mutation of the object $E_i$ in this collection is the object $R^1 E_i = R_{E_{i+1}} E_i$ (resp. $L^1 E_i = L_{E_{i-1}} E_i$); the
corresponding mutated collection

\[ R_{E_i}^k \sigma := (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, \ldots, E_n) \]

(and the analogous collection \( L_{E_i}^k \sigma \)) is exceptional. The multiple mutations of the objects and of the collection are defined inductively:

\[ R^k E_i = R E_{i+k} R^{k-1} E_i, \quad R^k \sigma = R_{R^k-1 E_i} (R^{k-1} \sigma), \quad k \leq n - i \]

(and in the same way for left mutations).

**Definition 18.5.** A helix of the period \( n \) is an infinite sequence \( \{E_i\}_{i \in \mathbb{Z}} \) such that for each \( i \in \mathbb{Z} \) the collection \( (E_i, \ldots, E_{i+n}) \) is exceptional, and moreover \( R^{n-1} E_i = E_{i+n} \).

If \( \sigma = (E_1, \ldots, E_n) \) is an exceptional collection then it naturally extends to a helix by the conditions

\[ E_{i+n} = R^{n-1} E_i, \quad i \geq 1, \quad E_{i-n} = L^{n-1} E_i, \quad i \leq n. \]

In this case the helix is said to be generated by the collection \( \sigma \).

If the helix is generated by the full exceptional collection then it satisfies the property of the partial periodicity: \( \Phi(E_i) \cong E_{i-n} \), where \( \Phi = F[1-n] \) is the composition of the Serre functor \( F \) and the multiple shift \([1-n]\).

**Definition 18.6.** ([BP]) A helix \( \mu = \{E_i\} \) is called geometric if for each pair \((i, j) \in \mathbb{Z}^2\) such that \( i \leq j \) one has

\[ \text{Hom}^k(E_i, E_j) = 0 \text{ for } k \neq 0. \]

**Definition 18.7.** ([BP]) An exceptional collection is called geometric if it generates a geometric helix.

**Proposition 18.8.** ([BP]) Each sub-collection of a geometric exceptional collection is again geometric.

**Proposition 18.9.** ([BP]) A full exceptional collection of the length \( m \) of coherent sheaves on a smooth projective variety \( X \) of dimension \( n \) is geometric iff \( m = n + 1 \).

**Definition 18.10.** ([BP]) The endomorphism \( \mathbb{Z} \)-algebra \( \mathcal{A} = \text{End}(S) \) of a helix \( S = \{E_i\} \) is defined by the formula

\[ A_{ij} = \text{Hom}(E_i, E_j) \]

with natural composition.

**Theorem 18.11.** ([BP]) If \( S \) is a geometric helix generated by an exceptional collection of length \( n \) then the endomorphism \( \mathbb{Z} \)-algebra \( \mathcal{A} \) of \( S \) is Koszul, co-Koszul and of finite global homological dimension \( n \).

**Definition 18.12.** ([BP]) Koszul co-Koszul \( \mathbb{Z} \)-algebra of finite homological dimension \( n \) is called a geometric \( \mathbb{Z} \)-algebra of the period \( n \).

Let \( \mathcal{A} \) be a geometric \( \mathbb{Z} \)-algebra of the period \( n \). Let \( K \subset D(\text{Mod-}\mathcal{A}) \) be the full triangulated subcategory generated by the modules \( P_i \). Note that \( S_i \in K \). Let \( F \in K \) be the full triangulated subcategory generated by the modules \( S_i \).

**Theorem 18.13.** ([BP]) Let \( \mathcal{A} \) be a geometric \( \mathbb{Z} \)-algebra of the period \( n \). Then \( F \) is a thick subcategory in \( K \); the images of modules \( P_i \) in \( K/F \) form a geometric helix \( S \) of the period \( n \), and moreover the \( \mathbb{Z} \)-algebra of \( S \) is equivalent to \( \mathcal{A} \).
Now we prove the main theorem of this section. It is closely related to the previous one but unfortunately cannot be deduced from it.

**Theorem 18.14.** Let \( \mathcal{A} \) be a geometric helix of the period \( n \). Put \( B = \mathcal{A}[1,n] = \bigoplus_{1 \leq i,j \leq n} A_{ij} \). Then there is an equivalence of categories \( D^*(\text{QMod}(\mathcal{A})) \cong D^*(\text{Mod-}B) \).

**Proof.** The proof is in two main steps. First we prove that the category \( D^*(\text{QMod}(\mathcal{A})) \) is naturally equivalent to the quotient category of \( D^*(\text{Mod-}\mathcal{A}_{\leq n}) \) by the full thick triangulated subcategory \( D^*_{\text{Tors}}(\mathcal{A}_{\leq n}) \) which consists of complexes with torsion cohomology. Then we construct mutually inverse exact equivalences between the categories \( D^*(\text{Mod-}\mathcal{A}_{\leq n})/D^*_{\text{Tors}}(\mathcal{A}_{\leq n}) \) and \( D^*(\text{Mod-}B) \) given by DG bimodules.

**Lemma 18.15.** The categories \( D^*(\text{QMod}(\mathcal{A})) \) and \( D^*(\text{Mod-}\mathcal{A}_{\leq n})/D^*_{\text{Tors}}(\mathcal{A}_{\leq n}) \) are naturally equivalent.

**Proof.** First recall the functor \( \mathcal{A} \rightarrow \text{Mod-}\mathcal{A}_{\leq n} \). It induces a fully faithful functor

\[
K^*(\omega_{\leq n}) : K^*(\text{QMod}(\mathcal{A}_{\leq n})) \rightarrow K^*(\text{Mod-}\mathcal{A}_{\leq n})
\]

between homotopy categories, which is right adjoint to \( K^*(\pi_{\leq n}) \). It follows that \( K^*(\text{QMod}(\mathcal{A}_{\leq n})) \) is equivalent to the quotient category \( K^*(\text{Mod-}\mathcal{A}_{\leq n})/K^*(\pi_{\leq n})^{-1}(0) \). Let \( K^*_{\text{Tors}}(\mathcal{A}_{\leq n}) \subset K^*(\text{Mod-}\mathcal{A}_{\leq n}) \) be the full subcategory which consists of complexes with torsion cohomology. Then acyclic complexes in the category \( K^*(\text{QMod}(\mathcal{A}_{\leq n})) \) correspond to the classes of complexes with torsion cohomology in \( K^*(\text{Mod-}\mathcal{A}_{\leq n})/K^*(\pi_{\leq n})^{-1}(0) \). Thus, \( D^*(\text{QMod}(\mathcal{A}_{\leq n})) \) is equivalent to the quotient of \( K^*(\text{Mod-}\mathcal{A}_{\leq n})/K^*(\pi_{\leq n})^{-1}(0) \) by \( K^*_{\text{Tors}}(\mathcal{A}_{\leq n})/K^*(\pi_{\leq n})^{-1}(0) \). This quotient is further equivalent to \( D^*(\text{Mod-}\mathcal{A}_{\leq n})/D^*_{\text{Tors}}(\mathcal{A}_{\leq n}) \). Lemma is proved.

Denote by \( (Q_1, \ldots, Q_n) \) the exceptional collection of indecomposable projective \( B^\text{op} \)-modules. By Theorem 18.13 the helix \( \{Q_i\}_{i \in \mathbb{Z}} \) generated by \( (Q_1, \ldots, Q_n) \) is geometric. It follows from its partial periodicity that for \( i \leq 0 \) we have \( Q_i \cong H^{n-1}(Q_i)[1-n] \). Thus, we may and will assume that \( Q_i \) is concentrated in degree \( n-1 \) for \( i \leq 0 \). Put \( M_0 = \bigoplus_{i \leq 0} Q_i[n-1] \). Since \( \text{Hom}_B(Q_i, Q_j) = A_{ij} = A_{(i+n)(j+n)} \), \( M_0 \) is naturally an \( \mathcal{A}_{\leq n} \otimes B^\text{op} \)-module. Further, the functor \( \Phi^{-1} = F^{-1}[1-n] \) can be given by the object \( B^i[1-n] \in D(B \otimes B^\text{op}) \), where \( B^i = \mathcal{R} \text{Hom}_{B^\text{op} \otimes B}(B, B \otimes B) \). Since \( Q_i \otimes_{B^\text{op}} B^i[1-n] \) are pure modules for \( i = 1, \ldots, n, \) \( B^i[1-n] \) is a pure bimodule. We define the object \( M_1 \in D^b(\mathcal{A}_{\leq n} \otimes B^\text{op}) \) by the formula

\[
M_1 = M_0 \otimes_B B^i[2-2n].
\]

We have that

\[
P_i \otimes_{\mathcal{A}_{\leq n}} M_1 \cong Q_i
\]

for \( i \leq n \). Since \( \mathcal{A}_{\leq n} \) has finite left homological dimension, we have a well defined functor

\[
\otimes_{\mathcal{A}_{\leq n}} M_1 : D^*(\text{Mod-}\mathcal{A}_{\leq n}) \rightarrow D^*(\text{Mod-}B).
\]

**Lemma 18.16.** For each \( K \in D_{\text{Tors}}(\mathcal{A}_{\leq n}^\text{op}) \) we have \( K \otimes_{\mathcal{A}_{\leq n}} M_1 = 0 \).

**Proof.** Clearly, it suffices to prove the Lemma for \( M_0 \) instead of \( M_1 \). We have that the complex \( K_i \) from section 16 is a projective resolution of \( S_i \) for \( l \leq n \). Further, \( S_i \otimes_{\mathcal{A}_{\leq n}} M_0 \cong K_i \otimes_{\mathcal{A}_{\leq n}} M_0 \) and the last complex is up to shift of the following form
\[
\cdots 0 \rightarrow A^*_{l-n-2n} \otimes Q_{l-2n}[n-1] \rightarrow \cdots \rightarrow A^*_{l-n-l-n-1} \otimes Q_{l-n-1}[n-1] \rightarrow Q_{l-n}[n-1] \rightarrow 0 \rightarrow \ldots
\]

This complex is acyclic since it corresponds to the image of \( K_{l-n} \) in \( K/F \) under the equivalence of Theorem 18.13, and the image of \( K_{l-n} \) in \( K/F \) is zero.

Further, the torsion modules \( T_{km}, k < m \leq n \) have finite filtrations with subquotients being direct sums of \( S_l \). Thus, we have \( T_{km} \otimes_{A_{\leq n}} M_0 = 0 \).

Since each torsion \( A^\text{op}_{\leq n} \)-module has a left resolution by the direct sums of \( T_{km} \), it follows that the statement of the Lemma holds if \( K^- \) is a pure torsion \( A^\text{op}_{\leq n} \)-module. Finally, since \( M_0 \) is quasi-isomorphic to a finite complex of bimodules which are projective as left \( A \)-modules, the statement of the Lemma holds for each \( K^- \in D_{Tors}(A^\text{op}_{\leq n}) \).

By the previous Lemma, the formula \( - \otimes_{A_{\leq n}} M_1 \) defines a functor

\[
\Phi : D^*(\text{Mod-}A_{\leq n})/D^i_{Tors}(A_{\leq n}) \rightarrow D^*(\text{Mod-B}).
\]

Further, \( M_2 := \bigoplus_{1 \leq i \leq n} P_i \) is naturally an \( A^\text{op}_{\leq n} \otimes B \)-module. Consider the functor

\[
\Psi : D^*(\text{Mod-B}) \rightarrow D^*(\text{Mod-}A_{\leq n})/D^i_{Tors}(A_{\leq n})
\]

defined by the formula

\[
\Psi(-) = \pi_{tors}(- \otimes_{B} M_2),
\]

where \( \pi_{tors} : D^*(\text{Mod-}A_{\leq n}) \rightarrow D^*(\text{Mod-}A_{\leq n})/D^i_{Tors}(A_{\leq n}) \) is the projection.

**Lemma 18.17.** The functors \( \Phi \) and \( \Psi \) are mutually inverse equivalences.

**Proof.** First, the isomorphism

\[
M_2 \otimes_{A_{\leq n}} M_1 \rightarrow B
\]

in \( D(B \otimes B^{\text{op}}) \) induces the isomorphism of functors \( \Phi \circ \Psi \cong \text{Id} \).

Further, we claim that \( H^0(M_1 \otimes_{B} M_2) \cong A_{\leq n} \), \( H^{n-1}(M_1 \otimes_{B} M_2) \) is torsion as \( A^\text{op} \)-module and \( H^i(M_1 \otimes_{B} M_2) = 0 \) for \( i \neq 0, n-1 \). Indeed, since \( K_l \) is a resolution of \( S_l \) it follows from Lemma 18.16 by decreasing induction on \( l \leq n \) that

\[
H^i(P_l \otimes_{A_{\leq n}} M_1 \otimes_{B} M_2) = \begin{cases} P_l & \text{for } i = 0; \\ \text{is torsion} & \text{for } i \geq n-1 \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, it remains to note that \( H^k(M_1) = 0 \) for \( k \geq n \) and \( M_2 \) is pure.

Finally, we have the natural morphism \( A_{\leq n} \rightarrow M_1 \otimes_{B} M_2 \) in \( D(A_{\leq n} \otimes A^\text{op}_{\leq n}) \) and for each \( K^- \in D(\text{Mod-}A_{\leq n}) \) we have that

\[
\text{Cone}(K^- \rightarrow K^- \otimes_{A_{\leq n}} M_1 \otimes_{B} M_2) \cong K^- \otimes_{A_{\leq n}} \text{Cone}(A_{\leq n} \rightarrow M_1 \otimes_{B} M_2) \in D_{Tors}(A^\text{op}_{\leq n})
\]

Thus, \( \Psi \circ \Phi \cong \text{Id} \). Lemma is proved.

Theorem follows from Lemmas 18.17 and 18.15.
Now we apply the above theorem to noncommutative Grassmanians introduced in section 17. By Propositions 18.8 and 18.9 we have that the exceptional collection

\[ \sigma = (\mathcal{O}_{\mathbb{P}(V)}(m-n), \ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}) \]

of coherent sheaves on \( \mathbb{P}(V) \) is geometric. Let \( S = \{ E_i \} \) be the helix generated by \( \sigma \), so that \( E_i = \mathcal{O}_{\mathbb{P}(V)}(i) \) for \( i = m - n, \ldots, -1, 0 \).

**Proposition 18.18.** The endomorphism \( \mathbb{Z} \)-algebra \( \mathcal{A} \) of the helix \( S \) is equivalent to \( \mathcal{A}^{m,V} \).

**Proof.** Note that both \( \mathcal{A} \) and \( \mathcal{A}^{m,V} \) are quadratic and \((n - m + 1)\)-periodic. It remains to show that the space \( \mathcal{A}_{i,i+1} \) is isomorphic to \( \mathcal{A}_{i,i+1}^{m,V} \) for \( i = m - n, \ldots, -1, 0 \), and the quadratic relations \( I_{i,i+2} \in \mathcal{A}^{i+1,i+2} \otimes \mathcal{A}_{i,i+1} \) coincide with that of \( \mathcal{A}^{m,V} \) for \( i = m - n, \ldots, 0 \).

All of this is clear for \( i = m - n, \ldots, -1 \). Further, the object \( E_1 \) is isomorphic to the complex

\[ \cdots \to 0 \to \mathcal{O}_{\mathbb{P}(V)}(m-n) \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n+1) \to \cdots \to \Lambda^{n-m}V \otimes \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \cdots, \]

where the last non-zero term is in degree zero. It follows that

\[ \mathcal{A}_{0,1} = \text{Hom}(E_0, E_1) = \Lambda^{n-m}V = \mathcal{A}_{0,1}^{m,V}. \]

Furthermore, the quadratic relation \( I_{-1,1} \subset \mathcal{A}^{0,1} \otimes \mathcal{A}_{-1,0} \) coincides with the subspace \( \Lambda^{n-m-1}V \subset \Lambda^{n-m}V \otimes V^* \), as in \( \mathcal{A}^{m,V} \). Finally, \( E_2 \) is the convolution of the complex

\[ \cdots \to 0 \to E_{m-n+1} \to V \otimes E_{m-n+2} \to \cdots \to \Lambda^{n-m-1}V \otimes E_0 \to V^* \otimes E_1 \to 0 \to \cdots, \]

where the last non-zero term is in degree zero. It follows that the quadratic relation \( I_{0,2} \subset \mathcal{A}^{1,2} \otimes \mathcal{A}_{0,1} \) coincides with the subspace \( \Lambda^{n-m-1}V \subset V^* \otimes \Lambda^{n-m}V \), as in \( \mathcal{A}^{m,V} \). \( \square \)

Let \( B^{m,V} \) be the endomorphism algebra of \( \sigma \). As a corollary of the above results we obtain the following

**Theorem 18.19.** The derived category \( D^*(\text{QMod}(\mathcal{A}^{m,V})) \) is equivalent to the derived category \( D^*(B^{m,V}) \). The objects \( \pi(P_i) \) in \( D^*(\text{QMod}(\mathcal{A}^{m,V})) \) form a geometric helix of the period \( \dim V - m + 1 \).

**Proof.** Indeed, by Proposition 18.18 and Theorem 18.11 the \( \mathbb{Z} \)-algebra \( \mathcal{A}^{m,V} \) is geometric. Thus, the first statement follows from Theorem 18.14. After that, the second statement follows from Theorem 18.13. \( \square \)

Now we introduce the perfect derived category.

**Definition 18.20.** Let \( \mathcal{A} \) be a positively oriented \( \mathbb{Z} \)-algebra. The perfect derived category \( D_{\text{perf}}(\text{QMod}(\mathcal{A})) \) is the minimal full thick triangulated subcategory of \( D^b(\text{QMod}(\mathcal{A})) \) which contains the objects \( \pi(P_i) \). We will call the objects of \( D_{\text{perf}}(\text{QMod}(\mathcal{A})) \) perfect complexes.

We will also write below \( \text{Perf}(\text{NGr}(m,V)) \) instead of \( D_{\text{perf}}(\text{QMod}(\mathcal{A}^{m,V})) \).

**Proposition 18.21.** Let \( \mathcal{A} \) be a geometric \( \mathbb{Z} \)-algebra of the period \( n \), and \( B = \bigoplus_{1-n \leq i,j \leq 0} \mathcal{A}_{ij} \). Then the category \( D_{\text{perf}}(\text{QMod}(\mathcal{A})) \) is equivalent to \( D^b(\text{mod}_{\text{finite}}B) \).

**Proof.** By Theorem 18.14 \( (\pi(P_{-n}), \ldots, \pi(P_{-1}), \pi(P_0)) \) is full strong exceptional collection in \( D_{\text{Perf}}(\text{QMod}(\mathcal{A})) \). Further, the category \( D_{\text{Perf}}(\text{QMod}(\mathcal{A})) \) is enhanced and \( \text{End}(\bigoplus_{1-n \leq i \leq 0} \pi(P_i)) = B \). Hence \( D_{\text{Perf}}(\text{QMod}(\mathcal{A})) \) is equivalent to \( D^b(\text{mod}_{\text{finite}}B) \). \( \square \)
Corollary 18.22. The category $D_{Perf}(QMod(A^{m,V}))$ is equivalent to the full triangulated subcategory $T^{m,V} \subset D^b_{coh}(\mathbb{P}(V))$ generated by the exceptional collection 

$$(\mathcal{O}_{\mathbb{P}(V)}(m-n), \ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}) .$$

Under this equivalence the exceptional collection $(\pi(P_{m-n}), \ldots, \pi(P_0))$ corresponds to the exceptional collection $(\mathcal{O}_{\mathbb{P}(V)}(m-n), \ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)})$.

Remark 18.23. Notice that by [Mi] the $\mathbb{Z}$-algebra $A^{dimV-1,V}$ is coherent. Further, the $\mathbb{Z}$-algebra $A^{1,V}$ is Noetherian and hence is coherent. It should be plausible that all the $\mathbb{Z}$-algebras $A^{m,V}$ (and, more generally, all geometric $\mathbb{Z}$-algebras) are coherent, but it is not clear how to prove this statement. However, if $A$ and $B$ are as in Theorem 18.14, and $A$ is coherent, then the subcategory $D_{Perf}(QMod(A)) \subset D^b(QMod(A))$ coincides with the subcategory $D^b_{qmod}(QMod(A))$ which consists of complexes with cohomology lying in $qmod$. This category is further equivalent to $D^b(qmod(A))$. Therefore, in this case we also have an equivalence 

$$D^b(qmod(A)) \cong D^b(mod_{finite,B}).$$

The coherence of geometric $\mathbb{Z}$-algebra $A$ of period $n$ is equivalent to some statement about t-structures. Namely, let $(\tau_{\leq 0}, \tau_{\geq 1})$ be a t-structure on $D^b(Mod_{A^{[1,n]}})$ induced by the equivalence of Theorem 18.14. It can be shown that $A$ is coherent iff the t-structure $(\tau_{\leq 0}, \tau_{\geq 1})$ induces a t-structure on $D^b(mod_{finite,A^{[1,n]}})$.

19. The $k$-points of noncommutative Grassmanians

To discuss the $k$-points of noncommutative Grassmanians defined above we should first relate the following two approaches to noncommutative geometry.

The first one is to think of noncommutative stacks as of $Proj(A)$, where $A$ is a $\mathbb{Z}$-algebra. The special case of graded algebras, i.e. 1-periodic $\mathbb{Z}$-algebras is studied in [M], [V1], [V2], [AZ] and other papers. However, it seems to be more reasonable to consider $\mathbb{Z}$-algebras. Note that our noncommutative Grassmanians are naturally defined as $Proj$ of a $\mathbb{Z}$-algebra but not a graded algebra.

The other approach is to think of a noncommutative stacks as of (equivalence classes of) presheaves of (small) groupoids $X$ on the category $Alg^{op}_k$ opposite to the category of unital associative algebras. Morally the groupoid $X(A)$ should be thought of as the groupoid of maps from the affine noncommutative scheme $Sp(A)$ to $X$. This approach is studied in [Or] in the case of sets (trivial groupoids). In this case we have the category of quasi-coherent sheaves which is not always abelian (it always has cokernels but may not admit kernels), and the structure sheaf.

In the second approach we obviously have the groupoid of $k$-points $X(k)$.

From this moment we assume that the $\mathbb{Z}$-algebra $A$ is positively oriented but not necessarily connected. We will make an attempt to define the presheaf of groupoids of morphisms $Sp(A) \to Proj(A)$, $A \in Alg_k$. First note that a morphism $f : Sp(A) \to Proj(A)$ must give a $k$-linear additive functor $f^* : QMod(A) \to Mod-A$ together with an isomorphism $f^*(\pi(P_0)) \cong A$. Moreover, $f^*$ must commute with colimits.

Notice that if $C$ is a $k$-linear abelian category with infinite direct sums and $\mathcal{F} : A \to C$ is a $k$-linear functor then we have the tensor product functor 

$$- \otimes_A \mathcal{F} : Mod-A \to C$$

given by the formula
Proof. $M \otimes_{\mathcal{A}} \mathcal{F} = \text{Coker}(b; M \otimes \mathcal{A} \otimes \mathcal{F} \to M \otimes \mathcal{F})$,
where $b = \mu_{M} \otimes 1_{\mathcal{A}} - 1_{M} \otimes \mu_{\mathcal{F}}$ (we identify $\mathcal{F}$ with $\bigoplus_{i \in \mathbb{Z}} \mathcal{F}(i) \in \mathcal{C}$). Clearly, the functor $- \otimes_{\mathcal{A}} \mathcal{F}$ commutes with colimits. We denote by $\text{Tor}^A(-, \mathcal{F})$ its left derived functors.

**Definition 19.1.** Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a $k$-linear abelian category with infinite direct sums together with a distinguished object $Y \in \mathcal{C}$.

We denote by $G_{1}(\mathcal{A}, \mathcal{C}, Y)$ the groupoid of pairs $(f^{*}, \theta)$, where $f^{*}: \text{QMod}(\mathcal{A}) \to \mathcal{C}$ is a $k$-linear functor commuting with colimits and $\theta: f^{*}(\pi(P_{0})) \to Y$ is an isomorphism.

We denote by $G_{2}(\mathcal{A}, \mathcal{C}, Y)$ the groupoid of pairs $(\mathcal{F}, \sigma)$, where $\mathcal{F}: \mathcal{A} \to \mathcal{C}$ is a $k$-linear functor such that $\text{Tor}^{A}(T, \mathcal{F}) = \text{Tor}^{A}(T, \mathcal{F}) = 0$ for each torsion $\mathcal{A}^{\text{op}}$-module $T$, and $\sigma: \mathcal{F}(0) \to Y$ is an isomorphism.

**Theorem 19.2.** Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a $k$-linear abelian category with infinite direct sums together with a distinguished object $Y \in \mathcal{C}$. Then the groupoids $G_{1}(\mathcal{A}, \mathcal{C}, Y)$ and $G_{2}(\mathcal{A}, \mathcal{C}, Y)$ are equivalent.

**Proof.** We define the functor $\Phi: G_{1}(\mathcal{A}, \mathcal{C}, Y) \to G_{2}(\mathcal{A}, \mathcal{C}, Y)$ as follows. Let $(f^{*}, \theta) \in G_{1}(\mathcal{A}, \mathcal{C}, Y)$. The functor $\Phi(f^{*}): \mathcal{A} \to \mathcal{C}$ is defined by the formulas

$$\Phi(f^{*})(i) = f^{*}(\pi(P_{i})).$$

and for $x \in \mathcal{A}_{ij}$

$$\Phi(f^{*})(x) = f^{*}(\pi(x)).$$

We claim that the pair $(\Phi(f^{*}), \theta)$ is an object of $G_{2}(\mathcal{A}, \mathcal{C}, Y)$. Indeed, let $T \in \text{Tors}(\mathcal{A})$. Since the sequence

$$\pi(T \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(T \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(T \otimes \mathcal{A}) \to 0$$

is exact in $\text{QMod}(\mathcal{A})$ it follows that the sequence

$$T \otimes \mathcal{A} \otimes \mathcal{A} \otimes \Phi(f^{*})(\mathcal{A}) \to T \otimes \mathcal{A} \otimes \Phi(f^{*})(\mathcal{A}) \to T \otimes \Phi(f^{*})(\mathcal{A}) \to 0$$

is exact in $\mathcal{C}$, i.e.

$$\text{Tor}^{A}(S_{i}, \Phi(f^{*})) = \text{Tor}^{A}(S_{i}, \Phi(f^{*})) = 0.$$

Thus, the functor $\Phi$ is defined on objects. It obviously extends to morphisms.

Now we define the functor $\Psi: G_{2}(\mathcal{A}, \mathcal{C}, Y) \to G_{1}(\mathcal{A}, \mathcal{C}, Y)$ as follows. Let $(\mathcal{F}, \sigma) \in G_{2}(\mathcal{A}, \mathcal{C}, Y)$. We claim that the formula

$$\Psi(\mathcal{F})(\pi(M)) = M \otimes_{\mathcal{A}} \mathcal{F}$$

well defines a functor $\Psi(\mathcal{F}): \text{QMod}(\mathcal{A}) \to \mathcal{C}$ which is right exact and commutes with infinite direct sums. Indeed it follows from the condition

$$\text{Tor}^{A}(T, \mathcal{F}) = \text{Tor}^{A}(T, \mathcal{F}) = 0$$

for torsion $\mathcal{A}^{\text{op}}$-modules $T$.

Hence, the pair $(\Psi(\mathcal{F}), \sigma)$ is an object of $G_{1}(\mathcal{A}, \mathcal{C}, Y)$. This defines the functor $\Psi$ on objects and it obviously extends to morphisms.
It is clear that the composition $\Phi \circ \Psi$ is isomorphic to the identity functor. To see this for the composition $\Psi \circ \Phi$, it remains to note that each functor $f^*$ from the pair in $\mathcal{G}_i(A, C, Y)$ can be reconstructed from the functor $\Phi(f^*)$ using exact sequences

$$\pi(M \otimes A \otimes A) \to \pi(M \otimes A) \to \pi(M) \to 0.$$  

\[\square\]

Notice that it follows from the above theorem that each functor $f^*: \text{QMod}(A) \to \mathcal{C}$ commuting with direct sums and right exact has the right adjoint $f_*: \mathcal{C} \to \text{QMod}(A)$ given by the formula $f_*(X) = \pi(f_*(X))$

$f_*(X)(i) = \text{Hom}_\mathcal{C}(f^*(\pi(P_i)), X),$

and for $\phi \in \text{Hom}_\mathcal{C}(f^*(\pi(P_j)), X)$, $x \in A_{ij}$

$$\phi \cdot x = \phi \cdot f^*(\pi(x)).$$

Indeed, this follows from the formula

$$f^*(\pi(M)) = M \otimes_A \mathcal{F}.$$

It is clear that $X_i(A) = \langle A \mapsto \mathcal{G}_i(A, \text{Mod-A}, A) \rangle$, $i = 1, 2$, are presheaves of groupoids on the category $\text{Alg}_k^{op}$, and the equivalence from the above theorem extends to the equivalence of these presheaves.

However, not all functors $f^*$ commuting with colimits should come from true morphisms $f: \text{Sp}(A) \to \text{Proj}(A)$. Although a true presheaf of groupoids should be defined as a full (small) subpresheaf of $X_2(A)$. We are going to make an attempt in this direction. Our motivation is the following Proposition.

**Proposition 19.3.** Let $A$ be a $\mathcal{Z}$-algebra. Further, let $\mathcal{C}$ be a $k$-linear abelian category with infinite direct sums and with the distinguished object $Y$. Let $(f^*, \theta) \in \mathcal{G}_i(A, C, Y)$ and $(F, \sigma) \in \mathcal{G}_2(A, C, Y)$ be objects which correspond to each other under the equivalence of Theorem 19.2. The following conditions are equivalent:

(i) there exists a left derived functor $L^f^*: D^-(\text{QMod}(A)) \to D^-(\mathcal{C})$, and $L^if^*(\pi(P_j)) = 0$ for $i \neq 0$ and all $j$;

(ii) we have $\text{Tor}^A_i(T, \mathcal{F}) = 0$ for all $i \geq 0$, $T \in \text{Tors}(A)$.

**Proof.** Prove that (i) implies (ii). We have that the functor $f^* \cong - \otimes \mathcal{F}$ maps acyclic right bounded complexes of direct sums of $\pi(P_i)$ to acyclic complexes. Applying this to the projection of the free resolution of a torsion module $T$, we obtain that $\text{Tor}^A_i(T, \mathcal{F}) = 0$ for $i \geq 0$.

Prove that (ii) implies (i). Since each object in $\text{QMod}(A)$ can be covered by a direct sum of $\pi(P_i)$, it suffices to prove that $f^*$ maps right bounded acyclic complexes of direct sums of $\pi(P_i)$ to acyclic complexes.

Since the kernel and the cokernel of the morphism $P_j \to \omega(\pi(P_j))$ are torsion, we have $\text{Tor}^A_i(\omega(\pi(P_j)), \mathcal{F}) = 0$ for $i > 0$. Further, if $K^*$ is a right bounded acyclic complex of direct sums of $\pi(P_i)$ then $\omega(K^*)$ has torsion cohomology. Therefore,

$$f^*(K^*) = \omega(K^*) \otimes_A \mathcal{F} = \omega(K^*) \otimes_A \mathcal{F},$$

and the last complex is acyclic since $\text{Tor}^A_i(T, \mathcal{F}) = 0$ for $i \geq 0$, $T \in \text{Tors}(A)$.

For each $\mathcal{Z}$-algebra $A$ we define the presheaf $X_A$ of groupoids on the category $\text{Alg}_k^{op}$ as follows. It is a full subpresheaf of $X_2(A)$ and the groupoid $X_A(A) \subset X_2(A)(A)$ consists of pairs $(F, \sigma) \in X_2(A)(A)$ such that:

1) we have $\text{Tor}^A_i(T, \mathcal{F}) = 0$ for all $i$ and $T \in \text{Tors}(A)$;

2) the $A^{op}$-modules $\mathcal{F}(i)$ are flat.
It is clear that \( X_A \) is indeed a subpresheaf of \( X_2(A) \). For \( f = (\mathcal{F}, \sigma) \in X_A(A) \) we denote by \( f^* : \text{QMod}(A) \to \text{Mod-}A \) the corresponding functor \(- \otimes_A \mathcal{F} \). We also regard the objects \( f \in X_A(A) \) as maps from \( Sp(A) \) to \( \text{Proj}(A) \), where \( Sp(A) \) is a noncommutative affine scheme corresponding to \( A \).

The following Lemma simplifies the complicated condition on \( \text{Tor}_i \). Recall the torsion \( \mathcal{A}^{\text{op}} \)-modules \( T_{p,q} \) from Section 16.

**Lemma 19.4.** Let \( \mathcal{A} \) be a positively oriented connected \( \mathbb{Z} \)-algebra and let \( \mathcal{C} \) be a \( k \)-linear abelian category with infinite direct sums. Let \( \mathcal{F} : \mathcal{A} \to \mathcal{C} \) be a \( k \)-linear functor.

Suppose that \( \text{Tor}_i^\mathcal{A}(T_{j,j}, \mathcal{F}) = 0 \) for all \( i \) and \( j \). Then \( \text{Tor}_i^\mathcal{A}(T, \mathcal{F}) = 0 \) for all \( i \) and all torsion \( \mathcal{A}^{\text{op}} \)-modules \( T \).

**Proof.** First note that if \( T = T1_j \), then \( T \) has a left resolution by direct sums of \( T_{j,j} \). Hence, Lemma holds for such \( T \).

Further, the torsion modules \( T_{p,q} \), \( p \leq q \), have finite filtrations with subquotients \( T_m \) such that \( T_m = T_m1_m \), \( p \leq m \leq q \). Hence \( \text{Tor}_i^\mathcal{A}(T_{p,q}, \mathcal{F}) = 0 \) for all \( i \), and \( p \leq q \). Now Lemma follows from the observation that each torsion module has a left resolution by direct sums of modules \( T_{pq} \).

**Definition 19.5.** We say that a positively oriented \( \mathbb{Z} \)-algebra \( \mathcal{A} \) satisfies the condition (***) if the following holds:

(i) the algebra \( \mathcal{A} \) is generated by its subspaces \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \);

(ii) for each \( i \in \mathbb{Z} \), the object \( \pi(P_i) \) has a finite right resolution by direct sums of \( \pi(P_j) \) with \( j > i \).

The next Proposition motivates the condition (***)

**Proposition 19.6.** Let \( \mathcal{A} \) be a \( \mathbb{Z} \)-algebra satisfying (***) and \( A \in \text{Alg}_k \). Then for each \( f = (\mathcal{F}, \sigma) \in X_A(A) \) we have \( \text{Aut}(f) = \{1\} \).

**Proof.** Let \( g \in \text{Aut}(f) \). Clearly, \( g(0) : \mathcal{F}(0) \to \mathcal{F}(0) \) is the identity morphism. Further, for each \( i \in \mathbb{Z} \) the surjection \( \mathcal{A}_{i,i+1} \otimes \pi(P_i) \to \pi(P_{i+1}) \) is mapped by \( f^* \) to the surjection \( \mathcal{A}_{i,i+1} \otimes \mathcal{F}(i) \to \mathcal{F}(i+1) \). Hence, we obtain by increasing induction over \( i \) that \( g(i) : \mathcal{F}(i) \to \mathcal{F}(i) \) is the identity for \( i \geq 0 \).

Finally, since \( \mathcal{A} \) satisfies (***) it follows from Proposition 19.3 that there exists an injection of the form \( \pi(P_i) \to \bigoplus \pi(P_{j_\alpha}) \) with \( j_\alpha > i \) which is mapped by \( f^* \) to the injection \( \mathcal{F}(i) \to \bigoplus \mathcal{F}(j_\alpha) \). Hence, we obtain by decreasing induction on \( i \) that \( g(i) : \mathcal{F}(i) \to \mathcal{F}(i) \) is the identity for all \( i \in \mathbb{Z} \).

Therefore, if \( \mathcal{A} \) satisfies (***) we may and will replace \( X_A \) by the equivalent presheaf of trivial groupoids \( \pi_0(X_A) \). It is easily seen from the proof of the above Proposition that \( \pi_0(X_A(A)) \) is a set. Thus, \( X_A \) is a presheaf of sets.

Now we would like to compare our definition of morphisms from \( Sp(A) \) to \( \text{Proj}(\mathcal{A}) \) with the morphisms from commutative Noetherian \( k \)-schemes to commutative projective \( k \)-schemes.

Note that we can restrict the presheaf \( X_A \) to the full subcategory of \( \text{Alg}_k \) which consists of commutative Noetherian \( k \)-algebras. Further, we can extend this restricted presheaf onto the category \( \text{Noeth}_k \) of all commutative Noetherian \( k \)-schemes.

**Definition 19.7.** Let \( \mathcal{A} \) be a positively oriented \( \mathbb{Z} \)-algebra. We define the presheaf \( X_{\text{true}(\mathcal{A})} : \text{Noeth}_k^{\text{op}} \to \text{Gpd} \) as follows. The groupoid \( X_{\mathcal{A}}(Y) \) is a full sub-groupoid of \( G_2(\mathcal{A}, \text{QCoh}(Y), \mathcal{O}_Y) \) which consists of objects \( (\mathcal{F}, \sigma) \) such that the following conditions hold:
Proof. Noeth\textsubscript{k}.

We also regard the objects of the groupoid $X_\mathcal{A}(Y)$ as maps from $Y$ to $\text{Proj}(\mathcal{A})$. The analogue of Proposition 19.6 obviously holds for Noetherian $k$-schemes instead of associative algebras. For each commutative Noetherian $k$-scheme $Y$ we denote by $Y^\vee : \text{Noeth}\textsuperscript{op}_k \to \text{Set}$ the presheaf of sets represented by $Y$.

Now let $Z \in \mathbb{P}(V)$ be a closed subscheme and let $\mathcal{A}$ be a $Z$-algebra associated to its homogeneous coordinate ring $\bigoplus_{d \geq 0} S^dV^*/I$.

**Proposition 19.8.** The $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition (**). The presheaves of sets $Z^\vee$ and $X_\mathcal{A} : \text{Noeth}\textsuperscript{op}_k \to \text{Set}$ on the category $\text{Noeth}\textsuperscript{op}_k$ are isomorphic.

**Proof.** Recall that the category $\text{QCoh}(Z)$ is equivalent to $\text{Proj}(\mathcal{A})$ by Serre Theorem. The sheaves $\mathcal{O}_Z(i)$ correspond under this equivalence to $\pi(P_i)$.

Let $f : Y \to Z$ be a morphism. Then the sheaves $f^*(\mathcal{O}_Z(i))$ are invertible and hence are locally flat. Further, $f^*$ maps acyclic right bounded complexes of direct sums of $\mathcal{O}_Z(i)$ to acyclic complexes. Finally, we have an isomorphism $f^*(\mathcal{O}_Z) \cong \mathcal{O}_Y$. Thus, we have a morphism of presheaves $Z^\vee \to X_\mathcal{A}$.

Conversely, let $Y \in \text{Noeth}_k$ and $g \in X_\mathcal{A}(Y)$. Notice that for each $i \in \mathbb{Z}$ we have an acyclic Koszul complex on $\mathbb{P}(V)$ twisted by $\mathcal{O}_{\mathbb{P}(V)}(i)$, and we can restrict it to $Z$:

$$0 \to \mathcal{O}_Z(i) = \Lambda^nV^* \otimes \mathcal{O}_Z(i) \to \Lambda^{n-1}V^* \otimes \mathcal{O}_Z(i+1) \to \cdots \to \mathcal{O}_Z(i+n) \to 0.$$ 

In particular, the $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition (**).

Hence, we have acyclic complexes

$$(19.1) \quad 0 \to g^*(\pi(P_1)) = \Lambda^nV^* \otimes g^*(\pi(P_1)) \to \Lambda^{n-1}V^* \otimes g^*(\pi(P_{i+1})) \to \cdots \to g^*(\pi(P_{i+n})) \to 0.$$ 

In particular, we have surjections $V^* \otimes g^*(\pi(P_1)) \to g^*(\pi(P_{i+1}))$ and injections $g^*(\pi(P_i)) \to V \otimes g^*(\pi(P_{i+1}))$.

Since $g^*(\pi(P_0)) \cong \mathcal{O}_Y$, we obtain by increasing and decreasing inductions on $i$ that all the sheaves $g^*(\pi(P_i))$ are coherent and are non-zero on each connected component of $Y$. Since they are locally flat, they are locally free.

Further, put $\mathcal{L} = g^*(\pi(P_1))$. Clearly, $g$ can be reconstructed from the surjective morphism $\phi : V^* \otimes A \to \mathcal{L}$ using the exact sequences

$$\Lambda^2V^* \otimes g^*(\pi(P_{i-1})) \to V^* \otimes g^*(\pi(P_i)) \to g^*(\pi(P_{i+1})) \to 0$$

and

$$0 \to g^*(\pi(P_{i-1})) \to V \otimes g^*(\pi(P_i)) \to \Lambda^2V \otimes g^*(\pi(P_{i+1}))$$

from complexes (19.1). Suppose that $\text{rank}(\mathcal{L}|_{Y_0}) \geq 2$ on some connected component $Y_0 \subset Y$. Then it is easy to see that the morphism $\Lambda^{n-1}V^* \otimes \mathcal{O}_{Y_0} \to \Lambda^{n-2}V^* \otimes \mathcal{L}|_{Y_0}$ is injective. Hence $g^*(\pi(P-1))|_{Y_0} = 0$, a contradiction. Thus, $\mathcal{L}$ is an invertible sheaf.

The surjective morphism $\phi$ above defines a morphism $\tilde{g} : Y \to \mathbb{P}(V)$. It follows that $\tilde{g}^* \cong g^*\iota^*$, where $\iota : Z \to \mathbb{P}(V)$ is the embedding. Hence, $g^*(\pi(P_i)) \cong \mathcal{L}^{\otimes i}$. Further, the induced morphism of graded algebras

$$\bigoplus_{d \geq 0} S^dV^* \to \bigoplus_{i \geq 0} H^0(\mathcal{L}^{\otimes i})$$

passes through $\bigoplus_{d \geq 0} S^dV^*/I$, thus $\tilde{g}$ passes through $Z$, and we obtain a morphism $Y \to Z$. Hence we have a morphism of presheaves $X_\mathcal{A} \to Z^\vee$. 


The constructed morphisms of presheaves are inverse to each other. Proposition is proved. □

Now we want to describe the \( k \)-points of noncommutative Grassmanians.

**Lemma 19.9.** Let \( \mathcal{A} \) be a geometric \( \mathbb{Z} \)-algebra of period \( n \). Then it satisfies the condition (**).

**Proof.** By definition, \( \text{Alg}_k \mathcal{A} \) is generated by \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \). Further, the projections \( \pi(K) \) of Koszul complexes are acyclic. The first non-zero term of \( \pi(K_{i+n}) \) equals to \( \pi(P_i) \) (since \( \mathcal{A}^i \) is Frobenious). Therefore, each \( \pi(P_i) \) has the required right resolution. □

Denote by \( \text{pr}^{m,V}_r : D^b_{\text{coh}}(\mathbb{P}(V)) \to T^{m,V} \) the functor which is right adjoint to the inclusion \( \iota : T^{m,V} \to D^b_{\text{coh}}(\mathbb{P}(V)) \). The next theorem describes the \( k \)-points of noncommutative Grassmanians and the objects in \( T^{m,V} \) corresponding to their structure sheaves under the equivalence of Corollary 18.22.

**Theorem 19.10.** The \( k \)-points of the noncommutative Grassmanian \( \text{NGr}(m,V) \), i.e. the elements \( f \in X_{\mathcal{A}^m,V}(k) \), naturally correspond to vector subspaces \( W \subset V \) of dimension \( 1 \leq \dim W \leq m \). Further, if \( f \) corresponds to \( W \) then \( f_i(k) \) is a perfect complex, and it corresponds to the object \( \text{pr}^{m,V}_r(\mathcal{O}_W) \in T^{m,V} \) under the equivalence \( \text{Derf}(\text{QMod}(\mathcal{A})) \cong T^{m,V} \).

**Proof.** Let \( f \in X_{\mathcal{A}}(k) \). For each \( i \in \mathbb{Z} \) we have the natural acyclic complex

\[
0 \to f^*(\mathcal{O}(P_i)) \cong \mathcal{A}^{m,V}_{i+n+1} \otimes f^*(\mathcal{O}(P_i)) \to \mathcal{A}^{m,V}_{i+n, i+1} \otimes f^*(\mathcal{O}(P_{i+1})) \to \cdots \to f^*(\mathcal{O}(P_{n})) \to 0.
\]

In particular, we have surjections \( V^* \otimes f^*(\mathcal{O}(P_i)) \to f^*(\mathcal{O}(P_{i+1})) \) and injections \( f^*(\mathcal{O}(P_i)) \to V \otimes f^*(\mathcal{O}(P_{i+1})) \). Since \( f^*(\mathcal{O}(P_0)) \cong k \), we obtain by increasing and decreasing inductions over \( i \) that all the spaces \( f^*(\mathcal{O}(P_i)) \) are non-zero and finite-dimensional. Further, using the exact sequences

\[
\Lambda^2 V^* \otimes f^*(\mathcal{O}(P_{i-1})) \to V^* \otimes f^*(\mathcal{O}(P_i)) \to f^*(\mathcal{O}(P_{i+1})) \to 0
\]

and

\[
0 \to f^*(\mathcal{O}(P_{i+1})) \to V \otimes f^*(\mathcal{O}(P_i)) \to \Lambda^2 V \otimes f^*(\mathcal{O}(P_{i+1})),
\]

one can reconstruct \( f \) from the injection \( f^*(\mathcal{O}(P_{i+1})) \to V \otimes f^*(\mathcal{O}(P_0)) \cong V \). Thus, we can associate a non-zero vector subspace \( W \subset V \) to each \( f \in X_{\mathcal{A}}(k) \) and \( f \) can be reconstructed from the subspace \( W \). We will show that \( W \subset V \) gives a \( k \)-point iff \( 1 \leq \dim W \leq m \).

First suppose that \( \dim W > m \) and \( W \) gives a \( k \)-point \( f \). Then

\[
f^*(\mathcal{O}(P_1)) = \text{Coker}(\Lambda^{n-m-1}V \otimes W \to \Lambda^{n-m}V),
\]

and the last space is zero since \( \dim W > m \). But \( f^*(\mathcal{O}(P_1)) \neq 0 \), a contradiction.

Now let \( 1 \leq \dim W = d \leq m \). Let \( S = \{E_i\} \) be a geometric helix in \( T^{m,V} \) of period \( n-m-1 \) such that \( E_j = \mathcal{O}_{P(V)}(j) \) for \( m-n \leq j \leq 0 \). Then the endomorphism \( \mathbb{Z} \)-algebra of \( S \) is equivalent to \( \mathcal{A}^{m,V} \). We define the functor \( F : \mathcal{A} \to \text{k-Vect} \) by the formula

\[
F(i) = \text{Hom}(E_i, \mathcal{O}_{P(V)}) = \text{Hom}(E_i, \text{pr}^{m,V}_r(\mathcal{O}_{P(V)})) \text{^\vee}.
\]

We put \( f = (F, \text{id}) \).

Now we prove that \( f \in X_{\mathcal{A}^m,V}(k) \). By Lemma 19.4, it suffices to show that \( \text{Tor}(S_j, F) = 0 \) for \( i > 0 \), \( j \in \mathbb{Z} \). Since the complexes

\[
0 \to E_i \cong \mathcal{A}^{m,V}_{i+n, i} \otimes E_i \to \mathcal{A}^{m,V}_{i+n, i+1} \otimes E_{i+1} \to \cdots \to E_{i+n} \to 0
\]
of objects in $D^b_{\text{coh}}(\mathbb{P}(V))$ have zero convolutions, it suffices to prove that

$$\text{Hom}^i(E_j, pr^m_{r}(\mathcal{O}_{\mathbb{P}(W)})) = \text{Hom}^i(E_i, \mathcal{O}_{\mathbb{P}(W)}) = 0$$

for $i \neq 0$, $j \in \mathbb{Z}$. It is clear that this holds for $m - n \leq j \leq 0$. Further, we have a Koszul resolution of the sheaf $\mathcal{O}_{\mathbb{P}(W)}$ on $\mathbb{P}(V)$:

$$0 \to \Lambda^{n-d}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(d-n) \to \cdots \to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)}.$$ 

Thus, $pr^m_{r}(\mathcal{O}_{\mathbb{P}(W)})$ is isomorphic to the complex

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n) \to \cdots \to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0 \cdots.$$ 

Since the helix $\{E_i\}$ is geometric, we have $\text{Hom}^i(E_j, pr^m_{r}(\mathcal{O}_{\mathbb{P}(W)})) = 0$ for $i > 0$, $j \leq 0$.

Recall that $E_{j+n-m+1} = \Phi^{-1}(E_j)$, where $\Phi = F|m-n|$, and $F$ is a Serre functor on $T^m,V$. We have that

$$\text{(19.2)} \quad F^{-1}(K) \cong pr^m_{r}(K \otimes \mathcal{O}_{\mathbb{P}(V)}(n)[1-n]),$$

where $pr^m_{r} : D^b_{\text{coh}}(\mathbb{P}(V)) \to T^m,V$ is the functor which is left adjoint to the inclusion $\iota : T^m,V \to D^b_{\text{coh}}(\mathbb{P}(V))$.

**Lemma 19.11.** The functor $pr^m_{r} : D^b_{\text{coh}}(\mathbb{P}(V)) \to T^m,V$ maps $\text{Ob}(D^b_{\text{coh}}(\mathbb{P}(V)))$ to $\text{Ob}(D^b_{\text{coh}}(\mathbb{P}(V)) \cap T^m,V)$. The functor $\Phi^{-1}$ preserves $\text{Ob}(D^b_{\text{coh}}(\mathbb{P}(V)) \cap T^m,V)$.

**Proof.** The second statement follows from the first one by the isomorphism $(19.2)$. To prove the first statement, it suffices to note that

$$\text{pr}^m_{r}(\mathcal{O}(i)) = L\mathcal{O}_{\mathbb{P}(V)}(1) \cdots L\mathcal{O}_{\mathbb{P}(V)}(m-1)(X)[m-1].$$

Since $\text{pr}^m_{r}(\mathcal{O}(i)) = 0$ for $i = 1, \ldots, m-1$, we have

$$\Phi^{-1}(\text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)})) = \text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)}) \otimes \mathcal{O}_{\mathbb{P}(V)}[1-m] = \text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)})(n)[1-m],$$

and the last object belongs to $\text{Ob}(D^b_{\text{coh}}(\mathbb{P}(V)) \cap T^m,V)$ by Lemma 19.11. Again by Lemma 19.11 we have that $\Phi^{-1}(\text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)}))$ lies in $\text{Ob}(D^b_{\text{coh}}(\mathbb{P}(V)) \cap T^m,V)$ for $l > 0$. Thus, we have

$$\text{Hom}^i(E_{j-(n-m+1)k}, \text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)})) = \text{Hom}^i(E_j, \Phi^{-k}(\text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)}))) = 0$$

for $i < 0$, $n - m \leq j \leq 0$, and $k > 0$. Therefore,

$$\text{Hom}^i(E_j, \text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)})) = 0$$

for $i \neq 0$, $j \leq 0$.

To prove the same for $j > 0$, note that $\text{Hom}^i(E_j, \text{pr}^m_{r}(\mathcal{O}_{\mathbb{P}(W)}))$ is the $i$-th cohomology of the complex

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \text{Hom}^{n-m}(E_j, \mathcal{O}_{\mathbb{P}(V)}(m-n)) \to \cdots$$

$$\to (V/W)^* \otimes \text{Hom}^{n-m}(E_j, \mathcal{O}_{\mathbb{P}(V)}(-1)) \to \text{Hom}^{n-m}(E_j, \mathcal{O}_{\mathbb{P}(V)}) \to 0 \cdots,$$

where the left non-zero term is in degree zero. This complex is dual to the complex

$$\cdots \to 0 \to \text{Hom}(\mathcal{O}_{\mathbb{P}(V)}, E_{j-n+m-1}) \to (V/W) \otimes \text{Hom}(\mathcal{O}_{\mathbb{P}(V)}(-1), E_{j-n+m-1}) \to \cdots$$

$$\to \Lambda^{n-m}(V/W) \otimes \text{Hom}(\mathcal{O}_{\mathbb{P}(V)}(m-n), E_{j-n+m-1}) \to 0 \to \cdots,$$
Proposition 19.12. Let $V$ be a finite-dimensional vector space and let $1 \leq d \leq m \leq \dim V = n$. Then there exists a natural morphism $f_{d,m,V} : \text{Gr}(d,V) \to \text{NGr}(m,V)$ such that the derived inverse image functor $L f_{d,m,V}^*$ induces a full embedding

$$\text{Perf} \big( \text{NGr}(m,V) \big) \to D^b_{\text{coh}} \big( \text{Gr}(d,V) \big).$$

Proof. For each $W \in \text{Gr}(d,V)$ denote by $f_W$ the corresponding $k$-point of $\text{NGr}(m,V)$. It is clear that there exist vector bundles $\mathcal{F}(i)$ on $\text{Gr}(d,V)$ such that the fiber of $\mathcal{F}(i)$ over the point corresponding to $W$ is naturally identified with $f_W^*(\pi(P_i))$ (in particular, $\mathcal{F}(-1)$ is a tautological bundle). So we have a natural functor $\mathcal{F} : \mathcal{A} \to \text{QCoh} \big( \text{Gr}(m,V) \big)$. Also by Theorem 19.10 we have that the complexes of vector bundles

$$0 \to \mathcal{F}(i) \cong \mathcal{A}_{i+n,V}^{m,V} \otimes \mathcal{F}(i) \to \mathcal{A}_{i+n+1,V}^{m,V} \otimes \mathcal{F}(i+1) \to \cdots \to \mathcal{F}(i+n) \to 0$$

are acyclic in the fibers over closed points (if the residue field of a point is greater than $k$ we can make an extension of scalars). Hence, these complexes are acyclic themselves. It follows from Lemma 19.4 that the pair $(\mathcal{F}, id)$ defines a map $f_{d,m,V} : \text{Gr}(d,V) \to \text{NGr}(m,V)$.

Further, for $m-n \leq j \leq 0$ we have that $L f_{d,m,V}^*(\pi(P_j)) = f_{d,m,V}^*(\pi(P_j)) = S^{-j}E$, where $E$ is a tautological bundle. The collection $(S^{m-n}E, \ldots, E, \mathcal{O}_{\text{Gr}(d,V)})$ is a sub-collection of the full strong exceptional collection on $\text{Gr}(d,V)$ constructed by Kapranov [Ka]. Moreover, the functor $L f_{d,m,V}^*$ induces isomorphisms

$$\text{Hom}(\pi(P_j), \pi(P_j)) \to \text{Hom}(S^{-j}E, S^{-j}E).$$
for $m - n \leq i \leq j \leq 0$. Thus, the induced functor

$$L_{f_{d,m,V}^*} : \text{Perf}(\text{NGr}(m, V)) \to D^b_{\text{coh}}(\text{Gr}(d, V))$$

is a full embedding. □

Notice that the full embedding $L_{f_{1,m,V}^*} : \text{Perf}(\text{NGr}(m, V)) \to D^b_{\text{coh}}(\mathbb{P}(V))$ coincides with the composition of the equivalence of Corollary 18.22 with the tautological embedding $T^{m,V} \hookrightarrow D^b_{\text{coh}}(\mathbb{P}(V))$.

References


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