DEFORMATIONS OF OBJECTS IN DERIVED CATEGORIES AND NONCOMMUTATIVE GRASSMANIANS

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Abstract. This paper is the part of our joint papers with D.O. Orlov and V.A. Lunts [ELOI], [ELOII], [ELOIII]. All results in this paper are obtained by the author.

In [ELOI] we developed deformation theory of objects in homotopy and derived categories of DG categories. In [ELOII] we extended the (derived) deformation functors of an object E to an appropriate bicategory of artinian DG algebras. In this paper we prove (under some assumptions) that these extended functors are pro-representable in a strong sense by complete pro-artinian DG algebra. It is defined in terms of the minimal A_∞-structure on $\mathrm{Ext}(E,E)$. One of the main ingredients in the proof is Maurer-Cartan pseudo-functor for A_∞-algebras. Moreover, using A_∞-methods, we obtain some results on comparison of the homotopy and derived deformation functors.

We also construct noncommutative Grassmanians as examples of noncommutative moduli spaces of objects in derived categories. The derived categories of quasi-coherent sheaves on these noncommutative Grassmanians are described. Further, we make an attempt to relate two different approaches to noncommutative algebraic geometry. After that, we describe the k-points of noncommutative Grassmanians.

Contents

1. Introduction 2

Part 1. A_∞-structures and the bar complex 4

2. Coalgebras 4
2.1. Coalgebras and comodules 4
2.2. From comodules to modules 4
3. Preliminaries on A_∞-algebras, A_∞-categories and A_∞-modules 5
3.1. A_∞-algebras and A_∞-modules 5
3.2. Strictly unital A_∞-algebras 7
3.3. Minimal models of A_∞-algebras. 8
3.4. Perfect A_∞-modules and A_∞-bimodules. 9
3.5. A_∞-categories 9
3.6. The tensor product of an A_∞-algebra and a DG algebra 9
3.7. The category of $A_{\mathcal{C}}$-modules for an A_∞-algebra A and a DG algebra $\mathcal{C}$ 10
3.8. The bar complex 12
4. Admissible A_∞-algebras. 12

Part 2. Maurer-Cartan pseudo-functor for A_∞-algebras 13

5. The definition 13
6. Obstruction theory 15

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7. Invariance Theorems 17
8. Twisting cochains 20

Part 3. The pseudo-functors DEF **and** coDEF 22
9. The bicategory 2-adgalg and deformation pseudo-functor coDEF 22
10. Deformation pseudo-functor coDEF for an augmented A_∞-algebra 26
11. The bicategory $2'$-adgalg and deformation pseudo-functor DEF 27
12. Deformation pseudo-functor DEF for an augmented A_∞-algebra 30
13. Comparison of pseudo-functors coDEF_- and DEF_- 31

Part 4. Pro-representability theorems 33
14. Pro-representability of the pseudo-functor coDEF_- 33
15. Pro-representability of the pseudo-functor DEF_- 35
15.1. Explicit equivalence $\mathrm{DEF}_-(E) \cong h'_{\hat{S}}$ 38

Part 5. Noncommutative Grassmanians 39
16. Preliminaries on $\mathbb{Z}$-algebras 39
17. The definition of noncommutative Grassmanians 42
18. The derived categories of noncommutative Grassmanians 43
19. The k-points of noncommutative Grassmanians 48
References 56

1. Introduction

In the paper [ELOI] we developed a general deformation theory of objects in homotopy and derived categories of DG categories. The corresponding deformation pseudo-functors are defined on the category of artinian DG algebras dgart and take values in the 2-category **Gpd** of groupoids. More precisely if $\mathcal{A}$ is a DG category and E is a right DG module over $\mathcal{A}$ we defined four pseudo-functors

$$\mathrm{Def}^{\mathrm{h}}(E), \mathrm{coDef}^{\mathrm{h}}(E), \mathrm{Def}(E), \mathrm{coDef}(E) : \mathrm{dgart} \to \mathbf{Gpd}.$$

The first two are the *homotopy* deformation and co-deformation pseudo-functors, i.e. they describe deformations (and co-deformations) of E in the homotopy category of DG $\mathcal{A}^{op}$-modules; and the last two are their *derived* analogues. The pseudo-functors $\mathrm{Def}^{\mathrm{h}}(E)$, $\mathrm{coDef}^{\mathrm{h}}(E)$ are equivalent and depend only on the quasi-isomorphism class of the DG algebra $\mathrm{End}(E)$. The derived pseudo-functors $\mathrm{Def}(E)$, $\mathrm{coDef}(E)$ need some boundedness conditions to give the "right" answer and in that case they are equivalent to $\mathrm{Def}^{\mathrm{h}}(F)$ and $\mathrm{coDef}^{\mathrm{h}}(F)$ respectively for an appropriately chosen h-projective or h-injective DG module F which is quasi-isomorphic to E (one also needs to restrict the pseudo-functors to the category dgart_- of negative artinian DG algebras).

In this paper we prove pro-representability of these pseudo-functors. Recall that "classically" one defines representability only for functors with values in the category of sets (since the collection of morphisms between two objects in a category is a set). For example, given a moduli problem in the form of a pseudo-functor with values in the 2-category of goupoids one then composes it with the functor π_0 to get a set valued functor,

which one then tries to (pro-) represent. This is certainly a loss of information. But in order to represent the original pseudo-functor one needs the source category to be a bicategory.

It turns out that there is a natural bicategory $2\text{-}\mathrm{adgalg}$ of augmented DG algebras. It is defined in [ELOII]. (Actually we consider two versions of this bicategory, $2\text{-}\mathrm{adgalg}$ and $2'\text{-}\mathrm{adgalg}$, but then show that they are equivalent). The derived deformation functors can be naturally extended to pseudo-functors

$$\mathrm{coDEF}(E) : 2\text{-}\mathrm{dgart} \to \mathbf{Gpd}, \quad \mathrm{DEF}(E) : 2'\text{-}\mathrm{dgart} \to \mathbf{Gpd}.$$

Then (under some finiteness conditions on the graded algebra $\mathrm{Ext}(E,E) = H(\mathcal{C})$, where $\mathcal{C} = \mathbf{R}\,\mathrm{Hom}(E,E)$), we prove pro-representability of these pseudo-functors restricted to negative artinian DG algebras. The pro-representing DG algebra $\hat{S} = (B\bar{A})^*$ is the linear dual of the bar construction $B\bar{A}$ of the minimal A_∞-model of $\mathcal{C}$ (Theorems 14.1, 14.2, 15.1, 15.2).

This pro-representability appears to be more "natural" for the pseudo-functor coDEF_-, because the bar complex $B\bar{A} \otimes_{\tau_A} A$ is the "universal co-deformation" of A considered as an A_∞-module over A^{op}. The pro-representability of the pseudo-functor DEF_- may then be formally deduced from that of coDEF_-, but we can find the corresponding "universal deformation" (of A) only under an additional assumption on A (Theorem 15.9). We also make the equivalence $\mathrm{DEF}_-(E) \cong 1\text{-}\mathrm{Hom}(\hat{S}, -)$ explicit in this case (Corollary 15.12).

We define and investigate the noncommutative Grassmanians $\mathrm{NGr}(m,V)$. The noncommutative Grassmanian $\mathrm{NGr}(m,V)$ is the true global noncommutative moduli space of the sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$, where $W \subset V$ are vector subspaces of dimension $\dim W = m$.

The paper is organized as follows.

The first part of the paper is devoted to preliminaries on A_∞-algebras, A_∞-modules and A_∞-categories. The only non-standard point here is the DG category of A_∞ $A_{\mathcal{C}}$-modules for an A_∞-algebra A and a DG algebra $\mathcal{C}$, and the corresponding derived category $D_\infty(A_{\mathcal{C}})$. We also recollect some facts on admissible A_∞-algebras from [ELOII].

Second part is devoted to the Maurer-Cartan pseudo-functor $\mathcal{MC}(A) : \mathrm{dgart} \to \mathbf{Gpd}$ for a strictly unital A_∞-algebra A. The Maurer-Cartan groupoid $\mathcal{MC}_{\mathcal{R}}(A)$ can be defined by means of some A_∞-category with the same objects, which are solutions of the generalized Maurer-Cartan equation. We develop the obstruction theory for this Maurer-Cartan pseudo-functor (Proposition 6.1). Further, we show the invariance of (quasi-) equivalence classes of the constructed A_∞-categories and Maurer-Cartan pseudo-functors under the quasi-isomorphisms of A_∞-algebras (Theorems 7.1, 7.2).

In the third part we recall the bicategories $2\text{-}\mathrm{adgalg}$ and $2'\text{-}\mathrm{adgalg}$ from [ELOII] and the pseudo-functors coDEF_- and DEF_- and prove the theorem on their relation (Theorem 13.4). We also obtain here some results on the equivalences between the homotopy and derived (co-)deformation functors (Lemma 9.7, Theorem 11.7).

In the fourth part we prove the pro-representability theorems (Theorems 14.1, 14.2, 15.1, 15.2).

The fifth part is devoted to noncommutative Grassmanians. First we recall some facts and notions concerning $\mathbb{Z}$-algebras (Section 16). Then we define noncommutative Grassmanians $\mathrm{NGr}(m,V)$, $1 \le m \le \dim V - 1$ as $\mathrm{Proj}(\mathcal{A}^{m,V})$, where $\mathcal{A}^{m,V}$ is a certain (periodic) $\mathbb{Z}$-algebra. We explain what means that $\mathrm{NGr}(m,V)$ is a true noncommutative moduli space of the sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$, where $W \subset V$ are subspaces of dimension $\dim W = m$ (Section 17). Then we describe the derived categories of quasi-coherent sheaves on $\mathrm{NGr}(m,V)$ (Theorem 18.19, Corollary 18.22). In Section 19 we describe the k-points of noncommutative Grassmanians. To do that, we first associate to each (poisitively oriented) $\mathbb{Z}$-algebra $\mathcal{A}$ a presheaf of groupoids $X_{\mathcal{A}}$ on the category Alg_k^{op} dual to the category of unital associative algebras. This relates two

approaches to noncommutative geometry. One approach is to think of noncommutative stacks as of Proj of some noncommutative graded ring, see [M], [V1], [V2], [AZ]. We deal with its improvement using $\mathbb{Z}$-algebras, see [BP], [P]. The other approach is to think of noncommutative stacks as of presheaves of groupoids on the category of noncommutative affine schemes, see [Or]. In particular, we obtain the groupoid of k-points of $\mathrm{Proj}(\mathcal{A})$. It turns out that in the case of noncommutative Grassmanians all groupoids are trivial. There is a bijection between the set $\mathrm{NGr}(m,V)(k) = X_{\mathcal{A}^{m,V}}(k)$ of k-points of $\mathrm{NGr}(m,V)$ and the set of all vector subspaces $W \subset V$ such that $1 \leq \dim W \leq m$ (Theorem 19.10).

We freely use the notation and results of [ELOI].

Part 1. A_∞-structures and the bar complex

2. Coalgebras

2.1. **Coalgebras and comodules.** We will consider DG coalgebras. For a DG coalgebra $\mathcal{G}$ we denote by $\mathcal{G}^{\mathrm{gr}}$ the corresponding graded coalgebra obtained from $\mathcal{G}$ by forgetting the differential. Recall that if $\mathcal{G}$ is a DG coalgebra, then its graded dual $\mathcal{G}^*$ is naturally a DG algebra. Also given a finite dimensional DG algebra $\mathcal{B}$ its dual $\mathcal{B}^*$ is a DG coalgebra.

A morphism of DG coalgebras $k \to \mathcal{G}$ (resp. $\mathcal{G} \to k$) is called a co-augmentation (resp. a co-unit) of $\mathcal{G}$ if it satisfies some obvious compatibility condition. We denote by $\overline{\mathcal{G}}$ the cokernel of the co-augmentation map.

Denote by $\overline{\mathcal{G}}_{[n]}$ the kernel of the n-th iterate of the co-multiplication map $\Delta^n : \overline{\mathcal{G}} \to \overline{\mathcal{G}}^{\otimes n}$. The DG coalgebra $\mathcal{G}$ is called *co-complete* if

$$\overline{\mathcal{G}} = \bigcup_{n \geq 2} \overline{\mathcal{G}}_{[n]}.$$

A $\mathcal{G}$*-comodule* means a left DG comodule over $\mathcal{G}$.

A $\mathcal{G}^{\mathrm{gr}}$-comodule is *cofree* if it is isomorphic to $\mathcal{G} \otimes V$ with the obvious comodule structure for some graded vector space V.

Denote by $\mathcal{G}^{op}$ the DG coalgebra with the opposite co-multiplication.

Let $g : \mathcal{H} \to \mathcal{G}$ be a homomophism of DG coalgebras. Then $\mathcal{H}$ is a DG $\mathcal{G}$-comodule with the co-action $g \otimes 1 \cdot \Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{G} \otimes \mathcal{H}$ and a DG $\mathcal{G}^{op}$-comodule with the co-action $1 \otimes g \cdot \Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$.

Let M and N be a right and left DG $\mathcal{G}$-comodules respectively. Their cotensor product $M \square_{\mathcal{G}} N$ is defined as the kernel of the map

$$\Delta_M \otimes 1 - 1 \otimes \Delta_N : M \otimes N \to M \otimes \mathcal{G} \otimes N,$$

where $\Delta_M : M \to M \otimes \mathcal{G}$ and $\Delta_N : N \to \mathcal{G} \otimes N$ are the co-action maps.

A DG coalgebra $\mathcal{G}$ is a left and right DG comodule over itself. Given a DG $\mathcal{G}$-comodule M the co-action morphism $M \to \mathcal{G} \otimes M$ induces an isomorphism $M = \mathcal{G} \square_{\mathcal{G}} M$. Similarly for DG $\mathcal{G}^{op}$-modules.

Definition 2.1. *The dual $\mathcal{R}^*$ of an artinian DG algebra $\mathcal{R}$ is called an artinian DG coalgebra.*

Given an artinian DG algebra $\mathcal{R}$, its augmentation $\mathcal{R} \to k$ induces the co-augmentation $k \to \mathcal{R}^*$ and its unit $k \to \mathcal{R}$ induces the co-unit $\mathcal{R}^* \to k$.

2.2. **From comodules to modules.** If P is a DG comodule over a DG coalgebra $\mathcal{G}$, then P is naturally a DG module over the DG algebra $(\mathcal{G}^*)^{op}$. Namely, the $(\mathcal{G}^*)^{op}$-module structure is defined as the composition

$$P \otimes \mathcal{G}^* \xrightarrow{\Delta_P \otimes 1} \mathcal{G} \otimes P \otimes \mathcal{G}^* \xrightarrow{T \otimes 1} P \otimes \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{1 \otimes \mathrm{ev}} P,$$

where $T : \mathcal{G} \otimes P \to P \otimes \mathcal{G}$ is the transposition map.

Similarly, if Q is a DG $\mathcal{G}^{op}$-comodule, then Q is a DG module over $\mathcal{G}^*$.

Let P and Q be a left and right DG $\mathcal{G}$-comodules respectively. Then $P\otimes Q$ is a DG $\mathcal{G}^*$-bimodule, i.e. a DG $\mathcal{G}^*\otimes\mathcal{G}^{*0}$-module by the above construction. Note that its center

$$Z(P\otimes Q):=\{x\in P\otimes Q \mid ax=(-1)^{\bar a\bar x}xa \ \text{ for all } \ a\in\mathcal{G}^*\}$$

is isomorphic to the cotensor product $Q\Box_{\mathcal{G}}P$.

3. Preliminaries on A_∞-algebras, A_∞-categories and A_∞-modules

3.1. **A_∞-algebras and A_∞-modules.** Our basic reference for A_∞-algebras, A_∞-modules and A_∞-categories is [Le-Ha].

Let $A=\bigoplus_{n\in\mathbb{Z}}A^n$ be a $\mathbb{Z}$-graded k-vector space. Put $BA=T(A[1])=\bigoplus_{n\geq 0}A[1]^{\otimes n}$. Then the graded vector space BA has natural structure of a graded coalgebra with counit:

$$\Delta(a_1\otimes\cdots\otimes a_n)=\sum_{m=0}^{n}(a_1\otimes\cdots\otimes a_m)\otimes(a_{m+1}\otimes\cdots\otimes a_n),$$

$$\varepsilon(a_1\otimes\cdots\otimes a_n)=\begin{cases}0 & \text{for } n\geq 1;\\ 1 & \text{for } n=0.\end{cases}$$

Here we put $a_1\otimes\cdots\otimes a_n=1$ for $n=0$. Put also $\overline{BA}=BA/k$. Then $\overline{BA}$ is also a graded coalgebra, but it is non-counital. The most effective way to define the notion of a $\mathbb{Z}$-graded (non-unital) A_∞-algebra is the following:

Definition 3.1. *A structure of a (non-unital) A_∞-algebra on $\mathbb{Z}$-graded vector space A is a coderivation $b:\overline{BA}\to\overline{BA}$ of degree 1 such that $b^2=0$, i.e. a structure of a DG coalgebra on the graded coalgebra $\overline{BA}$.*

Such a coderivation is equivalent to a sequence of maps $b_n=b_n^A:A[1]^{\otimes n}\to A[1]$, $n\geq 1$, of degree 1 satisfying for each $n\geq 1$ the following identity:

$$\sum_{r+s+t=n} b_{r+1+t}(\mathbf{1}^{\otimes r}\otimes b_s\otimes\mathbf{1}^{\otimes t})=0. \tag{3.1}$$

Note that the coderiveation $b:\overline{BA}\to\overline{BA}$ naturally extends to a coderivation $b:BA\to BA$ (which we denote by the same letter), thus BA also becomes a DG coalgebra, and $\varepsilon\cdot b=0$. Thus, its dual $\hat S=(BA)^*$ is naturally a DG algebra.

Let $s:A\to A[1]$ be the translation map. Identify $A^{\otimes n}$ with $A[1]^{\otimes n}$ via the map $s^{\otimes n}$, and A with $A[1]$ via the map s. Let $m_n=m_n^A:A^{\otimes n}\to A$ be the maps corresponding to b_n. Then m_n has degree $(2-n)$ and this sequence of maps satisfies for each $n\geq 1$ the following identity:

$$\sum_{r+s+t=n}(-1)^{r+st}m_{r+1+t}(\mathbf{1}^{\otimes r}\otimes m_s\otimes\mathbf{1}^{\otimes t})=0. \tag{3.2}$$

In particular, m_1 is a differential on A, hence A is a complex. Further, m_2 is a morphisms of complexes and it is associative up to homotopy given by m_3. Thus, the cohomology $H(A)$ is naturally a (possibly non-unital) graded algebra. Further, if $m_n=0$ for $n\geq 3$, then A is a (possibly non-unital) DG algebra.

Let A_1, A_2 be (non-unital) A_∞-algebras. The most effective way to define the notion of an A_∞-morphism between them is the following:

Definition 3.2. *An A_∞-morphism $f : A_1 \to A_2$ is a (counital) homomorphism of DG coalgebras $f : BA_1 \to BA_2$ (which we denote by the same letter).*

Thus, the assignment $A \mapsto BA$ is the full embedding of the category of (non-unital) A_∞-algebras and A_∞-morphisms to the category of counital DG coalgebras.

An A_∞-morphism $f : A_1 \to A_2$ is equivalent to a sequence of maps $\widetilde{f_n} : A[1]^{\otimes n} \to A[1]$, $n \geq 1$, of degree zero satisfying for each $n \geq 1$ the following identity:

$$\sum_{i_1+\dots+i_s=n} b_s^{A_2}(\widetilde{f_{i_1}} \otimes \dots \otimes \widetilde{f_{i_s}}) = \sum_{r+s+t} \widetilde{f_{r+1+t}}(\mathbf{1}^{\otimes r} \otimes b_s^{A_1} \otimes \mathbf{1}^{\otimes t}). \tag{3.3}$$

Let $f_n : A_1^{\otimes n} \to A_2$ be the maps corresponding to $\widetilde{f_n}$ with respect to our identifications. Then f_n has degree $(1-n)$ and this sequence of maps satisfies for each $n \geq 1$ the following identity:

$$\sum_{i_1+\dots+i_s=n} (-1)^{\epsilon(i_1,\dots,i_{s-1},s)} m_s^{A_2}(f_{i_1} \otimes \dots \otimes f_{i_s}) = \sum_{r+s+t} (-1)^{r+st+s} f_{r+1+t}(\mathbf{1}^{\otimes r} \otimes m_s^{A_1} \otimes \mathbf{1}^{\otimes t}), \tag{3.4}$$

where $\epsilon(i_1, \dots, i_{s-1}, s) = (s-1)i_1 + \dots + i_{s-1} + \frac{s(s+1)}{2}$.

In particular, f_1 is a morphism of complexes, and $H(f) : H(A_1) \to H(A_2)$ is a morphism of (non-unital) graded associative algebras. An A_∞-morphism f is called quasi-isomorphism if f_1 is a quasi-isomorphism of complexes.

Further, we define the DG category $A\text{-mod}_\infty$ of A_∞ A-modules for an A_∞-algebra A.

Definition 3.3. *A structure of an A_∞-module over A on the graded vector space M is the differential $b^M : BA \otimes M[1] \to BA \otimes M[1]$ of degree 1, which defines a structure of a DG BA-comodule on the graded cofree $(BA)^{gr}$-comodule $BA \otimes M[1]$.*

Such a structure is equivalent to a sequence of maps $b_n = b_n^M : A[1]^{\otimes(n-1)} \otimes M[1] \to M[1]$, $n \geq 1$, of degree 1, satisfying for each $n \geq 1$ the identity (3.1), where b_i is interpreted as b_i^A or b_i^M, according to the type of its arguments. It is also equivalent to the sequence of maps $m_n = m_n^M : A^{\otimes(n-1)} \otimes M \to M$, $n \geq 1$, of degree $(2-n)$ satisfying for each $n \geq 1$ the identity (3.2), where m_i is interpreted as m_i^A or m_i^M, according to the type of its arguments. In particular, $(m_1^M)^2 = 0$, hence M is a complex. Again, m_2^M is a morphism of complexes and m_2 is associative up to a homotopy given by m_3^M. Thus, $H(M)$ is naturally a graded $H(A)$-module.

If M and N are A_∞ A-modules, then we put

$$\mathrm{Hom}_{A\text{-mod}_\infty}(M, N) := \mathrm{Hom}_{BA\text{-comod}}(BA \otimes M[1], BA \otimes N[1]).$$

More explicitly,

$$\mathrm{Hom}^n_{A\text{-mod}_\infty}(M, N) = \prod_{m \geq 1} \mathrm{Hom}^n_k(A[1]^{\otimes(m-1)} \otimes M[1], N[1]),$$

and for $\phi = (\phi_m) \in \mathrm{Hom}^n_{A\text{-mod}_\infty}(M, N)$ one has

$$(d\phi)_m = \sum_{1 \leq i \leq m} b^N_{m-i+1}(\mathbf{1}^{\otimes(l-i)} \otimes \phi_i) - (-1)^n \sum_{r+s+t=m} \phi_{r+1+t}(\mathbf{1}^{\otimes r} \otimes b_s \otimes \mathbf{1}^{\otimes t}), \tag{3.5}$$

where b_s in the RHS is interpreted as b_s^A or b_s^M, according to the type of its arguments. If $\phi = (\phi_m) \in \mathrm{Hom}_{A\text{-mod}_\infty}(M, N)$ and $\psi = (\psi_m) \in \mathrm{Hom}_{A\text{-mod}_\infty}(N, L)$, then

$$(\psi \cdot \phi)_m = \sum_{1 \le i \le m} \psi_{m-i+1}(\mathbf{1}^{\otimes(m-i)} \otimes \phi_i). \tag{3.6}$$

We will write $\mathrm{Hom}_A(M,N)$ instead of $\mathrm{Hom}_{A\text{-mod}_\infty}(M,N)$.

The closed morphism $\phi \in \mathrm{Hom}^0(A\text{-mod}_\infty)$ is called quasi-isomorphism if ϕ_1 is a quasi-isomorphism of complexes.

The homotopy category $K_\infty(A)$ is defined as $\mathrm{Ho}(A\text{-mod}_\infty)$. It is always triangulated. It turns out that all acyclic A_∞ A-modules in $K_\infty(A)$ are already null-homotopic. Hence the corresponding derived category $D_\infty(A)$ is the same as $K_\infty(A)$. However, we will write $D_\infty(A)$ instead of $K_\infty(A)$.

Let $f : A_1 \to A_2$ be an A_∞-morphism. Then we have the DG functor $f_* : A_2\text{-mod}_\infty \to A_1\text{-mod}_\infty$, which we call the "restriction of scalars". Namely, if $M \in A_2\text{-mod}_\infty$, then $f_*(M)$ coincides with M as a graded vector space, and the differential on $BA_1 \otimes f_*(M)[1]$ coincides with the differential on $BA_1 \Box_{BA_2}(BA_2 \otimes M[1])$ after the natural identification

$$BA_1 \otimes f_*(M)[1] \cong BA_1 \Box_{BA_2}(BA_2 \otimes M[1]).$$

We also have the resulting exact functor $f_* : D_\infty(A_2) \to D_\infty(A_1)$. If f is a quasi-isomorphism, then the DG functor $f_* : A_2\text{-mod}_\infty \to A_1\text{-mod}_\infty$ is quasi-equivalence, and hence the functor $f_* : D_\infty(A_2) \to D_\infty(A_1)$ is an equivalence.

We would like also to define the A_∞-bimodules.

Definition 3.4. *Let A_1 and A_2 be A_∞-algebras. A structure of an A_∞ A_1-A_2-bimodule on the graded vector space M is a differential $b^M : BA_1 \otimes M[1] \otimes BA_2 \to BA_1 \otimes M[1] \otimes BA_2$ which defines the structure of a DG comodule over $BA_1 \otimes (BA_2)^{op}$ on the $(BA_1 \otimes (BA_2)^{op})^{gr}$-bicomodule $BA_1 \otimes M[1] \otimes BA_2$.*

Such a differential is given by a sequence of maps

$$b_{i,j} : A_1[1]^{\otimes i} \otimes M[1] \otimes A_2[1]^{\otimes j} \to M[1]$$

satisfying analogous equations. In particlular, we have a regular A_1-A_2-bimodule $A_1 \otimes A_2$. In the case when $A_1 = A_2$, we have a diagonal bimodule A. The DG category $A_1\text{-mod-}A_2$ of A_∞ A_1-A_2-bimodules is defined analogously (see also [KoSo]). Again, we define $K_\infty(A_1\text{-}A_2)$ as the homotopy category $\mathrm{Ho}(A_1\text{-mod-}A_2)$. All acyclic A_∞-bimodules in $K_\infty(A_1\text{-}A_2)$ are null-homotopic and hence the corresponding derived category $D_\infty(A_1\text{-}A_2)$ coincides with $K_\infty(A_1\text{-}A_2)$.

3.2. Strictly unital A_∞-algebras.

Definition 3.5. *An A_∞-algebra is called strictly unital if there exists an element $1_A \in A$ of degree zero satisfying the following properties:*
(U1) $m_1(1_A) = 0$;
(U2) $m_2(a, 1_A) = m_2(1_A, a) = a$ for each $a \in A$;
(U3) for $n \ge 3$, $m_n(a_1, \dots, a_n)$ vanishes if at least one of a_i equals to 1_A.
Such an element 1_A is called a strict unit.

Clearly, if a strict unit exists then it is unique. An A_∞-morphism $f : A_1 \to A_2$ of strictly unital A_∞-algebras is called strictly unital if $f_1(1_{A_1}) = 1_{A_2}$, and for $n \ge 2$ $f_n(a_1, \dots, a_n)$ vanishes if at least one of a_i equals to 1_{A_1}. Further, an A_∞-module $M \in A\text{-mod}_\infty$ is called strictly unital if $m_2^M(1_A, m) = m$ for each

$m \in M$ and for $n \geq 3$ $m_n^M(a_1, \dots, a_{n-1}, m) = 0$ if at least one of a_i equals to 1_A. If A is strictly unital then we denote by $D_\infty^{su}(A) \subset D_\infty(A)$ the full subcategory which consists of strictly unital A_∞ A-modules.

Analogously, if A_1 and A_2 are strictly unital A_∞-algebras, then we have a notion of strictly unital A_∞ A_1-A_2-bimodules, and we define $D_\infty^{su}(A_1\text{-}A_2) \subset D_\infty(A_1\text{-}A_2)$ as the full subcategory which consists of strictly unital A_∞-bimodules.

If $\mathcal{C}$ is a DG algebra then it is also a strictly unital A_∞-algebra with $m_n = 0$ for $n \geq 3$. We have an obvious DG functor $\mathcal{C}\text{-mod} \to \mathcal{C}\text{-mod}_\infty$. It induces an equivalence

$$D(\mathcal{C}) \xrightarrow{\sim} D_\infty^{su}(\mathcal{C}).$$

Let A be an arbitrary A_∞-algebra. Then its unitization $A_+ := k \cdot 1_+ \oplus A$, which is a strictly unital A_∞-algebra, is defined as follows:

$$m_n^{A_+}(a_1, \dots, a_n) = m_n^A(a_1, \dots, a_n) \text{ for any } a_1, \dots, a_n \in A,$$
$$m_1(1_+) = 0,$$
$$m_2^{A_+}(1_+, a) = m_2^{A_+}(a, 1_+) = a \text{ for each } a \in A_+,$$
$$m_n^{A_+}(a_1, \dots, a_n) = 0 \text{ if at least one of } a_i \text{ equals to } 1_+.$$

Clearly, the assignment $A \mapsto A_+$ defines faithful functor from the category of A_∞-algebras and A_∞-morphisms to the category of strictly unital A_∞-algebras and strictly unital A_∞-morphisms. Further, we have an obvious faithful DG functor $A\text{-mod}_\infty \to A_+\text{-mod}_\infty$. Its image consists of strictly unital A_∞-modules. The induced functor $D_\infty(A) \to D_\infty^{su}(A_+)$ is an equivalence.

We call A_∞-algebras of the form A_+ augmented A_∞-algebras. We also use the notation $A = \overline{A_+}$.

Definition 3.6. *Let A be an augmented A_∞-algebra. Its bar-cobar contruction $U(A)$, which is a DG algebra, together with a strictly unital A_∞ quasi-isomorphism $f_A : A \to U(A)$ are defined by the following universal property. If $\mathcal{B}$ is a DG algebra, and $f : A \to \mathcal{B}$ is a strictly unital A_∞-morphism then there exists a unique morphism of DG algebras $\varphi : U(A) \to \mathcal{B}$ such that $f = \varphi \cdot f_A$.*

More explicitly, $U(A)$ equals to $T(\overline{B\bar{A}}[-1])$ as a graded algebra, and the differential comes from the differential and comultiplication on $\overline{B\bar{A}}$. The A_∞-morphism f_A is the obvious one.

3.3. **Minimal models of A_∞-algebras.** An A_∞-algebra A is called minimal if $m_1^A = 0$. Each (strictly unital) A_∞-algebra is quasi-isomorphic to the minimal (strictly unital) A_∞-algebra.

Proposition 3.7. *(*[Le-Ha]*, Corollaire 1.4.1.4, Proposition 3.2.4.1) Let A be an A_∞-algebra. There exists an A_∞-algebra structure on $H(A)$ such that*

a) $m_1 = 0$ and m_2 is induced by m_2^A;

b) there exists an A_∞-quasi-isomorphism of A_∞-algebras $f : H(A) \to A$ such that f_1 induces the identity in cohomology.

Moreover, if A is strictly unital then this A_∞-structure on $H(A)$ and the quasi-isomorphism can be chosen to be strictly unital.

3.4. **Perfect A_∞-modules and A_∞-bimodules.** Let A be a strictly unital A_∞-algebra. The category $\mathrm{Perf}(A)$ of perfect A_∞ A-modules is the minimal full thick triangulated subcategory of $D^{su}_\infty(A)$ which contains A.

Further, if A_1 and A_2 are strictly unital A_∞-algebras then the category $\mathrm{Perf}(A_1\text{-}A_2)$ of perfect A_∞ A_1-A_2-bimodules is the minimal full thick triangulated subcategory of $D^{su}_\infty(A_1\text{-}A_2)$ which contains $A_1 \otimes A_2$.

3.5. **A_∞-categories.** The notion of an A_∞-category is a straightforward generalization of the notion of an A_∞-algebra. Namely, a non-unital A_∞-category $\mathcal{A}$ is the following data:

- the class of of objects of $\mathcal{A}$;
- for each two objects X_1, X_2 the graded vector space $\mathrm{Hom}(X_1, X_2)$;
- for each finite sequence of objects $X_0, X_1, \dots, X_n \in \mathcal{A}$, $n \geq 1$, the map

$$m_n^{\mathcal{A}(X_0,\dots,X_n)} : \mathrm{Hom}(X_{n-1}, X_n) \otimes \dots \otimes \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(X_0, X_n)$$

of degree $(2-n)$, such that for any $Y_1, \dots, Y_m \in \mathcal{A}$ the graded vector space $\bigoplus\limits_{1\leq i,j\leq m} \mathrm{Hom}(Y_i, Y_j)$ becomes an A_∞-algebra.

If $\mathcal{A}$ is an A_∞-category then $\mathrm{Ho}(\mathcal{A})$ is a pre-category, i.e. a "category" which may not have identity morphisms.

An element $\mathbf{1}_X \in \mathrm{Hom}(X, X)$ of degree zero is called a strict identity morphism if it satisfies the conditions $(U1), (U2), (U3)$ from Definition 3.5, where a and a_i are arbitrary morphisms such that the equalities make sense. An A_∞-category is called strictly unital if each object has a strict identity morphism. If $\mathcal{A}$ is a strictly unital A_∞-category then $\mathrm{Ho}(\mathcal{A})$ is a true category.

A (strictly unital) A_∞-algebra can be thought of as a (strictly unital) A_∞-category with one object.

Let $\mathcal{A}_1, \mathcal{A}_2$ be A_∞-categories. An A_∞-functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ is the following data:

- an object $F(X) \in \mathcal{A}_2$ for each object $X \in \mathcal{A}_1$;
- for each finite sequence of objects $X_0, X_1, \dots, X_n \in \mathcal{A}_1$, $n \geq 1$, the map

$$F(X_0, \dots, X_n) : \mathrm{Hom}(X_{n-1}, X_n) \otimes \dots \otimes \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(F(X_0), F(X_n))$$

of degree $(1-n)$, such that for any $Y_1, \dots, Y_m \in \mathcal{A}_1$ we obtain an A_∞-morphism between A_∞-algebras

$$\bigoplus_{1\leq i,j\leq m} \mathrm{Hom}(Y_i, Y_j) \to \bigoplus_{1\leq i,j\leq m} \mathrm{Hom}(F(Y_i), F(Y_j)).$$

The definition of a strictly unital A_∞-functor between strictly unital A_∞-categories is analogous to the definition of a strictly unital A_∞-morphism between strictly unital A_∞-algebras.

A strictly unital A_∞-functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ between strictly unital A_∞-categories is called quasi-equivalence if the following conditions hold:

- the map $F(X, Y) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is a quasi-isomorphism of complexes for any $X, Y \in \mathcal{A}_1$;
- the induced functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{A}_1) \to \mathrm{Ho}(\mathcal{A}_2)$ is an equivalence.

3.6. **The tensor product of an A_∞-algebra and a DG algebra.** Let A be an A_∞-algebra and $\mathcal{C}$ be a DG algebra. Then their tensor $A \otimes \mathcal{C}$ is naturally an A_∞-algebra with the following multiplications:

$$m_1^{A\otimes\mathcal{C}} = m_1^A \otimes \mathbf{1}_\mathcal{C} + \mathbf{1}_\mathcal{A} \otimes d_\mathcal{C};$$
$$m_n^{A\otimes\mathcal{C}}(a_1 \otimes c_1, \dots, a_n \otimes c_n) = (-1)^\epsilon m_n^A(a_1, \dots, a_n) \otimes (c_1 \dots c_n) \text{ for } n \geq 2,$$

where $\epsilon = \sum_{i<j} \bar{a}_j \bar{c}_i$ (all a_i and c_i are homogeneous). If A is strictly unital, then $A \otimes \mathcal{C}$ is also strictly unital and $\mathbf{1}_{A\otimes\mathcal{C}} = \mathbf{1}_A \otimes \mathbf{1}_{\mathcal{C}}$.

Remark 3.8. *The constructed tensor product is a specialization of the complicated construction of the tensor product of A_∞-algebras which was first proposed in* [SU]. *We also remark that in the case when A_1 and A_2 are strictly unital A_∞-algebras, there is a canonical DG model for $A_1 \otimes A_2$:*

$$A_1 \text{"}\otimes\text{"} A_2 = \mathrm{End}_{A_1\text{-}mod\text{-}A_2^{op}}(A_1 \otimes A_2),$$

see [KoSo].

3.7. **The category of $A_{\mathcal{C}}$-modules for an A_∞-algebra A and a DG algebra $\mathcal{C}$.** Let A be an A_∞-algebra and let $\mathcal{C}$ be a DG algebra. We want to define the DG category of A_∞ $A_{\mathcal{C}}$-modules which is analogue of the category of $(A \otimes \mathcal{C})$-modules in the case when A is a DG algebra.

Definition 3.9. *A structure of an A_∞ $A_{\mathcal{C}}$-module on the graded vector space M is the following data:*

1) A structure of a $\mathcal{C}^{gr}$-module on M;

2) A differential $b^M : BA \otimes M[1] \to BA \otimes M[1]$ of degree 1 which makes $BA \otimes M[1]$ into a DG comodule over BA and into a DG module over $\mathcal{C}$.

If we are already given with the structure of a $\mathcal{C}^{gr}$-module on M then such a differential b^M is equivalent to the sequence of maps $b_n = b_n^M : A[1]^{\otimes(n-1)} \otimes M[1] \to M[1]$, $n \geq 1$, satisfying the following properties:
1) The maps b_n^M satisfy the identities (3.1) (in the same sense as for A_∞ A-modules);
2) The differential b_1^M makes $M[1]$ into a DG module over $\mathcal{C}$;
3) The maps b_n^M are $\mathcal{C}^{gr}$-linear for $n \geq 2$.

Further, the corresponding maps $m_n = m_n^M : A^{\otimes(n-1)} \otimes M \to M$ have to satisfy the following properties:
1) The maps m_n^M satisfy the identities (3.2) (in the same sense as for A_∞ A-modules);
2) The differential m_1^M makes M into a DG module over $\mathcal{C}$;
3) The maps m_n^M are $\mathcal{C}^{gr}$-linear for $n \geq 2$.

If M, N are A_∞ $A_{\mathcal{C}}$-modules then we put

$$\mathrm{Hom}_{A_{\mathcal{C}}\text{-mod}_\infty}(M,N) := \mathrm{Hom}_{BA\text{-comod}} \cap \mathrm{Hom}_{\mathcal{C}\text{-mod}}(BA \otimes M[1], BA \otimes N[1]).$$

More explicitly,

$$\mathrm{Hom}^n_{A_{\mathcal{C}}\text{-mod}_\infty}(M,N) = \prod_{m\geq 1} \mathrm{Hom}^n_{\mathcal{C}^{gr}}(A[1]^{\otimes(m-1)} \otimes M[1], N[1]),$$

the differential and the compositions are defined by the formulas (3.5) and (3.6) respectively.

We will write $\mathrm{Hom}_{A_{\mathcal{C}}}(M,N)$ instead of $\mathrm{Hom}_{A_{\mathcal{C}}\text{-mod}_\infty}(M,N)$.

Again, the homotopy category $K_\infty(A_{\mathcal{C}})$ is defined as $\mathrm{Ho}(A_{\mathcal{C}}\text{-mod}_\infty)$. The acyclic A_∞ $A_{\mathcal{C}}$-module in $K_\infty(A_{\mathcal{C}})$ are not null-homotopic in general, hence we define the derived category $D_\infty(A_{\mathcal{C}})$ as the Verdier quotient of $K_\infty(A_{\mathcal{C}})$ by the subcategory of acyclic A_∞ $A_{\mathcal{C}}$-modules.

Remark 3.10. *Notice that the structure of an A_∞ $A_{\mathcal{C}}$-module is not equivalent to the structure of an A_∞ $A\otimes\mathcal{C}$-module. Moreover, there is a natural DG functor $A_{\mathcal{C}}\text{-}mod_\infty \to A_+ \otimes \mathcal{C}\text{-}mod_\infty$ which induces an equivalence $D_\infty(A_{\mathcal{C}}) \xrightarrow{\sim} D^{su}_\infty(A_+ \otimes \mathcal{C})$. Also, in the case when A is strictly unital, the DG functor $A_{\mathcal{C}}\text{-}mod_\infty \to A\otimes\mathcal{C}\text{-}mod_\infty$ induces an equivalence $D^{su}_\infty(A_{\mathcal{C}}) \xrightarrow{\sim} D^{su}_\infty(A \otimes \mathcal{C})$.*

Definition 3.11. *An A_∞ $A_{\mathcal{C}}$-module M is called h-projective (resp. h-injective) if for each acyclic $N \in A_{\mathcal{C}}$-mod$_\infty$ the complex $\mathrm{Hom}_{A_{\mathcal{C}}}(M,N)$ (resp. $\mathrm{Hom}_{A_{\mathcal{C}}}(N,M)$) is acyclic.*

It turns out that an A_∞ $A_{\mathcal{C}}$-module is h-projective (resp. h-injective) iff it is such as a DG $\mathcal{C}$-module.

Proposition 3.12. *Let M be an A_∞ $A_{\mathcal{C}}$-module. Suppose that M is h-projective (resp. h-injective) as a DG $\mathcal{C}$-module. Then M is also h-projective (resp. h-injective) as an A_∞ $A_{\mathcal{C}}$-module.*

Proof. We will prove Proposition for h-projectives. The proof for h-injectives is analogous.

So let $M \in A_{\mathcal{C}}$-mod$_\infty$ and suppose that M is h-projective as a DG $\mathcal{C}$-module. Let N be an acyclic A_∞ $A_{\mathcal{C}}$-module. The complex $K^{\cdot} = \mathrm{Hom}_{A_{\mathcal{C}}}(M,N)$ admits a decreasing filtration by subcomplexes

$$F^pK^{\cdot} = \prod_{n\geq p} \mathrm{Hom}_{\mathcal{C}^{gr}}(A^{\otimes n}\otimes M, N).$$

The subquotients

$$F^pK^{\cdot}/F^{p+1}K^{\cdot} = \mathrm{Hom}_{\mathcal{C}}(A^{\otimes p}\otimes M, N)$$

are acyclic since the DG modules $A^{\otimes p}\otimes M$ are h-projective. Since

$$K^{\cdot} = \lim_{\leftarrow} K^{\cdot}/F^pK^{\cdot},$$

the complex $K^{\cdot}$ is also acyclic. Therefore, M is h-projective as an A_∞ $A_{\mathcal{C}}$-module. □

We denote by $K^P_\infty(A_{\mathcal{C}}) \subset K_\infty(A_{\mathcal{C}})$ (resp. by $K^I_\infty(A_{\mathcal{C}}) \subset K_\infty(A_{\mathcal{C}})$) the full subcategory which consists of h-projective (resp. h-injective) A_∞ $A_{\mathcal{C}}$-modules.

Theorem 3.13. *For each $M \in A_{\mathcal{C}}$-mod$_\infty$, there exist quasi-isomorphisms $M \to I$, $P \to M$, where $I \in A_{\mathcal{C}}$-mod$_\infty$ is h-injective and $P \in A_{\mathcal{C}}$-mod$_\infty$ is h-projective. The natural functor $K^P_\infty(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$ (resp. $K^I_\infty(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$) is an equivalence.*

Proof. First we construct a quasi-isomorphism $pM \to M$ with h-projective P. Namely, let pM be the total complex of the bicomplex

$$\cdots \to \mathcal{C}^{\otimes n}\otimes M \xrightarrow{d^n} \mathcal{C}^{\otimes n-1}\otimes M \to \cdots \to \mathcal{C}\otimes M,$$

where d^n is the bar differential. Then pM is naturally an A_∞ $A_{\mathcal{C}}$-module. A quasi-isomorphism of complexes $pM \to M$ is a quasi-isomorphism in $A_{\mathcal{C}}$-mod$_\infty$ (with zero components $f_n : A^{\otimes n-1}\otimes M \to M$ for $n \geq 2$). Further, pM satisfies property (P) ([ELOI], Definition 3.2) as a DG $\mathcal{C}$-module. Hence, pM is an h-projective $A_{\mathcal{C}}$-module.

The construction $M \to pM$ extends to the functor $p : K_\infty(A_{\mathcal{C}}) \to K^P_\infty(A_{\mathcal{C}})$ which is right adjoint to the inclusion $K^P_\infty(A_{\mathcal{C}}) \to K_\infty(A_{\mathcal{C}})$. The kernel of p consists of acyclic $A_{\mathcal{C}}$-modules. Thus, the functor $K^P_\infty(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$ is an equivalence.

Analogously, one can construct a functor $i : K_\infty(A_{\mathcal{C}}) \to K^I_\infty(A_{\mathcal{C}})$ which is left adjoint to the inclusion $K^I_\infty(A_{\mathcal{C}}) \to K_\infty(A_{\mathcal{C}})$, and the kernel of i also consists of acyclic complexes. Thus, the functor $K^I_\infty(A_{\mathcal{C}}) \to D_\infty(A_{\mathcal{C}})$ is an equivalence. Theorem is proved. □

Notice that if $G : K_\infty(A_{\mathcal{C}}) \to \mathcal{T}$ is an exact functor between triangulated categories then we can define its left and right derived functors

$$\mathbf{L}G : D_\infty(A_{\mathcal{C}}) \to \mathcal{T}, \quad \mathbf{R}G : D_\infty(A_{\mathcal{C}}) \to \mathcal{T}.$$

Namely, for each $M \in A_{\mathcal{C}}\text{-mod}_\infty$ choose quasi-isomorphisms $P \to M$, $M \to I$ with h-projective P and h-injective I, and put

$$\mathbf{L}G(M) = G(P), \quad \mathbf{R}G(M) = G(I).$$

Proposition 3.14. *The derived categories $D_\infty(A_{\mathcal{C}})$ and $D(U(A_+) \otimes \mathcal{C})$ are naturally equivalent.*

Proof. Indeed, the "restriction of scalars" DG functor

$$f_{A*} : (U(A_+) \otimes \mathcal{C})\text{-mod} \to A_{\mathcal{C}}\text{-mod}_\infty$$

admits a right adjoint DG functor

$$f_A^! : A_{\mathcal{C}}\text{-mod}_\infty \to (U(A_+) \otimes \mathcal{C})\text{-mod},$$

given by the formula

$$f_A^!(M) = \mathrm{Hom}_A(U(A_+), M).$$

For any $M \in (U(A_+) \otimes \mathcal{C})\text{-mod}$, $N \in A_{\mathcal{C}}\text{-mod}_\infty$, the adjunction morphisms $M \to f_A^! f_{A*} M$, $f_{A*} f_A^! N \to N$ are quasi-isomorphisms. Moreover, both f_{A*} and $f_A^!$ preserve acyclic modules. Thus, the induced functors

$$f_{A*} : D(U(A_+) \otimes \mathcal{C}) \to D_\infty(A_{\mathcal{C}}), \quad f_A^! : D_\infty(A_{\mathcal{C}}) \to D(U(A) \otimes \mathcal{C})$$

are mutually inverse equivalences. □

3.8. **The bar complex.** Let A be an A_∞-algebra. The graded vector space $BA \otimes A[1] \otimes BA$ carries a natural differential which makes it into a DG bicomodule over BA. Namely, such a differential is determined by its components

$$b_{i,j} : A[1]^{\otimes i} \otimes A[1] \otimes A[1]^{\otimes j} \to A[1],$$

and we put $b_{i,j} = b^A_{i+j+1}$.

In particular, $BA \otimes A$ is an A_∞-module over A^{op}. It is called the bar complex and is denoted by $BA \otimes_{\tau_A} A$.

Now let A be an augmented A_∞-algebra, and put $\hat{S} = (B\bar{A})^*$. The graded vector space $B\bar{A} \otimes A[1] \otimes B\bar{A}$ also carries a natural differential which makes it into a DG bicomodule over $B\bar{A}$. In particular, $B\bar{A} \otimes A$ is an A_∞-module over $\bar{A}^{op}$. It is also called the bar complex and is denoted by $B\bar{A} \otimes_{\tau_A} A$. Note that $B\bar{A} \otimes_{\tau_A} A$ is a $B\bar{A}$-comodule, and hence is a $\hat{S}^{op}$-module. This makes it into an object of $\bar{A}^{op}_{\hat{S}^{op}}\text{-mod}_\infty$.

Analgously, we have an A_∞ $\bar{A}_{\hat{S}}$-module $A \otimes_{\tau_A} B\bar{A}$.

4. Admissible A_∞-algebras.

Here we recollect some facts about admissible A_∞-algebras from [ELOII].

Definition 4.1. *(*[ELOII]*) An augmented A_∞-algebra $\mathcal{C}$ is called*

a) nonnegative if $\mathcal{C}^i = 0$ for $i < 0$;

b) connected if $\mathcal{C}^0 = k$;

c) locally finite if $\dim_k \mathcal{C}^i < \infty$ for all i.

We say that $\mathcal{C}$ is admissible if it satisfies a), b), c).

Fix an augmented A_∞-algebra $\mathcal{C}$. Consider the bar construction $B\bar{\mathcal{C}}$, the corresponding DG algebra $\hat{S} = (B\bar{\mathcal{C}})^*$ and the A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module $B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}$ (the bar complex). If $\mathcal{C}$ is connected and nonnegative, then $B\bar{\mathcal{C}}$ is concentrated in nonnegative degrees and consequently $\hat{S}$ is concentrated in nonpositive degrees.

Let $\mathcal{B}$ be a DG algebra. Denote by $D_f(\mathcal{B}^{op}) \subset D(\mathcal{B}^{op})$ the full triangulated subcategory consisting of DG modules with finite dimensional cohomology.

Lemma 4.2. *([ELOII]) Assume that DG algebra $\mathcal{B}$ is augmented and local and complete. Also assume that $\mathcal{B}^i = 0$ for $i > 0$. Then the category $D_f(\mathcal{B}^{op})$ is the triangulated envelope of the DG $\mathcal{B}^{op}$-module k.*

Choose a quasi-isomorphism of A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-modules $B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C} \to J$, where J is h-injective.

Proposition 4.3. *([ELOII]) Under the assumptions of the above theorem the following holds.*

a) The complex $\mathbf{R}\operatorname{Hom}_{\hat{S}^{op}}(k,k)$ is quasi-isomorphic to $\mathcal{C}$.

b) The natural morphism of complexes $\operatorname{Hom}_{\hat{S}^{op}}(k, B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \to \operatorname{Hom}_{\hat{S}^{op}}(k, J)$ is a quasi-isomorphism.

Remark 4.4. *Notice that for any augmented A_∞-algebra $\mathcal{C}$ we have $\operatorname{Hom}_{\hat{S}^{op}}(k, B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) = \mathcal{C}$. Thus the A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module $B\bar{\mathcal{C}} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}$ is a "homotopy $\hat{S}$-co-deformation" of $\mathcal{C}$. The Proposition 4.3 implies that for an admissible $\mathcal{C}$ this A_∞ $\bar{\mathcal{C}}^{op}_{\hat{S}^{op}}$-module is a "derived $\hat{S}$-co-deformation" of $\mathcal{C}$. (Of course we have only defined co-deformations along artinian DG algebras.)*

Lemma 4.5. *([ELOII]) Let K be a DG $\hat{S}^{op}$-module such that $\dim_k K < \infty$. Then the natural morphism of complexes*

$$\operatorname{Hom}_{\hat{S}^{op}}(K, B\mathcal{C} \otimes_{\tau_{\mathcal{C}}} \mathcal{C}) \to \operatorname{Hom}_{\hat{S}^{op}}(K, J)$$

is a quasi-isomorphism.

Part 2. Maurer-Cartan pseudo-functor for A_∞-algebras

5. The definition

Let A be a strictly unital A_∞-algebra, and $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Recall that $A \otimes \mathcal{R}$ is naturally a strictly unital A_∞-algebra (see subsection 3.6). We define the set $MC(A \otimes \mathfrak{m})$ as the set of $\alpha \in (A \otimes \mathfrak{m})^1$ such that the generalized Maurer-Cartan equation holds:

$$\sum_{n \geq 1} (-1)^{\frac{n(n+1)}{2}} m_n(\alpha, \dots, \alpha) = 0. \tag{5.1}$$

This equation is well defined since $\mathfrak{m} \subset \mathcal{R}$ is nilpotent ideal. Below for convenience we will write α^n instead of $\underbrace{\alpha, \dots, \alpha}_{n}$.

There is a natural A_∞-category $\mathcal{MC}^{\mathcal{R}}_\infty(A)$ with the set of objects $MC(A \otimes \mathfrak{m})$. Namely, for $\alpha_1, \alpha_2 \in MC(A \otimes \mathfrak{m})$ we define

$$\operatorname{Hom}_{\mathcal{MC}^{\mathcal{R}}_\infty(A)}(\alpha_1, \alpha_2) := (A \otimes \mathcal{R})^{gr}$$

as a graded vector space. Further, for $\alpha_0, \alpha_1, \dots, \alpha_m \in MC(A \otimes \mathfrak{m})$ and for homogeneous $x_1 \in \operatorname{Hom}(\alpha_0, \alpha_1), \dots, x_n \in \operatorname{Hom}(\alpha_{n-1}, \alpha_n)$ we define

$$m_n^{\mathcal{MC}^{\mathcal{R}}_\infty(A)(\alpha_0,\dots,\alpha_n)}(x_n, \dots, x_1) = \sum_{i_0,\dots,i_n \geq 0} (-1)^\epsilon m^{A \otimes \mathcal{R}}_{n+i_0+\dots+i_n}(\alpha_n^{i_n}, x_n, \alpha_{n-1}^{i_{n-1}}, \dots, \alpha_1^{i_1}, x_1, \alpha_0^{i_0}),$$

where

$$\epsilon = \sum_{n \geq k > j \geq 0} (\bar{x_k} + i_k) i_j + \sum_{k=0}^{n} \frac{i_k(i_k+1)}{2} + \sum_{k=1}^{n} k i_k.$$

One checks without difficulties that this indeed defines an A_∞-category and that $\mathbf{1} \in (A \otimes \mathcal{R})^{gr} = \operatorname{Hom}(\alpha, \alpha)$ is a strict identity for each $\alpha \in \mathcal{MC}^{\mathcal{R}}_\infty(A)$. Below we will write $m_n^{\alpha_0,\dots,\alpha_n}$ instead of $m_n^{\mathcal{MC}^{\mathcal{R}}_\infty(A)(\alpha_0,\dots,\alpha_n)}$

Remark 5.1. *The Maurer-Cartan equation and the formulas for higher multiplications are the same as in the definition of the A_∞-category of one-sided twisted complexes, see* [Ko]. *Note that in the case of one-sided twisted complexes all the solutions of Maurer-Cartan equation are automatically "nilpotent".*

Now we define the Maurer-Cartan pseudo-functor $\mathcal{MC}(A) : \mathrm{dgart} \to \mathbf{Gpd}$ as follows. Let $\mathcal{R}$ and $\mathfrak{m}$ be as above. The objects of the groupoid $\mathcal{MC}_{\mathcal{R}}(A)$ are the same as the objects of $\mathcal{MC}^{\mathcal{R}}_{\infty}(A)$. For $\alpha, \beta \in \mathcal{MC}^{\mathcal{R}}_{\infty}(A)$, let $G(\alpha, \beta)$ be the set of elements $g \in \mathbf{1} + (A \otimes \mathfrak{m})^0$ such that

$$m_1^{\alpha,\beta}(g) = \sum_{i_0, i_1 \geq 0} (-1)^{i_0 i_1 + \frac{i_0(i_0+1)}{2} + \frac{i_1(i_1-1)}{2}} m^{A\otimes\mathcal{R}}_{1+i_0+i_1}(\beta^{i_1}, g, \alpha^{i_0}) = 0.$$

Then we have an obvious action of the group $(A \otimes \mathfrak{m})^{-1}$ on the set $G(\alpha, \beta)$:

$$h : g \mapsto g + m_1^{\alpha,\beta}(h) = g + \sum_{i_0, i_1 \geq 0} (-1)^{i_0 i_1 + \frac{i_0(i_0-1)}{2} + \frac{i_1(i_1-1)}{2}} m^{A\otimes\mathcal{R}}_{1+i_0+i_1}(\beta^{i_1}, g, \alpha^{i_0}).$$

We define $\mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\alpha, \beta)$ as the set of orbits $G(\alpha, \beta)/(A \otimes \mathfrak{m})^{-1}$. The composition of morphisms in $\mathcal{MC}_{\mathcal{R}}(A)$ is induced by $m_2^{\mathcal{MC}^{\mathcal{R}}_{\infty}(A)}$. It follows from the axioms of A_∞-structures that we obtain a well defined category.

Proposition 5.2. *The category $\mathcal{MC}_{\mathcal{R}}(A)$ is a groupoid.*

Proof. Let $g \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\alpha, \beta)$. Prove that it has a left inverse $g' \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\beta, \alpha)$.

Let $\tilde{g} \in G(\alpha, \beta)$ be a lift of g. First prove that there exists $\tilde{g'} \in 1 + (A \otimes \mathfrak{m})^0$ such that

$$m_2^{\alpha,\beta,\alpha}(\tilde{g'}, \tilde{g}) = 1. \tag{5.2}$$

Let n be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over n.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m \geq 1$. Prove it for $n = m + 1$. From the induction hypothesis it follows that there exists $\tilde{g'} \in \mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\beta, \alpha)$ such that $m_2^{\alpha,\beta,\alpha}(\tilde{g'}, \tilde{g}) = 1 + x$, where $x \in (A \otimes \mathfrak{m}^{n-1})^0$. Then we obviously have

$$m_2^{\alpha,\beta,\alpha}(\tilde{g'} - x, \tilde{g}) = 1.$$

Thus, the induction hypothesis is proved for $n = m + 1$.The statement is proved.

Further, take $\tilde{g'} \in 1 + (A \otimes \mathfrak{m})^0$ such that (5.2) holds. To prove that g has a left inverse it suffices to prove that

$$m_1^{\beta,\alpha}(\tilde{g'}) = 0.$$

From the equality (5.2), and since $m_1^{\alpha,\beta}(g) = 0$, we obtain that

$$m_2^{\alpha,\beta,\alpha}(m_1^{\beta,\alpha}(\tilde{g'}), \tilde{g}) = 0.$$

Suppose that $m_1^{\beta,\alpha}(\tilde{g'}) \neq 0$. Take the maximal positive integer m such that $m_1^{\beta,\alpha}(\tilde{g'}) \in (A \otimes \mathfrak{m}^m)^0$. Then we obviously obtain that $m_2^{\alpha,\beta,\alpha}(m_1^{\beta,\alpha}(\tilde{g'}), \tilde{g}) \in (A \otimes \mathfrak{m}^m)^0 \setminus (A \otimes \mathfrak{m}^{m+1})^0$, this leads to contradiction.

Thus, g has a left inverse. Analogously, it has a right inverse, hence g is invertible. Therefore, the category $\mathcal{MC}_{\mathcal{R}}(A)$ is a groupoid. □

Clearly, the assignment $\mathcal{R} \mapsto \mathcal{MC}_{\mathcal{R}}(A)$ defines a pseudo-functor from dgart to **Gpd**. We denote this pseudo-functor by $\mathcal{MC}(A)$ and call it Maurer-Cartan pseudo-functor.

Notice that if A is a DG algebra, i.e. $m_n^A = 0$ for $n \geq 3$, then $\mathcal{MC}^{\mathcal{R}}_{\infty}(A)$ is a DG category. Further, for $\phi \in \mathrm{Hom}(\alpha, \beta)$ we have

$$d^{\mathcal{MC}^\infty_{\mathcal{R}}(A)}(x) = d^{A\otimes\mathcal{R}}(x) + \beta x - (-1)^{\bar{x}} x\alpha,$$

and the composition in $\mathcal{MC}^\infty_{\mathcal{R}}(A)$ is just the product in $A\otimes\mathcal{R}$. It follows that the constructed Maurer-Cartan pseudo-functor coincides in this case with that constructed in [ELOI], Section 5.

Remark 5.3. *The Maurer-Cartan groupoid $\mathcal{MC}_{\mathcal{R}}(A)$ can be extended to a ∞-groupoid $\mathcal{MC}^\infty_{\mathcal{R}}(A)$ so that $\mathcal{MC}_{\mathcal{R}}(A) = \pi_0(\mathcal{MC}^\infty_{\mathcal{R}}(A))$. Further, the assignment $\mathcal{R} \to \mathcal{MC}^\infty_{\mathcal{R}}(A)$ defines a pseudo- functor $\mathcal{MC}^\infty(A)$:* dgart $\to$ **Gpd**$^\infty$ *, where* **Gpd**$^\infty$ *is a ∞-category of ∞-groupoids.*

6. Obstruction theory

Fix a strictly unital A_∞-algebra A.

Let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Further, let n be the minimal positive integer such that $\mathfrak{m}^{n+1} = 0$. Put $\mathcal{I} = \mathfrak{m}^n$, $\bar{\mathcal{R}} = \mathcal{R}/\mathcal{I}$, and $\pi : \mathcal{R} \to \bar{\mathcal{R}}$ — the projection morphism. The next Proposition describes the obstruction theory for lifting of objects and morphisms along the functor

$$\pi^* : \mathcal{MC}_{\mathcal{R}}(A) \to \mathcal{MC}_{\bar{\mathcal{R}}}(A).$$

Proposition 6.1. *1). There exists a map $o_2 : Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A)) \to H^2(A\otimes\mathcal{I})$ such that $\alpha \in \mathcal{MC}_{\bar{\mathcal{R}}}(A)$ is in the image of π^* if and only if $o_2(\alpha) = 0$. Furthermore, if $\alpha, \beta \in \mathcal{MC}_{\bar{\mathcal{R}}}(A)$ are isomorphic then $o_2(\alpha) = 0$ iff $o_2(\beta) = 0$.*

2). Let $\xi \in Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A))$ be such that the fiber $(\pi^*)^{-1}(\xi)$ is non-empty. Then there exists a simply transitive action of the group $Z^1(A\otimes\mathcal{I})$ on the set $Ob((\pi^*)^{-1}(\xi))$. Let $\xi_1, \xi_2 \in Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A))$ be isomorphic objects such that both fibers $(\pi^*)^{-1}(\xi_1)$, $(\pi^*)^{-1}(\xi_2)$ are non-empty, and let $f : \xi_1 \to \xi_2$ be a morphism. Take the action of $Z^1(A\otimes\mathcal{I})$ on $Ob((\pi^*)^{-1}(\xi_2))$ as above and the action on $Ob((\pi^*)^{-1}(\xi_1))$ which is inverse to the above action. Then there is a (non-canonical) $Z^1(A\otimes\mathcal{I})$-equivariant map

$$\widetilde{o_1} : Ob((\pi^*)^{-1}(\xi_1)) \times Ob((\pi^*)^{-1}(\xi_2)) \to Z^1(A\otimes\mathcal{I}),$$

such that the composition of it with the projection

$$Z^1(A\otimes\mathcal{I}) \to H^1(A\otimes\mathcal{I}),$$

which we denote by o_1^f, is canonically defined and satisfies the following property: for $\alpha_1 \in Ob((\pi^*)^{-1}(\xi_1))$, $\alpha_2 \in Ob((\pi^*)^{-1}(\xi_2))$ there exists a morphism $\gamma : \alpha_1 \to \alpha_2$ such that $\pi^*(\gamma) = f$ iff $o_1^f(\alpha_1, \alpha_2) = 0$.

3) Let $\tilde{\alpha}, \tilde{\beta} \in \mathcal{MC}_{\mathcal{R}}(A)$ be objects and let $f : \alpha \to \beta$ be a morphism from $\alpha = \pi^*(\tilde{\alpha})$ to $\beta = \pi^*(\tilde{\beta})$. Suppose that the set $(\pi^*)^{-1}(f)$ of morphisms $\tilde{f} : \tilde{\alpha} \to \tilde{\beta}$ such that $\pi^*(\tilde{f}) = f$ is non-empty. Then there is a simple transitive action of the group $\mathrm{Im}(H^0(A\otimes\mathcal{I}) \to H^0(A\otimes\mathfrak{m}, m_1^{\alpha,\beta}))$ on the set $(\pi^*)^{-1}(f)$. In particular, the difference map

$$o_0 : (\pi^*)^{-1}(f) \times (\pi^*)^{-1}(f) \to \mathrm{Im}(H^0(A\otimes\mathcal{I}) \to H^0(A\otimes\mathfrak{m}, m_1^{\alpha,\beta}))$$

satisfies the following property: if $\tilde{f}, \tilde{f}' \in (\pi^*)^{-1}(f)$ then $\tilde{f} = \tilde{f}'$ iff $o_0(\tilde{f}, \tilde{f}') = 0$.

Proof. 1). Let $\alpha \in \mathcal{MC}_{\bar{\mathcal{R}}}(A)$. Take some $\tilde{\alpha} \in (A\otimes\mathfrak{m})^1$ such that $\pi(\tilde{\alpha}) = \alpha$. Then we have

$$\sum_{n\geq 1} (-1)^{\frac{n(n+1)}{2}+1} m_n^{A\otimes\mathcal{R}}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \in (A\otimes\mathcal{I})^2.$$

A straightforward applying of (3.2) shows that

$$\sum_{n\geq 1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\alpha},\dots,\tilde{\alpha})\in Z^2(A\otimes\mathcal{I}).$$

Further, if $\tilde{\alpha'}\in A\otimes\mathfrak{m}$ is another lift of α then

$$\sum_{n\geq 1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\alpha'},\dots,\tilde{\alpha'})-\sum_{n\geq 1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\alpha},\dots,\tilde{\alpha})=m_1^{A\otimes\mathcal{R}}(\tilde{\alpha'}-\tilde{\alpha}). \tag{6.1}$$

Hence, we obtain the well defined element $o_2(\alpha)\in H^2(A\otimes\mathcal{I})$ and therefore the map $o_2: Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A))\to H^2(A\otimes\mathcal{I})$. The first property of o_2 is obviously satisfied.

Further, let $\alpha,\beta\in\mathcal{MC}_{\bar{\mathcal{R}}}(A)$, and $f:\alpha\to\beta$ be a morphism. Suppose that $o_2(\alpha)=0$. Take some $\tilde{\alpha}\in(\pi^*)^{-1}(\alpha)$. Further, take some $\tilde{f}\in 1+(A\otimes\mathfrak{m})^0$ such that $\pi(\tilde{f})$ represents f, and $\tilde{\beta}\in(A\otimes\mathfrak{m})^1$ such that $\pi(\tilde{\beta})=\beta$. We have that

$$\sum_{i_0,i_1\geq 0}(-1)^{i_0i_1+\frac{i_0(i_0+1)}{2}+\frac{i_1(i_1+3)}{2}}m_{1+i_0+i_1}^{A\otimes\mathcal{R}}(\tilde{\beta}^{i_1},\tilde{f},\tilde{\alpha}^{i_0})\in(A\otimes\mathcal{I})^1$$

A straightforward applying of (3.2) shows that

$$m_1^{A\otimes\mathcal{R}}(\sum_{i_0,i_1\geq 0}(-1)^{i_0i_1+\frac{i_0(i_0+1)}{2}+\frac{i_1(i_1+3)}{2}}m_{1+i_0+i_1}^{A\otimes\mathcal{R}}(\tilde{\beta}^{i_1},\tilde{f},\tilde{\alpha}^{i_0}))=$$

$$=m_2^{A\otimes\mathcal{R}}(\sum_{n\geq 1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\beta},\dots,\tilde{\beta}),\tilde{f})=\sum_{n\geq 1}(-1)^{\frac{n(n+1)}{2}+1}m_n^{A\otimes\mathcal{R}}(\tilde{\beta},\dots,\tilde{\beta}).$$

Therefore, $o_2(\beta)=0$. This proves 1).

2). Let $\eta\in Z^1(A\otimes\mathcal{I})$. It follows from (6.1) that the formula

$$\eta:\alpha\mapsto\alpha+\eta$$

defines a simply transitive action of the group $Z^1(A\otimes\mathcal{I})$ on the set $Ob((\pi^*)^{-1}(\xi))$. Let ξ_1, ξ_2, f be as in Proposition. Take some $\tilde{f}\in 1+(A\otimes\mathfrak{m})^0$ such that $\overline{\pi(\tilde{f})}=f$. Define $\widetilde{o_1}$ by the formula

$$\widetilde{o_1}(\alpha,\beta)=o_1^{\tilde{f}}(\alpha,\beta)=m_1^{\alpha,\beta}(\tilde{f}).$$

It is easy to see that the image of $o_1^{\tilde{f}}$ lies in $Z^1(A\otimes\mathcal{I})$ and that $o_1^{\tilde{f}}$ is $Z^1(A\otimes\mathcal{I})$-equivariant.

If $\tilde{f}'$ is another lift of f, then there exists $h\in(A\otimes\mathfrak{m})^{-1}$ such that

$$v=\tilde{f}'-\tilde{f}-m_1^{\alpha,\beta}(h)\in(A\otimes\mathcal{I})^0.$$

Further,

$$o_1^{\tilde{f}'}(\alpha,\beta)-o_1^{\tilde{f}}(\alpha,\beta)=m_1^{A\otimes\mathcal{R}}(v),$$

hence the map o_1^f is canonically defined.

Suppose that $o_1^f(\alpha,\beta)=0$ for some $\alpha\in(\pi^*)^{-1}(\xi_1),\beta\in(\pi^*)^{-1}(\xi_2)$. Let $\tilde{f}$ be as above. Then there exists $x\in(A\otimes\mathcal{I})^0$ such that

$$m_1^{\alpha,\beta}(\tilde{f})=m_1^{A\otimes\mathcal{R}}(x).$$

We have $\tilde{f}-x\in G(\alpha,\beta)$, and $\pi^*(\overline{\tilde{f}-x})=f$.

Conversely, suppose that there exists a morphism $\gamma\in\mathrm{Hom}_{\mathcal{MC}_{\mathcal{R}}(A)}(\alpha,\beta)$ for some $\alpha\in(\pi^*)^{-1}(\xi_1)$, $\beta\in(\pi^*)^{-1}(\xi_2)$, such that $\pi^*(\gamma)=f$. Let $\tilde{\gamma}\in G(\alpha,\beta)$ be a representative of γ. Then we have $o_1^{\tilde{\gamma}}(\alpha,\beta)=0$, hence $o_1^f(\alpha,\beta)=0$. This proves 2).

3). First we define the action of the group $Z^0(A\otimes\mathcal{I})$ on the set $(\pi^*)^{-1}(f)$ by the formula

$$\eta:\bar{\tilde{f}}\to\overline{\tilde{f}+\eta},$$

where $\eta\in Z^0(A\otimes\mathcal{I})$, and $\tilde{f}\in G(\alpha,\beta)$ is such that $\pi^*(\bar{\tilde{f}})=f$. Clearly, this is correct. Further, if $\eta=m_1^{A\otimes\mathcal{R}}(\zeta)$ for some $\zeta\in(A\otimes\mathfrak{m})^{-1}$, then

$$\eta(\bar{\tilde{f}})=\overline{\tilde{f}+m_1^{\alpha,\beta}(\zeta)}=\bar{\tilde{f}}.$$

Hence, we have an action of $\mathrm{Im}(H^0(A\otimes\mathcal{I})\to H^0(A\otimes\mathfrak{m},m_1^{\alpha,\beta}))$ on the set $(\pi^*)^{-1}(f)$.

Tautologically, this action is simple.

Prove that it is transitive. Let $\tilde{f},\tilde{f}'\in G(\alpha,\beta)$ be such that $\pi^*(\bar{\tilde{f}})=\pi^*(\bar{\tilde{f}}')=f$. Then, by definition, there exists $h\in(A\otimes\mathfrak{m})^{-1}$ such that

$$\tilde{f}'-\tilde{f}-m_1^{\alpha,\beta}(h)\in(A\otimes\mathcal{I})^0.$$

Replacing $\tilde{f}$ by $\tilde{f}+m_1^{\alpha,\beta}(h)$, we obtain $\tilde{f}'=\tilde{f}+\eta$, where $\eta\in(A\otimes\mathcal{I})^0$. Since $\tilde{f},\tilde{f}'\in G(\alpha,\beta)$, we have that $\eta\in Z^0(A\otimes\mathcal{I})$. This shows transitivity and proves 3).

Proposition is proved. □

Remark 6.2. *One can also construct the obstruction theory for lifting of objects and all k-morphisms along the ∞-functor*

$$\pi^*:\mathcal{MC}^{\infty}_{\mathcal{R}}(A)\to\mathcal{MC}^{\infty}_{\mathcal{R}}(A).$$

7. Invariance Theorems

Let A_1,A_2 be strictly unital A_∞-algebras and $f:A_1\to A_2$ be a strictly unital A_∞-morphism between them given by a sequence of maps

$$f_n:A_1^{\otimes n}\to A_2.$$

Further, let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$.

Then we have a (strictly unital) A_∞-functor

$$f^*_{\mathcal{R}}:\mathcal{MC}^{\mathcal{R}}_{\infty}(A_1)\to\mathcal{MC}^{\mathcal{R}}_{\infty}(A_2)$$

defined by the formulas

$$f^*_{\mathcal{R}}(\alpha)=\sum_{n\geq1}(-1)^{\frac{n(n-1)}{2}}f_n(\alpha,\dots,\alpha);$$

$$f^*_{\mathcal{R}}(\alpha_0,\dots,\alpha_n)(x_1,\dots,x_n)=\sum_{i_0,\dots,i_n\geq0}(-1)^{\epsilon}f_{n+i_0+\dots+i_n}(\alpha_n^{i_n},x_n,\alpha_{n-1}^{i_{n-1}},\dots,\alpha_1^{i_1},x_1,\alpha_0^{i_0}),$$

where

$$\epsilon=\sum_{n\geq k>j\geq0}(\bar{x_k}+i_k)i_j+\sum_{k=0}^{n}\frac{i_k(i_k-1)}{2}+\sum_{k=1}^{n}ki_k.$$

One checks without difficulties that these formulas indeed define a strictly unital A_∞-functor.

It induces a functor $f^*_{\mathcal{R}}:\mathcal{MC}_{\mathcal{R}}(A_1)\to\mathcal{MC}_{\mathcal{R}}(A_2)$ and we obtain a morphism of pseudo-functors

$$f^*:\mathcal{MC}(A_1)\to\mathcal{MC}(A_2).$$

The following theorems show that for quasi-isomorphic strictly unital A_∞-algebras the corresponding Maurer-Cartan A_∞-categories (resp. Maurer-Cartan pseudo-functors) are quasi-equivalent (resp. equivalent).

Theorem 7.1. *Let $f : A_1 \to A_2$ be a strictly unital quasi-isomorphism of strictly unital A_∞-algebras and let $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $\mathfrak{m}$. Then the A_∞-functor*

$$f^*_{\mathcal{R}} : \mathcal{MC}^{\mathcal{R}}_\infty(A_1) \to \mathcal{MC}^{\mathcal{R}}_\infty(A_2)$$

is a quasi-equivalence.

Proof. 1). Prove that for any $\alpha, \beta \in \mathcal{MC}^{\mathcal{R}}_\infty(A_1)$ the morphism of complexes

$$f^*_{\mathcal{R}}(\alpha, \beta) : \mathrm{Hom}_{\mathcal{MC}^{\mathcal{R}}_\infty(A_1)}(\alpha, \beta) \to \mathrm{Hom}_{\mathcal{MC}^{\mathcal{R}}_\infty(A_2)}(f^*(\alpha), f^*(\beta))$$

is quasi-isomorphism. Note that both complexes have finite filtrations by subcomplexes $A_1 \otimes \mathfrak{m}^i$ and $A_2 \otimes \mathfrak{m}^i$. The morphism $f^*_{\mathcal{R}}(\alpha, \beta)$ is compatible with these filtrations and induces quasi-isomorphisms on the subquotients. Hence, it is quasi-isomorphism.

2). Now we prove that the functor

$$\mathrm{Ho}(f^*) : \mathrm{Ho}(\mathcal{MC}^{\mathcal{R}}_\infty(A_1)) \to \mathrm{Ho}(\mathcal{MC}^{\mathcal{R}}_\infty(A_2))$$

is an equivalence. We have already proved that it is full faithful, hence it remains to prove that it is essentially surjective. We will prove the stronger statement: the functor

$$f^*_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(A_1) \to \mathcal{MC}_{\mathcal{R}}(A_2)$$

is essentially surjective.

Let n be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over n.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m$. Prove it for $n = m + 1$. Let $\mathcal{I}$, $\bar{\mathcal{R}}$, $\bar{\mathfrak{m}}$, π be as above. A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A_1)) & \xrightarrow{f^*_{\bar{\mathcal{R}}}} & Ob(\mathcal{MC}_{\bar{\mathcal{R}}}(A_2)) \\ {\scriptstyle o_2}\downarrow & & {\scriptstyle o_2}\downarrow \\ H^2(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^2(A_2 \otimes \mathcal{I}), \end{array} \tag{7.1}$$

where the map o_2 is defined in Prpoosition 6.1.

Let $\alpha \in \mathcal{MC}_{\mathcal{R}}(A_2)$. By the induction hypothesis, there exists $\beta \in \mathcal{MC}_{\bar{\mathcal{R}}}(A_1)$ such that $f^*_{\bar{\mathcal{R}}}(\beta)$ is isomorphic to $\pi^*(\alpha)$ in $\mathcal{MC}_{\bar{\mathcal{R}}}(A_2)$. Since the diagram (7.1) commutes, we have that $o_2(\beta) = 0$. Thus, by Proposition 6.1, the fiber $(\pi^*)^{-1}(\beta)$ is nonempty. Fix some $\tilde{\beta} \in (\pi^*)^{-1}(\beta)$. Let $\gamma : f^*_{\bar{\mathcal{R}}}(\beta) \to \pi^*(\alpha)$ be a morphism. A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} Ob((\pi^*)^{-1}(\beta)) & \xrightarrow{f^*_{\mathcal{R}}} & Ob((\pi^*)^{-1}(f^*_{\bar{\mathcal{R}}}(\beta))) \\ \downarrow{\scriptstyle o_1^{id_\beta}(*,\tilde{\beta})} & & \downarrow{\scriptstyle o_1^{\gamma}(f^*_{\mathcal{R}}(*),\alpha) - o_1^{\gamma}(f^*_{\mathcal{R}}(\tilde{\beta}),\alpha)} \\ H^1(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^1(A_2 \otimes \mathcal{I}), \end{array} \tag{7.2}$$

where the vertical arrows are defined in Proposition 6.1. Since the map $o_1^{id_\beta}(*, \tilde{\beta})$ is surjective and the diagram (7.2) commutes, there exists an object $\tilde{\beta}' \in Ob((\pi^*)^{-1}(\beta))$ such that $o_1^{\gamma}(f^*_{\mathcal{R}}(\tilde{\beta}'), \alpha) = 0$. Then, by Proposition 6.1, there exists a morphism $\tilde{\gamma} : f^*_{\mathcal{R}}(\tilde{\beta}') \to \alpha$ (such that $\pi^*(\tilde{\gamma}) = \gamma$). Therefore, the functor $f^*_{\mathcal{R}}$ is essentially surjective, and the induction hypothesis is proved for $n = m + 1$. The statement is proved.

Theorem is proved. □

Theorem 7.2. *Let $f : A_1 \to A_2$ be a strictly unital quasi-isomorphism of strictly unital A_∞-algebras. Then the morphism of pseudo-functors*

$$f^* : \mathcal{MC}(A_1) \to \mathcal{MC}(A_2)$$

is an equivalence.

Proof. Fix an artinian DG algebra $\mathcal{R}$ with the maximal ideal $\mathfrak{m}$. We must prove that the functor

$$f^*_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(A_1) \to \mathcal{MC}_{\mathcal{R}}(A_2)$$

is an equivalence.

In the proof of the previous Theorem we have already shown that it is essentially surjective. So it remains to prove that it is full and faithful.

Let n be the minimal positive integer such that $\mathfrak{m}^n = 0$. The proof is by induction over n.

For $n = 1$, there is nothing to prove.

Suppose that the induction hypothesis holds for $n = m \geq 1$. Prove it for $n = m + 1$.

Full. Let $\alpha, \beta \in \mathcal{MC}_{\mathcal{R}}(A_1)$ and let $\gamma : f^*_{\mathcal{R}}(\alpha) \to f^*_{\mathcal{R}}(\beta)$ be a morphism. By induction hypothesis, there exists a morphism $g : \pi^*(\alpha) \to \pi^*(\beta)$ such that

$$f^*_{\bar{\mathcal{R}}}(g) = \pi^*(\gamma).$$

A straightforward checking shows that the following diagram commutes

$$\begin{array}{ccc} (\pi^*)^{-1}(\pi^*(\alpha)) \times (\pi^*)^{-1}(\pi^*(\beta)) & \longrightarrow & (\pi^*)^{-1}(\pi^*(f^*_{\bar{\mathcal{R}}}(\alpha))) \times (\pi^*)^{-1}(\pi^*(f^*_{\bar{\mathcal{R}}}(\beta))) \\ \Big\downarrow o_1^g & & \Big\downarrow o_1^{\pi^*(\gamma)} \\ H^1(A_1 \otimes \mathcal{I}) & \xrightarrow{\sim} & H^1(A_2 \otimes \mathcal{I}). \end{array} \tag{7.3}$$

By Proposition 6.1 and since the diagram (7.3) commutes there exists a morphism $\tilde{g} : \alpha \to \beta$ such that $\pi^*(\tilde{g}) = g$. Further, a straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} (\pi^*)^{-1}(g) & \xrightarrow{o_0(*,\tilde{g})} & \mathrm{Im}(H^0(A_1 \otimes \mathcal{I}) \to H^0(A_1 \otimes \mathfrak{m}, m_1^{\alpha,\beta})) \\ \Big\downarrow f^*_{\mathcal{R}} & & \sim\Big\downarrow \\ (\pi^*)^{-1}(\pi^*(\gamma)) & \xrightarrow{o_0(f^*_{\mathcal{R}}(*),\gamma) - o_0(f^*_{\mathcal{R}}(\tilde{g}),\gamma)} & \mathrm{Im}(H^0(A_2 \otimes \mathcal{I}) \to H^0(A_2 \otimes \mathfrak{m}, m_1^{f^*_{\mathcal{R}}(\alpha), f^*_{\mathcal{R}}(\beta)})) \end{array} \tag{7.4}$$

Since the upper arrow is surjective, there exists a morphism $\tilde{g}' \in (\pi^*)^{-1}(g)$ such that

$$o_0(f^*_{\mathcal{R}}(\tilde{g}'), \gamma) = 0,$$

i.e. $f^*_{\mathcal{R}}(\tilde{g}') = \gamma$. Hence, the functor $f^*_{\mathcal{R}}$ is full.

Faithful. Let $\gamma_1, \gamma_2 : \alpha \to \beta$ be two morphisms in $\mathcal{MC}_{\mathcal{R}}(A_1)$. Suppose that $f^*_{\mathcal{R}}(\gamma_1) = f^*_{\mathcal{R}}(\gamma_2)$. Then we have also $f^*_{\mathcal{R}}(\pi^*(\gamma_1)) = f^*_{\mathcal{R}}(\pi^*(\gamma_2))$, hence by induction hypothesis $\pi^*(\gamma_1) = \pi^*(\gamma_2)$. A straightforward checking shows that the following diagram commutes:

$$\begin{array}{ccc} (\pi^*)^{-1}(\pi^*(\gamma_1)) \times (\pi^*)^{-1}(\pi^*(\gamma_1)) & \xrightarrow{o_0} & \mathrm{Im}(H^0(A_1 \otimes \mathcal{I}) \to H^0(A_1 \otimes \mathfrak{m}, m_1^{\alpha,\beta})) \\ \Big\downarrow f^*_{\mathcal{R}} & & \sim\Big\downarrow \\ (\pi^*)^{-1}(\pi^*(f^*_{\mathcal{R}}(\gamma_1))) \times (\pi^*)^{-1}(\pi^*(f^*_{\mathcal{R}}(\gamma_1))) & \xrightarrow{o_0} & \mathrm{Im}(H^0(A_2 \otimes \mathcal{I}) \to H^0(A_2 \otimes \mathfrak{m}, m_1^{f^*_{\mathcal{R}}(\alpha), f^*_{\mathcal{R}}(\beta)})). \end{array} \tag{7.5}$$

By Proposition 6.1 and since the diagram (7.5) commutes we have that

$$o_0(\gamma_1, \gamma_2) = 0,$$

hence $\gamma_1 = \gamma_2$. Thus, the functor $f^*_{\mathcal{R}}$ is full.

The induction hypothesis is proved for $n = m + 1$. The statement is proved.

Theorem is proved. □

Remark 7.3. *It can be proved that an A_∞-quasi-isomorphism $f : A_1 \to A_2$ induces an equivalence of ∞-groupoids $f^*_{\mathcal{R}} : \mathcal{MC}^\infty_{\mathcal{R}}(A_1) \to \mathcal{MC}^\infty_{\mathcal{R}}(A_2)$.*

8. Twisting cochains

Let $\mathcal{G}$ be a co-augmented DG coalgebra. Let A be an arbitrary A_∞-algebra. Then the graded vector space $Hom_k(\mathcal{G}, A)$ has natural structure of an A_∞-algebra. If $\dim \mathcal{G} < \infty$ or $\dim A < \infty$, then $Hom_k(\mathcal{G}, A)$ is canonically identified with $A \otimes \mathcal{G}^*$ as an A_∞-algebra.

Suppose that the DG coalgebra $\mathcal{G}$ is co-complete. The map $\tau : \mathcal{G} \to A$ of degree 1 is called a twisting cochain if it passes through $\bar{\mathcal{G}}$ and satisfies the generalized Maurer-Cartan equation (5.1) as an element of A_∞-algebra $Hom_k(\mathcal{G}, A)$. This is well defined since $\mathcal{G}$ is co-complete. If $\mathcal{R}$ is an artinian DG algebra and A is strictly unital, then we have natural bijection between the set of twisting cochains $\tau : \mathcal{R}^* \to A$ and the set $MC(A \otimes \mathfrak{m})$. In the case when A is augmented, the twisting cochain is called admissible if it passes through $\bar{A}$. Tautologically, admissible twisting cochains $\mathcal{G} \to A$ are in one-to-one correspondence with twisting cochains $\mathcal{G} \to \bar{A}$.

Proposition 8.1. *Let A be an A_∞-algebra The composition $\tau_A : BA \to A$ of the natural projection $BA \to A[1]$ with the shift map $A[1] \to A$ is the universal twisting cochain. That is, if $\mathcal{G}$ is a co-augmented co-complete DG coalgebra and $\tau : \mathcal{G} \to A$ is a twisting cochain then there exists a unique homomorphism $g_\tau : \mathcal{G} \to BA$ of co-augmented DG coalgebras, such that $\tau_A \cdot g_\tau = \tau$.*

It follows that if A is augmented then the composition of $\tau_{\bar{A}}$ with the embedding $\bar{A} \hookrightarrow A$, which we also denote by τ_A, is the universal admissible twisting cochain in the same sense.

Proof. A straightforward checking. □

Further, if $\mathcal{G}$ is a co-augmented co-complete DG coalgebra, and $\tau : \mathcal{G} \to A$ is a twisting cochain then

$$\mathcal{G} \otimes_\tau A := \mathcal{G} \Box_{BA} (BA \otimes_{\tau_A} A)$$

is an object of $A^{op}_{(\mathcal{G}^*)^{op}}\text{-mod}_\infty$.

Proposition 8.2. *Let $f : A_1 \to A_2$ be an A_∞-quasi-isomorphism of A_∞-algebras, $\mathcal{G}$ be a co-augmented co-complete DG coalgebra, and $\tau : \mathcal{G} \to A_1$ be a twisting cochain. Then there is a natural homotopy equivalence in $A^{op}_{1(\mathcal{G}^*)^{op}}$-mod$_\infty$:*

$$\mathcal{G} \otimes_\tau A_1 \to f_*(\mathcal{G} \otimes_{f \cdot \tau} A_2).$$

Proof. We have a natural homotopy equivalence of DG bicomodules over BA_1:

$$BA_1 \otimes A_1[1] \otimes BA_1 \to BA_1 \Box_{BA_2} (BA_2 \otimes A_2[1] \otimes BA_2) \Box_{BA_2} BA_1.$$

Co-tensoring it on the left by $\mathcal{G}$, we obtain the required homotopy equivalence. □

Now let A be an augmented A_∞-algebra. If $\mathcal{G}$ is a co-augmented co-complete DG coalgebra and $\tau : \mathcal{G} \to A$ is an admissible twisting cochain then

$$\mathcal{G} \otimes_\tau A := \mathcal{G} \square_{B\bar{A}} (B\bar{A} \otimes_{\tau_A} A)$$

is an object of $\bar{A}^{op}_{(\mathcal{G}^*)^{op}}\text{-mod}_\infty$.

Proposition 8.3. *Let $f : A_1 \to A_2$ be an A_∞-quasi-isomorphism of augmented A_∞-algebras, $\mathcal{G}$ be a co-augmented co-complete DG coalgebra and $\tau : \mathcal{G} \to A_1$ be an admissible twisting cochain. Then there is a natural homotopy equivalence in $\bar{A}_1{}^{op}_{(\mathcal{G}^*)^{op}}$-mod$_\infty$:*

$$\mathcal{G} \otimes_\tau A_1 \to f_*(\mathcal{G} \otimes_{f\cdot\tau} A_2).$$

Proof. We have a natural homotopy equivalence of DG bicomodules over $B\bar{A}_1$:

$$B\bar{A}_1 \otimes A_1[1] \otimes B\bar{A}_1 \to B\bar{A}_1 \square_{B\bar{A}_2} (B\bar{A}_2 \otimes A_2[1] \otimes B\bar{A}_2) \square_{B\bar{A}_2} B\bar{A}_1.$$

Co-tensoring it on the left by $\mathcal{G}$, we obtain the required homotopy equivalence. □

Let $\mathcal{R}$ be an artinian DG algebra, and $\tau : \mathcal{R}^* \to A$ be an admissible twisting cochain. Then by Proposition 8.1 we have a natural morphism of DG coalgebras $g_\tau : \mathcal{R}^* \to B\bar{A}$. Further, we have the dual morphism of DG algebras $g_\tau^* : \hat{S} \to \mathcal{R}$. In particular, $\mathcal{R}$ becomes a DG $\hat{S}^{op}$-module.

Lemma 8.4. *In the above notation A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-modules $\mathrm{Hom}_{\hat{S}^{op}}(\mathcal{R}, B\bar{A} \otimes_{\tau_A} A)$ and $\mathcal{R}^* \otimes_\tau A$ are isomorphic.*

Proof. Evident. □

If $\tau : \mathcal{R}^* \to A$ is an admissible twisting cochain and $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ is the corresponding object, then we will write also $A \otimes_\alpha \mathcal{R}^*$ instead of $\mathcal{R}^* \otimes_\tau A$.

Further, for $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ corresponding to an admissible twisting cochain we put

$$A \otimes_\alpha \mathcal{R} := \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^*, A \otimes_\alpha \mathcal{R}^*).$$

This is an object of $\bar{A}^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$. Its $(\mathcal{R}^{op})^{gr}$-module structure is obvious and A_∞-module structure can also be given by the explicit formulas:

$$m_n^{A\otimes_\alpha \mathcal{R}}(m, a_1, \dots, a_{n-1}) = m_n^{0,\dots,0,\alpha}(m, a_1 \otimes \mathbf{1}_{\mathcal{R}}, \dots, a_{n-1} \otimes \mathbf{1}_{\mathcal{R}}). \tag{8.1}$$

Proposition 8.5. *Let $f : A_1 \to A_2$ be an A_∞-quasi-isomorphism of augmented A_∞-algebras, $\mathcal{R}$ be an artinian DG algebra and let $\alpha \in \mathcal{MC}_{\mathcal{R}}(A_1)$. Then there is a natural homotopy equivalence in $\bar{A}_1{}^{op}_{\mathcal{R}^{op}}$-mod$_\infty$:*

$$A_1 \otimes_\alpha \mathcal{R} \to f_*(A_2 \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}).$$

Proof. The required homotopy equivalence is obtained by applying the functor $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^*, -)$ to the homotopy equivalence

$$A_1 \otimes_\alpha \mathcal{R}^* \to f_*(A_2 \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}^*)$$

from Proposition 8.3. □

Note that if A is a DG algebra, then $A\otimes_\alpha \mathcal{R}^*$ and $A\otimes_\alpha \mathcal{R}$ are the DG modules from $\mathrm{coDef}^h_{\mathcal{R}}(A)$ and $\mathrm{Def}^h_{\mathcal{R}}(A)$ respectively, which correspond to α.

Finally, if A is a strictly unital but not necessarily augmented A_∞-algebra, $\mathcal{R}$ is an artinian DG algebra, α is an object of $\mathcal{MC}_{\mathcal{R}}(A)$ and $\tau:\mathcal{R}^*\to A$ is the corresponding twisting cochain then we also write $A\otimes_\alpha \mathcal{R}^*$ instead of $\mathcal{R}^*\otimes_\tau A$. Further, we put

$$A\otimes_\alpha \mathcal{R}=\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^*, A\otimes_\alpha \mathcal{R}^*).$$

This is the object of $A^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$. Again, its $(\mathcal{R}^{op})^{gr}$-module structure is obvious and the A_∞-module structure is given by the formulas 8.1. The following Proposition is absolutely analogous to the previous one and we omit the proof.

Proposition 8.6. *Let $f:A_1\to A_2$ be a strictly unital A_∞-morphism of strictly unital A_∞-algebras, $\mathcal{R}$ be an artinian DG algebra and let $\alpha\in\mathcal{MC}_{\mathcal{R}}(A_1)$. Then there is a natural homotopy equivalence in $A^{op}_{1\mathcal{R}^{op}}\text{-mod}_\infty$:*

$$A_1\otimes_\alpha \mathcal{R}\to f_*(A_2\otimes_{f^*_{\mathcal{R}}(\alpha)}\mathcal{R}).$$

Part 3. The pseudo-functors DEF and coDEF

9. The bicategory 2-adgalg and deformation pseudo-functor coDEF

Let $\mathcal{E}$ be a bicategory and $F,G:\mathcal{E}\to\mathbf{Gpd}$ two pseudo-functors. A morphism $\epsilon:F\to G$ is called an equivalence if for each $X\in Ob\mathcal{E}$ the functor $\epsilon_X:F(X)\to G(X)$ is an equivalence of categories.

Definition 9.1. *([ELOII]) We define the bicategory* 2-adgalg *of augmented DG algebras as follows. The objects are augmented DG algebras. For DG algebras $\mathcal{B},\mathcal{C}$ the collection of 1-morphisms* $1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$ *consists of pairs (M,θ), where*

- *$M\in D(\mathcal{B}^{op}\otimes\mathcal{C})$ is such that there exists an isomorphism (in $D(\mathcal{C})$) $\mathcal{C}\to\nu_*M$ (where $\nu_*:D(\mathcal{B}^{op}\otimes\mathcal{C})\to D(\mathcal{C})$ is the functor of restriction of scalars corresponding to the natural homomorphism $\nu:\mathcal{C}\to\mathcal{B}^{op}\otimes\mathcal{C}$);*
- *and $\theta:k\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}M\to k$ is an isomorphism in $D(\mathcal{B}^{op})$.*

The composition of 1-morphisms

$$1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})\times 1\text{-}\mathrm{Hom}(\mathcal{C},\mathcal{D})\to 1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{D})$$

is defined by the tensor product $\cdot\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}\cdot$. Given 1-morphisms $(M_1,\theta_1),(M_2,\theta_2)\in 1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$ a 2-morphism $f:(M_1,\theta_1)\to(M_2,\theta_2)$ is an isomorphism (in $D(\mathcal{B}^{op}\otimes\mathcal{C})$) $f:M_2\to M_1$ (not from M_1 to M_2!) such that $\theta_1\cdot k\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}(f)=\theta_2$. So in particular the category $1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$ is a groupoid. Denote by 2-dgart *the full subbicategory of* 2-adgalg *consisting of artinian DG algebras. Similarly we define the full subbicategories* $2\text{-}\mathrm{dgart}_+$, $2\text{-}\mathrm{dgart}_-$, 2-art, 2-cart *([ELOI], Definition 2.3).*

Remark 9.2. *Assume that augmented DG algebras $\mathcal{B}$ and $\mathcal{C}$ are such that $\mathcal{B}^i=\mathcal{C}^i=0$ for $i>0$, $\dim\mathcal{B}^i,\dim\mathcal{C}^i<\infty$ for all i and $\dim H(\mathcal{C})<\infty$. Denote by $\langle k\rangle\subset D(\mathcal{B}^{op}\otimes\mathcal{C})$ the triangulated envelope of the DG $\mathcal{B}^{op}\otimes\mathcal{C}$-module k. Let $(M,\theta)\in 1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$. Then by [ELOI], Corollary 3.22 $M\in\langle k\rangle$.*

For any augmented DG algebra $\mathcal{B}$ we obtain a pseudo-functor $h_{\mathcal{B}}$ between the bicategories 2-adgalg and **Gpd** defined by $h_{\mathcal{B}}(\mathcal{C})=1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$.

Note that a usual homomorphism of augmented DG algebras $\gamma : \mathcal{B} \to \mathcal{C}$ defines the structure of a DG $\mathcal{B}^{op}$-module on $\mathcal{C}$ with the canonical isomorphism of DG $\mathcal{B}^{op}$-modules $\mathrm{id} : k \overset{\mathbf{L}}{\otimes}_{\mathcal{C}} \mathcal{C} \to k$. Hence, we have a natural pseudo-functor $\mathcal{F} : \mathrm{adgalg} \to 2\text{-}\mathrm{adgalg}$.

The next proposition from [ELOII] asserts that the deformation functor coDef has a natural "lift" to the bicategory 2-dgart. We present its proof for completeness.

Proposition 9.3. *([ELOII]) There exist a pseudo-functor* $\mathrm{coDEF}(E)$ *from* 2-dgart *to* **Gpd** *which is an extension to* 2-dgart *of the pseudo-functor* coDef *i.e. there is an equivalence of pseudo-functors* $\mathrm{coDef}(E) \simeq \mathrm{coDEF}(E) \cdot \mathcal{F}$.

Proof. Given artinian DG algebras $\mathcal{R}, \mathcal{Q}$ and $M = (M, \theta) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ we need to define the corresponding functor

$$M^! : \mathrm{coDef}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{Q}}(E).$$

Let $S = (S, \sigma) \in \mathrm{coDef}_{\mathcal{R}}(E)$. Put

$$M^!(S) := \mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}(M, S) \in D(\mathcal{A}^{op}_{\mathcal{Q}}).$$

We claim that $M^!(S)$ defines an object in $\mathrm{coDef}_{\mathcal{Q}}(E)$, i.e. $\mathcal{R}\,\mathrm{Hom}_{\mathcal{Q}^{op}}(k, M^!(S))$ is naturally isomorphic to E (by the isomorphisms θ and σ).

Indeed, choose quasi-isomorphisms $P \to k$ and $S \to I$ for $P \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{Q}})$ and $I \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$. Then

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{Q}^{op}}(k, M^!(S)) = \mathrm{Hom}_{\mathcal{Q}^{op}}(P, \mathrm{Hom}_{\mathcal{R}^{op}}(M, I)).$$

By [ELOI], Lemma 3.17 the last term is equal to $\mathrm{Hom}_{\mathcal{R}^{op}}(P \otimes_{\mathcal{Q}} M, I)$. Now the isomorphism θ defines an isomorphism between $P \otimes_{\mathcal{Q}} M = k \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M$ and k, and we compose it with the isomorphism $\sigma : E \to \mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}(k, I) = i^! S$.

So $M^!$ is a functor from $\mathrm{coDef}_{\mathcal{R}}(E)$ to $\mathrm{coDef}_{\mathcal{Q}}(E)$.

Given another artinian DG algebra $\mathcal{Q}'$ and $M' \in 1\text{-}\mathrm{Hom}(\mathcal{Q}, \mathcal{Q}')$ there is a natural isomorphism of functors

$$(M' \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M)^!(-) \simeq M'^! \cdot M^!(-).$$

(This follows again from [ELOI], Lemma 3.17).

Also a 2-morphism $f \in 2\text{-}\mathrm{Hom}(M, M_1)$ between objects $M, M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ induces an isomorphism of the corresponding functors $M^! \xrightarrow{\sim} M_1^!$.

Thus we obtain a pseudo-functor $\mathrm{coDEF}(E) : 2\text{-}\mathrm{dgart} \to \mathbf{Gpd}$, such that $\mathrm{coDEF}(E) \cdot \mathcal{F} = \mathrm{coDef}(E)$. □

We denote by $\mathrm{coDEF}_+(E)$, $\mathrm{coDEF}_-(E)$, $\mathrm{coDEF}_0(E)$, $\mathrm{coDEF}_{\mathrm{cl}}(E)$ the restriction of the pseudo-functor $\mathrm{coDEF}(E)$ to subbicategories $2\text{-}\mathrm{dgart}_+$, $2\text{-}\mathrm{dgart}_-$, 2-art and 2-cart respectively.

Proposition 9.4. *([ELOII]) A quasi-isomorphism* $\delta : E_1 \to E_2$ *of DG* $\mathcal{A}^{op}$*-modules induces an equivalence of pseudo-functors*

$$\delta^* : \mathrm{coDEF}(E_2) \to \mathrm{coDEF}(E_1)$$

defined by $\delta^*(S, \sigma) = (S, \sigma \cdot \delta)$.

Proof. This is clear. □

Proposition 9.5. *([ELOII]) Let $F:\mathcal{A}\to\mathcal{A}'$ be a DG functor which induces a quasi-equivalence $F^{pre\text{-}tr}:\mathcal{A}^{pre\text{-}tr}\to\mathcal{A}'^{pre\text{-}tr}$ (this happens for example if F is a quasi-equivalence). Then for any $E\in D(\mathcal{A}^{op})$ the pseudo-functors $\mathrm{coDEF}_-(E)$ and $\mathrm{coDEF}_-(\mathbf{R}F^!(E))$ are equivalent (hence also $\mathrm{coDEF}(F_*(E'))$ and $\mathrm{coDEF}_-(E')$ are equivalent for any $E'\in D(\mathcal{A}'^0)$).*

Corollary 9.6. *([ELOII]) Assume that DG algebras $\mathcal{B}$ and $\mathcal{C}$ are quasi-isomorphic. Then the pseudo-functors $\mathrm{coDEF}_-(\mathcal{B})$ and $\mathrm{coDEF}_-(\mathcal{C})$ are equivalent.*

Proof. We may assume that there exists a homomorphism of DG algebras $\mathcal{B}\to\mathcal{C}$ which is a quasi-isomorphism. Then put $\mathcal{A}=\mathcal{B}$ and $\mathcal{A}'=\mathcal{C}$ in the last proposition. □

The following Lemma is stronger then [ELOI], Corollary 11.15 for the pseudo-functors coDef_- and coDef^h_-. It is proved by A_∞-methods.

Lemma 9.7. *Let $\mathcal{B}$ be a DG algebra. Suppose that the following conditions hold:*

a) $H^{-1}(\mathcal{B})=0$;

b) the graded algebra $H(\mathcal{B})$ is bounded below.

Then the pseudo-functors $\mathrm{coDef}_-(\mathcal{B})$ and $\mathrm{coDef}^h_-(\mathcal{B})$ are equivalent.

Proof. Fix some negative artinian DG algebra $\mathcal{R}\in\mathrm{dgart}_-$. Take some $(T,id)\in\mathrm{coDef}^h_{\mathcal{R}}(\mathcal{B})$. Due to [ELOI], Corollary 11.4 b) it suffices to prove that $i^!T=\mathbf{R}i^!T$ Let A be a strictly unital minimal model of $\mathcal{B}$, and let $f:A\to\mathcal{B}$ be a strictly unital A_∞ quasi-isomorphism. By our assumption on $H(\mathcal{B})$, A is bounded below.

By Theorem 7.2 there exists an object $\alpha\in\mathcal{MC}_{\mathcal{R}}(A)$ such that $S\cong\mathcal{B}\otimes_{f^*_{\mathcal{R}}(\alpha)}\mathcal{R}^*$. The DG $\mathcal{R}^{op}$-modules $\mathcal{B}\otimes_{f^*_{\mathcal{R}}(\alpha)}\mathcal{R}^*$ and $f_*(\mathcal{B}\otimes_{f^*_{\mathcal{R}}(\alpha)}\mathcal{R}^*)$ are naturally identified. Further, by Proposition 8.6 we have natural homotopy equivalence (in $A^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$)

$$\gamma:A\otimes_\alpha\mathcal{R}^*\to f_*(\mathcal{B}\otimes_{f^*_{\mathcal{R}}(\alpha)}\mathcal{R}^*)$$

Thus, it remains to prove that

$$i^!(A\otimes_\alpha\mathcal{R}^*)=\mathbf{R}i^!(A\otimes_\alpha\mathcal{R}^*).$$

We claim that $A\otimes_\alpha\mathcal{R}^*$ is h-injective. Indeed, since A is bounded below and $\mathcal{R}\in\mathrm{dgart}_-$, this DG $\mathcal{R}^{op}$-module has a decreasing filtration by DG $\mathcal{R}^{op}$-submodules $A^{\geq i}\otimes\mathcal{R}^*$ with subquotients being cofree DG $\mathcal{R}^{op}$-modules $A^i\otimes\mathcal{R}^*$. Thus $A\otimes_\alpha\mathcal{R}^*$ satisfies property (I) as DG $\mathcal{R}^{op}$-module ([ELOI], Definition 3.3) and hence is h-injective. Lemma is proved. □

The next result implies stronger statement for pseudo-functor coDef_- then [ELOI], Proposition 11.16.

Proposition 9.8. *Let $E\in\mathcal{A}^{op}$-mod. Assume that*

a) $\mathrm{Ext}^{-1}(E,E)=0$;

b) the graded algebra $\mathrm{Ext}(E,E)$ is bounded below;

b) there exists a bounded below h-projective or h-injective DG $\mathcal{A}^{op}$-module F which is quasi-isomorphic to E.

Put $\mathcal{B}=\mathrm{End}(F)$. Then the pseudo-functors $\mathrm{coDEF}_-(\mathcal{B})$ and $\mathrm{coDEF}_-(E)$ are equivalent.

Proof. Consider the DG functor

$$\mathcal{L}:=\Sigma^F\cdot\psi^*:\mathcal{B}^{op}\text{-mod}\to\mathcal{A}^{op}\text{-mod},\quad \mathcal{L}(N)=N\otimes_{\mathcal{C}}F$$

as in [ELOI], Remark 11.17. It induces the equivalence of pseudo-functors

$$\mathrm{coDef}^{\mathrm{h}}(\mathcal{L}) : \mathrm{coDef}^{\mathrm{h}}_{-}(\mathcal{B}) \xrightarrow{\sim} \mathrm{coDef}^{\mathrm{h}}_{-}(F),$$

i.e. for every artinian DG algebra $\mathcal{R} \in \mathrm{dgart}_{-}$ the corresponding DG functor

$$\mathcal{L}_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to \mathcal{A}^{op}_{\mathcal{R}}\text{-mod}$$

induces the equivalence of groupoids $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}) \xrightarrow{\sim} \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(F)$ ([ELOI], Propositions 9.2, 9.4). By [ELOI], Theorem 11.6 b) there is a natural equivalences of pseudo-functors

$$\mathrm{coDef}^{\mathrm{h}}_{-}(F) \simeq \mathrm{coDef}_{-}(E).$$

By Lemma 9.7 there is an equivalence of pseudo-functors

$$\mathrm{coDef}^{\mathrm{h}}_{-}(\mathcal{B}) \simeq \mathrm{coDef}_{-}(\mathcal{B}).$$

Hence the functor $\mathbf{L}\mathcal{L}$ induces the equivalence

$$\mathbf{L}\mathcal{L} : \mathrm{coDef}_{-}(\mathcal{B}) \xrightarrow{\sim} \mathrm{coDef}_{-}(E).$$

Fix $\mathcal{R}, \mathcal{Q} \in 2\text{-}\mathrm{dgart}_{-}$ and $M \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. We need to show that there exists a natural isomorphism

$$\mathbf{L}\mathcal{L}_{\mathcal{Q}} \cdot M^{!} \simeq M^{!} \cdot \mathbf{L}\mathcal{L}_{\mathcal{R}}$$

between functors from $\mathrm{coDef}_{\mathcal{R}}(\mathcal{B})$ to $\mathrm{coDef}_{\mathcal{Q}}(E)$.

Since the cohomology of M is finite dimensional, and the DG algebra $\mathcal{R} \otimes \mathcal{Q}$ has no components in positive degrees, by [ELOI], Corollary 3.21 we may assume that M is finite dimensional.

Lemma 9.9. *Let* (S, id) *be an object in* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B})$ *or in* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(F)$. *Then* S *is acyclic for the functor* $\mathrm{Hom}_{\mathcal{R}^{op}}(M, S)$, *i.e.* $M^{!}(S) = \mathrm{Hom}_{\mathcal{R}^{op}}(M, S)$.

Proof. In the proof of Lemma 9.7 (resp. in [ELOI], Lemma 11.8) we showed that S is h-injective when considered as a DG $\mathcal{R}^{op}$-module. □

Choose $(S, \mathrm{id}) \in \mathrm{coDef}^{\mathrm{h}}(\mathcal{B})$. By the above lemma $M^{!}(S) = \mathrm{Hom}_{\mathcal{R}^{op}}(M, S)$.

We claim that the DG $\mathcal{B}^{op}$-module $\mathrm{Hom}_{\mathcal{R}^{op}}(M, S)$ is h-projective. Indeed, first notice that the graded $\mathcal{R}^{op}$-module S is injective being isomorphic to a direct sum of copies of shifted graded $\mathcal{R}^{op}$-module $\mathcal{R}^{*}$ (the abelian category of graded $\mathcal{R}^{op}$-modules is locally notherian, hence a direct sum of injectives is injective). Second, the DG $\mathcal{R}^{op}$-module M has a (finite) filtration with subquotients isomorphic to k. Thus the DG $\mathcal{B}^{op}$-module $\mathrm{Hom}_{\mathcal{R}^{op}}(M, S)$ has a filtration with subquotients isomorphic to $\mathrm{Hom}_{\mathcal{R}^{op}}(k, S) = i^{!}S \simeq \mathcal{B}$. So it satisfies property (P).

Hence $\mathbf{L}\mathcal{L} \cdot M^{!}(S) = \mathrm{Hom}_{\mathcal{R}^{op}}(M, S) \otimes_{\mathcal{B}} F$. For the same reasons $M^{!} \cdot \mathbf{L}\mathcal{L}_{\mathcal{R}}(S) = \mathrm{Hom}_{\mathcal{R}^{op}}(M, S \otimes_{\mathcal{B}} F)$. The isomorphism

$$\mathrm{Hom}_{\mathcal{R}^{op}}(M, S) \otimes_{\mathcal{B}} F = \mathrm{Hom}_{\mathcal{R}^{op}}(M, S \otimes_{\mathcal{B}} F)$$

follows from the fact that S as a graded module is a tensor product of graded $\mathcal{C}^{op}$ and $\mathcal{R}^{op}$ modules and also because $\dim_k M < \infty$. □

10. Deformation pseudo-functor coDEF for an augmented A_∞-algebra

Let A be an augmented A_∞-algebra. We are going to define the pseudo-functor $\mathrm{coDEF}(A) : 2\text{-}\mathrm{dgart} \to \mathbf{Gpd}$.

Let $\mathcal{R}$ be an artinian DG algebra. An object of the groupoid $\mathrm{coDEF}_{\mathcal{R}}(A)$ is a pair (S,σ), where $S \in D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}})$, and σ is an isomorphism (in $D_\infty(\bar{A}^{op})$)

$$\sigma : A \to \mathbf{R}i^!(S).$$

A morphism $f : (S,\sigma) \to (T,\tau)$ in $\mathrm{coDEF}_{\mathcal{R}}(A)$ is an isomorphism (in $D(\bar{A}^{op}_{\mathcal{R}^{op}})$) $f : S \to T$ such that

$$\mathbf{R}i^!(f) \circ \sigma = \tau.$$

This defines the pseudo-functor $\mathrm{coDEF}(A)$ on objects. Further, let $(M,\theta) \in 1\text{-}\mathrm{Hom}(\mathcal{R},\mathcal{Q})$. Define the corresponding functor

$$M^! : \mathrm{coDEF}_{\mathcal{R}}(A) \to \mathrm{coDEF}_{\mathcal{Q}}(A)$$

as follows. For an object $(S,\sigma) \in \mathrm{coDEF}_{\mathcal{R}}(A)$ put

$$M^!(S) = \mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}(M,S) \in D_\infty(\bar{A}^{op}_{\mathcal{Q}^{op}}).$$

Then we have natural isomorphisms in $D_\infty(\bar{A})$:

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{Q}^{op}}(k, M^!(S)) \cong \mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}(k \otimes^{\mathbf{L}}_{\mathcal{R}^{op}} M, S) \cong \mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}(k,S) = \mathbf{R}i^!(S)$$

(the second isomorphism is induced by θ). Thus, $M^!$ is a functor form $\mathrm{coDEF}_{\mathcal{R}}(A)$ to $\mathrm{coDEF}_{\mathcal{Q}}(A)$.

If $\mathcal{Q}'$ is another artinian DG algebra and $(M',\theta') \in 1\text{-}\mathrm{Hom}(\mathcal{Q},\mathcal{Q}')$ then there is a natural isomorphism of functors

$$(M' \otimes^{\mathbf{L}}_{\mathcal{Q}} M)^! \cong M'^! \cdot M^!.$$

Further, if $f \in 2\text{-}\mathrm{Hom}((M,\theta),(M,\theta_1))$ is a 2-morphism between objects $(M,\theta),(M,\theta_1) \in 1\text{-}\mathrm{Hom}(\mathcal{R},\mathcal{Q})$ then it induces an isomorphism between the corresponding functors $M^! \simeq M_1^!$.

Thus we obtain a pseudo-functor $\mathrm{coDEF}(A) : 2\text{-}\mathrm{dgart} \to \mathbf{Gpd}$. We denote by $\mathrm{coDEF}_{\_}(A)$ its restriction to the sub-2-category $2\text{-}\mathrm{dgart}_{\_}$.

Proposition 10.1. *Let A be an augmented A_∞-algebra and $U(A)$ its bar-cobar construction. Then there is a natural equivalence of pseudo-functors* $\mathrm{coDEF}(U(A)) \cong \mathrm{coDEF}(A)$.

Proof. Let $f_A : A \to U(A)$ be the universal strictly unital A_∞-morphism. Let $\mathcal{R}$ be an artinian DG algebra. Recall that by Proposition 3.14 we have an equivalence

$$f_{A*} : D((U(A) \otimes \mathcal{R})^{op}) \to D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}).$$

Moreover, the following diagram of functors commutes up to an isomorphism:

$$\begin{array}{ccc} D((U(A)\otimes\mathcal{R})^{op}) & \xrightarrow{f_{A*}} & D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}) \\ \Big\downarrow \mathbf{R}i^! & & \Big\downarrow \mathbf{R}i^! \\ D(U(A)^{op}) & \xrightarrow{f_{A*}} & D_\infty(\bar{A}^{op}). \end{array}$$

Hence, the functor f_{A*} induces an equivalence of groupoids $\mathrm{coDEF}_{\mathcal{R}}(U(A)) \to \mathrm{coDEF}_{\mathcal{R}}(A)$ and we obtain the required equivalence of pseudo-functors. $\square$

Corollary 10.2. *Let A be an augmented A_∞-algebra and let $\mathcal{B}$ be a DG algebra quasi-isomorphic to A. Then the pseudo-functor* $\mathrm{coDEF}(A)$ *and* $\mathrm{coDEF}(\mathcal{B})$ *are equivalent.*

Proof. Indeed, by Proposition 10.1 the pseudo-functors $\mathrm{coDEF}(A)$ and $\mathrm{DEF}(U(A))$ are equivalent, and by Corollary 9.6 the pseudo-functors $\mathrm{coDEF}(U(A))$ and $\mathrm{coDEF}(\mathcal{B})$ are equivalent. □

Corollary 10.3. *Let A be an admissible A_∞-algebra, and $\mathcal{R}$ be an artinian negative DG algebra. Then for any $(S,\sigma)\in\mathrm{coDEF}_{\mathcal{R}}(A)$ there exists a morphism of DG algebras $\hat{S}\to\mathcal{R}$ such that the pair (T,id), where $T=\mathrm{Hom}_{\hat{S}^{op}}(\mathcal{R},B\bar{A}\otimes_{\tau_A}A)$, defines an object of $\mathrm{coDEF}_{\mathcal{R}}(A)$ which is isomorphic to (S,σ).*

Proof. This follows easily from Proposition 10.1, the proof of Lemma 9.7 in the case $\mathcal{B}=U(A)$, and Lemma 8.4. □

11. The bicategory $2'$-adgalg and deformation pseudo-functor DEF

It turns out that the deformation pseudo-functor Def lifts naturally to a different version of a bicategory of augmented DG algebras. We denote this bicategory $2'$-adgalg. It differs from 2-adgalg in two respects: the 1-morphisms are objects in $D(\mathcal{B}\otimes\mathcal{C}^{op})$ (instead of $D(\mathcal{B}^{op}\otimes\mathcal{C})$) and 2-morphisms go in the opposite direction. We will relate the bicategories 2-adgalg and $2'$-adgalg (and the pseudo-functors coDEF and DEF) in section 13 below.

Definition 11.1. *(*[ELOII]*) We define the bicategory $2'$-adgalg of augmented DG algebras as follows. The objects are augmented DG algebras. For DG algebras $\mathcal{B},\mathcal{C}$ the collection of 1-morphisms $1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$ consists of pairs (M,θ), where*

- *$M\in D(\mathcal{B}\otimes\mathcal{C}^{op})$ and there exists an isomorphism (in $D(\mathcal{C}^{op})$) $\mathcal{C}\to\nu_*M$ (where $\nu_*:D(\mathcal{B}\otimes\mathcal{C}^{op})\to D(\mathcal{C}^{op})$ is the functor of restriction of scalars corresponding to the natural homomorphism $\nu:\mathcal{C}^{op}\to\mathcal{B}\otimes\mathcal{C}^{op}$);*
- *and $\theta:M\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}k\to k$ is an isomorphism in $D(\mathcal{B})$.*

The composition of 1-morphisms

$$1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})\times 1\text{-}\mathrm{Hom}(\mathcal{C},\mathcal{D})\to 1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{D})$$

is defined by the tensor product $\cdot\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}\cdot$. Given 1-morphisms $(M_1,\theta_1),(M_2,\theta_2)\in 1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$ a 2-morphism $f:(M_1,\theta_1)\to(M_2,\theta_2)$ is an isomorphism (in $D(\mathcal{B}\otimes\mathcal{C}^{op})$) $f:M_1\to M_2$ such that $\theta_1=\theta_2\cdot((f)\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}k)$. So in particular the category $1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$ is a groupoid. Denote by $2'$-dgart the full subbicategory of $2'$-adgalg consisting of artinian DG algebras. Similarly we define the full subbicategories $2'\text{-}\mathrm{dgart}_+$, $2'\text{-}\mathrm{dgart}_-$, $2'$-art, $2'$-cart ([ELOI]*, Definition 2.3).*

Remark 11.2. *The exact analogue of Remark 9.2 holds for the bicategory $2'$-adgalg.*

For any augmented DG algebra $\mathcal{B}$ we obtain a pseudo-functor $h'_{\mathcal{B}}$ between the bicategories $2'$-adgalg and **Gpd** defined by $h'_{\mathcal{B}}(\mathcal{C})=1\text{-}\mathrm{Hom}(\mathcal{B},\mathcal{C})$.

Note that a usual homomorphism of DG algebras $\gamma:\mathcal{B}\to\mathcal{C}$ defines the structure of a $\mathcal{B}$-module on $\mathcal{C}$ with the canonical isomorphism of DG $\mathcal{B}$-modules $\mathcal{C}\overset{\mathbf{L}}{\otimes}_{\mathcal{C}}k$. Hence, we have a natural pseudo-functor $\mathcal{F}':\mathrm{adgalg}\to 2\text{-}\mathrm{adgalg}$.

The next Proposition from [ELOII] asserts that the pseudo-functor Def has a natural lift on the bicategory $2'$-adgalg. We present its proof for completeness.

Proposition 11.3. *There exist a pseudo-functor* $\mathrm{DEF}(E)$ *from* $2'$-dgart *to* **Gpd** *and which is an extension to* $2'$-dgart *of the pseudo-functor* $\mathrm{Def}(E)$ *, i.e. there is an equivalence of pseudo-functors* $\mathrm{Def}(E) \simeq \mathrm{DEF}(E) \cdot \mathcal{F}'$ *.*

Proof. Let $\mathcal{R}$, $\mathcal{Q}$ be artinian DG algebras. Given $(M, \theta) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ we define the corresponding functor

$$M^* : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{Q}}(E)$$

as follows

$$M^*(S) := S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} M$$

for $(S, \sigma) \in \mathrm{Def}_{\mathcal{R}}(E)$. Then we have the canonical isomorphism

$$M^*(S) \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} k = S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} (M \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} k) \xrightarrow{\theta} S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sigma} E.$$

So that $M^*(S) \in \mathrm{Def}_{\mathcal{Q}}(E)$ indeed.

Given another artinian DG algebra $\mathcal{Q}'$ and $M' \in 1\text{-}\mathrm{Hom}(Q, Q')$ there is a natural isomorphism of functors

$$M'^* \cdot M^* = (M \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} M')^*.$$

Also a 2-morphism $f \in 2\text{-}\mathrm{Hom}(M, M_1)$ between $M, M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ induces an isomorphism of corresponding functors $M^* \xrightarrow{\sim} M_1^*$.

Thus we obtain a pseudo-functor $\mathrm{DEF}(E) : 2'\text{-}\mathrm{dgart} \to \mathbf{Gpd}$, such that $\mathrm{DEF}(E) \cdot \mathcal{F}' = \mathrm{Def}(E)$. □

We denote by $\mathrm{DEF}_+(E)$, $\mathrm{DEF}_-(E)$, $\mathrm{DEF}_0(E)$, $\mathrm{DEF}_{\mathrm{cl}}(E)$ the restriction of the pseudo-functor $\mathrm{DEF}(E)$ to subbicategories $2'\text{-}\mathrm{dgart}_+$, $2'\text{-}\mathrm{dgart}_-$, $2'\text{-}\mathrm{art}$ and $2'\text{-}\mathrm{cart}$ respectively.

Proposition 11.4. *(*[ELOII]*) A quasi-isomorphism* $\delta : E_1 \to E_2$ *of DG* $\mathcal{A}^{op}$ *-modules induces an equivalence of pseudo-functors*

$$\delta_* : \mathrm{DEF}(E_1) \to \mathrm{DEF}(E_2)$$

defined by $\delta_*(S, \sigma) = (S, \delta \cdot \sigma)$ *.*

Proof. This is clear. □

Proposition 11.5. *(*[ELOII]*) Let* $F : \mathcal{A} \to \mathcal{A}'$ *be a DG functor which induces a quasi-equivalence* $F^{\text{pre-tr}} : \mathcal{A}^{\text{pre-tr}} \to \mathcal{A}'^{\text{pre-tr}}$ *(this happens for example if* F *is a quasi-equivalence). Then for any* $E \in D(\mathcal{A}^{op})$ *the pseudo-functors* $\mathrm{DEF}_-(E)$ *and* $\mathrm{DEF}_-(\mathbf{L}F^*(E))$ *are equivalent (hence also* $\mathrm{DEF}_-(F_*(E'))$ *and* $\mathrm{DEF}_-(E')$ *are equivalent for any* $E' \in D(\mathcal{A}'^0)$ *).*

Corollary 11.6. *(*[ELOII]*) Assume that DG algebras* $\mathcal{B}$ *and* $\mathcal{C}$ *are quasi-isomorphic. Then the pseudo-functors* $\mathrm{DEF}_-(\mathcal{B})$ *and* $\mathrm{DEF}_-(\mathcal{C})$ *are equivalent.*

Proof. We may assume that there exists a morphism of DG algebras $\mathcal{B} \to \mathcal{C}$ which is a quasi-isomorphism. Then put $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}' = \mathcal{C}$ in the last proposition. □

The following Theorem is stronger then [ELOI], Corollary 11.15 for the pseudo-functors Def_- and Def_-^h .

Theorem 11.7. *Let* $E \in \mathcal{A}^{op}$*-mod be a DG module. Suppose that the following conditions hold:*

a) $\mathrm{Ext}^{-1}(E, E) = 0$ *;*

b) the graded algebra $\mathrm{Ext}(E, E)$ *is bounded above.*

Let $F \to E$ *be a quasi-isomorphism with h-projective* F *. Then the pseudo-functors* $\mathrm{Def}_-(E)$ *and* $\mathrm{Def}_-^h(F)$ *are equivalent.*

Proof. Replace the pseudo-functor $\mathrm{Def}(E)$ by the equivalent pseudo-functor $\mathrm{Def}(F)$. Fix some negative artinian DG algebra $\mathcal{R} \in \mathrm{dgart}_-$.

Due to [ELOI], Corollary 11.4 a) it suffices to prove that for each $(S, id) \in \mathrm{Def}^h_{\mathcal{R}}(F)$ one has $i^*(S) = \mathbf{L}i^*(S)$. Consider the DG algebra $\mathcal{B} = \mathrm{End}(F)$. First we will prove the following special case:

Lemma 11.8. *The pseudo-functors* $\mathrm{Def}_-(\mathcal{B})$ *and* $\mathrm{Def}^h_-(\mathcal{B})$ *are equivalent.*

Proof. Take some $(S, \sigma) \in \mathrm{Def}^h(\mathcal{B})$. Let A be a strictly unital minimal model of $\mathcal{B}$, and let $f : A \to \mathcal{B}$ be a strictly unital A_∞ quasi-isomorphism. By our assumption on $Ext(E, E) \cong H(\mathcal{B})$, A is bounded above.

By Theorem 7.2 there exists an object $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ such that $S \cong \mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}$. The DG $\mathcal{R}^{op}$-modules $\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R}$ and $f_*(\mathcal{B} \otimes_{f^*_{\mathcal{R}}(\alpha)} \mathcal{R})$ are naturally identified. Further, by Proposition 8.6 we have natural homotopy equivalence (in $A^{op}_{\mathcal{R}^{op}}\text{-mod}_\infty$)

$$\gamma : \mathcal{R} \otimes_\alpha A \to f_*(\mathcal{B} \otimes_{f^*(\alpha)} \mathcal{R}).$$

Thus, it remains to prove that

$$i^*(A \otimes_\alpha \mathcal{R}) = \mathbf{L}i^*(A \otimes_\alpha \mathcal{R}).$$

We claim that $\mathcal{R} \otimes_\alpha A$ is h-projective. Indeed, since A is bounded above and $\mathcal{R} \in \mathrm{dgart}_-$, this DG $\mathcal{R}^{op}$-module has an increasing filtration by DG $\mathcal{R}^{op}$-submodules $A^{\geq i} \otimes \mathcal{R}$ with subquotients being free DG $\mathcal{R}^{op}$-modules $A^i \otimes \mathcal{R}$. Thus $A \otimes_\alpha \mathcal{R}$ satisfies property (P) as DG $\mathcal{R}^{op}$-module and hence is h-projective. Lemma is proved. □

Now take some $(S, id) \in \mathrm{Def}^h(F)$. We claim that S is h-projective. Recall the DG functor

$$\Sigma_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to \mathcal{A}^{op}_{\mathcal{R}}\text{-mod}, \quad \Sigma(M) = M \otimes_{\mathcal{B}} F.$$

From [ELOI], Proposition 9.2 e) we know that $S \cong \Sigma_{\mathcal{R}}(S')$ for some $(S', id) \in \mathrm{Def}^h(\mathcal{B})$. By the above Lemma and [ELOI], Proposition 11.2, DG $(\mathcal{B} \otimes \mathcal{R})^{op}$-module S' is h-projective. Since the DG functor $\Sigma_{\mathcal{R}}$ preserves h-projectives, it follows that S is also h-projective. Theorem is proved. □

The next proposition is the analogue of Proposition 9.8 for the pseudo-functor DEF_-. Note that here we do not need boundedness assumptions on the h-projective DG module.

Proposition 11.9. *Let* $E \in \mathcal{A}^{op}$*-mod be a DG module. Suppose that the following conditions hold:*

a) $\mathrm{Ext}^{-1}(E, E) = 0$*;*

b) the graded algebra $\mathrm{Ext}(E, E)$ *is bounded above.*

Put $\mathcal{B} = \mathbf{R}\,\mathrm{Hom}(E, E)$. *Then pseudo-functors* $\mathrm{DEF}_-(\mathcal{B})$ *and* $\mathrm{DEF}_-(E)$ *are equivalent.*

Proof. Take some h-projective F quasi-isomorphic to E and replace $\mathrm{DEF}_-(E)$ by the equivalent pseudo-functor $\mathrm{DEF}_-(F)$. We may assume that $\mathcal{B} = \mathrm{End}(F)$.

By [ELOI], Proposition 9.2 e) the DG functor $\Sigma = \Sigma^F : \mathcal{B}^{op}\text{-mod} \to \mathcal{A}^{op}\text{-mod}$, $\Sigma(N) = N \otimes_{\mathcal{B}} F$ induces an equivalence of pseudo-functors

$$\mathrm{Def}^h(\Sigma) : \mathrm{Def}^h(\mathcal{B}) \to \mathrm{Def}^h(F).$$

By Lemma 11.7 we have that the pseudo-functors $\mathrm{Def}_-(F)$ and $\mathrm{Def}^h_-(F)$ (resp. $\mathrm{Def}_-(\mathcal{B})$ and $\mathrm{Def}^h_-(\mathcal{B})$) are equivalent. We conclude that Σ also induces an equivalence of pseudo-functors

$$\mathrm{Def}_-(\Sigma) : \mathrm{Def}_-(\mathcal{B}) \to \mathrm{Def}_-(F).$$

Let us prove that it extends to an equivalence

$$\mathrm{DEF}_{\_}(\Sigma) : \mathrm{DEF}_{\_}(\mathcal{B}) \to \mathrm{DEF}_{\_}(F).$$

Let $\mathcal{R}, \mathcal{Q} \in \mathrm{dgart}_{\_}$, $M \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. We need to show that the functorial diagram

$$\begin{array}{ccc} \mathrm{DEF}_{\mathcal{R}}(\mathcal{B}) & \xrightarrow{\mathrm{DEF}_{\mathcal{R}}(\Sigma)} & \mathrm{DEF}_{\mathcal{R}}(F) \\ {\scriptstyle M^*}\big\downarrow & & {\scriptstyle M^*}\big\downarrow \\ \mathrm{DEF}_{\mathcal{Q}}(\mathcal{B}) & \xrightarrow{\mathrm{DEF}_{\mathcal{Q}}(\Sigma)} & \mathrm{DEF}_{\mathcal{Q}}(F). \end{array}$$

commutes. This follows from the natural isomorphism

$$N \otimes_{\mathcal{B}} F \otimes_{\mathcal{R}} M \cong N \otimes_{\mathcal{R}} M \otimes_{\mathcal{B}} F.$$

□

12. Deformation pseudo-functor DEF for an augmented A_∞-algebra

Let A be an augmented A_∞-algebra. We are going to define the pseudo-functor $\mathrm{DEF}(A) : 2'\text{-}\mathrm{dgart} \to \mathbf{Gpd}$.

Let $\mathcal{R}$ be an artinian DG algebra. An object of the groupoid $\mathrm{DEF}_{\mathcal{R}}(A)$ is a pair (S, σ) , where $S \in D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}})$, and σ is an isomorphism (in $D_\infty(\bar{A}^{op})$)

$$\sigma : \mathbf{L}i^*(S) \to A.$$

A morphism $f : (S, \sigma) \to (T, \tau)$ in $\mathrm{DEF}_{\mathcal{R}}(A)$ is an isomorphism (in $D(\bar{A}^{op}_{\mathcal{R}^{op}})$) $f : S \to T$ such that

$$\tau \circ \mathbf{L}i^*(f) = \sigma.$$

This defines the pseudo-functor $\mathrm{DEF}(A)$ on objects. Further, let $(M, \theta) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. Define the corresponding functor

$$M^* : \mathrm{DEF}_{\mathcal{R}}(A) \to \mathrm{DEF}_{\mathcal{Q}}(A)$$

as follows. For an object $(S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$ put

$$M^*(S) = S \otimes^{\mathbf{L}}_{\mathcal{R}} M \in D_\infty(\bar{A}^{op}_{\mathcal{Q}^{op}}).$$

Then we have natural isomorphisms in $D_\infty(\bar{A})$:

$$M^*(S) \otimes^{\mathbf{L}}_{\mathcal{Q}} k = S \otimes^{\mathbf{L}}_{\mathcal{R}} (M \otimes^{\mathbf{L}}_{\mathcal{Q}} k) \cong S \otimes^{\mathbf{L}}_{\mathcal{R}} k \cong A$$

(the second isomorphism is induced by θ). Thus, M^* is a functor form $\mathrm{DEF}_{\mathcal{R}}(A)$ to $\mathrm{DEF}_{\mathcal{Q}}(A)$.

If $\mathcal{Q}'$ is another artinian DG algebra and $(M', \theta') \in 1\text{-}\mathrm{Hom}(\mathcal{Q}, \mathcal{Q}')$ then there is a natural isomorphism of functors

$$(M' \otimes^{\mathbf{L}}_{\mathcal{Q}} M)^* \cong M'^* \cdot M^*.$$

Further, if $f \in 2\text{-}\mathrm{Hom}((M, \theta), (M, \theta_1))$ is a 2-morphism between objects $(M, \theta), (M, \theta_1) \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$ then it induces an isomorphism between corresponding functors $M^* \to M_1^*$.

Thus we obtain a pseudo-functor $\mathrm{DEF}(A) : 2\text{-}\mathrm{dgart} \to \mathbf{GPd}$. We denote by $\mathrm{DEF}_{\_}(A)$ its restriction to the sub-2-category $2\text{-}\mathrm{dgart}_{\_}$.

Proposition 12.1. *Let A be an augmented A_∞-algebra and $U(A)$ its universal DG algebra. Then there is a natural equivalence of pseudo-functors* $\mathrm{DEF}(U(A)) \cong \mathrm{DEF}(A)$.

Proof. The proof is the same as of Proposition 10.1 and we omit it. □

Corollary 12.2. *Let A be an augmented A_∞-algebra and let $\mathcal{B}$ be a DG algebra quasi-isomorphic to A. Then the pseudo-functor* $\mathrm{DEF}(A)$ *and* $\mathrm{DEF}(\mathcal{B})$ *are equivalent.*

Proof. Indeed, by Proposition 10.1 the pseudo-functors $\mathrm{DEF}(A)$ and $\mathrm{DEF}(U(A))$ are equivalent, and by Corollary 11.6 the pseudo-functors $\mathrm{DEF}(U(A))$ and $\mathrm{DEF}(\mathcal{B})$ are equivalent. □

Corollary 12.3. *Let A be an admissible A_∞-algebra, and $\mathcal{R}$ be an artinian negative DG algebra. Then for any $(S,\sigma)\in \mathrm{DEF}_{\mathcal{R}}(A)$ there exists an $\alpha\in \mathcal{MC}_{\mathcal{R}}(A)$ such that the pair (T,id), where $T=A\otimes_\alpha \mathcal{R}$, defines an object of $\mathrm{DEF}_{\mathcal{R}}(A)$ which is isomorphic to (S,σ).*

Proof. This follows easily from Proposition 12.1 and the proof of Lemma 11.7 in the case $\mathcal{B}=U(A)$. □

13. Comparison of pseudo-functors coDEF_- and DEF_-

We have proved in [ELOI], Corollary 11.9 that under some conditions on E the pseudo-functors $\mathrm{coDef}_-(E)$ and $\mathrm{Def}_-(E)$ from dgart_- to **Gpd** are equivalent. Note that we cannot speak about an equivalence of pseudo-functors $\mathrm{coDEF}_-(E)$ and $\mathrm{DEF}_-(E)$ since they are defined on different bicategories. So our first goal is to establish an equivalence of the bicategories $2\text{-}\mathrm{adgalg}$ and $2'\text{-}\mathrm{adgalg}$ in the following sense: we will construct pseudo-functors

$$\mathcal{D}: 2\text{-}\mathrm{adgalg}\to 2'\text{-}\mathrm{adgalg},$$

$$\mathcal{D}': 2'\text{-}\mathrm{adgalg}\to 2\text{-}\mathrm{adgalg},$$

which have the following properties

1) $\mathcal{D}$ (resp. $\mathcal{D}'$) is the identity on objects;
2) for each $\mathcal{B},\mathcal{C}\in Ob(2\text{-}\mathrm{adgalg})$ they define mutually inverse equivalences of groupoids

$$\mathcal{D}: \mathrm{Hom}_{2\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C})\to \mathrm{Hom}_{2'\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C}),$$

$$\mathcal{D}': \mathrm{Hom}_{2'\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C})\to \mathrm{Hom}_{2\text{-}\mathrm{adgalg}}(\mathcal{B},\mathcal{C}).$$

Fix augmented DG algebras $\mathcal{B}$, $\mathcal{C}$ and let M be a DG $\mathcal{C}\otimes\mathcal{B}^{op}$-module. Define the DG $\mathcal{B}\otimes\mathcal{C}^{op}$-module $\mathcal{D}(M)$ as

$$\mathcal{D}(M):=\mathbf{R}\,\mathrm{Hom}_{\mathcal{C}}(M,\mathcal{C}).$$

Further, let N be a DG $\mathcal{B}\otimes\mathcal{C}^{op}$-module. Define the DG $\mathcal{B}^{op}\otimes\mathcal{C}$-module $\mathcal{D}'(N)$ as

$$\mathcal{D}'(N)=\mathbf{R}\,\mathrm{Hom}_{\mathcal{C}^{op}}(N,\mathcal{C}).$$

Proposition 13.1. *([ELOII]) The operations $\mathcal{D}$, $\mathcal{D}'$ as above induces the pseudo-functors*

$$\mathcal{D}: 2\text{-}\mathrm{adgalg}\to 2'\text{-}\mathrm{adgalg},$$

$$\mathcal{D}': 2'\text{-}\mathrm{adgalg}\to 2\text{-}\mathrm{adgalg},$$

so that the properties 1) and 2) hold.

Corollary 13.2. *([ELOII]) For any augmented DG algebra $\mathcal{B}$ the pseudo-functor $\mathcal{D} : 2\text{-}\mathrm{adgalg} \to 2'\text{-}\mathrm{adgalg}$ induces a morphism of pseudo-functors*

$$h_{\mathcal{B}} \to h'_{\mathcal{B}} \cdot \mathcal{D},$$

which is an equivalence.

Similarly, the pseudo-functor $\mathcal{D}' : 2'\text{-}\mathrm{adgalg} \to 2\text{-}\mathrm{adgalg}$ induces an equivalence of pseudo-functors

$$h'_{\mathcal{B}} \to h_{\mathcal{B}} \cdot \mathcal{D}'.$$

Proof. This is clear. □

Lemma 13.3. *([ELOII]) Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \in Ob(2\text{-}\mathrm{adgalg})$, $M_1 \in 1\text{-}\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$, $M_2 \in 1\text{-}\mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_3)$. Assume that M_1 and M_2 are h-projective as DG $\mathcal{B}_2 \otimes \mathcal{B}_1^{op}$- and $\mathcal{B}_3 \otimes \mathcal{B}_2^{op}$-modules respectively. Then*

a) The DG $\mathcal{B}_2^{op}$-module $\mathrm{Hom}_{\mathcal{B}_2}(M_1, \mathcal{B}_2)$ is h-projective.

b) The DG $\mathcal{B}_3$-module $M_2 \otimes_{\mathcal{B}_2} M_1$ is h-projective.

The next theorem is related to [ELOII], Theorem 13.4. It asserts the stronger statement in the case when E is a DG algebra considered as a DG module over itself.

Theorem 13.4. *Let $\mathcal{B}$ be a DG algebra. Suppose that the following conditions hold:*

a) $H^{-1}(\mathcal{B}) = 0$;

b) the cohomology algebra $H(\mathcal{B})$ is bounded above and bounded below. Then the pseudo-functors $\mathrm{coDEF}_{-}(\mathcal{B})$ and $\mathrm{DEF}_{-}(\mathcal{B}) \cdot \mathcal{D}$ from $2\text{-}\mathrm{dgart}_{-}$ to **Gpd** *are equivalent.*

Proof. Let $\mathcal{R}$ be a negative artinian DG algebra. Recall the DG functors

$$\epsilon_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod}, \quad \epsilon_{\mathcal{R}}(M) = M \otimes_{\mathcal{R}} \mathcal{R}^*,$$

$$\eta_{\mathcal{R}} : (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod} \to (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod}, \quad \eta_{\mathcal{R}}(M) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^*, M).$$

By [ELOI], Proposition 4.7 they induce quasi-inverse equivalences

$$\epsilon_{\mathcal{R}} : \mathrm{Def}^h_{\mathcal{R}}(\mathcal{B}) \to \mathrm{coDef}^h_{\mathcal{R}}(\mathcal{B}),$$

$$\eta_{\mathcal{R}} : \mathrm{coDef}^h_{\mathcal{R}}(\mathcal{B}) \to \mathrm{Def}^h_{\mathcal{R}}(\mathcal{B}).$$

By Theorem 11.7 the pseudo-functors $\mathrm{Def}_{-}(\mathcal{B})$ and $\mathrm{Def}^h_{-}(\mathcal{B})$ are equivalent. By Lemma 9.7 the pseudo-functors $\mathrm{coDef}_{-}(\mathcal{B})$ and $\mathrm{coDef}^h_{-}(\mathcal{B})$. It follows that the derived functors $\mathbf{L}\epsilon_{\mathcal{R}}$, $\mathbf{R}\eta_{\mathcal{R}}$ induce mutually inverse equivalences

$$\mathbf{L}\epsilon_{\mathcal{R}} : \mathrm{Def}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{coDef}_{\mathcal{R}}(\mathcal{B}),$$

$$\mathbf{R}\eta_{\mathcal{R}} : \mathrm{coDef}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{Def}_{\mathcal{R}}(\mathcal{B}).$$

Let now $\mathcal{Q} \in \mathrm{dgart}_{-}$ and $M \in 1\text{-}\mathrm{Hom}(\mathcal{R}, \mathcal{Q})$. It suffices to prove that the functorial diagram

$$\begin{array}{ccc} \mathrm{Def}_{\mathcal{R}}(\mathcal{B}) & \xrightarrow{\mathbf{L}\epsilon_{\mathcal{R}}} & \mathrm{coDef}_{\mathcal{R}}(\mathcal{B}) \\ {\scriptstyle \mathcal{D}(M)^*}\downarrow & & {\scriptstyle M^!}\downarrow \\ \mathrm{coDef}_{\mathcal{Q}}(\mathcal{B}) & \xrightarrow{\mathbf{L}\epsilon_{\mathcal{Q}}} & \mathrm{coDef}_{\mathcal{Q}}(\mathcal{B}) \end{array}$$

naturally commutes.

Choose a bounded above h-projective or h-injective P quasi-isomorphic to E. By [ELOI], Theorem 11.6 a) the groupoids $\mathrm{Def}_{\mathcal{R}}(E)$ and $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ are equivalent. Hence given $(S,\mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ it suffices to prove that there exists a natural isomorpism of objects in $D(\mathcal{A}^{op}_{\mathcal{Q}})$

$$M^! \cdot \mathbf{L}\epsilon_{\mathcal{R}}(S) \simeq \mathbf{L}\epsilon_{\mathcal{Q}} \cdot \mathcal{D}(M)^*(S),$$

i.e.

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{R}^{op}}(M, S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \mathcal{R}^*) \simeq S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \mathbf{R}\,\mathrm{Hom}_{\mathcal{Q}}(M,\mathcal{Q}) \overset{\mathbf{L}}{\otimes}_{\mathcal{Q}} \mathcal{Q}^*.$$

We may and will assume that the DG $\mathcal{Q}\otimes\mathcal{R}^{op}$-module M is h-projective. In the proof of [ELOI], Lemma 11.7 we showed that the DG $\mathcal{A}^{op}_{\mathcal{R}}$-module S is h-projective as a DG $\mathcal{R}^{op}$-module. Therefore by Lemma 13.3 a) it suffices to prove that the morphism of DG $\mathcal{A}^{op}_{\mathcal{Q}}$-modules

$$\eta : S \otimes_{\mathcal{R}} \mathrm{Hom}_{\mathcal{Q}}(M,\mathcal{Q}) \otimes_{\mathcal{Q}} \mathcal{Q}^* \to \mathrm{Hom}_{\mathcal{R}^{op}}(M, S\otimes_{\mathcal{R}} \mathcal{R}^*)$$

defined by

$$\eta(s\otimes f\otimes g)(m)(r) = sg(f(mr))$$

is a quasi-isomorphism.

It suffices to prove that η is a quasi-isomorphism of DG $\mathcal{Q}^{op}$-modules. Notice that just the $\mathcal{R}^{op}$-module structure on S is important for us. Furthermore we may assume that S satisfies property (P) as DG $\mathcal{R}^{op}$-module. Thus it suffices to prove that η is a quasi-isomorphism if $S=\mathcal{R}$. Then

$$\eta : \mathrm{Hom}_{\mathcal{Q}}(M,\mathcal{Q}) \otimes_{\mathcal{Q}} \mathcal{Q}^* \to \mathrm{Hom}_{\mathcal{R}^{op}}(M,\mathcal{R}^*).$$

We have the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{R}^{op}}(M, \mathrm{Hom}_k(\mathcal{R},k)) = \mathrm{Hom}_k(M\otimes_{\mathcal{R}}\mathcal{R},k) = M^*.$$

Also, since the DG $\mathcal{Q}^{op}$-module M is homotopy equivalent to $\mathcal{Q}$, we have the homotopy equivalences

$$\mathrm{Hom}_{\mathcal{Q}}(M,\mathcal{Q})\otimes_{\mathcal{Q}}\mathcal{Q}^* \simeq \mathrm{Hom}_{\mathcal{Q}}(\mathcal{Q},\mathcal{Q})\otimes_{\mathcal{Q}}\mathcal{Q}^* \simeq \mathcal{Q}^* \simeq M^*.$$

□

Part 4. Pro-representability theorems

14. Pro-representability of the pseudo-functor coDEF_-

The next theorem claims that under some conditions on the DG algebra $\mathcal{C}$ that the functor $\mathrm{coDEF}_-(\mathcal{C})$ is pro-representable.

Theorem 14.1. *Let $\mathcal{C}$ be a DG algebra such that the cohomology algebra $H(\mathcal{C})$ is admissible finite-dimensional. Let A be a strictly unital minimal model of $\mathcal{C}$. Then the pseudo-functor $\mathrm{coDEF}_-(\mathcal{C})$ is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$. That is, there exists an equivalence of pseudo-functors $\mathrm{coDEF}_-(\mathcal{C}) \simeq h_{\hat{S}}$ from $2\text{-}\mathrm{dgart}_-$ to* **Gpd**.

As a corollary, we obtain the following

Theorem 14.2. *Let $E\in\mathcal{A}^{op}$-mod. Assume that the following conditions hold:*

a) the graded algebra $\mathrm{Ext}(E,E)$ is admissible finite-dimensional;

b) E is quasi-isomorphic to a bounded below F which is h-projective or h-injective.

Then the pseudo-functor $\mathrm{coDEF}_-(E)$ is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$, where A is a strictly unital minimal model of $\mathbf{R}\,\mathrm{Hom}(E,E)$.

Proof. By Proposition 9.8 the pseudo-functors $\mathrm{coDEF}_-(E)$ and $\mathrm{coDEF}_-(\mathbf{R}\operatorname{Hom}(E,E))$ are equivalent. So it remains to apply Theorem 14.1. □

Proof. Note that we have natural quasi-isomorphism of DG algebras $U(A) \to \mathcal{C}$, hence $\mathrm{coDEF}_-(\mathcal{C}) \simeq \mathrm{coDEF}_-(U(A))$. Further, by Proposition 10.1 we have $\mathrm{coDEF}_-(U(A)) \simeq \mathrm{coDEF}_-(A)$. We will construct an equivalence of pseudo-functors $\Theta : h_{\hat{S}} \to \mathrm{coDEF}_-(A)$.

Consider the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $B\bar{A} \otimes A$. Choose a quasi-isomorphism $B\bar{A} \otimes A \to J$, where J is an h-injective A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module. Note that J is also h-injective as a DG $\hat{S}^{op}$-module.

Given an artinian DG algebra $\mathcal{R}$ and a 1-morphism $(M,\theta) \in 1\text{-}\operatorname{Hom}(\hat{S},\mathcal{R})$ we define

$$\Theta(M) := \operatorname{Hom}_{\hat{S}^{op}}(M,J).$$

We have $\mathbf{R}\operatorname{Hom}_{\mathcal{R}^{op}}(k,\operatorname{Hom}_{\hat{S}^{op}}(M,J)) = \mathbf{R}\operatorname{Hom}_{\hat{S}^{op}}(k \otimes^{\mathbf{L}}_{\mathcal{R}} M, J)$. Hence the quasi-isomorphism $\theta : k \otimes^{\mathbf{L}}_{\mathcal{R}} M \to k$ induces a quasi-isomorphism

$$\mathbf{R}\operatorname{Hom}_{\mathcal{R}^{op}}(k,\Theta(M)) \simeq \mathbf{R}\operatorname{Hom}_{\hat{S}^{op}}(k,J) = \operatorname{Hom}_{\hat{S}^{op}}(k,J),$$

and by Proposition 4.3 the last term is canonically quasi-isomorphic to A as an A_∞ A^{op}-module.

If we are given with another artinian DG algebra $\mathcal{Q}$ and a 1-morphism $(N,\delta) \in 1\text{-}\operatorname{Hom}(\mathcal{R},\mathcal{Q})$, then the object $\Theta(N \otimes^{\mathbf{L}}_{\mathcal{R}} M)$ is canonically quasi-isomorphic to the object $\mathbf{R}\operatorname{Hom}(N,\Theta(M))$. Thus, Θ is a morphism of pseudo-functors.

It remains to prove that for each $\mathcal{R} \in 2\text{-}\mathrm{dgart}_-$ the induced functor $\Theta_{\mathcal{R}} : 1\text{-}\operatorname{Hom}(\hat{S},\mathcal{R}) \to \mathrm{coDEF}_{\mathcal{R}}(A)$ is an equivalence of groupoids. So fix a DG algebra $\mathcal{R} \in 2\text{-}\mathrm{dgart}_-$.

Surjective on isomorphism classes. Let (S,σ) be an object of $\mathrm{coDEF}_{\mathcal{R}}(A)$. By Corollary 10.3, there exists a morphism of DG algebras $\phi : \hat{S} \to \mathcal{R}$ such that the pair (T, id), where $T = \operatorname{Hom}_{\hat{S}^{op}}(\mathcal{R}, B\bar{A} \otimes A)$, defines an object of $\mathrm{coDEF}_{\mathcal{R}}(A)$ which is isomorphic to (S,σ). Further, by Proposition 4.5 the morphism $\operatorname{Hom}_{\hat{S}^{op}}(\mathcal{R}, B\bar{A} \otimes A) \to \operatorname{Hom}_{\hat{S}^{op}}(\mathcal{R},J)$ is quasi-isomorphism. Therefore, the object (T, id) is isomorphic to $\Theta(M)$, where $M = \mathcal{R}$ is DG $\hat{S}^{op} \otimes \mathcal{R}$-module via the homomorphism ϕ.

Full and Faithful. Consider the above Θ as a contravariant DG functor from $\mathcal{R} \otimes \hat{S}^{op}$-mod to $\bar{A}^{op}_{\mathcal{R}^{op}}$-$\mathrm{mod}_\infty$. Define the contravariant DG functor $\Phi : \bar{A}^{op}_{\mathcal{R}^{op}}\text{-}\mathrm{mod}_\infty \to \mathcal{R} \otimes \hat{S}^{op}\text{-mod}$ defined by the similar formula:

$$\Phi(N) = \operatorname{Hom}_{\bar{A}^{op}}(N,J).$$

These DG functors induce the corresponding DG functors between derived categories

$$\Theta : D(\mathcal{R} \otimes \hat{S}^{op}) \to D_\infty(\bar{A}^{op}_{\hat{S}^{op}}), \quad \Phi : D_\infty(\bar{A}^{op}_{\hat{S}^{op}}) \to D(\mathcal{R} \otimes \hat{S}^{op}).$$

Denote by $\langle k \rangle \subset D(\mathcal{R} \otimes \hat{S}^{op})$ and $\langle A \rangle \subset D_\infty(\bar{A}^{op}_{\hat{S}^{op}})$ the triangulated envelopes of the DG $\mathcal{R} \otimes \hat{S}^{op}$-module k and A_∞ $\bar{A}^{op}_{\mathcal{R}^{op}}$-module A respectively.

Lemma 14.3. *The functors Θ and Φ induce mutually inverse anti-equivalences of the triangulated categories $\langle k \rangle$ and $\langle A \rangle$.*

Proof. For $M \in \mathcal{R} \otimes \hat{S}^{op}$-mod, and $N \in \bar{A}^{op}_{\mathcal{R}^{op}}$-$\mathrm{mod}_\infty$ we have the functorial closed morphisms

$$\beta_M : M \to \Phi(\Theta(M)), \quad \beta_M(x)_1(f) = (-1)^{|f||x|} f(x), \quad \beta_M(x)_n = 0 \text{ for } n \geq 2;$$

$$\gamma_N : N \to \Theta(\Phi(N)), \quad (\gamma_N)_n(a_1,\dots,a_{n-1},y)(f) = (-1)^{n(|a_1|+\dots+|a_{n-1}|+|y|)} f_n(a_1,\dots,a_{n-1},y).$$

By Proposition 4.3 the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $\Theta(k)$ is quasi-isomorphic to A. Further, $\Phi(A)$ is quasi-isomorphic to J and hence to k. Therefore, β_k and γ_A are quasi-isomorphisms, and Lemma is proved. □

Note that for $(M,\theta) \in 1\text{-}\operatorname{Hom}(\hat{S},\mathcal{R})$ (resp. for $(S,\sigma) \in \operatorname{coDEF}_{\mathcal{R}}(A)$) $M \in \langle k \rangle$ (resp. $S \in \langle A \rangle$). Hence the functor $\Theta_{\mathcal{R}} : 1\text{-}\operatorname{Hom}(\hat{S},\mathcal{R}) \to \operatorname{coDEF}_{\mathcal{R}}(A)$ is fully faithful. This proves the theorem. □

15. Pro-representability of the pseudo-functor DEF_-

Pro-representability Theorems 14.1 and 14.2 imply analogous results for the pseudo-functor DEF_-. Namely, we have the following Theorems.

Theorem 15.1. *Let $\mathcal{C}$ be a DG algebra such that the cohomology algebra $H(\mathcal{C})$ is admissible finite-dimensional. Let A be a strictly unital minimal model of $\mathcal{C}$. Then the pseudo-functor $\operatorname{DEF}_-(\mathcal{C})$ is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$. That is, there exists an equivalence of pseudo-functors $\operatorname{DEF}_-(\mathcal{C}) \simeq h'_{\hat{S}}$ from $2'\text{-}\operatorname{dgart}_-$ to* **Gpd**.

Proof. By Theorem 14.1 we have the equivalence

$$\operatorname{coDEF}_-(\mathcal{C}) \simeq h_{\hat{S}}$$

of pseudo-functors from $2\text{-}\operatorname{dgart}$ to **Gpd**.

By Theorem 13.4 we have the equivalence

$$\operatorname{coDEF}_-(\mathcal{C}) \simeq \operatorname{DEF}_-(\mathcal{C}) \cdot \mathcal{D}$$

of pseudo-functors from $2\text{-}\operatorname{dgart}$ to **Gpd**.

Further, by Corollary 13.2

$$h_{\hat{S}} \simeq h'_{\hat{S}} \cdot \mathcal{D}.$$

Hence $\operatorname{DEF}_-(\mathcal{C}) \cdot \mathcal{D} \simeq h'_{\hat{S}} \cdot \mathcal{D}$ and therefore

$$\operatorname{DEF}_-(\mathcal{C}) \simeq h'_{\hat{S}}.$$

□

We get the following corollary.

Theorem 15.2. *Let $E \in \mathcal{A}^{op}$-mod. Assume that the graded algebra $\operatorname{Ext}(E,E)$ is admissible finite-dimensional. Then the pseudo-functor $\operatorname{DEF}_-(E)$ is pro-representable by the DG algebra $\hat{S} = (B\bar{A})^*$, where A is a strictly unital minimal model of the DG algebra $\mathbf{R}\operatorname{Hom}(E,E)$.*

Proof. Indeed, by Proposition 11.9 the pseudo-functors $\operatorname{DEF}_-(E)$ and $\operatorname{DEF}_-(\mathbf{R}\operatorname{Hom}(E,E))$ are equivalent. And by Theorem 15.1 the pseudo-functors $\operatorname{DEF}_-(\mathbf{R}\operatorname{Hom}(E,E))$ and $h'_{\hat{S}}$ are equivalent. □

In the proof of Theorem 14.1 we showed that the bar complex $B\bar{A} \otimes_{\tau_A} A$ is the "universal co-deformation" of the A_∞ $\bar{A}^{op}$-module A. However, Theorem 15.1 is deduced from Theorem 14.1 without finding the analogous "universal deformation" of the A_∞ $\bar{A}^{op}$-module A. We do not know if this "universal deformation" exists in general (under the assumptions of Theorem 15.1). But we can find it and hence give a direct proof of Theorem 15.1 if the minimal model A of $\mathcal{C}$ satisfies an extra assumption (*) below.

For the rest of this section we assume that A is an augmented A_∞-algebra.

Definition 15.3. *Let A be an augmented A_∞-algebra. Consider k as a left A_∞ A-module. We say that A satisfies the condition (*) if the canonical morphism*

$$k \to \mathrm{Hom}_{\bar{A}^{op}}(\mathrm{Hom}_{\bar{A}}(k, A), A)$$

of left A_∞ $\bar{A}$-modules is a quasi-isomorphism.

Example 15.4. *Let A be an augmented A_∞-algebra. If k lies in $\mathrm{Perf}(A)$ then A satisfies the condition (*).*

In particular, suppose that A is homologically smooth and compact. That is, the diagonal A_∞ $A-A$-bimodule A lies in $\mathrm{Perf}(A-A)$ (smoothness), and $\dim H(A) < \infty$ (compactness). Then the A_∞ A-module is perfect iff it has finite-dimensional total cohomology. Thus, $k \in \mathrm{Perf}(A)$ and A satisfies the condition (*).

Example 15.5. *Let A be an augmented A_∞-algebra which is left and right Gorenstein of dimension d. This means that*

$$\mathrm{Ext}^p_{\bar{A}}(k, A) = \begin{cases} k, & \text{if } p{=}d \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Ext}^p_{\bar{A}^{op}}(k, A) = \begin{cases} k, & \text{if } p{=}d \\ 0, & \text{otherwise.} \end{cases}$$

Then A satisfies the condition (*).

For the rest of this section assume that A is admissible, finite-dimensional and satisfies the condition (*).

Denote by $\mathcal{E}$ the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module

$$\mathcal{E} := \mathrm{Hom}_A(k, A).$$

This A_∞-module is isomorphic to $A \otimes \hat{S}$ as a graded $(\hat{S}^{op})^{gr}$-module and can be given explicitly by the formula

$$m^{\mathcal{E}}_n(m, a_1, \dots, a_{n-1}) = m_n^{\mathcal{MC}^{\hat{S}}_\infty(A)(0,\dots,0,\tau_A)}(m, a_1 \otimes \mathbf{1}_{\hat{S}}, \dots, a_{n-1} \otimes \mathbf{1}_{\hat{S}}). \tag{15.1}$$

Remark 15.6. *The definition of the A_∞-category $\mathcal{MC}^{\hat{S}}_\infty(A)$ is the same as if $\hat{S}$ would be artinian. It is correct because $\hat{S}$ is complete in $\mathfrak{m}$-adic topology and A is finite-dimensional. In the above formula τ_A is considered as an element of $A \otimes \hat{S} = \mathrm{Hom}_k(B\bar{A}, A)$. We denote the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $\mathcal{E}$ by $A \otimes_{\tau_A} \hat{S}$.*

We claim that $\mathcal{E}$ is the "universal deformation" of A. This is justified by Theorem 15.9 below. Let us start with a few lemmas.

Lemma 15.7. *The object $\mathcal{E}$ considered as a DG $\hat{S}^{op}$-module is h-projective.*

Proof. Notice that the stupid filtration $A^{\geq i}$ of the complex A is finite. Since A is admissible it follows that the differential $m^{\mathcal{E}}_1$ preserves the $(\hat{S}^{op})^{gr}$-submodule $A^{\geq i} \otimes \hat{S}$. Hence the DG $\hat{S}^{op}$-module $\mathcal{E} = A \otimes_{\tau_A} \hat{S}$ has a finite filtration by DG $\hat{S}^{op}$-submodules $A^{\geq i} \otimes \hat{S}$ with subquotients being free $\hat{S}^{op}$-modules $A^i \otimes \hat{S}$. Thus the DG $\hat{S}^{op}$-module $\mathcal{E}$ is h-projective. □

Lemma 15.8. *The A_∞ $\bar{A}^{op}$-module $\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k$ is canonically quasi-isomorphic to A.*

Proof. By Lemma 15.7 and Remark 15.6 we have

$$\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k = \mathcal{E} \otimes_{\hat{S}} k = A \otimes_{\tau_A} \hat{S} \otimes_{\hat{S}} k,$$

and the last A_∞ A^{op}-module is isomorphic to A since $\tau_A \in A \otimes \mathfrak{m}$, where $\mathfrak{m} \subset \hat{S}$ is the augmentation ideal. □

Now we are ready to define a morphism of pseudo-functors

$$\Psi : h'_{\hat{S}} \to \mathrm{DEF}_{\_}(A).$$

Let $\mathcal{R} \in \mathrm{dgart}_{\_}$ and $M = (M, \theta) \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$. We put

$$\Psi(M) := \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M \in D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}).$$

Notice that the structure isomorphism $\theta : M \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \to k$ defines an isomorphism

$$\mathbf{L}i^*\Psi(M) = \Psi(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k = \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sim} \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k,$$

and the last term is canonically quasi-isomorphic to A as an A_∞ $\bar{A}^{op}$-module by Lemma 15.8. Hence $\Psi(M)$ is indeed an object in the groupoid $\mathrm{DEF}_{\mathcal{R}}(A)$.

If $\delta : M \to N$ is a 2-morphism, where $M, N \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$, then $\Psi(\delta) : \Psi(M) \to \Psi(N)$ is a morphism of objects in the groupoid $\mathrm{DEF}_{\mathcal{R}}(A)$. Thus Ψ is indeed a morphism of pseudo-functors.

Theorem 15.9. *The morphism* $\Psi : h'_{\hat{S}} \to \mathrm{DEF}_{\_}(A)$ *is an equivalence.*

Proof. It remains to show that for each $\mathcal{R} \in \mathrm{dgart}_{\_}$ the induced functor

$$\Psi : 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R}) \to \mathrm{DEF}_{\mathcal{R}}(A)$$

is an equivalence of groupoids.

We fix $\mathcal{R}$.

Surjective on isomorphism classes.

Let $(S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$. By Corollary 12.3 there exists an element $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$ such that the pair (T, id), where $T = A \otimes_\alpha \mathcal{R}$, defines an object of $\mathrm{DEF}_{\mathcal{R}}(A)$ and (T, id) is isomorphic to (S, σ) in $\mathrm{DEF}_{\mathcal{R}}(A)$. The element α corresponds to a (unique) admissible twisting cochain $\tau : \mathcal{R}^* \to A$, which in turn corresponds to a homomorphism of DG coalgebras $g_\tau : \mathcal{R}^* \to B\bar{A}$ (Proposition 8.1). By dualizing we obtain a homomorphism of DG algebras $g_\tau^* : \hat{S} \to \mathcal{R}$ and hence the corresponding object $M_\alpha = ({}_{\hat{S}}\mathcal{R}_{\mathcal{R}}, \mathrm{id}) \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$.

Lemma 15.10. *The object* $\Psi(M_\alpha) \in \mathrm{Def}_{\mathcal{R}}(A)$ *is isomorphic to* (T, id).

Proof. By Remark 15.6

$$\mathcal{E} = A \otimes_{\tau_A} \hat{S},$$

and hence by Lemma 15.7

$$\Psi(M_\alpha) = (A \otimes_{\tau_A} \hat{S}) \otimes_{g_\tau^*} \mathcal{R}.$$

Notice that the image of τ_A under the map

$$\mathbf{1}_A \otimes g_\tau^* : A \otimes \hat{S} \to A \otimes \mathcal{R}$$

coincides with τ. Thus $\Psi(M_\alpha) = T$. □

Full and faithful.

Let us define a functor $\Pi : \mathrm{DEF}_{\mathcal{R}}(A) \to 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$ as follows: for $S = (S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$ we put

$$\Pi(S) := \mathrm{Hom}_{\bar{A}^{op}}(\mathcal{E}, S) \in D(\hat{S}^{op} \otimes \mathcal{R}).$$

We claim that $\Pi(S)$ is an object in $1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$, i. e. it is quasi-isomorphic to $\mathcal{R}$ as a DG $\mathcal{R}^{op}$-module and the isomorphism σ defines an isomorphism $\Pi(S) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sim} k$.

Indeed, again by Corollary 12.3 we may and will assume that $(S, \sigma) = (T, \mathrm{id})$, where $T = A \otimes_\alpha \mathcal{R}$, $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$. We have

$$\Pi(T) = \mathrm{Hom}_{A^{op}}(\mathcal{E}, A \otimes_\alpha \mathcal{R}) = \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(k, A), A) \otimes_\alpha \mathcal{R}.$$

Since the A_∞-algebra A satisfies the condition (*) the last term as a DG $\mathcal{R}^{op}$-module is canonically quasi-isomorphic to $k \otimes \mathcal{R} = \mathcal{R}$. Thus we have a canonical isomorphism $\Pi(S) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} k \xrightarrow{\sim} k$.

Note that the functors Ψ and Π are adjoint:

$$\mathrm{Hom}_{\mathrm{DEF}_{\mathcal{R}}(A)}(\Psi(M), S) = \mathrm{Hom}_{1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})}(M, \Pi(S)).$$

Now let us consider Ψ and Π as functors simply between the derived categories $D(\hat{S} \otimes \mathcal{R}^{op})$ and $D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}$. (They remain adjoint). Denote by $\langle k \rangle \subset D(\hat{S} \otimes \mathcal{R}^{op})$ and $\langle A \rangle \subset D_\infty(\bar{A}^{op}_{\mathcal{R}^{op}}$ the triangulated envelopes of the DG module k and the A_∞ $\bar{A}^{op}_{\mathcal{R}^{op}}$-module A respectively. Let $(S, \sigma) \in \mathrm{DEF}_{\mathcal{R}}(A)$. By Corollary 12.3 we may and will assume that $(S, \sigma) = (T, \mathrm{id})$, where $T = A \otimes_\alpha \mathcal{R}$, $\alpha \in \mathcal{MC}_{\mathcal{R}}(A)$. Hence $S \in \langle A \rangle$. Choose $(M, \theta) \in 1\text{-}\mathrm{Hom}(\hat{S}, \mathcal{R})$. Since the DG algebra $\hat{S} \otimes \mathcal{R}^{op}$ is local and complete by Lemma 4.2 we have $M \in \langle k \rangle$. Therefore it suffices to prove the following lemma.

Lemma 15.11. *The functors Ψ and Π induce mutually inverse equivalences of triangulated categories $\langle k \rangle$ and $\langle A \rangle$.*

Proof. It suffices to prove that the adjunction maps $k \to \Pi\Psi(k)$ and $\Psi\Pi(A) \to A$ are isomorphisms.

We have $\Pi\Psi(k) = \mathrm{Hom}_{\bar{A}^{op}}(\mathcal{E}, \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k) = \mathrm{Hom}_{A^{op}}(\mathcal{E}, A)$ (Lemma 15.8). Hence $k \to \Pi\Psi(k)$ is a quasi-isomorphism because A satisfies property (*).

Vice versa, $\Psi\Pi(A) = \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} (\mathrm{Hom}_{\bar{A}^{op}}(\mathcal{E}, A)) = \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k$, since A satisfies property (*). But $\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\hat{S}} k = A$ by Lemma 15.8.

This proves the lemma. □

Theorem is proved. □

15.1. **Explicit equivalence** $\mathrm{DEF}_-(E) \cong h'_{\hat{S}}$**.** Let $E \in \mathcal{A}^{op}$-mod. Suppose that the graded algebra $\mathrm{Ext}(E, E)$ is admissible and finite-dimensional. Let A be a strictly unital minimal model of the DG algebra $\mathbf{R}\,\mathrm{Hom}(E, E)$. Suppose that A satisfies the condition (*) above. Further, let $F \to E$ be a quasi-isomorphism with h-projective F, $\mathcal{C} = \mathrm{End}(F)$ and let $f : A \to \mathcal{C}$ be a strictly unital A_∞-quasi-isomorphism. By Theorem 15.9, the A_∞ $\bar{A}^{op}_{\hat{S}^{op}}$-module $\mathrm{Hom}_{\bar{A}}(k, A)$ is the "universal deformation" of the A_∞ $\bar{A}^{op}$-module A. It follows from the equivalence $\mathrm{DEF}_-(A) \cong \mathrm{DEF}_-(\mathcal{C})$ (Corollary 12.2) that that the $(\mathcal{C} \otimes \hat{S})^{op}$-module

$$\mathrm{Hom}_{\bar{A}}(k, \mathcal{C}) = \mathcal{C} \otimes_{f^*(\tau_A)} \hat{S}$$

is the "universal deformation" of the DG $\mathcal{C}^{op}$-module $\mathcal{C}$.

Put

$$\mathcal{F} = \mathrm{Hom}_{\bar{A}}(k, \mathcal{C}) \otimes_{\mathcal{C}} F = (\mathcal{C} \otimes_{f^*(\tau_A)} \hat{S}) \otimes_{\mathcal{C}} F.$$

Then $\mathcal{F}$ is a DG $\mathcal{A}_{\hat{S}}^{op}$-module. We claim that it is a "universal deformation" of the DG module E. More precisely, we get the following

Corollary 15.12. *Let E and $\mathcal{F}$ be as above. Then the functors $\Phi_{\mathcal{R}} : D(\hat{S} \otimes \mathcal{R}^{op}) \to D(\mathcal{A}_{\mathcal{R}}^{op})$,*

$$\Phi_{\mathcal{R}}(M) = \mathcal{F} \overset{\mathbf{L}}{\otimes}_{\hat{S}} M,$$

induce the equivalence of pseudo-functors

$$\Phi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(E)$$

from dgart$_{-}$ *to* **Gpd**.

Proof. Indeed, the morphism $\Phi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(E)$ is isomorphic to the composition of the equivalence

$$\Psi : h'_{\hat{S}} \to \mathrm{DEF}_{-}(A)$$

from Theorem 15.9, the equivalence

$$\mathrm{DEF}_{-}(A) \cong \mathrm{DEF}_{-}(\mathcal{C})$$

from Corollary 12.2, and the equivalence

$$\mathrm{DEF}_{-}(\Sigma) : \mathrm{DEF}_{-}(\mathcal{C}) \to \mathrm{DEF}_{-}(E)$$

from the proof of Proposition 11.9. □

Part 5. Noncommutative Grassmanians

16. Preliminaries on $\mathbb{Z}$-algebras

In this section we define the notion of a $\mathbb{Z}$-algebra and associate to it an abelian category which should be thought of as a category of quasi-coherent sheaves on the corresponding noncommutative stack. We also define Koszul $\mathbb{Z}$-algebras.

Definition 16.1. *A $\mathbb{Z}$-algebra $\mathcal{A}$ over the field k is a k-linear category with the set of objects $\mathbb{Z}$. For $i, j \in \mathbb{Z}$, we write $\mathcal{A}_{ij}$ instead of $\mathrm{Hom}_{\mathcal{A}}(i, j)$. Sometimes we will identify a $\mathbb{Z}$-algebra $\mathcal{A}$ with the corresponding ordinary non-unital algebra $\mathrm{Alg}_{\mathcal{A}}$.*

Further, if $\mathcal{A}$ is a $\mathbb{Z}$-algebra, then we define the abelian category Mod-$\mathcal{A}$ as the category $\mathrm{Fun}(\mathcal{A}^{op}, k\text{-Vect})$ of contravariant functors from $\mathcal{A}$ to k-vector spaces. Equivalently Mod-$\mathcal{A}$ is the full subcategory of Mod- $\mathrm{Alg}_{\mathcal{A}}$ which consists of right $\mathrm{Alg}_{\mathcal{A}}$-modules M such that

$$M = \bigoplus_{i \in \mathbb{Z}} M \cdot \mathbf{1}_i$$

(quasi-unital modules). We call the objects of Mod-$\mathcal{A}$ $\mathcal{A}^{op}$-modules. For each $i \in \mathbb{Z}$ put

$$P_i := \mathrm{Hom}(-, i) = \mathbf{1}_i Alg_{\mathcal{A}} \in \text{Mod-}\mathcal{A}.$$

By Ioneda Lemma, for each $M \in$ Mod-$\mathcal{A}$ we have

$$\mathrm{Hom}_{\mathcal{A}^{op}}(P_i, M) = M(i),$$

hence P_i are projectives. Clearly, each $M \in$ Mod-$\mathcal{A}$ can be covered by a direct sum of P_i's, hence the abelian category Mod-$\mathcal{A}$ has enough projectives.

Definition 16.2. *Let $M \in$ Mod-$\mathcal{A}$ be an $\mathcal{A}^{op}$-module. An element $x \in M(i)$ is called torsion if we have $x\mathcal{A}_{ji} = 0$ for $j << i$. Torsion elements form a submodule of M which we denote by $\tau(M)$. An $\mathcal{A}^{op}$-module M is called torsion if we have $M = \tau(M)$. We denote by $\mathrm{Tors}(\mathcal{A})$ the full subcategory of Mod-$\mathcal{A}$ which consists of torsion $\mathcal{A}^{op}$-modules.*

The category $\mathrm{QMod}(\mathcal{A})$ is defined as the quotient category Mod-$\mathcal{A}/\mathrm{Tors}(\mathcal{A})$. We denote by $\pi :$ Mod-$\mathcal{A} \to \mathrm{QMod}(\mathcal{A})$ the projection functor.

If M, N are $\mathcal{A}^{op}$-modules then

$$\mathrm{Hom}_{\mathrm{QMod}(\mathcal{A})}(\pi(M), \pi(N)) = \varinjlim \mathrm{Hom}_{\mathcal{A}^{op}}(M', N/\tau(N)),$$

where M' runs over the quasi-directed category of submodules $M' \subset M$ such that M/M' is torsion.

The category $\mathrm{QMod}(\mathcal{A})$ should be thought of as the category $\mathrm{QCoh}(\mathrm{Proj}(\mathcal{A}))$ of quasi-coherent sheaves on the noncommutative projective stack $\mathrm{Proj}(\mathcal{A})$. Furthermore, the object $\pi(P_0) \in \mathrm{QMod}(\mathcal{A})$ should be thought of as the structure sheaf $\mathcal{O}_{\mathrm{Proj}(A)}$.

Remark 16.3. *Let $A = \bigoplus_{i\in\mathbb{Z}} A^i$ be a $\mathbb{Z}$-graded (unital) algebra. Then one can associate to it a $\mathbb{Z}$-algebra $\mathcal{A}$ with $\mathcal{A}_{ij} = A^{j-i}$ so that the composition in $\mathcal{A}$ comes from the multiplication in A. Recall that in* [AZ] *there defined the category $\mathrm{QGr}(A)$ as the quotient category $\mathrm{Gr}\,A/\mathrm{Tors}$ of the category $\mathrm{Gr}\,A$ of graded A-modules by the subcategory Tors of torsion modules. It is clear that the categories $\mathrm{Gr}\,A$, Tors, $\mathrm{QGr}\,A$ are equivalent to Mod-$\mathcal{A}$, $\mathrm{Tors}(\mathcal{A})$, $\mathrm{QMod}(\mathcal{A})$ respectively.*

Notice that it can happen that the graded algebras A_1 and A_2 are not isomorphic but the associated $\mathbb{Z}$-algebras are equivalent. Thus, it is more reasonable to consider $\mathbb{Z}$-algebras but not graded algebras.

The projection functor $\pi : \mathrm{Mod}\text{-}\mathcal{A} \to \mathrm{QMod}(\mathcal{A})$ admits a right adjoint functor $\omega : \mathrm{QMod}(\mathcal{A}) \to \mathrm{Mod}\text{-}\mathcal{A}$ defined by the formula

$$\omega(X)(i) = \mathrm{Hom}(\pi(P_i), X).$$

The adjunction morphism $\pi\omega \to \mathrm{id}$ is an isomorphism.

Definition 16.4. *A $\mathbb{Z}$-algebra $\mathcal{A}$ is called*

a) positively (resp. negatively) oriented if $\mathcal{A}_{ij} = 0$ for $i > j$ (resp. for $i < j$);

b) connected if $\mathcal{A}_{ii} \cong k$ for each $i \in \mathbb{Z}$;

c) locally finite if $\dim \mathcal{A}_{ij} < \infty$ for any $i, j \in \mathbb{Z}$.

Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. We denote by $\mathcal{A}_{\leq i}$ the full subcategory of $\mathcal{A}$ such that $Ob(A_{\leq i}) = \{j : j \leq i\}$. Clearly, we also have the categories Mod-$\mathcal{A}_{\leq i}$ and $\mathrm{Tors}(\mathcal{A}_{\leq i})$. It is easy to see that the quotient category Mod-$\mathcal{A}_{\leq i}/\mathrm{Tors}(\mathcal{A}_{\leq i})$ is equivalent to $\mathrm{QMod}(\mathcal{A})$. We denote by $\pi_{\leq i} :$ Mod-$\mathcal{A}_{\leq i} \to \mathrm{QMod}(\mathcal{A})$ and $\omega_{\leq i} : \mathrm{QMod}(\mathcal{A}) \to$ Mod-$\mathcal{A}_{\leq i}$ the projection functor and its right adjoint respectively.

If $\mathcal{A}$ is a positively oriented $\mathbb{Z}$-algebra then we put

$$T_{ij} = P_j/(P_j)_{\leq i}$$

, where

$$(P_j)_{\leq i} = \bigoplus_{k\leq i} A_{kj}.$$

Clearly, the $\mathcal{A}^{op}$-modules T_{ij} are torsion.

If $\mathcal{A}$ is a positively or negatively oriented connected $\mathbb{Z}$-algebra then we denote by S_n the simple $\mathcal{A}^{op}$-modules defined by the formula

$$S_n(i) = \begin{cases} k & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $\mathcal{A}$ is positively oriented then $S_n = T_{n,n}$.

Definition 16.5. *A connected positively (resp. negatively) oriented $\mathbb{Z}$-algebra is called quadratic if the algebra $Alg_{\mathcal{A}}$ is generated by the subspaces $\mathcal{A}_0 = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{ii}$ and $\mathcal{A}_1 = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{i,i+1}$ (resp. $\mathcal{A}_{-1} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_{i+1,i}$) and is determined by the quadratic relations $I_{i,i+2} \in \mathcal{A}_{i+1,i+2} \otimes \mathcal{A}_{i,i+1}$ (resp. $I_{i+2,i} \in \mathcal{A}_{i+1,i} \otimes \mathcal{A}_{i+2,i+1}$).*

For a locally finite positively (resp. negatively) oriented quadratic $\mathbb{Z}$-algebra $\mathcal{A}$ one can define the dual quadratic $\mathbb{Z}$-algebra $\mathcal{A}^!$ with the opposite orientation. It is defined by the dual generators $\mathcal{A}^!_{i+1,i} = \mathcal{A}^*_{i,i+1}$ (resp. $\mathcal{A}^!_{i,i+1} = \mathcal{A}^*_{i+1,i}$) and the dual quadratic relations $S(I^\perp_{i,i+2}) \subset \mathcal{A}^*_{i,i+1} \otimes \mathcal{A}^*_{i+1,i+2}$ (resp. $S(I^\perp_{i+2,i}) \subset \mathcal{A}^*_{i+2,i+1} \otimes \mathcal{A}^*_{i+1,i}$), where $I^\perp_{i,i+2} \subset \mathcal{A}^*_{i+1,i+2} \otimes \mathcal{A}^*_{i,i+1}$ (resp. $I^\perp_{i+2,i} \subset \mathcal{A}^*_{i+1,i} \otimes \mathcal{A}^*_{i+2,i+1}$) is the dual subspace and S is the transposition of factors.

Further, one can define a Koszul complex

$$K := \mathcal{A}^{!^*} \otimes_{\mathcal{A}_0} \mathcal{A} = \bigoplus \mathcal{A}^{!^*}_{kj} \otimes_k \mathcal{A}_{ij}.$$

Here $\mathcal{A}^{!^*} = \bigoplus_{i,j} \mathcal{A}^{!^*}_{ij}$ is a bounded dual of $\mathcal{A}^!$. It is an $\mathcal{A}^!$-bimodule.

The differential $d : K \to K$ is defined as follows. Suppose that $\mathcal{A}$ is positively oriented. We have the natural maps

$$\mathcal{A}_{j,j+1} \otimes \mathcal{A}_{ij} \to \mathcal{A}_{i,j+1}, \quad \text{and} \quad \mathcal{A}^*_{j,j+1} \otimes \mathcal{A}^{!^*}_{kj} \to \mathcal{A}^{!^*}_{k,j+1}.$$

In particular, we have the maps

$$\psi_{ijk} : \mathcal{A}^*_{j,j+1} \otimes \mathcal{A}_{j,j+1} \to \mathrm{Hom}_k(\mathcal{A}^{!^*}_{kj} \otimes_k \mathcal{A}_{ij}, \mathcal{A}^{!^*}_{k,j+1} \otimes_k \mathcal{A}_{i,j+1}).$$

The non-zero components of d are the maps $d_{ijk} = \psi_{ijk}(\mathbf{1}_{A_{jj+1}})$. Note that d is $\mathcal{A}_0$-linear and $\mathcal{A}^! \otimes \mathcal{A}^{op}$-linear. Thus, $K_n = \mathbf{1}_n K$ and $K^m_n = K_n \mathbf{1}_m$ are d-invariant. The complex K_n is of the form

$$\cdots \to \mathcal{A}^{!^*}_{n,n-2} \otimes P_{n-2} \to \mathcal{A}^{!^*}_{n,n-1} \otimes P_{n-1} \to P_n \to 0,$$

and the complex K^m_n is of the form

$$\cdots \to \mathcal{A}^{!^*}_{n,n-2} \otimes \mathcal{A}_{m,n-2} \to \mathcal{A}^{!^*}_{n,n-1} \otimes \mathcal{A}_{m,n-1} \to \mathcal{A}_{m,n} \to 0.$$

In particular, $K^n_n \cong k$.

Analogously for negatively oriented $\mathbb{Z}$-algebras.

For the rest of this section we assume that $\mathcal{A}$ is positively oriented.

Definition 16.6. *A quadratic locally finite $\mathbb{Z}$-algebra is called Koszul if the complex K^m_n is acyclic for $n \neq m$, or, equivalently, K_n is a resolution of S_n.*

We refer to [BP] for the definition of co-Koszul and Gorenstein $\mathbb{Z}$-algebras. We will not need these definitions but we will need the following proposition:

Proposition 16.7. *([BP]) Let $\mathcal{A}$ be a Koszul (positively oriented) $\mathbb{Z}$-algebra of finite homoogical dimension n. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ is co-Koszul;

(ii) $\mathcal{A}$ is Gorenstein;

(iii) $\mathcal{A}^!$ is Frobenious, i.e. $\mathcal{A}^!_{i+n,i} = k$ for all i, and the multiplication $\mathcal{A}_{ji} \otimes \mathcal{A}_{i+n,j} \to k$ is a perfect pairing for all i and j.

Now we define the notion of a coherent $\mathbb{Z}$-algebra and the category $\mathrm{qmod}(\mathcal{A})$ for a coherent $\mathbb{Z}$-algebra $\mathcal{A}$.

Definition 16.8. *Let $\mathcal{A}$ be a $\mathbb{Z}$-algebra. An $\mathcal{A}^{op}$-module M is called finitely generated if there exists a surjective morphism (in Mod-$\mathcal{A}$)*

$$\bigoplus_{j=1}^{m} P_{i_j} \to M,$$

where $i_1, \dots, i_m \in \mathbb{Z}$. Further, a finitely generated $\mathcal{A}^{op}$-module M is called coherent if for each (not necessarily surjective) morphism (in Mod-$\mathcal{A}$)

$$\phi : \bigoplus_{j=1}^{m} P_{i_j} \to M$$

the $\mathcal{A}^{op}$-module $\ker(\phi)$ is finitely generated. A $\mathbb{Z}$-algebra $\mathcal{A}$ is called coherent if for each $i \in \mathbb{Z}$ the module P_i is coherent.

If $\mathcal{A}$ is coherent then we denote by $\mathrm{qmod}(\mathcal{A})$ the full (abelian) subcategory of $\mathrm{QMod}(\mathcal{A})$ which consists of the images of coherent $\mathcal{A}^{op}$-modules.

The category $\mathrm{qmod}(\mathcal{A})$ should be thought of as the category $\mathrm{Coh}(\mathrm{Proj}(\mathcal{A}))$. By definition, we have that the structure sheaf $\mathcal{O}_{\mathrm{Proj}(\mathcal{A})}$ is coherent.

17. The definition of noncommutative Grassmanians

Let V be a finite-dimensional k-vector space of dimension $n > 0$. Let m be an integer such that $1 \leq m \leq n-1$. We define the noncommutative Grassmanians by the formula

$$\mathrm{NGr}(m, V) := \mathrm{Proj}(\mathcal{A}^{m,V}),$$

where $\mathcal{A}^{m,V}$ is the following quadratic $\mathbb{Z}$-algebra:

$$\mathcal{A}^{m,V}_{i,i+1} = \begin{cases} V^* & \text{for } (n-m+1) \nmid i, \\ \Lambda^{n-m} V & \text{otherwise,} \end{cases}$$

and the quadratic relations are defined by the natural exact sequences

$$\begin{cases} \Lambda^2 V^* \to \mathcal{A}^{m,V}_{i+1,i+2)} \otimes \mathcal{A}^{m,V}_{i,i+1} \to \mathcal{A}^{m,V}_{i,i+2} \to 0 & \text{for } (n-m+1) \nmid i, i+1, \\ \Lambda^{n-m-1} V \to \mathcal{A}^{m,V}_{i+1,i+2} \otimes \mathcal{A}^{m,V}_{i,i+1} \to \mathcal{A}^{m,V}_{i,i+2} \to 0 & \text{otherwise.} \end{cases}$$

Notice that if we fix a volume form $\omega \in \Lambda^n V$, then the $\mathcal{A}^{1,V}$ is naturally equivalent to the $\mathbb{Z}$-algebra associated to the symmetric algebra $\bigoplus_{l \geq 0} S^l V^*$, where $\deg(V^*) = 1$. Hence, the stack $\mathrm{NGr}(1, V)$ is isomorphic to the commutative projective space $\mathbb{P}(V)$.

We claim that $\mathrm{NGr}(m, V)$ is a true noncommutative moduli space of structure sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$, where $W \subset V$ are vector subspaces of dimension $\dim W = m$. Namely, it satisfies the following properties:

(i) there is a natural fully faithful functor Φ from the perfect derived category $\mathrm{Perf}(\mathrm{NGr}(m,V)) = D_{perf}(\mathrm{QMod}(\mathcal{A}^{m,V}))$ (Definition 18.20) to $D^b_{coh}(\mathbb{P}(V))$. Its image is the double orthogonal to the family of objects $\mathcal{O}_{\mathbb{P}(W)}$, i.e. the full subcategory generated by objects $\mathcal{O}_{\mathbb{P}(V)}(m-n),\dots,\mathcal{O}_{\mathbb{P}(V)}(-1),\mathcal{O}_{\mathbb{P}(V)}$. This is Corollary 18.22 below;

(ii) There is a k-point $x_W \in \mathrm{NGr}(m,V)(k) = X_{\mathcal{A}^{m,V}}(k)$ (see Section 19 below) for each subspace $W \subset V$ of dimension $\dim W = m$. Further, $(x_W)_*(k)$ lies in $\mathrm{Perf}(\mathrm{NGr}(m,V))$ and $\Phi(x_*(k)) \cong \mathcal{O}_{\mathbb{P}(W)}$. This is a part of Theorem 19.10 below.

Further, according to our pro-representability Theorems (namely, Theorem 15.2), the completion of the local ring of the k-point x_W must be isomorphic to $H^0(\hat{S})$, where $\hat{S}$ is dual to the bar construction of the minimal A_∞-structure on $\mathrm{Ext}(\mathcal{O}_{\mathbb{P}(W)},\mathcal{O}_{\mathbb{P}(W)})$. Indeed, it can be shown that the DG algebra $\mathbf{R}\,\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(W)},\mathcal{O}_{\mathbb{P}(W)})$ is formal and the graded algebra $\mathrm{Ext}(\mathcal{O}_{\mathbb{P}(W)},\mathcal{O}_{\mathbb{P}(W)})$ is quadratic Koszul, and hence the projection $\hat{S} \to H^0(\hat{S})$ is a quasi-isomorphism. However, we do not have a suitably defined notion of a (completion of) the local ring at the k-point of noncommutative stack. Thus, we can not establish and prove the corresponding result at the moment.

Furthermore, we do not have a moduli functor of our family of objects $\mathcal{O}_{\mathbb{P}(W)}$, which should be defined on the category of noncommutative affine schemes. However, the properties (i) and (ii) suggest that $\mathrm{NGr}(m,V)$ is a true moduli space of this family of objects, in our context of deformations of objects in derived categories.

It is remarkable that there is a natural morphism from the commutative Grassmanian $\mathrm{Gr}(m,V)$ to noncommutative one $\mathrm{NGr}(m,V)$. Moreover, the functor $\Phi : \mathrm{Perf}(\mathrm{NGr}(m,V)) \to D^b_{coh}(\mathbb{P}(V))$ above coincides with $\mathbf{L}f^*_{1,m,V}$, where $f_{1,m,V} : \mathbb{P}(V) \to \mathrm{NGr}(m,V)$ is a natural morphism. Both these statements are parts of Proposition 19.12 below.

18. The derived categories of noncommutative Grassmanians

Before we formulate and prove results on the derived categories of quasi-coherent sheaves on noncommutative Grassmanians $\mathrm{NGr}(m,V)$ we need to remind some notions and results from [BP].

Let D be a k-linear enhanced triangulated category.

Definition 18.1. *An object $E \in Ob(D)$ is called exceptional if $\mathrm{Hom}^i(E,E) = 0$ for $i \neq 0$, and $\mathrm{Hom}^0(E,E) = k$.*

Definition 18.2. *A collection $(E_1,\dots,E_m)$ of exceptional objects in D is called exceptional if $\mathrm{Hom}^*(E_i,E_j) = 0$ for $i > j$.*

Definition 18.3. *A full exceptional collection of objects in the category D is a collection which generates D as triangulated category.*

Definition 18.4. *An exceptional collection $(E_1,\dots,E_n)$ is called strong exceptional if it satisfies the additional assumption $\mathrm{Hom}^i(E_k,E_l) = 0$ for $i \neq 0$ and all k and l.*

Let (E,F) be an exceptional pair. Define the objects L_EF and R_FE by the exact triangles

$$L_EF \to \mathrm{Hom}^{\cdot}(E,F)\otimes E \to F, \quad E \to \mathrm{Hom}^{\cdot}(E,F)^\vee \otimes F \to R_FE.$$

Let $\sigma = (E_1,\dots,E_n)$ be an exceptional collection. If $1 \le i \le n-1$ (resp. $2 \le i \le n$), then the right (resp. left) mutation of the object E_i in this collection is the object $R^1E_i = R_{E_{i+1}}E_i$ (resp. $L^1E_i = L_{E_{i-1}}E_i$); the

corresponding mutated collection

$$R^1_{E_i}\sigma := (E_1, \dots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \dots, E_n)$$

(and the analogous collection $L^1_{E_i}\sigma$) is exceptional. The multiple mutations of the objects and of the collection are defined inductively:

$$R^k E_i = R_{E_{i+k}} R^{k-1} E_i, \quad R^k_{E_i}\sigma = R^1_{R^{k-1}E_i}(R^{k-1}_{E_i}\sigma), \quad k \le n-i$$

(and in the same way for left mutations).

Definition 18.5. *A helix of the period n is an infinite sequence $\{E_i\}_{i\in\mathbb{Z}}$ such that for each $i \in \mathbb{Z}$ the collection $(E_i, \dots, E_{i+n})$ is exceptional, and moreover $R^{n-1}E_i = E_{i+n}$.*

If $\sigma = (E_1, \dots, E_n)$ is an exceptional collection then it naturally extends to a helix by the conditions

$$E_{i+n} = R^{n-1}E_i, \quad i \ge 1, \qquad E_{i-n} = L^{n-1}E_i, \quad i \le n.$$

In this case the helix is said to be generated by the collection σ .

If the helix is generated by the full exceptional collection then it satisfies the property of the partial periodicity: $\Phi(E_i) \cong E_{i-n}$, where $\Phi = F[1-n]$ is the composition of the Serre functor F and the multiple shift $[1-n]$.

Definition 18.6. *(*[BP]*) A helix $\mu = \{E_i\}$ is called geometric if for each pair $(i,j) \in \mathbb{Z}^2$ such that $i \le j$ one has*

$$\mathrm{Hom}^k(E_i, E_j) = 0 \text{ for } k \ne 0.$$

Definition 18.7. *(*[BP]*) An exceptional collection is called geometric if it generates a geometric helix.*

Proposition 18.8. *(*[BP]*) Each sub-collection of a geometric exceptional collection is again geometric.*

Proposition 18.9. *(*[BP]*) A full exceptional collection of the length m of coherent sheaves on a smooth projective variety X of dimension n is geometric iff $m = n+1$.*

Definition 18.10. *(*[BP]*) The endomorphism $\mathbb{Z}$ -algebra $\mathcal{A} = \mathrm{End}(S)$ of a helix $S = \{E_i\}$ is defined by the formula*

$$\mathcal{A}_{ij} = \mathrm{Hom}(E_i, E_j)$$

with natural composition.

Theorem 18.11. *(*[BP]*) If S is a geometric helix generated by an exceptional collection of length n then the endomorphism $\mathbb{Z}$ -algebra $\mathcal{A}$ of S is Koszul, co-Koszul and of finite global homological dimension n .*

Definition 18.12. *(*[BP]*) Koszul co-Koszul $\mathbb{Z}$ -algebra of finite homological dimension n is called a geometric $\mathbb{Z}$ -algebra of the period n .*

Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of the period n . Let $K \subset D(\text{Mod-}\mathcal{A})$ be the full triangulated subcategory generated by the modules P_i . Note that $S_i \in K$. Let $F \in K$ be the full triangulated subcategory generated by the modules S_i .

Theorem 18.13. *(*[BP]*) Let $\mathcal{A}$ be a geometric $\mathbb{Z}$ -algebra of the period n . Then F is a thick subcategory in K ; the images of modules P_i in K/F form a geometric helix S of the period n , and moreover the $\mathbb{Z}$ -algebra of S is equivalent to $\mathcal{A}$.*

Now we prove the main theorem of this section. It is closely related to the previous one but unfortunately cannot be deduced from it.

Theorem 18.14. *Let $\mathcal{A}$ be a geometric helix of the period n. Put $B = \mathcal{A}_{[1,n]} = \bigoplus_{1\le i,j\le n} \mathcal{A}_{ij}$. Then there is an equivalence of categories $D^*(\mathrm{QMod}(\mathcal{A})) \cong D^*(Mod\text{-}B)$.*

Proof. The proof is in two main steps. First we prove that the category $D^*(\mathrm{QMod}(\mathcal{A}))$ is naturally equivalent to the quotient category of $D^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})$ by the full thick triangulated subcategory $D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})$ which consists of complexes with torsion cohomology. Then we construct mutually inverse exact equivalences between the categories $D^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})/D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})$ and $D^*(\mathrm{Mod}\text{-}\mathcal{B})$ given by DG bimodules.

Lemma 18.15. *The categories $D^*(\mathrm{QMod}(\mathcal{A}))$ and $D^*(Mod\text{-}\mathcal{A}_{\le n})/D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})$ are naturally equivalent.*

Proof. First recall the functor $\omega_{\le n} : \mathrm{QMod}(\mathcal{A}) \to \mathrm{Mod}\text{-}\mathcal{A}_{\le n}$. It induces a fully faithful functor

$$K^*(\omega_{\le n}) : K^*(\mathrm{QMod}(\mathcal{A}_{\le n})) \to K^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})$$

between homotopy categories, which is right adjoint to $K^*(\pi_{\le n})$. It follows that $K^*(\mathrm{QMod}(\mathcal{A}_{\le n}))$ is equivalent to the quotient category $K^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})/K^*(\pi_{\le n})^{-1}(0)$. Let $K^*_{\mathrm{Tors}}(\mathcal{A}_{\le n}) \subset K^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})$ be the full subcategory which consists of complexes with torsion cohomology. Then acyclic complexes in the category $K^*(\mathrm{QMod}(\mathcal{A}_{\le n}))$ correspond to the classes of complexes with torsion cohomology in $K^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})/K^*(\pi_{\le n})^{-1}(0)$. Thus, $D^*(\mathrm{QMod}(\mathcal{A}_{\le n}))$ is equivalent to the quotient of $K^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})/K^*(\pi_{\le n})^{-1}(0)$ by $K^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})/K^*(\pi_{\le n})^{-1}(0)$. This quotient is further equivalent to $D^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n})/D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})$. Lemma is proved. □

Denote by $(Q_1,\dots,Q_n)$ the exceptional collection of indecomposable projective B^{op}-modules. By Theorem 18.13 the helix $\{Q_i\}_{i\in\mathbb{Z}}$ generated by $(Q_1,\dots,Q_n)$ is geometric. It follows from its partial periodicity that for $i \le 0$ we have $Q_i \cong H^{n-1}(Q_i)[1-n]$. Thus, we may and will assume that Q_i is concentrated in degree $n-1$ for $i \le 0$. Put $M_0 = \bigoplus_{i\le 0} Q_i[n-1]$. Since $\mathrm{Hom}_B(Q_i,Q_j) = \mathcal{A}_{ij} = \mathcal{A}_{(i+n)(j+n)}$, M_0 is naturally an $\mathcal{A}_{\le n}\otimes B^{op}$-module. Further, the functor $\Phi^{-1} = F^{-1}[1-n]$ can be given by the object $B^![1-n] \in D(B\otimes B^{op})$, where $B^! = \mathbf{R}\,\mathrm{Hom}_{B^{op}\otimes B}(B, B\otimes B)$. Since $Q_i \otimes^{\mathbf{L}}_B B^![1-n]$ are pure modules for $i = 1,\dots,n$, $B^![1-n]$ is a pure bimodule. We define the object $M_1 \in D^b(\mathcal{A}_{\le n}\otimes B^{op})$ by the formula

$$M_1; = M_0 \overset{\mathbf{L}}{\otimes}_B B^![2-2n].$$

We have that

$$P_i \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\le n}} M_1 \cong Q_i$$

for $i \le n$. Since $\mathcal{A}_{\le n}$ has finite left homological dimension, we have a well defined functor

$$-\otimes^{\mathbf{L}}_{\mathcal{A}_{\le n}} M_1 : D^*(\mathrm{Mod}\text{-}\mathcal{A}_{\le n}) \to D^*(\mathrm{Mod}\text{-}B).$$

Lemma 18.16. *For each $K^\cdot \in D_{\mathrm{Tors}}(\mathcal{A}^{op}_{\le n})$ we have*

$$K^\cdot \otimes^{\mathbf{L}}_{\mathcal{A}_{\le n}} M_1 = 0.$$

Proof. Clearly, it suffices to prove the Lemma for M_0 instead of M_1. We have that the complex K_l from section 16 is a projective resolution of S_l for $l \le n$. Further, $S_l \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\le n}} M_0 \cong K_l \otimes_{\mathcal{A}_{\le n}} M_0$ and the last complex is up to shift of the following form

$$\dots 0 \to A^{!*}_{l-n,l-2n} \otimes Q_{l-2n}[n-1] \to \dots \to A^{!*}_{l-n,l-n-1} \otimes Q_{l-n-1}[n-1] \to Q_{l-n}[n-1] \to 0 \to \dots$$

This complex is acyclic since it corresponds to the image of K_{l-n} in K/F under the equivalence of Theorem 18.13, and the image of K_{l-n} in K/F is zero.

Further, the torsion modules T_{km}, $k \le m \le n$ have finite filtrations with subquotients being direct sums of S_l. Thus, we have $T_{km} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\le n}} M_0 = 0$.

Since each torsion $\mathcal{A}^{op}_{\le n}$-modules has a left resolution by the direct sums of T_{km}, it follows that the statement of the Lemma holds if $K^{\cdot}$ is a pure torsion $\mathcal{A}^{op}_{\le n}$-module. Finally, since M_0 is quasi-isomorphic isomorphic to a finite complex of bimodules which are projective as left $\mathcal{A}$-modules, the statement of the Lemma holds for each $K^{\cdot} \in D_{\mathrm{Tors}}(\mathcal{A}^{op}_{\le n})$. □

By the previous Lemma, the formula $- \otimes^{\mathbf{L}}_{\mathcal{A}_{\le n}} M_1$ defines a functor

$$\Phi : D^*(\text{Mod-}\mathcal{A}_{\le n})/D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n}) \to D^*(\text{Mod-}B).$$

Further, $M_2 := \bigoplus_{1 \le i \le n} P_i$ is naturally an $\mathcal{A}^{op}_{\le n} \otimes B$-module. Consider the functor

$$\Psi : D^*(\text{Mod-}B) \to D^*(\text{Mod-}\mathcal{A}_{\le n})/D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})$$

defined by the formula

$$\Psi(-) = \pi_{tors}(- \otimes^{\mathbf{L}}_B M_2),$$

where $\pi_{tors} : D^*(\text{Mod-}\mathcal{A}_{\le n}) \to D^*(\text{Mod-}\mathcal{A}_{\le n})/D^*_{\mathrm{Tors}}(\mathcal{A}_{\le n})$ is the projection.

Lemma 18.17. *The functors Φ and Ψ are mutually inverse equivalences.*

Proof. First, the isomorphism

$$M_2 \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\le n}} M_1 \to B$$

in $D(B \otimes B^{op})$ induces the isomorphism of functors $\Phi \circ \Psi \cong Id$.

Further, we claim that $H^0(M_1 \otimes^{\mathbf{L}}_B M_2) \cong \mathcal{A}_{\le n}$, $H^{n-1}(M_1 \otimes^{\mathbf{L}}_B M_2)$ is torsion as $\mathcal{A}^{op}$-module and $H^i(M_1 \otimes^{\mathbf{L}}_B M_2) = 0$ for $i \ne 0, n-1$. Indeed, since K_l is a resolution of S_l it follows from Lemma 18.16 by decreasing induction on $l \le n$ that

$$H^i(P_l \otimes^{\mathbf{L}}_{\mathcal{A}_{\le n}} M_1 \otimes^{\mathbf{L}}_B M_2) = \begin{cases} P_l & \text{for } i = 0; \\ \text{is torsion} & \text{for } i \ge n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, it remains to note that $H^k(M_1) = 0$ for $k \ge n$ and M_2 is pure.

Finally, we have the natural morphism $\mathcal{A}_{\le n} \to M_1 \otimes^{\mathbf{L}}_B M_2$ in $D(A_{\le n} \otimes A^{op}_{\le n})$ and for each $K^{\cdot} \in D(\text{Mod-}\mathcal{A}_{\le n})$ we have that

$$Cone(K^{\cdot} \to K^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\le n}} M_1 \overset{\mathbf{L}}{\otimes}_B M_2) \cong K^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\le n}} Cone(\mathcal{A}_{\le n} \to M_1 \overset{\mathbf{L}}{\otimes}_B M_2) \in D_{\mathrm{Tors}}(\mathcal{A}^{op}_{\le n})$$

. Thus, $\Psi \circ \Phi \cong Id$. Lemma is proved. □

Theorem follows from Lemmas 18.17 and 18.15. □

Now we apply the above theorem to noncommutative Grassmanians introduced in section 17. By Propositions 18.8 and 18.9 we have that the exceptional collection

$$\sigma = (\mathcal{O}_{\mathbb{P}(V)}(m-n), \dots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)})$$

of coherent sheaves on $\mathbb{P}(V)$ is geometric. Let $S = \{E_i\}$ be the helix generated by σ, so that $E_i = \mathcal{O}_{\mathbb{P}(V)}(i)$ for $i = m-n, \dots, -1, 0$.

Proposition 18.18. *The endomorphism $\mathbb{Z}$-algebra $\mathcal{A}$ of the helix S is equivalent to $\mathcal{A}^{m,V}$.*

Proof. Note that both $\mathcal{A}$ and $\mathcal{A}^{m,V}$ are quadratic and $(n-m+1)$-periodic. It remains to show that the space $\mathcal{A}_{i,i+1}$ is isomorphic to $\mathcal{A}^{m,V}_{i,i+1}$ for $i = m-n, \dots, -1, 0$, and the quadratic relations $I_{i,i+2} \in \mathcal{A}^{i+1,i+2} \otimes \mathcal{A}_{i,i+1}$ coincide with that of $\mathcal{A}^{m,V}$ for $i = m-n, \dots, 0$.

All of this is clear for $i = m-n, \dots, -1$. Further, the object E_1 is isomorphic to the complex

$$\dots \to 0 \to \mathcal{O}_{\mathbb{P}(V)}(m-n) \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n+1) \to \dots \to \Lambda^{n-m} V \otimes \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \dots,$$

where the last non-zero term is in degree zero. It follows that

$$\mathcal{A}_{0,1} = \mathrm{Hom}(E_0, E_1) = \Lambda^{n-m} V = \mathcal{A}^{m,V}_{0,1}.$$

Furthermore, the quadratic relation $I_{-1,1} \subset \mathcal{A}^{0,1} \otimes \mathcal{A}_{-1,0}$ coincides with the subspace $\Lambda^{n-m-1} V \subset \Lambda^{n-m} V \otimes V^*$, as in $\mathcal{A}^{m,V}$. Finally, E_2 is the convolution of the complex

$$\dots \to 0 \to E_{m-n+1} \to V \otimes E_{m-n+2} \to \dots \to \Lambda^{n-m-1} V \otimes E_0 \to V^* \otimes E_1 \to 0 \to \dots,$$

where the last non-zero term is in degree zero. It follows that the quadratic relation $I_{0,2} \subset \mathcal{A}^{1,2} \otimes \mathcal{A}_{0,1}$ coincides with the subspace $\Lambda^{n-m-1} V \subset V^* \otimes \Lambda^{n-m} V$, as in $\mathcal{A}^{m,V}$. □

Let $B^{m,V}$ be the endomorphism algebra of σ. As a corollary of the above results we obtain the following

Theorem 18.19. *The derived category $D^*(\mathrm{QMod}(\mathcal{A}^{m,V}))$ is equivalent to the derived category $D^*(B^{m,V})$. The objects $\pi(P_i)$ in $D^*(\mathrm{QMod}(\mathcal{A}^{m,V}))$ form a geometric helix of the period $\dim V - m + 1$.*

Proof. Indeed, by Proposition 18.18 and Theorem 18.11 the $\mathbb{Z}$-algebra $\mathcal{A}^{m,V}$ is geometric. Thus, the first statement follows from Theorem 18.14. After that, the second statement follows from Theorem 18.13. □

Now we introduce the perfect derived category.

Definition 18.20. *Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. The perfect derived category $D_{perf}(\mathrm{QMod}(\mathcal{A}))$ is the minimal full thick triangulated subcategory of $D^b(\mathrm{QMod}(\mathcal{A}))$ which contains the objects $\pi(P_i)$. We will call the objects of $D_{perf}(\mathrm{QMod}(\mathcal{A}))$ perfect complexes.*

We will also write below $\mathrm{Perf}(\mathrm{NGr}(m,V))$ instead of $D_{perf}(\mathrm{QMod}(\mathcal{A}^{m,V}))$.

Proposition 18.21. *Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of the period n, and $B = \bigoplus_{1-n \le i,j \le 0} \mathcal{A}_{ij}$. Then the category $D_{perf}(\mathrm{QMod}(\mathcal{A}))$ is equivalent to $D^b(mod_{finite}\text{-}B)$.*

Proof. By Theorem 18.14 $(\pi(P_{1-n}), \dots, \pi(P_{-1}), \pi(P_0))$ is full strong exceptional collection in $D_{Perf}(\mathrm{QMod}(\mathcal{A}))$. Further, the category $D_{Perf}(\mathrm{QMod}(\mathcal{A}))$ is enhanced and $\mathrm{End}(\bigoplus_{1-n \le i \le 0} \pi(P_i)) = B$. Hence $D_{Perf}(\mathrm{QMod}(\mathcal{A}))$ is equivalent to $D^b(\mathrm{mod}_{finite}\text{-}B)$. □

Corollary 18.22. *The category* $D_{Perf}(\mathrm{QMod}(\mathcal{A}^{m,V}))$ *is equivalent to the full triangulated subcategory* $\mathcal{T}^{m,V} \in D^b_{coh}(\mathbb{P}(V))$ *generated by the exceptional collection*

$$(\mathcal{O}_{\mathbb{P}(V)}(m-n), \dots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}).$$

Under this equivalence the exceptional collection $(\pi(P_{m-n}), \dots, \pi(P_0))$ *corresponds to the exceptional collection* $(\mathcal{O}_{\mathbb{P}(V)}(m-n), \dots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)})$.

Remark 18.23. *Notice that by* [Mi] *the* $\mathbb{Z}$ *-algebra* $\mathcal{A}^{\dim V-1,V}$ *is coherent. Further, the* $\mathbb{Z}$ *-algebra* $\mathcal{A}^{1,V}$ *is Noetherian and hence is coherent. It should be plausible that all the* $\mathbb{Z}$ *-algebras* $\mathcal{A}^{m,V}$ *(and, more generally, all geometric* $\mathbb{Z}$ *-algebras) are coherent, but it is not clear how to prove this statement. However, if* $\mathcal{A}$ *and* B *are as in Theorem 18.14, and* $\mathcal{A}$ *is coherent, then the subcategory* $\mathcal{D}_{perf}(\mathrm{QMod}(\mathcal{A})) \subset D^b(\mathrm{QMod}(\mathcal{A}))$ *coincides with the subcategory* $D^b_{\mathrm{qmod}}(|QMod(\mathcal{A}))$ *which consists of complexes with cohomology lying in* qmod . *This category is further equivalent to* $D^b(\mathrm{qmod}(\mathcal{A}))$. *Therefore, in this case we also have an equivalence*

$$D^b(\mathrm{qmod}(\mathcal{A})) \cong D^b(mod_{finite}\text{-}B).$$

The coherence of geometric $\mathbb{Z}$ *-algebra* $\mathcal{A}$ *of period* n *is equivalent to some statement about t-structures. Namely, let* $(\tau_{\leq 0}, \tau_{\geq 1})$ *be a t-structure on* $D^b(Mod\text{-}\mathcal{A}_{[1,n]})$ *induced by the equivalence of Theorem 18.14. It can be shown that* $\mathcal{A}$ *is coherent iff the t-structure* $(\tau_{\leq 0}, \tau_{\geq 1})$ *induces a t-structure on* $D^b(mod\text{-}_{finite}(\mathcal{A}_{[1,n]}))$.

19. The k -points of noncommutative Grassmanians

To discuss the k -points of noncommutative Grassmanians defined above we should first relate the following two approaches to noncommutative geometry.

The first one is to think of noncommutative stacks as of $\mathrm{Proj}(\mathcal{A})$, where $\mathcal{A}$ is a $\mathbb{Z}$ -algebra. The special case of graded algebras, i.e. 1 -periodic $\mathbb{Z}$ -algebras is studied in [M], [V1], [V2], [AZ] and other papers. However, it seems to be more reasonable to consider $\mathbb{Z}$ -algebras. Note that our noncommutative Grassmanians are naturally defined as Proj of a $\mathbb{Z}$ -algebra but not a graded algebra.

The other approach is to think of a noncommutative stacks as of (equivalence classes of) presheaves of (small) groupoids X on the category Alg_k^{op} opposite to the category of unital associative algebras. Morally the groupoid $X(A)$ should be thought of as the groupoid of maps from the affine noncommutative scheme $Sp(A)$ to X . This approach is studied in [Or] in the case of sets (trivial groupoids). In this case we have the category of quasi-coherent sheaves which is not always abelian (it always has cokernels but may not admit kernels), and the structure sheaf.

In the second approach we obviously have the groupoid of k -points $X(k)$.

From this moment we assume that the $\mathbb{Z}$ -algebra $\mathcal{A}$ is positively oriented but not necessarily connected. We will make an attempt to define the presheaf of groupoids of morphisms $Sp(A) \to \mathrm{Proj}(\mathcal{A})$, $A \in \mathrm{Alg}_k$. First note that a morphism $f : Sp(A) \to \mathrm{Proj}(\mathcal{A})$ must give a k -linear additive functor $f^* : \mathrm{QMod}(\mathcal{A}) \to \mathrm{Mod}\text{-}A$ together with an isomorphism $f^*(\pi(P_0)) \cong A$. Moreover, f^* must commute with colimits.

Notice that if $\mathcal{C}$ is a k -linear abelian category with infinite direct sums and $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ is a k -linear functor then we have the tensor product functor

$$- \otimes_{\mathcal{A}} \mathcal{F} : \mathrm{Mod}\text{-}\mathcal{A} \to \mathcal{C}$$

given by the formula

$$M \otimes_{\mathcal{A}} \mathcal{F} = \mathrm{Coker}(b; M \otimes \mathcal{A} \otimes \mathcal{F} \to M \otimes \mathcal{F}),$$

where $b = \mu_M \otimes \mathbf{1}_{\mathcal{A}} - \mathbf{1}_M \otimes \mu_{\mathcal{F}}$ (we identify $\mathcal{F}$ with $\bigoplus_{i\in\mathbb{Z}} \mathcal{F}(i) \in \mathcal{C}$). Clearly, the functor $- \otimes_{\mathcal{A}} \mathcal{F}$ commutes with colimits. We denote by $\mathrm{Tor}_i^{\mathcal{A}}(-, \mathcal{F})$ its left derived functors.

Definition 19.1. *Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums together with a distinguished object $Y \in \mathcal{C}$.*

We denote by $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ the groupoid of pairs (f^*, θ), where $f^* : \mathrm{QMod}(\mathcal{A}) \to \mathcal{C}$ is a k-linear functor commuting with colimits and $\theta : f^*(\pi(P_0)) \to Y$ is an isomorphism.

We denote by $\mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ the groupoid of pairs $(\mathcal{F}, \sigma)$, where $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ is a k-linear functor such that $\mathrm{Tor}_0^{\mathcal{A}}(T, \mathcal{F}) = \mathrm{Tor}_1^{\mathcal{A}}(T, \mathcal{F}) = 0$ for each torsion $\mathcal{A}^{op}$-module T, and $\sigma : \mathcal{F}(0) \to Y$ is an isomorphism.

Theorem 19.2. *Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums together with a distinguished object $Y \in \mathcal{C}$. Then the groupoids $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ and $\mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ are equivalent.*

Proof. We define the functor $\Phi : \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y) \to \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ as follows. Let $(f^*, \theta) \in \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$. The functor $\Phi(f^*) : \mathcal{A} \to \mathcal{C}$ is defined by the formulas

$$\Phi(f^*)(i) = f^*(\pi(P_i)),$$

and for $x \in \mathcal{A}_{ij}$

$$\Phi(f^*)(x) = f^*(\pi(x)).$$

We claim that the pair $(\Phi(f^*), \theta)$ is an object of $\mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$. Indeed, let $T \in \mathrm{Tors}(\mathcal{A})$. Since the sequence

$$\pi(T \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(T \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(T \otimes \mathcal{A}) \to 0$$

is exact in $\mathrm{QMod}(\mathcal{A})$ it follows that the sequence

$$T \otimes \mathcal{A} \otimes \mathcal{A} \otimes \Phi(f^*)(\mathcal{A}) \to T \otimes \mathcal{A} \otimes \Phi(f^*)(\mathcal{A}) \to T \otimes \Phi(f^*)(\mathcal{A}) \to 0$$

is exact in $\mathcal{C}$, i.e.

$$\mathrm{Tor}_0^{\mathcal{A}}(S_i, \Phi(f^*)) = \mathrm{Tor}_0^{\mathcal{A}}(S_i, \Phi(f^*)) = 0.$$

Thus, the functor Φ is defined on objects. It obviously extends to morphisms.

Now we define the functor $\Psi : \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y) \to \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ as follows. Let $(\mathcal{F}, \sigma) \in \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$. We claim that the formula

$$\Psi(\mathcal{F})(\pi(M)) = M \otimes_{\mathcal{A}} \mathcal{F}$$

well defines a functor $\Psi(\mathcal{F}) : \mathrm{QMod}(\mathcal{A}) \to \mathcal{C}$ which is right exact and commutes with infinite direct sums. Indeed it follows from the condition

$$\mathrm{Tor}_0^{\mathcal{A}}(T, \mathcal{F}) = \mathrm{Tor}_1^{\mathcal{A}}(T, \mathcal{F}) = 0$$

for torsion $\mathcal{A}^{op}$-modules T.

Hence, the pair $(\Psi(\mathcal{F}), \sigma)$ is an object of $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$. This defines the functor Ψ on objects and it obviously extends to morphisms.

It is clear that the composition $\Phi\circ\Psi$ is isomorphic to the identity functor. To see this for the composition $\Psi\circ\Phi$, it remains to note that each functor f^* from the pair in $\mathcal{G}_1(\mathcal{A},\mathcal{C},Y)$ can be reconstructed from the functor $\Phi(f^*)$ using exact sequences

$$\pi(M\otimes\mathcal{A}\otimes\mathcal{A})\to\pi(M\otimes\mathcal{A})\to\pi(M)\to 0.$$

□

Notice that it follows from the above theorem that each functor $f^*:\mathrm{QMod}(\mathcal{A})\to\mathcal{C}$ commuting with direct sums and right exact has the right adjoint $f_*:\mathcal{C}\to\mathrm{QMod}(\mathcal{A})$ given by the formula $f_*(X)=\pi(\widetilde{f_*(X)})$

$$\widetilde{f_*(X)}(i)=\mathrm{Hom}_{\mathcal{C}}(f^*(\pi(P_i)),X),$$

and for $\phi\in\mathrm{Hom}_{\mathcal{C}}(f^*(\pi(P_j)),X)$, $x\in\mathcal{A}_{ij}$

$$\phi\cdot x=\phi\cdot f^*(\pi(x)).$$

Indeed, this follows from the formula

$$f^*(\pi(M))=M\otimes_{\mathcal{A}}\mathcal{F}.$$

It is clear that $X_i(\mathcal{A})=(A\mapsto\mathcal{G}_i(\mathcal{A},\text{Mod-}A,A))$, $i=1,2$, are presheaves of groupoids on the category Alg_k^{op}, and the equivalence from the above theorem extends to the equivalence of these presheaves.

However, not all functors f^* commuting with colimits should come from true morphisms $f:Sp(A)\to\mathrm{Proj}(\mathcal{A})$. Although a true presheaf of groupoids should be defined as a full (small) subpresheaf of $X_2(\mathcal{A})$. We are going to make an attempt in this direction. Our motivation is the following Proposition.

Proposition 19.3. *Let $\mathcal{A}$ be a $\mathbb{Z}$-algebra. Further, let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums and with the distinguished object Y. Let $(f^*,\theta)\in\mathcal{G}_1(\mathcal{A},\mathcal{C},Y)$ and $(\mathcal{F},\sigma)\in\mathcal{G}_2(\mathcal{A},\mathcal{C},Y)$ be objects which correspond to each other under the equivalence of Theorem 19.2. The following conditions are equivalent:*

(i) there exists a left derived functor $\mathbf{L}f^*:D^-(\mathrm{QMod}(\mathcal{A}))\to D^-(\mathcal{C})$, and $\mathbf{L}^if^*(\pi(P_j))=0$ for $i\neq 0$ and all j;

(ii) we have $\mathrm{Tor}_i^{\mathcal{A}}(T,\mathcal{F})=0$ for all $i\geq 0$, $T\in\mathrm{Tors}(\mathcal{A})$.

Proof. Prove that (i) implies (ii). We have that the functor $f^*\cong-\otimes\mathcal{F}$ maps acyclic right bounded complexes of direct sums of $\pi(P_i)$ to acyclic complexes. Applying this to the projection of the free resolution of a torsion module T, we obtain that $\mathrm{Tor}_i^{\mathcal{A}}(T,\mathcal{F})=0$ for $i\geq 0$.

Prove that (ii) implies (i). Since each object in $\mathrm{QMod}(\mathcal{A})$ can be covered by a direct sum of $\pi(P_i)$, it suffices to prove that f^* maps right bounded acyclic complexes of direct sums of $\pi(P_i)$ to acyclic complexes.

Since the kernel and the cokernel of the morphism $P_j\to\omega\pi(P_j)$ are torsion, we have $\mathrm{Tor}_i^{\mathcal{A}}(\omega\pi(P_j),\mathcal{F})=0$ for $i>0$. Further, if $K^{\cdot}$ is a right bounded acyclic complex of direct sums of $\pi(P_i)$ then $\omega(K^{\cdot})$ has torsion cohomology. Therefore,

$$f^*(K^{\cdot})=\omega(K^{\cdot})\otimes_{\mathcal{A}}\mathcal{F}=\omega(K^{\cdot})\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}\mathcal{F},$$

and the last complex is acyclic since $\mathrm{Tor}_i^{\mathcal{A}}(T,\mathcal{F})=0$ for $i\geq 0$, $T\in\mathrm{Tors}(\mathcal{A})$. □

For each $\mathbb{Z}$-algebra $\mathcal{A}$ we define the presheaf $X_{\mathcal{A}}$ of groupoids on the category Alg_k^{op} as follows. It is a full subpresheaf of $X_2(\mathcal{A})$ and the groupoid $X_{\mathcal{A}}(A)\subset X_2(\mathcal{A})(A)$ consists of pairs $(\mathcal{F},\sigma)\in X_2(\mathcal{A})(A)$ such that:

1) we have $\mathrm{Tor}_i^{\mathcal{A}}(T,\mathcal{F})=0$ for all i and $T\in\mathrm{Tors}(\mathcal{A})$;

2) the A^{op}-modules $\mathcal{F}(i)$ are flat.

It is clear that $X_{\mathcal{A}}$ is indeed a subpresheaf of $X_2(\mathcal{A})$. For $f = (\mathcal{F}, \sigma) \in X_{\mathcal{A}}(A)$ we denote by $f^* : \mathrm{QMod}(\mathcal{A}) \to \text{Mod-}\mathcal{A}$ the corresponding functor $- \otimes_{\mathcal{A}} \mathcal{F}$. We also regard the objects $f \in X_{\mathcal{A}}(A)$ as maps from $Sp(A)$ to $\mathrm{Proj}(\mathcal{A})$, where $Sp(A)$ is a noncommutative affine scheme corresponding to A.

The following Lemma simplifies the complicated condition on Tor_i. Recall the torsion $\mathcal{A}^{op}$-modules $T_{p,q}$ from Section 16.

Lemma 19.4. *Let $\mathcal{A}$ be a positively oriented connected $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums. Let $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ be a k-linear functor.*

Suppose that $\mathrm{Tor}_i^{\mathcal{A}}(T_{j,j}, \mathcal{F}) = 0$ for all i and j. Then $\mathrm{Tor}_i^{\mathcal{A}}(T, \mathcal{F}) = 0$ for all i and all torsion $\mathcal{A}^{op}$-modules T.

Proof. First note that if $T = T\mathbf{1}_j$, then T has a left resolution by direct sums of $T_{j,j}$. Hence, Lemma holds for such T.

Further, the torsion modules $T_{p,q}$, $p \le q$, have finite filtrations with subquotients T_m such that $T_m = T_m \mathbf{1}_m$, $p \le m \le q$. Hence $\mathrm{Tor}_i^{\mathcal{A}}(T_{p,q}, \mathcal{F}) = 0$ for all i, and $p \le q$. Now Lemma follows from the observation that each torsion module has a left resolution by direct sums of modules T_{pq}. □

Definition 19.5. *We say that a positively oriented $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition (**) if the following holds:*

(i) the algebra $\mathcal{A}$ is generated by its subspaces $\mathcal{A}_0$ and $\mathcal{A}_1$;

(ii) for each $i \in \mathbb{Z}$, the object $\pi(P_i)$ has a finite right resolution by direct sums of $\pi(P_j)$ with $j > i$.

The next Proposition motivates the condition (**).

Proposition 19.6. *Let $\mathcal{A}$ be a $\mathbb{Z}$-algebra satisfying (**), and $A \in \mathrm{Alg}_k$. Then for each $f = (\mathcal{F}, \sigma) \in X_{\mathcal{A}}(A)$ we have $\mathrm{Aut}(f) = \{1\}$.*

Proof. Let $g \in \mathrm{Aut}(f)$. Clearly, $g(0) : \mathcal{F}(0) \to \mathcal{F}(0)$ is the identity morphism. Further, for each $i \in \mathbb{Z}$ the surjection $\mathcal{A}_{i,i+1} \otimes \pi(P_i) \to \pi(P_{i+1})$ is mapped by f^* to the surjection $\mathcal{A}_{i,i+1} \otimes \mathcal{F}(i) \to \mathcal{F}(i+1)$. Hence, we obtain by increasing induction over i that $g(i) : \mathcal{F}(i) \to \mathcal{F}(i)$ is the identity for $i \ge 0$.

Finally, since $\mathcal{A}$ satisfies (**), it follows from Proposition 19.3 that there exists an injection of the form $\pi(P_i) \to \bigoplus_\alpha \pi(P_{j_\alpha})$ with $j_\alpha > i$ which is mapped by f^* to the injection $\mathcal{F}(i) \to \bigoplus_\alpha \mathcal{F}(j_\alpha)$. Hence, we obtain by decreasing induction on i that $g(i) : \mathcal{F}(i) \to \mathcal{F}(i)$ is the identity for all $i \in \mathbb{Z}$. □

Therefore, if $\mathcal{A}$ satisfies (**) we may and will replace $X_{\mathcal{A}}$ by the equivalent presheaf of trivial groupoids $\pi_0(X_{\mathcal{A}})$. It is easily seen from the proof of the above Proposition that $\pi_0(X_{\mathcal{A}}(A))$ is a set. Thus, $X_{\mathcal{A}}$ is a presheaf of sets.

Now we would like to compare our definition of morphisms from $Sp(A)$ to $\mathrm{Proj}(\mathcal{A})$ with the morphisms from commutative Noetherian k-schemes to commutative projective k-schemes.

Note that we can restrict the presheaf $X_{\mathcal{A}}$ to the full subcategory of Alg_k which consists of commutative Noetherian k-algebras. Further, we can extend this restricted presheaf onto the category $Noeth_k$ of all commutative Noetherian k-schemes.

Definition 19.7. *Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. We define the presheaf $X_{true(\mathcal{A})} : Noeth_k^{op} \to \mathbf{Gpd}$ as follows. The groupoid $X_{\mathcal{A}}(Y)$ is a full sub-groupoid of $\mathcal{G}_2(\mathcal{A}, \mathrm{QCoh}(Y), \mathcal{O}_Y)$ which consists of objects $(\mathcal{F}, \sigma)$ such that the following conditions hold:*

1) we have $\mathrm{Tor}_i^{\mathcal{A}}(T,\mathcal{F})=0$ *for all* i *and* $T\in\mathrm{Tors}(\mathcal{A})$ *;*
2) the sheaves $\mathcal{F}(i)$ *are locally flat.*

We also regard the objects of the groupoid $X_{\mathcal{A}}(Y)$ as maps from Y to $\mathrm{Proj}(\mathcal{A})$. The analogue of Proposition 19.6 obviously holds for Noetherian k-schemes instead of associative algebras. For each commutative Noetherian k-scheme Y we denote by $Y^\vee : Noeth_k^{op}\to\mathrm{Set}$ the presheaf of sets represented by Y.

Now let $Z\in\mathbb{P}(V)$ be a closed subscheme and let $\mathcal{A}$ be a $\mathbb{Z}$-algebra associated to its homogeneous coordinate ring $\bigoplus\limits_{d\geq 0} S^dV^*/I$.

Proposition 19.8. *The* $\mathbb{Z}$*-algebra* $\mathcal{A}$ *satisfies the condition (**). The presheafs of sets* $Z^\vee$ *and* $X_{\mathcal{A}}$ *:* $Noeth_k^{op}\to\mathrm{Set}$ *on the category* $Noeth_k^{op}$ *are isomorphic.*

Proof. Recall that the category $\mathrm{QCoh}(Z)$ is equivalent to $\mathrm{Proj}(\mathcal{A})$ by Serre Theorem. The sheaves $\mathcal{O}_Z(i)$ correspond under this equivalence to $\pi(P_i)$.

Let $f:Y\to Z$ be a morphism. Then the sheaves $f^*(\mathcal{O}_Z(i))$ are invertible and hence are locally flat. Further, f^* maps acyclic right bounded complexes of direct sums of $\mathcal{O}_Z(i)$ to acyclic complexes. Finally, we have an isomorphism $f^*(\mathcal{O}_Z)\cong\mathcal{O}_Y$. Thus, we have a morphism of presheaves $Z^\vee\to X_{\mathcal{A}}$.

Conversely, let $Y\in Noeth_k$ and $g\in X_{\mathcal{A}}(Y)$. Notice that for each $i\in\mathbb{Z}$ we have an acyclic Koszul complex on $\mathbb{P}(V)$ twisted by $\mathcal{O}_{\mathbb{P}(V)}(i)$, and we can restrict it to Z:

$$0\to\mathcal{O}_Z(i)=\Lambda^nV^*\otimes\mathcal{O}_Z(i)\to\Lambda^{n-1}V^*\otimes\mathcal{O}_Z(i+1)\to\cdots\to\mathcal{O}_Z(i+n)\to 0.$$

In particular, the $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition (**).

Hence, we have acyclic complexes

$$0\to g^*(\pi(P_i))=\Lambda^nV^*\otimes g^*(\pi(P_i))\to\Lambda^{n-1}V^*\otimes g^*(\pi(P_{i+1}))\to\cdots\to g^*(\pi(P_{i+n}))\to 0. \tag{19.1}$$

In particular, we have surjections $V^*\otimes g^*(\pi(P_i))\to g^*(\pi(P_{i+1}))$ and injections $g^*(\pi(P_i))\to V\otimes g^*(\pi(P_{i+1}))$. Since $g^*(\pi(P_0))\cong\mathcal{O}_Y$, we obtain by increasing and decreasing inductions on i that all the sheaves $g^*(\pi(P_i))$ are coherent and are non-zero on each connected component of Y. Since they are locally flat, they are locally free.

Further, put $\mathcal{L}=g^*(\pi(P_1))$. Clearly, g can be reconstructed from the surjective morphism $\phi:V^*\otimes A\to\mathcal{L}$ using the exact sequences

$$\Lambda^2V^*\otimes g^*(\pi(P_{i-1}))\to V^*\otimes g^*(\pi(P_i))\to g^*(\pi(P_{i+1}))\to 0$$

and

$$0\to g^*(\pi(P_{i-1}))\to V\otimes g^*(\pi(P_i))\to\Lambda^2V\otimes g^*(\pi(P_{i+1}))$$

from commlexes (19.1). Suppose that $\mathrm{rank}(\mathcal{L}_{|Y_0})\geq 2$ on some connected component $Y_0\subset Y$. Then it is easy to see that the morphism $\Lambda^{n-1}V^*\otimes\mathcal{O}_{Y_0}\to\Lambda^{n-2}V^*\otimes\mathcal{L}_{|Y_0}$ is injective. Hence $g^*(\pi(P_{-1}))_{|Y_0}=0$, a contradiction. Thus, $\mathcal{L}$ is an invertible sheaf.

The surjective morphism ϕ above defines a morphism $\widetilde{g}:Y\to\mathbb{P}(V)$. It follows that $\widetilde{g}^*\cong g^*\iota^*$, where $\iota:Z\to\mathbb{P}(V)$ is the embedding. Hence, $g^*(\pi(P_i))\cong\mathcal{L}^{\otimes i}$. Further, the induced morphism of graded algebras

$$\bigoplus_{d\geq 0}S^dV^*\to\bigoplus_{i\geq 0}H^0(\mathcal{L}^{\otimes i})$$

passes through $\bigoplus\limits_{d\geq 0}S^dV^*/I$, thus $\widetilde{g}$ passes through Z, and we obtain a morphism $Y\to Z$. Hence we have a morphism of presheaves $X_{\mathcal{A}}\to Z^\vee$.

The constructed morphisms of presheaves are inverse to each other. Proposition is proved. □

Now we want to describe the k-points of noncommutative Grassmanians.

Lemma 19.9. *Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of period n. Then it satisfies the condition (**).*

Proof. By definition, $\mathrm{Alg}_{\mathcal{A}}$ is generated by $\mathcal{A}_0$ and $\mathcal{A}_1$. Further, the projections $\pi(K_i)$ of Koszul complexes are acyclic. The first non-zero term of $\pi(K_{i+n})$ equals to $\pi(P_i)$ (since $\mathcal{A}^!$ is Frobenious). Therefore, each $\pi(P_i)$ has the required right resolution. □

Denote by $\mathrm{pr}_r^{m,V} : D^b_{coh}(\mathbb{P}(V)) \to \mathcal{T}^{m,V}$ the functor which is right adjoint to the inclusion $\iota : \mathcal{T}^{m,V} \to D^b_{coh}(\mathbb{P}(V))$. The next theorem describes the k-points of noncommutative Grassmanians and the objects in $\mathcal{T}^{m,V}$ corresponding to their structure sheaves under the equivalence of Corollary 18.22.

Theorem 19.10. *The k-points of the noncommutative Grassmanian $\mathrm{NGr}(m,V)$, i.e. the elements $f \in X_{\mathcal{A}^{m,V}}(k)$, naturally correspond to vector subspaces $W \subset V$ of dimension $1 \le \dim W \le m$. Further, if f corresponds to W then $f_*(k)$ is a perfect complex, and it corresponds to the object $\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}) \in \mathcal{T}^{m,V}$ under the equivalence $D_{perf}(\mathrm{QMod}(\mathcal{A})) \cong \mathcal{T}^{m,V}$.*

Proof. Let $f \in X_{\mathcal{A}}(k)$. For each $i \in \mathbb{Z}$ we have the natural acyclic complex

$$0 \to f^*(\pi(P_i)) \cong \mathcal{A}^{m,V!*}_{i+n,i} \otimes f^*(\pi(P_i)) \to \mathcal{A}^{m,V!*}_{i+n,i+1} \otimes f^*(\pi(P_{i+1})) \to \cdots \to f^*(\pi(P_{i+n})) \to 0.$$

In particular, we have surjections $V^* \otimes f^*(\pi(P_i)) \to f^*(\pi(P_{i+1}))$ and injections $f^*(\pi(P_i)) \to V \otimes f^*(\pi(P_{i+1}))$. Since $f^*(\pi(P_0)) \cong k$, we obtain by increasing and decreasing inductions over i that all the spaces $f^*(\pi(P_i))$ are non-zero and finite-dimensional. Further, using the exact sequences

$$\Lambda^2 V^* \otimes f^*(\pi(P_{i-1})) \to V^* \otimes f^*(\pi(P_i)) \to f^*(\pi(P_{i+1})) \to 0$$

and

$$0 \to f^*(\pi(P_{i-1})) \to V \otimes f^*(\pi(P_i)) \to \Lambda^2 V \otimes f^*(\pi(P_{i+1})),$$

one can reconstruct f from the injection $f^*(\pi(P_{-1})) \hookrightarrow V \otimes f^*(\pi(P_0)) \cong V$. Thus, we can associate a non-zero vector subspace $W \subset V$ to each $f \in X_{\mathcal{A}}(k)$ and f can be reconstructed from the subspace W. We will show that $W \subset V$ gives a k-point iff $1 \le \dim W \le m$.

First suppose that $\dim W > m$ and W gives a k-point f. Then

$$f^*(\pi(P_1)) = \mathrm{Coker}(\Lambda^{n-m-1} V \otimes W \to \Lambda^{n-m} V),$$

and the last space is zero since $\dim W > m$. But $f^*(\pi(P_1)) \ne 0$, a contradiction.

Now let $1 \le \dim W = d \le m$. Let $S = \{E_i\}$ be a geometric helix in $\mathcal{T}^{m,V}$ of period $n-m+1$ such that $E_j = \mathcal{O}_{\mathbb{P}(V)}(j)$ for $m-n \le j \le 0$. Then the endomorphism $\mathbb{Z}$-algebra of S is equivalent to $\mathcal{A}^{m,V}$. We define the functor $\mathcal{F} : \mathcal{A} \to k\text{-Vect}$ by the formula

$$\mathcal{F}(i) = \mathrm{Hom}(E_i, \mathcal{O}_{\mathbb{P}(W)})^\vee = \mathrm{Hom}(E_i, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))^\vee.$$

We put $f = (\mathcal{F}, id)$.

Now we prove that $f \in X_{\mathcal{A}^{m,V}}(k)$. By Lemma 19.4, it suffices to show that $\mathrm{Tor}_i(S_j, \mathcal{F}) = 0$ for $i > 0$, $j \in \mathbb{Z}$. Since the complexes

$$0 \to E_i \cong \mathcal{A}^{m,V!*}_{i+n,i} \otimes E_i \to \mathcal{A}^{m,V!*}_{i+n,i+1} \otimes E_{i+1} \to \cdots \to E_{i+n} \to 0$$

of objects in $D^b_{coh}(\mathbb{P}(V))$ have zero convolutions, it suffices to prove that

$$\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = \mathrm{Hom}^i(E_i, \mathcal{O}_{\mathbb{P}(W)}) = 0$$

for $i \neq 0$, $j \in \mathbb{Z}$. It is clear that this holds for $m - n \leq j \leq 0$. Further, we have a Koszul resolution of the sheaf $\mathcal{O}_{\mathbb{P}(W)}$ on $\mathbb{P}(V)$:

$$0 \to \Lambda^{n-d}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(d-n) \to \dots \to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)}.$$

Thus, $\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$ is isomorphic to the complex

$$\dots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n) \to \dots \to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \dots.$$

Since the helix $\{E_i\}$ is geometric, we have $\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = 0$ for $i > 0$, $j \leq 0$.

Recall that $E_{j+n-m+1} = \Phi^{-1}(E_j)$, where $\Phi = F[m-n]$, and F is a Serre functor on $\mathcal{T}^m, V$. We have that

$$F^{-1}(K^\cdot) \cong \mathrm{pr}_l^{m,V}(K^\cdot \otimes \mathcal{O}_{\mathbb{P}(V)}(n)[1-n]), \tag{19.2}$$

where $\mathrm{pr}_l^{m,V} : D^b_{coh}(\mathbb{P}(V)) \to \mathcal{T}^{m,V}$ is the functor which is left adjoint to the inclusion $\iota : \mathcal{T}^{m,V} \to D^b_{coh}(\mathbb{P}(V))$.

Lemma 19.11. *The functor* $\mathrm{pr}_l^{m,V} : D^b_{coh}(\mathbb{P}(V)) \to \mathcal{T}^{m,V}$ *maps* $Ob(D^{\geq i}_{coh}(\mathbb{P}(V)))$ *to* $Ob(D^{\geq i-m+1}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$. *The functor* Φ^{-1} *preserves* $Ob(D^{\geq i}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$.

Proof. The second statement follows from the first one by the isomorphism (19.2). To prove the first statement, it suffices to note that

$$\mathrm{pr}_l^{m,V}(X) \cong L_{\mathcal{O}_{\mathbb{P}(V)}(1)} \cdot \dots \cdot L_{\mathcal{O}_{\mathbb{P}(V)}(m-1)}(X)[m-1].$$

□

Since $\mathrm{pr}_l^{m,V}(\mathcal{O}(i)) = 0$ for $i = 1, \dots, m-1$, we have

$$\Phi^{-1}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = \mathrm{pr}_l^{m,V}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}) \otimes \mathcal{O}_{\mathbb{P}(V)}(n))[1-m] = \mathrm{pr}_l^{m,V}(\mathcal{O}_{\mathbb{P}(W)}(n))[1-m],$$

and the last object belongs to $Ob(D^{\geq 0}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$ by Lemma 19.11. Again by Lemma 19.11 we have that $\Phi^{-l}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))$ lies in $Ob(D^{\geq 0}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$ for $l > 0$. Thus, we have

$$\mathrm{Hom}^i(E_{j-(n-m+1)k}, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = \mathrm{Hom}^i(E_j, \Phi^{-k}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))) = 0$$

for $i < 0$, $n - m \leq j \leq 0$, and $k > 0$. Therefore,

$$\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = 0$$

for $i \neq 0$, $j \leq 0$.

To prove the same for $j > 0$, note that $\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))$ is the i-th cohomology of the complex

$$\dots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \mathrm{Hom}^{n-m}(E_j, \mathcal{O}_{\mathbb{P}(V)}(m-n)) \to \dots$$
$$\to (V/W)^* \otimes \mathrm{Hom}^{n-m}(E_j, \mathcal{O}_{\mathbb{P}(V)}(-1)) \to \mathrm{Hom}^{n-m}(E_j, \mathcal{O}_{\mathbb{P}(V)}) \to 0 \to \dots,$$

where the left non-zero term is in degree zero. This complex is dual to the complex

$$\dots \to 0 \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(V)}, E_{j-n+m-1}) \to (V/W) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(V)}(-1), E_{j-n+m-1}) \to \dots$$
$$\to \Lambda^{n-m}(V/W) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(V)}(m-n), E_{j-n+m-1}) \to 0 \to \dots,$$

and the last one is isomorphic to the complex

$$\cdots \to 0 \to \mathrm{Hom}(E_{1-j}, \mathcal{O}_{\mathbb{P}(V)}(m-n)) \to (V/W) \otimes \mathrm{Hom}(E_{1-j}, \mathcal{O}_{\mathbb{P}(V)}(m-n+1)) \to \dots \\ \to \Lambda^{n-m}(V/W) \otimes \mathrm{Hom}(E_{1-j}, \mathcal{O}_{\mathbb{P}(V)}) \to 0 \to \dots . \tag{19.3}$$

The complex (19.3) calculates $\mathrm{Hom}^i(E_{1-j}, X_0)$, where X_0 is the complex

$$\dots 0 \to \mathcal{O}_{\mathbb{P}(V)}(m-n) \to (V/W) \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n+1) \to \cdots \to \Lambda^{n-m}(V/W) \otimes \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \dots .$$

Thus, it remains to show that $\mathrm{Hom}^i(E_j, X_0) = 0$ for $i < 0$, $j \le 0$.

If we prove this for $n - m \le j \le 0$, then the rest of the proof will be analogous to the proof of the same vanishing for $\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$ instead of X_0. So let $i < 0$, $n - m \le j \le 0$. We have the chain of isomorphisms

$$\mathrm{Hom}^i(\mathcal{O}_{\mathbb{P}(V)}(j), X_0) \cong \mathrm{Hom}^{n-m+i}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}), \mathcal{O}_{\mathbb{P}(V)}(m-n-j)) \cong \\ \mathrm{Ext}^{n-m+i}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(V)}(m-n-j)) \cong \\ \mathrm{Ext}^{m-i-1}(\mathcal{O}_{\mathbb{P}(V)}(m-j), \mathcal{O}_{\mathbb{P}(W)})^\vee = H^{m-i-1}(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(j-m))^\vee,$$

and the last space is zero since $m - i - 1 > d - 1 = \dim \mathbb{P}(W)$.

Thus, f is indeed a k-point. Furthermore, we have that the complex

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \pi(P_{m-n}) \to \cdots \to (V/W)^* \otimes \pi(P_{-1}) \to \pi(P_0) \to 0 \to \dots$$

is a resolution of $f_*(k)$. Thus, $f_*(k)$ is a perfect complex and it corresponds to $\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$ under the equivalence $D_{perf}(\mathrm{QMod}(\mathcal{A}^{m,V})) \cong \mathcal{T}^{m,V}$. Theorem is proved. □

It turns out that the embedding of k-points $\mathrm{Gr}(d,V)(k) \hookrightarrow \mathrm{NGr}(m,V)(k)$ for $1 \le d \le m$ can be extended to a morphism $\mathrm{Gr}(d,V) \to \mathrm{NGr}(m,V)$.

Proposition 19.12. *Let V be a finite-dimensional vector space and let $1 \le d \le m \le \dim V = n$. Then there exists a natural morphism $f_{d,m,V} : \mathrm{Gr}(d,V) \to \mathrm{NGr}(m,V)$ such that the derived inverse image functor $\mathbf{L}f^*_{d,m,V}$ induces a full embedding*

$$\mathrm{Perf}(\mathrm{NGr}(m,V)) \to D^b_{coh}(\mathrm{Gr}(d,V)).$$

Proof. For each $W \in \mathrm{Gr}(d,V)$ denote by f_W the corresponding k-point of $\mathrm{NGr}(m,V)$. It is clear that there exist vector bundles $\mathcal{F}(i)$ on $\mathrm{Gr}(d,V)$ such that the fiber of $\mathcal{F}(i)$ over the point corresponding to W is naturally identified with $f_W^*(\pi(P_i))$ (in particular, $\mathcal{F}(-1)$ is a tautological bundle). So we have a natural functor $\mathcal{F} : \mathcal{A} \to \mathrm{QCoh}(\mathrm{Gr}(m,V))$. Also by Theorem 19.10 we have that the complexes of vector bundles

$$0 \to \mathcal{F}(i) \cong \mathcal{A}^{m,V!*}_{i+n,i} \otimes \mathcal{F}(i) \to \mathcal{A}^{m,V!*}_{i+n,i+1} \otimes \mathcal{F}(i+1) \to \cdots \to \mathcal{F}(i+n) \to 0$$

are acyclic in the fibers over closed points (if the residue field of a point is greater than k we can make an extension of scalars). Hence, these complexes are acyclic themselves. It follows from Lemma 19.4 that the pair $(\mathcal{F}, id)$ defines a map $f_{d,m,V} : \mathrm{Gr}(d,V) \to \mathrm{NGr}(m,V)$.

Further, for $m-n \le j \le 0$ we have that $\mathbf{L}f^*_{d,m,V}(\pi(P_j)) = f^*_{d,m,V}(\pi(P_j)) = S^{-j}E$, where E is a tautological bundle. The collection $(S^{n-m}E, \dots, E, \mathcal{O}_{\mathrm{Gr}(d,V)})$ is a sub-collection of the full strong exceptional collection on $\mathrm{Gr}(d,V)$ constructed by Kapranov [Ka]. Moreover, the functor $\mathbf{L}f^*_{d,m,V}$ induces isomorphisms

$$\mathrm{Hom}(\pi(P_i), \pi(P_j)) \to \mathrm{Hom}(S^{-i}E, S^{-j}E)$$

for $m-n \le i \le j \le 0$. Thus, the induced functor

$$\mathbf{L}f^*_{d,m,V} : \mathrm{Perf}(\mathrm{NGr}(m,V)) \to D^b_{coh}(\mathrm{Gr}(d,V))$$

is a full embedding. □

Notice that the full embedding $\mathbf{L}f^*_{1,m,V} : \mathrm{Perf}(\mathrm{NGr}(m,V)) \to D^b_{coh}(\mathbb{P}(V))$ coincides with the composition of the equivalence of Corollary 18.22 with the tautological embedding $\mathcal{T}^{m,V} \hookrightarrow D^b_{coh}(\mathbb{P}(V))$.

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