CLASSIFICATION OF LINKS AND KNOTTED TORI IN THE 2-METASTABLE DIMENSION

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Abstract. This paper is devoted to the classification of embeddings of higher dimensional manifolds. We present a short proof of an explicit formula for the group of links in the 2-*metastable* dimension. This improves a result of Haefliger from 1966. Denote by $L^m_{p,q}$ (resp. K^m_p) the group of smooth embeddings $S^p \sqcup S^q \to S^m$ (resp. $S^p \to S^m$) up to smooth isotopy.

Theorem 1. Suppose that $1 \le p \le q \le m-3$ and $2p+2q \le 3m-6$; then

$$L^m_{p,q} \cong \pi_p(S^{m-q-1}) \oplus \pi_{p+q+2-m}(SO/SO_{m-p-1}) \oplus K^m_p \oplus K^m_q.$$

We study classification of embeddings $S^p \times S^q \to S^m$ (*knotted tori*). This is a natural generalization of link theory. In *metastable* dimension knotted tori up to isotopy were classified by Haefliger, Zeeman and A. Skopenkov. We give an explicit criterion for the finiteness of this isotopy classes set in the *2-metastable* dimension:

Theorem 2. Assume that $p + \frac{4}{3}q + 2 \le m < p + \frac{3}{2}q + 2$ and $m \ge 2p+q+3$. Then the set of smooth embeddings $S^p \times S^q \to S^m$ up to isotopy is infinite if and only if either $q+1$ or $p+q+1$ is divisible by 4.

Our approach is based on a reduction of the classification of links and knotted tori to the classification of link maps, and on a suspension theorem for link maps. We obtain a new short proof of the classification of link maps due to Habegger and Kaiser. Denote by $LM^m_{p,q}$ the group of link maps $S^p \sqcup S^q \to S^m$ up to link homotopy.

Theorem 3. Suppose that $1 \le p, q \le m-3$ and $2p+2q \le 3m-5$; then $LM^m_{p,q} \cong \pi^S_{p+q+1-m}$.

0. Introduction

This paper is devoted to the classification of embeddings of higher dimensional manifolds. This subject was actively studied in the sixties [Zee62, Hud63, Hae66C] and there has been a renewed interest for it in the last years [CRS04, CeRe05, Sko06"].

This problem generalizes the subject of classical knot theory. In contrast to the classical situation of simple closed curves in $\mathbb{R}^3$, in higher dimensions a complete answer can sometimes be obtained. E. g., for *knots* $S^q \to S^m$ there is known an explicit classification in some dimensions, and a complete rational classification in codimension at least 3:

The Haefliger Theorem. [Hae66A, Corollary 6.7] *Assume that* $q+3 \le m < \frac{3}{2}q+2$. *Then up to isotopy the set of smooth embeddings* $S^q \to S^m$ *is infinite if and only if* $q+1$ *is divisible by* 4.

The classification of *links* $S^p \sqcup S^q \to S^m$ is the next natural problem after knots. Denote by $L^m_{p,q}$ (respectively, K^m_p) the set of smooth embeddings $S^p \sqcup S^q \to S^m$ (respectively, $S^p \to S^m$) up to smooth isotopy. For $p, q \le m-3$ this set of equivalence classes is a group with respect to 'componentwise connected sum' operation [Hae66C].

Our first result is an explicit formula for the group $L^m_{p,q}$ in terms of the groups K^m_p in the *2-metastable* (*quadruple point free*) dimension:

Theorem 0.1. *Suppose that* $1 \le p \le q \le m-3$ *and* $2p+2q \le 3m-6$*; then*

$$L^m_{p,q} \cong \pi_p(S^{m-q-1}) \oplus \pi_{p+q+2-m}(V_{M+m-p-1,M}) \oplus K^m_p \oplus K^m_q.$$

Here $V_{M+l,M}$ is the *Stiefel manifold* of M-frames at the origin of $\mathbb{R}^{M+l}$, where M is large. Many of the groups $\pi_n(V_{M+l,M})$ are known [Pae56].

Theorem 0.1 is the most general explicit classification of links available. However, for arbitrary $p, q \le m-3$ there is a famous exact sequence involving the groups $L^m_{p,q}$, certain homotopy groups and maps between them including Whitehead products [Hae66C].

Theorem 0.1 was proved in [Hae66C, Theorems 10.7 and 2.4] under stronger restrictions $p \le q$ and $p+3q \le 3m-7$. Our inequality in Theorem 0.1 is sharp (see an example in §5). We not only improve the result of [Hae66C], but give a simpler proof of this classical result. However, the Haefliger argument can be extended to cover our dimension range (see §5).

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The main subject of this paper is the classification of *knotted tori* $S^p \times S^q \to S^m$. A link $S^q \sqcup S^q \to S^m$ is a specific case of knotted torus (Figure 1a). The investigation of knotted tori is a natural next step (after link theory) towards the classification of embeddings of arbitrary manifolds [Web67, Sko05], by the handle decomposition theorem.

Figure 1 approximately here

There was known an explicit formula for the set of knotted tori in the *metastable* dimension $m \geq p + \frac{3}{2}q + 2$, $p \leq q$ [Hae62T, Zee62, Sko02] (Figure 2, range I). The knotting problem for any $(p-1)$-connected $(p+q)$-manifold is easier in the metastable dimension, and this restriction is a natural limit for classical classification methods [Sko06"]. Little is known below the metastable dimension: all known explicit results concern knots, links (see above), knotted tori in dimension $m = p + \frac{3}{2}q + \frac{3}{2}$ [Sko06'], 3-manifolds in $\mathbb{R}^6$ [Sko06] and 4-manifolds in $\mathbb{R}^7$ (A. Skopenkov).

Our main result is an explicit finiteness criterion for the set of knotted tori in the *2-metastable* dimension (Figure 2, range II):

Theorem 0.2. *Assume that $p + \frac{4}{3}q + 2 \leq m < p + \frac{3}{2}q + 2$ and $m \geq 2p + q + 3$. Then the set of smooth embeddings $S^p \times S^q \to S^m$ up to isotopy is infinite if and only if either $q + 1$ or $p + q + 1$ is divisible by 4.*

Example. *The set of knotted tori $S^1 \times S^5 \to S^{10}$ is finite. All the dimensions, in which the set of knotted tori is infinite for $p = 1$, are shown in Figure 3 (obtained by combining Theorem 0.2 with the results of [Sko02, Sko06, Sko06']).*

Figures 2 and 3 approximately here

Our approach to the classification of links and knotted tori is based on studying of almost embeddings (see §3.1) and link maps, which is an interesting problem in itself [Sco68, Kos90, HaKa98].

A link map is a continuous map $X \sqcup Y \to Z$ such that $fX \cap fY = \varnothing$. *A link homotopy* is a continuous family of link maps $f_t : X \sqcup Y \to Z$. Denote by $LM^m_{p,q}$ the set of link maps $S^p \sqcup S^q \to S^m$ up to link homotopy. For $p, q \leq m - 3$ this group admits a natural commutative group structure with respect to 'componentwise connected sum' operation (see §1.1).

The third result of this paper is a new short proof of the following theorem:

Theorem 0.3. [HaKa98, Theorem I] *Suppose that $1 \leq p, q \leq m - 3$ and $2p + 2q \leq 3m - 5$; then*

$$LM^m_{p,q} \cong \pi^S_{p+q-m+1}.$$

The isomorphism is given by the α-invariant (defined in §2.3). The inequality is sharp [HaKa98, §6].

Theorem 0.3 is the most general known explicit classification of link maps in codimension at least 3. However, under slightly weaker dimension restriction there is a powerful exact sequence involving the groups $LM^m_{p,q}$ and certain bordism groups [Kos90].

Our approach to the classification is based on the suspension map. *The suspension map* $\Sigma : LM^m_{p,q} \to LM^{m+1}_{p+1,q}$ is defined by suspending the p-component and including the q-component. It is not difficult to see that eventually (after iterated suspension) the set $LM^{m+M}_{p+M,q}$ is in bijection with the group $\pi^S_{p+q-m+1}$ [Kos88]. Thus Theorem 0.3 follows from the following assertion:

Theorem 0.3' (Suspension theorem for link maps). [HaKa98] *Suppose that $1 \leq p, q \leq m - 3$; then the suspension map Σ is bijective for $2p + 2q \leq 3m - 5$ and surjective for $2p + 2q \leq 3m - 4$.*

This theorem has been known earlier only as a corollary of Theorem 0.3. In this paper we present a short and direct proof of Theorem 0.3' analogous to the proof of the Freudental suspension theorem and to Zeeman's proof of the higher-dimensional Poincaré conjecture. Our proof is almost self-contained, we use only 'concordance implies isotopy in codimension at least 3' theorem.

The paper is organized as follows. In §1 we prove Theorems 0.3'. In §2 we deduce Theorem 0.1 from Theorem 0.3'. In §3 we deduce Theorem 0.2 from a lemma proved in §2. These 3 sections can be read independently in the sense that if in one section we use a result proved in another section, then we do not use the proof but only the statement. We assume piecewise linear category throughout §1 and §2 and smooth category throughout §3 except otherwise indicated. In §4 we give some remarks not used in the rest of the paper. In Appendix we give an approach for an alternative proof of our main results hopefully interesting in itself.

1. Classification of link maps

1.1. Preliminaries.

Let us introduce some notations and conventions.

An embedding $f : X \times I \to S^m \times I$ is *a concordance* if $X \times 0 = f^{-1}(S^m \times 0)$ and $X \times 1 = f^{-1}(S^m \times 1)$. We tacitly use the facts that in codimension at least 3 *concordance implies isotopy* and *any concordance or isotopy is ambient* [Hud69, Hud70].

Define a *link concordance* to be a continuous map $f:(X\sqcup Y)\times I\to S^m\times I$ such that $f(X\times I)\cap f(Y\times I)=\emptyset$, $(X\sqcup Y)\times 0=f^{-1}(S^m\times 0)$ and $(X\sqcup Y)\times 1=f^{-1}(S^m\times 1)$. In codimension at least 3 *link concordance implies link homotopy,* which was announced in [KrTa97, Kos97, Mel00] and proved in [Mel, cf. BaTe99].

A consequence of these facts is the group structure on the sets $L^m_{p,q}$ and $LM^m_{p,q}$ for $p,q\le m-3$. More precisely, these facts allow to construct the additive inverses in these semigroups [Sco68, p. 187, Kos88, Remark 2.4].

Recall some easy notions from piecewise linear topology. Let K be a simplicial complex and L a subcomplex. Denote by (K',L') the *derived* triangulation of (K,L), i. e., the barycentric subdivision of (K,L). *The supplement* of L in K, denoted by $K\div L$, is the subcomplex of K' spanned by all the vertices of $K'-L'$; i. e., a simplex of K' belongs to $K\div L$ if and only if none of its vertices is in L'.

It is not difficult to check that *each simplex of K' that is not a simplex of L' or of $K\div L$, is the join of a simplex of L' with a simplex of $K\div L$,* and *if L contains the n-dimensional skeleton of K, then $dim(K\div L)\le dim\,K-n-1$* [BM52].

For a map $f:X\to Y$ denote $S(f)=Cl\{\,x\in X:|f^{-1}fx|\ge 2\,\}$.

Let us introduce the main notion used in our proof of Theorem 0.3' and state main lemma of the proof.

Definition of a standardized link map. (Figure 4) Denote by $S^k=D^k_+\cup(S^{k-1}\times I)\cup D^k_-$ the standard decomposition of the sphere. Let $pr:S^{k-1}\times I\to S^{k-1}\times 0$ be the obvious projection. We say that a link map $f:S^p\sqcup S^q\to S^m$ is *standardized* if the following 3 conditions hold:
(i) $fD^p_+\subset D^m_+$, $fD^p_-\subset D^m_-$, $f(S^{p-1}\times I)\subset S^{m-1}\times I$;
(ii) $fS^q\subset S^{m-1}\times I$;
(iii) $pr\,f(S^{p-1}\times I)=f(S^{p-1}\times 0)$, i. e. $f(S^{p-1}\times I)$ is *'straight'*.

Figure 4 approximately here.

Lemma 1.1. *Suppose that $p\le q+1$ and $2p+2q\le 3m-5$; then any generic link map $f:S^p\sqcup S^q\to S^m$ is link homotopic to a standardized link map. Moreover, there exist homeomorphisms $h_p:S^p\to S^p$ and $h_m:S^m\to S^m$ such that $h_m^{-1}\circ f\circ(h_p\sqcup id)$ is standardized.*

We are going first to check the surjectivity in Theorem 0.3' in case $p\le q$, then the injectivity in case $p\le q$ by a similar argument, and finally we deduce case $p>q$ of Theorem 0.3' from case $p\le q$.

1.2 Proof of the surjectivity in Suspension theorem for link maps in case $p\le q$.

Proof of the surjectivity in Theorem 0.3' for $p\le q$ modulo Lemma 1.1. (Conical construction) Take an arbitrary link map $f:S^{p+1}\sqcup S^q\to S^{m+1}$. Let us modify it to a suspension by a link homotopy.

By Lemma 1.1 we may assume that f is standardized. We may also assume that the disks D^{p+1}_+ and D^{m+1}_+ are the upper half-spheres of S^{p+1} and S^{m+1} respectively.

Push the image of the q-component along the fibers of $S^m\times I$ until it lies in $S^{m-1}\times 0=\partial D^{m+1}_+$. Then transform fD^{p+1}_+ and $f(S^{p+1}-D^{p+1}_+)$ to the cones over $f\partial D^{p+1}_+$ in D^{m+1}_+ and $S^{m+1}-Int\,D^{m+1}_+$ respectively. (This is done by a link homotopy, which is rectilinear inside both D^{m+1}_+ and $S^{m+1}-Int\,D^{m+1}_+$.) The link map obtained is the suspension of a link map $S^p\sqcup S^q\to S^m$. $\square$

Now we proceed to the proof of Lemma 1.1. We are going first to give the construction of the required homeomorphisms without any indication why all our steps are possible. Reading the next 4 paragraphs is sufficient to understand main ideas of the proof. Then we present the technical details required to check the possibility of the construction.

Construction of the homomorphism $h_p:S^p\to S^p$ in Lemma 1.1 for $p\le q$. (Figure 5) It is in 2 steps:

(1) Construction of certain decomposition $S^p=\bar D^p_+\cup(S^{p-1}\times I)\cup\bar D^p_-$. (The Zeeman engulfing) Triangulate S^p, S^q and S^m to make $f=f_1\sqcup f_2:S^p\sqcup S^q\to S^m$ a non-degenerate simplicial map. Let A_+ be the skeleton of $S(f_1)$ formed by the simplices of dimension not greater than $\frac12\,dim\,S(f_1)$. Let $A_-=S(f_1)\div A_+$ be the supplement of A_+ in $S(f_1)$. Embed the cones over A_+ and A_- into S^p. Denote these cones by B_+ and B_- respectively. Generically $B_+\cap B_-=\varnothing$. Let $\bar D^p_+$ and $\bar D^p_-$ be the second derived neighborhoods in S^p of B_+ and B_- respectively. Perform a homeomorphism $h'_p:S^p\to S^p$ taking the balls $\bar D^p_+$ and $\bar D^p_-$ to the balls D^p_+ and D^p_- of the standard decomposition $S^p=D^p_+\cup(S^{p-1}\times I)\cup D^p_-$.

(2) Straightening of $S(f_1)\cap(S^{p-1}\times I)$. (The Alexander trick) It can be shown that $S(f_1)\cap(S^{p-1}\times I)\cong(S(f_1)\cap(S^{p-1}\times 0))\times I$. Perform a homeomorphism $h''_p:S^p\to S^p$ making $pr\,S(f_1)\cap(S^{p-1}\times I)=S(f_1)\cap(S^{p-1}\times 0)$. The homeomorphism $h_p=h''_p\circ h'_p$ is the required.

Figure 5 approximately here.

Construction of the homomorphism $h_m:S^m\to S^m$ in Lemma 1.1. We again do 2 steps similar to the above:

(3) Construction of certain decomposition $S^m=\bar D^m_+\cup(S^{m-1}\times I)\cup\bar D^m_-$ for $p\le q$. Embed the cones over fB_+ and fB_- into S^m. Denote these cones by C_+ and C_- respectively. Generically by the assumption $2p+2q\le 3m-5$ we have $C_+\cap fS^q=\varnothing$, $C_-\cap fS^q=\varnothing$ and $C_+\cap C_-=\varnothing$. Let $\bar D^m_+$ and $\bar D^m_-$ be the second derived neighborhoods

of C_+ and C_- respectively in $S^m - fS^q$. Perform a homeomorphism $h'_m : S^m \to S^m$ taking the balls $\bar D^m_+$ and $\bar D^m_-$ to the balls of the standard decomposition $S^m = D^m_+ \cup (S^{m-1} \times I) \cup D^m_-$.

(4) Straightening of $fS^p \cap (S^{m-1} \times I)$. It can be shown that $fS^p \cap (S^{m-1} \times I) \cong (fS^p \cap (S^{m-1} \times 0)) \times I$. Perform a homeomorphism $h''_m : S^m \to S^m$ making $pr\, fS^p \cap (S^{m-1} \times I) \cong fS^p \cap (S^{m-1} \times 0)$. The homeomorphism $h_m = h''_m \circ h'_m$ is the required.

Details of the proof of Lemma 1.1. Let us check that all steps of the above construction are indeed possible.

(1) Construction of the decomposition $S^p = \bar D^p_+ \cup (S^{p-1} \times I) \cup \bar D^p_-$ for $p \le q$. We need to check that the inclusions $A_\pm \hookrightarrow S^p$ extend to embeddings $B_\pm \hookrightarrow S^p$, and that $B_+ \cap B_- = \varnothing$. We are going to show also that $B_\pm \cap S(f_1) = A_\pm$, which is required for step (2).

Generically $dim\, S(f_1) \le 2p - m$, thus $dim\, A_\pm \le p - \lfloor \frac{m+1}{2} \rfloor$. So a generic extension $g : CA_+ \to S^p$ of the inclusion $A_+ \hookrightarrow S^p$ is an embedding, because $dim\, S(g) \le 2(p - \lfloor \frac{m+1}{2} \rfloor + 1) - p < 0$ by the assumption $p \le m - 3$. Put $B_+ = g(CA_+)$, define B_- analogously. Then $dim(B_+ \cap B_-) \le 2(p - \lfloor \frac{m+1}{2} \rfloor + 1) - p < 0$, thus $B_+ \cap B_- = \varnothing$. Further, $(B_+ - A_+) \cap S(f_1)$ has dimension $(p - \lfloor \frac{m+1}{2} \rfloor + 1) + (2p - m) - p \le \frac{1}{2}(2q + 2p - 3m + 4) < 0$, which follows from the assumptions $p \le q + 1$ and $2p + 2q \le 3m - 5$. So $B_+ \cap S(f_1) = A_+$ and similarly $B_- \cap S(f_1) = A_-$.

(2) Construction of the homeomorphism $h''_p : S^p \to S^p$. We need to check that $S(f_1) \cap (S^{p-1} \times I) \cong (S(f_1) \cap (S^{p-1} \times 0)) \times I$ and that there exists a homeomorphism $h''_p : S^p \to S^p$ as required.

Take the derived triangulation of the triangulation from step (1). Then each simplex $\sigma \subset S(f_1)$ such that $\sigma \not\subset A_+ \cup A_-$ is the join of a simplex $\sigma_+ \subset A_+$ and a simplex $\sigma_- \subset A_-$. Since $B_\pm \cap S(f_1) = A_\pm$, it follows that $\sigma \cap \bar D^p_\pm$ are the second derived neighborhoods of $\sigma_\pm$ in σ. So there is a natural homeomorphism $\sigma \cap (S^{p-1} \times I) \to \sigma_+ \times \sigma_- \times I$. Combining such homeomorphisms for all simplices $\sigma \subset S(f_1)$ such that $\sigma \not\subset A_+ \cup A_-$, we get $S(f_1) \cap (S^{p-1} \times I) \cong (S(f_1) \cap (S^{p-1} \times 0)) \times I$.

The latter homeomorphism can be thought as a concordance between two embeddings of the polyhedron $S(f_1) \cap (S^{p-1} \times 0)$ into $S^{p-1} \times 0$ and $S^{p-1} \times 1$ respectively. Since any concordance in codimension at least 3 is ambient isotopic to an isotopy, by a homeomorphism of S^p one can transform the concordance $(S(f_1) \cap (S^{p-1} \times 0)) \times I \to S^{p-1} \times I$ to an isotopy. Since any isotopy in codimension at least 3 extends to an ambient isotopy, the constructed isotopy extends to a homeomorphism $S^{p-1} \times I \to S^{p-1} \times I$. Extend arbitrarily the latter homeomorphism to a homeomorphism $h''_p : S^p \to S^p$. The obtained homeomorphism is the required.

(3) Construction of the decomposition $S^m = \bar D^m_+ \cup (S^{m-1} \times I) \cup \bar D^m_-$ for $p \le q$. The construction of the embeddings $C_\pm \hookrightarrow S^m$ and checking the property $C_+ \cap C_- = \varnothing$ are analogous to step (1). To satisfy the properties (i) and (ii) of Definition of a standardized link map from §2 it remains to check that $C_\pm \cap fS^q = \varnothing$ and $f^{-1} C_\pm = B_\pm$.

Generically $dim(C_+ \cap fS^q) \le (p - \lfloor \frac{m+1}{2} \rfloor + 2) + q - m < 0$, which is equivalent to the assumption $2p + 2q \le 3m - 5$. (This is the only place in the proof where the restriction $2p + 2q \le 3m - 5$ is sharp.) Further, generically $dim(C_+ - fB_+) \cap fS^p \le (p - \lfloor \frac{m+1}{2} \rfloor + 2) + p - m < 0$, which follows from the assumptions $p \le q$ and $2p + 2q \le 3m - 5$.

(4) Construction of the homeomorphism $h''_m : S^m \to S^m$. We need to check that $fS^p \cap (S^{m-1} \times I) \cong (fS^p \cap (S^{m-1} \times 0)) \times I$ and that there exist a homeomorphism $h''_m : S^m \to S^m$ as required. This is sufficient to satisfy the property (iii) from §2.

In step (2) we constructed a homeomorphism $S(f_1) \cap (S^{p-1} \times I) \to (S(f_1) \cap (S^{p-1} \times 0)) \times I$ and its extension to a homeomorphism $h : S^{p-1} \times I \to S^{p-1} \times I$. Taking the quotient of the homeomorphism h we get $fS^p \cap (S^{m-1} \times I) \cong (fS^p \cap (S^{m-1} \times 0)) \times I$. The construction of the homeomorphism $h''_m : S^m \to S^m$ is completely analogous to step (2).

(5) Modifications necessary in case $S(f_1) = \varnothing$. In this case we take $h_p = id$, take arbitrary decomposition $S^m = \bar D^m_+ \cup C^m \cup \bar D^m_-$ satisfying the properties (i),(ii) from §2, and then argue as in step (4) above.

(6) Modifications necessary in case $p = q + 1$. (The Irwin trick) In this case the set $f^{-1}(C_+ - fB_+)$ may be nonempty. If this set is nonempty, then by step (3) above it consists of finitely many isolated points, not belonging to $S(f_1)$. Join each of these points with B_+ by a general position arc in S^p. Let B'_+ be the union of these arcs and the cone B_+. Adding appropriate cones over $f(B'_+ - B_+)$ to C_+, we get a subcomplex $C'_+ \subset S^m$ such that $dim(C'_+ - C_+) \le 2$. Now by general position $f^{-1}(C'_+ - fB'_+) = \varnothing$. We define $\bar D^p_+$ and $\bar D^m_+$ to be the second derived neighborhoods of B'_+ in S^p and of C'_+ in $S^m - fS^q$ respectively. The balls $\bar D^p_-$ and $\bar D^m_-$ are defined analogously. The rest is similar to steps (2) and (4) above. $\square$

Remark. In fact we have proved the surjectivity in Theorem 0.3' in case $p \le q$ without the assumption $q \le m - 3$.

1.3 Completion of the proof of suspension theorem for link maps.

The proof of the injectivity in Theorem 0.3' is based on the following relative version of Lemma 1.1:

Lemma 1.1'. *Suppose that $p \le q + 1$ and $2p + 2q \le 3m - 5$; then any generic link map $f : D^p \sqcup D^q \to D^m$, whose restriction to the boundary is a suspension, is link homotopic relatively the boundary to a standardized link map.*

We omit the obvious, but rather long and technical definition of a standardized link map $f : D^p \sqcup D^q \to D^m$.

Proof of the injectivity in Theorem 0.3' for $p \le q$ modulo Lemma 1.1'. We need to prove that if the suspension of a link map $f_0 : S^p \sqcup S^q \to S^m$ is null homotopic then the link map f_0 is null homotopic. Take an arbitrary null homotopy $f : D^{p+2} \sqcup D^{q+1} \to D^{m+2}$ of Σf_0. By Lemma 1.1' we may assume that the link map $f : D^{p+2} \sqcup D^{q+1} \to D^{m+2}$ is standardized.

Push the $(q+1)$-component along the fibers of $D^{m+1} \times I$ toward ∂D^{m+2}_+ until it lies in $\partial D^{m+2}_+ - \partial D^{m+2}$. Let $f' : D^{p+2} \sqcup D^{q+1} \to D^{m+2}$ be the obtained null homotopy. The restriction $f' : (\partial D^{p+2}_+ - \partial D^{p+2}) \sqcup D^{q+1} \to \partial D^{m+2}_+ - \partial D^{m+2}$ is a null link concordance of the link map f_0. Since link concordance implies link homotopy in codimension at least 3, it follows that f_0 is null homotopic. $\square$

Although the statement of Lemma 1.1' is analogous to Lemma 1.1, the proof is not completely analogous (because there is no appropriate relative version of the supplement to a subcomplex). Let us present the necessary modifications.

Modifications necessary in the proof of Lemma 1.1 to prove Lemma 1.1'. Attach the cylinders $\partial D^k \times I$ to the balls D^k for $k = p$, q and m, along $\partial D^k \times 0$. Denote by $\hat D^p$, $\hat D^q$ and $\hat D^m$ the obtained balls and by $\hat f : \hat D^p \sqcup \hat D^q \to \hat D^m$ the obvious extension of the link map $f : D^p \sqcup D^q \to D^m$. Clearly, it suffices to make the link map $\hat f$ standardized.

Denote by $D^{p-1}_\pm$ the upper and downer half-spheres of ∂D^p, and by $D^{m-1}_\pm$ the upper and downer half-spheres of ∂D^m. Denote by $f_0 : S^{p-2} \sqcup S^{q-1} \to S^{m-2}$ be the restriction of f to the equators.

(1) Construction of certain decomposition $D^p = \bar D^p_+ \cup (D^{p-1} \times I) \cup \bar D^p_-$ for $p \le q$. Let A_+ be the union of $S(f_0)$ and the skeleton of $S(f_1)$ formed by the simplices of dimension not greater than $\frac12 \, dim\, S(f_1)$. Put $A_- = S(f) \div A_+$. Take a general position homotopy $g_t : A_+ \to D^p$ constant on $A_+ \cap \partial D^p$ and such that $g_0 : A_+ \to D^p$ is the inclusion, $g_1 A_+ \subset \partial D^p$. Let $B_+ \subset D^p$ be the trace of the homotopy. Denote by $\bar D^p_+$ the second derived neighborhood of $B_+ \cup ((B_+ \cap D^{p-1}_+) \times I) \cup (D^{p-1}_+ \times 1)$ relatively ∂D^{p-1}_+ in $\hat D^p$. Define $\bar D^p_-$ analogously.

(2) Construction of certain decomposition $D^m = \bar D^m_+ \cup (D^{m-1} \times I) \cup \bar D^m_-$ for $p \le q$. Take a general position homotopy $g_t : fB_+ \to D^m$ constant on $fB_+ \cap \partial D^m$ such that $g_0 : fB_+ \to D^m$ is the inclusion, and $g_1 fB_+ \subset \partial D^m$. Let $C_+ \subset D^m$ be the trace of the homotopy. Define $\bar D^m_+$ to be the second derived neighborhood of $C_+ \cup ((C_+ \cap D^{m-1}_+) \times I) \cup (D^{m-1}_+ \times 1)$ relatively ∂D^{m-1}_+ in $\hat D^m$. Define $\bar D^m_-$ analogously.

Completion of the constructions of the homeomorphisms $h_p : D^p \to D^p$ and $h_m : D^m \to D^m$, and modifications in case $p = q+1$ are completely analogous to steps (2), (4) and (6) from the proof of Lemma 1.1.$\square$

Proof of Theorem 0.3'. In the above we have proved Theorem 0.3' in case $p \le q$. Now assume $p > q$. For example, let us prove the surjectivity. First suspend the second component (q-component) many times until its dimension becomes equal to p. By the case $p \le q$ of Theorem 0.3' this iterated suspension is surjective. Now suspend once the first component. Since the dimensions of the components are now equal, one can apply Theorem 0.3' and conclude that this single suspension is surjective. Finally, desuspend backwards the second component $p-q$ times using that the corresponding suspension maps are bijective by case $p \le q$ of Theorem 0.3'. The composition of all the considered suspensions and desuspensions is a single suspension of the first component. We have shown that it is surjective. The injectivity is proved analogously. $\square$

2. Classification of links

2.1 Reduction to the classification of disc link maps.

Let us introduce some notation.

Let $\widehat L^m_{p,q}$ be the group of piecewise linear embeddings $S^p \sqcup S^q \to S^m$ up to piecewise linear isotopy.

An *almost link* is a link map $f : S^p \sqcup S^q \to S^m$ whose restriction to S^q is an embedding. An *almost isotopy* is a link homotopy $f_t : S^p \sqcup S^q \to S^m$ whose restriction to S^q is an isotopy. Let $\overline L^m_{p,q}$ be the set of almost links up to almost isotopy. For $p, q \le m-3$ this set admits a natural commutative group structure defined analogously to those of $L^m_{p,q}$ and $LM^m_{p,q}$.

It is not difficult to see that $\overline L^m_{p,q} \cong \pi_p(S^{m-q-1})$ (cf. definition of the isomorphism $\lambda : \overline L^m_{p,q} \to \pi_p(S^{m-q-1})$ in §2.3). Notice that the obvious map $\widehat L^m_{p,q} \to \overline L^m_{p,q}$ can be identified with the iterated suspension map $\widehat L^m_{p,q} \to \widehat L^{m+M}_{p,q+M}$ for M large.

A *disc link map* [cf. Hae66A, Ne84] is a proper link map $D^p \sqcup D^q \to D^m$ whose restriction to $\partial S^p \sqcup S^q$ is an embedding. A *disc link homotopy* is a homotopy through disc link maps, whose restriction to $\partial S^p \sqcup S^q$ is an isotopy. Let $\widehat{DM}^m_{p,q}$ be the set of disc link maps up to disc link homotopy. For $p, q \le m-3$ it admits a natural group structure, defined analogously to the above.

Theorem 2.1 (Geometric EHP sequence for links). (A. Skopenkov, cf. [Ne84], see Figure 6) *For $p, q \le m-3$ there is an exact sequence:*

$$\dots \longrightarrow \widehat L^m_{p,q} \xrightarrow{e} \overline L^m_{p,q} \xrightarrow{h} \widehat{DM}^m_{p,q} \xrightarrow{p} \widehat L^{m-1}_{p-1,q-1} \longrightarrow \dots$$

Figure 6 approximately here

The easy proof is presented below. We are going to use tacitly that *concordance implies isotopy*, *link concordance implies link homotopy*, *almost concordance implies almost isotopy*, and *any concordance or isotopy is ambient* [Hud69, Hud70, Mel00, Mel].

Proof of Theorem 2.1. Construction of the homomorphisms. Let e be the obvious map. Let p be the 'restriction to the boundary' map. The map h is the 'cutting' homomorphism defined as follows. Take an almost link $f : S^p \sqcup S^q \to S^m$. By an almost isotopy of f we may assume that there are points $x \in S^p$ and $y \in S^q$ such that the restrictions of f onto their neighborhoods B^p_x and B^q_y are standard embeddings. Since $p, q \le m-3$, we can take a path l from fx to fy intersecting $f(S^p \sqcup S^q)$ only at ∂l. Set $h(f)$ to be the restriction of f to a map $(S^p - Int\, B^p) \sqcup (S^q - Int\, B^q) \to S^m - Int\, B^m$, where B^m is a neighborhood of l.

Proof of the exactness. We have $\mathrm{Im}\, p = \mathrm{Ker}\, e$ because a link $f : S^p \sqcup S^q \to S^m$ extends to a disc link map $D^{p+1} \sqcup D^{q+1} \to D^{m+1}$ if and only if it is null almost isotopic.

We have $\mathrm{Im}\, h = \mathrm{Ker}\, p$ because a disc link map $f : D^p \sqcup D^q \to D^m$ extends without adding new self-intersections to an almost link $S^p \sqcup S^q \to S^m$ if and only if the restriction of f to the boundary is isotopic to the standard link.

To prove $\mathrm{Im}\, e \subset \mathrm{Ker}\, h$, take $f \in \widehat{DM}^m_{p,q}$ which is an embedding. Take a pair of points $x \in D^p$ and $y \in D^q$. Join fx and fy by an arc l intersecting $f(D^p \sqcup D^q)$ only at ∂l. Denote by B^p, B^q and B^m small neighborhoods of these fx, fy and l respectively. Clearly, the restriction $f : B^p \sqcup B^q \to B^m$ is trivial. The restriction $f : (D^p - B^p) \sqcup (D^q - B^q) \to (D^m - B^m)$ can be thought as a concordance between the restriction of f to the boundary and the trivial embedding. Since a concordance is ambient isotopic to an isotopy in codimension at least 3, we may assume that this restriction is level-preserving. Then it is obvious that the embedding f is ambient isotopic to the restriction $f : B^p \sqcup B^q \to B^m$. Hence $f = 0$ and $h \circ e = 0$.

To prove $\mathrm{Ker}\, h \subset \mathrm{Im}\, e$, take $f \in \overline{L}^m_{p,q}$ such that $h(f) = 0$. Then, by definition, there exist a disc link homotopy h_t between $h(f)$ and the trivial embedded disc link map. By the isotopy extension theorem [Hud70] the restriction of h_t to the boundary extends to an ambient isotopy of $S^m - D^m$. So h_t can be extended to an almost isotopy link homotopy of f without adding new self-intersections. The latter is a homotopy between f and a link $f' \in \widehat{L}^m_{p,q}$. Hence $f = e(f')$. $\square$

Corollary 2.1'. *For $p \le q \le m-3$ we have $L^m_{p,q} \cong K^m_p \oplus K^m_q \oplus \pi_p(S^{m-q-1}) \oplus \widehat{DM}^{m+1}_{p+1,q+1}$.*

Proof. By [Hae66C, Theorem 2.4] we have $L^m_{p,q} \cong \widehat{L}^m_{p,q} \oplus K^m_p \oplus K^m_q$. So it suffices to show that for $p \le q$ the homomorphism $e : \widehat{L}^m_{p,q} \to \overline{L}^m_{p,q}$ in Theorem 2.1 has a right inverse $e' : \overline{L}^m_{p,q} \to \widehat{L}^m_{p,q}$. The required right inverse $e' : \pi_p(S^{m-q-1}) \to \widehat{L}^m_{p,q}$ takes the class of a map $\phi : S^p \to S^{m-q-1}$ to a link $f : S^p \sqcup S^q \to D^{p+1} \times S^{m-q-1} \sqcup D^{p+1} \times 0 \subset S^m$ given by the formula $f(x \sqcup y) = (x, \phi x) \sqcup (y, 0)$ (see the details in [Hae66C, Zeeman Theorem 10.1]). $\square$

2.2 Simplification of the group of disc link maps.

Define $\overline{DM}^m_{p,q}$ to be the group of proper link maps $f : D^p \sqcup D^q \to D^m$ whose restriction to ∂D^p is an embedding (up to link homotopy whose restriction to ∂D^p is an isotopy).

Lemma 2.2. *The natural map $\widehat{DM}^m_{p,q} \to \overline{DM}^m_{p,q}$ is bijective for $2p+2q \le 3m-5$ and surjective for $2p+2q \le 3m-4$.*

Proof of the surjectivity. Take any general position link map $f \in \overline{DM}^m_{p,q}$. Let us check that the pair $(D^m - fD^p, \partial D^m - f\partial D^p)$ is $(2m-2p-3)$-connected. Indeed, in codimension at least 3 the pair $(D^m - fD^p, \partial)$ is 1-connected. Identify D^m with the upper half-sphere of S^m. By the homology excision theorem $H_i(D^m - fD^p, \partial) \cong H_i(S^m - fD^p)$. By the Alexander duality $H_i(S^m - fD^p) \cong H^{m-i-1}(fD^p)$. It is not difficult to see that fD^p is homotopy equivalent to the mapping cone of the restriction $f : S(f) \to fS(f)$. (Indeed, both of these spaces are obtained from $\mathrm{Cyl}(S(f) \to fS(f)) \cup_{S(f) \subset D^p} D^p$ by appropriate contractions.) Generically the dimension of the mapping cone is at most $2p-m+1$. So $H^{m-i-1}(fD^p) = 0$ for $i \le 2m-2p-3$. Thus the pair $(D^m - fD^p, \partial D^m - f\partial D^p)$ is $(2m-2p-3)$-connected by the Hurewicz theorem.

By the condition $2p+2q \le 3m-4$ and the embedding theorem moving the boundary [Hud70] the restriction $f|_{D^q} : (D^q, \partial D^q) \to (D^m - fD^p, \partial D^m - f\partial D^p)$ is homotopic to an embedding. So f belongs to the image of the natural map $\widehat{DM}^m_{p,q} \to \overline{DM}^m_{p,q}$.

Proof of the injectivity. Take a general position link homotopy $f : (D^p \sqcup D^q) \times I \to D^m \times I$, whose restriction to $(\partial D^p \sqcup D^q) \times \partial I \cup \partial D^p \times I$ is an embedding. It suffices to remove the self-intersections of $D^q \times I$ by a link homotopy fixed on $(D^p \sqcup D^q) \times \partial I$.

Analogously to the proof of the surjectivity one can check that the pair $(D^m \times I - f(D^p \times I), \partial D^m \times I - f(\partial D^p \times I))$ is $(2m-2p-3)$-connected. So the self-intersections of $D^q \times I$ can be removed by the following embedding theorem proved analogously to [Hud70] $\square$.

Embedding theorem moving a part of the boundary. *Let M be a compact $(m+1)$-dimensional manifold, B a codimension zero submanifold of ∂M, A a codimension zero submanifold of ∂D^p. Let $f : (D^{q+1}, A) \to (M, B)$ be a proper map such that $f|_{\partial D^{q+1} - A}$ is an embedding. If $m \ge q+3$ and $(M; B)$ is $(2q-m+2)$-connected, then f is properly link homotopic rel $\partial D^{q+1} - A$ to an embedding.*

2.3 Classification of disc link maps.

Theorem 0.1 follows from Theorem 0.3', Corollary 2.1', Lemma 2.2, 5-lemma and the following result:

Theorem 2.3 (Geometric EHP sequence for almost links). (A. Skopenkov) *For $3p+q \le 3m-5$ there is the following diagram with exact lines, commutative up to sign:*

$$\begin{tikzcd}
\overline{L}^m_{q,p} \arrow[r, "e"] \arrow[d, "\lambda"] & LM^m_{p,q} \arrow[r, "h"] \arrow[d, "\alpha"] & \overline{DM}^m_{p,q} \arrow[r, "p"] \arrow[d, "\beta"] & \overline{L}^{m-1}_{q-1,p-1} \arrow[r] \arrow[d, "\lambda"] & \dots \\
\pi_q(S^{m-p-1}) \arrow[r, "E"] & \pi^S_{p+q+1-m} \arrow[r, "H"] & \pi_{p+q+1-m}(V_{M+m-p-1,M}) \arrow[r, "P"] & \pi_{q-1}(S^{m-p-1}) \arrow[r] & \dots
\end{tikzcd}$$

The top line in Theorem 2.3 is defined analogously to Theorem 2.1 (with similar proof of the exactness). The bottom line is the stable James EHP sequence [Jam54, KoSa77], which is exact for $3p+q \le 3m-5$. The linking number $\lambda = \lambda_{12}$ and the map α are defined in [Hae66C] and [Kos90, §1] respectively. The map β can be defined analogously to the β-invariant $LM^m_{p,q} \to \pi_{p+q+1-m}(V_{M+m-p-1,M})$ of [Kos90, §1]. The new part of Theorem 2.3 is the (anti)commutativity of the right-hand square, the (anti)commutativity of the other squares is known [Ker59, Lemma 5.1, Kos88, Theorem 4.8].

In the rest of §2 we work in smooth category. Clearly, the group $\overline{L}^m_{q,p}$ is isomorphic to the group of link maps $S^p \sqcup S^q \to S^m$, whose restriction to S^p is an unknotted smooth embedding (up to link homotopy whose restriction to S^p is a smooth isotopy). Analogously, $\overline{DM}^m_{p,q}$ is isomorphic to the group of proper link maps $D^p \sqcup D^q \to D^m$ whose restriction to ∂D^p is an unknotted smooth embedding (up to proper link homotopy whose restriction to ∂D^p is a smooth isotopy).

Let us give geometric definitions of all the maps from the diagram.

Construction of the EHP sequence. [KoSa77, §1 and §4, cf. Szu76, Ecc80] Denote $n = p+q+1-m$. Let Emb^q_n and Imm^q_n be the groups of framed embeddings and immersions, respectively, of closed n-manifolds into S^q (up to framed cobordism).

A *proper immersion* is a proper framed immersion of an n-manifold into D^q, whose restriction to the boundary is an embedding. A *proper cobordism* is a proper framed immersion $c : N^{n+1} \to D^q \times I$, whose restriction to $c^{-1}(S^{q-1} \times I)$ is an embedding. Let PIm^q_n be the group of proper immersions up to proper cobordism.

Let $E : Emb^q_n \to Imm^n_q$ and $P : PIm^q_n \to Emb^{q-1}_{n-1}$ be the obvious maps. Let the map $H : Imm^q_n \to PIm^q_n$ be cutting a small neighborhood of a nonsingular point belonging to a given immersed n-manifold.

By a beautiful result [KoSa77, Main Theorem and Proposition 4.1] for $3p+q \le 3m-5$ the bottom line in Theorem 2.3 is isomorphic up to sign to the exact sequence

$$Emb^q_n \xrightarrow{E} Imm^q_n \xrightarrow{H} PIm^q_n \xrightarrow{P} Emb^{q-1}_{n-1}. \qquad (*)$$

Further replace the bottom line in Theorem 2.3 by its geometric form $(*)$. This form of the theorem in some sense reflects duality between link maps and immersions: link maps (resp., immersions) are not embeddings because they have self-intersections of 'close' (resp., 'distant') points.

Construction of the vertical homomorphisms. Remove a point from S^m and identify the result with $\mathbb{R}^m$. For a link map $f : X \sqcup Y \to \mathbb{R}^m$ define the map $\tilde{f} : X \times Y \to S^{m-1}$ by the formula

$$\tilde{f}(x,y) = \frac{fx - fy}{|fx - fy|}.$$

Denote also by $\mathrm{pr} : X \times Y \to Y$ the obvious projection.

Definition of α. Let $f : S^p \sqcup S^q \to \mathbb{R}^m$ be a general position smooth link map. Take a regular value $v \in S^{m-1}$ of the map $\tilde{f}$. Then $\tilde{f}^{-1}v$ is a framed manifold, and the map pr induces a framed immersion $\tilde{f}^{-1}v \to S^q$. Let $\alpha(f)$ be the class of this framed immersion in $Imm^q_{p+q+1-m}$.

Definition of λ. Take a link map $f : S^p \sqcup S^q \to \mathbb{R}^m$, whose restriction to S^p is an unknotted embedding. Since fS^p is unknotted, it follows that the complement $S^m - fS^p$ retracts to a small sphere S^{m-p-1} bounding a normal disc to fS^p. Thus by an appropriate link homotopy $\mathrm{rel}\, S^p$ one can put the image fS^q into the sphere S^{m-p-1}. Perform an ambient isotopy to make the sphere S^{m-p-1} standard. We may assume that after this isotopy fS^p is in general position. Take a regular value $v \in S^{m-p-1}$ of the map $\tilde{f}$. Then the map pr induces a framed embedding of $\tilde{f}^{-1}v$ to S^q. Let $\lambda(f)$ be the class of this framed embedding in $Emb^q_{p+q+1-m}$.

Definition of β. Take a proper disc link map $f : D^p \sqcup D^q \to \mathbb{R}^m_+$, where $\mathbb{R}^m_+$ is the upper semispace. By a proper link homotopy, restricting to an isotopy of ∂D^p, one can put the image $f\partial D^q$ into the standard sphere S^{m-p-1}. Assume that fD^q is in general position. Take a regular value $v \in S^{m-p-1}$ of $\tilde{f}$. Then the map pr induces a proper immersion of $\tilde{f}^{-1}v$ to D^q. Let $\beta(f)$ be the class of this proper immersion in $PIm^q_{p+q+1-m}$.

The commutativity up to sign in Theorem 2.3 in its geometric form is checked directly. The proof of Theorem 0.1 is completed.

Let us conclude §2 by a corollary of Theorem 2.3, which will be used later. Denote by $DM^m_{p,q}$ the group of proper link maps $D^p \sqcup D^q \to D^m$ whose restriction to ∂D^p is a smooth embedding (possibly knotted), up to proper link homotopy whose restriction to ∂D^p is a smooth isotopy.

Corollary 2.3'. *For $3p+4q \le 3m-5$ we have $DM^m_{p+q,q} \cong \pi_{p+2q+1-m}(V_{M+m-p-q-1,M}) \oplus K^{m-1}_{q-1}$.*

Proof. By [Hae66C, Theorem 2.4] it follows that $DM^m_{p+q,q} \cong \widehat{DM}^m_{p+q,q} \oplus K^{m-1}_{q-1}$. By Lemma 2.2, Theorem 2.3 and 5-lemma we have $\widehat{DM}^m_{p+q,q} \cong \pi_{p+2q+1-m}(V_{M+m-p-q-1,M})$. $\square$

Remark. Notice that link maps in the 2-metastable dimension were originally classified using the Haefliger classification of links, so the results of §2 without §1 *do not* give a shorter proof of Theorem 0.1.

3. Classification of knotted tori

3.1. Preliminaries.

Our approach to the classification of embeddings is based on an exact sequence (Theorem 3.1 below) reducing this problem to an easier classification of *almost embeddings*.

Informally, an *almost embedding* is a map admitting only 'local' self-intersections (see Figure 7). To give a formal definition, fix a codimension 0 ball B in a manifold M, where $B \cap \partial M = \emptyset$, if M has boundary (see Figure 1b). A smooth map $F : M \to N$ into a manifold N is an *almost embedding*, if the following two conditions hold:

(i) F is a smooth embedding outside B; and
(ii) $FB \cap F(M-B) = \emptyset$.

An *almost isotopy* is defined analogously, only the ball B is replaced by $B \times I$.

Figure 7 is approximately here

Denote by $KT^m_{p,q}$ the set of all smooth embeddings $S^p \times S^q \to S^m$ up to smooth isotopy. Denote by $\overline{KT}^m_{p,q}$ the set of all almost embeddings $S^p \times S^q \to S^m$ up to almost isotopy. By [Sko06] for $m \ge 2p+q+3$ the 'S^p-parametric connected sum' operation gives a natural group structure on the sets $KT^m_{p,q}$ and $\overline{KT}^m_{p,q}$ (see Figure 8).

Figure 8 approximately here

Now let us state the main theoretical result of §3.

Theorem 3.1. [cf. Kos88, Theorem A, HaKa98, Theorem IV] *For every $p+\frac{4}{3}q+2 \le m < p+\frac{3}{2}q+2$ and $m \ge 2p+q+3$ there is an exact sequence*

$$KT^m_{p,q} \to \overline{KT}^m_{p,q} \to \pi_{p+2q-m+1}(V_{M+m-p-q-1,M}) \oplus K^{m-1}_{p+q-1} \to KT^{m-1}_{p,q-1} \to \overline{KT}^{m-1}_{p,q-1} \to \dots$$

We are going first to prove Theorem 3.1 using the classification of proper almost embeddings (= disc link maps) $D^{p+q} \sqcup D^q \to D^m$ from §2, Corollary 2.3', and then to deduce Theorem 0.2 from Theorem 3.1 using classification of almost embeddings $S^p \times S^q \to S^m$.

3.2 Relation between embeddings and almost embeddings.

Theorem 3.1 follows from Corollary 2.3' above and Lemmas 3.2 and 3.3, which we are going to state now.

A smooth map $f : M \to N$ is *proper*, if $f^{-1}\partial M = \partial N$ and fM is transversal to ∂N. Denote by $\overline{PT}^m_{p,q}$ the group of proper almost embeddings $S^p \times D^q \to D^m$ up to proper almost isotopy (see Figure 9). Fix a codimension 0 ball $B \subset D^{p+q}$ and denote by $DM^m_{p+q,q}$ the group of proper almost embeddings $D^{p+q} \sqcup D^q \to D^m$ up to proper almost isotopy.

Figure 9 approximately here

Lemma 3.2. *(cf. Theorem 2.1 and Figure 6 above) For $m \ge 2p+q+3$ there is an exact sequence*

$$KT^m_{p,q} \xrightarrow{e} \overline{KT}^m_{p,q} \xrightarrow{h} \overline{PT}^m_{p,q} \xrightarrow{p} KT^{m-1}_{p,q-1} \to \dots$$

Lemma 3.3. *For $m \ge 2p+q+3$ we have $\overline{PT}^m_{p,q} \cong DM^m_{p+q,q}$.*

The proofs of both Lemmas 3.2 and 3.3 are based on the following notion.

Definition of the web D^{p+1}. Mark a point $* \in S^q$. A *web* of an almost embedding $f : S^p \times S^q \to S^m$ is a framed disc $D^{p+1} \subset S^m$ satisfying the following 3 conditions:

(i) $\partial D^{p+1} = f(S^p \times *)$;
(ii) $\operatorname{Int} D^{p+1} \cap \operatorname{Im} f = \emptyset$; and
(iii) the first q vector fields of the framing of ∂D^{p+1} form the standard framing of $f(S^p \times *)$ in $f(S^p \times S^q)$.

A *web* of an almost isotopy $f_t : S^p \times S^q \to S^m$ and of a proper almost embedding $f : S^p \times D^q \to D^m$ is defined analogously.

Proposition 3.4. *(A. Skopenkov) If $m \geq 2p+q+2$ then for any almost embedding $f : S^p \times S^q \to S^m$ there exist a web. If $m \geq 2p+q+3$, then for any almost isotopy $f_t : S^p \times S^q \to S^m$ there exists a web extending given webs of f_0 and f_1.*

Proof. (A. Skopenkov) The bundle $\nu(f)|_{S^p\times *}$ is stably trivial and $m-p-q \geq p$, hence this bundle is trivial. Take a $(m-p-q)$-framing ξ of this bundle.

Take the section formed by the first vectors of ξ. Since $m \geq 2p+q+2 \geq 2p+2$, it follows that $f|_{S^p\times *}$ is unknotted in S^m. So there is an embedding $\bar{f} : D^{p+1} \subset S^m$ satisfying property (i) from the Definition of the web above. Since $m \geq 2p+q+2$, by general position we may assume also property (ii).

By deleting the first vector from ξ we obtain a $(m-p-q-1)$-framing ξ_1 on $\bar{f}(\partial D^{p+1})$ normal to $\bar{f}(D^{p+1})$. Denote by η the standard normal q-framing of $f(S^p \times *)$ in $f(S^p \times S^q)$. Then (ξ_1, η) is a normal $(m-p-1)$-framing on $\bar{f}(\partial D^{p+1})$ normal to $\bar{f}(D^{p+1})$. Since $p < m-p-q-1$, the map $\pi_p(SO_{m-p-q-1}) \to \pi_p(SO_{m-p-1})$ is epimorphic. Hence we can change ξ_1 (and thus ξ) so that (ξ_1, η) extends to a normal framing on $\bar{f}(D^{p+1})$. By construction it satisfies property (iii).

The second assertion is proved analogously [cf. Sko06, Proof of Standardization Lemma 2.1 in §3] $\square$

An important consequence of this proposition is the group structure on the set of knotted tori [Sko06']. This also allows us to prove Lemma 3.2 analogously to Theorem 2.1. We need the following proposition.

Proposition 3.5. *For each $m \geq 2p+q+3$ all proper embeddings $S^p \times D^q \to D^m$ are properly ambient isotopic, and all proper almost embeddings $S^p \times D^q \to D^m$ are properly almost isotopic.*

Proof. Take a proper embedding $f : S^p \times D^q \to D^m$. Let $* \in \operatorname{Int} D^q$ be the marked point. Take a web D^{p+1} of f. Let $\bar{D}^m$ be the tubular neighborhood of D^{p+1}. Clearly, the restriction $f : f^{-1}\bar{D}^m \to \bar{D}^m$ is isotopic to standard embedding $S^p \times D^q \to D^m$. The restriction $f : (S^p \times D^q - f^{-1}\bar{D}^m) \to (D^m - \bar{D}^m)$ can be thought as a concordance between the restriction of f to the boundary and the standard embedding. Since a concordance is ambient isotopic to an isotopy in codimension at least 3, we may assume that this restriction is level-preserving. Then by the Alexander trick the embedding f is ambient isotopic to the trivial embedding.

The second assertion is similarly deduced from almost concordance implies almost isotopy in codimesion at least 3 (proved similarly to [Mel]). $\square$

Proof of Lemma 3.2. (cf. Proof of Theorem 2.1 in §2) *(1) Construction of the homomorphisms.* Let e be the obvious map. Let p be the 'restriction to the boundary' map. The homomorphism h is the 'cutting' map defined as follows. Take an almost embedding $f : S^p \times S^q \to S^m$. By Proposition 3.4 there exist a web $D^{p+1} \subset S^m$. Let $\bar{D}^m$ be a tubular neighborhood of D^{p+1}. Set $h(f)$ to be the restriction of f to a map $(S^p \times S^q - f^{-1} \operatorname{Int} D^m) \to S^m - \operatorname{Int} D^m$.

(2) Exactness at $KT^m_{p,q}$. The sequence is exact at $KT^m_{p,q}$ because an embedding $f : S^p \times S^q \to S^m$ extends to a proper almost embedding $S^p \times D^{q+1} \to D^{m+1}$ if and only if f is almost isotopic to the standard embedding (cf. Proposition 3.5).

(3) Exactness at $\overline{PT}^m_{p,q}$. The sequence is exact at $\overline{PT}^m_{p,q}$ because a proper almost embedding $f : S^p \times D^q \to D^m$ extends without adding new self-intersections to an almost embedding $S^p \times S^q \to S^m$ if and only if the restriction of f to the boundary is isotopic to the standard embedding (by Proposition 3.5).

(4) Exactness at $\overline{KT}^m_{p,q}$. The inclusion $\operatorname{Im} e \subset \operatorname{Ker} h$ follows from Proposition 3.5. To prove $\operatorname{Ker} h \subset \operatorname{Im} e$, take $f \in \overline{KT}^m_{p,q}$ such that $h(f) = 0$. Then, by definition, there exist a proper almost isotopy h_t between $h(f)$ and the standard proper embedding. By the isotopy extension theorem [Hud69] the restriction of h_t to the boundary extends to an ambient isotopy of $S^m - D^m$. So h_t can be extended to an almost isotopy of f without adding new self-intersections. The latter is an almost isotopy between f and an embedding $f' \in KT^m_{p,q}$. Hence $f = e(f')$. $\square$

Now we proceed to the proof of Lemma 3.3. The proof is based on surgery over the torus $S^p \times S^q$ along the meridian. To make our approach more clear, let us first give an easier example of using this surgery.

Proposition 3.6. *For $m \geq 2p+q+3$ the natural map $K^m_{p+q} \to KT^m_{p,q}$ is injective.*

In fact this proposition implies case '$p+q+1$ divisible by 4' of Theorem 0.2, by Haefliger Theorem in §0. The restriction $m \geq 2p+q+3$ is essential, e. g., Proposition 3.6 is not true for the natural action $K^6_3 \to KT^6_{1,2}$ [Sko06].

Proof of Proposition 3.6. Let us define the natural map $\xi : K^m_{p+q} \to KT^m_{p,q}$. To a knot $f : S^{p+q} \to S^m$ assign the connected sum of f and the standard embedding $S^p \times S^q \to S^m$. (The connected sum is made along an arc l joining the images of f and the standard embedding). To prove the proposition it suffices to construct a left inverse $\xi' : KT^m_{p,q} \to K^m_{p+q}$ of ξ.

The map $\xi' : KT^m_{p,q} \to K^m_{p+q}$ is defined as follows. Take an embedding $f : S^p \times S^q \to S^m$. By Proposition 3.4 it admits a web $D^{p+1} \subset S^m$. The boundary sphere of this web is endowed with the standard framing in $f(S^p \times S^q)$ by property (iii) from Definition of the web. Perform an embedded surgery along this framed sphere inside a tubular neighborhood of the web. Let $\xi'(f) : S^{p+q} \to S^m$ be the map obtained by the surgery.

The element $\xi'(f)$ is well-defined by the second assertion of Proposition 3.4. Indeed, assume that $f_0 : S^p \times S^q$ and $f_1 : S^p \times S^q \to S^m$ are concordant and two webs D^{p+1}_0 and D^{p+1}_1 are chosen. Take the web D^{p+2} of the concordance

f_t given by Proposition 3.4. Surgery along ∂D^{p+2} transforms the concordance f_t a concordance between $\xi'(f_0)$ and $\xi'(f_1)$.

To prove $\xi' \circ \xi = id$, notice that generically the web D^{p+1} misses the arc l. Then there is an obvious ambient isotopy joining f and $\xi' \circ \xi(f)$ (cf. Proposition 3.7 below). $\square$

Lemma 3.3 is proved by certain relative version of this surgery. The crucial point of the above argument is the existence of an ambient isotopy between f and $\xi' \circ \xi(f)$. Further instead of an explicit construction of required ambient isotopies we are going to use the following easy general result. We say that a ball $\bar{D}^k \subset D^k$ is *admissible*, if $\bar{D}^k \cap \partial D^k$ is a $(k-1)$-ball and $\partial \bar{D}^k - \partial D^k$ is transversal to ∂D^k (thus $\bar{D}^k$ is a smooth manifold with corners).

Proposition 3.7. *Let $\bar{D}^k \subset D^k$ be an admissible ball and $D^{k-1} = \bar{D}^k \cap \partial D^k$. If $m \geq n+3$, then any two proper embeddings $f, g : (\bar{D}^n, D^{n-1}) \to (\bar{D}^m, D^{m-1})$, which coincide on $\partial \bar{D}^n - D^{n-1}$, are ambient isotopic relatively $\partial \bar{D}^n - D^{n-1}$.*

Proof. The map $f : \bar{D}^n \to \bar{D}^m$ can be considered as a concordance relatively the boundary between the two embeddings $f : D^{n-1} \to D^{m-1}$ and $f : \bar{D}^n - D^{n-1} \to \bar{D}^m - D^{m-1}$. Since concordance implies ambient isotopy in codimension at least 3, it follows that f 'extends' to an ambient isotopy $h_f : \bar{D}^m \to \bar{D}^m$ fixed on $\bar{D}^m - D^{m-1}$. Define analogously an ambient isotopy $h_g : \bar{D}^m \to \bar{D}^m$. Then $g = h_g h_f^{-1} f$. $\square$

Proof of Lemma 3.3. To prove this lemma, we construct two mutually inverse homomorphisms $h_1 : \overline{PT}^m_{p,q} \to DM^m_{p+q,q}$ and $h_2 : DM^m_{p+q,q} \to \overline{PT}^m_{p,q}$. Fix a point $* \in S^p$.

(1) Construction of a homomorphism $h_1 : \overline{PT}^m_{p,q} \to DM^m_{p+q,q}$. Take an arbitrary map $f \in \overline{PT}^m_{p,q}$. Take a nonzero vector field on $* \times D^q$ normal to $f(S^p \times D^q)$. Moving the disc $* \times D^q$ toward this vector field, we get a proper embedding $f_1 : D^q \to D^m$ missing $f(S^p \times D^q)$.

By Proposition 3.4 the restriction of f to the boundary admits a web $\bar{f} : D^p \to \partial D^m$, i. e. there is a framed embedding $\bar{f} : D^p \to \partial D^m$ satisfying properties (i)–(iii) from Definition of the web. Generically for $m \geq 2p+q+3$ this web misses $f_1(\partial D^q)$. Perform an embedded surgery of the map $f : S^p \times D^q \to D^m$ along the sphere $\bar{f}\partial D^p$ inside a tubular neighborhood of $\bar{f}D^p$. Let $f_2 : D^{p+q} \to D^m$ be the proper map obtained by the surgery. By definition, put $h_1(f) = f_1 \sqcup f_2 \in DM^m_{p+q,q}$.

(2) Construction of a homomorphism $h_2 : DM^m_{p+q,q} \to \overline{PT}^m_{p,q}$. Take an arbitrary map $f \in DM^m_{p+q,q}$. Extend the restriction $f : D^q \to D^m$ to a torus $f_1 : S^p \times D^q \to D^m$ with the image inside a tubular neighborhood of fD^q. Join a point $x \in f\partial D^{p+q}$ with a point $y \in f_1(S^p \times \partial D^q)$ by an arc $l \subset \partial D^m$ missing the images of $\partial D^{p+q} - \{x\}$ and $S^p \times \partial D^q - \{y\}$. Perform an embedded surgery of $f : D^{p+q} \to D^m$ and $f_1 : S^p \times D^q \to D^m$ along the 0-sphere ∂l inside a tubular neighborhood of l (i. e., make a connected sum of fD^{p+q} and $f_1(S^p \times D^q)$). Denote by $h_2(f) \in \overline{PT}^m_{p,q}$ the map obtained by the surgery.

The homomorphisms h_1 and h_2 are well-defined by the second assertion of Proposition 3.4.

(3) Proof that $h_1 \circ h_2 = id$. Let us prove that any map $f \in DM^m_{p+q,q}$ is ambient isotopic to $h_1 \circ h_2(f)$. It suffices to construct 2 admissible balls $\bar{D}^{p+q} \subset D^{p+q}$ and $\bar{D}^m \subset D^m$ satisfying the following 2 properties:
(i) $f^{-1}\bar{D}^m = h_1 \circ h_2(f)^{-1}\bar{D}^m = \bar{D}^{p+q}$;
(ii) $f = h_1 \circ h_2(f)$ outside $\bar{D}^{p+q}$.

Then one can combine f and $h_1 \circ h_2(f)$ on D^{p+q} by the ambient isotopy given by Proposition 3.7. Moving $h_1 \circ h_2(f)D^q$ backwards along the vector field constructed in step (1), we combine the maps f and $h_1 \circ h_2(f)$.

Construction of the balls $\bar{D}^p$ and $\bar{D}^m$. We may assume that the web $\bar{f}D^p$ of the restriction of $h_1(f)$ to the boundary is contained in a tubular neighborhood of fD^q. The required ball $\bar{D}^m$ is obtained from the union of tubular neighborhoods of fD^q and the arc l constructed in step (2) by cutting off a tubular neighborhood of the web $\bar{f}D^p$. Put $\bar{D}^{p+q} = f^{-1}\bar{D}^m$. By the construction $\bar{D}^{p+q} \subset D^{p+q}$ and $\bar{D}^m \subset D^m$ are admissible balls satisfying (i) and (ii).

(4) Proof that $h_2 \circ h_1 = id$. Let us show that any map $f \in \overline{PT}^m_{p,q}$ is ambient isotopic to $h_2 \circ h_1(f)$. Again it suffices to construct 2 admissible balls $\bar{D}^{p+q} \subset D^{p+q}$ and $\bar{D}^m \subset D^m$ satisfying the properties (i) and (ii) from step (3).

Construction of the balls $\bar{D}^p$ and $\bar{D}^m$. Let U be a tubular neighborhood of $f(* \times D^q)$. We may assume that the arc l constructed in step (2) is contained in U. Let $z \in D^q$ be the point such that $\partial l \cap f(S^p \times D^q) = f(* \times z)$. Modify $h_2 \circ h_1(f)$ by an ambient isotopy inside U to make $h_2 \circ h_1(f) = f$ in some tubular neighborhood V of the wedge $* \times D^q \cup S^p \times z$. Now the maps f and $h_2 \circ h_1(f)$ coinside outside the union of $Cl(U-V)$ and a tubular neighborhood W of the web $\bar{f}D^p$. The balls $\bar{D}^m = Cl(U-V) \cup W$ and $\bar{D}^{p+q} = f^{-1}\bar{D}^m$ are the required. $\square$

3.3 Classification of almost embeddings.

To prove Theorem 0.2 we need the classification of almost embeddings $S^p \times S^q \to S^m$. Denote by $FK^m_{p,q}$ the set of smooth embeddings $D^p \times S^q \to S^m$ up to smooth isotopy. This set admits a natural commutative group structure analogous to $KT^m_{p,q}$. The classification of almost embeddings is given by the following result.

Theorem 3.8. *For $m \geq 2p+q+3$ there exist exact sequences*

$$\begin{array}{llllllll} (i) & \pi_{p+q}(S^{m-q-1}) & \to & \overline{KT}^m_{p,q} & \to & FK^m_{p,q} & \to & \pi_{p+q-1}(S^{m-q-1}) \to \dots \\ (ii) & \pi_q(V_{m-q,p}) & \to & FK^m_{p,q} & \to & K^m_q & \to & \pi_{q-1}(V_{m-q,p}) \to \dots \end{array}$$

Theorem 3.8 (ii) is proved by direct checking analogously to [Hae66A, Corollary 5.9]. Theorem 3.8 (i) is due to A. Skopenkov [cf. Sko06, Restriction Lemma 5.2]. Our proof of assertion (i) consist of several steps similar to the proof of Theorem 3.1.

Proof of assertion (i) in 3.8. (1) Definition of the groups $FK^m_{p,q}$ and $PT^m_{p,q}$. A smooth map is $S^p \times S^q \to S^m$ said to be a *weak almost embedding*, if it is a smooth embedding outside $B \subset S^p \times S^q$. A *weak almost isotopy* is defined analogously. Identify $FK^m_{p,q}$ with the group of weak almost embeddings up to weak almost isotopy (clearly, these groups are isomorphic for $m \geq 2p+q+3$).

Fix a $(p+q)$-ball $B \subset S^p \times D^q$ meeting the boundary transversely by a $(p+q-1)$-ball. A proper piecewise linear map $f : S^p \times D^q \to D^m$ is said to be a *proper weak almost embedding*, if the following two conditions hold:
(i) f is a smooth embedding outside $B \subset S^p \times D^q$; and
(ii) $f(S^p \times \partial D^q \cap B) \cap F(S^p \times \partial D^q - B) = \emptyset$.
A *proper weak almost isotopy* is defined analogously. Denote by $PT^m_{p,q}$ the group of proper weak almost embeddings up to proper weak almost isotopy.

(2) For every $m \geq 2p+q+3$ there exist an exact sequence:

$$\overline{KT}^m_{p,q} \xrightarrow{e} FK^m_{p,q} \xrightarrow{h} PT^m_{p,q} \xrightarrow{p} \overline{KT}^{m-1}_{p,q-1} \to \dots$$

Here e, h and p are the obvious forgetful, cutting and restriction homomorphisms respectively. This assertion is proved completely analogously to Lemma 3.2.

(3) Definition of the homomorphism $\lambda : PT^m_{p,q} \to \pi_{p+q-1}(S^{m-q-1})$. Take a proper weak almost embedding $f : S^p \times D^q \to D^m$. By definition $f\partial B \cap f(* \times D^q) = \emptyset$. Notice that $D^m - f(* \times D^q) \simeq S^{m-q-1}$. Let $\lambda(f)$ be the homotopy class of the restriction $f : \partial B \to D^m - f(* \times D^q)$.

(4) λ is injective. Take a proper weak almost embedding $f : S^p \times D^q \to D^m$ such that $\lambda(f) = 0$. Then $f|_{\partial B}$ extends to a smooth map $g : B \to D^m$ missing $f(* \times D^q)$. Generically for $m \geq 2p+q+3$ the map g misses $f(S^p \times 0)$. Thus we may assume that g misses $f(S^p \times D^q - B)$. Perform a proper weak almost isotopy which replaces $f|_B$ by g. Thus we get a map $S^p \times D^q \to D^m$ which is a proper almost isotopy. Then by Proposition 3.5 f is properly weak almost isotopic to the standard proper embedding $S^p \times D^q \to D^m$.

(5) λ is surjective. Take an element $x \in \pi_{p+q-1}(S^{m-q-1})$. Take the standard proper embedding $f : S^p \times D^q \to D^m$. Realize the element x by a smooth map $g : S^{p+q-1} \to \partial D^m - f(* \times D^q)$. Without loss of generality for $m \geq 2p+q+3$ we may assume that g misses $f(S^p \times \partial D^q - B)$. Extend the map g to a smooth map $g' : D^{p+q} \to D^m$. Let $\mu(x)$ be the connected sum (relatively the boundary) of g' and $f|_{S^p \times D^q - B}$. Clearly, $\lambda(\mu(x)) = x$. This completes the proof of assertion (i). $\square$

3.4 Rational classification of knotted tori.

In order to prove Theorem 0.2 we need to know which groups from Theorems 3.1 and 3.8 are finite.

Theorem 3.9. *Assume that $p + \frac{4}{3}q + 2 \leq m < p + \frac{3}{2}q + 2$, $m \geq 2p+q+3$ and $m \geq n+3$. Then*
(i) K^m_n is infinite if and only if $2m \leq 3n+3$, $n+1$ is divisible by 4.
(ii) $\pi_{p+q}(S^{m-q-1})$ is infinite if and only if $m = \frac{p}{2} + \frac{3}{2}q + \frac{3}{2}$, $p+q+1$ is divisible by 4.
(iii) $\pi_q(V_{m-q,p})$ is infinite if and only if $p \geq 1$, $\frac{3}{2}q + \frac{3}{2} \leq m \leq p + \frac{3}{2}q + \frac{1}{2}$ and $q+1$ is divisible by 4.
(iv) $\pi_{p+2q-m+2}(V_{M+m-p-q-1,M})$ is infinite if and only if $m = p + \frac{3}{2}q + \frac{3}{2}$ and $q+1$ is divisible by 4.

Theorem 3.9 can be easily reduced to known results. Assertion (i) is the Haefliger theorem [Hae66A, Corollary 6.7]. Assertion (ii) of is a specific case of well-known Serre theorem. Assertions (iii) and (iv) are proved using the exact homotopy sequence of the 'forgetting the last vector' bundle $S^{m-p-q} \to V_{m-q,p} \to V_{m-q,p-1}$:

Proof of assertion (iii) in Theorem 3.9. The assumptions $m \geq 2p+q+3$ and $m < p + \frac{3}{2}q + 2$ together imply that $m \leq 2q$. We are going to prove assertion (iii) by induction over p under the only assumption $m \leq 2q$.

(1) Case $q+1$ not divisible by 4. Since $m \leq 2q$, it follows that $\pi_q(V_{m-q,1}) \cong \pi_q(S^{m-q-1})$ is finite. Using the homotopy exact sequence of the 'forgetting the last vector' bundle $S^{m-p-q} \to V_{m-q,p} \to V_{m-q,p-1}$ tensored by $\mathbb{Q}$, we get inductively that $\pi_q(V_{m-q,p})$ is finite.

(2) Case $q+1$ divisible by 4, and either $m < \frac{3}{2}q + \frac{3}{2}$ or $m > p + \frac{3}{2}q + \frac{1}{2}$. In this case the groups $\pi_q(S^{m-q-i})$ are still finite for each $i = 1, 2, \dots, p$. Similarly to the above we get $\pi_q(V_{m-q,p})$ finite.

(3) Case $q+1$ divisible by 4, and $\frac{3}{2}q + \frac{3}{2} \leq m \leq p + \frac{3}{2}q + \frac{1}{2}$. Take i such that $m = i + \frac{3}{2}q + \frac{1}{2}$. Consider the above exact homotopy sequence for $p = i$. Analogously to the above it can be shown that for $q+1$ divisible by 4 and $m \leq 2q$ the group $\pi_{q+1}(V_{m-q,i-1})$ is finite. Thus the group $\pi_q(V_{m-q,i})$ is infinite. By induction $\pi_q(V_{m-q,p})$ is also infinite. $\square$

Proof of the assertion (iv) in Theorem 3.9. Denote by $s = 2p+3q-2m+3$ and $l = m-p-q-1$. Then the group in question is $\pi_{s+l}(V_{M+l,M})$. Our restriction $p+\frac{4}{3}q+2 \le m < p+\frac{3}{2}q+2$ is equivalent to the restriction $0 \le s \le l-2$.

(1) Case $s=0$. By [Pae56] the group $\pi_l(V_{M+l,M})$ is infinite if and only if $l+1$ is divisible by 2. Together with condition $s=0$ this is equivalent to the conditions $m = p+\frac{3}{2}q+\frac{3}{2}$ and $q+1$ divisible by 4.

(2) Case $s \ge 1$. Let us prove by induction over s that the group $\pi_{s+l}(V_{M+l,M})$ is finite in this case provided that $s \le l-2$. Indeed, for $s=1$ this is proved in [Pae56]. For $s>1$ consider the homotopy exact sequence of the 'forgetting the last vector' bundle $S^l \to V_{M+l,M} \to V_{M+l,M-1}$ tensored by $\mathbb{Q}$. In this sequence $\pi_{s+l}(S^l)$ is finite because $1 \le s \le l-2$. By inductive hypothesis the group $\pi_{l+s}(V_{M+l,M-1}) \cong \pi_{s-1+l+1}(V_{M+l+1,M})$ is also finite. So is the group $\pi_{s+l}(V_{M+l,M})$. $\square$

The dimension restrictions in Theorem 3.9 are assumed to simplify both the statement and the proof. Similarly, the upper bound for the dimension m in Theorem 0.2 is not essential, it is assumed by ethtetic reasons (to simplify the statement).

To prove Theorem 0.2 consider the following 5 cases:

(1) $p+q+1$ is divisible by 4;
(2) $q+1$ is divisible by 4, $m \le \frac{3}{2}q+\frac{3}{2}$, $p \ge 1$;
(3) $q+1$ is divisible by 4, $\frac{3}{2}q+\frac{3}{2} \le m \le p+\frac{3}{2}q+\frac{1}{2}$, $p \ge 1$;
(4) $q+1$ is divisible by 4, $m = p+\frac{3}{2}q+\frac{3}{2}$, $p \ge 1$; and
(5) $q+1, p+q+1$ are not divisible by 4.

Here case (1) is easy, it follows directly from Proposition 3.6. Cases (2) and (3) are also not hard, they can be treated using Theorem 3.8(ii) only. Cases (4) and (5) are more difficult, they require both Theorems 3.1 and 3.8. Case (4) was treated in [Sko06'] by different methods.

Proof of Theorem 0.2. (1) Case $p+q+1$ *is divisible by* 4. This follows directly from Proposition 3.6.

(2) Case $q+1$ *divisible by* 4, $m \le \frac{3}{2}q+\frac{3}{2}$. It suffices to construct a knotted torus $H : S^p \times S^q \to S^m$ having an infinite order in $KT^m_{p,q}$.

Construction of the Haefliger torus H. The group K^m_q is infinite in this case. Take an infinite order element x. The obstruction to existence of a $(p+1)$-frame on the knot x belongs to the finite group $\pi_{q-1}(V_{m-q,p+1})$. So for some $N \in \mathbb{N}$ the embedding Nx extends to the desired smooth embedding $H : S^p \times S^q \to S^m$. Clearly, H has an infinite order.

(3) Case $q+1$ *divisible by* 4, $\frac{3}{2}q+\frac{3}{2} \le m \le p+\frac{3}{2}q+\frac{1}{2}$, $p \ge 1$. It suffices to construct a knotted torus $T : S^p \times S^q \to S^m$ having an infinite order in $KT^m_{p,q}$.

Construction of the torus T. The group $\pi_q(V_{m-q,p})$ is infinite in this case. Take an infinite order element x of this group. Consider the map $\tau : \pi_q(V_{m-q,p}) \to FK^m_{p,q}$ from 3.8(ii). This map takes the element x to the canonical p-frame $D^p \times S^q \to S^m$ of the standard sphere $S^q \subset S^m$. The complete obstruction to extension of this p-frame to a $(p+1)$-frame belongs to $\pi_{q-1}(S^{m-p-q-1})$. The latter group is finite in our case. So for some $N \in \mathbb{N}$ the element $N\tau(x)$ can be extended to a smooth embedding $S^p \times S^q \to S^m$, which is the desired torus T.

Proof that T *has an infinite order.* It suffices to prove that the element $\tau(x) \in FK^m_{p,q}$, which is the restriction of T to $D^p \times S^q$, has an infinite order. Assume the converse. Then $N\tau(x) = 0$ for some $N \in \mathbb{N}$. So by Theorem 3.8(ii) Nx belongs to the image of the map $K^{m+1}_{q+1} \to \pi_q(V_{m-q,p})$. But the group K^{m+1}_{q+1} is finite in our case. This contradiction proves that T has infinite order.

(4) Case $q+1$ *divisible by* 4, $m = p+\frac{3}{2}q+\frac{3}{2}$.

Construction of the Whitehead torus W. The group $\pi_{p+2q-m+2}(V_{M+m-p-q-1,M})$ is infinite in this case. Take an infinite order element x of this group. Take the smooth embedding $\omega(x,0) : S^p \times S^q \to S^m$, where $\omega : \pi_{p+2q-m+2}(V_{M+m-p-q-1,M}) \oplus K^m_{p+q} \to KT^m_{p,q}$ is the map from Theorem 3.1.

Proof that W *has an infinite order.* By Theorem 3.1 it suffices to prove that $\overline{KT}^{m+1}_{p,q+1}$ is finite in our case.

Since $q+1$ is divisible by 4 it follows that $\pi_{q+1}(V_{m-q,p})$ and K^{m+1}_{q+1} are finite. The group $\pi_{p+q+1}(S^{m-q-1})$ is also finite in our case. So by Theorem 3.8 it follows that $\overline{KT}^{m+1}_{p,q+1}$ is finite.

(5) Case $q+1, p+q+1$ *not divisible by* 4. Recall that if $X \to Y \to Z$ is an exact sequence with finite X and Z, then Y is also finite. Applying this 2 times to the first 3 columns of Theorem 3.8 starting from the bottom, and then to the sequence from Theorem 3.1, we are done, because the groups in the first column of 3.8 and the groups $\pi_{p+2q-m+2}(V_{M+m-p-q-1,M})$, K^m_q and K^m_{p+q} are finite for $q+1, p+q+1$ not divisible by 4. $\square$

4. Concluding remarks

(i) The argument of [Hae66C] can be extended to cover at least the dimension range $2p_1+2p_2 \le 3m-7$ (we use the notation of [Hae66C] in this paragraph). Indeed, this restriction is sufficient in all steps of the Haefliger proof except [Hae66C, Proposition 10.2]. The groups $\Lambda^{(q)}_{(p_1)}$ and $\Pi^{(q)}_{m-2}$ in the proof of [Hae66C, Proposition 10.2] may have now a larger number of generators. But these generators can be written explicitly: the group $\Lambda^{(q)}_{p_1}$ is generated by all $\theta_k(i_1,i_2)$ for all integers $k\ge 0$ such that $kp_1+p_2\ge(k+1)(m-2)$, and the group $\Pi^{(q)}_{m-2}$ is generated by $[[i_2,i_1],i_2]$ and all $\theta_{k+1}(i_1,i_2)$ for the same k. Thus the proof of [Hae66C, Proposition 10.2] can be completed by the initial argument. A possible reason why Haefliger did not notice this improvement was that in contrast to $3p_1+p_2\le 3m-7$ the restriction $2p_1+2p_2\le 3m-7$ did not appear in his theory.

(ii) The dimension restriction $2p+2q\le 3m-6$ in Theorem 0.1 is best possible, the formula fails for $2p+2q=3m-5$. For example, take $m=p+4=4k-1$, $k\ge 5$, $q=2k+2$. Then the group $LM^{m+1}_{p+1,q+1}$ is infinite [Kos90, p. 755-756]. By Theorem 2.3 the rank of the group $\overline{DM}^{m+1}_{p+1,q+1}$ is greater than the rank of the group $\pi_{p+q+2-m}(V_{M+m-p-1,M})$. By Lemma 2.2 the natural homomorphism $\widehat{DM}^{m+1}_{p+1,q+1}\to\overline{DM}^{m+1}_{p+1,q+1}$ is still surjective. So the rank of the group $DM^{m+1}_{p+1,q+1}$ is also greater than the rank of the group $\pi_{p+q+2-m}(V_{M+m-p-1,M})$. Thus by Corollary 2.1' the rank of left-hand side in the formula of Theorem 0.1 is greater than the rank of right-hand side.

(iii) A natural question is to describe the kernel of the suspension map $\Sigma: LM^m_{p,q}\to LM^{m+1}_{p+1,q}$ at the boundary range $2p+2q=3m-4$. This may help to determine which groups $LM^m_{p,q}$ are finite in codimension 3.

(iv) It is interesting to weaken the dimension restrictions in the Theorem 0.2. As we have remarked before, the restriction $m<p+\frac{3}{2}q+2$ in Theorem 0.2 is only ethtetic. The 2-metastable restriction $m\ge p+\frac{4}{3}q+2$ is essential, but our methods provide much information outside this dimension range. Indeed, we have a collection of exact sequences in some sense reducing the classification of knotted tori to the classification of links and knots (see Lemmas 3.1 and 3.2, Theorems 2.1 and 3.8). Besides, the 'group structure' restriction $m\ge 2p+q+3$ in Theorem 0.2 is a natural limit for our approach. Probably it can be eliminated in piecewise linear category. For example, we conjecture that the set of piecewise linear embeddings $S^p\times S^q\to S^m$ up to piecewise linear isotopy admits a natural group structure in codimension at least 3.

Appendix. Surgery over the self-intersection manifold

Here we do surgery on the self-intersection manifold, which provides an alternative proof of our main results and hopefully is interesting in itself. Roughly speaking, this alternative argument replaces engulfing from §1 by surgery. None of the results proved here are used in the rest of the paper.

Let us state our particular problem and the results. Let $f: D^q\to M^m$ be a general position proper immersion whose restriction to the boundary is an embedding. The Embedding Theorem relatively boundary [Hud69, Theorem 10.2] allows to remove the self-intersection of f by a homotopy rel ∂D^q under certain conditions. In the dimension range where the embedding theorem is not true, we give an approach to simplify the self-intersection in some sense.

Consider the diagram

$$\begin{array}{ccc}\tilde\Delta(f) & \xrightarrow{\tilde i} & D^q\\ {\scriptstyle 2:1}\downarrow & & \downarrow{\scriptstyle f}\\ \Delta(f) & \xrightarrow{i} & M^m\end{array}$$

Here $\tilde\Delta(f)=\mathrm{Cl}\{\,(x,y)\in D^q\times D^q\,|\,x\ne y, fx=fy\,\}$ and $\Delta(f)=\tilde\Delta(f)/\mathbb{Z}_2$ are the *double point manifolds*. The immersions $\tilde i:\tilde\Delta(f)\to D^q$ and $i:\Delta(f)\to M^m$ are given by the formulas $\tilde i(x,y)=x$ and $i\{x,y\}=fx$. Denote by $\lambda(f)$ the line bundle associated with the double covering $\tilde\Delta(f)\to\Delta(f)$. Denote by $\bar\lambda(f):\Delta(f)\to P^\infty$ the map classifying the line bundle $\lambda(f)$.

The main results of Appendix are the following surgery theorems due to M. Cencelj and D. Repovs:

Theorem 5.1. [cf. HaKa98, Theorem 4.5] *Let M^m be an $(s+1)$-connected manifold. Let $f: D^q\to M^m$ be a proper self-transverse immersion such that $f|_{\partial D^q}$ is an embedding. Suppose that $2s\le 2q-m-2$ and $s\le m-q-3$. Then by a regular homotopy of f rel ∂D^q the classifying map $\Delta\to P^\infty$ can be made $(s+1)$-connected.*

Corollary 5.2. [cf. HaKa98, Corollary 4.4] *Assume that $m\ge p+\frac{4}{3}q+2$. Then any proper almost embedding F is properly almost isotopic to a proper almost embedding F' such that*

$$\pi_q(D^m-F'D^{p+q},\partial)\cong\Omega^m_{p,q}.$$

This isomorphism is given by the formula $G\mapsto\beta(F',G)$.

Here F is a proper almost embedding $F: D^{p+q}\sqcup D^q\to D^m$ (see §3), $\Omega^m_{p,q}\cong\pi_{p+2q-m+1}(V_{M+m-p-q-1,M})$ and the element $\beta(F',G)$ is the β-invariant of the proper link map $F'|_{D^{p+q}}\sqcup G: D^{p+q}\sqcup D^q\to D^m$.

First let us give an alternative proof of Corollary 2.3' modulo these results (our arguments in §§2–3 show that this corollary implies Theorems 0.1 and 0.2), then we prove Theorem 5.1 and Corollary 5.2 themselves.

An alternative proof of Corollary 2.3' modulo Corollary 5.2. It suffices to prove that $\beta:\overline{DM}^m_{p+q,q}\to\Omega^m_{p,q}$ is bijective.

The injectivity of β. Take $f\in\overline{DM}^m_{p+q,q}$ such that $\beta(f)=0$. By Corollary 5.2 one can assume that $f|_{D^q}$ is null-homotopic outside fD^{p+q}. Perform the following 2-step link homotopy:

(i) Modify $f|_{D^q}$ to an embedding into ∂D^m.
(ii) (Conic construction) Replace $f|_{D^{p+q}}$ by the 'conic' homeomorphism onto $C(f\partial D^{p+q})$. This is done via a 'rectilinear' link homotopy.
So f becomes a piecewise linear embedding. Thus it represents 0 in $\overline{DM}^m_{p,q}$ (by §2, proof that $\operatorname{Im} e \subset \operatorname{Ker} h$ in Theorem 3.1).

The surjectivity of β. Take any $v \in \pi_{p+2q+1-m}(V_{M+m-p-q-1,M})$. Start with any $f \in \overline{DM}^m_{p+q,q}$. By Corollary 5.2 there exist a proper link map $f' : D^{p+q} \sqcup D^q \to D^m$ and a proper map $g : D^q \to D^m - f'D^{p+q}$ such that $\beta(f'|_{D^{p+q}} \sqcup g) = v$. Thus β is surjective. □

5.1. The complement to an immersed disc.

In this subsection we prove Corollary 5.2 modulo Theorem 5.1. We shall use the following notions.

Denote by $\{X, Y\}$ the set of stable homotopy classes of maps $X \to Y$. The stabilization of the target group $\Sigma^\infty \pi_q(D^m - \operatorname{Im} f, \partial)$ is isomorphic to $\{S^q, S^m - \operatorname{Im} f\}$, because any pair (X, Y) is stably equivalent to the space $X \cup CY$. Hereafter fix the decomposition $S^m = D^m \cup CS^{m-1}$.

Denote by $T(\lambda)$ the mapping cone of the covering $\tilde\Delta \to \Delta$. Take a map $\bar i : C\tilde\Delta \to D^{p+q}$ extending the above map $\tilde i : \tilde\Delta \to D^{p+q}$. Define $\bar f : T(\lambda) \to S^m$ to be the quotient map of the composition $f\bar i$.

The following proposition shows an alternative way of getting the β-invariant:

Proposition 5.3. [cf. HaKa98, Proposition 3.2] *Under the composition*

$$\pi_p(D^m - fD^{p+q}, \partial) \xrightarrow{\Sigma} \{S^q, S^m - fD^{p+q}\} \xrightarrow{SWD}$$
$$\xrightarrow{SWD} \{fD^{p+q}, S^{m-p-q-1}\} \xrightarrow{\bar f_*} \{T(\lambda), S^{m-p-q-1}\} \xrightarrow{PT} \pi_{p+2q+1-m}(V_{M+m-p-q-1,M})$$

the class of a map $g : D^q \to (D^m - fD^{p+q}, \partial)$ *is sent to* $\beta(f \sqcup g)$.

Here the first arrow is the suspension map. The second map is the Spanier-Whitehead duality. The third arrow is induced by the map $\bar f$ defined above. The fourth map is given by the Pontryagin-Thom construction (see [Kos88'] for details). This proposition is proved by a direct checking analogously to [HaKa98, Proposition 3.2].

By this assertion it is useful to know the homotopy type of fD^{p+q}. It is given by:

Proposition 5.4. *Let C be the mapping cone of the restriction $f : \tilde i\tilde\Delta \to i\Delta$, then $fD^q \simeq C$.*

To prove this proposition note that both of these spaces are obtained from $\operatorname{Cyl}(\tilde i\tilde\Delta \to i\Delta) \cup_{\tilde i\tilde\Delta \subset D^{p+q}} D^{p+q}$ by appropriate contractions. Let us begin the study the homotopy type of the pair $(D^m - fD^{p+q}, \partial)$ with:

Proposition 5.5. [cf. HaKa98, Lemma 4.2] *The pair $(D^m - fD^{p+q}, \partial)$ is $(2m - 2p - 2q - 3)$-connected.*

Proof. Generically in codimension at least 3 the pair $(D^m - fD^{p+q}, \partial)$ is 1-connected. By the homology excision theorem $H_i(D^m - fD^{p+q}, \partial) \cong H_i(S^m - fD^{p+q})$. By the Alexander duality $H_i(S^m - fD^{p+q}) \cong H^{m-i-1}(fD^{p+q})$. By Proposition 5.4 $H^{m-i-1}(fD^{p+q}) = 0$ for $i \le 2m - 2p - 2q - 3$, because generically $\dim C = 2p + 2q - m + 1$. Thus Proposition 5.5 follows from the Hurewicz theorem. □

Proof of Corollary 5.2 modulo Theorem 5.1. By Theorem 5.1 we may assume that the classifying map $\Delta(f) \to P^\infty$ is $(s+1)$-connected. Let us show that under this assumption all maps in Proposition 5.3 are bijective.

(1) *The first map is bijective* by Proposition 5.5 and the suspension theorem, because the assumption $3p + 4q \le 3m - 6$ implies $q/2 \le 2m - 2p - 2q - 3$.

(2) *The second map is bijective* by Spanier-Whitehead duality.

(3) *The third map is bijective.* By Proposition 5.4 $\{\operatorname{Im} f, S^{m-p-q-1}\} \cong \{C, S^{m-p-q-1}\}$, it remains to check that $\{C, S^{m-p-q-1}\} \cong \{T(\lambda), S^{m-p-q-1}\}$. Denote by λ_T the restriction of λ to the triple points, by C_T the image of $T(\lambda_T)$ under the map $T(\lambda) \to C$, and by R the mapping cone of the map $T(\lambda_T) \to C_T$. Then R is a deformation retract of the mapping cone of the map $T(\lambda) \to C$. Consider the Puppe exact sequence of the pair $(\operatorname{Cyl}(T(\lambda) \to C), T(\lambda))$:

$$\{R, S^{m-p-q-1}\} \to \{C, S^{m-p-q-1}\} \to \{T(\lambda), S^{m-p-q-1}\} \to \{R, S^{m-p-q}\}.$$

Since R has dimension $\le 2p + 3q - 2m + 2$, by the assumption $3p + 4q \le 3m - 6$ it follows that $\{C, S^{m-p-q-1}\} \cong \{T(\lambda), S^{m-p-q-1}\}$.

(4) *The fourth map is bijective,* if the classifying map $\bar\lambda : \Delta \to P^\infty$ is $(s+1)$-connected. This is proved by the standard tools of bordism theory. Indeed, by the Thom-Pontryagin construction $\{T(\lambda), S^{m-p-q-1}\} \cong \Omega_s(\Delta, (m-p-q-1)\lambda)$. By homotopy lifting the induced map $\Omega_s(\Delta, (m-p-q-1)\lambda) \to \Omega_s(P^\infty, (m-p-q-1)\lambda)$ is an isomorphism. Finally, $\Omega_s(P^\infty, (m-p-q-1)\lambda) \cong \pi_{p+2q+1-m}(V_{M+m-p-q-1,M})$. (The definition of the groups $\Omega_s(\Delta, l\lambda)$, $\Omega_s(P^\infty, l\lambda)$ and the details can be found in [Kos88'] or [HaKa98,§3]). □

5.2 Surgery on the double point manifold.

Sketch of the proof of Theorem 5.1. Hereafter omit f from the notation of $\Delta(f)$, $\tilde\Delta(f)$, $\lambda(f)$ and $\bar\lambda(f)$. Making the map $\bar\lambda:\Delta\to P^\infty$ $(s+1)$-connected proceeds in 2 steps:
Step 1. Making Δ connected and $\pi_1(\Delta)\to\pi_1(P^\infty)$ surjective (i. e. $\tilde\Delta$ connected).
Step 2. Killing the elements of $\mathrm{Ker}(\pi_i(\Delta)\to\pi_i(P^\infty))$ for $1\le i\le s$.
These steps are sufficient because the map $\pi_i(\Delta)\to\pi_i(P^\infty)$ is automatically surjective for $i>0$.

In both steps 1 and 2 we make the following Whitney-Haefliger trick, performing surgery on Δ.

Let us begin with Step 2. Take a map $g:S^i\to\Delta$ representing an element of the kernel of $\pi_i(\Delta)\to\pi_i(P^\infty)$. Generically for $2s\le 2q-m-2$ and $s\le m-q-3$ it is an embedding missing the triple points of f. Since the composition $\bar\lambda\circ g:S^i\to P^\infty$ is trivial, it follows that $g(S^i)$ is trivially covered in $\tilde\Delta$. Denote by S^i_+ and S^i_- the two copies of S^i in $\tilde\Delta$. Span these spheres by two disjoint balls D^{i+1}_+ and D^{i+1}_- in D^q with the interiors missing $\tilde\Delta$. Extend the embedding of $S^{i+1}=D^{i+1}_+\cup D^{i+1}_-$ to an embedding of D^{i+2} into M^m with the interior missing fD^q. Pushing one of the two caps $D^{i+1}_\pm$ across a neighborhood of D^{i+2} performs a surgery on Δ killing the element g.

In Step 1 we have a choice of the spheres $S^0_\pm$, because they are disconnected. After an appropriate choice our surgery will connect distinct components of $\tilde\Delta$, because $\dim\Delta=2q-m\ge 2s+2\ge 2$.

To make this argument precise we need to construct the standard model for doing the surgery and to define all the framings required to embed this standard model into M^m.

Proof of Theorem 5.1. Standard model for doing surgery. [HaKa98] We will make use of the model manifold $\mathbb{R}^m=\mathbb{R}\times\mathbb{R}^{i+1}\times\mathbb{R}^{2q-m-i}\times\mathbb{R}^{m-q-1}\times\mathbb{R}^{m-q-1}$ and of two embeddings g_+ and g_- of $\mathbb{R}^q=\mathbb{R}^{i+1}\times\mathbb{R}^{2q-m-i}\times\mathbb{R}^{2q-m-1}$ into $\mathbb{R}^m$ intersecting transversally along $0\times S^i\times\mathbb{R}^{2q-m-i}\times 0\times 0$. For example, one may take $g_-(x,y,z)=(|x|^2-1,x,y,0,z)$ and $g_+(x,y,z)=(1-|x|^2,x,y,z,0)$. The sphere S^i bounds a ball $D^{i+1}\subset\mathbb{R}^{i+1}\subset\mathbb{R}^q$. Denote by $D^{i+1}_\pm=g_\pm(D^{i+1})$. The sphere $S^{i+1}=D^{i+1}_+\cup D^{i+1}_-$ bounds a ball $D^{i+2}\subset\mathbb{R}\times\mathbb{R}^{i+1}\subset\mathbb{R}^m$ with corners along S^i. Pushing one of the two 'caps' $D^{i+1}_\pm$ across D^{i+2} performs the surgery. More precisely, the double points of the resulting regular homotopy are the trace of this surgery.

Step 2 (killing the elements of the kernel of $\pi_i(\Delta)\to\pi_i(P^\infty)$*).* Assume that $g:S^i\to\Delta$ represents an element of the kernel of the map $\pi_i(\Delta)\to\pi_i(P^\infty)$, $1\le i\le s$. Since $2i\le\dim\Delta-1$ (because $2s\le 2q-m-2$), we may assume that g is an embedding. By general position the triple point set has dimension $\le 3q-2m$. Since $s\le m-q-3$, it follows that $i+3q-2m\le\dim\Delta-1$, so generically $\mathrm{Im}\,g$ does not contain triple points. Since the composition $\bar\lambda(f)\circ g:S^i\to P^\infty$ is trivial, it follows that $g(S^i)$ is trivially covered in $\tilde\Delta$. Denote by S^i_+ and S^i_- the two copies of S^i in $\tilde\Delta$.

Let us construct trivializations of the normal bundles $N(\tilde\Delta,D^q)$ and $N(S^i,\Delta)$. Take vector fields $\{e^k\}_{k=1}^{m-q}$ forming a trivialization of $N(D^q,M^m)$. Denote by e^k_x the vector of the k-th field at the point $x\in D^q$. Then the projections of the vectors e^k_y at the point $(x,y)\in\tilde\Delta$ to the normal bundle $N(\tilde\Delta,D^q)$ form a trivialization of this bundle.

Further, for each point $\{x,y\}\in\Delta$ we have a decomposition $N(\Delta,M^m)_{\{x,y\}}\cong N(\tilde\Delta,D^q)_{(x,y)}\oplus N(\tilde\Delta,D^q)_{(y,x)}$. Thus the vectors $\{e^k_x,e^k_y\}$ form a 'skew framing' of Δ: interchanging of the points x and y implies interchanging of the vectors e^k_x and e^k_y. Thus $N(\Delta,M^m)$ can be decomposed into the sum of all line bundles $\langle e^k_x+e^k_y\rangle\cong\epsilon$ and $\langle e^k_x-e^k_y\rangle\cong\lambda$. This gives an isomorphism $N(\Delta,M^m)\cong(m-q)\lambda\oplus\epsilon^{m-q}$. Since $S^i\to P^\infty$ is trivial, it follows that $(m-q)\lambda\,|_{S^i}$ is trivial. Thus the restriction $N(\Delta,M^m)\,|_{S^i}$ is trivial. Since M^m is $(s+1)$-connected, it follows that $T(M^m)\,|_{S^i}$ is also trivial. Thus $N(S^i,\Delta)$ is stably trivial and hence trivial. Denote this trivial bundle by η.

Surgery on S^i. First let us span the spheres S^i_+ and S^i_- by two disjoint balls D^{i+1}_+ and D^{i+1}_- in D^q. To do it push S^i_+ and S^i_- along the first vector field of the trivialized bundle $N(\tilde\Delta,D^q)$ and then extend to embeddings of the discs missing the image of $\tilde\Delta$. This is possible provided that $i+1+\dim\Delta\le q-1$ and $2(i+1)\le q-1$, which follows from $s\le m-q-3$ and $3s\le q-5$. Here the second dimension restriction is obtained by summing $s\le m-q-3$ and $2s\le 2q-m-2$.

Now take a natural decomposition $N(D^{i+1}_+,D^q)\Big|_{S^i_+}=\eta\oplus\epsilon^{m-q-1}$ (recall that $\eta=N(S^i,\Delta)$). We wish to extend this partial $(m-q-1)$-framing over D^{i+1}_+. The complete obstruction lies in $\pi_i(V_{q-i-1,m-q-1})=0$, provided that $2i\le 2q-m-1$, which follows from $2s\le 2q-m-2$. Thus we obtain a decomposition $N(D^{i+1}_+,D^q)=\eta_+\oplus\epsilon^{m-q-1}$, where η_+ is the complementary extension of η over D^{i+1}_+. Define the bundle η_- analogously.

Next we extend the embedding of $S^{i+1}=D^{i+1}_+\cup D^{i+1}_-$ to an embedding of D^{i+2} into M^m. To do it push D^{i+1}_+ and D^{i+1}_- along the first vector field of the trivialized bundle $N(D^q,M^m)$. Thus we obtain an embedding of a collar neighborhood of the sphere S^{i+1} into M^m. It can be extended to an embedding of the disc D^{i+2}, provided that $i+2+q\le m-1$ and $2(i+2)\le m-1$ (which follows from $q+s+3\le m$, because $s\le m-1$).

Finally, consider the following partial $(m-q-1)$-framing of the sphere S^{i+1}. On the disc D^{i+1}_+ take the $(m-q-1)$-framing complementary to η_+. On the disc D^{i+1}_- take the $(m-q-1)$-framing obtained from the trivialization of $N(D^q,M^m)$ by forgetting the first vector field. By the above construction it follows that these two partial framings coincide on S^i. Thus we obtain a $(m-q-1)$-framing of S^{i+1}. Let us extend it to D^{i+2}. The complete obstruction lies in $\pi_{i+1}(V_{m-i-2,m-q-1})$, so it vanishes for $2i+2\le q-1$ (this condition is satisfied, see above). Now set η' to be the complementary bundle to the obtained $(m-q-1)$-framing over D^{i+2}. On D^{i+1}_- we have the splitting $\eta' = \eta_- \oplus \epsilon^{m-q-1}$. Pushing the remaining $(m-q-1)$-framing across D^{i+2} yields a splitting $N(D^{i+2}, M^m) = \eta'' \oplus \epsilon^{m-q-1} \oplus \epsilon^{m-q-1}$ for some bundle η''.

Thus the relevant framing information along D^{i+2} agrees with that of the standard model, so there is a homeomorphism of a neighborhood of D^{i+2} and the standard model. So one can perform our surgery and kill the spheroid $g : S^i \to \Delta$.

Step 1 (making Δ and $\tilde{\Delta}$ connected). If $\Delta = \emptyset$, then first create a self-intersection (for example, by a 'finger Whitney trick'). Take a pair of points $(a, b), (c, d)$ belonging to distinct components of $\tilde{\Delta}$. One can assume that they are outside the triple point set. Consider the spheres $S^0 = \{\{a, b\}, \{c, d\}\}$, $S^0_+ = \{(a, b), (c, d)\}$, and $S^0_- = \{(b, a), (d, c)\}$, and let η be the trivial normal bundle $N(S^0, \Delta)$. The surgery on S^0 (see Step2 above) will connect the components of $\tilde{\Delta}$, because $\dim \Delta \geq 2$. $\square$

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References

[BaTe99] A. Bartels and P. Teichner, *All two dimensional links are null homotopic*, Geom. Topol. **3** (1999), 235–252.

[BM52] A. Blakers and W. Massey, *Homotopy groups of a triad II*, . Ann. Math. (1952).

[CeRe03] M. Cencelj and D. Repovš, *On embeddings of tori in Euclidean spaces*, Acta Math. Sinica (Engl. Ser.) **21:2** (2005), 435–438..

[CRS04] M. Cencelj, D. Repovs and A. Skopenkov, *On the Browder-Levine-Novikov embedding theorems*, Trudy Math. Inst. Ross. Akad. Nauk **247** (2004), 280–290 (in Russian); English transl., Proc. of the Steklov Inst. Math. **247** (2004).

[Gro86] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer Verlag, Berlin-New York, 1986.

[Ecc80] P. Eccles, *Multiple points of codimension one immersions*, Lecture Notes in Math. **788** (1980), 23–38.

[HaKa98] N. Habegger and U. Kaiser, *Link homotopy in the 2-metastable range*, Topology **37:1** (1998), 75–94.

[Hae62T] A. Haefliger, *Differentiable links*, Topology **1** (1962), 241–244.

[Hae66A] A. Haefliger, *Differentiable embeddings of S^n in S^{n+q} for $q > 2$*, Ann. Math. **83:3** (1966), 402–436.

[Hae66C] A. Haefliger, *Enlacements de spheres en codimension superiure a 2*, Comment. Math. Helv. **41** (1966-67), 51–72. (in French)

[HaQu74] A. Hatcher and F. Quinn, *Bordism invariants of intersections of submanifolds*, Trans. Amer. Math. Soc. **200** (1974), 327–344.

[Hud63] J. F. P. Hudson, *Knotted tori*, Topology **2** (1963), 11-22.

[Hud69] J. F. P. Hudson, *Piecewise-linear topology*, Benjamin, New York-Amsterdam, 1969.

[Hud70] J. F. P. Hudson, *Concordance, isotopy and diffeotopy*, Ann. Math. **91:3** (1970), 425–448.

[Irw65] M. C. Irwin, *Embeddings of polyhedral manifolds*, Ann. Math **82:2** (1965), 1–14.

[Jam54] I. M. James, *On the iterated suspension*, Quart. J. Math. Oxford **5** (1954), 1–10.

[Ker59] M. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant*, Ann. of Math. **69** (1959), 345–362.

[KeMi63] M. Kervaire, J. Milnor, *Groups of homotopy spheres: I*, Ann. Math. **77:3** (1963), 504–537.

[Kos88] U. Koschorke, *Link maps and the geometry of their invariants*, Manuscripta Math. **61:4** (1988), 383–415.

[Kos88'] U. Koschorke, *Multiple point invariants of link maps*, Lect. Notes Math., Springer-Verlag **1350** (1988), 44–86.

[Kos90] U. Koschorke, *On link maps and their homotopy classification*, Math. Ann. **286:4** (1990), 753–782.

[Kos97] U. Koschorke, *A generalization of Milnor's μ-invariants to higher dimensional link maps*, Topology **36:2** (1997), 301–324.

[Kos03] U. Koschorke, *Nielsen coincidence theory in arbitrary codimensions*, Fach. Math., Univ. Siegen **306** (2003), 1–23.

[KoSa77] U. Koschorke and B. Sanderson, *Geometric interpretation of the generalized Hopf invariant*, Math. Scand. **41** (1977), 199–217.

[KrTa97] V. Krushkal and P. Teichner, *Alexander duality, gropes and link homotopy*, Geom. Topol. **1** (1997), 51–69.

[Mel00] S. Melikhov, *Link concordance implies link homotopy in codimension ≥ 3*, Uspekhi Mat. Nauk **55:3** (2000), 183–184 (in Russian); *English transl:*, Russian Math. Surv. **55:3** (2000).

[Mel] S. Melikhov, *Link concordance implies link homotopy in codimension ≥ 3*, preprint.

[MiRe71] R. J. Milgram and E. Rees, *On the normal bundle to an embedding*, Topology **10** (1971), 299-308.

[Ne84] V. Nezhinsky, *A suspension sequence in link theory*, Izv. Akad. Nauk **48:1** (1984), 126–143. (in Russian)

[Pae56] Paechter, *On the groups $\pi_r(V_{mn})$*, Quart. J. Math. Oxford, Ser.2; *I* **7:28** (1956), 249–265; *II* **9:33** (1958), 8–27; *III* **10:37** (1959), 17–37; *IV* **10:40** (1959), 241–260; *V* **11:41** (1960), 1–16.

[ReSk99] D. Repovš and A. Skopenkov, *New results on embeddings of polyhedra and manifolds into Euclidean spaces*, Uspekhi Mat. Nauk **54:6** (1999), 61–109 (in Russian); English transl., Russ. Math. Surv. **54:6** (1999), 1149–1196.

[Sco68] G. P. Scott, *Homotopy links*, Abh. Math. Sem. Univ. Hamburg **32** (1968), 186–190.

[Sko02] A. Skopenkov, *On the Haefliger-Hirsh-Wu invariants for embeddings and immersions*, Comm. Math. Helv. **77** (2002), 78–124.

[Sk05] A. Skopenkov, *A new invariant and parametric connected sum of embeddings*, subm. to Fund. Math.; http://arxiv.org/abs/math/0509621.

[Sk06] A. Skopenkov, *Classification of smooth embeddings of 3-manifolds in the 6-space*, submitted; http://arxiv.org / math.GT/0603429.

[Sk06'] A. Skopenkov, *Classification of embeddings below the metastable dimension*, preprint; http://arxiv.org /abs/math.GT/0607422.

[Sko06"] A. Skopenkov, *Embedding and knotting of higher-dimensional manifolds in Euclidean spaces*, London Math. Soc. Lect. Notes, to appear.

[Szu76] A. Szücs, *Cobordism group of l-immersions (in Russian)*, Acta Math. Hungar. **28** (1976), 93-102.

[Web67] C. Weber, *Plongements de polyédres dans le domain metastable*, Comment. Math. Helv. **42** (1967), 1–27. (in French)

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