The Möbius contest paper: Braid group actions on triangulated categories

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Abstract. The main purpose of this paper is to investigate one special type of braid group actions on triangulated categories. It is obtained from the general construction invented recently by R. Anno. It can be also considered as a generalization of actions invented by Seidel and Thomas. The main result is faithfulness under some assumptions.

1 Introduction

This paper is devoted to studying of braid group actions on triangulated categories. There are many known and conjectural examples of such actions, e.g.

1. Actions generated by a chain of spherical objects [STh, HKh].
2. Actions on the derived categories of constructible sheaves of vector spaces on flag varieties [Ro1, Ro2] and the related actions on $D^b(\mathcal{O}_0)$, where $\mathcal{O}_0$ is a regular block of the highest weight category for $\mathfrak{sl}_n$, and on its subcategories [St].
3. Actions on categories of complexes of matrix factorizations [KR].
4. Actions on Fukaya-Floer categories of various symplectic manifolds [SS].
5. Affine braid group action on $D^b(T^*Fl)$, where Fl is a full flag variety of n-dimensional $\mathbb{C}$-vector space [KhTh].

R. Anno [A] has recently invented the general construction of braid group actions using spherical functors. In this construction the important difficulty is that cones are not functorial. To avoid this difficulty one can work either with enhanced triangulated categories (see [BoK2]) or with algebraic triangulated categories (following Keller). We choose the second way which is more convenient for us.

Recall that in [STh] there defined the notions of spherical object in triangulated category which gives an exact autoequivalence (more precisely, they work not in arbitrary categories because they need some cones to be functorial), and of (A_m)-configuration of such objects which give action of braid group B_{m+1} on the category by corresponding autoequivalences, i.e. satisfy braid relations. These spherical objects correspond to spherical functors from the bounded derived category of finite-dimensional $\mathbb{F}$-vector spaces $D^b(\mathbb{F}\text{-}Vect_{f.d.})$ to our triangulated category, and an (A_m)-configuration of spherical objects correspond to an (A_m)-configuration of spherical functors.

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Certainly one can consider something more complicated then $D^b(\mathbb{F}\text{-}Vect)$. For example, Khovanov and Thomas [KhTh] in their construction of action of affine braid group on $D^b(T^*Fl)$ in fact deal with the spherical functors from $D^b(T^*FL_i)$ to $D^b(T^*Fl)$, where Fl_i are partial flag varieties.

The construction in [STh] was motivated by occurence of braid group actions in symplectic geometry (see [Khos, S]), and Kontsevich's homological mirror symmetry conjecture. These actions are generated by Dehn twists along Lagrangian spheres. As it is pointed out in [Ro1], where the notion of spherical functor is defined, there should exist autoequivalences of the derived categories of Calabi-Yau varieties with respect to spherical functors, which naturally correspond to symplectomorphisms associated to Lagrangian submanifolds more complicated than spheres.

The main purpose of this paper is to study the case when the spherical functors are from $D^b(V_k)$ to the given triangulated category $\mathcal{T}$. Here V_k is the quiver with k vertices and without arrows. To define such a functor is the same as to set a sequence of k objects in $\mathcal{T}$ satisfying some properties, see subsection 4.2. We define the notion of (A_m^k)-configuration of such objects which corresponds to (A_m)-configuration of spherical functors. Hence, we have an action of B_{m+1}. Further, we prove the faithfulness of this action under some assumptions (see subsection 4.3).

The paper is organized as follows. In section 2 we expose some general notions and facts including (algebraic) triangulated categories, adjoint functor, Serre functors and some notions concerning quivers.

In section 3 we expose the construction by R. Anno in our setting of algebraic triangulated categories. We illustrate this construction in the example of spherical objects studied by Seidel and Thomas.

The section 4 is the main one in this paper. First, in subsection 4.1 we show how to provide the bounded derived category of rather general abelian category with a structure of an algebraic triangulated category. Further, we work with such derived categories (although one can prove the same results for sufficiently well algebraic triangulated categories). Then, in subsection 4.2 we give the definition of multi-twist functors from given category to itself, with respect to a sequence of k objects in our category. This corresponds to twist functors with respect to functors from $D^b(V_k) \cong D^b(\mathbb{F}^k-mod)$ (not necessarily spherical) to our category. We obtain the corresponding notion of n-spherical sequences of k objects. Such sequences give autoequivalences. Further, in subsection 4.3 we construct B_{m+1}-actions generated by multi-twist functors with respect to n-spherical sequence. The rest part of section 4 is devoted to the proof of its faithfulness for $n \geq 2k$. To do that, we path from our category to the derived category of some dg-algebra and describe its cohomology algebra $A_{m,n}^k$ (subsection 4.4). Then, in subsection 4.5, we prove that this cohomology algebra is intrinsically formal for if $n \geq 2k$ (see definition there). This allows us to work with $D(A_{m,n}^k)$ (in sense of subsection 4.4) instead of the derived category of *dg*-algebra. Finally, in subsection 4.6 we construct B_{m+1}-action on its subcategory $D'(A_{m,n}^k)$ of modules with bounded cohomology (in fact, it extends to $D(A_{m,n}^k)$ but we do not need it), prove the faithfulness of this action, and prove as a consequence the main result on faithfulness of the initial action.

In section 5 we make some general remarks and give some examples.

Acknowledgements. I would like to thank Alexei Bondal, Alexander Kuznetsov and Dmitry Orlov for very useful discussions. I am also grateful to Alexander Kuznetsov for constant attention during the preparation of this version of the paper, which is especially for Möbius contest.

2 Preliminaries.

2.1 Triangulated categories.

A triangulated category is an additive category $\mathcal{T}$ equipped with an autoequivalence $T : \mathcal{T} \to \mathcal{T}$, $A \mapsto A[1]$, which is called "the shift" and a collection of distinguished triangles

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_1[1]$$

of morphisms in $\mathcal{T}$ satisfying certain axioms, see [Ve]. We write $A[i]$ instead of $T^i(A)$ and

$$\mathrm{Hom}^i(A_1, A_2) = \mathrm{Hom}(A_1, A_2[i]).$$

The main example of a triangulated category is the derived category $D(\mathcal{A})$ (or $D^*(A)$, where $* \in \{+, -, b\}$) of an abelian category $\mathcal{A}$. In this case the shift functor moves a complex to the left by one and changes the sign of the differential, and distinguished triangles are triangles which are isomorphic to the mapping cones of morphisms of complexes. For $A_1, A_2 \in \mathcal{A}$ we have $\mathrm{Hom}^i(A_1, A_2) = \mathrm{Ext}^i(A_1, A_2)$.

Another important example which we will also consider is the derived category (and its bounded variants) of a dg-algebra (see subsection 4.4). Other examples are provided by semiorthogonal decomposition (see subsection 2.4)

Definition 2.1. *Exact functor between triangulated categories $\mathcal{T}_1$, $\mathcal{T}_2$ is a pair (F, λ) which consists of an additive functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ and an isomorphism of functors $\lambda : F \circ [1] \to [1] \circ F$, such that for each distinguished triangle*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_1[1]$$

in $\mathcal{T}_1$ the triangle

$$F(A_1) \xrightarrow{F(f_1)} F(A_2) \xrightarrow{F(f_2)} F(A_3) \xrightarrow{\lambda(A_1)\circ F(f_3)} F(A_1)[1]$$

in $\mathcal{T}_2$ is also distinguished.

If for some additive $F : \mathcal{T}_1 \to \mathcal{T}_2$ such λ exists then we will say that F is exact. For example, derived functors between derived categories of abelian categories are exact.

2.2 Algebraic triangulated categories.

In this paper we will work with algebraic triangulated categories. They allow us to take cones of morphisms of exact functors. Here we follow [Ro1], subsection 8.1.1, and [Ro2], subsection 2.1.1.

Definition 2.2. *A Frobenius category is an exact category with enough injectives and projectives, and such that the classes of projectives and injectives coincide.*

It is well-known that the stable category of a Frobenius category (see [K4] for definition) is always triangulated (see [H]).

Definition 2.3. *A triangulated category is called algebraic if it is the stable category of some Frobenius category*

By $Kom_{acyc}(\mathcal{E} - proj)$ we denote the category of acyclic complexes of projective objects in $\mathcal{E}$. It is also a Frobenius category. By $Frob$ we denote the 2-category of Frobenius categories. 1-arrows in $Frob$ are exact functors which send injectives to projectives, and 2-arrows are natural transformations (i.e. morphisms in the usual sence) of functors. Similarly, we define the 2-category Tr of triangulated categories. 1-arrows in Tr are exact functors and 2-arrows in Tr are morphisms between functors.

We have a 2-functor from $Frob$ to Tr which sends $\mathcal{E}$ to its stable category $\bar{\mathcal{E}}$. This 2-functor factors through the endo-2-functor $\mathcal{E} \mapsto Kom_{acyc}(\mathcal{E} - proj)$.

The category of exact functors $Kom_{acyc}(\mathcal{E} - proj) \to Kom_{acyc}(\mathcal{E}' - proj)$, preserving projectives, is a Frobenius category. Let $AlgTr(\mathcal{E}, \mathcal{E}')$ be its stable category. For two exact functors $F, G : Kom_{acyc}(\mathcal{E} - proj) \to Kom_{acyc}(\mathcal{E}' - proj)$ we have that $\mathrm{Hom}_{AlgTr}(\mathcal{E}, \mathcal{E}')$ is the image of $\mathrm{Hom}(F, G)$ in $\mathrm{Hom}_{Fun(\bar{\mathcal{E}}, \bar{\mathcal{E}}')}(\bar{F}, \bar{G})$.

This defines the 2-category $AlgTr$ of algebraic triangulated categories. Note that its objects are Frobenius categories. The natural 2-functor from $AlgTr$ to Tr (which sends $\mathcal{E}$ to $\bar{\mathcal{E}}$) is 2-full and 2-faithful.

For exact functors $F, G : Kom_{acyc}(\mathcal{E} - proj) \to Kom_{acyc}(\mathcal{E}' - proj)$ and for the morphism of functors $\phi : F \to G$ we have a naturally defined $Cone(\phi)$ and natural morphims $G \to Cone(\phi)$ and $Cone(\phi) \to F[1]$. This allows us to take cones of morphisms of exact functors between algebraic triangulated categories.

2.3 Adjoint functors.

Let $F : \mathcal{T}_1 \to \mathcal{T}_2$, $G : \mathcal{T}_2 \to \mathcal{T}_1$ be two functors. We say that G is left adjoint to F, or F is right adjoint to G if there exists a bifunctorial isomorphism

$$\mathrm{Hom}(G(-), ?) \cong \mathrm{Hom}(-, F(?)).$$

If there exists a left (or right) adjoint to a given functor then it is unique up to an isomorphism. Clearly, a left (right) adjoint to the composition of functors is the composition of left (right) adjoint to them.

Further, if F is left adjoint to G then there exist natural morphisms of functors $\eta_1 : G \circ F \to Id_{\mathcal{T}_1}$, $\eta_2 : Id_{\mathcal{T}_2} \to F \circ G$ obtained by applying the adjunction isomorphism to the identity morphisms of F and G respectively.

The [GM] for explanation of the statement of the following Lemma.

Lemma 2.4. *Suppose that some morphisms of functors $\eta_1 : G \circ F \to Id_{\mathcal{T}_1}$, $\eta_2 : Id_{\mathcal{T}_2} \to F \circ G$ are given and the compositions*

$$F \xrightarrow{\eta_2(F)} F \circ G \circ F \xrightarrow{F(\eta_1)} F,$$

$$G \xrightarrow{G(\eta_1)} G \circ F \circ G \xrightarrow{\eta_2(G)} G$$

coincide with the identity morphisms of F and G respectively. Then the composition

$$\mathrm{Hom}(G(-), ?) \xrightarrow{F(-,?)} \mathrm{Hom}(FG(-), F(?)) \longrightarrow \mathrm{Hom}(-, F(?))$$

is isomorphism. Here the last map is the composition with η_2. In particular, G is left adjoint to F.

The next Proposition is proved in [BoK].

Proposition 2.5. *Let F and G be adjoint additive functors between triangulated categories. If F is exact, then G is also exact.*

2.4 Semiorthogonal decompositions.

Let $\mathcal{T}$ be a triangulated category, $\mathcal{S} \subset \mathcal{T}$ be its full triangulated subcategory. This means that $\mathcal{S}$ is closed under taking cones of morphisms. By $\mathcal{S}^\perp$ (${}^\perp\mathcal{S}$) we denote its right (resp. left) orthogonal category, i.e. the full subcategory which consists of objects $B \in \mathcal{T}$, such that $\mathrm{Hom}(A, B) = 0$ (resp. $\mathrm{Hom}(B, A) = 0$) for each $A \in \mathcal{S}$. From two standard long exact sequences of Hom's it follows that $\mathcal{S}^\perp$ and ${}^\perp\mathcal{S}$ are also triangulated.

Definition 2.6. *Let $\mathcal{S} \subset \mathcal{T}$ be as above. Then $\mathcal{S}$ is called right (resp. left) admissible if for each $X \in \mathcal{T}$ there exists a distinguished triangle $A \to X \to B$, where $A \in \mathcal{S}$, $B \in \mathcal{S}^\perp$ (resp. $C \to X \to A$, where $C \in^\perp \mathcal{S}$, $A \in \mathcal{S}$). A subcategory $\mathcal{S}$ is called admissible if it is both left and right admissible.*

The following Proposition is proved in [Bo].

Proposition 2.7. *Let $\mathcal{S} \subset \mathcal{T}$ be as above. The following conditions are equivalent:*
a) $\mathcal{S}$ is right (resp. left) admissible,
b) the embedding functor $\mathcal{S} \hookrightarrow \mathcal{T}$ admits right (resp. left) adjoint,
c) $\mathcal{T}$ is generated by $\mathcal{S}$ and $\mathcal{S}^\perp$ (resp. ${}^\perp\mathcal{S}$ and $\mathcal{S}$) as a triangulated category.

Now we define the notion of semiorthogonal decomposition.

Definition 2.8. *Let $\langle \mathcal{S}_1, \ldots, \mathcal{S}_m \rangle$ be a collection of admissible subcategories in $\mathcal{T}$. It is called semiorthogonal decomposition if $S_j \subset S_i^{\perp}$ for $i > j$ and $S_1, \ldots, S_m$ generate $\mathcal{T}$ as a triangulated category.*

The important example of semiorthogonal decomposition is the following. Suppose that $\mathcal{T}$ is $\mathbb{F}$-linear category with finite-dimensional Hom's and such that for any $A, B \in \mathcal{T}$ we have $Hom^k(A, B) = 0$ for almost all $k \in \mathbb{Z}$.

Definition 2.9. *An object $E \in \mathcal{T}$ is called exceptional if*

$$Hom^k(E, E) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following Proposition is proved in [Bo].

Proposition 2.10. *A subcategory generated (as triangulated subcategory) by an exceptional object is equivalent to the bounded derived category of finite-dimensional vector $\mathbb{F}$-spaces $D^b(\mathbb{F}\text{-}Vect_{f.d.})$ and is admissible.*

Definition 2.11. *A collection of exceptional objects $E_1, \ldots, E_k$ is called exceptional if $Hom^k(E_i, E_j) = 0$ for $i > j$, $k \in \mathbb{Z}$.*

Clearly if $\mathcal{T}$ is generated by an exceptional collection then it admits a semiorthogonal decomposition into subcategories derived which are equivalent to $D^b(\mathbb{F}\text{-}Vect_{f.d.})$.

2.5 Serre functors.

The Serre-Grothendieck duality on a smooth projective variety was axiomatized by Bondal and Kapranov [BoK] into the notion of Serre functor on a triangulated category. Let $\mathbb{F}$ be a field.

Definition 2.12. *Let $\mathcal{T}$ be an $\mathbb{F}$-linear category with finite-dimensional Hom's. An additive functor $S : \mathcal{T} \to \mathcal{T}$, commuting with the shift, is called Serre functor if it is an equivalence, and there exist bifunctorial isomorphisms*

$$\varphi_{A,B} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(B, S(A))$$

for any $A, B \in \mathcal{T}$.

Clearly, if a Serre functor exists then it is unique up to isomorphism of functors. The following three Propositions are proved in [BoK]:

Proposition 2.13. *Any Serre functor is exact.*

Proposition 2.14. *A triangulated category $\mathcal{T}$ admits a Serre functor iff all contravariant functors $\mathrm{Hom}(X, -)^\vee$ and covariant functors $\mathrm{Hom}(-, X)^\vee$ are representable.*

Proposition 2.15. *Let $\langle \mathcal{S}_1, \ldots, \mathcal{S}_m \rangle$ be a semiorthogonal decomposition in $\mathcal{T}$. Then $\mathcal{T}$ admits a Serre functor iff each of $\mathcal{S}_i$ does.*

In particular, if $\mathcal{T}$ is generated by an exceptional collection of objects and for any $A, B \in \mathcal{T}$ we have $Hom^k(A, B) = 0$ for almost all $k \in \mathbb{Z}$ then $\mathcal{T}$ admits a Serre functor.

2.6 Quivers.

A quiver $Q = (Q_0, Q_1)$ is a finite oriented graph with the set of vertices Q_0 and the set of arrows Q_1. For $\varphi \in Q_1$ denote by $t(\varphi)$ and $h(\varphi)$ its tail and its head respectively. An (oriented) path in Q is a sequence $\rho = (\rho_1, \ldots, \rho_l)$ of arrows satisfying $t(\rho_{i+1}) = h(\rho_i)$, $1 \le i \le l-1$ (l can be equal to zero). We write $t(\rho)$ and $h(\rho)$ for $t(\rho_1)$ and $h(\rho_l)$ respectively. Each vertex $i \in Q_0$ can be considered as a path of length zero. We denote this path by (i). Let $\mathbb{F}$ be a field. By $\mathbb{F}[Q]$ we denote the path algebra of Q. As $\mathbb{F}$-vector space it is freely generated by all paths $\rho \in Q$. The multiplication is given by

$$\rho \cdot \gamma = \begin{cases} \rho\gamma & \text{if } t(\gamma) = h(\rho) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{I} \in \mathbb{F}[Q]$ be a two-sided ideal in $\mathbb{F}[Q]$. A representation of $(Q, \mathcal{I})$ is the collection of $\mathbb{F}$-vector spaces M_i, $i \in Q_0$, and homomorphisms $M_\alpha : M_{t(\alpha)} \to M_{h(\alpha)}$, $\alpha \in Q_1$, satisfying the relations in $\mathcal{I}$. This means that $\sum\limits_{i_1,\ldots,i_l} c_{i_1,\ldots,i_l} M_{\alpha_{i_l}} \ldots M_{\alpha_{i_1}} = 0$ if $\sum\limits_{i_1,\ldots,i_l} c_{i_1,\ldots,i_l} \alpha_{i_1} \ldots \alpha_{i_l} \in \mathcal{I}$. A morphism of two representations N and M of $(Q, \mathcal{I})$ is a family of linear maps $f_i : N_i \to M_i$, $i \in Q_0$, satisfying $f_{h(\alpha)} N_\alpha = M_\alpha f_{t(\alpha)}$. It is well-known that the category $Rep(Q, \mathcal{I})$ of finite-dimensional representations of $(Q, \mathcal{I})$ is equivalent to the category $mod-(k[Q]/\mathcal{I})$ of finite-dimensional right $\mathbb{F}[Q]/\mathcal{I}$-modules. We write $D^*(Q, \mathcal{I})$ instead of $D^*(Rep(Q, \mathcal{I}))$.

Suppose that the quiver Q has not oriented cycles. Denote by S_i the simple representation of $(Q, \mathcal{I})$ in the i-th vertex. It has $(S_i)_j = \mathbb{F}^{\delta_{ij}}$ for $j \in Q_0$ and $(S_i)_\alpha = 0$ for each $\alpha \in Q_1$. Further, denote by P_i the projective representation in the i-th vertex. As a right $\mathbb{F}[Q]/\mathcal{I}$-module, it equals to $(i)\mathbb{F}[Q]/\mathcal{I}$. Analogously, I_i is an injective representation in the i-th vertex.

3 General spherical functors and braid group actions.

In this section we expose the results of R. Anno [A] working with algebraic triangulated categories.

3.1 Spherical functors.

Here we define the notion of a spherical functor between two triangulated categories and associate autoequivalences to such functors. Let $\Phi_* : \mathcal{S} \to \mathcal{T}$ be an exact functor between triangulated categories. Suppose that it admits a right adjoint functor $\Phi^! : \mathcal{T} \to \mathcal{S}$. Then we give the following

Definition 3.1. *In the above notation, the twist functor $T_{\Phi_*} : \mathcal{T} \to \mathcal{T}$ corresponding to Φ_* is defined as follows:*

$$T_{\Phi_*} = Cone(\Phi_* \Phi^! \to Id_{\mathcal{T}})$$

The functor T_{Φ_*} is always exact. The proof of the following Proposition is left to the reader.

Proposition 3.2. *Let $\Phi_* : \mathcal{S} \to \mathcal{T}$ be an exact functor and let $\Psi : \mathcal{S}' \to \mathcal{S}$ be an exact equivalence of triangulated categories. Then the functors T_{Φ_*} and $T_{\Phi_* \circ \Psi}$ are naturally isomorphic.*

Now suppose that Φ_* also admits a left adjoint functor $\Phi^* : \mathcal{T} \to \mathcal{S}$.

Definition 3.3. *In the above notation we define the dual twist functor T'_{Φ_*} as follows:*

$$T'_{\Phi_*} = Cone(Id_{\mathcal{T}} \to \Phi_* \Phi^*)[-1]$$

We want to know when T_{Φ_*} and T'_{Φ_*} are quasi-inverse autoequivalences. Define $\Gamma_{\Phi_*}, \quad \Gamma'_{\Phi_*} : \mathcal{S} \to \mathcal{S}$ as follows:

$$\Gamma_{\Phi_*} = Cone(Id_{\mathcal{S}} \to \Phi^!\Phi_*), \quad \Gamma' = Cone(\Phi^*\Phi_* \to Id_{\mathcal{S}}).$$

Note that we have the following natural morphisms of functors:

$$\Phi^! \to \Gamma_{\Phi_*}\Phi^*, \quad \Gamma'_{\Phi_*}\Phi^! \to \Phi^*.$$

Now we define the notion of spherical functor.

Definition 3.4. *An exact functor $\Phi_* : \mathcal{S} \to \mathcal{T}$ is called spherical if the following conditions hold:*

$(GS1)$ Φ_* *admits left and right adjoint functors*

$$\Phi^* : \mathcal{T} \to \mathcal{S}, \quad \Phi^! : \mathcal{T} \to \mathcal{S}$$

respectively;

$(GS2)$ *the functor* $\Gamma_{\Phi_*} : \mathcal{S} \to \mathcal{S}$ *is an equivalence;*

$(GS3)$ *the natural morphism* $\Phi^! \to \Gamma_{\Phi_*}\Phi^*$ *is an isomorphism.*

Proposition 3.5. *If the functor $\Phi_* : \mathcal{S} \to \mathcal{T}$ is spherical then the corresponding twist functor $T_{\Phi_*} : \mathcal{T} \to \mathcal{T}$ is an equivalence.*

Proof. This is essentially Proposition 8.1 in [Ro1] or Proposition 2.1 in [Ro2]. □

Note that if Φ_* is spherical then T'_{Φ_*} is also right adjoint to T_{Φ_*}.

3.2 Braid group actions

Let $\Phi_{1*} : \mathcal{S}_1 \to \mathcal{T}$, $\Phi_{2*} : \mathcal{S}_2 \to \mathcal{T}$ be two spherical functors. We are going to give some sufficient conditions for $T_{\Phi_{1*}}, T_{\Phi_{2*}}$ to commute and to braid with each other.

Proposition 3.6. *Let $\Phi_{1*}, \Phi_{2*} : \mathcal{S} \to \mathcal{T}$ be spherical. Suppose that at least one of the compositions $\Phi_1^!\Phi_{2*}$ and $\Phi_2^!\Phi_{1*}$ is isomorphic to zero. Then the corresponding twist functors commute (up to an isomorphism), i.e.*

$$T_{\Phi_{1*}}T_{\Phi_{2*}} \cong T_{\Phi_{2*}}T_{\Phi_{1*}}$$

Proof. First show that if one of the compositions mentioned above is isomorphic to zero then the other is zero as well. Let, for example, $\Phi_1^!\Phi_{2*} \cong 0$. Then $\Phi_2^*\Phi_{1*}$ is left adjoint to the functor which is isomorphic to zero and hence is also isomorphic to zero. Further, $\Phi_2^!\Phi_{1*} \cong \Gamma_{\Phi_{2*}}\Phi_2^*\Phi_{1*} \cong 0$.

Now prove the statement of Proposition. We have that $T_{\Phi_{1*}}T_{\Phi_{2*}}$ is the following convolution:

$$\left\{\begin{array}{ccc} \Phi_{1*}\Phi_1^!\Phi_{2*}\Phi_2^! & \longrightarrow & \Phi_{2*}\Phi_2^! \\ \downarrow & & \downarrow \\ \Phi_{1*}\Phi_1^! & \longrightarrow & Id_{\mathcal{T}} \end{array}\right\}$$

Here $Id_{\mathcal{T}}$ is in degree 0. From our assumptions we have that the functor in the top left corner is isomorphic to zero. Thus, we may replace it by zero:

$$\left\{\begin{array}{ccc} 0 & \longrightarrow & \Phi_{2*}\Phi_2^! \\ \downarrow & & \downarrow \\ \Phi_{1*}\Phi_1^! & \longrightarrow & Id_{\mathcal{T}} \end{array}\right\}$$

Analogously, we obtain that $T_{\Phi_{2*}}T_{\Phi_{1*}}(F)$ is the same convolution. This concludes the proof. □

Lemma 3.7. *Let Φ_{1*} and Φ_{2*} be spherical. Then the functor $T_{\Phi_{2*}} \circ \Phi_{1*}$ is also spherical and $T_{\Phi_{2*}} T_{\Phi_{1*}}$ is naturally isomorphic to $T_{(T_{\Phi_{2*}} \circ \Phi_{1*})} T_{\Phi_{2*}}$.*

Proof. The first statement follows from Proposition 3.2. As in the previous Proposition, we have that $T_{\Phi_{2*}} T_{\Phi_{1*}}$ is

$$Cone(T_{\Phi_{2*}} \Phi_{1*} \Phi_1^! \to T_{\Phi_{2*}}).$$

Clearly, the right adjoint functor to $T_{\Phi_{2*}} \circ \Phi_{1*}$ is $\Phi_1^! \circ T'_{\Phi_{2*}}$. We have the following commutative diagram of morphisms of functors:

$$\begin{array}{ccc} T_{\Phi_{2*}} \Phi_{1*} \Phi_1^! T'_{\Phi_{2*}} T_{\Phi_{2*}} & \longrightarrow & T_{\Phi_{2*}} \\ \downarrow & & \downarrow \\ T_{\Phi_{2*}} \Phi_{1*} \Phi_1^! & \longrightarrow & T_{\Phi_{2*}}. \end{array}$$

Here the upper horizontal arrow is the natural morphism from $G^* G_*$ to Id, applied to the functor $T_{\Phi_{2*}}$, where $G_* = \Phi_1^! T'_{\Phi_{2*}}$ and $G^* = T_{\Phi_{2*}} \Phi_{1*}$ is an adjoint pair. The left vertical arrow is the result of applying the functor $T_{\Phi_{2*}} \Phi_{1*} \Phi_1^!$ to the natural morphism $T'_{\Phi_{2*}} T_{\Phi_{2*}} \to Id$. From Proposition 3.5 it follows that the left vertical arrow is an isomorphism. Taking cones of horizontal arrows we obtain the natural morphism

$$T_{(T_{\Phi_{2*}} \circ \Phi_{1*})} T_{\Phi_{2*}}(F) \to T_{\Phi_{2*}} T_{\Phi_{1*}}$$

which is again an isomorphism. □

Now we give a sufficient condition for Φ_{1*}, Φ_{2*} to give autoequivalences which braid with each other.

Proposition 3.8. *Suppose that $\Phi_{1*} : \mathcal{S}_1 \to \mathcal{T}$, $\Phi_{2*} : \mathcal{S}_2 \to \mathcal{T}$ are spherical functors such that there exists an exact equivalence $\Psi : \mathcal{S}_1 \to \mathcal{S}_2$ and an isomorphism of functors*

$$T_{\Phi_{2*}} \circ \Phi_{1*} \cong T'_{\Phi_{1*}} \circ \Phi_{2*} \circ \Psi.$$

Then there exists the following natural isomorphism:

$$T_{\Phi_{1*}} T_{\Phi_{2*}} T_{\Phi_{1*}} \cong T_{\Phi_{2*}} T_{\Phi_{1*}} T_{\Phi_{2*}}$$

Proof. From Proposition 3.2, Proposition 3.5 and Lemma 3.7 we obtain the following chain of isomorphisms:

$$T_{\Phi_{1*}} T_{\Phi_{2*}} T_{\Phi_{1*}} \cong T_{\Phi_{1*}} T_{T_{\Phi_{2*}} \circ \Phi_{1*}} T_{\Phi_{2*}} \cong T_{\Phi_{1*}} T_{T'_{\Phi_{1*}} \circ \Phi_{2*}} T_{\Phi_{2*}} \cong T_{\Phi_{2*}} T_{\Phi_{1*}} T_{\Phi_{2*}}$$

□

It is useful to note the following special case.

Proposition 3.9. *If one of the functors $\Phi_1^! \Phi_{2*}$, $\Phi_1^! \Phi_{2*}$, $\Phi_1^! \Phi_{2*}$, $\Phi_1^! \Phi_{2*}$ is an equivalence then*

$$T_{\Phi_{1*}} T_{\Phi_{2*}} T_{\Phi_{1*}} \cong T_{\Phi_{2*}} T_{\Phi_{1*}} T_{\Phi_{2*}}.$$

Proof. Analogously to the proof of Proposition 3.6 we see that if one of these four composition is equivalence that all the other are equivalences as well. In particular, $\Phi_2^! \Phi_{1*}$ and $\Phi_1^* \Phi_{2*}$ are quasi-inverse equivalences. Further, we have

$$T_{\Phi_{2*}} \Phi_{1*} = Cone(\Phi_{2*} \Phi_2^! \Phi_{1*} \to \Phi_{1*}) \cong Cone(\Phi_{2*} \to \Phi_{1*} \Phi_1^* \Phi_{2*}) \Phi_2^! \Phi_{1*} \cong T'_{\Phi_{1*}} \Phi_{2*} \circ \Phi_2^! \Phi_{1*}[1].$$

Thus, the condition of the Proposition 3.8 is satisfied and this completes the proof. □

Propositions 3.6 and 3.9 lead us to the following

Definition 3.10. *A collection of spherical functors $\Phi_{1*}, \dots, \Phi_{m*} : \mathcal{S} \to \mathcal{T}$ is called an (A_m)-configuration if the following conditions hold:*

(A1) If $1 \le i, j \le m$ and $|i-j| > 1$ then $\Phi_i^! \Phi_{j*} \cong 0$;

(A2) If $1 \le i \le m-1$ then $\Phi_i^! \Phi_{(i+1)*}$ is an autoequivalence of $\mathcal{S}$

Theorem 3.11. *Let $\Phi_{1*}, \dots, \Phi_{m*} : \mathcal{S} \to \mathcal{T}$ be an (A_m)-configuration of spherical functors. Then the twist functors are exact autoequivalences and satisfy the relations of the braid group B_{m+1} (up to an isomorphism). Hence, they generate a homomorphism $\rho : B_{m+1} \to Aut(\mathcal{T})$.*

Example: spherical objects. Recall that an object $E \in \mathcal{T}$ in $\mathbb{F}$-linear triangulated category $\mathcal{T}$ is called n-spherical if $\mathrm{Hom}^*(E, E)$ is two-dimensional, is concentrated in degrees 0 and n and for each $F \in \mathcal{T}$ the composition $\mathrm{Hom}(E, F) \times \mathrm{Hom}^n(F, E) \to \mathrm{Hom}^n(E, E) \cong \mathbb{F}$ is a perfect pairing. Let $\mathcal{S} = D^b(\mathbb{F}\text{-}Vect_{f.d.})$-the derived category of finite-dimensional vector spaces. One can check that the spherical functors $\Phi_* : \mathcal{S} \to \mathcal{T}$ with $\Gamma_{\Phi_*} \cong [n]$ are in one-to-one correspondence with n-spherical objcets in $\mathcal{T}$ and (A_m)-configuration of such spherical functors correspond to (A_m)-configurations of spherical objects in sense of [STh].

In the next section we investigate the case when $\mathcal{S} = D^b(V_k)$ where V_k is aquiver with k vertices without arrows, in details.

4 One type of braid group actions and faithfullness.

4.1 Notation and assumptions.

Let $\mathbb{F}$ be a field. In this paragraph we consider only $\mathbb{F}$-linear categories. Let $\mathcal{A}$ be a category. By $Kom(\mathcal{A})$, $K(\mathcal{A})$, $D(\mathcal{A})$ we denote the category of cochain complexes in $\mathcal{A}$, the homotopy category and the derived category respectively. Their bounded variants are denoted by $Kom^+(\mathcal{A})$, $Kom^-(\mathcal{A})$, $Kom^b(\mathcal{A})$, etc.

If $\dots \to C_{k-1} \to C_k \to C_{k+1} \to \dots$ is a finite complex of objects in the category $Kom(\mathcal{A})$ (which is again abelian) then we write $\{\dots \to C_{k-1} \to C_k \to C_{k+1} \to \dots\}$ for the total complex of this bicomplex in $\mathcal{A}$. For example, $\{C_{-1} \to C_0\}$ corresponds to a cone of this map in the triangulated category $D(\mathcal{A})$ (or $K(\mathcal{A})$). If C_i are bicomplexes in $\mathcal{A}$ then we apply for them the same notation.

For $C, D \in Kom(\mathcal{A})$, by $\mathrm{RHom}(C, D)$ we denote the complex of $\mathbb{F}$-vector spaces with $\mathrm{RHom}(C, D)^i = \prod_{k \in \mathbb{Z}} \mathrm{Hom}(C^k, D^{k+i})$ and $d^i_{\mathrm{RHom}(C,D)}(\varphi) = d_D \varphi - (-1)^i \varphi d_C$. Clearly, $H^* \mathrm{RHom}(C, D) = \mathrm{Hom}^*_{K(\mathcal{A})}(C, D)$. Moreover, if D is a bounded below complex of injectives or C is a bounded above complex of projectives then $H^* \mathrm{RHom}(C, D) = \mathrm{Hom}^*_{D(\mathcal{A})}(C, D)$.

Now suppose that $\mathcal{A}$ admits infinite direct sums and products. Then for any objects $b \in Kom(\mathbb{F}\text{-}Vect)$, where $\mathbb{F}\text{-}Vect)$ is the category of all $\mathbb{F}$-vector spaces, and for $C \in Kom(\mathcal{A})$ we can define their tensor product $b \otimes C$ and the complex of linear maps $\mathrm{RHom}_{\mathbb{F}}(b, C)$ which are both objects of $Kom(\mathcal{A})$. The precise definition of $\mathrm{RHom}_{\mathbb{F}}(b, C)$ can be found in [STh] (it is denoted there by $lin(b, C)$). Clearly, if b is finite-dimensional then we do not need $\mathcal{A}$ to admit infinite direct sums and products. Further, there are the following natural morphisms in $Kom(\mathbb{F}\text{-}Vect)$ and $Kom(\mathcal{A})$:

$$b \otimes \mathrm{RHom}(C, D) \to \mathrm{RHom}(C, b \otimes D),$$

$$\mathrm{RHom}(C, D) \otimes b \to \mathrm{RHom}(\mathrm{RHom}_{\mathbb{F}}(b, C), D),$$

$$\mathrm{RHom}(B, \mathrm{RHom}_{\mathbb{F}}(b, C)) \otimes D \to \mathrm{RHom}_{\mathbb{F}}(b, \mathrm{RHom}(B, C) \otimes D).$$

$$b \otimes \mathrm{RHom}_{\mathbb{F}}(\mathrm{RHom}(B, C), D) \to \mathrm{RHom}_{\mathbb{F}}(\mathrm{RHom}(b \otimes B, C), D),$$

where $b \in Kom(\mathbb{F}\text{-}Vect)$, $B, C, D \in Kom(\mathcal{A})$. These morphisms are isomorphisms if b is finite-dimensional and quasi-isomorphisms if $H^*(b)$ is finite-dimensional.

From this moment we suppose that one of the following holds:
I. $\mathcal{A}$ contains either enough injectives or enough projectives, all Hom's between objects in $\mathcal{A}$ are finite-dimensional and $\mathcal{A}$ has finite homological dimension.
II. $\mathcal{A} \subset \mathcal{A}'$ is a full abelian subcategory and $\mathcal{A}, \mathcal{A}'$ satisfy the following conditions:

(C1) any subobject and any quotient of any object in $\mathcal{A}$ lies again in $\mathcal{A}$. Moreover, $\mathcal{A}$ is closed under extensions. In other words, $\mathcal{A}$ is a Serre subcategory of $\mathcal{A}'$;

(C2) $\mathcal{A}'$ admits infinite direct sums and products;

(C3) $\mathcal{A}'$ contains enough injectives and any direct sum of injectives is again injective.

(C4) for any epimorphism $f : A' \to A$ with $A' \in \mathcal{A}'$, $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$ and a morphism $g : B \to A'$ such that fg is epimorphism.

Proposition 4.1. *Let X be a Noetherian scheme over $\mathbb{F}$ and $\mathcal{A}' = Coh(X)$, $\mathcal{A}' = Qcoh(X)$ be the categories of coherent and quasi-coherent sheaves on X respectively. Then conditions (C1)-(C4) are satisfied.*

Proof. This is Lemma 2.1 in [STh]. □

Definition 4.2. *In the case 1, if $\mathcal{A}$ has enough injectives then $\mathcal{T} \subset K^+(\mathcal{A})$ is a full subcategory which consists of bounded below complexes of injectives with bounded cohomology;*
if $\mathcal{A}$ has enough projectives, then $\mathcal{T} \subset K^-(\mathcal{A})$ is a full subcategory which consists of bounded above complexes of projectives with bounded cohomology;
in the case 2, the category $\mathcal{T} \in K^+(\mathcal{A}')$ consists of bounded below complexes C with bounded cohomology and such that $H^i(C)$ lies in $\mathcal{A}$.

Note that $\mathcal{T}$ is a triangulated subcategory of $K^+(\mathcal{A})$ (resp. $K^-(A)$, $K^+(A')$).

Proposition 4.3. *In both cases, the categories $\mathcal{T}$ and $D^b(\mathcal{A})$ are equivalent as triangulated categories.*

Proof. In the case I this is well-known, see, for example, [GM]. In the case II, this is Proposition 2.4 in [STh]. □

Further, note that the category $\mathcal{T}$ is always algebraic triangulated category. Indeed, the corresponding Frobenius category has the same objects, its morphisms are usual morphisms of complexes. A morphism is infaltion (resp. conflation) iff it induces an injection (resp. surjection) in each term. The class of injective-projective objects is just the class of acyclic complexes (see [K4] for explanation).

4.2 Multi-twist functors and spherical sequences.

Let V_k be a quiver with k vertices and without any arrows between them. Let $\mathcal{S} = D^b(V_k)$ be the derived category of finite-dimensional $\mathbb{F}$-linear representations of this quiver. Recall that its simple representation in the i-th vertex by S_i. For convenience we put $S_{i+k} = S_i$. Let $\Gamma_{l_1,\dots,l_k} : \mathcal{S} \to \mathcal{S}$ be the autoequivalence which sends S_i to $S_{i+1}[-l_i]$ for $1 \leq i \leq k$. In this section we describe the twist functors $T_{\Phi_*} : \mathcal{T} \to \mathcal{T}$, where $\Phi_* : \mathcal{S} \to \mathcal{T}$, for which $\Gamma_{l_1,\dots,l_k} \cong Cone(Id_{\mathcal{S}} \to \Phi_! \Phi_*)[-1] = \Gamma_{\Phi_*}$, construct the corresponding braid group actions and prove their faithfulness under some assumptions.

Let $E^1, \dots, E^k \in \mathcal{T}$ be a sequence of objects and $\Phi_* : D^b(V_k) \to \mathcal{T}$ be an exact functor which sends S_i to E^i. Suppose that $E^1, \dots, E^k$ satisfy the following conditions:

(K1) E^i has a finite injective resolution,

(K2) For each $F \in \mathcal{T}$ the graded vector spaces $\mathrm{Hom}^*(E^i, F)$ and $\mathrm{Hom}^*(F, E^i)$ are finite-dimensional over $\mathbb{F}$.

Then the functor Φ_* admits right adjoint $\Phi^!(F) = \bigoplus_{j=1}^{k} \mathrm{Hom}^*(E^j, F) \otimes S_i$ and left adjoint $\Phi^*(F) = \bigoplus_{j=1}^{k} \mathrm{Hom}^*(F, E^j)^\vee \otimes S_i$. We denote the corresponding twist functor T_{Φ_*} by $T_{(E)}$ and the dual twist functor T'_{Φ_*} by $T'_{(E)}$. We will also call $T_{(E)}$ a multi-twist functor and $T'_{(E)}$ a dual multi-twist functor. The precise expression for $T_{(E)}$ is the following:

$$T_{(E)}(F) = \{ev : \bigoplus_{j=1}^{k} \mathrm{RHom}(E^j, F) \otimes E^j \to F\}. \tag{1}$$

Here ev is evaluation map on each summand and F is placed in degree zero. Further, the precise expression for $T'_{(E)}$ is

$$T'_{(E)}(F) = \{ev' : F \to \bigoplus_{j=1}^{k} \mathrm{RHom}_{\mathbb{F}}(\mathrm{RHom}(F, E^j), E^j).\} \tag{2}$$

Here the projection of ev' to each summand is coevaluation map; F is placed in degree zero.

Proposition 4.4. *If (E') is another sequence obtained by reordering and shifting of objects E^i then the multi-twist functor $T_{(E')}$ is isomorphic to $T_{(E)}$. If $(E_1^1, \ldots, E_1^k)$ and $(E_1^1, \ldots, E_1^k)$ are 2 sequences of objects in $\mathcal{T}$ and $E_1^i \cong E_2^i$ for $1 \le i \le k$. Then the functors $T_{(E_1)}$ and $T_{(E_2)}$ are isomorphic.*

Proof. Follows from Proposition 3.2. □

Now we are interested in those sequences (E) of objects for which $\Gamma_{l_1,\ldots,l_k} = \{Id_{\mathcal{T}} \to \Phi^!\Phi_*\}$. From this moment we put $E^{i+k} := E^i$. Moreover if i and j are indices for terms of the sequence (E) then we write $i = j$ instead of $i \equiv j \pmod k$.

Definition 4.5. *Assume that $k \ge 2$. The sequence $E^1, \ldots, E^k$ of objects in $\mathcal{T}$ satisfying $(K1), (K2)$ is called n-spherical for some $n \in \mathbb{Z}$ if there exist numbers $l_1, \ldots, l_k \in \mathbb{Z}$ with $\sum_{i=1}^{k} l_i = n$ such that*

(K3) $\dim_{\mathbb{F}}\mathrm{Hom}^p(E^i, E^j) = \begin{cases} 1 & \text{if } p = 0,\ i = j \text{ or } p = l_j,\ i = j+1 \\ 0 & \text{otherwise} \end{cases}$

(K4) For each $1 \le i \le k$ and for each $F \in \mathcal{T}$ the composition $\mathrm{Hom}^{l_i}(F, E^i) \times \mathrm{Hom}(E^{i+1}, F) \to \mathrm{Hom}^{l_i}(E^{i+1}, E^i) \cong \mathbb{F}$ is a perfect pairing.

One can easily check that if $(E^1, \ldots, E^k)$ is n-spherical sequence than sequences $(E^2, \ldots, E^k, E^1)$ and $(E^1[n_1], \ldots, E^k[n_k])$ are also n-spherical for each $n_1, \ldots, n_k \in \mathbb{Z}$ but maybe with another values of integers $l_1, \ldots, l_k$ from the above definition.

Proposition 4.6. *In the notation of Definition 4.5 we have $\Gamma_{l_1,\ldots,l_k} = \{Id_{\mathcal{T}} \to \Phi^!\Phi_*\}$*

Proof. Evident. □

Proposition 4.7. *If $(E^1, \ldots, E^k)$ is an n-spherical sequence of objects in $\mathcal{T}$ then $T_{(E)}$ is an autoeqivalence.*

Proof. As we know from Proposition 4.6, $\{Id_{\mathcal{T}} \to \Phi^!\Phi_*\} = \Gamma_{l_1,\ldots,l_k}$ and hence is an autoequivalence. Thus, to apply Theorem 3.5, we just need to check that the natural map $\Phi^!(F) \to \Gamma\Phi^*(F)$ is an isomorphism for each $F \in \mathcal{T}$. We have that

$$\Phi^!(F) \cong \bigoplus_{j=1}^{k} \mathrm{Hom}^*(E^j, F) \otimes S_j, \quad \Gamma_{l_1,\ldots,l_k}\Phi^*(F) \cong \bigoplus_{j=1}^{k} \mathrm{Hom}^*(F, E^{j-1})^\vee[-l_{j-1}] \otimes S_j.$$

The map $\Phi^!(F) \to \Gamma\Phi^*(F)$ at the j-th vertex and in degree d is given by the pairing

$$\mathrm{Hom}^d(E^j, F) \times \mathrm{Hom}^{l_{j-1}-d}(F, E^{j-1}) \to \mathrm{Hom}^{l_{j-1}}(E^j, E^{j-1}) \cong \mathbb{F},$$

which is perfect by the condition (K4) from Definition 4.5. Thus, this map is an isomorphism. □

4.3 The construction of braid group actions.

The next two propositions will lead us to construction of braid group actions.

Proposition 4.8. *Let $(E_1), (E_2)$ be sequences as in Lemma 2.10. Suppose that $\mathrm{Hom}^*(E_2^i, E_1^j) = 0$ for $1 \le i, j \le k$. Then $T_{(E_2)}T_{(E_1)} \cong T_{(E_1)}T_{(E_2)}$*

Proof. By Proposition **??** we have that

$$\Phi_1^!\Phi_{2*}(S_j) = \bigoplus_{i=1}^{k} \mathrm{Hom}^*(E_1^i, E_2^j) \otimes S_i = 0.$$

Thus, the statement follows from Proposition 3.6. □

Proposition 4.9. *Let $(E_1^1, \dots E_1^k), (E_2^1, \dots, E_2^k)$ be n-spherical sequences. Suppose that*

$$\sum_{j=1}^{k} \dim_{\mathbb{F}} \mathrm{Hom}^*(E_2^1, E_1^j) = 1.$$

Then

$$T_{(E_1)}T_{(E_2)}T_{(E_1)} \cong T_{(E_2)}T_{(E_1)}T_{(E_2)}.$$

Proof. Since cyclic permutations in sequence do not change the multi-twist functor, we may assume that $\dim_{\mathbb{F}}\mathrm{Hom}^*(E_2^1, E_1^1) = 1$. Note that $\mathrm{Hom}^*(E_2^1, E_1^j) = 0$ for $2 \le j \le k$. From the definition of n-spherical sequences we see that for $1 \le i, j \le k$ there is a natural isomorphism $\mathrm{Hom}^d(E_1^{i+1}, E_2^i) \cong \mathrm{Hom}^{l_i-d}(E_2^i, E_1^i)^\vee$. Thus, we have

$$\begin{gathered}
\dim_{\mathbb{F}}\mathrm{Hom}^*(E_2^i, E_1^i) = 1 \text{ for } 1 \le i \le k;\\
\dim_{\mathbb{F}}\mathrm{Hom}^*(E_1^{i+1}, E_2^i) = 1 \text{ for } 1 \le i \le k;\\
\dim_{\mathbb{F}}\mathrm{Hom}^*(E_1^j, E_2^i) = 0 \text{ for } j - i \ne 1;\\
\dim_{\mathbb{F}}\mathrm{Hom}^*(E_2^j, E_1^i) = 0 \text{ for } i \ne j.
\end{gathered}$$

Also, making shifts, we may assume that $\mathrm{Hom}^*(E_2^i, E_1^i)$ is concentrated in degree zero. Then we have that

$$\Phi_2^!\Phi_{1*}(S_i) \cong S_i$$

and hence is $\Phi_2^!\Phi_{1*}$ is an equivalence. Hence, from Proposition 3.9 we obtain the needed isomorphism. □

Now we will spell out the definition of n-spherical sequences of objects in the category $D^b(\mathcal{A})$.

Definition 4.10. *A sequence of objects $E^1, \dots, E^k$ in the category $D^b(\mathcal{A})$ is called n-spherical if it satisfies the following conditions for some integers $l_1, \dots, l_k$ such that $\sum_{i=1}^{k} l_i = n$. The first two of them are only for the case II:*

(S1) E^i has a finite resolution by $\mathcal{A}'$-injectives, for each $1 \le i \le k$;

(S2) For each $F \in D^b(\mathcal{A})$ and for each $1 \le i \le k$ graded vector spaces $\mathrm{Hom}^*(E^i, F)$ and $\mathrm{Hom}^*(F, E^i)$ are finite-dimensional over $\mathbb{F}$;

(S3) $\mathrm{Hom}^p(E^i, E^j) = \begin{cases} \mathbb{F} & \text{if } p = 0,\ i = j \text{ or } p = l_j,\ i = j+1 \\ 0 & \text{otherwise} \end{cases};$

(S4) For each $1 \le i \le k$ *and for each* $F \in D^b(\mathcal{A})$ *the composition* $\mathrm{Hom}^{l_i}(F, E^i) \times \mathrm{Hom}(E^{i+1}, F) \to \mathrm{Hom}^{l_i}(E^{i+1}, E^i) \cong \mathbb{F}$ *is a perfect pairing.*

Clearly, if the sequence $(E^1, \ldots, E^k) \in D^b(\mathcal{A})$ (resp. $D^b(\mathcal{A}')$) is n-spherical then each sequence formed by resolutions of its objects in $\mathcal{T}$ is also n-spherical (in sense of Definition 4.5). Thus, by Proposition 4.7, the multi-twist functor with respect to these resolutions is equivalence. Moreover, it does not depend on these resolutions (up to an isomorphism), by Proposition 4.4. Using the equivalence $D^b(\mathcal{A}) \cong \mathcal{T}$ we obtain the multi-twist functor $T_{(E)}$ from $D^b(\mathcal{A})$ to itself. This functor is an exact equivalence.

Definition 4.11. *An* (A^k_m)*-configuration of* n*-spherical* k*-sequences in* $D^b(\mathcal{A})$ *is a collection* $(E^1_1, \ldots, E^k_1), \ldots, (E^1_m, \ldots, E^k_m)$ *of sequences such that there exist numbers* $k_1, \ldots, k_{m-1} \in \mathbb{Z}$ *such that for* $i_1 \ne i_2$ *the following holds:*

$$\dim_{\mathbb{F}}\mathrm{Hom}^*(E^{j_1}_{i_1}, E^{j_2}_{i_2}) = \begin{cases} 1 & \text{if } i_2 = i_1+1,\ j_2 = j_1 + k_{i_1} \text{ or } i_2 = i_1 - 1,\ j_2 = j_1 + k_{i_1} - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The following Lemma is very simple and is left to the reader:

Lemma 4.12. *Let* $(E^1_1, \ldots, E^k_1), \ldots, (E^1_m, \ldots, E^k_m)$ *be a collection of* n*-spherical sequences. Then it is* (A^k_m)*-configuration iff for some indices* $j_1, \ldots, j_{m-1} \in \{1, 2, \ldots, k\}$ *and for each* $1 \le i \le m-1$ *we have*

$$\sum_{j=1}^{k} \dim_{\mathbb{F}}\mathrm{Hom}^*(E^{j_i}_i, E^j_{i+1}) = 1.$$

From Propositions 4.8 and 4.9 we obtain the following result:

Theorem 4.13. *Let* $(E^1_1, \ldots, E^k_1), \ldots, (E^1_m, \ldots, E^k_m)$ *be an* (A^k_m)*-configuration of* n*-spherical sequences. Then the multi-twist functors* $T_{(E_1)}, \ldots, T_{(E_m)}$ *satisfy the relations of the braid group* B_{m+1}*. Thus, they generate a homomorphism* $\rho : B_{m+1} \to Aut(D^b(\mathcal{A}))$*.*

In the rest part of this section we prove the following Theorem on faithfullness:

Theorem 4.14. *Suppose that* $n \ge 2k$*. Then the actionof the braid group* B_{m+1} *described in Theorem 4.13 is faihfull. Moreover, even if* $\rho(g)(E^{j_i}_i) \cong E^{j_i}_i$ *for each* $1 \le i \le m$ *and for some* $j_1, \ldots, j_m \in \{1, 2, \ldots, k\}$ *then* g *is the identity of* B_{m+1}*.*

Remark. It would be interesting to consider cases when $\mathcal{S}$ is the derived category of more complicated quivers, construct the corresponding braid group actions and prove an analogue of this result.

4.4 Dg-algebras, dg-modules and graded algebras $A^k_{m,n}$.

We refer to [K1], [K2], [K3] for the general theory of dg-algebras and modules.

First of all, we fix notation and assumptions. Let $R = \mathbb{F}^{mk}$ be the semisimple $\mathbb{F}$-algebra with generators e^j_i, where $1 \le i \le m$, $1 \le j \le k$, and with relations

$$e^{j_1}_{i_1} e^{j_2}_{i_2} = \begin{cases} e^{j_1}_{i_1} & \text{if } i_1 = i_2,\ j_1 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

In this section, by a graded algebra we mean a $\mathbb{Z}$-graded algebra equipped with a unital homomorphism of algebras $\iota_A : R \to A^0$. All morphisms $A \to B$ of graded algebras are required to be unital and commute with ι_A, ι_B. Note that each graded algebra is an R-bimodule; for $a \in A$ we write $e^j_i a$ and $a e^j_i$ instead of $\iota_A(e^j_i)a$ and $a\iota_A(e^j_i)$ respectively.

Definition 4.15. *A differential graded algebra (= dg-algebra) $\mathcal{A}$ is a graded algebra A equipped with a differential $d_A : A \to A$ (i.e. a homogeneous homomorphism of graded $\mathbb{F}$-spaces of degree 1 which satisfies $d_A^2 = 0$) which is also a derivation of A and satisfies $d_A \iota_A = 0$.*

By $H(\mathcal{A})$ we denote the cohomology of $\mathcal{A}$ considered as a graded algebra.

A morphism of dg-algebras $\mathcal{A} = (A, d_A)$ and $\mathcal{B} = (B, d_B)$ is a morphism $f : A \to B$ of graded algebras such that $fd_A = d_B f$. Morphism f is called quasi-isomorphism if it induces isomorphism in cohomology. Dg-algebras $\mathcal{A}$ and $\mathcal{B}$ are called quasi-isomorphic if there exists a chain of dg-algebras and quasi-isomorphisms $\mathcal{A} \leftarrow \mathcal{C}_1 \to \dots \leftarrow \mathcal{C}_k \to \mathcal{B}$.

In fact, $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic iff there exists a dg-algebra $\mathcal{C}$ and quasi-isomorphisms $\mathcal{C} \to \mathcal{A}$, $\mathcal{C} \to \mathcal{B}$ (see [K2]).

By a graded module over a graded algebra we always mean a right module. Each such module M is obviously a right R-module and for $m \in M$ we write me_i^j instead of $m\iota_A(e_i^j)$.

Definition 4.16. *A differential graded module (=dg-module) over a dg-algebra $\mathcal{A} = (A, d_A)$ is a pair $(M) = (M, d_M)$ which consists of a graded A-module M and a differential $d_M : M \to M$ such that $d_M(xa) = d_M(x)a + (-1)^{deg(x)} x d_A(a)$ for all homogeneous $x \in M$, $a \in A$.*

A morphism of dg-modules is a morphism of graded modules which is also a morphism of complexes.

We denote the category of dg-modules over a dg-algebra $\mathcal{A}$ by $Dgm(\mathcal{A})$. The homotopy category (which has the same objects and whose morphisms are homotopy classes of morphisms of dg-modules) is denoted by $K(\mathcal{A})$. It has an obvious structure of a triangulated category. The derived category $D(\mathcal{A})$ is, by definition, the localization of $K(\mathcal{A})$ by quasi-isomorphisms. It is also triangulated. Moreover, the category $D(A)$ is always algebraic (see [K1] for explanation).

One can easily define the convolution of finite complexes $\dots \to \mathcal{C}_i \to \mathcal{C}_{i+1} \to \dots$ of objects in $Dgm(\mathcal{A})$, we denote the result by $\{\dots \to \mathcal{C}_i \to \mathcal{C}_{i+1} \to \dots\}$. As usually, $\{C_{-1} \to C_0\}$ corresponds to a cone in $K(\mathcal{A})$ (and in $D(\mathcal{A})$).

Further, for each morphism $f : \mathcal{A} \to \mathcal{B}$ of dg-algebras we have obvious functor $f^* : Dgm(\mathcal{B}) \to Dgm(\mathcal{A})$, which is the restriction of scalars. It is easy to see that it descends to exact functors $K(\mathcal{B}) \to K(\mathcal{A})$ and further $D(\mathcal{B}) \to D(\mathcal{A})$. These functors will be also denoted by f^*. The following Proposition is proved in [BeL].

Proposition 4.17. *If $f : \mathcal{A} \to \mathcal{B}$ is quasi-isomorphism, then $f^* : D(\mathcal{B}) \to D(\mathcal{A})$ is an equivalense.*

Now we spell out the definition of the standard multi-twist functors t_i which generalize standard twist functors defined in [STh]. We denote dg-modules $e_i^j \mathcal{A}$ by $\mathcal{P}_i^j$.

Definition 4.18. *The standard multi-twist functors $t_i : Dgm(\mathcal{A}) \to Dgm(\mathcal{B})$ are defined as follows:*

$$t_i(\mathcal{M}) = \{\bigoplus_{j=1}^{k} \mathcal{M} e_i^j \otimes (P)_i^j \to (M)\}$$

Here the map is the multiplication map on each component.

One can see that t_i is isomorphic to $T_{\mathcal{P}_i^1, \dots, \mathcal{P}_i^k}$ in sence of formula (1), because $\mathcal{M}e_i^j$ is isomorphic to $RHom(\mathcal{P}_i^j, \mathcal{M})$, but in general $\mathcal{P}_i^1, \dots, \mathcal{P}_i^k$ do not satisfy thecondition analogous to (K2).

The following Lemma is left to the reader and is analogous to Lemma 4.2 in [STh].

Lemma 4.19. *Let $f : \mathcal{A} \to \mathcal{B}$ be a quasi-isomorphism of dg-algebras. Then the following diagram commutes (up to isomorphism):*

$$\begin{array}{ccc} D(\mathcal{B}) & \xrightarrow{t_i} & D(\mathcal{B}) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ D(\mathcal{A}) & \xrightarrow{t_i} & D(\mathcal{A}) \end{array}$$

Now let $\mathcal{T}$ be the category defined in subsection 4.1. Let E_i^j, where $1 \le i \le m$, $1 \le j \le k$, be objects in $\mathcal{T}$. Put $E = \bigoplus_{1\le i\le m, 1\le j\le k} E_i^j$. Then the complex of vector spaces

$$\mathrm{REnd}(E) = \mathrm{RHom}(E,E) = \bigoplus_{1\le i_1,i_2\le m, 1\le j_1,j_2\le k} \mathrm{RHom}(E_{i_1}^{j_1}, E_{i_2}^{j_2})$$

has natural structure of a dg-algebra. The definition of the composition is obvious, and $\iota_{end(E)}(e_i^j) = Id_{E_i^j} \in \mathrm{RHom}(E_i^j, E_i^j)$ for $1 \le i \le m$, $1 \le j \le k$. In particular, $\mathcal{P}_i^j = \mathrm{RHom}(E, E_i^j)$. Further, we have that $\mathrm{RHom}(E,F)$ is a dg-module over $REnd(E)$ for each $F \in \mathcal{T}$ and we obtain an exact functor $\Psi_E : \mathrm{RHom}(E,-) : \mathcal{T} \to D(REnd(E))$. The following Lemma is left to the reader and is analogous to Lemma 4.3 in [STh].

Lemma 4.20. *Suppose that the objects E_i^j, where $1 \le i \le m$, $1 \le j \le k$ satisfy conditions (K1), (K2) of Definition 2.4. Then the following diagram commutes (up to an isomorphism)*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{T_{(E_i)}} & \mathcal{T} \\ {\scriptstyle \Psi_E}\downarrow & & {\scriptstyle \Psi_E}\downarrow \\ D(REnd(E)) & \xrightarrow{t_i} & D(REnd(E)) \end{array}$$

From now on we assume that $(E_1^1, \dots, E_1^k), \dots, (E_m^1, \dots, E_m^k)$ is an (A_m^k)-configuration of n-spherical sequences in the category $\mathcal{T}$. We want to describe the cohomology algebra of $REnd(E)$. To do that, it is convenient first to shift and make cyclic permutations of objects in these sequences (recall that if $(E^1, \dots, E^k)$ is n-spherical sequence then sequences $(E^2, \dots, E^k, E^1)$ and $(E^1[n_1], \dots, E^k[n_k])$ are also n-spherical).

Put

$$d_i = \left[\frac{in}{2k}\right] - \left[\frac{(i-1)n}{2k}\right], i \in \mathbb{N}.$$

Using cyclic permutations and shifting we may assume that for $1 \le i \le m-1$, $1 \le j \le k$

$$\dim_{\mathbb{F}} \mathrm{Hom}^{d_{2j+i-2}}(E_{i+1}^j, E_i^j) = \dim_{\mathbb{F}} \mathrm{Hom}^{d_{2j+i-1}}(E_i^{j+1}, E_{i+1}^j) = 1. \quad (3)$$

Let $\Gamma_{m,n}^k$ be the graded quiver shown in Figure 1. It has mk vertices which we denote by (i^j), where $1 \le i \le m$, $1 \le j \le k$. For convenience, we put $i^j = i^{j+k}$. Further, for each $1 \le i \le m-1$, $1 \le j \le k$ there is one arrow from (i^j) to $((i+1)^j)$ of degree d_{2j+i-2}, and one arrow from $((i+1)^j)$ to (i^{j+1}) of degree d_{2j+i-1}. The path starting at $i_0^{j_0}$, pathing through $i_1^{j_1}, i_2^{j_2}, \dots, i_{s-1}^{j_{s-1}}$ and stopping at $i_s^{j_s}$, is denoted by $(i_0^{j_0}|i_1^{j_1}|\dots|i_s^{j_s})$. Let $\mathbb{F}[\Gamma_{m,n}^k]$ be the path algebra of $\Gamma_{m,n}^k$. Put $\iota_{\mathbb{F}[\Gamma_{m,n}^k]}(e_i^j) = (i^j)$. Note that $\mathbb{F}[\Gamma_{m,n}^k]$ is a graded algebra as defined above.

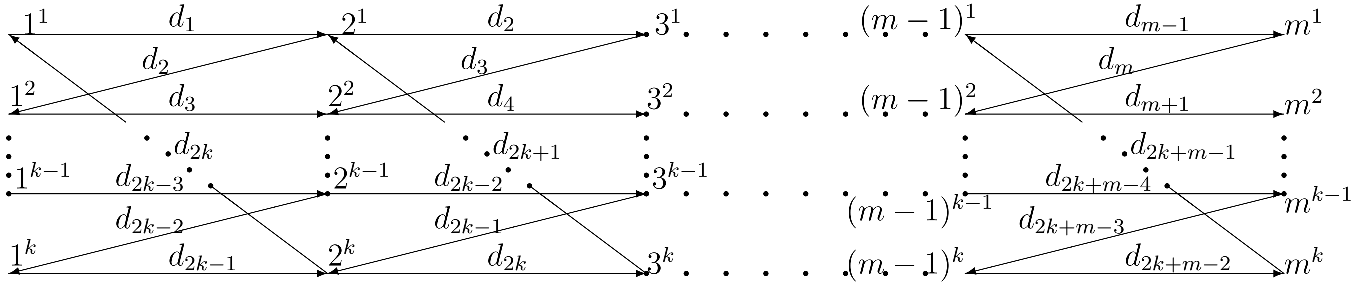

Figure 1.

For $k = 1$, $\Gamma_{m,n}^1$ is a graded quiver defined in [STh]. It is shown in Figure 2. Here

$$q_i = \begin{cases} \frac{n}{2} & \text{for } n \text{ even} \\ \frac{n+(-1)^i}{2} & \text{for } n \text{ odd.} \end{cases}$$

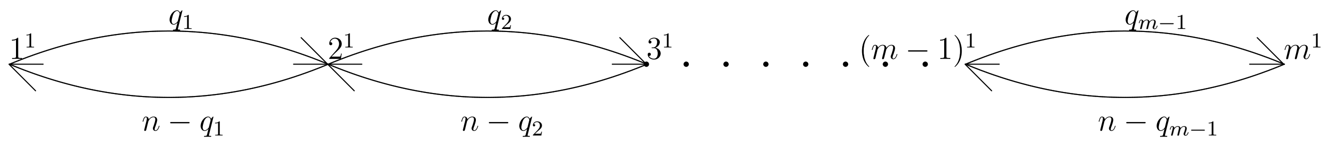

Figure 2.

Consider the two-sided ideal $J^k_{m,n} \subset \mathbb{F}[\Gamma^k_{m,n}]$ generated by the elements $(i^j|(i+1)^j|i^{j+1}) - (i^j|(i-1)^j|i^{j+1})$, $((i-1)^j|i^j|(i+1)^j)$, $((i+1)^{j-1}|i^j|(i-1)^{j+1})$, where $2 \le i \le m-1$, $1 \le j \le k$, for $m \ge 3$, and is generated by the elements $(1^j|2^j|1^{j+1}|2^{j+1})$, $(2^{j-1}|1^j|2^j|1^{j+1})$, where $1 \le j \le k$, for $m = 2$.

Definition 4.21. *In the above notation we define* $A^k_{m,n} = \mathbb{F}[\Gamma^k_{m,n}]/J^k_{m,n}$.

Further, for $k = 1$, the algebra $A^1_{m,n}$ is defined similarly and coincides with the algebra $A_{m,n}$ defined in [STh].

Since $J^k_{m,n}$ is homogeneous, $A^k_{m,n}$ is a graded algebra. Moreover, it is finite-dimensional over $\mathbb{F}$ of dimension $(4m-2)k$. Here is its (homogeneous) basis:

$$\begin{cases} (i^j), 1 \le i \le m, 1 \le j \le k, \\ (i^j|(i+1)^j), 1 \le i \le m-1, 1 \le j \le k, \\ ((i+1)^j|i^j), 1 \le i \le m-1, 1 \le j \le k, \\ (1^j|2^j|1^{j+1}), (2^j|3^j|2^{j+1}) = (2^j|1^{j+1}|2^{j+1}), \ldots, ((m-1)^j|m^j|(m-1)^{j+1}) = \\ = (m-1)^j|(m-2)^j|(m-1)^{j+1}, (m^j|(m-1)^{j+1}|m^{j+1}), 1 \le j \le k. \end{cases}$$

Note that we use the same notation for paths in $\mathbb{F}[\Gamma^k_{m,n}]$ and their classes in $A^k_{m,n}$. The following Lemma is left to the reader and is analogous to Lemma 4.10 in [STh].

Lemma 4.22. *Suppose that our* (A^k_m)*-configuration satisfies* (3). *Then* $H(end(E)) \cong A^k_{m,n}$.

Such a complicated choice of grading on $A^k_{m,n}$ will be very useful in the next subsection.

4.5 Intrinsic formality of $A^k_{m,n}$

This subsection motivates our dealing with $H(end(E))$. Note that each graded algebra A can be considered as a dg-algebra with zero differential. Further, if a dg-algebra $\mathcal{A}$ is quasi-isomorphic to $H(\mathcal{A})$ then $\mathcal{A}$ is called formal.

Definition 4.23. *Graded algebra* A *is called intrinsically formal if each dg-algebra* $\mathcal{B}$ *with* $H(\mathcal{B}) \cong A$ *is formal.*

In this subsection we prove the following Lemma:

Lemma 4.24. *Algebra* $A^k_{m,n}$ *is intrinsically formal for* $m \ge 2$, $n \ge 2k$.

First of all, we need to say few words about Hochschild cohomology. An augmented graded algebra is a graded algebra A equipped with a homomorphism $\epsilon_A : A \to R$ of graded algebras (grading on R is trivial) such that $\epsilon_A \iota_A = Id_R$. Its kernel A^+ is called the augmentation ideal. Let M be a graded bimodule over A. The Hochschild cohomology $HH^*(A, M)$ is the cohomology of the cochain complex $C^q(A, M) = \mathrm{Hom}_{R-R}((A^+)^{\otimes_R q}, M)$ with the differential

$$d^q(\varphi)(a_1, \ldots, a_{q+1}) = (-1)^{\epsilon} a_1 \varphi(a_2, \ldots, a_{q+1}) + \\ + \sum_{i=1}^{q} (-1)^{\epsilon_i} \varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{q+1}) - (-1)^{\epsilon_q} \varphi(a_1, \ldots, a_q) a_{q+1},$$

where $\epsilon = qdeg(a_1)$, $\epsilon_i = deg(a_1) + \cdots + deg(a_i) - i$. Further, if M is an A-bimodule then $M\langle s\rangle$ is its shift by s in the usual sense, and the bimodule structure on $M\langle s\rangle$ is defined as follows: $m \cdot a = ma$, $a \cdot m = (-1)^{sdeg(a)}am$.

Now we state a sufficient condition for intrinsic formality of graded algebra.

Theorem 4.25. *Let A be a graded algebra. If $HH^q(A, A\langle 2-q\langle) = 0$ for each $q > 2$ then A is intrinsically formal.*

Proof. This is Theorem 4.7 in [STh]. □

Note that $A^k_{m,n}$ has an obvious augmentation which sends (i^j) to e^j_i and is zero on all other basis elements. If $n \geq 2k$ then the image of $\iota_{A^k_{m,n}}$ coincides with $(A^k_{m,n})^0$ there are no other augmentations. $(A^k_{m,n})^+$ consists of elements of positive degree. Our proof of Lemma 4.24 is by straightforward computation of Hochschild cohomology.

Proof of Lemma 4.24. For convenience, we put $A = A^k_{m,n}$. First we make some basic remarks. We have that the degree of a path in $\Gamma^k_{m,n}$ depends only on its vertex of beginning and on its length. If it begins in the vertex (i^j) and has length l then its degree equals to $\sum\limits_{p=0}^{l-1} d_{2j+i+p-2} = [\frac{(2j+i+l-3)n}{2k}] - [\frac{(2j+i-3)n}{2k}]$. Since $[a+b] \geq [a]+[b]$ for $a, b \in \mathbb{R}$, we have that the degree of each path of length l is $\geq [\frac{nl}{2k}]$. Further, note that path of length $l > 2$ is zero in $A^k_{m,n}$. Thus, $A\langle 2-q\rangle$ is concentrated in degrees $\leq \lceil \frac{n}{k}\rceil + q - 2$, because $[a] - [b] \leq \lceil a-b\rceil$.

The vector $\mathbb{F}$-space $(A^+)^{\otimes_R q}$ is generated by the elements of the form

$$(i_{1,0}^{j_{1,0}}|\dots|i_{1,l_1}^{j_{1,l_1}}) \otimes \dots \otimes (i_{q,0}^{j_{q,0}}|\dots|i_{q,l_q}^{j_{q,l_q}}).$$

Such product is nonzero only if we have $i_{p+1,0} = i_{p,l_p}$, $j_{p+1,0} = j_{p,l_p}$ for $0 \leq p \leq q-1p$ (since the tensor product is over R). Thus, its degree is $\geq [\frac{n(l_1+\dots+l_q)}{2k}]$. In particular, $(A^+)^{\otimes_R q}$ is concentrated in degrees $\geq [\frac{nq}{2k}]$.

Now we want to know when $C^q(A, A\langle 2-q\rangle) \neq 0$. Let $\varphi \in C^q(A, A < 2-q >) = \mathrm{Hom}_{R-R}((A^+)^{\otimes_R q}, M)$, and let $c = (i_{1,0}^{j_{1,0}}|\dots|i_{1,l_1}^{j_{1,l_1}}) \otimes \dots \otimes (i_{q,0}^{j_{q,0}}|\dots|i_{q,l_q}^{j_{q,l_q}})$. Then $(i_{1,0}^{j_{1,0}})\varphi(c) = \varphi(c) = \varphi(c)(i_{q,l_q}^{j_{q,l_q}})$. Suppose that $\varphi(c) \neq 0$. It is possible only in one of the following cases:

$$\begin{cases} 1) i_{1,0} = i_{q,l_q}, \quad j_{1,0} = j_{q,l_q}, \quad \varphi(c) \in \mathbb{F}(i_{1,0}^{j_{1,0}}); \\ 2) i_{1,0} = i_{q,l_q} - 1, \quad j_{1,0} = j_{q,l_q}, \quad \varphi(c) \in \mathbb{F}(i_{1,0}^{j_{1,0}}|(i_{1,0}+1)^{j_{1,0}}); \\ 3) i_{1,0} = i_{q,l_q} + 1, \quad j_{1,0} = j_{q,l_q} - 1, \quad \varphi(c) \in \mathbb{F}(i_{1,0}^{j_{1,0}}|(i_{1,0}-1)^{j_{1,0}+1}); \\ 4) i_{1,0} = i_{q,l_q}, \quad j_{1,0} = j_{q,l_q} - 1, \quad \varphi(c) \in \mathbb{F}(i_{1,0}^{j_{1,0}}|(i_{1,0}+1)^{j_{1,0}}|i_{1,0}^{j_{1,0}+1}) \text{ for } i_{1,0} \leq m-1 \text{ and} \\ \varphi(c) \in \mathbb{F}(m^{j_{1,0}}|(m-1)^{j_{1,0}+1}|m^{j_{1,0}+1}) \text{ for } i_{1,0} = m. \end{cases}$$

The case 1. We have $deg(\varphi(c)) = q-2$ and $deg(c) \geq [\frac{nq}{2k}] \geq q$. This leads to the contradiction.

The case 2. Since $q > 2$ and all nonzero paths in A are of length ≤ 2 and each nonzero path can have not more than one arrow in non-horizontal direction, we have that $q \geq k+1$. Further, $deg(c) \geq [\frac{nq}{2k}]$, $deg(\varphi(c)) \leq \lceil \frac{n}{2k}\rceil + q - 2$. Thus,

$$0 = deg(c) - deg(\varphi(c)) \geq \left[\frac{nq}{2k}\right] - \left\lceil \frac{n}{2k}\right\rceil - q + 2 > \frac{n(q-1)}{2k} - q =$$
$$= \left(\frac{n}{2k} - 1\right)q - \frac{n}{2k} \geq \left(\frac{n}{2k} - 1\right)(k+1) - \frac{n}{2k} = \frac{n-2k-2}{2}$$

Hence, we have that $n \le 2k+1$. Then $\deg(c) \ge \left[\frac{nq}{2k}\right] \ge q > \lceil\frac{n}{2k}\rceil + q - 2 \ge deg(\varphi(c))$, and this leads to a contradiction.

The case 3. This case is absolutely analogous to the previous one.

The case 4. As in case 2, we have $q \ge k+1$. Further, $deg(c) \ge [\frac{nq}{2k}]$, $deg(\varphi(c)) \le \lceil\frac{n}{k}\rceil + q - 2$. Thus,

$$0 = deg(c) - deg(\varphi(c)) \ge \left[\frac{nq}{2k}\right] - \left\lceil\frac{n}{k}\right\rceil - q + 2 > \frac{n(q-2)}{2k} - q =$$
$$= (\frac{n}{2k} - 1)q - \frac{n}{k} \ge \left(\frac{n}{2k} - 1\right)(k+1) - \frac{n}{k} = \frac{(n-2k-4)(k-1)-4}{2k}$$

Hence, we have the following possibilities:

$$\begin{cases} k \ge 5, 2k \le n \le 2k+4 \\ k = 4, 8 \le n \le 13 \\ k = 3, 6 \le n \le 11 \\ k = 2, 4 \le n \le 11. \end{cases}$$

We start with the consideration of the cases $n = 2k, 2k+1$, $k \ge 2$.

The case $n = 2k$. We have that $(A^+)^{\otimes_R p}$ is concentrated in degrees $\ge p$, and $A\langle 2-q\rangle$ is concentrated in degrees $\le q$. For $p = q+1$, we see that $C^{q+1}(A, A\langle 2-q\rangle) = 0$. Thus, we need to proof that the map $d^{q-1} : C^{q-1}(A, A\langle 2-q\rangle) \to C^q(A, A\langle 2-q\rangle)$ is surjective. For each path $(i_0^{j_0}|\dots|i_q^{j_q}) \in \mathbb{F}[\Gamma_{m,n}^k]$ with $i_q = i_0$, $j_q = j_0+1$ let $\varphi_{i_0^{j_0},\dots,i_q^{j_q}} \in C^q(A, A\langle 2-q\rangle)$ be the homomorphism which sends $(i_0^{j_0}|i_1^{j_1})\otimes\dots\otimes(i_{q-1}^{j_{q-1}}|i_q^{j_q})$ to $(i_0^{j_0}|i_{q-1}^{j_{q-1}}|i_q^{j_q})$, and sends the other basis elements of $(A^+)^{\otimes_R q}$ to zero. Then the homomorphisms $\varphi_{i_0^{j_0},\dots,i_q^{j_q}}$ form a basis of $C^q(A, A\langle 2-q\rangle)$. Indeed, for grading reasons we have

$$\varphi((i_{1,0}^{j_{1,0}}|\dots|i_{1,l_1}^{j_{1,l_1}}) \otimes \dots \otimes (i_{q,0}^{j_{q,0}}|\dots|i_{q,l_q}^{j_{q,l_q}})) = 0$$

whenever $l_p > 1$ for at least one p.

Now we take some elements in $C^{q-1}(A, A\langle 2-q\rangle)$. For each path $(i_0^{j_0}|\dots|i_{q-1}^{j_{q-1}}) \in \mathbb{F}[\Gamma_{m,n}^k]$ with either $i_{q-1} = i_0+1, j_{q-1} = j_0$ or $i_{q-1} = i_0-1, j_{q-1} = j_0+1$ let $\varphi'_{i_0^{j_0},\dots,i_{q-1}^{j_{q-1}}} \in C^{q-1}(A, A\langle 2-q\rangle)$ be the homomorphism which sends $(i_0^{j_0}|i_1^{j_1})\otimes\dots\otimes(i_{q-1}^{j_{q-1}}|i_{q-1}^{j_{q-1}})$ to $(i_0^{j_0}|i_{q-1}^{j_{q-1}})$ and sends other basis elements to zero. By the very definition, we have that $d^{q-1}(\varphi'_{i_0^{j_0},\dots,i_{q-1}^{j_{q-1}}}) = \pm\varphi_{i_0^{j_0},\dots,i_{q-1}^{j_{q-1}},i_0^{j_0+1}} \pm \varphi_{i_{q-1}^{j_{q-1}-1},i_0^{j_0},\dots,i_{q-1}^{j_{q-1}}}$. Further, for each path $(i_0^{j_0}|\dots|i_q^{j_q}) \in \mathbb{F}[\Gamma_{m,n}^k]$ with $i_2 = i_0$ (and hence $j_2 = j_0+1$), $i_q = i_0$, $j_q = j_0+1$. Let $\varphi''_{(i_0^{j_0}|\dots|i_q^{j_q})} \in C^{q-1}(A, A < 2-q >)$ be homomorphism which sends $(i_0^{j_0}|i_1^{j_1}|i_2^{j_2})\otimes(i_2^{j_2}|i_3^{j_3})\dots\otimes\dots(i_{q-1}^{j_{q-1}}|i_q^{j_q})$ to $(i_0^{j_0}|i_1^{j_1}|i_q^{j_q})$ and sends other basis elements to zero. Then we have that

$$d^{q-1}(\varphi''_{i_0^{j_0},\dots,i_q^{j_q}}) = \begin{cases} \pm\varphi_{i_0^{j_0},(i_0+1)^{j_0},i_2^{j_2},\dots,i_q^{j_q}} \pm \varphi_{i_0^{j_0},(i_0-1)^{j_0+1},i_2^{j_2},\dots,i_q^{j_q}} & \text{for } 2 \le i_0 \le m-1; \\ \pm\varphi_{i_0^{j_0},(i_0+1)^{j_0},i_2^{j_2},\dots,i_q^{j_q}} & \text{for } i_0 = 1. \end{cases}$$

Therefore, we obtain the following relations in $HH^q(A, A\langle 2-q\rangle)$:
1) $[\varphi_{i_0^{j_0},\dots,i_q^{j_q}}] = \pm[\varphi_{i_1^{j_1},\dots,i_q^{j_q},i_1^{j_1+1}}]$;
2) $[\varphi_{i_0^{j_0},\dots,i_q^{j_q}}] = \pm[\varphi_{i_0^{j_0},(i_1-2)^{j_0+1},i_2^{j_2}\dots,i_q^{j_q}}]$ if $i_0 = i_2$, $i_1 = i_0+1$;
3) $[\varphi_{i_0^{j_0},\dots,i_q^{j_q}}] = 0$ if $i_0 = i_2 = 1$ (and hence $i_1 = 1$).

Using these relations it is easy to prove that $HH^q(A, A\langle 2-q\rangle) = 0$. Indeed, show that each class $[\varphi_{i_0^{j_0},\dots,i_q^{j_q}}]$ is zero. Using the relation 1), we may assume that i_1 is maximal of all i_p. Then $i_0 = i_2 = i_1 - 1$. For $i_1 = 2$ the relation 3) says that $[\varphi_{i_0^{j_0},\dots,i_q^{j_q}}] = 0$. For $i_1 = 2$ the relation 2) shows that our class coincides, up to a sign, with another such class, for which the sum $(i_1 + \cdots + i_q)$ is smaller. Thus, by induction over $(i_1 + \cdots + i_q)$ we obtain that all our classes are zero.

The case $n = 2k+1$. We prove that this case is impossible. Note that $deg(c) \geq [\mathfrak{n}(\mathfrak{l}_1 + \cdots + \mathfrak{l}_\mathfrak{q})2k] \geq [\frac{(2k+1)(2k+2)}{2k}] = 2k+3$. Further, $\lceil \frac{n}{k} \rceil + q - 2 = q + 1 \geq deg(\varphi(c))$. Hence, $q \geq 2k+2$. In the case $q = 2k+2$ we have that, on one side, since $deg(\varphi(c)) = 2k+3$, then the degree of $(i_{1,0}^{j_{1,0}}|i_{q,l_q-1}^{j_{q,l_q-1}}|i_{q,l_q}^{j_{q,l_q}})$ in A equals to 3, and, on the other side, since $deg(c) = 2k+3$, the degree of $(i_{1,0}^{j_{1,0}}|i_{q,l_q-1}^{j_{q,l_q-1}}|i_{q,l_q}^{j_{q,l_q}})$ in A equals to $deg(c) - 2k - 1 = 2$, this leads to a contradiction.

Suppose that $q > 2k+2$. Then we have that $l_1 + \cdots + l_q > 2k+2$, and hence $l_1 + \cdots + l_q \geq 4k+2$ and $deg(c) \geq [\frac{(2k+1)(4k+2)}{2k}] = 4k+4$, and, analogously, $q \geq 4k+3$. Thus,

$$0 = deg(c) - deg(\varphi(c)) \geq \left[\frac{(2k+1)q}{2k}\right] - \lceil \frac{2k+1}{k} \rceil - q + 2 \geq$$
$$\geq \left[\frac{(2k+1)(4k+3)}{2k}\right] + q - 4k - 3 - \lceil \frac{2k+1}{k} \rceil - q + 2 = \left[\frac{(2k+1)(4k+3)}{2k}\right] - 4k - 4 = \left[\frac{2k+3}{2k}\right] \geq 1,$$

this leads to a contradiction.

All the other cases. The other cases can be considered as follows. Let $n = 2k+t$. Using similar arguments one obtains that

$$\left\lceil \frac{t}{k} \right\rceil + q = \left\lceil \frac{n}{k} \right\rceil + q - 2 \geq deg(\varphi(c)) = deg(c) \geq \left[\frac{n(2k+2)}{2k}\right] \geq n + 2 + \left[\frac{t}{k}\right],$$

and hence $q \geq n + [\frac{n}{k}] - \lceil \frac{t}{k} \rceil$. Thus,

$$0 = deg(c) - deg(\varphi(c)) \geq \left[\frac{nq}{2k}\right] - \left\lceil \frac{n}{k} \right\rceil - q + 2 \geq \left[\frac{n(n + [\frac{n}{k}] - \lceil \frac{t}{k} \rceil)}{2k}\right] + q - n - \left[\frac{n}{k}\right] + \left\lceil \frac{t}{k} \right\rceil - \left\lceil \frac{n}{k} \right\rceil - q + 2 \geq$$
$$\geq \left[\frac{n(n + [\frac{n}{k}] - \lceil \frac{t}{k} \rceil)}{2k}\right] - n - \left[\frac{n}{k}\right]$$

Then one checks that the expression on the right is positive in our cases, and this leads to a contradiction. This finishes the proof. □

4.6 Faithfulness of action on $D'(A^k_{m,n})$ and proof of main theorem.

Due to Lemma 4.17, Lemma 4.19, Proposition 4.20 and Lemma 4.24, we may work with the category $D(A^k_{m,n})$ instead of $D(REnd(E))$. It will be more convenient for us to consider its full subcategory $D'(A^k_{m,n})$ which consists of dg-modules with finite-dimensional cohomology. Then one can check that the sequences $(\mathcal{P}^1_i, \ldots, \mathcal{P}^k_i)$ are n-spherical in $D'(A^k_{m,n})$. Further, the collection of sequences $(\mathcal{P}^1_1, \ldots, \mathcal{P}^k_1), \ldots, (\mathcal{P}^1_m, \ldots, \mathcal{P}^k_m)$ is an (A^k_m)-configuration. Thus, we obtain the following to lemmas.

Lemma 4.26. *The multi-twist functors* $t_i : D'(A) \to D'(A)$ *are autoequivalences.*

Lemma 4.27. *The autoequivalences* $t_i : D'(A) \to D'(A)$ *satisfy the relations of braid group* B_{m+1}*:*

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \text{ for } 1 \leq i \leq m-1,\ t_i t_j = t_j t_i \text{ for } |i-j| > 1.$$

It is useful to spell out the definition of the dual multi-twist functors which are quasi-inverse to t_i.

Definition 4.28. *The dual multi-twist functors* $t'_i : D'(A^k_{m,n}) \to D'(A^k_{m,n})$ *are given by*

$$t'_i(\mathcal{M}) = \{\eta_i : \mathcal{M} \to \bigoplus_{j=1}^{k} \mathcal{M}e_i^{j+1} \otimes \Omega_i^j\}.$$

Here $\mathcal{M}$ *is placed in degree zero,* $\Omega_i^j = \mathcal{P}_i^j[d_{2j-1} + d_{2j}]$*, and the composition of* η_i *and projection onto* j*-th component is*

$$\eta_i^j(x) = x(i^j|(i+1)^j|i^{j+1}) \otimes (i^j) + x((i+1)^j|i^{j+1}) \otimes (i^j|(i+1)^j) + \\ + x((i-1)^{j+1}|i^{j+1}) \otimes (i^j|(i-1)^{j+1}) + x(i^{j+1}) \otimes (i^j|(i+1)^j|i^{j+1})$$

The definition is correct because the element $(i^j|(i+1)^j|i^{j+1}) \otimes (i^j) + ((i+1)^j|i^{j+1}) \otimes (i^j|(i+1)^j) + ((i-1)^{j+1}|i^{j+1}) \otimes (i^j|(i-1)^{j+1}) + (i^{j+1}) \otimes (i^j|(i+1)^j|i^{j+1}) \in Ae_i^{j+1} \otimes \Omega_i^j$ is central, i.e left and right multiplications on it with any $a \in A^k_{m,n}$ give the same result.

Lemma 4.27 gives us the homomorphism $\rho^k_{m,n} : B_{m+1} \to Aut(D'(A^k_{m,n}))$. We are going to prove the following

Theorem 4.29. *Let* $g \in B_{m+1}$*. Suppose that* $\rho^k_{m,n}(g)(\mathcal{P}_i^j) \cong \mathcal{P}_i^j$ *for all* $1 \le i \le m$*,* $1 \le j \le k$*. Then* g *is central. Moreover, if* $n \ge 2k$ *then* g *is the identity element of* B_{m+1}*.*

Recall that the center of B_{m+1} is isomorphic to $\mathbb{Z}$ and is generated by the element $(g_1 \dots g_m)^{m+1}$, where g_i are the standard generators.

Lemma 4.30. *For each* $1 \le i \le m$*,* $1 \le j \le k$ *one has* $(t_1 \dots t_m)^{m+1}(\mathcal{P}_i^j) \cong \mathcal{P}_i^{j+m+1}[2m - d_{2j+i-2} - \dots - d_{2j+i+2m-1}]$*.*

Proof. Note that

$$t_{i_1}(\mathcal{P}_{i_2}^j) \cong \begin{cases} \mathcal{P}_i^{j+1}[1 - d_{2j+i-2} - d_{2j+i-1}] & \text{if } i_1 = i_2 = i, \\ \{\mathcal{P}_{i_2-1}^{j+1}[-d_{2j+i}] \to \mathcal{P}_{i_2}^j\} & \text{if } i_1 = i_2 - 1, \\ \{\mathcal{P}_{i_2+1}^{j}[-d_{2j+i}] \to \mathcal{P}_{i_2}^j\} & \text{if } i_1 = i_2 + 1, \\ \mathcal{P}_{i_2}^j & \text{if } |i_1 - i_2| > 1. \end{cases}$$

Using this observation, it is easy to obtain that

$$t_1 \dots t_m(\mathcal{P}_i^j) \cong \begin{cases} \mathcal{P}_{i+1}^j[1 - d_{2j+i-2}] & \text{for } 1 \le i \le m-1, \\ \mathcal{M}_0^j & \text{for } i = m, \end{cases}$$

where $\mathcal{M}_0^j = \{\mathcal{P}_1^{j+m}[1 - d_{m+2j-2} - \dots - d_{2m+2j-2}] \to \dots \to \mathcal{P}_m^{j+1}[1 - d_{m+2j-2} - d_{m+2j-1}]\}$ (the last term is placed in degree zero). Similarly,

$$t'_{i_1}(\mathcal{P}_{i_2}^j) \cong \begin{cases} \mathcal{P}_i^{j-1}[d_{2j+i-4} + d_{2j+i-3} - 1] & \text{if } i_1 = i_2 = i, \\ \{\mathcal{P}_{i_2}^j \to \mathcal{P}_{i_2-1}^j[d_{2j+i_2-3}]\} & \text{if } i_1 = i_2 - 1, \\ \{\mathcal{P}_{i_2}^j \to \mathcal{P}_{i_2+1}^{j-1}[d_{2j+i_2-3}]\} & \text{if } i_1 = i_2 + 1, \\ \mathcal{P}_{i_2}^j & \text{if } |i_1 - i_2| > 1. \end{cases}$$

Further, $t'_m \dots t'_1(\mathcal{P}_1^j) \cong \{\mathcal{P}_1^{j-1}[d_{2j-2} + d_{2j-3} - 1] \to \dots \to \mathcal{P}_m^{j-m}[d_{2j-2} + \dots + d_{2j-m-2} - 1]\} = \mathcal{M}_0^{j-m-1}[d_{2j-m-4} + \dots + d_{2j-2} - m - 1]$. Therefore, $t_1 \dots t_m(\mathcal{M}_0^j) = \mathcal{P}_1^{j+m+1}[m + 1 - d_{2j+m-2} - \dots d_{2j+2m}]$ Finally, we have that

$$t_1 \dots t_m(\mathcal{P}_1^j) = \mathcal{P}_2^j[1 - d_{2j-1}],$$
$$(t_1 \dots t_m)^2(\mathcal{P}_1^j) = \mathcal{P}_3^j[2 - d_{2j-1} - d_{2j}],$$
$$\dots$$
$$(t_1 \dots t_m)^{m-1}(\mathcal{P}_1^j) = \mathcal{P}_m^j[m - 1 - d_{2j-1} - d_{2j} - \dots - d_{2j+m-3}],$$
$$(t_1 \dots t_m)^m(\mathcal{P}_1^j) = \mathcal{M}_0^j[m - 1 - d_{2j-1} - \dots - d_{2j+m-3}],$$
$$(t_1 \dots t_m)^{m+1}(\mathcal{P}_1^j) = \mathcal{P}_1^{j+m+1}[2m - d_{2j-1} - \dots - d_{2j+m}].$$

The same argument for $\mathcal{P}_i^j$ instead of $\mathcal{P}_1^j$ finishes the proof. □

Now we need to recall some facts and notions concerning the topology of curves in 2-dimensional disk with $m+1$ marked points. We refer to [KhS, section 3] for the detailed exposition of the subject. Let D be a closed 2-disk, $\triangle \subset D \backslash \partial D$ be a subset with $|\triangle| = m+1$. Let $Diff(D, \partial D; \triangle)$ be the group of diffeomorphisms $f : D \to D$ such that $f|_{\partial D} = id$ and $f(\triangle) = \triangle$. There is an obvious notion of an isotopy within this group, and we write $f_0 \simeq f_1$ for isotopic diffeomorphisms. In this section by a curve in $(D, \triangle)$ we mean a subset $c \subset D \backslash \partial D$ which is the image of a smooth embedding $\gamma : [0; 1] \to D$ with $\gamma^{-1}(\triangle) = \{0; 1\}$. Note that our curves are not oriented. One can easily define the notion of isotopy for such curves and we also denote it by $c_0 \simeq c_1$. The geometric intersection number of curves c_0, c_1 is denoted by $I(c_0, c_1)$ and is defined as

$$I(c_0, c_1) = |(c_0' \cap c_1) \backslash \triangle| + \frac{1}{2}|c_0' \cap c_1 \cap \triangle|,$$

where c_0' is a curve in the isotopy class of c_0 which has minimal intersection with c_0 (see [KhS]). Note that $I(c_0, c_1) \in \frac{1}{2}\mathbb{Z}_{>0}$. Here are some basic properties of geometric intersection numbers:

(I1) $I(c_0, c_1)$ depends only on the isotopy classes of c_0, c_1;

(I2) $I(c_0, c_1) = I(f(c_0), f(c_1))$ for each $f \in Diff(D, \partial D; \triangle)$;

(I3) I is symmetric, i.e. $I(c_0, c_1) = I(c_1, c_0)$.

Fix an orientation on D. Then, $\pi_0(Diff(D, \partial D; \triangle))$ is identified with braid group B_{m+1} and standard generators $g_1, \dots, g_m \in B_{m+1}$ correspond to the connected components of positive half-twist (see [KhS] for definition) along the curves b_i, which are shown in Figure 3. For $f \in Diff(D, \partial D; \triangle$ we write $[f]$ for the element of B_{m+1} corresponding to the connected component of f.

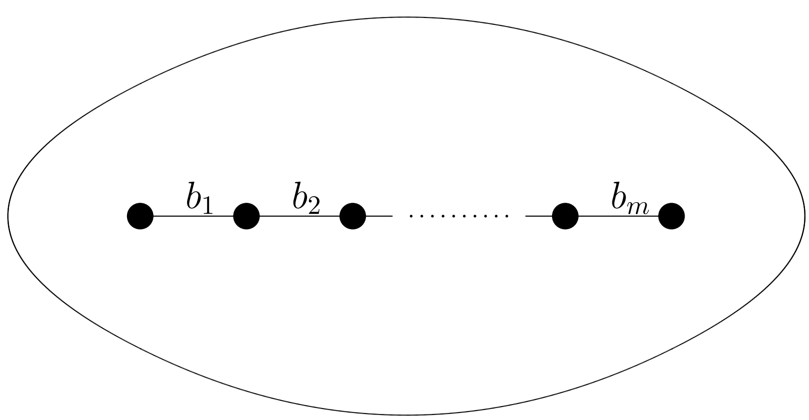

Figure 3.

Definition 4.31. *A curve d in $(D, \triangle)$ is called admissible if $d \simeq f(b_i)$ for some $f \in Diff(D, \partial D; \triangle)$, $1 \leq i \leq m$.*

Lemma 4.32. *Suppose that c_0, c_1 are admissible curves and $I(c_0, d) = I(c_1, d)$ for all admissible curves d. Then $c_0 \simeq c_1$.*

Lemma 4.33. *Let $f \in Diff(D, \partial D; \triangle)$ be a diffeomorphism such that $f(b_i) \simeq b_i$ for $1 \leq i \leq m$. Then $[f] \in B_{m+1}$ is central.*

Now we want to prove the Lemma which relates geometric intersection numbers and our braid group actions $\rho^k_{m,n}$.

Lemma 4.34. *Let $f \in Diff(D, \partial D; \triangle)$ be a diffeomorphism and $g = [f] \in B_{m+1}$ be the corresponding element. Then*

$$\sum_{1 \leq j \leq k, r \in \mathbb{Z}} \dim_{\mathbb{F}} \mathrm{Hom}_{D'(A^k_{m,n})}(\mathcal{P}^{j_1}_{i_1}, \rho^k_{m,n}(g)(\mathcal{P}^j_{i_2})[r]) = 2I(b_{i_1}, f(b_{i_2}))$$

for all $1 \leq i_1, i_2 \leq m$, $1 \leq j_1 \leq k$.

To prove this Lemma, we first show that the sum on the left side does not depend on n. Let *mod*-$A^k_{m,1}$ be the category of finitely generated right graded $A^k_{m,1}$-modules (without differential). We write $D^b(A^k_{m,1})$ for the bounded derived category of the abelian category *mod*-$A^k_{m,1}$. Denote the standard projective modules $e^j_i A^k_{m,1}$ by P^j_i. If M is a graded $A^k_{m,n}$-module then $M\langle 1\rangle$ denotes the result of shifting of its grading by one. This is obviously functorial in M and these functor descends to an exact autoequivalence of the category $D^b(A^k_{m,1})$. This autoequivalence should not be mixed with the usual shift [1] in the derived category. The following multi-twist functors are not the specialization of multi-twist functors with respect to n-spherical sequences.

Definition 4.35. *Multi-twist functors $\bar{t}_1, \ldots, \bar{t}_m$ from $D^b(A^k_{m,1})$ to itself are defined by*

$$\bar{t}_i(M) = \{\sum_{j=1}^{k} M(i^j) \otimes P^j_i \to M\}.$$

Here $M(i^j)$ is considered just as a complex of vector spaces, the tensor product is over $\mathbb{F}$ and the map is the multiplication map. The following Lemma is analogous to Proposition 2.4 and Theorem 2.5 in [KhS]. It can be also proved by applying the general caonstruction of braid group actions to the spherical functors from the bounded derived category of graded finite-dimensional $\mathbb{F}[V_k]$-modules (the grading on $\mathbb{F}[V_k]$ is trivial). We leave the details to the reader.

Lemma 4.36. *Functors $\bar{t}_1, \ldots, \bar{t}_m$ are exact autoequivalences and satisfy braid relations (up to isomorphism). Thus, they generate the braid group action $\bar{\rho}^k_m : B_{m+1} \to Aut(D^b(A^k_{m,1}))$.*

Let $\mathcal{B}$ be subcategory of *mod*-$A^k_{m,1}$ which consists of finite direct sums of objects P^j_i and let $K^b(\mathcal{B})$ be the full (triangulated) subcategory of $K^b($*mod*-$A^k_{m,1})$ which consists of finite complexes of objects in $\mathcal{B}$. From the very definition, $\bar{t}_i(K^b(\mathcal{B})) \subset K^b(\mathcal{B})$. The following Lemma relates the categories $K^b(\mathcal{B})$ and $D'(A^k_{m,n})$.

Lemma 4.37. *There exists an exact functor $G : K^b(\mathcal{B}) \to D'(A^k_{m,n})$ such that*

(1) $G(P^j_i)$ is isomorphic to $\mathcal{P}^j_i$ up to shifting;
(2) There exists an isomorphism of functors $G \circ < 1 > \cong [n] \circ G$;
(3) The natural map

$$\bigoplus_{r_2 + n r_1 = 0} \mathrm{Hom}_{K^b(\mathcal{B})}(X, Y\langle r_1\rangle[r_2]) \to \mathrm{Hom}_{D'(A^k_{m,n})}(G(X), G(Y))$$

is an isomorphism for all $X, Y \in K^b(\mathcal{B})$.

Proof. First put $G(P_i^j\langle r\rangle) = \mathcal{P}_i^j[\sigma_i^j + nr]$, where $\sigma_i^j = -d_1 - \cdots - d_{2j+i-3}$. Note that we have natural isomorphisms

$$\mathrm{Hom}_{K^b(\mathcal{B})}(P_{i_1}^{j_1}\langle r\rangle, P_{i_2}^{j_2}\langle s\rangle) \cong (i_2^{j_2})(A_{m,1}^k)^{s-r}(i_1^{j_1}) \cong$$
$$\cong (i_2^{j_2})(A_{m,n}^k)^{\sigma_{i_2}^{j_2} - \sigma_{i_1}^{j_1} + n(s-r)}(i_1^{j_1}) \cong \mathrm{Hom}_{D'(A_{m,n}^k)}(\mathcal{P}_{i_1}^{j_1}[\sigma_{i_1}^{j_1} + nr], \mathcal{P}_{i_2}^{j_2}[\sigma_{i_2}^{j_2} + ns]).$$

The isomorphisms $\mathrm{Hom}_{K^b(\mathcal{B})}(P_{i_1}^{j_1}\langle r\rangle, P_{i_2}^{j_2} \to \mathrm{Hom}_{D'(A_{m,n}^k)}(\mathcal{P}_{i_1}^{j_1}[\sigma_{i_1}^{j_1} + nr], \mathcal{P}_{i_2}^{j_2}[\sigma_{i_2}^{j_2} + ns])$ se isomorphisms are obviously compatible with the composition. Thus they define full and faithful functor from the full subcategory of $K^b(\mathcal{B})$, which consists of the objects $P_i^j\{r\}$ $(1 \le i \le m, 1 \le j \le k, r \in \mathbb{Z})$, to $D'(A_{m,n}^k)$. Extend this functor to the direct sums of $P_i^j\{r\}$ and then to the whole category $K^b(\mathcal{B})$ using convolutions. Note that the constructed functor is exact. Conditions (1) and (2) of the Lemma are obviously satisfied. Since, the modules $P_i^j[r]$ are projective, the condition (3) is already known for $X = \mathcal{P}_{i_1}^{j_1}\langle r\rangle, Y = \mathcal{P}_{i_2}^{j_2}\langle s\rangle$. Then, using Five Lemma, it is easy to obtain that it holds for all $X, Y \in K^b(\mathcal{B})$. □

Corollary 4.38. *The value of the expression*

$$\sum_{r\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{D'(A_{m,n}^k)}(\mathcal{P}_{i_1}^{j_1}, \rho_{m,n}^k(g)(\mathcal{P}_{i_2}^{j_2})[r])$$

depends only on $m, k \in \mathbb{N}, g \in B_{m+1}, 1 \le i_1, i_2 \le m, 1 \le j_1 \le k$, and does not depend on n.

Proof. From the definition of the functor G it follows that $G\circ\bar{t}_i|K^b(\mathcal{B}) \cong t_i\circ G$. Thus, for each $g \in B_{m+1}$ we have $G \circ \bar{\rho}_m^k(g)|K^b(\mathcal{B}) \cong \rho_{m,n}^k(g) \circ G$. Hence,

$$\sum_{r\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{D'(A_{m,n}^k)}(\mathcal{P}_{i_1}^{j_1}, \rho_{m,n}^k(g)(\mathcal{P}_{i_2}^{j_2})[r]) = \sum_{r_2+nr_1=0, r\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{K^b(\mathcal{B})}(P_{i_1}^{j_1}, \bar{\rho}_m^k(P_{i_2}^{j_2})\langle r_1\rangle[r_2+r]) =$$
$$= \sum_{r_1,r_2\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{K^b(\mathcal{B})}(P_{i_1}^{j_1}, \bar{\rho}_m^k(P_{i_2}^{j_2})\langle r_1\rangle[r_2]).$$

The value of the last expression does not depend on n, the Corollary is proved. □

Before we prove Lemma 4.34, we need the following result:

Lemma 4.39. *[STh, Lemma 4.17.] Let $f \in Diff(D, \partial D; \triangle)$ be a diffeomorphism and $g \in B_{m+1}$ be the corresponding element. Then*

$$\sum_{r\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{D'(A_{m,n}^1)}(\mathcal{P}_{i_1}^1, \rho_{m,n}^1(g)(\mathcal{P}_{i_2}^1)[r]) = 2I(b_{i_1}, f(b_{i_2}))$$

for all $1 \le i_1, i_2 \le m$.

Proof of Lemma 4.34. Due to Corollary 4.38, it is sufficient to prove Lemma for some certain value of n. We will prove it for $n = k$.

In this case we have that

$$d_i = \begin{cases} 0 & \text{for } i \text{ odd}, \\ 1 & \text{for } i \text{ even}. \end{cases}$$

We are going to construct exact functors

$$\Psi^* : D(A_{m,k}^k) \to D(A_{m,1}^1),\ \Psi_* : D(A_{m,1}^1) \to D(A_{m,k}^k),$$

satisfying the following properties:

(Ψ1) $\Psi^*(\mathcal{P}_i^j)$ is isomorphic to $\mathcal{P}_i^1$;

(Ψ2) $\Psi_*(\mathcal{P}_i^1)$ is isomorphic to $\bigoplus_{j=1}^{k}\mathcal{P}_i^j$;

(Ψ3) Ψ^* is left adjoint to Ψ_*.

Let $\mathcal{M}\in D(A_{m,k}^k)$. Define $\Psi^*(\mathcal{M})$ as follows. As a complex of vector $\mathbb{F}$-spaces, it equals to $\mathcal{M}$. Note that

$$\mathcal{M}=\sum_{1\le i\le m,1\le j\le k}\mathcal{M}e_i^j.$$

Now define the right multiplication in $\Psi^*(\mathcal{M})$ by the elements of $A_{m,1}^1$:
1) The multiplication by (i^1) acts by zero on $\mathcal{M}e_{i_1}^j$ for $i_1\ne i$, $1\le j\le k$, and acts as the identity on $\mathcal{M}e_i^j$ for $1\le j\le k$;
2) The multiplication by $(i^1|(i+1)^1)$ acts by zero on $\mathcal{M}e_{i_1}^j$ for $i_1\ne i$, $1\le j\le k$, and maps $\mathcal{M}e_i^j$ to $\mathcal{M}e_{i+1}^j$ as follows:

$$x(i^j)\cdot(i^1|(i+1)^1)=x(i^j|(i+1)^j);$$

3) The multiplication by $(i^1|(i-1)^1)$ acts by zero on $\mathcal{M}e_{i_1}^j$ for $i_1\ne i$, $1\le j\le k$, and maps $\mathcal{M}e_i^j$ to $\mathcal{M}e_{i-1}^{j+1}$ as follows:

$$x(i^j)\cdot(i^1|(i-1)^1)=x(i^j|(i-1)^{j+1}).$$

It is easy to check that this indeed defines on $\Psi^*(\mathcal{M})$ the structure of a right dg-module over $A_{m,1}^1$. Define Ψ^* on morphisms (in $D(A_{m,k}^k)$) in the tautological way. Then Ψ^* becomes an exact functor from $D(A_{m,k}^k)$ to $D(A_{m,1}^1)$. Note that $\Psi^*(\mathcal{P}_i^j)\cong\mathcal{P}_i^1$.

Now we want to define the functor $\Psi_*:D(A_{m,1}^1)\to D(A_{m,k}^k)$. Let $\mathcal{N}$ be an object of $D(A_{m,1}^1)$. As a complex of vector $\mathbb{F}$-spaces, $\Psi_*(\mathcal{N})$ equals to the direct sum $\mathcal{N}^{\oplus k}$. Denote the j-th copy of $\mathcal{N}$ in this sum by $\mathcal{N}^j$. For our convenience we put $\mathcal{N}^{j+k}=\mathcal{N}^j$. Define the right multiplication in $\Psi_*(\mathcal{N})$ by the elements of $A_{m,k}^k$ as follows:
1) The multiplication by (i^j) acts by zero on $\mathcal{N}^{j_1}e_{i_1}^1$ for $(i_1,j_1)\ne(i,j)$ and acts as the identity on $\mathcal{N}^je_i^1$;
2) The multiplication by $(i^j|(i+1)^j)$ acts by zero on $\mathcal{N}^{j_1}e_{i_1}^1$ for $(i_1,j_1)\ne(i,j)$ and maps $\mathcal{N}^je_i^1$ to $\mathcal{N}^je_{i+1}^1$ as follows:

$$x^j(i^1)\cdot(i^j|(i+1)^j)=x^j(i^1|(i+1)^1);$$

3) The multiplication by $(i^j|(i-1)^{j+1})$ acts by zero on $\mathcal{N}^{j_1}e_{i_1}^1$ for $(i_1,j_1)\ne(i,j)$ and maps $\mathcal{N}^je_i^1$ to $\mathcal{N}^{j+1}e_{i-1}^1$ as follows:

$$x^j(i^1)\cdot(i^j|(i-1)^{j+1})=x^{j+1}(i^1|(i-1)^1).$$

Similarly, one can check that this defines on $\Psi_*(\mathcal{N})$ the structure of right dg-module over $A_{m,k}^k$. Define Ψ_* first on morphisms in $Dgm(A_{m,1}^1)$. Let $\Psi_*(f)$ be the direct sum of k copies of f. It is easy to check that this is well-defined. Since Ψ_* preserves quasi-isomorphisms, it descends to an exact functor $\mathcal{C}_{m,1}^1\to D'(A_{m,k}^k)$. Note that $\Psi_*(\mathcal{P}_i^1)\cong\sum_{j=1}^{k}\mathcal{P}_i^j$.

As we have already noticed, the properties (Ψ1) and (Ψ2) are satisfied. Prove the property (Ψ3). We have from the very definitions of Ψ^* and Ψ_* that $\Psi_*\Psi^*(\mathcal{M})$ equals to $\mathcal{M}^{\oplus k}$ as a complex of vector spaces. Again, denote the j-th copy of $\mathcal{M}$ by $\mathcal{M}^j$. The right multiplication in $\Psi_*\Psi^*(\mathcal{M})$ by the elements of $A_{m,k}^k$ is the following:
1) The multiplication by (i^j) acts by zero on $\mathcal{M}^{j_1}e_{i_2}^{j_2}$ for $(i_2,j_1)\ne(i,j)$ and acts as the identity on $\mathcal{M}^je_i^{j_1}$ for $1\le j_1\le k$;
2) The multiplication by $(i^j|(i+1)^j)$ acts by zero on $\mathcal{M}^{j_1}e_{i_2}^{j_2}$ for $(i_2,j_1)\ne(i,j)$ and maps $\mathcal{M}^je_i^{j_1}$ to $\mathcal{M}^je_{i+1}^{j_1}$ as follows:

$$x^j(i^{j_1})\cdot(i^j|(i+1)^j)=x^j(i^{j_1}|(i+1)^{j_1});$$

3) The multiplication by $(i^j|(i-1)^{j+1})$ acts by zero on $\mathcal{M}^{j_1}e_{i_2}^{j_2}$ for $(i_1,j_1)\neq(i,j)$ and maps $\mathcal{M}^j e_i^{j_1}$ to $\mathcal{M}^{j+1}e_{i-1}^{j_1+1}$ as follows:

$$x^j(i^{j_1})\cdot(i^j|(i-1)^{j+1}) = x^{j+1}(i^{j_1}|(i-1)^{j_1+1}).$$

We have the following morphism of dg-modules $\phi_{\mathcal{M}}:\mathcal{M}\to\Psi_*\Psi^*(\mathcal{M})$, defined as $\phi_{\mathcal{M}}(x(i^j)) = x^j(i^j)$. Clearly, it is well-defined and is functorial in $\mathcal{M}$, so that it defines the morphism of functors $\pi: Id_{D(A^k_{m,k})}\to\Psi_*\Psi^*$.

Now describe $\Psi^*\Psi_*(\mathcal{M})$ as a dg-module. As a complex of vector spaces it equals again to $\mathcal{M}^{\oplus k}$ and we denote the j-th copy by $\mathcal{M}^j$. The right multiplication by the elements of $A^k_{m,1}$ is the following:
1) The multiplication by (i^1) acts by zero on $\mathcal{M}^j e^1_{i_1}$ for $i_1\neq i$ and acts as the identity on $\mathcal{M}^j e^1_i$ for $1\leq j\leq k$;
2) The multiplication by $(i^1|(i+1)^1)$ acts by zero on $\mathcal{M}^j e^1_{i_1}$ for $i_1\neq i$ and maps $\mathcal{M}^j e^1_i$ to $\mathcal{M}^j e^1_{i+1}$ as follows:

$$x^j(i^1)\cdot(i^1|(i+1)^1) = x^j(i^1|(i+1)^1);$$

3) The multiplication by $(i^1|(i-1)^1)$ acts by zero on $\mathcal{M}^j e^1_{i_1}$ for $i_1\neq i$ and maps $\mathcal{M}^j e^1_i$ to $\mathcal{M}^{j+1}e^1_{i-1}$ as follows:

$$x^j(i^1)\cdot(i^1|(i-1)^1) = x^{j+1}(i^1|(i-1)^1).$$

Similarly, we have the following morphism of dg-modules $\psi_{\mathcal{M}}:\Psi^*\Psi_*(\mathcal{M})\to\mathcal{M}$, defined as $\psi_{\mathcal{M}}(x^j) = x$. Clearly, this is also functorial in $\mathcal{M}$ and defines the morphism of functors $\psi:\Psi_*\Psi^*\to Id_{D(A^k_{m,1})}$.

A straightforward checking shows that the compositions

$$\Psi_* \xrightarrow{\phi(\Psi_*)} \Psi_*\circ\Psi^*\circ\Psi_* \xrightarrow{\Psi_*(\psi)} \Psi_*,$$
$$\Psi^* \xrightarrow{\Psi^*(\phi)} \Psi^*\circ\Psi_*\circ\Psi^* \xrightarrow{\psi(\Psi^*)} \Psi^*$$

coincide with the identity morphisms of Ψ_* and Ψ^* respectively. Then Proposition 2.4 shows that Ψ^* is left adjoint to Ψ_*

Note that both Ψ^* and Ψ_* commute with $t_1,\dots,t_m$ and hence are equivariant with respect to B_{m+1}-actions. Thus, we have the following chain of isomorphisms:

$$\sum_{1\leq j\leq k, r\in\mathbb{Z}} \mathrm{Hom}_{D(A^k_{m,k})}(\mathcal{P}^{j_1}_{i_1}, \rho^k_{m,k}(g)(\mathcal{P}^j_{i_2})[r]) \cong \sum_{r\in\mathbb{Z}} \mathrm{Hom}_{D(A^k_{m,k})}(\mathcal{P}^{j_1}_{i_1}, \rho^k_{m,k}(g)(\Psi_*(\mathcal{P}^1_{i_2}))[r]) \cong$$
$$\cong \sum_{r\in\mathbb{Z}} \mathrm{Hom}_{D(A^k_{m,k})}(\mathcal{P}^{j_1}_{i_1}, \Psi_*(\rho^k_{m,k}(g)(\mathcal{P}^1_{i_2}))[r]) \cong \sum_{r\in\mathbb{Z}} \mathrm{Hom}_{D(A^1_{m,1})}(\Psi^*(\mathcal{P}^{j_1}_{i_1}), \rho^k_{m,k}(g)(\mathcal{P}^1_{i_2})[r]) \cong$$
$$\cong \sum_{r\in\mathbb{Z}} \mathrm{Hom}_{D(A^1_{m,1})}(\mathcal{P}^1_{i_1}, \rho^k_{m,k}(g)(\mathcal{P}^1_{i_2})[r]).$$

Applying Lemma 4.39 finishes the proof. □

Proof of Theorem 4.29. Let $f\in Diff(D,\partial D;\triangle)$, and $g=[f]$. Take some another analogous pair f_1, g_1. Lemma 4.34 shows that

$$I(f_1(b_{i_1}), f(b_{i_2})) = I(b_{i_1}, f_1^{-1}f(b_{i_2})) = \frac{1}{2}\sum_{1\leq j\leq k, r\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{D'(A^k_{m,k})}(\mathcal{P}^{j_1}_{i_1}, \rho^k_{m,n}(g_1^{-1})\rho^k_{m,n}(g)(\mathcal{P}^j_{i_2})[r]) =$$
$$= \frac{1}{2}\sum_{1\leq j\leq k, r\in\mathbb{Z}} \dim_{\mathbb{F}}\mathrm{Hom}_{D'(A^k_{m,k})}(\mathcal{P}^{j_1}_{i_1}, \rho^k_{m,n}(g_1^{-1})(\mathcal{P}^j_{i_2})[r]) = I(b_{i_1}, f_1^{-1}(b_{i_2})) = I(f_1(b_{i_1}), b_{i_2})$$

Since we can choose i_1 and f_1 arbitrary, for any admissible curve d in $(D,\triangle)$ and for each $1\leq i_2\leq m$ we have $I(d, f(b_{i_2})) = I(d, b_{i_2})$. Then from Lemma 4.33 follows that g is central, i.e. $g=(g_1\dots g_m)^{\nu(m+1)}$

for some $\nu \in \mathbb{Z}$. We may assume that $\nu \geq 0$. From Lemma 4.30 we have that $\rho^k_{m,n}(g)(\mathcal{P}^j_i) \cong \mathcal{P}^{j+\nu(m+1)}_i[2\nu m - d_{2j+i-2} - \cdots - d_{2j+i+(2m+2)\nu-3}]$. Thus, $\nu(m+1)$ is divisible by k and this object equals to $\mathcal{P}^j_i[2\nu m - \frac{\nu n(m+1)}{k}]$, $2\nu m = \frac{\nu n(m+1)}{k}$. Hence, if $\nu > 0$ then $n < 2k$. □

Proof of Theorem 4.14. First note that if $\rho(g)(E^j_i) = E^j_i$ for some i, j then $\rho(g)(E^{j_1}_i) = E^{j_1}_i$ for each $1 \leq j_1 \leq k$. These follows from the definition of n-spherical sequence and from the standard facts about representable functors. Consider the following cases.
The case $m = 1$. In this case it is easy to see that $T_{(E_1)}(E^j_1) = E^{j+1}_1[1 - d_{2j-1} - d_{2j}]$. Suppose that $T^r_{(E_1)}(E^j_1) = E^j_1$. Then r is divisible by k and $r = d_{2j-1} + \cdots + d_{2j+2r-2} = \frac{rn}{k}$. This contradicts to our assumption that $n \geq 2k$.
The case $m \geq 2$. Recall that in our assumptions $H(end(E)) \cong A^k_{m,n}$. Suppose that $\rho(g)(E^j_i) = E^j_i$ for $1 \leq i \leq m$, $1 \leq j \leq k$. let $D'(REnd(E))$ be the full subcaategory of $D(REnd(E)$, which consists of dg-modules with bounded cohomology. Note that the image of the functor Ψ_E contains in $D'(REnd(E))$. Then $\Psi'_E : \mathcal{T} \to D'(A^k_{m,n})$ be the composition of Ψ_E and the equivalence $D'(REnd(E)) \to D'(A^k_{m,n})$, which is induced by the equivalence $D(REnd(E)) \to D(A^k_{m,n})$. Then $\Psi'_E(E^j_i) \cong \mathcal{P}^j_i$. Note that the functor Ψ'_E is equivariant with respect to our actions ρ and $\rho^k_{m,n}$. Thus, we have $\rho^k_{m,n}(g)(\mathcal{P}^j_i) = \mathcal{P}^j_i$ and Lemma 4.29 shows that g is the identity element. □

Note that we obtain the following
Corollary of Lemma 4.37 and Theorem 4.29. *The action of B_{m+1} on $D^b(A^k_{m,1})$ is faithful.*

Proof. It is sufficient to note again that the functor G from Lemma 4.37 is equivariant with respect to our B_{m+1}-actions. □

Remark. In the notation of Lemma 4.34, put

$$c^j_{i_1,i_2}(g) = \sum_{r \in \mathbb{Z}} \dim_{\mathbb{F}} \mathrm{Hom}_{D'(A^k_{m,n})}(\mathcal{P}^1_{i_1}, \rho^k_{m,n}(g)(\mathcal{P}^j_{i_2})[r]).$$

From the statement of the Lemma it follows that $\sum_{j=1}^{k} c^j_{i_1,i_2}(g) = I(b_{i_1}, f(b_{i_2}))$. This motivates us to find the topological meaning of numbers $c^k_{i_1,i_2}$. To do that, one should consider the curves on k-covering of D branched exactly at the points of $\triangle$. We will prove the corresponding result in another paper.

5 Applications.

5.1 General remarks.

In almost all situations, we need the category $\mathcal{T}$ to admit a Serre functor. If it exists then each n-spherical sequence is up to a shift of the following kind:

$$(E, S(E), S^2(E), \ldots, S^{k-1}(E)),$$

where the following conditions on E hold:
(E1) E is exceptional;
(E2) $S^k(E) \cong E[n]$;
(E3) $\mathrm{Hom}^*(E, S^j(E)) = 0$ for $2 \leq j \leq k-1$

Denote the corresponding multi-twist functor by R_E. Clearly, we are interested in categories which admit a Serre functor and an object E satisfying (E1),(E2),(E3).

5.2 Smooth projective varieties.

Let Y be a smooth projective variety and $\mathcal{A} = Coh(Y)$. Then, as it was mentioned in subsection 2.5, the Serre functor on $D^b(\mathcal{A}) = D^b(Y)$ equals to $S(-) = (-) \otimes \omega_Y[\mathrm{dim} Y]$. Let $E^1, \ldots, E^k$ be objects in $D^b(Y)$. Then the corresponding multi-twist functor $T_{(E)}$ can be easily given by a kernel on $Y \times Y$. The following Lemma is left to the reader:

Lemma 5.1. *The functor $T_{(E)}$ is isomorphic to the Fourier- Mukai transform $\Phi_{\mathcal{P}}$, where*

$$\mathcal{P} = Cone(\eta : \bigoplus_{j=1}^{k} E^j \boxtimes E^j \to \mathcal{O}_{\triangle}).$$

Now suppose that $\omega_Y^k \cong \mathcal{O}$ for some k (minimal with this property), or, equivalently, $S^k \cong [k\mathrm{dim}(Y)]$. Then (E2) is satisfied for each object in $D^b(Y)$. We want to describe the relation between the multi-twist functors by the sequences given by such exceptional objects and the twist functors investigated by Seidel and Thomas [STh]. The canonical bundle gives us an unramified $G = \mathbb{Z}/k\mathbb{Z}$-covering $\pi : \tilde{Y} \to Y$, where $Y = Spec_Y(\mathcal{O} \oplus \omega_Y \oplus \cdots \oplus \omega_Y^{k-1})$ Then the category $D^b(Y)$ is equivalent to the category $D^b(\tilde{Y})_G$ of G-equivariant objects in $D^b(Y)$.

Proposition 5.2. *Let E be an object in $D^b(Y)$. Then E satisfies (E1),(E2),(E3) iff $\pi^*(E) \in D^b(\tilde{Y})$ is spherical. In this case, $\pi^*(E)$ is $(\mathrm{dim} Y)$-spherical, and E gives a $(k\mathrm{dim} Y)$-spherical sequence.*

Proof. For each $E \in D^b(Y)$ we have

$$\pi_*\pi^*(E) \cong \bigoplus_{j=0}^{k-1} E \otimes \omega_Y^j.$$

Further,

$$\mathrm{Hom}^p(\pi^*(E), \pi^*(E)) \cong \mathrm{Hom}^p(E, \pi_*\pi^*(E)) \cong \bigoplus_{l=0}^{k-1} \mathrm{Hom}^{p+l\mathrm{dim} Y}(E, S^j(E)) \tag{4}$$

Hence, $\pi^*(E)$ is spherical iff $\sum_{j=0}^{k-1} \mathrm{dim}_{\mathbb{F}} \mathrm{Hom}^*(E, S^j(E)) = 2$. But we always have (for $E \neq 0$) $Hom(E, E) \neq 0$, $Hom(E, S(E)) \neq 0$. Thus, $\pi^*(E)$ is spherical iff E satisfies (E1), (E3). As it is mentioned above, E always satisfies (E2).

Finally, from (4) easily follows that if $\pi^*(E)$ is spherical then it is $(\mathrm{dim} Y)$-spherical and the sequence $(E, S(E), \ldots, S^{k-1}(E))$ is $(k\mathrm{dim} Y)$-spherical. □

The following Proposition relates the multi-twist functors and twist functors in the setting of this subsection. It is left to the reader.

Proposition 5.3. *Let E be an object satisfying (E1), (E2), (E3). Then have the following commutative (up to isomorphism) diagram of functors:*

$$\begin{array}{ccc} D^b(Y) & \xrightarrow{R_E} & D^b(Y) \\ \pi^* \downarrow & & \pi^* \downarrow \\ D^b(\tilde{Y}) & \xrightarrow{T_{\pi^*(E)}} & D^b(\tilde{Y}). \end{array}$$

5.3 Enriques surfaces.

Enriques surface is a surface Y with $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$ and $\omega_Y^2 = 0$. For each such surface there exist a $K3$-cover $\pi : \tilde{Y} \to Y$. Clearly, each exceptional object in $D^b(Y)$ satisfies (E1),(E2),(E3) and hence gives an autoequivalence R_E. Unfortunately, we cannot construct braid group action of our type because the Serre functor acts trivially on the numerical Grothendieck group of Y.

5.4 Quivers.

Let $Q = (Q_0, Q_1)$ be a quiver and $\mathcal{I} \subset \mathbb{F}[Q]$ be a two-sided ideal. Suppose that Q has not oriented cycles. Let $|Q_0| = m$. Order the vertices in Q_0 in such way that there are no arrows from i-th vertex to j-th vertex for $i \geq j$. Then $(P_m, \ldots, P_1)$ and $(S_1, \ldots, S_m)$ are exceptional colloections in $D^b(Q, \mathcal{I})$ which both generate this category. Hence, $D^b(Q, I)$ admits a Serre functor. It is easy to see that $S(P_i) = I_i$. We will now give several examples of quivers for which there exist invertible multi-twist functors and for which there exist braid group actions corresponding to (A_m^k)-configurations. Recall that, by Gabriel's theorem [G], for each quiver of Dynkin type has only finitely many indecomposable representations. Since the homological dimension of $Rep(Q)$ is not more than one for any quiver, than the same holds for the derived category of any quiver of Dynkin type.

1. Let $Q = A_m$ and $\mathcal{I} = 0$. Denote its paths by $(i|(i+1)| \ldots |j)$ Then $D^b(Q, \mathcal{I})$ has, up to shifting, $\frac{m(m+1)}{2}$ indecomposable objects

$$M_{ij} = \begin{cases} P_i/(i|(i+1)| \ldots |j+1)P_{j+1} & \text{for } 1 \leq i \leq j \leq m-1 \\ P_i & \text{for } 1 \leq i \leq m, j = m. \end{cases}$$

   Each object in $D^b(Q, \mathcal{I})$ is a direct sum of shifted copies of M_{ij}. Further, the Serre functor S acts on these objects as follows:

$$S(M_{ij}) = \begin{cases} M_{(i+1)(j+1)}[1] & \text{for } j \leq m-1 \\ M_{1i} & \text{for } j = m \end{cases})$$

   Further, we have $S^{m+1} \cong [m-1]$. The only multi-twist functor in this category is $R_{P_1} \cong S^{-1} \circ [1]$.

2. Let Q_0 be of card 3, and let Q_1 consist of two arrows from the 1-th vertex to the 2-nd (u_1^1, u_1^2) and two arrows from the 2-nd vertex to the 3-rd (u_2^1, u_2^2), see Figure 4.

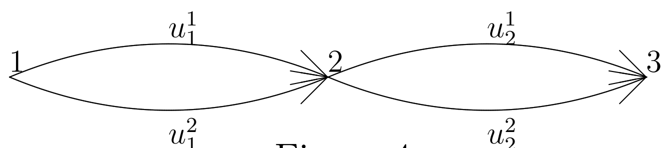

Figure 4.

   Let $\mathcal{I}$ be an ideal generated by $u_1^1 u_2^2$ and $u_1^2 u_2^1$. Let M_j, $j = 1, 2$ be a representation of $(Q, \mathcal{I})$ with $(M_j)_i = \mathbb{F}$ and

$$(M_j)_{u_p^q} = \begin{cases} Id_{\mathbb{F}} & \text{for } q = j \\ 0 & \text{otherwise.} \end{cases}$$

   Taking projective resolution of M_j which is $P_3 \to P_2 \to P_1$ one sees that M_j is exceptional. Further, it can be checked that $S(M_j) = M_{3-j}[2]$. Hence, (M_1, M_2) is 4-spherical sequence and it gives an equivalence $T_{(M)}$.

3. Let Q_0 be of card 4, Q_1 consist of four arrows: from i-th to $(i+1)$-th (u_i), where $1 \le i \le 3$, and from 2-nd to 4-th (v), see Figure 5.

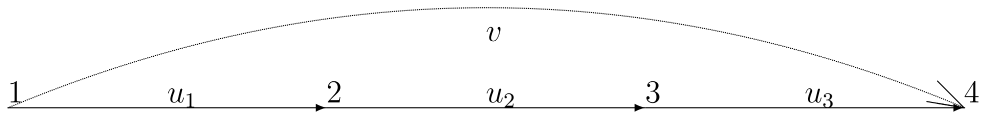

Figure 5.

Let $\mathcal{I}$ be generated by u_1u_2 and u_2u_3. Then one can check that $S^2(P_1) = P_1[3]$, $S^2(P_3) = P_3[1]$. Thus, they give autoequivalences R_{P_1}, R_{P_3}.

4. Let $Q = D_4$, and $\mathcal{I} = 0$. Then we have that

$$S(P_3) = I_3 = \{P_4 \to P_1\}, S^2(P_3) = \{P_3 \to P_2\}, S^3(P_3) = P_3[2],$$

and

$$S(P_4) = I_4 = \{P_3 \to P_1\}, S^2(P_4) = \{P_4 \to P_2\}[1], S^3(P_4) = P_4[2].$$

It is easy to check that both P_3 and P_4 satisfy (E1), (E2), (E3). Further, $(P_3, S(P_3), S^2(P_3))$ and $(P_4, S(P_4), S^2(P_4))$ form the (A_2^3)-configuration of 2-spherical sequences. Hence we have a B_3-action on $D^b(Q)$. This action is obviously not faithful (because there are finitely many indecomposable objects). In particular, the algebra $A^3_{2,2}$ is not intrinsically formal.

5. Let $Q = A_4$. Denote the arrows by u_1, u_2, u_3 as in Figure 6.

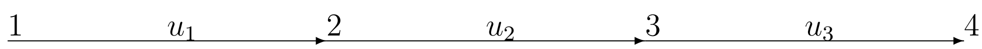

Figure 6.

Let $\mathcal{I}$ be generated by $u_1u_2u_3$. Then

$$S(P_1) = I_1 = \{P_4 \to P_2 \to P_1\}, S^2(P_1) = P_3[2], S^3(P_1) = P_1[2],$$

and

$$S(P_2) = I_2 = \{P_4 \to P_3 \to P_1\}, S^2(P_2) = P_4[2], S^3(P_2) = P_2[2].$$

As in the previous example, one can check that P_1 and P_2 satisfy (E1), (E2), (E3) and $(P_1, S(P_1), S^2(P_1))$, $(P_2, S(P_2), S^2(P_2))$ form the (A_2^3)-configuration of 2-spherical sequences. We are not able to say if the corresponding B_3-action is faithful.

It would be interesting to construct examples of faithful braid group actions given by (A_m^k)-configurations which are more complicated then our action on $D'(A^k_{m,n})$.

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