# NOTES ON SIMPLICIAL $BF$ THEORY

P. MNËV

ABSTRACT. In this work we discuss the construction of "simplicial $BF$ theory", the field theory with finite-dimensional space of fields, associated to a triangulated manifold, that is in a sense equivalent to topological $BF$ theory on the manifold (with infinite-dimensional space of fields). This is done in framework of simplicial program — program of constructing discrete topological field theories. We also discuss the relation of these constructions to homotopy algebra.

## CONTENTS

## 1. **Introduction**

This work contains first results obtained by the author in framework of simplicial program proposed to him by Andrei Losev, also some preliminary arguments are included. The idea of simplicial program is to take some topological quantum field theory $\mathcal{T}_M$ on manifold $M$ and formulate discrete field theory $\mathcal{T}_\Xi$ on triangulation $\Xi$ of $M$, which is (in some sense) equivalent to $\mathcal{T}_M$. Having succeeded in constructing $\mathcal{T}_\Xi$, we can use it to compute quantities associated to $\mathcal{T}_M$, like state sum or vacuum expectation values of observables, in terms of finite-dimensional integrals (since space of fields of $\mathcal{T}_\Xi$ is finite-dimensional), instead of functional integrals. Thus, having a discrete theory $\mathcal{T}_\Xi$, we leave behind all subtleties of functional integrals, like regularization and possibility of renormalization.

To our knowledge the simplicial program was successfully completed for abelian Chern-Simons theory (see [1], [15]).

We consider a special topological theory, "extended $BF$ theory", which is the ordinary non-abelian $BF$ theory with fields promoted to extended fields — non-homogeneous differential forms (also called "super fields" in physical literature). From physical point of view, extended $BF$ theory is ordinary $BF$ theory with ghosts, ghosts for ghosts etc. taken into account, and arises naturally in process of quantization. This theory perfectly fits into Batalin-Vilkovisky formalism as a special case of what we call "abstract $BF$ theory". The latter is a class of theories associated to differential graded Lie algebras, for extended $BF$ case — associated to algebra of differential forms on manifold $M$ with values in a gauge Lie algebra $\mathfrak{g}$. The action of extended $BF$ theory is a generating function for structure constants of commutator and differential on the algebra of $\mathfrak{g}$-valued differential forms, and Batalin-Vilkovisky master equation is equivalent to the three quadratic relations for forms: $d^2 = 0$, Leibniz identity and Jacobi identity.

For any abstract $BF$ theory, associated to some DGLA $\mathcal{G}$, if $\mathcal{G}$ is decomposed into sum of two subcomplexes $\mathcal{G} = \mathcal{G}' \oplus \mathcal{G}''$ with $\mathcal{G}''$ acyclic, we construct induced (in physical terminology, "effective") theory, associated to $\mathcal{G}'$. The action of this induced theory is constructed via an integral "over $\mathcal{G}''$" (more precisely, an integral over Lagrangian submanifold in $\Pi\mathcal{G}'' \oplus [\mathcal{G}'']^*$) — the Batalin-Vilkovisky integral. The question of convergence of the latter is quite subtle for infinite-dimensional $\mathcal{G}''$; we hope that it is at least perturbatively well-defined, and our calculations of induced action for $BF$ theory on simplex

confirm this conjecture (but do not prove it in full generality, of course). The tree part of effective action is the generating function for $L_\infty$ algebra operations on $\mathcal{G}'$, and quadratic relations on the operations follow from classical master equation. The general theorem states that $\mathcal{G}'$ and $\mathcal{G}$ are quasi-isomorphic as $L_\infty$ algebras. In terms of BV integral, the quasi-isomorphism between $\mathcal{G}'$ and $\mathcal{G}$ is explicitly constructed via "expectation value map" (theorem 2).

By Koszul duality, to define $L_\infty$ structure on $\mathcal{G}'$ is the same as to define cohomological vector field $Q$ (i.e. an odd vector field satisfying $Q^2 = 0$) on parity-reversed space $\Pi\mathcal{G}'$. The 1-loop part of effective action defines a $Q$-invariant measure (volume form) on $\Pi\mathcal{G}'$. The $Q$-invariance follows from quantum master equation. And there are no higher-loop contributions to BV integrals of abstract $BF$ theory. Thus there are classical (given by tree approximation) effects in BV integral — the induced $L_\infty$ operations on $\mathcal{G}'$ and the quasi-isomorphism $\mathcal{G}' \to \mathcal{G}$, and quantum effect (given by 1-loop contributions) — the $Q$-invariant measure on $\Pi\mathcal{G}'$. Calculation of the quantum effect is much more involved: values of 1-loop Feynman diagrams for effective action are expressed as certain super traces over $\mathcal{G}''$ and may contain divergencies if $\mathcal{G}''$ is infinite-dimensional. The induced theory associated to $\mathcal{G}'$ is in some sense "homotopic" to initial theory associated to $\mathcal{G}$. By "homotopic" we mean that $L_\infty$ structure on $\mathcal{G}'$ generated by tree part of effective action is quasi-isomorphic to DGLA structure on $\mathcal{G}$. Whether the notion of homotopy of $L_\infty$ algebras can be extended to $L_\infty$ algebras with $Q$-invariant measure, is an interesting question.

We use the name "classical higher operations" for terms of Taylor expansion of tree part of effective action — these correspond to $L_\infty$ operations on $\mathcal{G}'$, and name "quantum higher operations" for terms of Taylor expansion of 1-loop part of effective action (which is the logarithm of density of $Q$-invariant measure on $\Pi\mathcal{G}'$).

For the sake of simplicial program we are interested in constructing induced theory for extended $BF$ theory on manifold $M$, associated to subcomplex of $\mathfrak{g}$-valued differential forms, consisting of $\mathfrak{g}$-valued Whitney forms on triangulation $\Xi$ of $M$. We call this induced theory "simplicial $BF$ theory" on $\Xi$. From general arguments, this theory on triangulation, with finite-dimensional space of fields, is "homotopic" to extended $BF$ theory on $M$.

Technically, to write down simplicial $BF$ action for any triangulation $\Xi$, it is sufficient to solve the problem for single simplex in each dimension (this property is formulated as the factorization of BV integral for simplicial action on triangulation — theorem 5). Thus only one universal calculation is needed in each dimension. In dimension $D = 0$ it is trivial, for $D = 1$ — not quite trivial, but can be done exactly, and explicit formula for simplicial $BF$ action on 1-simplex is written (theorem 6). For dimensions $D > 1$ we do not know closed expression for effective action on $D$-simplex, but we have computed first classical and quantum higher operations (values of simplest Feynman diagrams for BV integral for the effective action are computed). We would like to emphasize that calculation of effective action on simplex is absolutely universal and done once and for all time — having it, we have defined simplicial $BF$ theory on any triangulation of any manifold, and may conduct further calculations starting from this discrete topological field theory, via finite-dimensional BV integrals.

One immediate use of simplicial $BF$ theory is as follows. We can construct induced $BF$ theory on de Rham cohomologies of manifold $M$. The tree part of effective action is the generating function for Massey operations on cohomologies, and 1-loop part provides a $Q$-invariant measure on parity-reversed space of cohomologies. But now instead of calculating this effective action via functional BV integral, starting from extended $BF$ theory on $M$, we may induce theory on cohomologies from simplicial $BF$ theory on

triangulation $\Xi$, via finite-dimensional integral. The 1-loop part of effective action on cohomologies may be integrated to give the state sum of extended $BF$ theory, which therefore again can be computed from simplicial $BF$ theory, avoiding functional integrals.

Other possible uses of simplicial $BF$ theory include the construction of knot (and higher-dimensional knot) invariants in terms of vacuum expectation values of certain observables in simplicial $BF$ theory. Another possible use is combinatorial construction of characteristic classes. We intend to elaborate on these points in the future.

## 1.1. **Main results.**

- The fact that simplicial $BF$ action on triangulation $\Xi$ decomposes into sum over simplices of $\Xi$ of some local contributions, which we call "reduced effective actions" on simplices (theorem 5). This statement is the reason why calculation of simplicial $BF$ action on one simplex in each dimension is universal and allows to define simplicial $BF$ action on any triangulation.
- Explicit expression for reduced effective action on 1-simplex (theorem 6), obtained by direct computation of corresponding BV integral. This result allows us to fully construct simplicial $BF$ theory on 1-dimensional simplicial complexes. We also use it to illustrate simplicial approach by computing state sum of $BF$ theory on circle $Z(S^1)$ starting from simplicial $BF$ action on discretized circle and calculating finite-dimensional BV integral. We show that $Z(S^1)$ equals the volume of gauge group. The expression for reduced effective action on 1-simplex is also an important ingredient for defining simplicial $BF$ action in dimensions $D > 1$.
- Explicit expressions for first classical higher operations on simplex of arbitrary dimension (theorem 7).
- Result of direct calculation of simplest non-trivial quantum operation $q^{(2)}$ for 2-simplex and for 3-simplex (theorem 9). This computation is quite long and contains divergent quantities in intermediate stages, and thus requires regularization. Not quite surprisingly, the final answers are finite.
- Partial result for first quantum operation $q^{(2)}$ on simplex of arbitrary dimension: symmetry allows only two possible terms for $q^{(2)}$. We recover coefficient for one of the terms from quantum master equation and known classical higher operations (theorem 8), while the other term is $Q$-exact, and thus the corresponding coefficient cannot be recovered in this way. This result agrees with theorem 9 in dimensions 2 and 3, but is much cheaper, in a sense that it does not require hard calculations.

Another important general construction (not specific to the simplicial setting) is the construction of $L_\infty$ quasi-isomorphism between DGLA (or more generally, $L_\infty$ algebra) $\mathcal{G}$ and induced $L_\infty$ structure on its subcomplex $\mathcal{G}'$ via BV integral (theorem 2).

## 1.2. **Open problems.** The following is the beginning of long list of interesting questions one can ask about simplicial $BF$ theory:

- Question of Wilson renormalization of simplicial $BF$ action. If $\Xi'$ is a triangulation and $\Xi$ is some subdivision of $\Xi'$, we can induce effective action $\tilde{S}_{\Xi'}$ on $\Xi'$ from simplicial $BF$ action $S_\Xi$ on $\Xi$. The question is: does $\tilde{S}_{\Xi'}$ obtained in this way differ from simplicial $BF$ action $S_{\Xi'}$ on $\Xi'$ (obtained by standard induction from extended $BF$ theory on manifold), and if yes, what is the difference? General arguments indicate that this difference ("renormalization") should be BV-exact, in a sense that exponentials $e^{\tilde{S}_{\Xi'}/\hbar}$ and $e^{S_{\Xi'}/\hbar}$ belong to the same cohomology class of BV-Laplacian. We checked that in dimension $D = 1$ there is no such renormalization, while already for $D = 2$ the first quantum operation gets renormalized under

barycentric aggregation of triangulation (while first classical higher operations are not renormalized).

- More general setting for the previous question is as follows. Let $\mathcal{G}$ be a DGLA and $\mathrm{Ret}_{\mathcal{G}}$ be the category of retracts, where objects are subcomplexes $\mathcal{G}' \subset \mathcal{G}$, containing all cohomology of $\mathcal{G}$ and morphisms are retractions. As we explain in this paper, information contained in induced $BF$ theory ($BF_\infty$ theory) on subcomplex $\mathcal{G}' \in \mathrm{Ret}_{\mathcal{G}}$ is equivalent to the pair $(Q, \rho)$ where $Q$ is a cohomological vector field on $\Pi\mathcal{G}'$ and $\rho$ is the $Q$-invariant measure on $\Pi\mathcal{G}'$. Then operation of induction of $BF_\infty$ theory from $\mathcal{G}'$ to $\mathcal{G}'' \subset \mathcal{G}'$ with Lagrangian manifold for BV integral defined by given chain homotopy operator $K$, may be regarded as "parallel transport" of $(Q, \rho)$-structure on objects of $\mathrm{Ret}_{\mathcal{G}}$ along morphism $K$. Then general setting for question about Wilson renormalization is: what can we say about holonomy of this parallel transport?
- Problem of constructing observables for simplicial $BF$ theory. Particularly we are interested in observables, associated to knots and higher dimensional knots. These should be some discrete analogs of observables constructed in [4].
- Simplicial $BF$ action defines curvature of a discretized superconnection. A natural question is: how to use it to write local combinatorial formulae for characteristic classes?
- Question of relation between state-sum of extended $BF$ theory on a manifold (calculated via simplicial $BF$ theory on triangulation) and Turaev-Viro-type invariants of manifolds, calculated as sum over colorings of triangulation (see [16]).
- Extension of our simplicial constructions to Poisson sigma model (see [12]), and their application to deformation quantization and Kontsevich integrals (see [8], [5]).

1.3. **Sources and literature.** The simplicial program for topological field theories was inspired by problem of constructing combinatorial version of Chern-Simons theory. This problem was proposed by M. Atiyah in [3]. Our main sources for geometric interpretation of Batalin-Vilkovisky formalism are [2] and [14]. Our source on infinity-algebras is [11]. One of key constructions for our treatment of simplicial $BF$ theory — the construction of Dupont's chain homotopy between differential forms on manifold $M$ and Whitney forms, associated to triangulation of $M$, is borrowed from [6]. The construction of effective action for $BF$ theory is explained in [10]. In unpublished paper [9] an alternative treatment of simplicial program is given. Paper [13] explains induction of effective action on cohomologies on tree level in mathematical rigor.

## 2. Extended $BF$ theory in Batalin-Vilkovisky formalism

Ordinary $BF$ theory on $D$-dimensional manifold $M$ with compact gauge group $G$ is defined by classical action

$$(1) \qquad\qquad S(\omega, B) = \mathrm{tr} \int_M B \wedge F$$

where $F = d\omega + \omega \wedge \omega$ is the curvature of connection 1-form $\omega$ on $M$ with values in Lie algebra $\mathfrak{g}$ of group $G$, and $B$ is $(D-2)$-form on $M$ with values in $\mathfrak{g}$. Trace is taken in some representation of $\mathfrak{g}$.

2.1. **Extended $BF$ theory: fields, action.** Now we move to the extended $BF$ theory. Let $\Omega(M) = \Omega^0(M) \oplus \cdots \oplus \Omega^D(M)$ be the commutative differential graded algebra (cDGA) of differential forms on $M$ (with $\Omega^k(M)$ being the subspace of $k$-forms)— the de Rham algebra with de Rham differential and wedge product. We denote by $\mathcal{G} = \mathfrak{g} \otimes \Omega(M) = \mathcal{G}^0 \oplus \cdots \oplus \mathcal{G}^D$ the differential graded Lie algebra (DGLA) of $\mathfrak{g}$-valued differential forms on $M$. Let $\{e_\alpha\}$ be some basis in $\mathcal{G}$ and $\{e^\alpha\}$ be the dual basis in $\mathcal{G}^*$. We also suppose that each basis element $e_\alpha$ is homogeneous and denote its degree by $|\alpha|$, which means that $e_\alpha \in \mathcal{G}^{|\alpha|}$. Parity of $e_\alpha$ (and parity of $e^\alpha$) is equal to parity of integer $|\alpha|$.

The extended $BF$ action is defined as

$$(2) \qquad S(\omega, p) = < p, d\omega + \frac{1}{2}[\omega, \omega] >$$

where $< \bullet, \bullet >$ denotes the canonical pairing of $\mathcal{G}$ and $\mathcal{G}^*$. The fields $\omega$ and $p$ are

$$(3) \qquad \omega = \sum_\alpha \omega^\alpha e_\alpha$$

$$(4) \qquad p = \sum_\alpha e^\alpha p_\alpha$$

where $\{\omega^\alpha\}$ are variables of parity opposite to parity of $|\alpha|$, while $\{p_\alpha\}$ are variables of parity coinciding with parity of $|\alpha|$. Thus $\omega$ is totally odd and $p$ is totally even. Field $\omega$ belongs to space $\overline{\Pi\mathcal{G}}$, the totally odd version of $\mathcal{G}$ (which is no longer a DGLA, but just a vector super space):

$$\omega \in \overline{\Pi\mathcal{G}} := [\mathbb{R}^{1|1} \otimes \mathcal{G}]^{\mathrm{odd}} = \Pi\mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \Pi\mathcal{G}^2 \oplus \cdots \oplus \Pi^{D+1}\mathcal{G}^D$$

while $p$ belongs to $\overline{\mathcal{G}^*}$, the totally even version of $\mathcal{G}^*$:

$$p \in \overline{\mathcal{G}^*} := [\mathcal{G}^* \otimes \mathbb{R}^{1|1}]^{\mathrm{even}} = [\mathcal{G}^0]^* \oplus \Pi[\mathcal{G}^1]^* \oplus [\mathcal{G}^2]^* \oplus \cdots \oplus \Pi^D[\mathcal{G}^D]^*$$

Here $\Pi$ is the parity reversing operation on vector super spaces, $\Pi^k = \Pi$ for $k$ odd and $\Pi^k = \mathrm{id}$ for $k$ even. We also use the traditional notation $\mathbb{R}^{k|l} = \mathbb{R}^k \oplus \Pi\mathbb{R}^l$, and for a vector super space $X$ we denote its even and odd subspaces by $[X]^{\mathrm{even}}$ and $[X]^{\mathrm{odd}}$ respectively.

The action

$$(5) \quad S(\omega, p) = < p, d\omega + \frac{1}{2}[\omega, \omega] > =$$

$$= \sum_{\alpha,\beta} (-1)^{|\beta|+1} < e^\alpha, d\,e_\beta > p_\alpha\,\omega^\beta + \frac{1}{2}\sum_{\alpha,\beta,\gamma} (-1)^{|\beta|\,(|\gamma|+1)} < e^\alpha, [e_\beta, e_\gamma] > p_\alpha\,\omega^\beta\omega^\gamma$$

belongs to the space $\mathbb{R}[[\{\omega^\alpha\}, \{p_\alpha\}]]$ of formal power series of variables $\{\omega^\alpha\}$ and $\{p_\alpha\}$. This space is an associative commutative super algebra freely generated by variables $\{\omega^\alpha\}$ and $\{p_\alpha\}$. Thus we may think of it as of algebra of functions $\mathrm{Fun}(\mathcal{F})$ on space $\mathcal{F} = \Pi\mathcal{G} \oplus \mathcal{G}^*$. The latter is called "space of fields". To match more general constructions of Batalin-Vilkovisky formalism (see [2]), we may identify $\mathcal{F}$ with cotangent bundle to $\Pi\mathcal{G}$ with reversed parity in fibers:

$$(6) \qquad \mathcal{F} = \Pi T^*(\Pi\mathcal{G})$$

In this picture $\omega^\alpha$ are coordinate functions on base of $\mathcal{F}$, while $p_\alpha$ are coordinate functions on fibers of $\mathcal{F}$. Our initial fields $\omega$ and $p$ are then "generating functions" for coordinate functions on base and on fibers of $\mathcal{F}$ respectively, useful to write short formulae for various objects of extended $BF$ theory (as in (5)).

Less formally, extended $BF$ action (2) is obtained from (1) by promoting $\omega$ and $B$ to extended fields (also called "super fields") and introducing a new field $p$, related to $B$ by

lowering an index: $< p, \bullet >= \mathrm{tr} \int_M B \wedge \bullet$. Field $\omega$ is a non-homogeneous differential form on $M$ with values in gauge algebra $\mathfrak{g}$:

$$\omega = \omega^{(0)} + \cdots + \omega^{(D)}$$

with $\omega^{(k)}$ being $\mathfrak{g}$-valued $k$-form with parity opposite to parity of integer $k$. Field $p$ is decomposed as

$$p = p^{(0)} + \cdots + p^{(D)}$$

with $p^{(k)}$ a $k$-coform (element of $[\Omega^k(M)]^*$, or equivalently $(D-k)$-form with lowered index) with values in coalgebra $\mathfrak{g}^*$ and with parity equal to parity of $k$. Thus $\omega$ is a non-homogeneous $\mathfrak{g}$-valued form of total parity 1, while $p$ is a non-homogeneous $\mathfrak{g}^*$-valued coform of total parity 0.

**Remark.** One can give an alternative description of extended $BF$ theory in terms of $\mathbb{Z} \oplus \mathbb{Z}$ grading: we may prescribe "ghost numbers" to $\{\omega^\alpha\}$ and $\{p_\alpha\}$ instead of just parities: $\mathrm{gh}(\omega^\alpha) = 1 - |\alpha|$, $\mathrm{gh}(p_\alpha) = -2 + |\alpha|$. Then we say that fields $\omega$ and $p$ belong to spaces $[\mathrm{Gr} \otimes \mathcal{G}]^{\deg + \mathrm{gh} = +1}$ and $[\mathcal{G}^* \otimes \mathrm{Gr}]^{\deg + \mathrm{gh} = -2}$ respectively, where $\mathrm{Gr} = \bigoplus_{k=-\infty}^{\infty} \mathbb{R}^{[+k]}$ — the vector super space graded by ghost number gh, and deg is the grading on $\mathcal{G}$. The space of functions $\mathrm{Fun}(\mathcal{F})$ becomes $\mathbb{Z}$-graded commutative associative algebra with grading given by ghost number. The action (2) has ghost number zero.

**Remark.** Extended $BF$ theory arises naturally in process of quantizing ordinary $BF$ theory by incorporating ghosts, ghosts for ghosts etc. and anti-fields for all these (and original fields). This set of fields is then organized into a pair of "superfields" — non-homogeneous differential forms $\omega$ and $p$, and the action may be written in simple form (2).

2.2. **$P$-structure on space of fields of extended $BF$ theory.** Space $\mathcal{F}$ has a structure of $QP$-manifold. The $P$-structure (odd simplectic structure) is defined by Batalin-Vilkovisky 2-form (odd simplectic form)

(7) $$\Omega_{\mathrm{BV}} = < \delta\omega, \delta p >$$

so that $\omega$ and $p$ are canonically conjugated w.r.t. $\Omega_{\mathrm{BV}}$. The odd simplectic structure on $\mathcal{F}$ induces on space of functions of fields $\mathrm{Fun}(\mathcal{F})$ the structure of anti-bracket algebra (odd Poisson algebra) with usual pointwise associative (super)commutative product and the anti-bracket, defined as

$$\{f, g\} = f \left( < \frac{\overleftarrow{\partial}}{\partial p}, \frac{\overrightarrow{\partial}}{\partial \omega} > - < \frac{\overleftarrow{\partial}}{\partial \omega}, \frac{\overrightarrow{\partial}}{\partial p} > \right) g$$

for a pair of functions $f, g \in \mathrm{Fun}(\mathcal{F})$. Following properties hold for anti-bracket:
- symmetry property

$$\{f, g\} = -(-1)^{(\epsilon(f)+1)(\epsilon(g)+1)}\{g, f\}$$

- Leibniz rule for anti-bracket

$$\{f, g \cdot h\} = \{f, g\} \cdot h + (-1)^{(\epsilon(f)+1)\,\epsilon(g)} g \cdot \{f, h\}$$

- Jacobi identity

$$\{\{f, g\}, h\} + (-1)^{(\epsilon(h)+1)(\epsilon(f)+\epsilon(g))}\{\{h, f\}, g\} + (-1)^{(\epsilon(f)+1)(\epsilon(g)+\epsilon(h))}\{\{g, h\}, f\} = 0$$

Here $\epsilon$ is notation for parity and we assume that $f, g, h \in \mathrm{Fun}(\mathcal{F})$ are functions of definite parity. Batalin-Vilkovisky Laplacian $\Delta_{\mathrm{BV}}$ on $\mathrm{Fun}(\mathcal{F})$ is defined as

$$\Delta_{\mathrm{BV}} = < \frac{\partial}{\partial \omega}, \frac{\partial}{\partial p} >$$

It satisfies
- the nilpotence property
$$\Delta_{\mathrm{BV}}^2 = 0$$
- the property relating $\Delta_{\mathrm{BV}}$ and the anti-bracket
$$\Delta_{\mathrm{BV}}(f \cdot g) = \Delta_{\mathrm{BV}} f \cdot g + (-1)^{\epsilon(f)} f \cdot \Delta_{\mathrm{BV}} g + (-1)^{\epsilon(f)}\{f, g\}$$

### 2.3. $Q$-structure on space of fields of extended $BF$ theory; master equation.
The $Q$-structure on $\mathcal{F}$ is induced from cohomological vector field $Q$ on base $\Pi\mathcal{G}$ defined as

$$Q = < d\omega + \frac{1}{2}[\omega, \omega], \frac{\partial}{\partial \omega} >$$

Action (2) on $\mathcal{F}$ is then $S = < p, Q\ \omega >$ and cohomological vector field on $\mathcal{F}$ is a Hamiltonian vector field generated by $S$:

$$Q_{\mathcal{F}} = \{S, \bullet\}$$

Action (2) satisfies the quantum master equation

(8) $$\Delta_{\mathrm{BV}} e^{S/\hbar} = 0$$

where $\hbar$ is formal infinitesimal parameter. Since

$$\Delta_{\mathrm{BV}} e^{S/\hbar} = (\hbar^{-2}\frac{1}{2}\{S, S\} + \hbar^{-1}\Delta_{\mathrm{BV}} S)e^{S/\hbar}$$

equation (8) is equivalent to the pair of equations

(9) $$\{S, S\} = 0$$

(the classical master equation) and

(10) $$\Delta_{\mathrm{BV}} S = 0$$

In terms of $Q$, (9) means $Q^2 = 0$, while (10) means that

(11) $$\mathrm{div}\, Q = 0$$

This is in turn equivalent to the fact that volume form $\eta = \prod_\alpha \delta\omega^\alpha$ on $\Pi\mathcal{G}$ is conserved by vector field $Q$.

### 2.4. Extended $BF$ action as generating function of DGLA structure on $\mathfrak{g}\otimes\Omega(M)$.
Action (2) may be viewed as the generating function for structure constants of differential and Lie bracket on $\mathcal{G}$. Classical master equation (9) is then equivalent to the three relations in DGLA $\mathcal{G}$: $d^2 = 0$, Leibniz identity and Jacobi identity:

$$\frac{1}{2}\{S, S\} = < p, -d^2\omega > + < p, -\frac{1}{2}\,d[\omega, \omega] + [d\omega, \omega] > + \frac{1}{2} < p, [[\omega, \omega], \omega] >= 0$$

Vanishing of linear in $\omega$ term here is equivalent to $d^2 = 0$, of quadratic term — to Leibniz identity, of cubic term — to Jacobi identity. Equation (10) reads

$$\Delta_{\mathrm{BV}} S = \mathrm{Str}_{\Pi\mathcal{G}}\,[\omega, \bullet] = 0$$

This follows from relation $f_{ab}^b = 0$ which we demand for structure constants of gauge algebra $\mathfrak{g}$. For example, it holds for semi-simple gauge algebras.

**2.5. Generalization from extended $BF$ to abstract $BF$ theory.** The construction of extended $BF$ theory admits a natural generalization to abstract $BF$ theory as follows: take any DGLA $\mathcal{G}$ instead of differential forms on manifold with values in a gauge algebra. We should only demand that structure constants of $\mathcal{G}$ satisfy $f_{\alpha\beta}^{\beta} = 0$. Then construct space of fields (6), which is again a $QP$ manifold. The action is again given by (2) and it again satisfies quantum master equation by virtue of general properties of DGLA: $d^2 = 0$, Leibniz identity, Jacobi identity, and the property $f_{\alpha\beta}^{\beta} = 0$ which we demanded for $\mathcal{G}$.

**2.6. Canonical transformations, gauge symmetry, symmetry under diffeomorphisms.** Infinitesimal canonical transformation is defined as follows: let $R \in \mathrm{Fun}(\mathcal{F})$ be some infinitesimal odd function of fields (the generator of canonical transformation). Then map

$$
\begin{aligned}
\phi_R^* : \mathrm{Fun}(\mathcal{F}) &\rightarrow \mathrm{Fun}(\mathcal{F}) \\
f &\mapsto f + \{f, R\}
\end{aligned}
$$

is an automorphism of anti-bracket algebra $\mathrm{Fun}(\mathcal{F})$ (in lowest order in $R$) due to Leibniz identity and Jacobi identity for the anti-bracket. Canonical transformation on functions $\phi_R^*$ may be understood as a pullback of simplectomorphism (in terminology of Hamiltonian formalism, canonical change of coordinates) $\phi_R : \mathcal{F} \rightarrow \mathcal{F}$, defined by $\omega \rightarrow \omega + \{\omega, R\}$ and $p \rightarrow p + \{p, R\}$. Action is transformed by $\phi_R^*$ as

$$
S \mapsto S + \{S, R\} + \hbar \Delta_{\mathrm{BV}} R
$$

the last term is due to the fact that action is not a scalar function but rather a logarithm of density of measure on space of fields, and thus transforms non-tensorially under change of coordinates. It may be more transparent to write canonical transformation of action in terms of exponentials:

$$
e^{S/\hbar} \mapsto e^{S/\hbar} + \Delta_{\mathrm{BV}}(e^{S/\hbar} R)
$$

Canonically transformed action leads of course to physically equivalent theory.

It turns out that two important symmetries of extended $BF$ theory, the gauge symmetry and symmetry under diffeomorphisms, may be regarded as special canonical transformations that leave action invariant. Namely, infinitesimal diffeomorphism, generated by vector field $v$ on manifold $M$, may be regarded as canonical transformation with generator

$$
R_v = <p, \mathcal{L}_v \omega>
$$

where $\mathcal{L}_v$ is the Lie derivative along $v$, acting on differential forms on $M$.

The gauge invariance may be formulated in more general setting of abstract $BF$ theory. The parameter of gauge transformation is totally even element $\alpha \in [\mathbb{R}^{1|1} \otimes \mathcal{G}]^{\mathrm{even}}$ (for extended $BF$ case, this is a totally even $\mathfrak{g}$-valued differential form). The gauge transformation acts on fields by

$$
\begin{aligned}
\omega &\mapsto \omega + [\omega, \alpha] + d\alpha \\
p &\mapsto p + [p, \alpha]
\end{aligned}
$$

where $[p, \alpha]$ is the right coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$. We see that this gauge transformation is actually a special canonical transformation with generator

$$
R_\alpha = - <p, d\alpha + [\omega, \alpha]>
$$

It is instructive to note that $R_\alpha$ may be obtained directly from the action:

$$
\tag{12} R_\alpha = <\alpha, \frac{\partial}{\partial \omega} S>
$$

Gauge invariance of the action follows then from master equation. This approach allows us to describe gauge transformation in even more general case of $BF_\infty$ theories, which we will introduce later.

In case of extended $BF$ theory gauge symmetry we described is rather a "super gauge symmetry", since it mixes components of $\omega$ and $p$ of different de Rham degree. The ordinary gauge transformations correspond to the case when gauge parameter $\alpha$ is just a $\mathfrak{g}$-valued function: $\alpha \in \mathcal{G}^0$. The form of gauge transformation of field $\omega$ allows us to call $\omega$ the "superconnection" in trivial $G$-bundle on $M$. Then $F = d\omega + \frac{1}{2}[\omega, \omega]$ is naturally the curvature of superconnection $\omega$.

## 3. Effective action for abstract $BF$ theory

We first describe a general construction of effective action for abstract $BF$ theory, and then specialize to differential forms (extended $BF$ case) in the next section.

### 3.1. Infrared and ultraviolet fields, chain homotopy, BV integral for effective action on infrared fields. Let DGLA $\mathcal{G}$ be split into sum of two subcomplexes

$$\mathcal{G} = \mathcal{G}' \oplus \mathcal{G}''$$

with $\mathcal{G}''$ acyclic. We call $\mathcal{G}'$ the infrared subcomplex and $\mathcal{G}''$ the ultraviolet subcomplex. Names "infrared" and "ultraviolet" come from physical construction of Wilson effective action in quantum field theory: one splits fields into low-frequency (infrared) and high-frequency (ultraviolet) parts and integrates out the ultraviolet fields to obtain effective action on infrared fields. Space of fields of $BF$ theory associated to $\mathcal{G}$ is then also split: $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$. Fields $\omega \in \overline{\Pi\mathcal{G}}$ and $p \in \overline{\mathcal{G}^*}$ are decomposed into infrared and ultraviolet parts: $\omega = \omega' + \omega''$, $p = p' + p''$ with $\omega' \in \overline{\Pi\mathcal{G}'}$, $\omega'' \in \overline{\Pi\mathcal{G}''}$, $p' \in \overline{[\mathcal{G}']^*}$, $p'' \in \overline{[\mathcal{G}'']^*}$. Let us denote projectors on $\mathcal{G}'$ and $\mathcal{G}''$ by $\mathcal{P}'$ and $\mathcal{P}''$ respectively.

Let also $K$ be a linear operator on $\mathcal{G}''$ satisfying

$$
\begin{align}
Kd + dK &= \mathcal{P}'' \tag{13}\\
K^2 &= 0 \tag{14}
\end{align}
$$

(a chain homotopy). It is continued to $\mathcal{G}$ by defining $K|_{\mathcal{G}'} = 0$. Starting from chain homotopy $K$ we construct a Lagrangian submanifold $\mathcal{L}_K \subset \mathcal{F}''$ as a vector subspace in $\mathcal{F}''$ defined by equations $K\omega'' = 0$, $p''K = 0$. In other words

$$\mathcal{L}_K = \Pi \ker K \oplus [\operatorname{coker} K]^*$$

Then we define the effective action $S'$ on $\mathcal{F}'$ via an integral over $\mathcal{L}_K$ ("BV integral"):

$$e^{\frac{1}{\hbar}S'(\omega',p';\hbar)} = \frac{1}{N} \int_{\mathcal{L}_K} e^{\frac{1}{\hbar}S(\omega'+\omega'',p'+p'')}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K} \tag{15}$$

Here $[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K}$ is the volume form on $\mathcal{L}_K$ and

$$N = \int_{\mathcal{L}_K} e^{\frac{1}{\hbar}<p'',d\omega''>}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K}$$

is the normalization factor.

### 3.2. Perturbative evaluation of BV integral for effective action.

We view (15) as a formal perturbative definition of $S'$. If we expand in ultraviolet fields the action in exponent in integrand of (15) we get

$$(16) \quad S(\omega' + \omega'', p' + p'') = S(\omega', p') + <p'', d\omega''> + <p'', \frac{1}{2}[\omega'', \omega'']> +$$

$$+ <p'', \frac{1}{2}[\omega', \omega']> + <p'', [\omega', \omega'']> + <p', [\omega', \omega'']> + <p', \frac{1}{2}[\omega'', \omega'']>$$

The first term here is constant (on $\mathcal{L}_K$), the second term we interpret as a free Gaussian action $S_0'' = <p'', d\omega''>$. The other terms are treated as perturbation of $S_0''$ (and hence as vertices in Feynman rules for (15)). Let us denote the normalized expectation value of $f \in \mathrm{Fun}(\mathcal{L}_K)$ with respect to $S_0''$ by

$$\ll f \gg_0 = \frac{1}{N} \int_{\mathcal{L}_K} e^{\frac{1}{\hbar}<p'', d\omega''>} f(\omega'', p'') \, [\mathcal{D}\omega'' \mathcal{D}p'']_{\mathcal{L}_K}$$

Then propagator $\ll \omega'' \otimes p'' \gg_0$ viewed as an element of $\mathcal{G}'' \otimes [\mathcal{G}'']^* = \mathrm{End}(\mathcal{G}'')$ is (up to constant factor) the chain homotopy operator:

$$\ll \omega'' \otimes p'' \gg_0 = -\hbar K$$

This implies that for constant vectors $\tilde{\omega} \in \overline{\Pi \mathcal{G}}$, $\tilde{p} \in \overline{\mathcal{G}^*}$ we have

$$\ll \frac{1}{\hbar}<\tilde{p}, \omega''> \cdot \frac{1}{\hbar}<p'', \tilde{\omega}> \gg_0 = -\frac{1}{\hbar}<\tilde{p}, K\tilde{\omega}>$$

Using this fact and Wick's theorem we obtain description of values of Feynman graphs for (15) in terms of iterated operation formalism.

First we need to introduce some general notation for a binary operation iterated on a rooted binary tree. By a rooted binary tree $T$ we mean an acyclic graph with one vertex of valence 2 (root), several vertices of valence 1 (leaves) and all other vertices of valence 3 (internal vertices). This graph comes with fixed planar structure, i.e. embedding $T \to \mathbb{R}^2$ modulo diffeomorphisms of $\mathbb{R}^2$, so that each non-leaf vertex has well-defined left and right children. Let $X$ be a vector super-space over $\mathbb{R}$ and $\mathcal{O} : X \times X \to X$ a bilinear map. For a rooted binary tree $T$ with $|T| = n$ leaves, define $n$-linear map

$$\mathrm{Iter}_{T, \mathcal{O}} : X^n \to X$$

by the following iterative procedure: for $(x_1, \ldots, x_n)$ the $n$-tuple of elements of $X$ decorate each leaf of $T$ with $x_i$ where $i$ is the number of leaf counted counterclockwise starting from root. Decorate each non-leaf vertex $v$ with $\mathcal{O}(x_{v_l}, x_{v_r})$ where $x_{v_l}$ and $x_{v_r}$ are elements of $X$ assigned to left and right children of $v$ respectively. We define $\mathrm{Iter}_{T, \mathcal{O}}(x_1, \ldots, x_n)$ as the value assigned to root of $T$ by this procedure. We also need the following modification of this definition: for $\mathcal{O}, \mathcal{O}' : X \times X \to X$ a pair of binary operations we define

$$\mathrm{Iter}_{T, \mathcal{O}, \mathcal{O}'} : X^n \to X$$

by the same procedure as above with the only difference that in the root we evaluate $\mathcal{O}'$ on children instead of $\mathcal{O}$ If we identify binary rooted trees with binary bracket structures, we have for example

$$\mathrm{Iter}_{((*(**))(**)), \, \mathcal{O}} \, (x_1, x_2, x_3, x_4, x_5) = \mathcal{O}(\mathcal{O}(x_1, \mathcal{O}(x_2, x_3)), \mathcal{O}(x_4, x_5))$$

and

$$\mathrm{Iter}_{((*(**))(**)), \, \mathcal{O}, \mathcal{O}'} \, (x_1, x_2, x_3, x_4, x_5) = \mathcal{O}'(\mathcal{O}(x_1, \mathcal{O}(x_2, x_3)), \mathcal{O}(x_4, x_5))$$

Here symbols $*$ denote leaves of the tree.

We also need the general notation for trace operator associated to a bilinear map and a binary 1-loop graph. Let $L$ be a graph with one oriented cycle (and fixed planar structure),

internal vertices of valence 3 and $|L| = n$ vertices of valence 1 (leaves). We define $n$-linear function $\text{Loop}_{L,\mathcal{O},X} : X^n \to \mathbb{R}$ as follows. Cutting any edge of the cycle of $L$ produces a tree $T$ with $n+1$ leaves and one leaf marked (it was connected to root before cutting). We define $\text{Loop}_{L,\mathcal{O},X}$ as the super-trace

$$\text{Loop}_{L,\mathcal{O},X}(x_1, \cdots, x_n) = \text{Str}_X \text{Iter}_{T,\mathcal{O}}(x_1, \cdots, x_{i-1}, \bullet, x_i, \cdots, x_n)$$

where $i$ is the number of the marked leaf (counted counterclockwise from the root, as before). If we denote 1-loop graphs as trees with one marked leaf, we have for example

$$\text{Loop}_{(((**)\bullet)*),\, \mathcal{O},\, X} = \text{Str}_X \mathcal{O}(\mathcal{O}(\mathcal{O}(x_1, x_2), \bullet), x_3)$$

Here symbols $*$ denote non-marked leaves and $\bullet$ denotes the marked leaf.

Let $\mathbf{T}$ be the set of binary rooted trees and $\mathbf{L}$ be the set of binary 1-loop graphs. Let also $\hat{\mathbf{T}}$ be set of binary rooted trees without planar structure (i.e. quotient of $\mathbf{T}$ over graph isomorphisms), and $\hat{\mathbf{L}}$ — the set binary 1-loop graphs without planar structure.

Having introduced the necessary notation we return to perturbation theory for (15). For every tree $T \in \hat{\mathbf{T}}$ we define a function $S_T \in \text{Fun}(\mathcal{F}')$ as

$$(17) \qquad S_T(\omega', p') = \frac{1}{|\text{Aut}(T)|} < p', \text{Iter}_{T, -K[\bullet,\bullet],\, [\bullet,\bullet]}(\omega', \ldots, \omega') >$$

Here $\text{Aut}(T)$ is the group of automorphisms of $T$ and $|\text{Aut}(T)|$ is its order (factor $\frac{1}{|\text{Aut}(T)|}$ is usually called "symmetry coefficient" of the Feynman graph). Expression (17) is linear in $p'$ and of degree $|T|$ in $\omega'$. It does not depend on planar structure on $T$ since binary operations we are iterating $\mathcal{O} = -K[\bullet, \bullet]$ and $\mathcal{O}' = [\bullet, \bullet]$ are commutative on $\overline{\Pi \mathcal{G}'}$, and so the result does not depend on which child of a vertex we call left and which we call right.

Analogously, for every binary 1-loop graph $L \in \hat{\mathbf{L}}$ we define a function $S_L \in \text{Fun}(\Pi \mathcal{G}')$ as the super-trace over $\Pi \mathcal{G}'$:

$$(18) \qquad S_L(\omega') = \frac{1}{|\text{Aut}(L)|} \text{Loop}_{L, -K[\bullet,\bullet],\, \overline{\Pi \mathcal{G}'}}(\omega', \ldots, \omega')$$

Here $|\text{Aut}(L)|$ is again the order of automorphism group of graph $L$. Expression (18) is of degree $|L|$ in $\omega'$. The independence of (18) on planar structure on $L$ is checked by the same argument as for trees.

Now we have all the ingredients to describe the perturbation series for $S'$ in powers of $\omega'$:

**Theorem 1 (Perturbation expansion for effective action of abstract $BF$ theory).** *Effective action is linear in $\hbar$:*

$$S'(\omega', p'; \hbar) = S'^{(0)}(\omega', p') + \hbar S'^{(1)}(\omega')$$

*$S'^{(0)}$ is represented as sum over rooted binary trees without planar structure*

$$(19) \quad S'^{(0)}(\omega', p') = S(\omega', p') + \sum_{T \in \hat{\mathbf{T}}:\, |T| \geq 3} S_T(\omega', p') =$$

$$= S(\omega', p') + \sum_{T \in \hat{\mathbf{T}}:\, |T| \geq 3} \frac{1}{|\text{Aut}(T)|} < p', \text{Iter}_{T, -K[\bullet,\bullet],\, [\bullet,\bullet]}(\omega', \ldots, \omega') >$$

*The first term here is a restriction of $BF$ action in full space $\mathcal{F}$ to subspace $\mathcal{F}'$. $S'^{(1)}$ is a sum over binary 1-loop graphs $L$ without planar structure*

$$(20) \qquad S'^{(1)}(\omega') = \sum_{L \in \hat{\mathbf{L}}} S_L(\omega') = \sum_{L \in \hat{\mathbf{L}}} \frac{1}{|\text{Aut}(L)|} \text{Loop}_{L, -K[\bullet,\bullet],\, \overline{\Pi \mathcal{G}'}}(\omega', \ldots, \omega')$$

12

*and does not depend on $p'$. First terms of perturbation expansions for $S'^{(0)}$ and $S'^{(1)}$ are:*

$$(21) \quad S'^{(0)}(\omega', p') = <p', d\omega'> + \frac{1}{2}<p', [\omega', \omega']> - \frac{1}{2}<p', [K[\omega', \omega'], \omega']> +$$

$$+ \frac{1}{2}<p', [K[K[\omega', \omega'], \omega'], \omega'] + \frac{1}{8}<p', [K[\omega', \omega'], K[\omega', \omega']]> + \mathcal{O}(p'\omega'^5)$$

*and*

$$(22) \quad S'^{(1)}(\omega') = -\operatorname{Str} K[\omega', \bullet] + \frac{1}{2}\operatorname{Str} K[K[\omega', \omega'], \bullet] + \frac{1}{2}\operatorname{Str} K[\omega', K[\omega', \bullet]] -$$

$$-\frac{1}{2}\operatorname{Str} K[K[K[\omega', \omega'], \omega'], \bullet] - \frac{1}{2}\operatorname{Str} K[K[\omega', \omega'], K[\omega', \bullet]] - \frac{1}{3}\operatorname{Str} K[\omega', K[\omega', K[\omega', \bullet]]] + \mathcal{O}(\omega'^4)$$

3.3. **Properties of effective theory on infrared fields: $QP$-structure on space of fields, master equation.** Space of infrared fields $\mathcal{F}' = \Pi T^*(\Pi \mathcal{G}')$ becomes equipped with $QP$-structure in the following way. The $P$ structure is provided by restriction of BV 2-form on $\mathcal{F}$ to $\mathcal{F}'$:

$$\Omega'_{\text{BV}} = \Omega_{\text{BV}}|_{\mathcal{F}'} = <\delta\omega', \delta p'>$$

Analogously, the BV Laplacian and anti-bracket on $\operatorname{Fun}(\mathcal{F}')$ are just restrictions of their counterparts on $\operatorname{Fun}(\mathcal{F})$ to $\operatorname{Fun}(\mathcal{F}')$. Effective action $S' \in \operatorname{Fun}(\mathcal{F}')$ automatically satisfies quantum master equation by virtue of general property of BV integrals:

$$<\frac{\partial}{\partial\omega'}, \frac{\partial}{\partial p'}> e^{\frac{1}{\hbar}S'(\omega', p'; \hbar)} = \frac{1}{N}\int_{\mathcal{L}_K}(\Delta_{\text{BV}} - <\frac{\partial}{\partial\omega''}, \frac{\partial}{\partial p''}>)e^{\frac{1}{\hbar}S(\omega, p)}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K} =$$

$$= -\frac{1}{N}\int_{\mathcal{L}_K}<\frac{\partial}{\partial\omega'''}, \frac{\partial}{\partial p''}> e^{\frac{1}{\hbar}S(\omega, p)}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K} = 0$$

In terms of $S'^{(0)}$ and $S'^{(1)}$ the quantum master equation means

$$(23) \qquad \{S'^{(0)}, S'^{(0)}\} = 0$$

(the classical master equation for $S'^{(0)}$) and

$$(24) \qquad \{S'^{(0)}, S'^{(1)}\} + \Delta_{\text{BV}} S'^{(0)} = 0$$

Hence tree effective action $S'^{(0)}$ provides a $Q$ structure to $\mathcal{F}'$ — the cohomological vector field

$$Q_{\mathcal{F}'} = \{S'^{(0)}, \bullet\}$$

Since $S'^{(0)}$ is linear in $p'$, vector field $Q_{\mathcal{F}'}$ is tangent to the base $\Pi\mathcal{G}'$, and defines on it the cohomological vector field $Q'$ (thus $Q'$ is a coderivation of $\operatorname{Fun}(\Pi\mathcal{G}')$). In terms of $Q'$ the classical master equation (23) is the cohomologicity condition

$$(25) \qquad Q'^2 = 0$$

while the equation (24) means

$$(26) \qquad \operatorname{div} Q' + Q'S'^{(1)} = 0$$

3.4. **Tree effective action on infrared fields as generating function of $L_\infty$ algebra structure.** A well known theorem from [2] states that $Q$ - structure on a manifold $\mathcal{N}$ generates an $L_\infty$ algebra structure on parity-reversed tangent space $\Pi T_a \mathcal{N}$ in the point $a \in \mathcal{N}$ where $Q$ vanishes. In our case of effective $BF$ theory we have $\mathcal{N} = \Pi\mathcal{G}'$, $a = 0$, due to linearity of space of fields we identify $\Pi T_0(\Pi\mathcal{G}')$ with $\mathcal{G}'$. Thus $Q'$ is a generating function for $L_\infty$ algebra structure on $\mathcal{G}'$.

This is also a special case of Koszul duality: to introduce a coderivation $Q'$ on commutative associative super algebra of functions $\operatorname{Fun}(\Pi\mathcal{G}')$ is equivalent to defining $L_\infty$ structure on $\mathcal{G}'$.

### 3.5. Construction of $L_\infty$ quasi-isomorphism between $\mathcal{G}'$ and $\mathcal{G}$ via expectation value map for BV integral; perturbative series.

We can construct an $L_\infty$ quasi-isomorphism $U$ of $L_\infty$ algebra $(\mathcal{G}', Q')$ and DGLA $(\mathcal{G}, Q)$:

$$U : \Pi\mathcal{G}' \to \Pi\mathcal{G}$$

Map $U$ is a non-linear deformation of the embedding $\iota : \mathcal{G}' \to \mathcal{G}$. The pullback $U^* : \text{Fun}(\Pi\mathcal{G}) \to \text{Fun}(\Pi\mathcal{G}')$ is constructed as expectation value map:

$$(27) \qquad U^*(f)(\omega') = \frac{\int_{\mathcal{L}_K} f(\omega)\, e^{\frac{1}{\hbar}S(\omega,p)}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K}}{\int_{\mathcal{L}_K} e^{\frac{1}{\hbar}S(\omega,p)}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K}}$$

for $f \in \text{Fun}(\Pi\mathcal{G})$. Map $U^*$ can be lifted to pre-$L_\infty$ morphism $U : \Pi\mathcal{G}' \to \Pi\mathcal{G}$ because $U^*$ is a homomorphism: $U^*(fg) = U^*(f)U^*(g)$. This is in turn a consequence of the fact that field $\omega$ is non self-interacting in $BF$ theory. To show that $U$ is a true $L_\infty$ morphism we need to check that for any function $f \in \text{Fun}(\Pi\mathcal{G})$ we have

$$(28) \qquad Q'U^*(f) = U^*(Qf)$$

This is ensured by the following argument:

$$\Delta'_{\text{BV}}(e^{S'/\hbar}U^*(f)) = e^{S'/\hbar}\left(\frac{1}{\hbar}\{S', U^*(f)\}' + \Delta'_{\text{BV}}U^*(f)\right) = \frac{1}{\hbar}e^{S'/\hbar}\{S', U^*(f)\}' = \frac{1}{\hbar}e^{S'/\hbar}Q'U^*(f)$$

(we put primes here on BV Laplacian and anti-bracket on $\text{Fun}(\mathcal{F}')$ to distinguish them from their counterparts on full space $\text{Fun}(\mathcal{F})$). On the other hand

$$\Delta'_{\text{BV}}(e^{S'/\hbar}U^*(f)) = \frac{1}{N}\Delta'_{\text{BV}}\left(\int_{\mathcal{L}_K} f\, e^{S/\hbar}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K}\right) =$$

$$= \frac{1}{N}\int_{\mathcal{L}_K} \Delta_{\text{BV}}(f\, e^{S/\hbar})[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K} = \frac{1}{N}\int_{\mathcal{L}_K} \frac{1}{\hbar}(Qf)\, e^{S/\hbar}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_K} = \frac{1}{\hbar}e^{S'/\hbar}U^*(Qf)$$

Hence (28) holds. Fact that $U$ is quasi-isomorphism is trivial since the embedding $\iota$ obviously induces an isomorphism of cohomologies $\iota : H^*(\mathcal{G}') \to H^*(\mathcal{G})$. The perturbation expansion for (27) gives an expression for $U$ as sum over binary rooted trees. We summarize these results in the following statement.

**Theorem 2.** *Map $U : \Pi\mathcal{G}' \to \Pi\mathcal{G}$ defined by (27) is an $L_\infty$ quasi-isomorphism between $L_\infty$ algebra $(\mathcal{G}', Q')$ and DGLA $(\mathcal{G}, Q)$, and may be expanded as the following sum over binary rooted trees*

$$(29) \quad U(\omega') = \omega' + \sum_{T \in \hat{\mathbf{T}}:\, |T|\geq 2} \frac{1}{|\text{Aut}(T)|}\text{Iter}_{T,\, -K[\bullet,\bullet]}(\omega', \ldots, \omega') =$$

$$= \omega' - \frac{1}{2}K[\omega',\omega'] + \frac{1}{2}K[K[\omega',\omega'],\omega'] - \frac{1}{2}K[K[K[\omega',\omega'],\omega'],\omega'] - \frac{1}{8}K[K[\omega',\omega'],K[\omega',\omega']] + \cdots$$

### 3.6. 1-loop effective action on infrared fields as logarithm of density on $\Pi\mathcal{G}'$.

The 1-loop part of effective action $S'^{(1)}$ has the following interpretation. Define function $\rho' \in \text{Fun}(\Pi\mathcal{G}')$ as exponential of $S'^{(1)}$:

$$\rho'(\omega') = e^{S'^{(1)}(\omega')}$$

Then $\rho'$ is a density on space $\Pi\mathcal{G}'$, such that the volume form

$$(30) \qquad \eta' = \rho' \prod_{\alpha'} \delta\omega^{\alpha'}$$

14

on $\Pi \mathcal{G}'$ is conserved by $Q'$ (in the sense that Lie derivative of $\eta'$ along $Q'$ vanishes). This conservation is equivalent to (26). Another formulation of this conservation property is hydrodynamical: substituting $S'^{(1)} = \log \rho'$ into (26) we obtain equation

$$\rho' \operatorname{div} Q' + Q' \rho' = 0$$

which is known in hydrodynamics as the equation of conservation of compressible fluid in a stationary flow, with $Q'$ the velocity field of the flow and $\rho'$ the density of the fluid.

3.7. **Dependence of effective action on chain homotopy.** Our definition of effective $BF$ action (15) depends on choice of chain homotopy operator $K : \mathcal{G}'' \to \mathcal{G}''$. We will include $K$ as a subscript in notation $S'_K(\omega', p'; \hbar)$ while we are interested in $K$-dependence. We formulate a statement on behaviour of $S'_K$ under infinitesimal changes of chain homotopy $K \mapsto K + \delta K$. As a consequence of (13,14), for $K + \delta K$ to be a chain homotopy (in first order in $\delta K$) the variation $\delta K$ needs to satisfy two properties: $d\, \delta K + \delta K\, d = 0$ and $K\, \delta K + \delta K\, K = 0$.

**Theorem 3.** *Effective action $S'_{K+\delta K}$ differs from $S'_K$ by an infinitesimal canonical transformation*

$$(31) \qquad S'_{K+\delta K} - S'_K = \{S'_K, R_{K,\delta K}\} + \hbar\, \Delta_{\mathrm{BV}}(R_{K,\delta K})$$

*and the generator of canonical transformation $R_{K,\delta K} \in \mathrm{Fun}(\mathcal{F}')$ is given by*

$$(32) \qquad R_{K,\delta K}(\omega', p'; \hbar) = \left.\frac{\partial}{\partial z}\right|_{z=0} S'_{K+zK\,\delta K}(\omega', p'; \hbar)$$

*where $z \in \mathbb{R}^{0|1}$ is an odd infinitesimal variable. Equivalently, the exponential of effective action changes under $K \mapsto K + \delta K$ by a $\Delta_{\mathrm{BV}}$-exact term:*

$$(33) \qquad e^{\frac{1}{\hbar} S'_{K+\delta K}} - e^{\frac{1}{\hbar} S'_K} = \Delta_{\mathrm{BV}}\left( \hbar\, \left.\frac{\partial}{\partial z}\right|_{z=0} e^{\frac{1}{\hbar} S'_{K+zK\,\delta K}} \right)$$

For the generator of infinitesimal canonical transformation $R_{K,\delta K}$ we obtain using (32) and series (21,22) the perturbative expansion

$$(34) \quad R_{K,\delta K}(\omega', p'; \hbar) =$$

$$= -\frac{1}{2} < p', [K\, \delta K[\omega', \omega'], \omega'] > + \frac{1}{2} < p', [K\, \delta K[K[\omega', \omega'], \omega'], \omega'] > -$$

$$- \frac{1}{2} < p', [K[K\, \delta K[\omega', \omega'], \omega'], \omega'] > + \frac{1}{4} < p', [K\, \delta K[\omega', \omega'], K[\omega', \omega']] > -$$

$$- \hbar\, \mathrm{Str}\, K\, \delta K[\omega', \bullet] + \hbar\, \mathrm{Str}\, K\, \delta K[\omega', K[\omega', \bullet]] + \mathcal{O}(p' \omega'^5) + \mathcal{O}(\hbar\, \omega'^3)$$

This expansion may be interpreted as a sum over binary rooted trees with one internal edge marked (we put operator $-K\, \delta K$ on the marked edge and $-K$ on the others, as usual) plus sum over binary 1-loop graphs with one internal edge marked (and the same rule for assigning operators to edges as for trees).

3.8. **Physical and mathematical interpretations of procedure of inducing effective action for abstract $BF$ theory.** We have two interpretations of construction for effective action $S' \in \mathrm{Fun}(\mathcal{F}')$ from abstract $BF$ action $S \in \mathrm{Fun}(\mathcal{F})$. The first is the physical interpretation: we construct effective action (in Wilson sense) for abstract $BF$ theory by integrating out ultraviolet degrees of freedom (15), ending up with an effective topological theory on the space of infrared fields $\mathcal{F}'$. The second is mathematical interpretation: starting from DGLA $\mathcal{G}$ (with additional property $f^{\beta}_{\alpha\beta} = 0$), we construct $L_\infty$ algebra structure on subcomplex $\mathcal{G}' \subset \mathcal{G}$, containing all the cohomologies of $\mathcal{G}$: $H^*(\mathcal{G}) \subset \mathcal{G}'$. The

$L_\infty$ operations on $\mathcal{G}'$ are generated by cohomological vector field $Q'$ on $\Pi\mathcal{G}'$. Additionally we get density function $\rho' = e^{S'^{(1)}} \in \mathrm{Fun}(\Pi\mathcal{G}')$, defining $Q'$-invariant measure (30) on space $\Pi\mathcal{G}'$. We have also built an $L_\infty$ quasi-isomorphism (27,29) between $\mathcal{G}'$ and $\mathcal{G}$.

Apparently, the physically-inspired tool, the BV integral, gives answers to questions that may be formulated in terms of homotopy algebra, but are not, to our knowledge, studied. Especially, not only we have the fact of existence of quasi-isomorphism between $\mathcal{G}$ and $\mathcal{G}'$, but we have expression for it in terms of BV integral. Further, the $Q'$-invariant measure $\rho'$ is a new object for homotopy algebra. The pair $(Q', \rho')$ of a cohomological vector field on $\Pi\mathcal{G}'$ and $Q'$-invariant measure on $\Pi\mathcal{G}'$ should be considered as defining a structure of "quantum $L_\infty$ algebra" on $\mathcal{G}'$.

3.9. **Generalization to $BF_\infty$ theories. Class of $BF_\infty$ theories as "closure" of class of abstract $BF$ theories with respect to procedure of inducing effective action.** Effective theory for abstract $BF$ theory belongs to a wider class of $BF$ theories, which we call $BF_\infty$. We define a $BF_\infty$ theory in the following way: let $(\mathcal{G}, Q, \rho)$ be any $L_\infty$ algebra with $\rho \in \mathrm{Fun}(\Pi\mathcal{G})$ a $Q$-invariant density on $\Pi\mathcal{G}$. Then the space of fields is (as for extended $BF$ and abstract $BF$ case)

$$\mathcal{F} = \Pi T^*(\Pi\mathcal{G})$$

and the action $S \in \mathrm{Fun}(\mathcal{F})$

(35) $$S(\omega, p; \hbar) = <p, Q\,\omega> + \hbar\,\log\rho(\omega)$$

We keep the notation $S^{(0)}$ for $<p, Q\,\omega>$ and $S^{(1)}$ for $\log\rho$. The $BF_\infty$ action automatically satisfies quantum master equation. This action is also invariant under gauge transformations — canonical transformations on $\mathcal{F}$ with generator

$$R_\alpha = <\alpha, \frac{\partial}{\partial\omega}\,S^{(0)}>$$

where gauge parameter $\alpha$ belongs to $[\mathbb{R}^{1|1} \otimes \mathcal{G}]^{\mathrm{even}}$. Invariance of action under gauge transformation follows directly from master equation. This argument is a straightforward generalization of argument from section 2.6.

If $\mathcal{G}$ is split into a sum of two subcomplexes $\mathcal{G} = \mathcal{G}' \oplus \mathcal{G}''$ with $\mathcal{G}''$ acyclic, and $K : \mathcal{G}'' \to \mathcal{G}''$ is the chain homotopy, we can use BV integral (15) to define effective action $S' \in \mathrm{Fun}(\mathcal{F}')$ on $\mathcal{F}' = \Pi T^*(\Pi\mathcal{G}')$. Then the effective theory on $\mathcal{F}'$ is again $BF_\infty$ theory. Class of $BF_\infty$ theories may be regarded as the closure of class of abstract $BF$ theories with respect to operation of inducing effective action.

3.10. **Perturbative expansion for effective action of $BF_\infty$ theory.** There are now more admissible Feynman graphs in perturbation expansion for $S'$ then for case of inducing from abstract $BF$ theory (subsection 3.2), due to the fact that action $S(\omega'+\omega'', p'+p''; \hbar)$ now contains vertices of order $\mathcal{O}(p''\omega''^3), \mathcal{O}(p''\omega''^4)$ etc. as well as vertices of order $\mathcal{O}(\hbar\omega'')$, $\mathcal{O}(\hbar\omega''^2)$ etc.

Let us introduce the the obvious generalization of Iter and Loop for case of rooted trees and 1-loop graphs without restriction on vertices to be trivalent (non-binary case). Let $X$ be a vector super-space over $\mathbb{R}$ and $\{\mathcal{O}_k\}_{k\geq 2} = \{\mathcal{O}_2, \mathcal{O}_3, \ldots\}$ be a collection of polylinear maps $\mathcal{O}_k : X^k \to X$. Let $T$ be a rooted tree with $|T| = n$ vertices of valence 1 (leaves), one root of valence $\geq 2$ and all other vertices (internal vertices) of valence $\geq 3$ (we also mean that $T$ comes with planar structure). We define the $n$-linear map $\mathrm{Iter}_{T, \{\mathcal{O}_k\}} : X^n \to X$ by the same iterative procedure as for binary trees, with only difference that we decorate each vertex (internal or root) with $k$ children by $\mathcal{O}_k$ evaluated on values assigned to

16

children. Map $\text{Iter}_{T,\{\mathcal{O}_k\},\{\mathcal{O}_k'\}} : X^n \to X$ is defined analogously where in the root we evaluate operator $\mathcal{O}_k'$ instead of $\mathcal{O}_k$, where $k$ is the valence of root. For example,

$$\text{Iter}_{((*(***))*(**)),\,\{\mathcal{O}_k\},\,\{\mathcal{O}_k'\}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \mathcal{O}_3'(\mathcal{O}_2(x_1, \mathcal{O}_3(x_2, x_3, x_4)), x_5, \mathcal{O}_2(x_6, x_7))$$

We also need a special case when operators $\mathcal{O}_k'$ take values in $\mathbb{R}$ instead of $X$. The definition of Iter does not change and it becomes an $\mathbb{R}$-valued map $\text{Iter}_{T,\{\mathcal{O}_k\},\{\mathcal{O}_k'\}} : X^n \to \mathbb{R}$. If we include the unary operator $\mathcal{O}_1'$ in the list of operators $\{\mathcal{O}_k'\}$ then we mean that trees with univalent root are allowed for this case.

Let $L$ be a 1-loop graph: a graph with one oriented cycle, $|L| = n$ vertices of valence $1$ — leaves, and with all other vertices (internal ones) of valence $\geq 3$. We define $\text{Loop}_{L,\,\{\mathcal{O}_k\},\,X} : X^n \to \mathbb{R}$ in complete analogy with binary case: we cut the cycle to transform $L$ into a rooted tree $T$ with one marked leaf and set

$$\text{Loop}_{L,\{\mathcal{O}_k\},X}(x_1, \ldots, x_n) = \text{Str}_X \text{Iter}_{T,\{\mathcal{O}_k\}}(x_1, \ldots, x_{i-1}, \bullet, x_i, \ldots, x_n)$$

where $i$ is the number of marked leaf. For example,

$$\text{Loop}_{((**)(*\bullet*)),\,\{\mathcal{O}_k\},\,X}(x_1, x_2, x_3, x_4) = \text{Str}_X\, \mathcal{O}_2(\mathcal{O}_2(x_1, x_2), \mathcal{O}_3(x_3, \bullet, x_4))$$

We also introduce notation $\mathbf{T}_\infty$, $\mathbf{L}_\infty$ for the sets of rooted trees and 1-loop graphs with planar structure, and notation $\hat{\mathbf{T}}_\infty$, $\hat{\mathbf{L}}_\infty$ for the corresponding sets factorized over graph isomorphisms (i.e. with planar structure forgotten).

Now we return to description of perturbation series for effective action of $BF_\infty$ theory. Let Taylor series for $Q$ be

$$Q = <\sum_{n=1}^\infty \frac{1}{n!}\, l^{(n)}(\omega, \ldots, \omega), \frac{\partial}{\partial\omega} >$$

with $l^{(n)} : (\Pi\mathcal{G})^{\otimes n} \to \mathcal{G}$ the set of super-antisymmetric polylinear maps (the $L_\infty$ algebra operations on $\mathcal{G}$) and Taylor series for $S^{(1)}$ be

$$S^{(1)} = \sum_{n=1}^\infty \frac{1}{n!}\, q^{(n)}(\omega, \ldots, \omega)$$

with $q^{(n)} \in \text{Fun}(\Pi\mathcal{G})$ the set of super-antisymmetric polylinear functions on $\Pi\mathcal{G}$. Let us formulate the generalization of theorem 1 for $BF_\infty$ case.

**Theorem 4.** *Effective action of $BF_\infty$ theory has the form*

$$S'(\omega', p'; \omega) = S'^{(0)}(\omega', p') + \hbar\, S'^{(1)}(\omega')$$

*with $S'^{(0)}$ expanded as a sum over rooted trees without planar structure as*

(36) $\quad S'^{(0)}(\omega', p') =$

$$= <p', l^{(1)}(\omega')> + \sum_{T\in\hat{\mathbf{T}}_\infty} \frac{1}{|\text{Aut}(T)|} <p', \text{Iter}_{T,\,\{-K\circ l^{(k)}\}_{k\geq 2},\,\{l^{(k)}\}_{k\geq 2}}(\omega', \ldots \omega')>$$

*and $S'^{(1)}$ is expanded as*

(37) $\quad S'^{(1)}(\omega') = \sum_{L\in\hat{\mathbf{L}}_\infty} \frac{1}{|\text{Aut}(L)|}\text{Loop}_{L,\,\{-K\circ l^{(k)}\}_{k\geq 2},\,\overline{\Pi\mathcal{G}'}}(\omega', \ldots, \omega') +$

$$+ \sum_{T\in\hat{\mathbf{T}}_\infty} \frac{1}{|\text{Aut}(T)|}\text{Iter}_{T,\{K\circ l^{(k)}\}_{k\geq 2},\{q^{(k)}\}_{k\geq 1}}(\omega', \ldots, \omega')$$

*First terms in (36) are given by*

$$(38) \quad S'^{(0)}(\omega', p') = <p', l^{(1)}(\omega')> + \frac{1}{2} <p', l^{(2)}(\omega', \omega')> + \frac{1}{6} <p', l^{(3)}(\omega', \omega', \omega')> -$$

$$- \frac{1}{2} <p', l^{(2)}(Kl^{(2)}(\omega', \omega'), \omega')> + \frac{1}{24} <p', l^{(4)}(\omega', \omega', \omega', \omega')> -$$

$$- \frac{1}{6} <p', l^{(2)}(Kl^{(3)}(\omega', \omega', \omega'), \omega')> - \frac{1}{4} <p', l^{(3)}(Kl^{(2)}(\omega', \omega'), \omega', \omega')> +$$

$$+ \frac{1}{2} <p', l^{(2)}(Kl^{(2)}(Kl^{(2)}(\omega', \omega'), \omega'), \omega')> + \frac{1}{8} <p', l^{(2)}(Kl^{(2)}(\omega', \omega'), Kl^{(2)}(\omega', \omega'))> + \mathcal{O}(p'\omega'^5)$$

*and the first terms in (37) are*

$$(39) \quad S^{(1)}(\omega') = q^{(1)}(\omega') - \mathrm{Str}\, Kl^{(2)}(\omega', \bullet) + \frac{1}{2} q^{(2)}(\omega', \omega') - \frac{1}{2} q^{(1)}(Kl^{(2)}(\omega', \omega')) +$$

$$+ \frac{1}{2} \mathrm{Str}\, Kl^{(2)}(Kl^{(2)}(\omega', \omega'), \bullet) + \frac{1}{2} \mathrm{Str}\, Kl^{(2)}(\omega', Kl^{(2)}(\omega', \bullet)) + \frac{1}{6} q^{(3)}(\omega', \omega', \omega') - \frac{1}{2} q^{(2)}(Kl^{(2)}(\omega', \omega'), \omega') -$$

$$- \frac{1}{6} q^{(1)}(Kl^{(3)}(\omega', \omega', \omega')) + \frac{1}{2} q^{(1)}(Kl^{(2)}(Kl^{(2)}(\omega', \omega'), \omega')) + \frac{1}{6} \mathrm{Str}\, Kl^{(2)}(Kl^{(3)}(\omega', \omega', \omega')), \bullet) -$$

$$- \frac{1}{2} \mathrm{Str}\, Kl^{(2)}(Kl^{(2)}(Kl^{(2)}(\omega', \omega'), \omega'), \bullet) - \frac{1}{2} \mathrm{Str}\, Kl^{(2)}(Kl^{(2)}(\omega', \omega'), Kl^{(2)}(\omega', \bullet)) -$$

$$- \frac{1}{3} \mathrm{Str}\, Kl^{(2)}(\omega', Kl^{(2)}(\omega', Kl^{(2)}(\omega', \bullet)))) + \mathcal{O}(\omega'^4)$$

*with the super traces taken in $\overline{\Pi \mathcal{G}'}$.*

3.11. **Effective action on $\Pi T^*(\Pi H^*(\mathcal{G}))$ as iterative limit. Case of limiting effective action for extended $BF$ theory, Massey operations on cohomologies.** The procedure of inducing effective action, starting from $BF_\infty$ theory built on $L_\infty$ algebra $\mathcal{G}$ can be iterated, and reaches the iterative limit on subcomplex $\mathcal{G}' = H^*(\mathcal{G})$ consisting of cohomologies of $\mathcal{G}$. Tree part of the corresponding effective action generates $L_\infty$ algebra structure on cohomologies $H^*(\mathcal{G})$. In particular, when we start from extended $BF$ theory on manifold $M$, so that $\mathcal{G} = \mathfrak{g} \otimes \Omega(M)$, the iterative limit of inducing effective action is reached on de Rham cohomologies of $M$ with values in $\mathfrak{g}$: $\mathcal{G}' = \mathfrak{g} \otimes H^*_{dR}(M)$. The induced $L_\infty$ algebra structure on $\mathfrak{g} \otimes H^*_{dR}(M)$ generates Massey operations on de Rham cohomologies $H^*_{dR}(M)$. The 1-loop part of effective action $S'^{(1)} \in \mathrm{Fun}(\mathfrak{g} \otimes \Pi H^*_{dR}(M))$ should then be interpreted as a generating function for "quantum Massey operations" on $H^*_{dR}(M)$.

3.12. **Iterated induction as parallel transport in category of retracts.** We now proceed to more formal description of iterated induction. Let $\mathcal{G}$ be a cochain complex, and suppose that $BF_\infty$ theory on $\mathcal{G}$ (that is, with space of fields $\Pi T^*(\Pi \mathcal{G})$) is defined by (35) by a pair $(Q, \rho)$ — a cohomological vector field on $\Pi \mathcal{G}$ and $Q$-invariant measure on $\Pi \mathcal{G}$. Let $\mathrm{Ret}_\mathcal{G}$ be the category of retracts of $\mathcal{G}$. Its objects are subcomplexes $\mathcal{G}' \subset \mathcal{G}$, containing all cohomology of $\mathcal{G}$. Objects constitute a partially ordered set w.r.t. inclusion: if $\mathcal{G}', \mathcal{G}'' \in \mathrm{Ret}_\mathcal{G}$ and $\mathcal{G}'' \subset \mathcal{G}'$, we say that $\mathcal{G}'$ is larger then $\mathcal{G}''$. Category $\mathrm{Ret}_\mathcal{G}$ possesses the largest object — full complex $\mathcal{G}$, and set of smallest objects, corresponding to different embeddings of cohomologies $H^*(\mathcal{G})$ into $\mathcal{G}$. Morphisms in $\mathrm{Ret}_\mathcal{G}$ are retractions: for $\mathcal{G}'' \subset \mathcal{G}'$ a pair of objects (subcomplexes), $\mathcal{P} : \mathcal{G}' \to \mathcal{G}''$ a projection and $K : \ker \mathcal{P} \to \ker \mathcal{P}$ a chain homotopy operator, contracting $\mathcal{G}'$ onto $\mathcal{G}''$, we associate to the pair $(\mathcal{P}, K)$ a morphism $m_{\mathcal{P}, K} : \mathcal{G}' \to \mathcal{G}''$. Thus morphisms are always from larger object to smaller one, and for such a pair of objects there are typically many morphisms. There are no nontrivial automorphism in $\mathrm{Ret}_\mathcal{G}$: the only automorphism for each object $\mathcal{G}'$ is the identity.

Now $BF_\infty$ theory on any object $\mathcal{G}' \in \mathrm{Ret}_\mathcal{G}$ is defined by a pair $(Q, \rho)_{\mathcal{G}'}$ — a cohomological vector field and $Q$-invariant measure on $\Pi\mathcal{G}'$. If $m_{\mathcal{P},K}$ is a morphism from (larger object) $\mathcal{G}'$ to (smaller object) $\mathcal{G}''$, the operation of induction of $BF_\infty$ theory from $\mathcal{G}'$ to $\mathcal{G}''$, using projection $\mathcal{P}$ to define separation of fields into infrared and ultraviolet parts, and using chain homotopy operator $K$ to define Lagrangian manifold $\mathcal{L}_K$ for BV integral, may be viewed as "parallel transport" of $(Q, \rho)$ structure from $\mathcal{G}'$ to $\mathcal{G}''$ along morphism $m_{\mathcal{P},K}$:

$$\mathcal{I}_{\mathcal{P},K} : (Q, \rho)_{\mathcal{G}'} \mapsto (Q, \rho)_{\mathcal{G}''}$$

where $\mathcal{I}_{\mathcal{P},K}$ denotes induction. Iterated induction is then interpreted as the parallel transport along a chain of morphisms. This parallel transport also respects composition of morphisms: if $m_{\mathcal{P}_2,K_2} \circ m_{\mathcal{P}_1,K_1} = m_{\mathcal{P}_3,K_3}$ then $\mathcal{I}_{\mathcal{P}_2,K_2} \circ \mathcal{I}_{\mathcal{P}_1,K_1} = \mathcal{I}_{\mathcal{P}_3,K_3}$. In particular this means that iterated induction can always be reduced to induction in one move.

Another equivalent picture may be useful. Category $\mathrm{Ret}_\mathcal{G}$ contains isomorphic objects that are different embeddings of the same complex into $\mathcal{G}$. We may factorize $\mathrm{Ret}_\mathcal{G}$ over chain isomorphisms. We denote the factorized category $\mathrm{Ret}_\mathcal{G}^\circ$ (one might call it "category of abstract retracts"). Its objects are abstract chain complexes $\mathcal{G}'$ that can be embedded into $\mathcal{G}$ and with cohomologies coinciding with cohomologies of $\mathcal{G}$. This category has only one smallest object — the complex of cohomologies $H^*(\mathcal{G})$, and one largest object — whole $\mathcal{G}$. A morphism $m_{\iota,\mathcal{P},K} : \mathcal{G}' \to \mathcal{G}''$ is now specified by a triple $(\iota, \mathcal{P}, K)$ where $\iota$ is the embedding (injective chain map) $\iota : \mathcal{G}'' \to \mathcal{G}'$, $\mathcal{P}$ is projection $\mathcal{P} : \mathcal{G}' \to \mathcal{G}''$ (surjective chain map satisfying $\mathcal{P} \circ \iota = \mathrm{id}_{\mathcal{G}''}$) and $K : \ker \mathcal{P} \to \ker \mathcal{P}$ is the chain homotopy, contracting $G'$ onto $\mathcal{G}''$. This category $\mathrm{Ret}_\mathcal{G}^\circ$ has fewer objects, but more morphisms between two given objects than in $\mathrm{Ret}_\mathcal{G}$. In particular, there are nontrivial automorphisms for objects of $\mathrm{Ret}_\mathcal{G}^\circ$, which correspond to chain automorphisms of complexes. We may understand operation of inducing $BF_\infty$ theory as parallel transport of $(Q, \rho)$ structure on objects of $\mathrm{Ret}_\mathcal{G}^\circ$ along morphisms in complete analogy with $\mathrm{Ret}_\mathcal{G}$.

Interpretation of induction of $BF_\infty$ theory in terms of category of retracts allows us to understand Wilson-type renormalization of simplicial $BF$ theory under aggregation of triangulation in terms of holonomy of the parallel transport $\mathcal{I}$.

3.13. **Towards state-sum for $BF_\infty$ theory.** The state-sum $Z(\mathcal{G})$ for $BF_\infty$ theory on $\Pi T^*(\Pi\mathcal{G})$ may be defined as follows: induce effective action $S'$ on $\mathcal{F}' = \Pi T^*(\Pi H^*(\mathcal{G}))$, then integrate the exponential of effective action along the base of $\mathcal{F}'$ (this is our choice of Lagrangian submanifold in $\mathcal{F}'$):

$$(40) \qquad Z(\mathcal{G}) = \int_{\Pi H^*(\mathcal{G})} e^{S'/\hbar} \mathcal{D}\omega' = \int_{\Pi H^*(\mathcal{G})} \rho' \mathcal{D}\omega'$$

with $\rho' = e^{S'^{(1)}}$ the induced density function on $\Pi H^*(\mathcal{G})$. The integral (40) over whole space $\Pi H^*(\mathcal{G})$ can diverge, and there should exist some "non-perturbative" reason, why we should regularize this integral. One possible regularization is to integrate over some domain in $\Pi H^*(\mathcal{G})$, for instance over connected component of support of $\rho'$, containing zero. However, we do not have a good explanation, why one should use this regularization for state-sum.

## 4. Effective action for extended $BF$ theory on a triangulation

We now proceed to specializing the construction of effective action to the case of constructing effective action of extended $BF$ theory on a triangulated manifold.

## 4.1. Whitney complex of a simplicial complex.

Let us recall the concept of Whitney complex of a simplicial complex (see [6]). Let $\Delta^n$ be a standard geometrical $n$-simplex with barycentric coordinates $t_0, \ldots, t_n$ subject to relation $t_0 + \cdots + t_n = 1$ and inequalities $t_0 \geq 0, \cdots, t_n \geq 0$. We introduce a set of special piecewise-linear differential forms on $\Delta^n$:

$$(41) \qquad \chi_{i_0 \cdots i_k} = k! \sum_{r=0}^{k} (-1)^r t_{i_r} dt_{i_0} \wedge \cdots \widehat{dt_{i_r}} \cdots \wedge dt_{i_k}$$

where hat means exclusion. Forms $\chi_\sigma$ are associated to subsets $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$ or, equivalently, to faces of $\Delta^n$. Following properties hold for forms $\chi_\sigma$:

- for $\sigma, \sigma'$ faces of $\Delta^n$

$$\int_{\sigma'} \chi_\sigma = \left\{ \begin{array}{ll} 1 & \text{if } \sigma = \sigma' \\ 0 & \text{otherwise} \end{array} \right.$$

- de Rham differential acts on forms $\chi_\sigma$ as

$$d\chi_{i_0 \ldots i_k} = \sum_{j=0}^{n} \chi_{j i_0 \cdots i_k}$$

Linear space spanned by forms $\chi_\sigma$ is closed under de Rham differential and is called Whitney complex of $\Delta^n$. We denote it $\Omega_W(\Delta^n)$. Elements of $\Omega_W(\Delta^n)$ (linear combinations of forms $\chi_\sigma$) are called Whitney forms. There is a natural isomorphism between Whitney complex $\Omega_W(\Delta^n)$ and complex $C^*(\Delta^n)$ of simplicial cochains on $\Delta^n$ that identifies basis cochains $e_\sigma$ with forms $\chi_\sigma$. The de Rham differential is identified with the coboundary operator on $C^*(\Delta^n)$. Canonical pairing between chains and cochains on $\Delta^n$ is interpreted as integral of Whitney form over a chain.

Let now $\Xi$ be a simplicial complex. The Whitney complex $\Omega_W(\Xi)$ on $\Xi$ is glued from Whitney complexes on simplices of $\Xi$ with $\Omega_W(\sigma)|_{\sigma \cap \sigma'}$ and $\Omega_W(\sigma')|_{\sigma \cap \sigma'}$ identified. The cocycle condition for this gluing is ensured by "compatibility" of Whitney complexes on a simplex $\sigma$ and its face $\sigma' \subset \sigma$:

$$\Omega_W(\sigma)|_{\sigma'} = \Omega_W(\sigma')$$

A Whitney form $\alpha \in \Omega_W(\Xi)$ is a differential form on $\Xi$ such that its restrictions to all simplices of $\Xi$ are Whitney forms: $\alpha|_\sigma \in \Omega_W(\sigma)$. Basis forms $\chi_\sigma$ are associated to each simplex $\sigma \in \Xi$. Form $\chi_\sigma$ is defined by (41) on each simplex $\sigma' \in \Xi$ containing $\sigma$ as a face, and by zero on all other simplices. Whitney complex $\Omega_W(\Xi)$ can again be identified with complex of simplicial cochains $C^*(\Xi)$, as in the case of one simplex $\Xi = \Delta^n$. By this identification we chose special representatives for simplicial cochains in de Rham algebra $\Omega(\Xi)$ — the Whitney forms.

Projection $\mathcal{P}_W : \Omega(\Xi) \to \Omega_W(\Xi)$ is defined as

$$\mathcal{P}_W(\alpha) = \sum_{\sigma \in \Xi} \left( \int_\sigma \alpha \right) \chi_\sigma$$

## 4.2. Chain homotopy between $\Omega(\Xi)$ and $\Omega_W(\Xi)$: Dupont's construction.

Consider first the case of one simplex $\Xi = \Delta^n$. We cite the Dupont's construction of chain homotopy between $\Omega(\Delta^n)$ and $\Omega_W(\Delta^n)$ from [6], adjusting it to our notations.

Given a vertex $[i]$ of the $n$-simplex $\Delta^n$, define the dilation map

$$\phi_i : [0,1] \times \Delta^n \to \Delta^n$$

by the formula

$$\phi_i(u, t_0, \ldots, t_n) = (u t_0, \ldots, u t_i + (1-u), \ldots, u t_n)$$

Let $\pi : [0,1] \times \Delta^n \to \Delta^n$ be the projection on the second factor, and let $\pi_* : \Omega^*([0,1] \times \Delta^n) \to \Omega^{*-1}(\Delta^n)$ be integration over the first factor. Define operators

$$h^i : \Omega^*(\Delta^n) \to \Omega^{*-1}(\Delta^n)$$

by the formula

$$h^i\alpha = \pi_*\phi_i^*\alpha$$

Let $\mathrm{ev}^i : \Omega(\Delta^n \to \mathbb{R})$ be evaluation at vertex $[i]$. Stokes's theorem implies that $h^i$ is the chain homotopy between the identity and $\mathrm{ev}^i$:

$$dh^i + h^i d = \mathrm{id} - \mathrm{ev}^i$$

Operators $h^i$ also satisfy

$$h^i h^j + h^j h^i = 0$$

The operator

$$(42) \qquad K_{\Delta^n} = \sum_{k=0}^{n-1}(-1)^k \sum_{0 \le i_0 < \cdots < i_k \le n} \chi_{i_0 \ldots i_k} h^{i_k} \cdots h^{i_0}$$

was introduced by Dupont. Dupont proved the following explicit form of de Rham theorem:

$$(43) \qquad dK_{\Delta^n} + K_{\Delta^n}d = \mathrm{id} - \mathcal{P}_W$$

Thus $K_{\Delta^n}$ is a chain homotopy between $\mathrm{id} : \Omega(\Delta^n) \to \Omega(\Delta^n)$ and $\mathcal{P}_W$. The following compatibility property holds: if $\sigma$ is a face of $\Delta^n$ and $\alpha \in \Omega(\Delta^n)$ then

$$(44) \qquad (K_{\Delta^n}\alpha)|_\sigma = K_\sigma(\alpha|_\sigma)$$

Now let $\Xi$ be any simplicial complex. We then define $K_\Xi : \Omega^*(\Xi) \to \Omega^{*-1}(\Xi)$ by

$$(45) \qquad (K_\Xi\alpha)|_\sigma = K_\sigma(\alpha|_\sigma)$$

for any simplex $\sigma \in \Xi$. This definition is self-consistent due to (44). Operator $K_\Xi$ is a chain homotopy between identity $\mathrm{id} : \Omega(\Xi) \to \Omega(\Xi)$ and projection $\mathcal{P}_W : \Omega(\Xi) \to \Omega_W(\Xi)$

$$(46) \qquad dK_\Xi + K_\Xi d = \mathrm{id} - \mathcal{P}_W$$

4.3. **Effective action of extended $BF$ theory on triangulation: factorization of BV integral, reducing the problem to single simplex.** Let $M$ be a $D$-dimensional manifold with corners, and let $\Xi$ be some triangulation of $M$. We split de Rham algebra $\Omega(M)$ into sum of two subcomplexes:

$$\Omega(M) = \Omega_W(\Xi) \oplus \Omega''(\Xi)$$

with Whitney complex playing the role of infrared subcomplex, $\Omega''(\Xi)$ the ultraviolet subcomplex. The latter consists of differential forms $\alpha''$ such that $\int_\sigma \alpha'' = 0$ for any simplex $\sigma \in \Xi$. The space of fields of extended $BF$ theory $\mathcal{F} = \Pi T^*(\Pi(g \otimes \Omega(M)))$ is then split into space of infrared fields $\mathcal{F}' = \Pi T^*(\Pi(g \otimes \Omega_W(\Xi)))$ and space of ultraviolet fields $\mathcal{F}'' = \Pi T^*(\Pi(g \otimes \Omega''(\Xi)))$. We use BV integral (15) to define effective action $S_\Xi$ on $\mathcal{F}'$. The Lagrangian manifold over which we integrate in (15) is defined by Dupont's chain homotopy operator $K_\Xi$.

Let us split the space of ultraviolet forms into subspaces enumerated by simplices of $\Xi$:

$$\Omega''(\Xi) = \bigoplus_{\sigma \in \Xi} \Omega''(M, \sigma)$$

where $\Omega''(M,\sigma)$ is the space of forms supported on the interior of $\sigma$ (and vanishing on its boundary), with zero integral over $\sigma$:

$$\Omega''(M,\sigma) = \{\alpha''_\sigma \in \Omega(M) : \alpha''_\sigma|_{M\backslash\sigma} = 0, \ \alpha''_\sigma|_{\partial\sigma} = 0, \int_\sigma \alpha''_\sigma = 0\}$$

Field $\omega \in \overline{\Pi(\mathfrak{g}\otimes\Omega(M))}$ is then decomposed as

$$(47) \qquad \omega = \sum_{\sigma\in\Xi}\omega^\sigma\chi_\sigma + \sum_{\sigma\in\Xi}\omega''_{(\sigma)} = \sum_{\sigma\in\Xi}\omega'_{(\sigma)} + \sum_{\sigma\in\Xi}\omega''_{(\sigma)}$$

where $\omega^\sigma \in \Pi\mathfrak{g}$ if $\sigma$ is even-dimensional and $\omega^\sigma \in \mathfrak{g}$ if $\sigma$ is odd-dimensional; $\omega''_{(\sigma)} \in \overline{\Pi(\mathfrak{g}\otimes\Omega''(M,\sigma))}$. Field $p \in [\Omega(M)]^* \otimes g^*$ is decomposed correspondingly:

$$(48) \qquad p = \sum_{\sigma\in\Xi}e^\sigma p_\sigma + \sum_{\sigma\in\Xi}p''_{(\sigma)} = \sum_{\sigma\in\Xi}p'_{(\sigma)} + \sum_{\sigma\in\Xi}p''_{(\sigma)}$$

where $p_\sigma \in \mathfrak{g}$ if $\sigma$ is even-dimensional, $p_\sigma \in \Pi\mathfrak{g}$ if $\sigma$ is odd-dimensional, $e^\sigma$ are basis simplicial chains on $\Xi$, $p''_{(\sigma)} \in \overline{[\Omega(M,\sigma)]^* \otimes \mathfrak{g}^*}$. We use here the identification of Whitney coforms on $\Xi$ and simplicial chains: $[\Omega_W(\Xi)]^* = C_*(\Xi)$. Substituting decompositions (47,48) into extended $BF$ action (2), we get (omitting terms with vanishing support)

$$(49) \quad S(\omega,p) = <p, d\omega + \frac{1}{2}[\omega,\omega]> =$$

$$= \sum_{\sigma\in\Xi}\left( <p'_{(\sigma)}, \sum_{\sigma_1\subset\sigma}d\omega'_{(\sigma_1)} + \frac{1}{2}\sum_{\sigma_1,\sigma_2\subset\sigma}[\omega'_{(\sigma_1)},\omega'_{(\sigma_2)}] + \sum_{\sigma_1\subset\sigma}[\omega'_{(\sigma_1)},\omega''_{(\sigma)}] + \frac{1}{2}[\omega''_{(\sigma)},\omega''_{(\sigma)}] > + \right.$$

$$\left. + <p''_{(\sigma)}, d\omega''_{(\sigma)} + \frac{1}{2}\sum_{\sigma_1,\sigma_2\subset\sigma}[\omega'_{(\sigma_1)},\omega'_{(\sigma_2)}] + \sum_{\sigma_1\subset\sigma}[\omega'_{(\sigma_1)},\omega''_{(\sigma)}] + \frac{1}{2}[\omega''_{(\sigma)},\omega''_{(\sigma)}] > \right) =$$

$$= \sum_{\sigma\in\Xi}S\left(\sum_{\sigma_1\subset\sigma}\omega'_{(\sigma_1)} + \omega''_{(\sigma)}, p'_{(\sigma)} + p''_{(\sigma)}\right)$$

Hence the BV integral (15) factorizes:

$$(50) \qquad \int_{\mathcal{L}_{K_\Xi}}e^{\frac{1}{\hbar}S(\omega,p)}[\mathcal{D}\omega''\mathcal{D}p'']_{\mathcal{L}_{K_\Xi}} = \prod_{\sigma\in\Xi}\int_{\mathcal{L}_{K_\sigma}}e^{\frac{1}{\hbar}S(\omega'|_\sigma+\omega''_{(\sigma)},p'_{(\sigma)}+p''_{(\sigma)})}[\mathcal{D}\omega''_{(\sigma)}\mathcal{D}p''_{(\sigma)}]_{\mathcal{L}_{K_\sigma}}$$

Here we use that $\omega'|_\sigma = \sum_{\sigma_1\subset\sigma}\omega'_{(\sigma_1)}$. The factorization of measure in (50) is due to "simplicial locality" of $K_\Xi$ (45). It follows that the effective action $S_\Xi$ on infrared fields splits into sum of contributions of individual simplices of $\Xi$.

**Theorem 5 (Separation of variables for $S_\Xi$).** *Effective action $S_\Xi$ on $\mathcal{F}' = \Pi T^*(\Pi(\mathfrak{g}\otimes\Omega_W(\Xi)))$ splits as*

$$(51) \qquad S_\Xi(\omega',p';\hbar) = \sum_{\sigma\in\Xi}\bar{S}_\sigma(\omega'|_\sigma, p'_{(\sigma)};\hbar)$$

*where functions $\bar{S}_\sigma$ are defined by following "elementary" BV integrals:*

$$(52) \qquad e^{\frac{1}{\hbar}\bar{S}_\sigma(\omega'|_\sigma,p'_{(\sigma)};\hbar)} = \frac{\int_{\mathcal{L}_{K_\sigma}}e^{\frac{1}{\hbar}S(\omega'|_\sigma+\omega''_{(\sigma)},p'_{(\sigma)}+p''_{(\sigma)})}[\mathcal{D}\omega''_{(\sigma)}\mathcal{D}p''_{(\sigma)}]_{\mathcal{L}_{K_\sigma}}}{\int_{\mathcal{L}_{K_\sigma}}e^{\frac{1}{\hbar}<p''_{(\sigma)},d\omega''_{(\sigma)}>}[\mathcal{D}\omega''_{(\sigma)}\mathcal{D}p''_{(\sigma)}]_{\mathcal{L}_{K_\sigma}}}$$

Thus the task of calculating effective action of extended $BF$ theory on any triangulated manifold $(M, \Xi)$ is reduced to the series of universal problems: calculate (52) for a simplex of each dimension $\sigma = \Delta^n$ with $n = 0, 1, 2, \ldots$

Notice that the elementary BV integral (52) does not define an effective action of $BF$-type theory, since $\bar{S}_\sigma$ is a function on space

$$\bar{\mathcal{F}}_\sigma = \Pi(\mathfrak{g} \otimes \Omega_W(\sigma)) \oplus \Pi^{|\sigma|} \mathfrak{g}^*$$

lacking canonical odd simplectic structure for $|\sigma| > 0$. We use notation $|\sigma|$ for the dimension of $\sigma$; symbol $\Pi^{|\sigma|}$ means "reverse parity if $\sigma$ is odd-dimensional". But instead we can think of $\bar{S}_\sigma$ as of honest effective $BF$ action on simplex $\sigma$, which according to (51) equals $S_\sigma(\omega', p'; \hbar) = \sum_{\sigma_1 \in \sigma} \bar{S}_{\sigma_1}(\omega'|_{\sigma_1}, p'_{(\sigma_1)}; \hbar)$, restricted from full space of infrared fields $\mathcal{F}'_\sigma = \Pi T^*(\Pi(\mathfrak{g} \otimes \Omega_W(\sigma))) = \Pi(\mathfrak{g} \otimes \Omega_W(\sigma)) \oplus [\Omega_W(\sigma)]^* \otimes \mathfrak{g}^*$ to the subspace $\bar{\mathcal{F}}_\sigma \subset \mathcal{F}'_\sigma$, so that $\bar{S}_\sigma = S_\sigma|_{\bar{F}_\sigma}$. Thus we may name $\bar{S}_\sigma$ the "reduced effective action" on simplex $\sigma$.

## 4.4. Simple cases of elementary BV integral on simplex: dimensions 0 and 1.
Task of computing (52) on 0-dimensional simplex $\sigma = \Delta^0$ is trivial, since the space of ultraviolet fields $\Pi T^*(\Pi(\mathfrak{g} \otimes \Omega''(\sigma, \sigma)))$ is empty in this case. Infrared fields are $\omega' = \omega'_{(0)} = \omega^0 \chi_0$, $p' = p'_{(0)} = e^0 p_0$ and the coordinates $\omega^0 \in \Pi\mathfrak{g}$, $p_0 \in \mathfrak{g}^*$. Hence

$$\bar{S}(\omega^0, p_0) = < p_0, \frac{1}{2}[\omega^0, \omega^0] >_{\mathfrak{g}}$$

which is indeed a extended $BF$ action on a point. Here $< \bullet, \bullet >_{\mathfrak{g}}$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Let us now turn to case of dimension 1 for (52). The 1-dimensional case $\sigma = \Delta^1$ turns out to be exactly solvable, due to the fact that on the Lagrangian submanifold $\mathcal{L}_K$ the action we are integrating in (52) becomes quadratic in ultraviolet fields.

Whitney forms on interval $\Delta^1$ are: $\chi_0 = t_0$, $\chi_1 = t_1$, $\chi_{01} = t_0 dt_1 - t_1 dt_0 = dt_1$. Let us expand infrared fields as

(53) $\qquad \omega' = \omega'_{(0)} + \omega'_{(1)} + \omega'_{(01)} = \omega^0 \chi_0 + \omega^1 \chi_1 + \omega^{01} \chi_{01}$ and $p' = p'_{(01)} = e^{01} p_{01}$

with the coordinates $\omega^0, \omega^1 \in \Pi\mathfrak{g}$, $\omega^{01} \in \mathfrak{g}$ and $p_{01} \in \Pi\mathfrak{g}^*$. Let us also expand ultraviolet fields according to de Rham degree:

$$\omega''_{(01)} = \omega''^0_{(01)} + \omega''^1_{(01)} \text{ and } p''_{(01)} = p''^0_{(01)} + p''^1_{(01)}$$

In these notations the superscript is the degree of form (or degree of coform for $p$), while the subscript is the simplex where the ultraviolet field is supported. Spaces where these components of ultraviolet fields belong are:

$$\begin{aligned}
\omega''^0_{(01)} &\in \Pi\mathfrak{g} \otimes \Omega''^0(\Delta^1, \Delta^1) \\
\omega''^1_{(01)} &\in \mathfrak{g} \otimes \Omega''^1(\Delta^1, \Delta^1) \\
p''^0_{(01)} &\in [\Omega''^0(\Delta^1, \Delta^1)]^* \otimes \mathfrak{g}^* \\
p''^1_{(01)} &\in [\Omega''^1(\Delta^1, \Delta^1)]^* \otimes \Pi\mathfrak{g}^*
\end{aligned}$$

Thus $\omega''^0_{(01)}$ is a $\Pi\mathfrak{g}$-valued function on interval $\Delta^1$ vanishing on the end-points, $\omega''^1_{01}$ is a $\mathfrak{g}$-valued 1-form on $\Delta^1$ with vanishing integral over $\Delta^1$, $p''^0_{(01)}$ is a $\mathfrak{g}^*$-valued 0-coform whose pairing with linear functions $\chi_0$, $\chi_1$ on $\Delta^1$ vanishes, $p''^1_{01}$ is a $\Pi\mathfrak{g}^*$-valued 1-coform whose pairing with constant 1-form $\chi_{01}$ vanishes.

Let us choose the homogeneous coordinate $t = t_1$, associated with right end-point of the interval as the parameter along $\Delta^1$. The chain homotopy operator (42) vanishes on

functions $\alpha \in \Omega^0(\Delta^1)$ and acts on 1-forms $\alpha = \alpha(t)dt \in \Omega^1(\Delta^1)$ as

(54) $\quad K(\alpha(t)dt) = \chi_0 h^0(\alpha) + \chi_1 h^1(\alpha) =$

$$= t_0 t_1 \int_0^1 du\, \alpha(ut_1) + t_1(t_1 - 1) \int_0^1 du\, \alpha(1 - u(1 - t_1)) = \int_0^t dt\, \alpha(t) - t \int_0^1 dt\, \alpha(t)$$

It is clearly seen from here that a form on interval is sent to zero by $K$, iff either it is a 0-form or a constant 1-form: $\{\alpha \in \Omega(\Delta^1) : \ K\alpha = 0\} = \Omega^0(\Delta^1) \oplus \Omega^1_W(\Delta^1)$. Thus the Lagrangian submanifold $\mathcal{L}_K$ is in our case

(55)
$$\mathcal{L}_K : \left\{ \begin{array}{l} \omega''^1_{(01)} = 0 \\ p''^0_{(01)} = 0 \end{array} \right.$$

Let us expand the action under integral in (52) on submanifold (55):

(56) $\quad S|_{\mathcal{L}_K} = <p'_{(01)}, d(\omega'_{(0)} + \omega'_{(1)}) + [\omega'_{(0)} + \omega'_{(1)}, \omega'_{(01)}] > + <p'_{(01)}, [\omega'_{(01)}, \omega''^0_{(01)}] > +$

$$+ <p''^1_{(01)}, [\omega'_{(0)} + \omega'_{(1)}, \omega'_{(01)}] > + <p''^1_{(01)}, [\omega'_{(01)}, \omega''^0_{(01)}] >$$

Elementary BV integral (52) can be interpreted as an integral of type (15), inducing effective action for extended $BF$ theory on one simplex $\sigma$ (i.e. a simplicial complex consisting of $\sigma$ and all its faces), and then restricting infrared field $p'$ to infrared coforms of highest degree. Thus we can use perturbative series (21,22) for (52). Absence of cubic terms in ultraviolet fields in (56) drastically reduces the number of possible Feynman diagrams for $\bar{S}(\omega', p'; \hbar)$, and series (21,22) for tree and 1-loop parts of $\bar{S}$ are simplified to

(57) $\quad \bar{S}^{(0)}(\omega', p') = <p'_{(01)}, d(\omega'_{(0)} + \omega'_{(1)}) + [\omega'_{(0)} + \omega'_{(1)}, \omega'_{(01)}] -$

$$- [K[\omega'_{(0)} + \omega'_{(1)}, \omega'_{(01)}], \omega'_{(01)}] + [K[K[\omega'_{(0)} + \omega'_{(1)}, \omega'_{(01)}], \omega'_{(01)}], \omega'_{(01)}] -$$

$$- [K[K[K[\omega'_{(0)} + \omega'_{(1)}, \omega'_{(01)}], \omega'_{(01)}], \omega'_{(01)}], \omega'_{(01)}] + \cdots >$$

and

(58) $\quad \bar{S}^{(1)}(\omega') = -\text{Str } K[\omega'_{(01)}, \bullet] + \frac{1}{2}\text{Str } K[\omega'_{(01)}, K[\omega'_{(01)}, \bullet]] -$

$$- \frac{1}{3}\text{Str } K[\omega'_{(01)}, K[\omega'_{(01)}, K[\omega'_{(01)}, \bullet]]] + \frac{1}{4}\text{Str } K[\omega'_{(01)}, K[\omega'_{(01)}, K[\omega'_{(01)}, K[\omega'_{(01)}, \bullet]]]] - \cdots$$

where the super traces are taken on $\Pi\mathfrak{g} \otimes \Omega''^0(\Delta^1, \Delta^1)$ or equivalently on the whole space of $\Pi\mathfrak{g}$-valued 0-forms $\Pi\mathfrak{g} \otimes \Omega^0(\Delta^1)$ (since the diagonal matrix elements of operators under super-traces vanish on Whitney 0-forms). The series (57,58) are evaluated using the two following lemmata.

**Lemma 1.** *On 1-dimensional simplex $\Delta^1$, for $n \geq 1$*

(59) $\qquad [K(\chi_{01} \wedge \bullet)]^n \circ \chi_1 = -[K(\chi_{01} \wedge \bullet)]^n \circ \chi_0 = \dfrac{B_{n+1}(t) - B_{n+1}}{(n+1)!}$

*where $B_n(t)$ is the $n$-th Bernoulli polynomial and $B_n = B_n(0)$ is $n$-th Bernoulli number. Also*

(60) $\qquad \int_{\Delta^1} \chi_{01}[K(\chi_{01} \wedge \bullet)]^n \circ \chi_1 = -\int_{\Delta^1} \chi_{01}[K(\chi_{01} \wedge \bullet)]^n \circ \chi_0 = -\dfrac{B_{n+1}}{(n+1)!}$

  **Proof.** Consider the generating function

(61) $\qquad\qquad f(x, t) = \sum_{n=0}^{\infty} x^n [K(\chi_{01} \wedge \bullet)]^n \circ \chi_1$

Applying $xK(\chi_{01} \wedge \bullet)$ to both sides and using (54) we get the integral equation

$$x \left( \int_0^t f dt - t \int_0^1 f dt \right) = f - t$$

and differentiating it w.r.t. $t$ we obtain

$$\frac{\partial}{\partial t} f - 1 = x \left( f - \int_0^1 f dt \right)$$

and hence $\frac{\partial}{\partial t} f = xf + C(x)$ where $C(x)$ is something not depending on $t$. Solving this as a differential equation in variable $t$ with boundary conditions $f(x, 0) = 0$, $f(x, 1) = 1$ (emerging from $n = 0$ term in (61), the other terms are vanishing on end-points of interval) yields unique solution

$$f(x, t) = \frac{e^{xt} - 1}{e^x - 1}$$

Since Bernoulli polynomials are defined by

$$\sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n = \frac{x e^{xt}}{e^x - 1}$$

we obtain

$$[K(\chi_{01} \wedge \bullet)]^n \circ \chi_1 = \frac{B_{n+1}(t) - B_{n+1}}{(n+1)!}$$

Fact that $K(\chi_{01} \wedge \bullet)]^n \circ \chi_1 = -[K(\chi_{01} \wedge \bullet)]^n \circ \chi_0$ is obvious from $K(\chi_{01} \wedge \bullet)]^n \circ \chi_1 + [K(\chi_{01} \wedge \bullet)]^n \circ \chi_0 = [K(\chi_{01} \wedge \bullet)]^n \circ 1 = 0$. Formula (60) follows directly from (59) and from the following property of Bernoulli polynomials: $\int_0^1 dt\, B_n(t) = 0$ for $n \geq 1$. $\square$

**Lemma 2.** *On 1-dimensional simplex $\Delta^1$ for $n \geq 2$*

(62)
$$\mathrm{Str}_{\Omega^0(\Delta^1)} [K(\chi_{01} \wedge \bullet)]^n = -\frac{B_n}{n!}$$

**Proof.** Let us calculate these super-traces (which are now just ordinary traces, as $\Omega^0(\Delta^1)$ is purely even vector space) in monomial basis $1, t, t^2, t^3, \ldots \in \Omega^0(\Delta^1)$. Denote for brevity the operator under super-trace in (62) by $\mathcal{M}^n$ with $\mathcal{M} = K(\chi_{01} \wedge \bullet)$ (letter $\mathcal{M}$ for monodromy). For small $n$ we may calculate (62) directly by finding all diagonal matrix elements on $\mathcal{M}^n$. Iterating operator $\mathcal{M}$ on a monomial $t^m$ we get:

(63) $\quad t^m \xrightarrow{\mathcal{M}} \dfrac{t^{m+1}}{m+1} - \dfrac{t}{m+1} \xrightarrow{\mathcal{M}}$

$$\xrightarrow{\mathcal{M}} \frac{t^{m+2}}{(m+1)(m+2)} - \frac{t^2}{2(m+1)} + \left( \frac{1}{2(m+1)} - \frac{1}{(m+1)(m+2)} \right) t \xrightarrow{\mathcal{M}} \cdots$$

It is clear that for general $m, n$ structure of $\mathcal{M}^n(t^m)$ is: $\mathcal{M}^n(t^m) = \frac{m!}{(m+n)!} t^{m+n} + P_n(t; m)$ where $P_n(t; m)$ is some polynomial of degree $n$ in $t$ with coefficients being some rational functions of $m$. Thus all matrix elements $< t^m | \mathcal{M}^n | t^m >$ vanish for $m > n$ and only first few contribute to Str, i.e. those with $1 \leq m \leq n$. For instance for $n = 2$ from (63) we obtain

(64) $\quad \mathrm{Str}\, \mathcal{M}^2 = < t | \mathcal{M}^2 | t > + < t^2 | \mathcal{M}^2 | t^2 > =$

$$= \left( \frac{1}{2(m+1)} - \frac{1}{(m+1)(m+2)} \right)_{m=1} + \left( -\frac{1}{2(m+1)} \right)_{m=2} = \left( \frac{1}{2 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \left( -\frac{1}{2 \cdot 3} \right) = -\frac{1}{12}$$

Yet for general $n$ we need to calculate somehow the diagonal matrix elements, and for this we need a generalization of generating function (61):

$$(65) \qquad f_m(x,t) = \sum_{n=0}^{\infty} x^n [K(\chi_{01} \wedge \bullet)]^n \circ t^m$$

We again obtain a differential equation for $f_m$:

$$\frac{\partial}{\partial t} f_m = x f_m + m t^{m-1} + C_m(x)$$

where $C_m(x)$ is something not depending on $t$. This equation with boundary conditions $f_m(x,0) = 0$, $f_m(x,1) = 1$ uniquely determine the solution

$$(66) \quad f_m(x,t) = \frac{e^{xt}-1}{e^x-1}\left(1 - e^x \int_0^1 d\bar{t}\, m\bar{t}^{m-1}e^{-x\bar{t}}\right) + e^{xt}\int_0^t d\bar{t}\, m\bar{t}^{m-1}e^{-x\bar{t}} =$$

$$= \frac{e^{xt}-1}{e^x-1}\sum_{k=0}^{m-1}\frac{m!}{(m-k)!}x^{-k} - \sum_{k=1}^{m-1}\frac{m!}{(m-k)!}t^{m-k}x^{-k}$$

Let us expand $f_m(x,t)$ in powers of $t$: $f_m(x,t) = \sum_{k=1}^{\infty} f_{m,k}(x)$. Extracting coefficient of $t^m$ from $f_m(x,t)$ we obtain the generation function for diagonal elements of powers of $\mathcal{M}$ in the following sense:

$$f_{m,m}(x) = <t^m|1|t^m> + x <t^m|\mathcal{M}|t^m> + x^2 <t^m|\mathcal{M}^2|t^m> + x^3 <t^m|\mathcal{M}^3|t^m> \cdots$$

From the explicit formula (66) we have

$$f_{m,m}(x) = 1 - \frac{1}{e^x-1}\sum_{k=m+1}^{\infty}\frac{x^k}{k!}$$

The unit term here is the matrix element of identity. Now, to get generating function for super-traces, we must evaluate the sum $\sum_{m=1}^{\infty}(f_{m,m}(x)-1)$:

$$(67) \quad \sum_{m=1}^{\infty}(f_{m,m}(x)-1) = x \operatorname{Str}\mathcal{M} + x^2 \operatorname{Str}\mathcal{M}^2 + x^3 \operatorname{Str}\mathcal{M}^3 + \cdots =$$

$$= -\frac{1}{e^x-1}\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty}\frac{x^k}{k!} = -\frac{1}{e^x-1}\sum_{k=2}^{\infty}\frac{k-1}{k!}x^k = 1 - x - \frac{x}{e^x-1} = -\frac{1}{2}x - \sum_{n=2}^{\infty}\frac{B_n}{n!}x^n$$

Thus we proved that $\operatorname{Str}\mathcal{M}^n = -\frac{B_n}{n!}$ for $n \geq 2$. $\square$

Now we have all the ingredients to obtain explicit expression for $\bar{S}$ on a interval $\Delta^1$: we just have to take series (57,58), plug there the decompositions of infrared fields (53), and use formulae (60,62). We should also take into account that the first term in (58) vanishes, since it is proportional to the contraction $f_{ab}^b = 0$ of structure constants of gauge algebra. The result is:

**Theorem 6.** *The reduced effective BF action on 1-dimensional simplex $\Delta^1$, as defined by (52), is*

$$\bar{S}(\omega^0, \omega^1, \omega^{01}, p_{01}; \hbar) = \bar{S}^{(0)}(\omega^0, \omega^1, \omega^{01}, p_{01}) + \hbar\, \bar{S}^{(1)}(\omega^{01})$$

*and the tree and 1-loop parts of $\bar{S}$ are:*

(68) $\quad \bar{S}^{(0)}(\omega^0, \omega^1, \omega^{01}, p_{01}) =$

$$= <p_{01}, -(\omega^1 - \omega^0) + \frac{1}{2}[\omega^0 + \omega^1, \omega^{01}] - \sum_{n=2}^{\infty} \frac{B_n}{n!}(\text{ad}_{\omega^{01}})^n(\omega^1 - \omega^0) >_{\mathfrak{g}} =$$

$$= <p_{01}, \frac{1}{2}[\omega^0 + \omega^1, \omega^{01}] - \left(\frac{\text{ad}_{\omega^{01}}}{2} \coth \frac{\text{ad}_{\omega^{01}}}{2}\right)(\omega^1 - \omega^0) >_{\mathfrak{g}}$$

*and*

(69) $$\bar{S}^{(1)}(\omega^{01}) = \sum_{n=2}^{\infty} \frac{1}{n} \frac{B_n}{n!} \text{tr}_{\mathfrak{g}}(\text{ad}_{\omega^{01}})^n = \text{tr}_{\mathfrak{g}} \log\left(\frac{\sinh \frac{\text{ad}_{\omega^{01}}}{2}}{\frac{\text{ad}_{\omega^{01}}}{2}}\right)$$

*where $\text{ad}_{\omega^{01}} = [\omega^{01}, \bullet]$ is the adjoint action of $\omega^{01}$, $<\bullet, \bullet>_{\mathfrak{g}}$ is canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, $\text{tr}_{\mathfrak{g}}$ is the trace over $\mathfrak{g}$.*

**Remark.** We can recognize in (68) a special case of Baker-Campbell-Hausdorff series:

$$\bar{S}^{(0)}(\omega^0, \omega^1, \omega^{01}, p_{01}) = <p_{01}, \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \log\left(e^{-\epsilon\omega^1} e^{-\omega^{01}} e^{\epsilon\omega^0}\right) >_{\mathfrak{g}}$$

where $\epsilon \in \Pi\mathbb{R}$ is an infinitesimal odd variable. The 1-loop part of reduced effective action on interval (69) has the following interpretation. The measure it defines on gauge Lie algebra $\mathfrak{g}$ is the pullback of Haar measure $\mu_G$ on gauge group $G$ w.r.t exponential map $\exp : \mathfrak{g} \to G$

$$e^{\bar{S}^{(1)}(\omega^{01})} \delta\omega^{01} = \det_g\left(\frac{\sinh \frac{\text{ad}_{\omega^{01}}}{2}}{\frac{\text{ad}_{\omega^{01}}}{2}}\right) \delta\omega^{01} = \exp^* \mu_G$$

(see e.g. [7]).

4.5. **Simplicial $BF$ theory on interval.** The simplest example of simplicial $BF$ theory (apart from trivial 0-dimensional case) is the case when the manifold $M$ is an interval $M = [0, 1]$ and triangulation $\Xi$ consists of one 1-dimensional simplex $[01]$ — the interval itself and two 0-dimensional simplices $[0], [1]$ — the end-points of interval. Infrared fields are $\omega' = \omega^0 \chi_0 + \omega^1 \chi_1 + \omega^{01} \chi_{01}$ and $p' = e^0 p_0 + e^1 p_1 + e^{01} p_{01}$. Here $\omega^0, \omega^1 \in \Pi\mathfrak{g}$, $\omega^{01} \in \mathfrak{g}$, $p_0, p_1 \in \mathfrak{g}^*$, $p_{01} \in \Pi\mathfrak{g}^*$. Space of fields $\omega'$ may be identified with space $\Pi\mathfrak{g} \otimes C^*(\Xi)$ of $\Pi\mathfrak{g}$-valued cochains on $\Xi$ and space of fields $p'$ with space $C_*(\Xi) \otimes \mathfrak{g}^*$ of $\mathfrak{g}^*$-valued chains. The effective action is

(70) $\quad S_\Xi(\omega', p'; \hbar) = \bar{S}_0(\omega^0, p_0) + \bar{S}_1(\omega^1, p_1) + \bar{S}_{01}(\omega^0, \omega^1, \omega^{01}, p_{01}; \hbar) =$

$$= <p_0, \frac{1}{2}[\omega^0, \omega^0] >_g + <p_1, \frac{1}{2}[\omega^1, \omega^1] >_g +$$

$$+ <p_{01}, \frac{1}{2}[\omega^0 + \omega^1, \omega^{01}] - \left(\frac{\text{ad}_{\omega^{01}}}{2} \coth \frac{\text{ad}_{\omega^{01}}}{2}\right)(\omega^1 - \omega^0) >_{\mathfrak{g}} + \hbar \, \text{tr}_{\mathfrak{g}} \log\left(\frac{\sinh \frac{\text{ad}_{\omega^{01}}}{2}}{\frac{\text{ad}_{\omega^{01}}}{2}}\right)$$

Action $S_\Xi$ satisfies quantum master equation by construction and thus defines cohomological vector field on $\Pi\mathfrak{g} \otimes C^*(\Xi)$:

(71) $\quad Q(\omega') = <\frac{1}{2}[\omega^0, \omega^0], \frac{\partial}{\partial\omega^0} >_{\mathfrak{g}} + <\frac{1}{2}[\omega^1, \omega^1], \frac{\partial}{\partial\omega^1} >_{\mathfrak{g}} +$

$$+ <\frac{1}{2}[\omega^0 + \omega^1, \omega^{01}] - \left(\frac{\text{ad}_{\omega^{01}}}{2} \coth \frac{\text{ad}_{\omega^{01}}}{2}\right)(\omega^1 - \omega^0), \frac{\partial}{\partial\omega^{01}} >_{\mathfrak{g}}$$

Vector field $Q$ generates $L_\infty$ algebra structure on the space of $\mathfrak{g}$-valued cochains $\mathfrak{g} \otimes C^*(\Xi)$. The 1-loop part of action $S_\Xi$ produces density function $\rho(\omega')$ on $\Pi\mathfrak{g} \otimes C^*(\Xi)$:

$$\rho(\omega') = e^{S_\Xi^{(1)}(\omega')} = \det{}_g \left( \frac{\sinh \frac{\mathrm{ad}_{\omega^{01}}}{2}}{\frac{\mathrm{ad}_{\omega^{01}}}{2}} \right)$$

Density $\rho$ is $Q$-invariant by construction. The $L_\infty$ quasi-isomorphism $U : \Pi\mathfrak{g} \otimes C^*(\Xi) \to \Pi\mathfrak{g} \otimes \Omega(\Delta^1)$ is easily found from (29) using (59):

$$U(\omega') = \omega^0 + \left( \frac{1 - e^{-t\,\mathrm{ad}_{\omega^{01}}}}{1 - e^{-\mathrm{ad}_{\omega^{01}}}} \right)(\omega^1 - \omega^0) + \omega^{01} dt$$

or equivalently in more symmetric form:

$$U(\omega') = \left( \frac{1 - e^{t_0\,\mathrm{ad}_{\omega^{01}}}}{1 - e^{\mathrm{ad}_{\omega^{01}}}} \right)\omega^0 + \left( \frac{1 - e^{-t_1\,\mathrm{ad}_{\omega^{01}}}}{1 - e^{-\mathrm{ad}_{\omega^{01}}}} \right)\omega^1 + \omega^{01} dt$$

Let us now take a triangulation $\Xi$ on the interval $I = [0,1]$, consisting of $N \geq 1$ 1-dimensional simplices $[01], [12], \ldots, [(N-1)N]$ and $N+1$ 0-dimensional simplices $[0], [1], \ldots, [N]$. Let the coordinate of $[i]$ on the interval $[0,1]$ be $\frac{i}{N}$, and let $\epsilon = \frac{1}{N}$ denote the spacing. The infrared fields are: $\omega' = \sum_{i=0}^N \omega^i \chi_i + \sum_{i=1}^N \omega^{i-1,i} \chi_{i-1,i}$, $p' = \sum_{i=0}^N e^i p_i + \sum_{i=1}^N e^{i-1,i} p_{i-1,i}$ and the effective action is

$$S_\Xi(\omega', p'; \hbar) = \sum_{i=0}^N \bar{S}_i(\omega^i, p_i) + \sum_{i=1}^N \bar{S}_{i-1,i}(\omega^{i-1}, \omega^i, \omega^{i-1,i}, p_{i-1,i}; \hbar)$$

Let us introduce normalized coordinates on space of infrared fields: $\omega^i = \tilde{\omega}^i$, $\omega^{i-1,i} = \epsilon \tilde{\omega}^{i-1,i}$, $p_i = \epsilon \tilde{p}_i$, $p_{i-1,i} = \tilde{p}_{i-1,i}$. Then the projector $\mathcal{P}'$ acts on smooth forms $\omega(t) \in \Pi\mathfrak{g} \otimes \Omega([0,1])$ as

$$\tilde{\omega}^i = \omega\left(\frac{i}{N}\right), \quad \tilde{\omega}^{i-1,i} = \frac{1}{\epsilon} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \omega$$

Thus $\tilde{\omega}$ is what we would call a "lattice approximation" of a smooth form. Expressing effective action $S_\Xi$ in terms of these normalized infrared fields, we obtain:

(72)  $S_\Xi(\omega', p'; \hbar) =$

$$\epsilon \left( -\sum_{i=1}^N <\tilde{p}_{i-1,i}, \frac{\tilde{\omega}^i - \tilde{\omega}^{i-1}}{\epsilon}>_\mathfrak{g} + \sum_{i=0}^N <\tilde{p}_i, \frac{1}{2}[\tilde{\omega}^i, \tilde{\omega}^i]>_\mathfrak{g} + \sum_{i=1}^N <\tilde{p}_{i-1,i}, [\frac{\tilde{\omega}^{i-1} + \tilde{\omega}^i}{2}, \tilde{\omega}^{i-1,i}]>_\mathfrak{g} \right) -$$

$$- \sum_{n=2}^\infty \epsilon^{n+1} \frac{B_n}{n!} \sum_{i=1}^N <p_{i-1,i}, (\mathrm{ad}_{\tilde{\omega}^{i-1,i}})^n \left( \frac{\tilde{\omega}^i - \tilde{\omega}^{i-1}}{\epsilon} \right)>_\mathfrak{g} + \hbar \sum_{n=2}^\infty \epsilon^n \frac{B_n}{n\,n!} \sum_{i=1}^N \mathrm{tr}_\mathfrak{g}(\mathrm{ad}_{\tilde{\omega}^{i-1,i}})^n$$

Expression (72) constitutes a subtle lattice version of ordinary extended $BF$ action (on smooth forms) on interval $S = <p, d\omega + \frac{1}{2}[\omega, \omega]>$. The first three terms in (72) are what we would call the naive lattice action for extended $BF$ theory on interval, while the other terms are corrections of higher order in spacing $\epsilon$. These additional terms make this lattice action satisfy quantum master equation (the naive lattice action does not satisfy QME).

4.6. **Simplicial $BF$ theory on circle, induction to cohomologies of circle, $BF$ state-sum on circle.** Now let $M = S^1$ be a circle, and $\Xi$ be a triangulation of circle, consisting of $N \geq 2$ 1-simplices $[01], [12], \ldots, [(N-1)N]$ and $N$ 0-simplices $[1], [2], \ldots, [N]$. Infrared fields are $\omega' = \sum_{i=1}^{N} \omega^i \chi_i + \sum_{i=1}^{N} \omega^{i-1,i} \chi_{i-1,i}$, $p' = \sum_{i=1}^{N} e^i p_i + \sum_{i=1}^{N} e^{i-1,i} p_{i-1,i}$ and effective action is

$$(73) \quad S_\Xi(\omega', p'; \hbar) = \sum_{i=1}^{N} < p_i, \frac{1}{2}[\omega^i, \omega^i] >_{\mathfrak{g}} +$$

$$+ \sum_{i=1}^{N} < p_{i-1,i}, \frac{1}{2}[\omega^{i-1} + \omega^i, \omega^{i-1,i}] - \left( \frac{\mathrm{ad}_{\omega^{i-1,i}}}{2} \coth \frac{\mathrm{ad}_{\omega^{i-1,i}}}{2} \right) (\omega^i - \omega^{i-1}) >_{\mathfrak{g}} +$$

$$+ \hbar \sum_{i=1}^{N} \mathrm{tr}_{\mathfrak{g}} \, \log \left( \frac{\sinh \frac{\mathrm{ad}_{\omega^{i-1,i}}}{2}}{\frac{\mathrm{ad}_{\omega^{i-1,i}}}{2}} \right)$$

We use here the convention $\omega^0 = \omega^N$.

Let us induce effective action on cohomologies of circle from action (73), considered as $BF_\infty$ type theory. For simplicity take $N = 2$, i.e. the simplest non-degenerate triangulation of the circle. Next we split the fields living on triangulation $\Xi$ into (new) infrared and ultraviolet parts:

$$\omega^1 = \omega^A - \frac{1}{2}\omega''^A, \; \omega^2 = \omega^A + \frac{1}{2}\omega''^A, \; \omega^{01} = \frac{\omega^B - \omega''^B}{2}, \; \omega^{12} = \frac{\omega^B + \omega''^B}{2},$$

$$p_1 = \frac{p_A - p''_A}{2}, \; p_2 = \frac{p_A + p''_A}{2}, \; p_{01} = p_B - \frac{1}{2}p''_B, \; p_{12} = p_B + \frac{1}{2}p''_B$$

Here $A$ and $B$ label the 0- and 1-dimensional cohomologies of circle: $e_A = 1$, $e_B = dt$. The Lagrangian submanifold $\mathcal{L}_K$ is: $\omega''^B = 0$, $p''_A = 0$. Restricted to $\mathcal{L}_K$ the action (73) gives

$$(74) \quad S_\Xi|_{\mathcal{L}_K} = < p_A, \frac{1}{2}[\omega^A, \omega^A] + \frac{1}{8}[\omega''^A, \omega''^A] >_{\mathfrak{g}} + < p_B, [\omega^A, \omega^B] >_{\mathfrak{g}} -$$

$$- < p''_B, \left( \frac{\mathrm{ad}_{\omega^B}}{4} \coth \frac{\mathrm{ad}_{\omega^B}}{4} \right) \omega''^A >_{\mathfrak{g}} + \hbar \, \mathrm{tr}_g \log \left( \frac{\sinh \frac{\mathrm{ad}_{\omega^B}}{4}}{\frac{\mathrm{ad}_{\omega^B}}{4}} \right)^2$$

The BV integral (15) is

(75)

$$e^{\frac{1}{\hbar}S_{H^*(S^1)}} = \int e^{S_\Xi/\hbar} \, \delta\omega''^A \delta p''_B = \exp \left( \frac{1}{\hbar}(< p_A, \frac{1}{2}[\omega^A, \omega^A] >_{\mathfrak{g}} + < p_B, [\omega^A, \omega^B] >_{\mathfrak{g}}) \right) \cdot$$

$$\cdot \det_{\mathfrak{g}} \left( \frac{\mathrm{ad}_{\omega^B}}{4} \coth \frac{\mathrm{ad}_{\omega^B}}{4} \right) \det_g \left( \frac{\sinh \frac{\mathrm{ad}_{\omega^B}}{4}}{\frac{\mathrm{ad}_{\omega^B}}{4}} \right)^2 =$$

$$= \exp \left( \frac{1}{\hbar}(< p_A, \frac{1}{2}[\omega^A, \omega^A] >_{\mathfrak{g}} + < p_B, [\omega^A, \omega^B] >_{\mathfrak{g}}) \right) \det_g \left( \frac{\sinh \frac{\mathrm{ad}_{\omega^B}}{2}}{\frac{\mathrm{ad}_{\omega^B}}{2}} \right)$$

Hence the effective action on cohomologies of circle (i.e. on space $\Pi T^*(\Pi\mathfrak{g} \otimes H^*(S^1))$) is

$$(76) \quad S_{H^*(S^1)}(\omega^A, \omega^B, p_A, p_B; \hbar) =$$

$$= < p_A, \frac{1}{2}[\omega^A, \omega^A] >_{\mathfrak{g}} + < p_B, [\omega^A, \omega^B] >_{\mathfrak{g}} + \hbar \, \mathrm{tr}_g \log \left( \frac{\sinh \frac{\mathrm{ad}_{\omega^B}}{2}}{\frac{\mathrm{ad}_{\omega^B}}{2}} \right)$$

This action coincides with (73) if we formally set $N = 1$ (although this corresponds to degenerate triangulation). Indeed we could derive (76) directly from continuous extended $BF$ theory on circle and arrive to the same answer. The difference is that inducing (76) from (73) we only need to calculate a finite-dimensional integral. Action (76) generates Lie algebra structure on $\mathfrak{g} \otimes H^*(S^1)$ and no higher homotopic operations (Massey operations), since circle is a formal manifold. The 1-loop part of (76) gives $Q$-invariant density function on $\Pi\mathfrak{g} \otimes H^*(S^1)$:

$$\rho(\omega^B) = \det_g \left( \frac{\sinh \frac{\mathrm{ad}_{\omega B}}{2}}{\frac{\mathrm{ad}_{\omega B}}{2}} \right)$$

The state-sum for extended $BF$ theory on circle, according to definition from (3.13) is then

(77) $$Z(\mathfrak{g} \otimes \Omega(S^1)) = \int_{\Pi H^*(S^1)} \rho(\omega^B) \, \delta\omega^A \delta\omega^B = \int_{\mathfrak{g}} \det_g \left( \frac{\sinh \frac{\mathrm{ad}_{\omega B}}{2}}{\frac{\mathrm{ad}_{\omega B}}{2}} \right) \delta\omega^B$$

As we observed before, the measure we are integrating is a pullback of Haar measure on the gauge Lie group $G$ under exponential map $\exp : \mathfrak{g} \to G$. Notice that integral (77) (if taken over the connected component of support of $\rho$, containing zero — the integral over whole $\mathfrak{g}$ diverges) gives the volume of the gauge group:

$$Z(\mathfrak{g} \otimes \Omega(S^1)) = \mathrm{vol}(G)$$

As we mentioned in section 3.13, we do not have a good explanation, why we should regularize the integral for state-sum in such a way. Notice also that the compactness of gauge group suddenly becomes important for finiteness of state-sum. This should be viewed as an essentially quantum phenomenon.

4.7. **Elementary BV integral on simplex of dimension $D \geq 2$: perturbative results.** Integral (52) on $D$-dimensional simplex $\Delta^D$ is no longer Gaussian if $D \geq 2$ and we do not know the closed expression for $\bar{S}_{\Delta^D}$. But we can use perturbative expansion (21,22) for $\bar{S}_{\Delta^D}$ and calculate its first terms explicitly.

We use the same notation for Taylor expansion of $\bar{S}$ as in subsection 3.10:

$$\bar{S}_{\Delta^D}(\omega, p_{\Delta^D}) = \sum_{n=1}^{\infty} < p_{\Delta^D}, \frac{1}{n!} \, \bar{l}_{\Delta^D}^{(n)}(\omega, \ldots, \omega) >_{\mathfrak{g}} + \hbar \sum_{n=1}^{\infty} \frac{1}{n!} \, \bar{q}_{\Delta^D}^{(n)}(\omega, \ldots, \omega)$$

where $\bar{l}_{\Delta^D}^{(n)}$ is a $n$-linear super-antisymmetric map

$$\bar{l}_{\Delta^D}^{(n)} : (\Pi\mathfrak{g} \otimes \Omega_W(\Delta^D))^{\otimes n} \to [\Omega_W^D(\Delta^D)]^* \otimes \mathfrak{g}^* \simeq \Pi^D \mathfrak{g}^*$$

and $\bar{q}_{\Delta^D}^{(n)}$ is a $n$-linear super-antisymmetric function

$$\bar{q}_{\Delta^D}^{(n)} : (\Pi\mathfrak{g} \otimes \Omega_W(\Delta^D))^{\otimes n} \to \mathbb{R}$$

and we put bars on $l$ and $q$ to indicate that they correspond to the reduced effective action on simplex.

Now introduce a set of functions $C_T$ on faces of $\Delta^D$ as follows: for every rooted binary tree $T$ with $|T| = n$ leaves and $\sigma_1, \ldots, \sigma_n$ faces of $\Delta^D$ we define

$$C_T(\sigma_1, \ldots, \sigma_n) = \int_{\Delta^D} \mathrm{Iter}_{T, K(\bullet \wedge \bullet), \bullet \wedge \bullet} (\chi_{\sigma_1}, \ldots, \chi_{\sigma_n})$$

For the trivial tree with one leaf we set

$$C_{(*)}(\sigma_1) = \int_{\Delta^D} d\chi_{\sigma_1}$$

We also introduce the sign functions $\epsilon_T$ taking values in $\{-1, 0, +1\}$, defined as

$$\epsilon_T(\sigma_1, \ldots, \sigma_n) = \begin{cases} +1 & \text{if } C_T(\sigma_1, \ldots, \sigma_n) > 0 \\ 0 & \text{if } C_T(\sigma_1, \ldots, \sigma_n) = 0 \\ -1 & \text{if } C_T(\sigma_1, \ldots, \sigma_n) < 0 \end{cases}$$

Functions $C_T$ have the following symmetry properties:

- Internal symmetry: for $\pi_1, \ldots, \pi_n$ permutations of vertices of simplices $\sigma_1, \ldots, \sigma_n$

(78) $$C_T(\pi_1 \sigma_1, \ldots \pi_n \sigma_n) = (-1)^{\pi_1} \cdots (-1)^{\pi_n} C_T(\sigma_1, \ldots, \sigma_n)$$

where $(-1)^{\pi_i}$ is the sign of permutation $\pi_i$.
- External symmetry: for $\pi$ a permutation of vertices of $\Delta^D$

(79) $$C_T(\pi \sigma_1, \ldots, \pi \sigma_n) = (-1)^{\pi} C_T(\sigma_1, \ldots, \sigma_n)$$

- Symmetry under tree isomorphisms: if trees $T$ and $T'$ are isomorphic as non-planar graphs and $\kappa : T \to T'$ is the isomorphism, then

(80) $$C_{T'}(\sigma_{\kappa(1)}, \ldots, \sigma_{\kappa(n)}) = \epsilon_\kappa(|\sigma_1|, \ldots, |\sigma_n|) \, C_T(\sigma_1, \ldots, \sigma_n)$$

where we understand that $\kappa$ maps leaves of $T$ into leaves of $T'$. The sign $\epsilon_\kappa(|\sigma_1|, \ldots, |\sigma_n|) = \pm 1$ depends only on dimensions of faces, not on faces themselves and is defined by (80). Important case of this symmetry is when $T' = T$ and $\kappa \in \mathrm{Aut}(T)$.

Examples of symmetry (80):

$$C_{(*(**))}(\sigma_3, \sigma_1, \sigma_2) = (-1)^{(|\sigma_1| + |\sigma_2| - 1) \, |\sigma_3|} C_{((**)*)}(\sigma_1, \sigma_2, \sigma_3)$$
$$C_{((**)*)}(\sigma_2, \sigma_1, \sigma_3) = (-1)^{|\sigma_1| \, |\sigma_2|} C_{((**)*)}(\sigma_1, \sigma_2, \sigma_3)$$

Obviously symmetries (78,79,80) also hold for sign functions $\epsilon_T$.

**Lemma 3.** *Values of $C_T$ for $|T| \leq 3$ are given by*

(81) $$C_{(*)}(\sigma_1) = \epsilon_{(*)}(\sigma_1),$$

(82) $$C_{(**)}(\sigma_1, \sigma_2) = \epsilon_{(**)}(\sigma_1, \sigma_2) \frac{|\sigma_1|! \, |\sigma_2|!}{(|\sigma_1| + |\sigma_2| + 1)!},$$

(83) $$C_{((**)*)}(\sigma_1, \sigma_2, \sigma_3) = \epsilon_{((**)*)}(\sigma_1, \sigma_2, \sigma_3) \frac{|\sigma_1|! \, |\sigma_2|! \, |\sigma_3|!}{(|\sigma_1| + |\sigma_2| + 1) \, (|\sigma_1| + |\sigma_2| + |\sigma_3| + 1)!}$$

*and signs $\epsilon_{(*)}$, $\epsilon_{(**)}$, $\epsilon_{((**)*)}$ are uniquely determined by symmetries (78,79,80), specific values*

(84) $$\epsilon_{(*)}([12 \cdots D]) = 1,$$
(85) $$\epsilon_{(**)}([0 \cdots a], [a \cdots D]) = 1 \text{ for } 0 \leq a \leq D,$$
(86) $$\epsilon_{((**)*)}([0 \cdots a], [a \cdots a + b], [a(a + b) \cdots D]) = (-1)^{a + b + 1}$$
$$\text{for } 0 \leq a \leq D - 1, \; 1 \leq b \leq D - a$$

*and non-vanishing conditions*

$$\epsilon_{(*)}(\sigma_1) \neq 0 \;\; \text{iff } |\sigma_1| = D - 1,$$
$$\epsilon_{(**)}(\sigma_1, \sigma_2) \neq 0 \;\; \text{iff } |\sigma_1| + |\sigma_2| = D \text{ and } \sigma_1 \cup \sigma_2 = \Delta^D,$$
$$\epsilon_{((**)*)}(\sigma_1, \sigma_2, \sigma_3) \neq 0 \text{ iff } |\sigma_1| + |\sigma_2| + |\sigma_3| = D + 1, \; \sigma_1 \cup \sigma_2 \cup \sigma_3 = \Delta^D$$
$$\text{and } \sigma_1 \cap \sigma_2 = \sigma_1 \cap \sigma_2 \cap \sigma_3 \text{ is a 0-simplex}$$

Here $\cup$ means union of simplices viewed as sets of vertices (or equivalently convex hull of geometric simplices), $\cap$ means intersection.

The absolute value of $C_T$ turns out to be a simple function of dimensions of simplices, non-vanishing condition is a combinatorial condition formulated in terms of dimensions and unions/intersections of simplices, while the sign $\epsilon_T$ of $C_T$ is the most tricky thing here, determined by reduction to standard cases (84,85,86) via symmetries (78,79,80).

We need coefficient functions $C_T$ for evaluating terms of perturbative expansion (21) for the reduced effective action:

$$(87) \quad < p, (-1)^{|T|}\mathrm{Iter}_{T,\,K[\bullet,\bullet],\,[\bullet,\bullet]}(\omega,\dots,\omega) >=$$

$$= \sum_{\sigma_1,\dots,\sigma_n \subset \Delta^D} < e^{\Delta^D} p_{\Delta^D}, (-1)^{|T|}\,\mathrm{Iter}_{T,\,K[\bullet,\bullet],\,[\bullet,\bullet]}(\omega^{\sigma_1}\chi_{\sigma_1},\dots,\omega^{\sigma_n}\chi_{\sigma_n}) >=$$

$$= \sum_{\sigma_1,\dots,\sigma_n \subset \Delta^D} \tilde{\epsilon}_T(|\sigma_1|,\dots,|\sigma_n|)\,C_T(\sigma_1,\dots,\sigma_n) < p_{\Delta^D}, \mathrm{Iter}_{T,\,[\bullet,\bullet]}(\omega^{\sigma_1},\dots,\omega^{\sigma_n}) >_{\mathfrak{g}}$$

Sign $\tilde{\epsilon}_T$ comes from interchanging coordinates $\omega$ and Whitney forms $\chi$ in (87), and is defined by

$$(88) \quad (-1)^{|T|}\,\mathrm{Iter}_{T,\,K[\bullet,\bullet],\,[\bullet,\bullet]}(\omega^{\sigma_1}\chi_{\sigma_1},\dots,\omega^{\sigma_n}\chi_{\sigma_n}) =$$

$$= \tilde{\epsilon}_T(|\sigma_1|,\dots,|\sigma_n|)\,C_T(\sigma_1,\dots,\sigma_n)\,\mathrm{Iter}_{T,\,[\bullet,\bullet]}(\omega^{\sigma_1},\dots,\omega^{\sigma_n})$$

For trees with $|T| \le 3$, $\tilde{\epsilon}_T$ is given by

$$(89) \qquad\qquad \tilde{\epsilon}_{(*)}(|\sigma_1|) = (-1)^{|\sigma_1|+1},$$

$$(90) \qquad\qquad \tilde{\epsilon}_{(**)}(|\sigma_1|,|\sigma_2|) = (-1)^{|\sigma_1|\,(|\sigma_2|+1)},$$

$$(91) \qquad \tilde{\epsilon}_{((**)*)}(|\sigma_1|,|\sigma_2|,|\sigma_3|) = (-1)^{|\sigma_1|\,|\sigma_2|+|\sigma_1|\,|\sigma_3|+|\sigma_2|\,|\sigma_3|+|\sigma_1|+|\sigma_3|}$$

Notice that (87) does not depend on planar structure of tree $T$: if $T$ and $T'$ are isomorphic as non-planar graphs, then $\bar{S}_{T'} = \bar{S}_T$. At the same time $\tilde{\epsilon}_T$, $C_T$ and $\mathrm{Iter}_{T,\,[\bullet,\bullet]}(\omega^{\sigma_1},\dots,\omega^{\sigma_n})$ separately do depend on planar structure of $T$.

The expansion (21) for tree part of reduced effective action on simplex $\Delta^D$ in terms of polylinear maps $\bar{l}_{\Delta^D}^{(n)}$ becomes

$$(92) \quad \bar{l}_{\Delta^D}^{(n)}(\omega,\dots,\omega) =$$

$$= n! \sum_{T:\,|T|=n} \frac{1}{\mathrm{Aut}(T)} \sum_{\sigma_1,\dots,\sigma_n \subset \Delta^D} \tilde{\epsilon}_T(|\sigma_1|,\dots,|\sigma_n|)\,C_T(\sigma_1,\dots,\sigma_n)\,\mathrm{Iter}_{T,\,[\bullet,\bullet]}(\omega^{\sigma_1},\dots,\omega^{\sigma_n})$$

where we sum over classes of isomorphic trees (or equivalently over trees without specified embedding into plane). Using Lemma 3 we obtain explicit expressions for $\bar{l}_{\Delta^D}^{(n)}$ with $n = 1,2,3$ (and thus expansion for $\bar{S}_{\Delta^D}^{(0)}$ up to order $\mathcal{O}(p\omega^3)$).

**Theorem 7.** *The first terms in tree part of reduced effective action $\bar{S}_{\Delta^D}^{(0)}$ are given by*

$$(93) \qquad \bar{l}_{\Delta^D}^{(1)}(\omega) \;=\; \sum_{\sigma_1 \subset \Delta^D} (-1)^{|\sigma_1|+1} \epsilon_{(*)}(\sigma_1)\, \omega^{\sigma_1},$$

$$(94) \qquad \bar{l}_{\Delta^D}^{(2)}(\omega,\omega) \;=\; \sum_{\sigma_1,\sigma_2 \subset \Delta^D} (-1)^{|\sigma_1|\,(|\sigma_2|+1)} \epsilon_{(**)}(\sigma_1,\sigma_2)\, \frac{|\sigma_1|!\,|\sigma_2|!}{(|\sigma_1|+|\sigma_2|+1)!}\, [\omega^{\sigma_1},\omega^{\sigma_2}],$$

$$(95) \bar{l}_{\Delta^D}^{(3)}(\omega,\omega,\omega) \;=\; 3 \sum_{\sigma_1,\sigma_2,\sigma_3 \subset \Delta^D} (-1)^{|\sigma_1|\,|\sigma_2|+|\sigma_1|\,|\sigma_3|+|\sigma_2|\,|\sigma_3|+|\sigma_1|+|\sigma_3|} \epsilon_{((**)*)}(\sigma_1,\sigma_2,\sigma_3) \cdot$$

$$\cdot \frac{|\sigma_1|!\,|\sigma_2|!\,|\sigma_3|!}{(|\sigma_1|+|\sigma_2|+1)\,(|\sigma_1|+|\sigma_2|+|\sigma_3|+1)!}\, [[\omega^{\sigma_1},\omega^{\sigma_2}],\omega^{\sigma_3}]$$

Let us now turn to the 1-loop part of reduced effective action $\bar{S}_{\Delta^D}^{(1)}(\omega)$. Similarly to what we did for tree part, for every loop graph $L$ with $|L| = n$ leaves we introduce a function $C_L$ on faces of $\Delta^D$:

$$C_L(\sigma_1,\ldots,\sigma_n) = \mathrm{Loop}_{L,\,K(\bullet \wedge \bullet),\,\Omega(\Delta^D)}(\chi_{\sigma_1},\ldots,\chi_{\sigma_n})$$

where the super-trace is taken over the space $\Omega(\Delta^D)$ of all differential forms on $\Delta^D$, and binary operator $K(\bullet \wedge \bullet)$ is acting on the same space. Obviously $C_L$ like $C_T$ possesses internal symmetry (78) and symmetry under graph isomorphisms (80) and the following form of external symmetry: for $\pi$ a permutation of vertices of $\Delta^D$

$$(96) \qquad C_L(\pi\sigma_1,\ldots,\pi\sigma_n) = C_L(\sigma_1,\ldots,\sigma_n)$$

only difference from the case of trees is the absence of sign $(-1)^\pi$.

Using $C_L$ we may evaluate the terms of expansion (22) for reduced effective action on $\Delta^D$ as follows

$$(97) \quad (-1)^{|L|}\, \mathrm{Loop}_{L,\,K[\bullet,\bullet],\,\Pi\mathfrak{g}\otimes\Omega(\Delta^D)}(\omega,\ldots,\omega) =$$

$$= \sum_{\sigma_1,\ldots,\sigma_n \subset \Delta^D} (-1)^{|L|}\, \mathrm{Loop}_{L,\,K[\bullet,\bullet],\,\Pi\mathfrak{g}\otimes\Omega(\Delta^D)}(\omega^{\sigma_1}\chi_{\sigma_1},\ldots,\omega^{\sigma_n}\chi_{\sigma_n}) =$$

$$= \sum_{\sigma_1,\ldots,\sigma_n \subset \Delta^D} \tilde{\epsilon}_L(|\sigma_1|,\ldots,|\sigma_n|)\, C_L(\sigma_1,\ldots,\sigma_n)\, \mathrm{Loop}_{L,[\bullet,\bullet],\mathfrak{g}}(\omega^{\sigma_1},\ldots,\omega^{\sigma_n})$$

The meaning of (97) is to separate super-trace over $\Pi\mathfrak{g} \otimes \Omega(\Delta^D)$ into trivial part — trace over $\mathfrak{g}$ and non-trivial part — super-trace over infinite-dimensional space $\Omega(\Delta^D)$. Signs $\tilde{\epsilon}_L$ come from interchanging $\omega^\sigma$ and $\chi_\sigma$ and are defined by (97). Plugging (97) into (22), we obtain

$$(98) \quad \bar{q}_{\Delta^D}^{(n)}(\omega,\ldots,\omega) =$$

$$= n! \sum_{L:\,|L|=n} \frac{1}{\mathrm{Aut}(L)} \sum_{\sigma_1,\ldots,\sigma_n \subset \Delta^D} \tilde{\epsilon}_L(|\sigma_1|,\ldots,|\sigma_n|)\, C_L(\sigma_1,\ldots,\sigma_n)\, \mathrm{Loop}_{L,[\bullet,\bullet],\mathfrak{g}}(\omega^{\sigma_1},\ldots,\omega^{\sigma_n})$$

Here we sum over classes of isomorphic 1-loop graphs. Graphs $L$ with cycle of length 1 do not contribute to (98) since for these graphs $\mathrm{Loop}_{L,[\bullet,\bullet],\mathfrak{g}}$ is proportional to the contraction $f_{ab}^b$ of structure constants of gauge algebra, and thus these terms vanish. For instance this means that $\bar{q}_{\Delta^D}^{(1)} = 0$. For $\bar{q}_{\Delta^D}^{(2)}$ the only contributing graph is $L = (*(*\bullet))$. Symmetries (78,96) for $C_{(*(*\bullet))}$ allow only two possible terms for $\bar{q}_{\Delta^D}^{(2)}$:

$$\bar{q}_{\Delta^D}^{(2)}(\omega,\omega) = \mathcal{A}_D \sum_{0 \le i < j \le D} \mathrm{tr}_\mathfrak{g}\,(\mathrm{ad}_{\omega^{ij}})^2 + \mathcal{B}_D \sum_{0 \le i < j < k \le D} \mathrm{tr}_\mathfrak{g}\,(\mathrm{ad}_{\omega^{jk}} - \mathrm{ad}_{\omega^{ik}} + \mathrm{ad}_{\omega^{ij}})^2$$

Here $\mathcal{A}_D$ and $\mathcal{B}_D$ are some coefficients, and symmetries tell nothing of their values. It turns out that value of $\mathcal{A}_D$ can be recovered from master equation for full effective action on simplex $\Delta^D$ (i.e. sum of reduced effective actions on all faces) and result (95) for tree part of effective action. Coefficient $\mathcal{B}_D$ on the other hand cannot be recovered from master equation since the canonical transformation

$$S_{\Delta^D} \mapsto S_{\Delta^D} + \hbar\, \alpha\, Q \left( \sum_{0 \leq i < j < k \leq D} \mathrm{tr}_{\mathfrak{g}} \left( \mathrm{ad}_{\omega^{jk}} - \mathrm{ad}_{\omega^{ik}} + \mathrm{ad}_{\omega^{ij}} \right) \cdot \mathrm{ad}_{\omega^{ijk}} \right)$$

shifts coefficient $\mathcal{B}_D$ by $\alpha$ (and gives indeed a solution to master equation). This also means that coefficient $\mathcal{B}_D$ is somehow less important then $\mathcal{A}_D$, since it stands in front of $Q$-exact term.

**Theorem 8.** *The first terms of 1-loop part of reduced effective action $\bar{S}_{\Delta^D}^{(1)}$ are given by*

$$\bar{q}_{\Delta^D}^{(1)}(\omega) = 0$$

*and*

$$(99) \quad \bar{q}_{\Delta^D}^{(2)}(\omega, \omega) = \mathcal{A}_D \sum_{0 \leq i < j \leq D} \mathrm{tr}_{\mathfrak{g}} \left( \mathrm{ad}_{\omega^{ij}} \right)^2 + \mathcal{B}_D \sum_{0 \leq i < j < k \leq D} \mathrm{tr}_{\mathfrak{g}} \left( \mathrm{ad}_{\omega^{jk}} - \mathrm{ad}_{\omega^{ik}} + \mathrm{ad}_{\omega^{ij}} \right)^2$$

*and coefficient $\mathcal{A}_D$ is*

$$(100) \quad \mathcal{A}_D = \frac{(-1)^{D+1}}{(D+1)^2\,(D+2)}$$

We also carried out an explicit calculation of super-trace $C_{(*(*\bullet))}$ in dimensions $D = 2, 3$ (not relying on master equation arguments) and found out the following:

**Theorem 9.** *In dimensions $D = 2, 3$ the lowest-order term in 1-loop part of reduced effective action $\bar{q}_{\Delta^D}^{(2)}$ is given by (99) with*

$$\mathcal{A}_2 = -\frac{1}{36},\ \mathcal{B}_2 = \frac{1}{270},$$
$$\mathcal{A}_3 = \frac{1}{80},\ \mathcal{B}_3 = -\frac{1}{648}$$

For the coefficient $\mathcal{B}_D$ we might try to guess some formula like

$$\mathcal{B}_D = \frac{(-1)^D}{9\,D(D+1)(D+3)}$$

from here, but this is just a guess, and our evidence is limited to only two points $D = 2, 3$.

Collecting our results for $D = 2$ we obtain

(101)  $\bar{S}_{[012]}(\omega, p; \hbar) =$

$=<p_{012}, (\omega^{01} + \omega^{12} + \omega^{20}) + \frac{1}{3}[\omega^0 + \omega^1 + \omega^2, \omega^{012}] + \frac{1}{6}([\omega^{01}, \omega^{12}] + [\omega^{12}, \omega^{20}] + [\omega^{20}, \omega^{12}]) +$

$+ \frac{1}{72}([[\omega^{01} + \omega^{12} + \omega^{20}, \omega^{01}], \omega^{01}] + [[\omega^{01} + \omega^{12} + \omega^{20}, \omega^{12}], \omega^{12}] + [[\omega^{01} + \omega^{12} + \omega^{20}, \omega^{20}], \omega^{20}]) -$

$- \frac{1}{24}([[\omega^1 - \omega^0, \omega^{01}], \omega^{012}] + [[\omega^2 - \omega^1, \omega^{12}], \omega^{012}] + [[\omega^0 - \omega^2, \omega^{20}], \omega^{012}]) -$

$- \frac{1}{36}([[\omega^1 - \omega^0, \omega^{012}], \omega^{01}] + [[\omega^2 - \omega^1, \omega^{012}], \omega^{12}] + [[\omega^0 - \omega^2, \omega^{012}], \omega^{20}]) >_{\mathfrak{g}} +$

$+ \hbar \, \mathrm{tr}_g \left( -\frac{1}{72} \left( (\mathrm{ad}_{\omega^{01}})^2 + (\mathrm{ad}_{\omega^{12}})^2 + (\mathrm{ad}_{\omega^{20}})^2 \right) + \frac{1}{540} (\mathrm{ad}_{\omega^{01}} + \mathrm{ad}_{\omega^{12}} + \mathrm{ad}_{\omega^{20}})^2 \right) +$

$+ \mathcal{O}(p\,\omega^4) + \mathcal{O}(\hbar \omega^3)$

Importance of effective action on 2-simplex is that its tree part restricted to Whitney 1-forms produces a formula for "simplicial curvature" of simplicial (i.e. Whitney) connection 1-form:

(102)  $F_{[012]}(\omega^{01}, \omega^{12}, \omega^{20}) = (\omega^{01} + \omega^{12} + \omega^{20}) + \frac{1}{6}([\omega^{01}, \omega^{12}] + [\omega^{12}, \omega^{20}] + [\omega^{20}, \omega^{12}]) +$

$+ \frac{1}{72}([[\omega^{01} + \omega^{12} + \omega^{20}, \omega^{01}], \omega^{01}] + [[\omega^{01} + \omega^{12} + \omega^{20}, \omega^{12}], \omega^{12}] + [[\omega^{01} + \omega^{12} + \omega^{20}, \omega^{20}], \omega^{20}]) + \mathcal{O}(\omega^4)$

**Remark on divergencies.** Calculating values of 1-loop Feynman graphs for effective action on simplex reduces essentially to calculating super-traces over infinite-dimensional space of differential forms. These might contain divergencies. As we have seen in section 4.4, this is not the case for dimension D=1: only finitely many terms of the monodromy matrix (written in monomial basis) are non-zero, and thus the super-trace is a sum of finitely many terms.

For dimension $D = 2$ we also carried out a calculation of super-trace for $q^{(2)}$ in monomial basis. For this case diagonal elements of monodromy matrix do not vanish on monomials of high degree. Moreover, super-traces of monodromy matrix on 0-forms and on 1-forms diverge if calculated separately. If we employ the regularization that is the monomial degree cut-off, i.e. we calculate super-trace of monodromy acting on monomials of total degree $< N$, these divergences are logarithmic: $\mathrm{Str}_{\Omega^0(\Delta^2)} \sim \log N$ and $\mathrm{Str}_{\Omega^1(\Delta^2)} \sim \log N$. But in the total super-trace over all differential forms these divergencies cancel, and the answer for $q^{(2)}$ is finite.

We also made a calculation of super-trace for $q^{(2)}$ in "coordinate representation", i.e. in basis of $\delta$-functions of coordinates on simplex, centered in different points (thus super-trace becomes an integral over the simplex). Here we also encounter divergencies, and a nice way to handle them is to introduce the following regularization: we change the Dupont's chain homotopy operator $K$ to a regularized one $K_\epsilon$, where $K_\epsilon$ is obtained by the same construction, described in section 4.2, where we redefine the dilation map $\phi_i$ to act on $[0, 1 - \epsilon] \times \Delta^D$ instead of $[0, 1] \times \Delta^D$. Here $\epsilon > 0$ is an infinitesimal parameter. This regularization immediately makes all answers in coordinate representation finite, and the result for $q^{(2)}$ coincides with one obtained in monomial basis.

For case $D = 3$ we calculated $q^{(2)}$ in coordinate representation only (these calculations are technically simpler than in monomial basis). In principle the corresponding super-trace could have not just logarithmic, but even a linear (in cut-off parameter) divergence. But, employing regularization $K \to K_\epsilon$, we obtain a finite answer.

## References

[1] D. H. Adams, *R-torsion and linking numbers from simplicial abelian gauge theories*, hep-th/9612009

[2] M. Aleksandrov, M. Kontsevich, A. Schwarz and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory,* hep-th/9502010

[3] M. Atiyah, *The Geometry and Physics of Knots*, Cambridge U. Press, Cambridge, 1990

[4] A. S. Cattaneo, P. Cotta-Ramusino, J. Fröhlich, M. Martellini, *Topological BF theories in 3 and 4 dimensions*, hep-th/9505027

[5] A. S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, math.QA/9902090

[6] E. Getzler, *Lie theory for nilpotent $L_\infty$-algebras,* math.AT/0404003

[7] A. A. Kirillov, *Elements of the Theory of Representations*, vol. 220 of Grundlehren Math. Wiss., Springer, Berlin, 1976

[8] M. Kontsevich, *Deformation quantization of Poisson manifolds*, q-alg/9709040

[9] A. Kozak, *Discretization of Chern-Simons-like theories*, unpublished draft in Russian, 2006

[10] D. Krotov, A. Losev, A. Gorodentsev, *Quantum Field Theory as Effective BV Theory from Chern-Simons,* hep-th/0603201

[11] M. Markl, S. Shnider, J. Stasheff, *Operads in Algebra, Topology and Physics*, volume 96 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, Rhode Island, 2002

[12] P. Schaller, T. Strobl, *Poisson structure induced (topological) field theories*, hep-th/9405110

[13] F. Schätz, *Higher homotopy structures and formal deformations of coisotropic submanifolds*, unpublished draft, 2006

[14] A. Schwarz, *Geometry of Batalin-Vilkovisky quantization,* hep-th/9205088

[15] S. Sen, S. Sen, J. C. Sexton and D. H. Adams, *A geometric discretization scheme applied to the Abelian Chern-Simons theory*, hep-th/0001030

[16] V. G. Turaev, O. Yu. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology 31 (1992) 865-902

PDMI RAS, 27, FONTANKA, 191023, SAINT-PETERSBURG, RUSSIA

*E-mail address*: pmnev@pdmi.ras.ru