Multiplicative adeles

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1 Introduction

There exists a general ideology according to which many geometric notions and constructions may be translated into the language of different *adelic* groups related to a scheme over a field and some additional data, for instance a coherent sheave on the scheme, for more details see, for example, [24]. This article gives several new examples to this approach. Namely, we study so-called *multiplicative adeles*, which should be related with algebraic cycles on the scheme.

The first example are idèles on an algebraic curve. They map to the Picard group of the curve. The Weil pairing of two elements $\mathcal{L}, \mathcal{M}$ from the m-torsion of the Jacobian of an algebraic curve C equals to the product of m-th powers of local Hilbert symbols of two idèles $\alpha, \beta \in \mathbf{A}_C^*$, corresponding to $\mathcal{L}$ and $\mathcal{M}$ in such a way that α^m and β^m are principal idèles:

$$(\mathcal{L}, \mathcal{M})_m = (\prod_{x \in C} \mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)} \overline{\alpha_x^{\mathrm{ord}_x(\beta_x)} \beta_x^{-\mathrm{ord}_x(\alpha_x)}}])^m.$$

First, it was proven for arbitrary genus in [13]; a more elemetary proof was given later by M. Mazo in [17]. On the other hand Arbarello, de Concini and Kac have constructed some central extension of the group of idèles on an algebraic curve C,

$$0 \to k^* \to \widetilde{\mathbf{A}}_C^* \to \mathbf{A}_C^* \to 0,$$

in which the commutator is also equal up to sign to the product of all local Hilbert symbols of two idèles $\alpha, \beta \in \mathbf{A}_C^*$:

$$(\alpha, \beta) = (-1)^{\deg(\alpha)\deg(\beta)} \prod_{x \in C} \mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)} \overline{\alpha_x^{\mathrm{ord}_x(\beta_x)} \beta_x^{-\mathrm{ord}_x(\alpha_x)}}],$$

We give an intrinsic explanation for the apparent similarity of these two formulas. In turns out that there exists a close relation between the central extension from [1] and a Poincaré biextension over $\mathrm{Jac}(C) \times \mathrm{Jac}(C)$, which defines the Weil pairing. In fact, Poincaré biextension is a quotient of a (trivial) biextension over the square of the group of idèles of zero degree $(\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0$, associated to $\widetilde{\mathbf{A}}_C^*$.

Next, let us look at the higher-dimensional case. Let X be a Noetherian separable scheme of finite type over a field. One should change idèles, and the sheaf $\mathcal{O}_X^*$, both related with K_1-functor, by certain higher-dimensional adeles and sheaves of K-groups, related with higher K-functors. Recall that for sheaves of K-groups $\mathcal{K}_n(\mathcal{O}_X)$ on X there have been constructed Gersten resolution (see section 3.1 for more details). It allows relate cohomology of sheaves of K-groups, called K-cohomology, with the (algebraic) geometry of X. In particular, a famous Bloch–Quillen formula says that $H^n(X, \mathcal{K}_n(\mathcal{O}_X)) = CH^n(X)$, see [26]. Further, there is a canonical product between sheaves of K-groups, induced by the product in K-groups themselves. However, this product structure cannot be prolonged to Gersten resolution: otherwise there would exist an intersection theory for algebraic cycles without taking them modulo rational equivalence.

On the other hand, the general theory of sheaves provides many multiplicative resolutions of sheaves, i.e. resolutions carrying the product structure, for example Chech resolution or Godement resolution, see [11]. But these resolutions seem to be too general to reflect the algebro-geometric structure of a scheme, e.g. direct image maps for a good class of morphisms, while Gersten resolution does have these properties.

Our aim is to construct a certain multiplicative resolution for sheaves of K-groups, called *adelic resolution*. Recall that Gersten resolution consists of direct sums over shematic points of fixed codimension. The main idea is to replace them by certain restricted products over flags of fixed length, i.e. sequences of schematic points $\eta_0 \ldots \eta_p$ on X such that $\eta_i \in \overline{\eta}_{i-1}$ and $\eta_i \neq \eta_{i-1}$ for all $1 \leq i \leq p$. This involves a simplicial structure, which allows define products in adelic resolution.

Note that the adelic resolution constructed below also does not have direct images, as general topological resolutions mentioned above do not. However, covariant Gersten resolution turns out to be a left dg-module over the dg- algebra of the adelic complex. It seems that there is no analogous module structure of the Gersten resolution over other multiplicative resolutions (such as Chech or Godement).

Analysis of the adelic resolution provides some explicit formulas for products and also Massey higher products in K-cohomology. In particular, we obtain a new direct proof of the coincidence of the intersection product in Chow groups and the natural product in K-cohomology. Another example is the triple product $m_3(\alpha, l, \mathcal{L})$, which occurs to be related to Weil pairing of α and $\mathcal{L}$, where $\alpha \in CH^d(X)_l$, $\mathcal{L} \in \mathrm{Pic}^0(X)_l$.

The paper is organized as follows. First, in section 2.1 we give an abstract construction of a pairing, associated to central extensions, and corresponding to the formula from [1]. The case of the idèle group is considered from this point of view (theorem 2.8). Then, in section 2.2 a construction of a quotient biextension is given (proposition 2.12). As a quotient of the trivial biextension associated with Arbarello, de Concini and Kac central extension appears Poincaré biextension over the Jacobian of a curve (theorem 2.16). The relation between two abstract constructions is given in section 2.3 (proposition 2.18), which implies, in particular, the adelic formula for the Weil pairing (corollary 2.20).

Next, in section 3.1 after a short overview of Gersten complex we give a detailed construction of higher-dimensional adelic groups, providing some examples, and discuss their general properties (propositions 3.9, 3.12, corollary 3.11). Then, in section 3.2 we treat some functorial properties: contravariancy of the adelic complex (proposition 3.15), dg-module structure of Gersten complex over the adelic complex (propositions 3.16, 3.19), and also a comparison morphism to the additive de Rham adelic complex (proposition 3.22).

Section 4 contains the main result (theorem 4.1), which states that the adelic complex is indeed a resolution of sheaves of K-groups for smooth varieties over infinite fields. Some consequences of this statement are discussed (corollaries 4.3, 4.6) and the first steps of the proof are done. After several technical lemmas the proof of the main theorem is reduced to some approximation statement (lemma 4.11). Then, in section 4.2 we develop a notion of *strongly locally effaceable pairs*, which is a globalization of the method from Quillen's proof of Gersten conjecture in the geometric case. In particular, we get some sort of a uniform version of local exactness of Gersten resolution, which might have interest in its own right (corollary 4.21). Finally, in section 4.3 we give a proof of lemma 4.11 using the developed technique of strongly locally effaceable pairs.

Section 5.1 is devoted to the explicit construction of classes in the adelic complex representing cocyles in Gersten resolution. Section 5.2 provides a construction of the Euler characteristic map from K-groups of the exact category of complexes of coherent sheaves on a scheme T, which are exact outside of a closed subscheme $S \subset T$, to K'-groups of S. The existence of such map also follows from a general construction of Waldhausen K-theory of perfect complexes in [29], but we tried to use an easier and a more explicit approach (though the results of this section do not pretend to be new). Next, explicit formulas for the products in K-cohomology in terms of Gersten cocycles are obtain in section 5.3 as a consequence of the product structure in the adelic resolution (theorem 5.16).

The case of triple products is treated in sections 6.1 and 6.2. After the easiest case of a curve we show the coincidence of a certain triple product of zero-cycles and divisors with Weil pairing (theorem 6.3). Next, we find an explicit formula for arbitrary codimension cycles (proposition 6.7) and say some comments about its coincidence with the Weil pairing between Griffiths intermediate Jacobians in the complex field case, providing its interpretation as a triple product in Deligne cohomology (lemma 6.11). The definition of Massey higher products is given in section 6.3.

The last section 7 contains several open questions and remarks about possible further investigations.

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2 Idèles on a curve

2.1 Central extensions

First abstract construction: pairing, related to the central extension of abelian groups

In all groups considered below the group law is written in a multiplicative manner.

Consider a central extension of an abelian group A by a group N (which, automatically, is also abelian):

$$1 \to N \to G \xrightarrow{\pi} A \to 1.$$

Let 1 be a unit in the group N, and e be a unit in the group A. For each element $a \in A$ denote by G_a "a fiber" $\pi^{-1}(a)$. For any two elements $g_a \in G_a$ and $g_b \in G_b$ their *commutator* $[g_a, g_b] = g_a g_b g_a^{-1} g_b^{-1} \in N$ depends only on a and b, since N is a central subgroup inside G. Let the bracket (a, b) denote the commutator $[g_a, g_b]$.

Remark 2.1. Suppose that the biextension corresponds to the cocycle $\alpha \in H^2(A, N)$, and let $\bar{\alpha}$: $A \times A \to N$ be lifting of this cocycle (that is not uniquely defined). Then the following identity holds:

$$(a,b) = \frac{\bar{\alpha}(a,b)}{\bar{\alpha}(b,a)}. \tag{$*$}$$

Evidently, the bracket $(\cdot,\cdot)$ is a *skew symmetric* one. Now let us show that $(\cdot,\cdot)$ is *bilinear*. For any group there is a following equality:

$$[fg,h] = [f,h][g,h]\mathrm{Ad}(g^{-1}h)(f^{-1})\mathrm{Ad}(hg^{-1})(f),$$

where $\mathrm{Ad}(g)(\cdot)$ is an internal automorphism corresponding to the conjugation by g. In our case the commutator of any two elements is central, hence $\mathrm{Ad}(g^{-1}h) = \mathrm{Ad}(hg^{-1})$, and

$$[fg,h] = [f,h][g,h].$$

Thus

$$(ab,c) = (a,c)(b,c)$$

for all a, b, $c \in A$.

Now let B be a subgroup in A such that the extension

$$1 \to N \to \pi^{-1}(B) \xrightarrow{\pi} B \to 1$$

splits, i.e. there exists an isomorphism $\varphi : \pi^{-1}(B) \cong N \times B$ under which π corresponds to the projection on the second multiple. In particular, it follows that $(b_1, b_2) = 1$ for all $b_1, b_1 \in B$.

Let us fix an isomorphism φ. Then B becomes isomorphic to a certain subgroup in G, whose elements will be denoted by $(b,1)$, $b \in B$. Let us remark that the multiplication on the right by $(b,1)$ *indetifies the fibers* G_a and $G_{a'}$ for $a' = a \cdot b$. It allows to define the map

$$G/B \to A/B,$$

where G/B has not only a structure of a pointed set, but also has a structure of a principal homogenous space of a relative group $N \times A/B$ over A/B.

In our notations the following equality holds:

$$\mathrm{Ad}(g)((b,1)) = [g,(b,1)](b,1) = (\pi(g),b)(b,1) = (b,(\pi(g),b)).$$

Thus B is normal in G, i.e. G/B is a *group*, if and only if $(a,b) = 1$ for all $a \in A$, $b \in B$.

Remark 2.2. On the other hand if G/B is a group, then the bracket $(\cdot,\cdot)$ should be defined for the quotient A/B. This is possible only if B is inside the kernel of the bilinear pairing $(\cdot,\cdot)$.

Now let make this condition to be satisfied in a "violent" way. Choose a natural number $m \in \mathcal{N}$ and change the extension G by $G^{\otimes m}$, which corresponds to the cocycle $\alpha^m \in H^2(A,N)$. It follows from the formula $(*)$ that the new bracket is equal to the m-th power of an old one, i.e. is equal to $(\cdot,\cdot)^m$. Moreover, the extension $G^{\otimes m}$ may be also obtained by changing all the fibers of G by their m-th tensor power (in the category of principal homogenous spaces over N), and also by raising to the m-th power the morphisms of fibers, defining the multiplication (this construction explains the notation "$G^{\otimes m}$").

Let us also replace the group A by a subgroup $\sqrt[m]{B}$, consisting of all elements $a \in A$, such that $a^m \in B$; the restriction of the new extension on it will be denoted by $G^{\otimes m}|_{\sqrt[m]{B}}$.

For the new extension

$$1 \to N \to G^{\otimes m}|_{\sqrt[m]{B}} \to \sqrt[m]{B} \to 1$$

the subgroup B is normal in $G^{\otimes m}|_{\sqrt[m]{B}}$:

$$(a,b)^m = (a,b^m) = 1 \qquad \text{for all} \quad a \in \sqrt[m]{B},\ b \in B.$$

Moreover, it is possible to extend B up to the subgroup $B \cdot A_m$ by "roots from unity of m-th degree" in the group A. Indeed, there is a canonical trivialization of the extension $G^{\otimes m}$ over A_m: $G_a^{\otimes m} \cong G_{a^m} = G_e = N \ni 1$, which defines a homomorphism of groups $A_m \to G^{\otimes m}$ by the commutativity of A_m.

What is important is that in fact two trivializations over $B \cap A_m$, coming from B and A_m, do coincide:

$$G_e \cong G_e^{\otimes m} \cong G_a^{\otimes m}$$
$$1 \mapsto 1^{\otimes m} \mapsto (a,1)^{\otimes m},$$

while

$$G_a^{\otimes m} \cong G_{a^m} = G_e$$
$$(a,1)^{\otimes m} \mapsto (a^m,1) = 1.$$

Besides, the elements of type $(a,1)$ and $(b,1)$, where $a \in A_m$, $b \in B$, commute with each other, since $(a,b)^m = (a^m,b) = (1,b) = 1$. This makes it possible to define a splitting over the subgroup $B \cdot A_m$ in $\sqrt[m]{B}$.

Thus we get a central extension

$$1 \to N \to (G^{\otimes m}|_{\sqrt[m]{B}})/(B \cdot A_m) \to \sqrt[m]{B}/(B \cdot A_m) \to 1.$$

Its commutator takes values in N_m — the subgroup of m-th torsion in N:

$$(a_1,a_2)^{m^2} = (a_1^m, a_2^m) = 1 \qquad \text{for all} \quad a_1, a_2 \in A.$$

So, starting with a subgroup B, over which the initial cetral extension splits, we obtain a skew symmetric bilinear pairing

$$\psi_m \colon \sqrt[m]{B}/(B \cdot A_m) \times \sqrt[m]{B}/(B \cdot A_m) \to N_m\,,$$

wich arises as a commutator in the quotientextension over $\sqrt[m]{B}/B$.

Remark 2.3. We also could say nothing about extensions and get the pairing ψ_m directly, just considering the bracket $(\cdot,\cdot)$, and using its bilinearity and triviality over B.

Example with idèles

Consider a nonsingular projective curve C of genus g over a perfect field k (maybe not an algebraically closed one). Let $K = k(C)$ be the field of rational functions on C, let $\mathbf{A}_C$ be the ring of adèles on C, $\mathcal{O} = \prod_{x\in C} \hat{\mathcal{O}}_x \subset \mathbf{A}_C$, and let $\mathbf{A}_C^*$ denote the group of idèles, i.e. $\mathcal{O}^* = \prod_{x\in C} \hat{\mathcal{O}}_x^* \subset \mathbf{A}_C^*$.

Remark 2.4. There is a natural surjective homomorphism from the group $\mathbf{A}_C^*$ to $\mathrm{Pic}(C)$. The kernel of this homomorphism is equal to the subgroup $K^* \cdot \mathcal{O}^*$.

For any idèle $\alpha \in \mathbf{A}_C^*$ the subspace $\alpha\mathcal{O} \subset \mathbf{A}_C$ is *commensurable* with $\mathcal{O}$, i.e. there exists a k-subspace $L \subset \mathbf{A}_C$ such that $L \subset \mathcal{O}$, $L \subset \alpha\mathcal{O}$, and the quotients $\mathcal{O}/L$ and $\alpha\mathcal{O}/L$ are finite dimensional. Thus there is a well-defined one-dimensional vector space $(\mathcal{O}|\alpha\mathcal{O})$ over k, which is equal to $\det_k(\alpha\mathcal{O}/L) \otimes \det_k(\mathcal{O}/L)$ (it is easy to show that this space doesn't depend on the choice of L). For instance, if $\alpha\mathcal{O} \supset \mathcal{O}$, then $(\mathcal{O}|\alpha\mathcal{O}) = \det_k(\alpha\mathcal{O}/\mathcal{O})$.

For each subspace $L \subset \mathbf{A}_C$, which is commensurable with $\mathcal{O}$, we associate an *adelic complex*

$$\mathcal{A}(L)\colon 0 \to K \oplus L \to \mathbf{A}_C \to 0,$$

where the differential d is given by the formula $d(f, (a_x)) = (f - a_x)$ for $f \in K$, $(a_x) \in L \subset \mathbf{A}_C$. Denote by $\mathcal{K}(L) = \det_k H^0(\mathcal{A}(L)) \otimes \det_k^{-1} H^1(\mathcal{A}(L))$ the determinant of cohomology of this complex.

There is a canonical isomorphism

$$\mathcal{K}(\mathcal{O}) \otimes (\mathcal{O}|\alpha\mathcal{O}) \cong \mathcal{K}(\alpha\mathcal{O}).$$

To obtain this we should consider two morphisms of adelic complexes $\mathcal{A}(L) \to \mathcal{A}(\mathcal{O})$ and $\mathcal{A}(L) \to \mathcal{A}(\alpha\mathcal{O})$, where $L \subset \mathcal{O}$, $L \subset \alpha\mathcal{O}$ is a subspace as described above. In other words, $(\mathcal{O}|\alpha\mathcal{O}) \cong \mathrm{Hom}(\mathcal{K}(\mathcal{O}), \mathcal{K}(\alpha\mathcal{O}))$.

Now let us construct a group extension, for which $A = \mathbf{A}_C^*$, $N = k^*$, and $G = \widetilde{\mathbf{A}}_C^*$ consists of pairs (α, r), where $\alpha \in \mathbf{A}_C^*$, $r \in (\mathcal{O}|\alpha\mathcal{O})$, $r \neq 0$. The multiplication in $\widetilde{\mathbf{A}}_C^*$ is given by mean of natural isomorphisms

$$(\mathcal{O}|\alpha_2\mathcal{O}) \overset{\alpha_1}{\cong} (\alpha_1\mathcal{O}|\alpha_1\alpha_2\mathcal{O})$$

and

$$(\mathcal{O}|\alpha_1\mathcal{O}) \otimes (\alpha_1\mathcal{O}|\alpha_1\alpha_2\mathcal{O}) \cong (\mathcal{O}|\alpha_1\alpha_2\mathcal{O}).$$

This extension was constructed and studied in [1]. In particular, there was proved the following result:

Theorem 2.5. *For all $\alpha, \beta \in \mathbf{A}_C^*$ the commutator of their liftings to the group $\widetilde{\mathbf{A}}_C^*$ is equal up to sign to the product of norm residue symbols:*

$$(\alpha, \beta) = (-1)^{\deg(\alpha)\deg(\beta)} \prod_{x\in C} \mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)} \overline{\alpha_x^{\mathrm{ord}_x(\beta_x)} \beta_x^{-\mathrm{ord}_x(\alpha_x)}}],$$

where $\mathrm{ord}_x(\gamma_x)$ *is an order of the idèle* $\gamma \in \mathbf{A}_C^*$ *at the point* $x \in C$, $\deg(\gamma) = \sum_{x\in C} \mathrm{ord}_x(\gamma_x)[k(x):k]$ *is the degree of the idèle* γ, $\overline{\gamma_x}$ *stays for the value of the idèle* γ *at the point* $x \in C$ *in the case* $\mathrm{ord}_x(\gamma_x) \geq 0$, *and* $\mathrm{Nm}_{k(x)/k}$ *is a usual Galois norm from the residue field* $k(x)$ *of the point* x *to the ground field* k.

The role of the subgroup B will be played by the subgroup of principal idèles $K^* \subset \mathbf{A}_C^*$. Indeed, the extension splits over this group, because there is a canonical isomorphism between the complexes $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(f\mathcal{O})$, given by multiplication by f (what is important is that it gives a trivialization of $\widetilde{\mathbf{A}}_C^*$ over K^* as a *group extension*, for more details see 2.2).

Remark 2.6. The subgroup K^* *is not inside the kernel* of the bracket $(\cdot,\cdot)$, hence the extension $\widetilde{\mathbf{A}}_C^*$ may not be factorized through the idèle class group.

Remark 2.7. The extension $\widetilde{\mathbf{A}}_C^*$ *does not split over the subgroup* $K^*\cdot\mathcal{O}^*$, since the bracket $(\cdot,\cdot)$ is not trivial on it. On the other hand, the extension $\widetilde{\mathbf{A}}_C^*$ splits over the subgroup $\mathcal{O}^*$, because all the spaces $(\mathcal{O}|\alpha\mathcal{O})$ coincide *by definition* with k for $\alpha \in \mathcal{O}^*$ ($\mathcal{O} = \alpha\mathcal{O}$), and the multiplication, defined in $\widetilde{\mathbf{A}}_C^*$ over $\mathcal{O}^*$, coincides with the natural multiplication in the direct product $\mathcal{O}^* \times k^*$. Nevertheless, two trivializations over $\mathcal{O}^* \cap K^* = k^*$, coming from $\mathcal{O}^*$ and K^*, are not the same: one trivialization corresponds to the trivial homomorphism $k^* \to \{1\} \subset k^*$, while another one corresponds to the homomorphism $k^* \to k^*$ defined by raising to the $(1-g)$-th power, respectively (see 2.2). Thus over the intersection of $\mathcal{O}^*$ and K^* two trivialization do not “glow” together. This is one of the reasons for the extension not to split over $\mathcal{O}^* \cdot K^*$.

Section 2.1 implies that there exists a skew symmetric bilinear pairing

$$\psi_m\colon\ \sqrt[m]{K^*}/(K^*\cdot(\mathbf{A}_C^*)_m) \times \sqrt[m]{K^*}/(K^*\cdot(\mathbf{A}_C^*)_m) \to \mu_m(k),$$

where $\mu_m(k)$ denotes the group of roots of m-th power from unity in the field k.

Let us remark the following indentities:

$$\sqrt[m]{K^*}/(\sqrt[m]{K^*}\cap K^*\cdot\mathcal{O}^*) = \sqrt[m]{K^*}/(K^*\cdot(\sqrt[m]{\mathcal{O}^*\cap K^*})_{\mathcal{O}^*}) =$$
$$= \sqrt[m]{K^*}/(K^*\cdot\mathcal{O}^*_m) = \sqrt[m]{K^*}/(K^*\cdot(\mathbf{A}_C^*)_m),$$

where $(\sqrt[m]{\mathcal{O}^*\cap K^*})_{\mathcal{O}^*}$ is the root of m-th power from $\mathcal{O}^*\cap K^* = k^*$ in the group $\mathcal{O}^*$.

The image of $\sqrt[m]{K^*}$ in $\mathrm{Pic}(C)$ is contained inside $\mathrm{Jac}(C)_m$. Let us denote it by $\mathrm{Jac}(C)'_m$.

We have obtained the following statement:

Theorem 2.8. *Over* $\mathrm{Jac}(C)'_m$ *there exists a canonical* k^**-extension, whose commutator will define a skew symmetric bilinear pairing*

$$\psi_m\colon \mathrm{Jac}(C)'_m \times \mathrm{Jac}(C)'_m \to \mu_m(k),$$

defined by the formula

$$\psi_m(\mathcal{L},\mathcal{M}) = (\prod_{x\in C} \mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)}\overline{\alpha_x^{\mathrm{ord}_x(\beta_x)}\beta_x^{-\mathrm{ord}_x(\alpha_x)}}])^m,$$

where $\alpha, \beta \in \mathbf{A}_C^*$ *are two idèles, corresponding to* $\mathcal{L}$ *and* $\mathcal{M}$ *from* $\mathrm{Jac}(C)'_m$*, such that* $\alpha^m, \beta^m \in K^*$*.*

Remark 2.9. If k is algebraically closed and $(\mathrm{char} k, m) = 1$, then $\mathrm{Jac}(C)'_m = \mathrm{Jac}(C)_m$.

Remark 2.10. If $\mathrm{char} k | m$, then the paring ψ_m may occur to be a trivial one. For example it holds, if k is a finite field, and $m = [k : \mathcal{F}_p]$, where $p = \mathrm{char} k$.

2.2 Biextensions

Construction of a quotient biextension

Let us return to the case of an arbitrary central extension of an abelian group (using the same notations as in section 2.1).

We could associate with each principal homogenous space over the relative group $N \times A$ a principal homogenous space $\mathrm{Sym}^2 G$ over the relative group $N \times (A \times A)$ by the formula $\mathrm{Sym}^2 G = \mathrm{Hom}(p_1^* G \otimes p_2^* G, m^* G) \to A \times A$, where p_1, p_2 and m denote, respectively, projections on the first mulptiple, second multiple and multiplication in the group A, while Hom and $\otimes$ are taken by fibers in the category of principal homogenous spaces over N.

Suppose that $G \to A$ is a group extension. Then $\mathrm{Sym}^2 G$ has a canonical trivialization, corresponding to the multiplication morphism

$$G_a \otimes G_b \to G_{ab}\,, \qquad \text{where} \quad a, b \in A.$$

In particular, this trivialization endows $\mathrm{Sym}^2 G$ with a structure of a trivial biextension over $A \times A$. Let us denote by $1_{(a,b)}$ the value at (a, b) of a constant "unit" section $A \times A \to \mathrm{Sym}^2 G$, induced by the trivialization described above. The biextension $\mathrm{Sym}^2 G$ is *symmetric*, i.e. there exists a canonical isomorphism of biextensions $t\colon i^*(\mathrm{Sym}^2 G) \to \mathrm{Sym}^2 G$, where $i\colon A \times A \to A \times A$ is a transposition of multiples. By definition, $t(1_{(b,a)}) = (a, b)1_{(a,b)}$.

Let $B, C \subset A$ be two subgroups, such that the extension G splits over each of them. Denote by $(b, 1_B)$ and $(c, 1_C)$ the elements of corresponding sections over B and C for $b \in B$, $c \in C$. The reason of this notation is that these sections could *not coincide* over the intersection $B \cap C$.

Suppose that for all $g \in G$ and for all $b \in B \cap C$ there is an equality

$$(b, 1_B)g = g(b, 1_C). \qquad (**)$$

Then it is possible to define a quotient $\widetilde{G} = B\backslash G/C$, that will be a principal homogenous space over the relative group $N \times (B\backslash A/C)$.

It is easy to check that $\mathrm{Sym}^2 \widetilde{G}$ also has a structure of a biextension, and, in fact, is a quotient $\widetilde{\mathrm{Sym}^2 G}$ of a trivial biextension $\mathrm{Sym}^2 G \to A \times A$ under the induced action of the group $(B \cdot C) \times (B \cdot C)$.

Moreover, it is possible to construct the quotient biextension over $B\backslash A/C$ not only when B and C satisfy the condition $(**)$, but also when for each $b \in B \cap C$ there exists $\varphi(b) \in N$ such that

$$(b, 1_B)g = \varphi(b)g(b, 1_C). \qquad (* * *)$$

Remark 2.11. In this case the map $\varphi\colon B\cap C\to N$ is automatically a homomorphism of groups.

If $(***)$ is satisfied, then, taking for g the unity from G, we get that $(b,1_B)=\varphi(b)(b,1_C)$ for $b\in B\cap C$. Taking into account this equality together with condition $(***)$ we see, that $B\cap C$ is inside the kernel of the bracket $(\cdot,\cdot)$.

In this case there is a well-defined action of $(B\cdot C)\times(B\cdot C)$ on $\mathrm{Sym}^2 G$ as if there were an action of $B\cdot C$ on G. Namely, for all $g,h\in G$ we have an identity

$$(b,1_B)g(c,1_C)\cdot(b',1_B)h(c',1_C)=(bb',1_B)gh(cc',1_C)\cdot(c,b')(\pi(g),b')(c,\pi(h)).$$

This gives an action of $(B\cdot C)\times(B\cdot C)$ on $\mathrm{Sym}^2 G$ by the formula

$$(bc,b'c')\circ(1_{(\pi(g),\pi(h))})=(c,b')(\pi(g),b')(c,\pi(h))(1_{(\pi(g)bc,\pi(h)b'c')}).$$

In spite of the non-existence of a well-defined action of $B\cdot C$ on G, the action on $\mathrm{Sym}^2 G$ does not depend on the choice of decompositions bc and $b'c'$ into the product of elements from B and from C. The last statement follows from the fact that $B\cap C$ is inside the kernel of the bracket $(\cdot,\cdot)$.

Let us remark that $(B\cdot C)\times(B\cdot C)$ acts on $\mathrm{Sym}^2 G$ not only as on a principal homogenous space but also as on a *biextension*, i.e. the action commutes the maps

$$(\mathrm{Sym}^2 G)_{(a_1,b)}\otimes(\mathrm{Sym}^2 G)_{(a_2,b)}\to(\mathrm{Sym}^2 G)_{(a_1a_2,b)},$$

$$(\mathrm{Sym}^2 G)_{(a,b_1)}\otimes(\mathrm{Sym}^2 G)_{(a,b_2)}\to(\mathrm{Sym}^2 G)_{(a,b_1b_2)}$$

for all $a_1,a_2,a,b_1,b_2,b\in A$. Moreover, it preserves the structure of a symmetric biextension, i.e. commutes with isomorphisms

$$t\colon(\mathrm{Sym}^2 G)_{(b,a)}\to(\mathrm{Sym}^2 G)_{(a,b)}$$

for all $a,b\in A$. All these properties follow from the explicit formula for the action of $(B\cdot C)\times(B\cdot C)$ on $\mathrm{Sym}^2 G$, and from the formulas for the maps mentioned above in terms of the constant "unit" section $1_{(a,b)}$.

So, we have shown the following result:

Proposition 2.12. *If a central extension $G\to A$ splits over the subgroups $B,C\subset A$, and the subgroups B and C satisfy the condition $(***)$, then there exists a symmetric biextension $\widetilde{\mathrm{Sym}^2}G$ over $B\backslash A/C\times B\backslash A/C$, which is a quotient of a trivial symmetric biextension $\mathrm{Sym}^2 G\to A\times A$.*

Second abstract construction: Weil pairing in terms of biextensions

Now suppose we are given a biextension $P\to A\times A$ over an arbitrary abelian group A, such that the fiber of P at (e,e) is isomorphic to the abelian group N. Let P be symmetric, i.e. there is an isomorphism of biextensions $t\colon i^*P\to P$. There exists a canonical trivialization of the biextension $P^{\otimes m}$ over $A_m\times A$, obtained by mean of the canonical isomorphisms $P_{(a,b)}^{\otimes m} \cong P_{(a^m,b)} = P_{(e,b)} \cong P_{(e,e)}$. Consider a restriction on $A_m \times A_m$ of the section s, induced under this trivialization from the constant "unit" section of a trivial biextension. The function $\phi_m(a,b) = t(s(b,a))/s(a,b)$ is defined on $A_m \times A_m$, takes values in N_m, and defines a skew symmetric bilinear pairing on A_m with values in N_m that we will call a *Weil pairing.*

Remark 2.13. If we choose another constant section (not a "unit" one), induced by the trivialization of $P^{\otimes m}$ over $A_m \times A$ as described above, then the function ϕ_m will not change.

Remark 2.14. If we replace the trivialization of $P^{\otimes m}$ over $A_m \times A_m$ by another one, obtained by raising to the m-th power the second multiple, then the Weil pairing will not change as well.

Remark 2.15. Let $s_1(a,b)$ and $s_2(a,b)$ be two sections of $P^{\otimes m}$, obtained from the constant "unit" section of the trivial biextension respectively by two different trivializations described in remark 2.14, then $s_1(a,b)/s_2(a,b) = \phi_m(a,b)$ (sometimes this is the definition of the Weil pairing).

The last two statements follow from the equivalency of two (non commutative!) diagrams, arising from the symmetry isomorphism $t\colon i^*P \to P$: the diagram

$$\begin{array}{ccccc} P_{(a,b)}^{\otimes m} & \to & P_{(a^m,b)} & \to & P_{(e,e)} \\ \downarrow & & & & \downarrow \\ P_{(a,b)}^{\otimes m} & \to & P_{(a,b^m)} & \to & P_{(e,e)}, \end{array}$$

is isomorphic to the diagram

$$\begin{array}{ccccc} P_{(b,a)}^{\otimes m} & \to & P_{(b,a^m)} & \to & P_{(e,e)} \\ \downarrow & & & & \downarrow \\ P_{(b,a)}^{\otimes m} & \to & P_{(b^m,a)} & \to & P_{(e,e)}. \end{array}$$

Relation with the Poincaré line bundle and the usual Weil pairing

Now let us go back to the example with idèles on the curve C over the perfect field k.

Since $(\mathcal{O}|\alpha\mathcal{O})$ is canonically isomorphic to $\mathrm{Hom}(\mathcal{K}(\mathcal{O}), \mathcal{K}(\alpha\mathcal{O}))$, for idèles $\alpha \in \mathbf{A}_C^*$ of degree $g-1$ (where g is the genus of the curve C) there is a canonical isomorphism $\varepsilon_{(f,u)}\colon (\mathcal{O}|\alpha\mathcal{O}) \to (\mathcal{O}|f\alpha u\mathcal{O})$, where $f \in K^*$, $u \in \mathcal{O}^*$, which is obtained from the multiplication by f, defining an isomorphism of complexes $\mathcal{A}(\alpha\mathcal{O})$ and $\mathcal{A}(f\alpha u\mathcal{O})$, and inducing a canonical isomorphism of their cohomologies (the independency on the choice of the decomposition fu by an element from K^* and from $\mathcal{O}^*$ follows from the fact that $\deg(\alpha) = g-1$, because the multiplication of a complex $\mathcal{A}(\alpha\mathcal{O})$ by a constant $c \in K^* \cap \mathcal{O}^* = k^*$ leads to the multiplication of $\mathcal{K}(\alpha\mathcal{O})$ by $c^{\chi(\mathcal{A}(\alpha\mathcal{O}))} = c^{\deg(\alpha)+1-g}$ since Riemann–Roch theorem).

Thus, we get a principal homogenous space of a relative group $k^* \times \mathrm{Pic}^{(g-1)}(C)$ over $\mathrm{Pic}^{(g-1)}(C)$, whose fibers are equal to $\mathcal{K}(\alpha\mathcal{O}) = \det R\Gamma(X,\mathcal{L})\backslash\{0\}$, where the invertible sheaf $\mathcal{L}$ corresponds to the idèle α by the following rule: $\mathcal{L}$ is a subsheaf in a constant sheaf K^*, consisting of all functions such that they have no poles after dividing them by α in the idèle group. The last statement follows from the existence of a canonical isomorphism of complexes

$$\mathcal{A}(\alpha\mathcal{O}) \cong \mathbf{A}_C(\mathcal{L}),$$

where $\mathbf{A}_C(\mathcal{L})$ denotes an *adelic complex*, associated to the invertible sheaf $\mathcal{L}$:

$$\mathbf{A}_C(\mathcal{L}) : 0 \to \mathcal{L}_\eta \oplus \prod_{x\in C} \hat{\mathcal{L}}_x \to \prod_{x\in C}{}' \hat{\mathcal{L}}_x \otimes_{\hat{\mathcal{O}}_x} K_x \to 0,$$

where η is a generic point on C, x runs over all the closed points on C, $\prod'$ denotes an adelic product, and ˆ stays for the completion (for a more detailed and general definition see [24]).

It is a well-known fact that the line bundle $\det R\Gamma(X, \mathcal{L})$ corresponds to the theta-divisor (wich is correctly defined on $\mathrm{Pic}^{(g-1)}(C)$), see [20].

In order to apply the abstract constructions introduced above we have to interpret this "cohomological" identification of fibers of the extension $\widetilde{\mathbf{A}}^*_C$ in group terms. Consider a commutative diagram:

$$\begin{array}{ccc} \mathcal{K}(\mathcal{O}) & \overset{\cdot f}{\cong} & \mathcal{K}(f\mathcal{O}) \\ \downarrow x & & \downarrow f_* x \\ \mathcal{K}(\alpha\mathcal{O}) & \overset{\cdot f}{\cong} & \mathcal{K}(f\alpha\mathcal{O}), \end{array}$$

where x is an arbitrary element from $(\mathcal{O}|\alpha\mathcal{O})\backslash\{0\}$, and $f \in K^*$. Comparing the corresponding adelic complexes it is easy to see, that $f_* x$ is equal to the image of x under the isomorphism $f\colon (\mathcal{O}|\alpha\mathcal{O}) \to (f\mathcal{O}|f\alpha\mathcal{O})$. Thus the product $(f, 1_{K^*}) \cdot x$ is equal to the diagonal of this commutative diagram. On the other hand, a "cohomological" identification of $(\mathcal{O}|\alpha\mathcal{O})$ with $(\mathcal{O}|f\alpha\mathcal{O})$ also maps x to the diagonal of this diagram. So, a "cohomological" identification coincides with the multiplication on the left by $(f, 1_{K^*})$.

On the other hand, for $u \in \mathcal{O}^*$ there exists an identification of $(\mathcal{O}|\alpha\mathcal{O})$ with $(\mathcal{O}|u\alpha\mathcal{O})$, coming from the coincidence of the spaces $\alpha\mathcal{O}$ and $u\alpha\mathcal{O}$. This identification, in turn, corresponds to the multiplication on the right by $(u, 1_{\mathcal{O}^*})$.

Now we are ready to apply the constructions from 2.2, taking for A the group of idèles of zero degree $(\mathbf{A}^*_C)^0$, for B the subgroup of principal idèles K^*, and for C the subgroup $\mathcal{O}^*$ (the extension G stays the same, i.e. $\widetilde{\mathbf{A}}^*_C$, but restricted to $(\mathbf{A}^*_C)^0$). There is an equality

$$(c, 1_{K^*})\eta = \eta(c, 1_{\mathcal{O}^*}) \cdot c^{1-g}$$

for all $\eta \in (\mathcal{O}|\alpha\mathcal{O})$, $\alpha \in (\mathbf{A}^*_C)^0$, $c \in K^* \cap \mathcal{O}^* = k^*$. In other words, the subgroups K^* and $\mathcal{O}^*$ satisfy the condition $(***)$, and by proposition 2.12 we get a certain symmetric biextension P over $\mathrm{Jac}(C) \times \mathrm{Jac}(C)$. The main theorem of this paper states following:

Theorem 2.16. *The symmetric biextension P, constructed above, is isomorphic to the symmetric biextension, induced by Poincaré line bundle.*

Proof. Recall that the action of $(K^* \cdot \mathcal{O}^*) \times (K^* \cdot \mathcal{O}^*)$ on the trivial biextension over $(\mathbf{A}^*_C)^0 \times (\mathbf{A}^*_C)^0$ corresponds to the "cohomological" indentification $(\mathcal{O}|\alpha\mathcal{O}) \cong \mathcal{K}(\alpha\mathcal{O}) \otimes \mathcal{K}(\mathcal{O})^{-1} \cong \det R\Gamma(\mathcal{L}) \otimes \det R\Gamma(\mathcal{O}_C)^{-1}$, where an invertible sheaf $\mathcal{L}$ correspond to the idèle α as described above. Thus P is canonically isomorphic to the Poincaré bundle $\mathcal{P} \to \mathrm{Jac}(C)\times\mathrm{Jac}(C)$ (with a removed zero section) as a principal homogenous space over the relative group $k^* \times (\mathrm{Jac}(C) \times \mathrm{Jac}(C))$, because of the formula $\mathcal{P}|_{(\mathcal{L},\mathcal{M})} = \det R\Gamma(\mathcal{L} \otimes \mathcal{M}) \otimes \det R\Gamma(\mathcal{L})^{-1} \otimes \det R\Gamma(\mathcal{M})^{-1} \otimes \det R\Gamma(\mathcal{O}_C)$. Besides, a "symmetric" structure of the biextensions P and $\mathcal{P}$ is also the same (this follows directly from the definition).

So we have to verify that $\mathcal{P}$ and P have the same biextension structure. For this it is enough to show that after we apply the isomorphism of principal homogenous spaces $P \cong \mathcal{P}$ the pull-back under the map $(\mathbf{A}_C^*)^0 \to \mathrm{Jac}(C)$ of the induced biextension structure from $\mathcal{P}$ to P is the same as the trivial biextension structure on $\mathrm{Sym}^2\widetilde{\mathbf{A}}_C^*$, described in 2.2. We use an explicit form of the biextension structure on $\mathcal{P}$, explained by Deligne in [8]. Namely we consider a pull-back of $\mathcal{P}$ on $\mathrm{Jac}(C) \times \mathrm{Div}(C)^0$, wich will be denoted by the same letter $\mathcal{P}$ for simplicity. There exists a canonical isomorphism $\varphi\colon \mathcal{P}|_{(\mathcal{L},D)} \cong \otimes_{x\in C}\mathcal{L}|_x^{\otimes \mathrm{ord}_x(D)}$, where $\mathcal{L}|_x$ denotes a fiber at the point x of the bundle, corresponding to the invertible sheaf $\mathcal{L}$. Moreover, a natural structure of biextension on $\otimes_{x\in C}\mathcal{L}|_x^{\otimes \mathrm{ord}_x(D)}$ induces a biextension structure on $\mathcal{P}$. Note that the preimage $\mathbb{D}$ of this biextension to $(\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0$ is a trivial biextension with the trivialization given by elements

$$\prod_{x\in C} \overline{\alpha_x}^{\mathrm{ord}_x(D)} \in \otimes_{x\in C}(\alpha_x\mathcal{O}_x/\mathfrak{m}_x\alpha_x\mathcal{O}_x)^{\otimes \mathrm{ord}_x(D)} = \pi^*[\otimes_{x\in C}\mathcal{L}|_x^{\otimes \mathrm{ord}_x(D)}],$$

where $\pi\colon (\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0 \to \mathrm{Jac}(C) \times \mathrm{Div}(C)^0$ is a natural projection.

It turns out, that the pull-back of the isomorphism φ maps the trivialization on $\mathrm{Sym}^2\tilde{\mathbf{A}}_C^*$ to the trivialization on $\mathbb{D}$.

To see this recall the explicit form of the Deligne isomorphism φ. First, suppose that D is effective. Then the exact sequences of sheaves

$$0 \to \mathcal{L} \to \mathcal{L}(D) \to \mathcal{L}(D)|_D \to 0,$$

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_C(D)|_D \to 0$$

lead to the isomorphism

$$\mu\colon\ \det R\Gamma(\mathcal{L}(D))\otimes\det R\Gamma(\mathcal{L})^{-1}\otimes\det R\Gamma(\mathcal{O}_C(D))^{-1}\otimes\det R\Gamma(\mathcal{O}_C) \cong \det(\mathcal{L}(D)|_D)\otimes\det(\mathcal{O}_C(D)|_D)^{-1}.$$

Further, by induction on the degree of D we establish a canonical isomorphism

$$\nu\colon\ \det(\mathcal{L}(D)|_D) \otimes \det(\mathcal{O}_C(D)|_D)^{-1} \cong \otimes_{x\in C}\mathcal{L}|_x^{\otimes \mathrm{ord}_x(D)}.$$

Moreover, if $s_x \in \mathcal{L}_x$ is a set of local sections, giving isomorphisms of the stalks of sheaves $s_x\colon \mathcal{O}_{C,x} \cong \mathcal{L}_x$, then the determinant of the isomorphisms $\prod_{x\in|D|} s_x\colon \mathcal{O}_C(D)|_D \to \mathcal{L}(D)|_D$ will be mapped under ν to the product $\prod_{x\in C}\overline{s_x}^{\otimes \mathrm{ord}_x(D)}$, where $\overline{s_x} \in \mathcal{L}|_x$ denotes the value at the point x of a section $s_x \in \mathcal{L}_x$, and $|D|$ is a support of the divisor D.

The isomorphism φ is just a composition $\nu \circ \mu$.

Now consider the pull-back of all these constructions under π^*. The exact sequences of sheaves will correspond to the exact sequences of complexes

$$0 \to \mathcal{A}(\alpha\mathcal{O}) \to \mathcal{A}(\alpha\beta\mathcal{O}) \to (\alpha\mathcal{O}|\alpha\beta\mathcal{O}) \to 0,$$

$$0 \to \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\beta\mathcal{O}) \to (\mathcal{O}|\beta\mathcal{O}) \to 0,$$

where the idèle α corresponds to the sheaf $\mathcal{L}$, and the idèle β is such that $\mathrm{div}(\beta) = -D$. Moreover, the pull-back of the isomorphism μ corresponds to such an isomorphism

$$\pi^*(\mu) \colon (\mathcal{O}|\alpha\beta\mathcal{O}) \otimes (\mathcal{O}|\alpha\mathcal{O})^{-1} \otimes (\mathcal{O}|\beta\mathcal{O})^{-1} \cong (\alpha\mathcal{O}|\alpha\beta\mathcal{O}) \otimes (\mathcal{O}|\beta\mathcal{O})^{-1},$$

under which the natural trivializations of a right hand side part and a left hand side part are mapped one to another. Here the trivialization of a right hand side part is given by the determinant of the isomorphism $\alpha \colon (\mathcal{O}|\beta\mathcal{O}) \cong (\alpha\mathcal{O}|\alpha\beta\mathcal{O})$. The pull-back of the isomorphism ν

$$\pi^*(\nu) \colon (\alpha\mathcal{O}|\alpha\beta\mathcal{O}) \otimes (\mathcal{O}|\beta\mathcal{O})^{-1} \cong \mathbb{D},$$

does map this trivialization to the trivialization on D as described above, defining on it a biextension structure. Thus we have treated the case of an effective divisor D.

The case of a negative divisor $-E$, $E \geq 0$ may be treated in an analogous way, considering two exact sequences of sheaves

$$0 \to \mathcal{L}(-E) \to \mathcal{L} \to \mathcal{L}|_E \to 0,$$

$$0 \to \mathcal{O}_C(-E) \to \mathcal{O}_C \to \mathcal{O}_C|_E \to 0.$$

The pull-back of these constructions will be letter by letter the same as before (we should take the idèle β such that $\mathrm{div}(\beta) = E$).

The case of an arbitrary divisor may be reduced to these two cases by mean of the inclusions of sheaves $\mathcal{L}(-E) \subset \mathcal{L}$ and $\mathcal{L}(-E) \subset \mathcal{L}(D-E)$ (resp. $\mathcal{O}_C(-E) \subset \mathcal{O}$ and $\mathcal{O}_C(-E) \subset \mathcal{O}_C(D-E)$), whose pull-back under π^* will correspond to the choice of a common subspace in the commensurable spaces $\alpha\mathcal{O}$ and $\alpha\beta\mathcal{O}$ (resp. $\mathcal{O}$ and $\beta\mathcal{O}$) when defining their bracket $(\alpha\mathcal{O}|\alpha\beta\mathcal{O})$ (resp. $(\mathcal{O}|\beta\mathcal{O})$). In this case also the pull-back of these constructions will be letter by letter the same as in two previous cases. □

Corollary 2.17. *If we apply the construction of a quotient biextension from 2.2 to the restriction of the trivial symmetric biextension* $\mathrm{Sym}^2(\widetilde{\mathbf{A}}_C^*)$ *on* $A \times A = (\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0$ *and to the subgroups* $B = K^*$, $C = \mathcal{O}^*$, *and consider a pairing from 2.2 on* $\mathrm{Jac}(C)_m \times \mathrm{Jac}(C)_m$, *obtained from this quotient biextension, then we get a "classical" Weil pairing on the torsion of the Jacobian of the curve.*

Proof. It follows explicitly from the definition of the Weil pairing in terms of the Poincaré biextension over $\mathrm{Jac}(C) \times \mathrm{Jac}(C)$, see [20]. □

2.3 Coincidence of two pairings

Suppose we are given an extenxion $G \to A$, and two subgroups $B, C \subset A$ satisfying conditions of proposition 2.12.

In section 2.1 there was constructed a pairing $\psi_m \colon \sqrt[m]{B} \times \sqrt[m]{B} \to N_m$, while in section 2.2 there was constructed a pairing $\phi_m \colon (B\backslash A/C)_m \times (B\backslash A/C)_m \to N_m$.

Proposition 2.18. *There is an equality* $\psi_m = \phi_m \circ p$, *where* $p \colon \sqrt[m]{B} \to (B\backslash A/C)_m$ *is a natural projection.*

Proof. We use notations of section 2.2. In particular, we use the section s, which defines a trivialization of the biextension over the torsion points (see 2.2). Let us denote the restrictions of biextensions on the subgroups by the same letters as the initial biextensions. Consider a commutative diagram

$$\begin{array}{ccc} (\mathrm{Sym}^2 G)^{\otimes m} & \longrightarrow & \sqrt[m]{B} \times \sqrt[m]{B} \\ \downarrow p & & \downarrow p \\ \widetilde{(\mathrm{Sym}^2 G)^{\otimes m}} & \longrightarrow & (B\backslash A/C)_m \times (B\backslash A/C)_m. \end{array}$$

Evidently, $(\mathrm{Sym}^2 G)^{\otimes m} = \mathrm{Sym}^2(G^{\otimes m})$. If we prove that $p^*(s)$ equals to a constant section of a trivial biextension $\mathrm{Sym}^2(G^{\otimes m})$, then we get the proposition. Indeed, in this case for $a, b \in \sqrt[m]{B}$ we have

$$\phi_m(p(a), p(b)) = t(s(p(b), p(a)))/s(p(a), p(b)) =$$

$$= t(p^*(s)(b), p^*(s)(a))/(p^*(s)(a), p^*(s)(b)) = (a, b)^m.$$

Now consider one more diagram

$$\begin{array}{ccc} \mathrm{Sym}^2 G & \longrightarrow & B \times \sqrt[m]{B} \\ \downarrow p & & \downarrow p \\ \widetilde{\mathrm{Sym}^2 G} & \longrightarrow & \{1\} \times (B\backslash A/C)_m. \end{array}$$

Both biextensions are trivial: the one "from below" is trivial since one of the multiples is equal to $\{1\}$, while the one "from above" is trivial since it comes from a group extension. Denote the constant "unit" sections, corresponding to these trivializations, by s_{down} and s_{up} respectively. Note that by definition of the Weil pairing from 2.2 we have the equality $(m, 1)^*(s_{down}) = s$. Besides, $(m, 1)^*(s_{up})$ coincides with the canonical section of the trivial biextension $\mathrm{Sym}^2(G^{\otimes m}) \to \sqrt[m]{B} \times \sqrt[m]{B}$. So, we have to check that $p^*(s_{down}) = s_{up}$.

By general properties of biextensions it is clear that $p^*(s_{down})$ is equal to s_{up} over $\{1\} \times \sqrt[m]{B}$. Moreover, it follows from the explicit formula for the action of $(B\cdot C)\times(B\cdot C)$ on $\mathrm{Sym}^2 G$ (see 2.2), that $B \times (\sqrt[m]{B} \cap B \cdot C)$ acts identically on s_{up}. Thus we get the desired equality $p^*(s_{down}) = s_{up}$. $\square$

Remark 2.19. The last argument of the proof would not be true if the group $\sqrt[m]{B}$ were replaced by a bigger group $\sqrt[m]{B} \cdot \sqrt[m]{C}$. It is related to the fact that in general the pairing $(\cdot, \cdot)^m$ is not correctly defined everywhere on this group.

Combining the results of sections 2.1, 2.2 with proposition 2.18, we obtain the adelic formula for the Weil pairing:

Corollary 2.20. *If for two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ from $\mathrm{Jac}(C)_m$ there exist two idèles $\alpha, \beta \in \mathbf{A}_C^*$ such that $\alpha^m, \beta^m \in K^*$, then the Weil pairing $(\mathcal{L}, \mathcal{M})_m$ of sheaves $\mathcal{L}$ and $\mathcal{M}$ may be given by the formula*

$$(\mathcal{L}, \mathcal{M})_m = (\prod_{x \in C} \mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)} \overline{\alpha_x^{\mathrm{ord}_x(\beta_x)} \beta_x^{-\mathrm{ord}_x(\alpha_x)}}])^m.$$

If in addition the divisors $D = \mathrm{div}(f)$ and $E = \mathrm{div}(g)$ do not intersect, where $f = \alpha^m$, $g = \beta^m$, then $(\mathcal{L}, \mathcal{M})_m = f(E) \cdot g^{-1}(D)$. The equivalence of this definition of Weil pairing with usual one was first shown by Howe in [13]. The coincidence of this formula with the definition of Weil pairing via biextensions was also directly explained by Mazo in [17].

3 Generalities on K-adeles

3.1 Adelic groups

Gersten resolution

Let X be a Noetherian separable scheme of Krull dimension d. Denote by $X^{(p)}$ the set of all schematic points in X of codimension p, and let $k(\eta)$ stay for the residue field of a schematic point $\eta \in X$.

For an integer $n \geq 0$ consider *sheaves of K-groups* $\mathcal{K}_n(\mathcal{O}_X)$ and $\mathcal{K}'_n(\mathcal{O}_X)$ associated with the Zarisky presheaves given by $U \mapsto K_n(U)$ and $U \mapsto K'_n(U)$, respectively. Here $K_n(U)$ and $K'_n(U)$ are Quillen K-groups of the exact categories of locally free sheaves and coherent sheaves on U, respectively. Zarisky cohomology of sheaves of K-groups $\mathcal{K}_n(\mathcal{O}_X)$ are sometimes called *K-cohomology*.

For an integer $p \geq 0$ define a group $(\mathbf{G}_n^X)^p$ to be equal to the direct sum $\oplus_{\eta \in X^{(p)}} K_{n-p}(k(\eta))$, if $p \leq d = \mathrm{dim} X$, and zero otherwise. Define also a flasque sheaf $(\underline{\mathbf{G}}_n^X)^p$ by formula $(\underline{\mathbf{G}}_n^X)^p(U) = (\mathbf{G}_n^U)^p$. In [26] there was constructed a complex of sheaves

$$0 \to (\underline{\mathbf{G}}_n^X)^0 \to \ldots \to (\underline{\mathbf{G}}_n^X)^n \to 0,$$

whose morphisms consist of canonical maps of type

$$\nu_{Ff} \colon K_m(F) \to K_{m-1}(f),$$

where F is a quotient field of a one-dimensional local ring R (maybe not a regular one), and f is the residue field of the maximal ideal in R. We refer the last maps as *residue maps* of K-groups, and the described above complex as *Gersten complex*.

Remark 3.1. In particular, when R is a discrete valuation ring, the residue map coincides with the discrete valuation for $m = 1$, and with Hilbert symbol for $m = 2$. If F is an n-dimensional local field and f denotes its last residue field, then, taking the composition of maps mentioned above and restricting it on Milnor K-groups, we get two canonical maps $K_n^M(F) \to K_0(f) = \mathbb{Z}$ and $K_{m+1}^M(F) \to K_1(f) = f^*$, which coincide correspondingly with the additive and the multiplicative higher symbols from higher-dimensional local class field theory introduced by Parshin, see [25].

Remark 3.2. The fact that the square of the differential in Gersten complex equals zero is a kind of reciprocity law. For instance, after we pass to global sections of sheaves in Gersten complex and restrict the differential on Milnor K-groups it leads we get some of Parshin's higher reciprocity laws.

There is a natural morphism from $\mathcal{K}'_n(\mathcal{O}_X)$ (considered as a complex concentrated in the zero term) to Gersten complex $(\underline{\mathbf{G}}_n^X)^\bullet$. Quillen has proved in [26] that for a regular scheme X of finite type over a field this morphism defines a flasque resolution of the sheaf $\mathcal{K}'_n(\mathcal{O}_X) = \mathcal{K}_n(\mathcal{O}_X)$. In this case the complex of global sections of Gersten complex is often called *Gersten resolution*. In fact, Quillen has proved that Gersten complex is a resolution for regular semi-local schemes X of geometric type, i.e. arising from regular schemes of finite type over a field. From Quillen's result one also deduces that $H^n(X, \mathcal{K}_n(\mathcal{O}_X)) = CH^n(X)$.

An important feature of sheaves of K-groups is the product structure. It is natural to construct flasque resolutions of sheaves of K-groups carrying this structure and also having algebro-geometric nature. It can be easily shown that the Gersten resolution does not carry the product structure. Indeed, this would mean that for any two cycles in an arbitrary position there is a well-defined cycle of right codimension, which should be a candidate for their intersection.

Definition of adeles

Let X be a separable Noetherian scheme of Krull dimension d. Denote by $\mathcal{K}_n^X$ a zero cohomology sheaf of Gersten complex $(\underline{\mathbf{G}}_n^X)^\bullet$. In particular, when X is integral, $\mathcal{K}_n^X$ is a subsheaf in the constant sheaf $K_n(k(X))$ defined by the following condition:

$$\mathcal{K}_n^X(U) = \{f \in K_n(k(X)) : \nu_D(f) = 0 \text{ for any irreducible divisor } D \subset U\},$$

where we consider the residue map $\nu_D : K_n(k(X)) \to K_{n-1}(k(D))$. In what follows, we always suppose X to be *irreducible*. Although everything can be stated for reducible scheme as well, we do not need their consideration. However, it would be useful for us to consider (irreducible) singular schemes (for instance, in proposition 3.12), that is why we have introduced this type of sheaves. Recall that the natural morphism of sheaves $\mathcal{K}'_n(\mathcal{O}_X) \to \mathcal{K}_n^X$ is an isomorphism for regular scheme X of finite type over a field.

When X is normal then $(\mathcal{K}_n^X)_\eta = \cap_D K_n(\mathcal{O}_D)$, where D runs over all irreducible divisors in X containing η. It follows from regularity of all discrete valuation rings $\mathcal{O}_D$ and the exactness of Gersten resolution for these rings.

Let $i_U : U \hookrightarrow X$ be an open embedding. We call sheaves of kind $(i_U)_*(\mathcal{K}_n^U)$ *K-sheaves of finite type*. We will denote them by the same expression $\mathcal{K}_n^U$ on the whole scheme X. In fact such sheaves are uniquely defined by the set of irreducible divisors in the complement $X \backslash U$ (this easily follows from definition).

For a closed subset $Z \subset X$ and a point $\eta \in X$ let $Z(\eta)$ denote the set of irreducible divisors on X, which are contained in Z and pass through η (as a subset in the set of all irreducible divisors on X). For $f \in K_n(k(X))$ take $\mathrm{div}(f)$ to be equal to the set of irreducible divisors D on X such that $\nu_D(f) \neq 0$.

For any schematic point $j_\eta : \mathrm{Spec}\, k(\eta) \hookrightarrow X$ consider the stalk-sheaf $(j_\eta)_*((\mathcal{K}_n^U)_\eta)$, which we will also denoted just by $(\mathcal{K}_n^U)_\eta$. There is a *canonical filtration* on this sheaf by K-sheaves of finite type in a natural way, namely,

$$(\mathcal{K}_n^U)_\eta = \varinjlim \mathcal{K}_n^{V \cap U},$$

where the limit is taken over all open sets $V \subset X$, containing η. In other words,

$$(\mathcal{K}_n^U)_\eta = \varinjlim \mathcal{K}_n^W,$$

where the limit is taken over all open subsets $W \subset X$ such that $(X \backslash W)(\eta) \subseteq (X \backslash U)(\eta)$.

We use Beilinson's simplicial approach to higher-dimensional adeles, first introduced by Parshin, see [3], [24]. For a separable Noetherian scheme X of Krull dimension d let $P(X)$ be the set of all its schematic points, $\overline{\eta}$ be the closure of $\eta \in P(X)$, and let

$$S(X)_p = \{(\eta_0, \eta_1 \dots, \eta_p) : \eta_i \in P(X), \eta_i \in \overline{\eta}_{i-1}\}$$

For $M \subset S(X)_p$, $\eta \in P(X)$ denote by ${}_\eta M$ the following set:

$$\eta M = \{(\eta_1, \dots, \eta_p) \in S(X)_{p-1} : (\eta, \eta_1, \dots, \eta_p) \in M\}.$$

We define inductively *adelic groups* in the following way. Consider a subset $M \subset P(X) = S(X)_0$ and a K-sheaf of finite type $\mathcal{F}$. We put by definition

$$\mathbf{A}_M(\mathcal{F}) = \prod_{\eta \in M} \mathcal{F}_\eta.$$

For a subset $M \subset S(X)_p$ we put

$$\mathbf{A}_M(\mathcal{F}) = \prod_{\eta \in P(X)} \mathbf{A}_{{}_\eta M}(\mathcal{F}_\eta),$$

and

$$\mathbf{A}_M(\mathcal{F}_\eta) = \varinjlim \mathbf{A}_M(\mathcal{F}_i),$$

where $\mathcal{F}_\eta = \varinjlim \mathcal{F}_i$ is the canonical filtration by K-sheaves of finite type. We refer elements of adelic groups as *K-adeles.*

Remark 3.3. It is a natural question whether the defined above adelic groups change if one replaces a canonical filtration of stalk-sheaves with an arbitrary filtration by K-sheaves of finite type. The answer is not known to the author.

Our main object is the *adelic complex* $\mathbf{A}_X(\mathcal{K}_n^X)^\bullet$, whose differential will be defined later, and whose terms are defined to be

$$\mathbf{A}_X(\mathcal{K}_n^X)^p = \prod_{0 \le i_0 < \ldots < i_p \le d} \mathbf{A}_{i_0 \ldots i_p}(\mathcal{K}_n^X),$$

where the expression $i_0 \dots i_p$ in the index stays for the set of all flags $\eta_0 \dots \eta_p$ such that for any $0 \le j \le p$ we have $\mathrm{codim}(\eta_j) = i_j$. We will say that such flags *are of type* $(i_0, \dots, i_p)$. Also, for a set of increasing natural numbers $(i_0 \dots i_p)$ define the *depth* $l(i_0 \dots i_p)$ to be the maximal number l such that $(i_0 \dots i_l) = (0 \dots l)$, and put $l(i_0 \dots i_p) = -1$ if $i_0 > 0$.

Remark 3.4. The adelic groups do not change after we reduce the scheme X. Thus, in what follows, we may always suppose the scheme X to be reduced.

Examples 3.5. 1) The adelic group $\mathbf{A}_p(\mathcal{K}_n^X)$ equals to the direct product $\prod_{\eta\in X^{(p)}}(\mathcal{K}_n^X)_\eta$.

2) Suppose $\dim X = 1$. Then

$$\mathbf{A}_{01}(\mathcal{K}_n^X) = \prod_{x\in X}{}' K_n(k(X)),$$

where $\prod'$ means that the we take a subset inside the corresponding direct product consisting of all collections $\{f_{Xx}\}$ for which $f_{Xx}\in K_n(\mathcal{O}_{X,x})$ for almost all $x\in X$. This is a natural generalization of classical adeles.

3) Suppose $\dim X = 2$ and X is regular. Let us describe explicitly the arising adelic groups. The adelic group $\mathbf{A}_{01}(\mathcal{K}_n^X)\subset \prod\limits_{C\subset X} K_n(k(X))$ consists of all collection $\{f_{XC}\}$ for which $f_{XC}\in K_n(\mathcal{O}_{X,C})$ for almost all irreducible curves $C\in X$. The adelic group $\mathbf{A}_{12}(\mathcal{K}_n^X)\subset \prod\limits_{x\in C} K_n(\mathcal{O}_{X,C})$ consists of all collections $\{f_{Cx}\}$ for which $f_{Cx}\in K_n(\mathcal{O}_{X,x})$ for almost all points $x\in C$ for a fixed C. The adelic group $\mathbf{A}_{02}(\mathcal{K}_n^X)\subset \prod\limits_{x\subset X} K_n(k(X))$ consists of all collections $\{f_{Xx}\}$ for which there exists a divisor $D\subset X$ such that $f_{Xx}\in K_n(\mathcal{O}_{X,x}\backslash D)$ for all closed points $x\in X$. The adelic group $\mathbf{A}_{012}(\mathcal{K}_n^X)\subset \prod\limits_{x\in C\subset X} K_n(k(X))$ consists of all collections $\{f_{XCx}\}$ satisfying the following condition. There exists a divisor $D\subset X$ and for each irreducible curve $C\subset X$ there is a divisor D_C such that $D_C(C) = D(C)$, and $f_{XCx}\in K_n(\mathcal{O}_{X,x}\backslash D_C)$ for all flags $x\in C\subset X$.

Basic properties of adelic groups

We will use the following explicit description of adelic groups.

Proposition 3.6. *For any subset $M\subset S(X)_p$ there is identity*

$$\mathbf{A}_M(\mathcal{K}_n^X) = \bigcup_{\{D_{\eta_0\cdots\eta_k}\},\,0\le k<p}\ \prod_{\eta_0\cdots\eta_p}(\mathcal{K}_n^{X\backslash D_{\eta_0\cdots\eta_{p-1}}})_{\eta_p},$$

where $\{D_{\eta_0\ldots\eta_k}\}$ is a system of divisors on X parameterized by "left parts" of flags from M such that for any $0<k<p$

$$D_{\eta_0\ldots\eta_{k-1}}(\eta_k)\supseteq D_{\eta_0\ldots\eta_{k-1}\eta_k}(\eta_k) \qquad (*)$$

for any flag $\eta_0\ldots\eta_k$ that can be prolonged to a flag from M. In particular, $D_{\eta_0}(\eta_0)=\emptyset$.

Proof. This follows immediately from the definition. □

Remark 3.7. It is far from being true that in general $(\mathcal{K}_n^{X\backslash D})_\eta = K_n(\mathcal{O}_{X,\eta}\backslash D)$ for a any regular scheme X, a divisor $D\subset X$ and a schematic point $\eta\in X$. In fact, any element $f\in(\mathcal{K}_n^{X\backslash D})_\eta$ belongs to the group $K_n(\mathcal{O}_{X,\eta}\backslash(D\cup S))$, where S is a closed subset in X whose all components have codimension at least two in X (see the first part of section 4.2).

Claim 3.8. *Condition* $(*)$ *implies that*

$$D_{\eta_0\ldots\eta_k}(\eta_j) \supseteq D_{\eta_0\ldots\eta_k\ldots\eta_{k+l}}(\eta_j)$$

for any $0 \le j \le k+1$, $0 \le l \le p-k-1$.

Proof. It is enough to show this for $l=1$. Each irreducible divisor $D \in D_{\eta_0\ldots\eta_k,\eta_{k+1}}(\eta_j)$ contains η_{k+1} and thus belongs to $D_{\eta_0\ldots\eta_k}(\eta_{k+1})$. Moreover, since D contains η_j it also belongs to $D_{\eta_0\ldots\eta_k}(\eta_j)$. □

It would be useful to always suppose that the system of divisors $\{D_{\eta_0\ldots\eta_k}\}$ also contains all (finitely many) divisor components of singularities X_{sing}, since the local rings of all other divisors are regular.

Consider two sets of indices $S_1 = \{i_0,\ldots,i_p\}$ and $S_2 = \{j_0\ldots,j_q\}$ inside $\{0,1,\ldots,p\}$ such that $S_1 \subseteq S_2$. There is a canonical inclusion

$$\prod_{\eta_0\ldots\eta_p} (\mathcal{K}_n^X)_{\eta_0} \hookrightarrow \prod_{\xi_0\ldots\xi_q} (\mathcal{K}_n^X)_{\xi_0},$$

where the flags $\eta_0\ldots\eta_p$ and $\xi_0\ldots\xi_q$ are of type $(i_0,\ldots,i_p)$ and $(j_0,\ldots,j_q)$, correspondingly. Namely, for two flags $F_1 = \{\eta_0\ldots\eta_p\}$ and $F_2 = \{\xi_0\ldots\xi_q\}$ the corresponding component of this map is the canonical inclusion $(\mathcal{K}_n^X)_{\eta_0} \hookrightarrow (\mathcal{K}_n^X)_{\xi_0}$ if $F_1 \subseteq F_2$, and equals zero otherwise. In what follows we will always imply this inclusion when comparing adelic groups with different indices.

Proposition 3.9. *In the above notations the following equality holds true:*

$$\mathbf{A}_{i_0\ldots i_p}(\mathcal{K}_n^X) = \mathbf{A}_{j_0\ldots j_q}(\mathcal{K}_n^X) \cap \prod_{\eta_0\ldots\eta_p} (\mathcal{K}_n^X)_{\eta_0}.$$

Proof. First, consider an element $(f_{\eta_0\ldots\eta_p})$ from the right hand side and a corresponding system of divisors $\{E_{\xi_0\ldots\xi_l}\}$ for $0 \le l \le q$. For each flag $\xi_0\ldots\xi_q$ containing a flag $\eta_0\ldots\eta_p$ we have

$$f_{\eta_0\ldots\eta_p} \in (\mathcal{K}_n^{X\setminus E_{\xi_0\ldots\xi_{q-1}}})_{\xi_q} \subseteq (\mathcal{K}_n^{X\setminus E_{\xi_0\ldots\xi_{l-1}}})_{\xi_l},$$

where $\xi_l = \eta_p$, since $E_{\xi_0\ldots\xi_{l-1}}(\xi_l) \supseteq E_{\xi_0\ldots\xi_{q-1}}(\xi_l)$. Define a system of divisors

$$D_{\eta_0\ldots\eta_k} = \bigcap_{\xi_0\ldots\xi_l} E_{\xi_0\ldots\xi_l},$$

where the intersection is taken over all flags $\xi_0\ldots\xi_l$ containing a given flag $\eta_0\ldots\eta_k$, such that $\xi_l = \eta_k$. We claim that for the system of divisors $\{D_{\eta_0\ldots\eta_k}\}$ the condition $(*)$ is satisfied, and $f_{\eta_0\ldots\eta_p} \in (\mathcal{K}_n^{X\setminus D_{\eta_0\ldots\eta_{p-1}}})_{\eta_p}$ for each flag $\eta_0\ldots\eta_p$. These two statements are implied by the same argument. Namely, using notations as above, consider an irreducible divisor

$$D \in \bigcap_{\xi_0\ldots\xi_{l-1}} E_{\xi_0\ldots\xi_{l-1}}(\xi_l)$$

(recall that $\xi_l = \eta_k$). Suppose $D \notin D_{\eta_0 \dots \eta_{k-1}}(\eta_k)$. In particular, $D \notin D_{\eta_0 \dots \eta_{k-1}}(\eta_{k-1})$, and also $\eta_{k-1} \neq \xi_{l-1}$. Thus we can construct a flag $\xi_0 \dots \xi_{l-1}$ such that it contains one of the irreducible components of $\overline{\eta}_{k-1} \cap D$. Moreover, after we choose suitable $\xi_0 \dots \xi_{l'}$, we get that $D_{\xi_0 \dots \xi_{l-1}}(\xi_{l'})$ is not a subset inside $D_{\xi_0 \dots \xi_{l'}}(\xi_{l'})$, where $\xi_{l'} = \eta_{k-1}$. So we are lead to contradiction, hence $D \in D_{\eta_0 \dots \eta_{k-1}}(\eta_k)$, and this ends the proof of the inclusion of the right hand side part into the left hand side part from the statement of proposition 3.9.

Now suppose $(f_{\eta_0 \dots \eta_p})$ belongs to the left hand side part and $\{D_{\eta_0 \dots \eta_k}\}$ is the corresponding system of divisors. For a given flag $\xi_0 \dots \xi_l$ let $\eta_0 \dots \eta_k$ be a maximal subflag of type $(i_0 \dots i_k)$. Then if $k < p$ define

$$E_{\xi_0 \dots \xi_l} = D_{\eta_0 \dots \eta_k},$$

while for $k = p$ we put

$$E_{\xi_0 \dots \xi_l} = D_{\eta_0 \dots \eta_p} \cup \mathrm{div}(f_{\eta_0 \dots \eta_p}).$$

It is a straightforward checking that the system $\{E_{\xi_0 \dots \xi_l}\}$ verifies condition $(*)$, and that $f_{\eta_0 \dots \eta_p} \in (\mathcal{K}_n^{X \setminus E_{\xi_0 \dots \xi_{q-1}}})_{\xi_q}$ for any flag $\xi_0 \dots \xi_q$. □

Now we can define the differential in the complex $\mathbf{A}_X(\mathcal{K}_n^X)^\bullet$. In evident notations the differential is given by a usual simplicial formula, which is correct due to proposition 3.9:

$$(df)_{\eta_0 \dots \eta_p} = \sum_{k=0}^{p} (-1)^k f_{\eta_0 \dots \hat{\eta}_k \dots \eta_p}$$

for $f \in \mathbf{A}_X(\mathcal{K}_n^X)^{p-1}$, where the hat stays for the forgetting an element in the flag. As usual, this formula defines a differential, i.e. $d^2 = 0$.

Proposition 3.10. *For any subset $M \subset S(X)_p$ the group $\oplus_{n \geq 0} \mathbf{A}_M(\mathcal{K}_n^X)$ carries a structure of a graded associative anticommutative ring, where the product is defined by formula*

$$(f \cdot g)_{\eta_0 \dots \eta_p} = \{f_{\eta_0 \dots \eta_p}, g_{\eta_0 \dots \eta_p}\}$$

for $f \in \mathbf{A}_M(\mathcal{K}_m^X)$, $g \in \mathbf{A}_M(\mathcal{K}_n^X)$ where $\{\cdot, \cdot\}$ denotes usual product in K-groups.

Proof. Let $\{D_{\eta_0 \dots \eta_k}\}$ and $\{E_{\eta_0 \dots \eta_k}\}$ be two systems of divisors, corresponding to f and g, respectively. Suppose also that they contain all divisor components of X_{sing}. Then, obviously, the union of two systems $\{F_{\eta_0 \dots \eta_k}\} = \{D_{\eta_0 \dots \eta_k} \cup E_{\eta_0 \dots \eta_k}\}$ verifies condition **??** and $(f \cdot g)_{\eta_0 \dots \eta_m} \in (\mathcal{K}_{m+n}^{X \setminus F_{\eta_0 \dots \eta_{p-1}}})_{\eta_p}$. It follows from the fact that $\nu_D(f) = 0$ if and only if $f \in K_n(\mathcal{O}_D)$ when D is outside of $F_{\eta_0 \dots \eta_{p-1}}$, and the existence of the product on K-groups $K_n(\mathcal{O}_D)$. □

In what follows we define the differential in the tensor product $A^\bullet \otimes B^\bullet$ of two complexes $A^\bullet$ and $B^\bullet$ by formula $d(a \otimes b) = da \otimes b + (-1)^{\deg a} a \otimes db$.

Corollary 3.11. *The group $\bigoplus\limits_{0 \leq p \leq d} \bigoplus\limits_{n \geq 0} \mathbf{A}_X(\mathcal{K}_n^X)^p$ has a natural structure of an associative dg-algebra, i.e. there are morphisms of complexes*

$$\mu : \mathbf{A}_X(\mathcal{K}_m^X)^\bullet \otimes \mathbf{A}_X(\mathcal{K}_n^X)^\bullet \longrightarrow \mathbf{A}_X(\mathcal{K}_{m+n}^X)^\bullet,$$

such that this multiplication is associative. Explicitly, the product is given by formula

$$(\mu(f\otimes g))_{\eta_0\ldots\eta_p\ldots\eta_{p+q}}=\{f_{\eta_0\ldots\eta_p},g_{\eta_p\ldots\eta_{p+q}}\}$$

for $f\in\mathbf{A}_X(\mathcal{K}_m^X)^p$, $g\in\mathbf{A}_X(\mathcal{K}_n^X)^q$.

Proof. The correctness of this product follows from propositions 3.9 and 3.10. Indeed, $\mu(f\otimes g)=f\cdot g$, where we apply to f and g two natural inclusions $\mathbf{A}_X(\mathcal{K}_m^X)^p\hookrightarrow\mathbf{A}_X(\mathcal{K}_m^X)^{p+q}$ and $\mathbf{A}_X(\mathcal{K}_n^X)^q\hookrightarrow\mathbf{A}_X(\mathcal{K}_n^X)^{p+q}$, respectively. Leibnitz rule for μ can be easily checked using the definition of differential in the adelic complex, and the associativity follows from that of the usual product in K-theory. □

Proposition 3.12. *For any* $1\le k\le d$, $k<i_{k+1}<\ldots<i_p\le d$ *there exists a well-defined map*

$$\nu_{0\ldots k}=\sum_{\eta_0\ldots\eta_k}\nu_{\eta_0\ldots\eta_k}\colon\mathbf{A}_{01\ldots ki_{k+1}\ldots i_p}(\mathcal{K}_n^X)\to\oplus_{\eta_k}\mathbf{A}_{0(i_{k+1}-k)\ldots(i_p-k)}(\mathcal{K}_{n-k}^{\overline{\eta}_k})$$

(the right hand side part is the sum of the adelic product groups on each $\overline{\eta}_k$*), i.e. the sum of residue maps over flags is finite for each element in* $\mathbf{A}_{01\ldots ki_{k+1}\ldots i_p}(\mathcal{K}_n^X)$.

Proof. We proceed by induction on k. The base and the induction step are provided by the following argument.

For any element $f_{\eta_0\ldots\eta_p}\in\mathbf{A}_{01\ldots ki_{k+1}\ldots i_p}(\mathcal{K}_n^X)$ the residue $\nu_{\eta_0\eta_1}(f_{\eta_0\ldots\eta_p})$ is zero for any divisor η_1 outside of D_{η_0}, because by claim 5.3 we have $D_{\eta_0\ldots\eta_p}(\eta_1)\subseteq D_{\eta_0}(\eta_1)$.

Now choose $\eta_1\in D_{\eta_0}$. For any flag $\xi_0\ldots\xi_l$ of type $(0,i_2-1,\ldots,i_p-1)$ on the scheme $Y=\overline{\eta}_1$ take the divisor $E_{\xi_0\ldots\xi_k}$ to be the intersection $Y\cap(D_{\eta_0\xi_0\ldots\xi_k}\backslash D_{\eta_0\xi_0\ldots\xi_k}(Y))$. The condition $(*)$ is evidently satisfied for this family of divisors on the scheme Y, and moreover $\nu_{\eta_0\eta_1}(f_{\eta_0\ldots\eta_p})\in\mathbf{A}_{0(i_2-1)\ldots(i_p-1)}(\mathcal{K}_{n-1}^{Y\backslash E_{\eta_1\ldots\eta_p}})$. Indeed, for any irreducible divisor $E\subset Y\backslash E_{\eta_1\ldots\eta_p}$ the only one divisor in X from $D_{\eta_0\eta_1\ldots\eta_p}$, that could contain E, is $Y=\overline{\eta}_1$. On the other hand by reciprocity law (which holds true for singular schemes as well)

$$\sum_{E\subset D\subset X}\nu_{XDE}(f_{\eta_0\eta_1\ldots\eta_p})=0.$$

Hence $\nu_{\eta_0\eta_1E}(f_{\eta_0\eta_1\ldots\eta_p})=0$ and we get the desired statement. □

Example 3.13. Take $i_{k+1}=\ldots=i_p=0$. Then we get a map

$$\nu_k=\sum_{\eta_0\ldots\eta_k}\nu_{\eta_0\ldots\eta_k}\colon\mathbf{A}_{01\ldots k}(\mathcal{K}_n^X)\to\oplus_{\eta\in X^{(k)}}K_{n-k}(k(\eta_k)).$$

Remark 3.14. There are variants of propositions 3.9, 3.10 and 3.12 for arbitrary K-sheaves of finite type, not just $\mathcal{K}_n^X$. The generalization of proposition 3.9 is straightforward. For proposition 3.10 we should consider product of adelic groups, associated with sheaves $\mathcal{K}_m^U$ and $\mathcal{K}_n^V$, with values in the adelic group, associated with $\mathcal{K}_{m+n}^{U\cap V}$. For proposition 3.12 the sum of residues from the adelic group, associated with $\mathcal{K}_n^U$, will take values in the sum of the adelic groups, associated with $\mathcal{K}_n^{U_{\eta_k}}$, where U_{η_k} is defined as follows: for each η_k take the union Z of all divisors from $X\backslash U$, which do not contain $\overline{\eta}_k$ (i.e. not from $(X\backslash U)(\overline{\eta}_k)$), let $V=X\backslash Z$, and then take $U_{\eta_k}=\overline{\eta}_k\cap V$.

3.2 Functorial properties of the adelic complex

Consider flasque sheaves associated to the adelic groups by a standard procedure. Namely, consider the sheaf $\underline{\mathbf{A}}_X(\mathcal{K}_n^X)^p$ defined by $\underline{\mathbf{A}}_X(\mathcal{K}_n^X)^p(U) = \mathbf{A}_U(\mathcal{K}_n^U)^p$ for any open $U \subset X$.

Contravariancy

Let $\varphi : X \to Y$ be a morphism of Noetherian separable schemes of finite Krull dimension such that Y is regular. There is a canonical map $\varphi^* : \mathcal{K}_n^Y = \mathcal{K}_n(\mathcal{O}_Y) \to \varphi_*(\mathcal{K}_n(\mathcal{O}_X)) \to \varphi_*(\mathcal{K}_n^X) = \varphi_*(\mathcal{K}_n(\mathcal{O}_X))$, and hence $\varphi^* : (\mathcal{K}_n^Y)_y \to (\mathcal{K}_n^X)_x$ for $x \in X$, $y \in Y$, $\varphi(x) = y$. Let us define a morphism of complexes

$$\varphi^* : \underline{\mathbf{A}}_Y(\mathcal{K}_n^Y)^\bullet \to \varphi_* \underline{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet,$$

giving explicitly each flagwise component of it over each open subset $U \subset Y$. Let $\eta_0 \ldots \eta_p$ be a flag on $\varphi^{-1}(U)$. If $\varphi(\eta_0) \ldots \varphi(\eta_p)$ is still a flag on Y, i.e. if $\varphi(\eta_i) \neq \varphi(\eta_{i+1})$ for all $0 \le i \le p-1$, then define the corresponding piece of the morphism of complexes just as $\varphi^* : (\mathcal{K}_n^Y)_{\varphi(\eta_0)} \to (\mathcal{K}_n^X)_{\eta_0}$. Put φ^* to be zero for all other pairs of flags on X and Y.

Proposition 3.15. *The defined above map*

$$\varphi^* : \underline{\mathbf{A}}_Y(\mathcal{K}_n^Y)^\bullet \to \varphi_* \underline{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet$$

is indeed a morphism of adelic complexes. Moreover, φ^* is the homomotphism of dg-algebras of adeles.

Proof. First, we should check that for all open subsets $U \subset Y$, $V = f^{-1}(U) \subset X$ the map $\varphi^* : \mathbf{A}_U(\mathcal{K}_n^U)^p \to \mathbf{A}_V(\mathcal{K}_n^V)^p$ is well-defined, i.e. that the adelic condition holds for $\varphi^*(f)$ for all $f \in \mathbf{A}_U(\mathcal{K}_n^U)^p$. Let $D_{\eta_0 \ldots \eta_k}$, $0 \le k \le p-1$, be a system of divisors associated to f. Let $E_{\xi_0 \ldots \xi_k}$ be the set of divisors in the closed subset $\varphi^{-1}(D_{\eta_0 \ldots \eta_k})$ if $\varphi(\xi_0 \ldots \xi_p) = (\eta_0 \ldots \eta_p)$ is a flag on Y, and let $E_{\xi_0 \ldots \xi_k} = \emptyset$ otherwise. Note that $\varphi^{-1}(D_{\eta_0 \ldots \eta_k})$ is never the whole scheme X since $D_{\eta_0 \ldots \eta_k}(\eta_0) = \emptyset$, and $\varphi(\xi_0) = \eta_0$ (we assume that $E_{\xi_0 \ldots \xi_k}$ is non-empty). Also the condition $E_{\xi_0 \ldots \xi_k}(\xi_{k+1}) \supset E_{\xi_0 \ldots \xi_{k+1}}(\xi_{k+1})$ is satisfied for $0 \le k \le p-1$. Finally, for each pair $\varphi(\xi_0 \ldots \xi_p) = \eta_0 \ldots \eta_p$ there is a well-defined map $\varphi^* : \mathcal{K}_n^{U'} \to \varphi_*(\mathcal{K}_n^{V'})$, where $U' = Y \backslash D_{\eta_0 \ldots \eta_{p-1}}$, $V' = Y \backslash E_{\xi_0 \ldots \xi_{p-1}}$. Hence $\varphi^*(f)$ satisfies the adelic condition with respect to the system of divisors $E_{\xi_0 \ldots \xi_k}$.

Now choose a flag $F = (\eta_0 \ldots \eta_{p+1})$ on $f^{-1}(U)$, and suppose first that $\varphi(F)$ is still a flag on U. Then, evidently, the image $\varphi(F')$ of any subflag F' inside F is also a flag on U. Hence just by coincidence of all ingredients inside the defining formulas we see that $d(\varphi^*(f))_F = \varphi^*(df)_F$. Now suppose that $\varphi(F)$ is not a flag on U, thus $\varphi^*(df)_F = 0$. If the same holds for all its subflags F' consisting of $p+1$ elements, then we still have the needed equality, since both sides are zero. Finally, suppose there is a subflag F' with $p+1$ elements such that $\varphi(F')$ is also a flag. Let F' correspond to the number l, $0 \le l \le p+1$, i.e. $F' = (\eta_0 \ldots \eta_{l-1} \eta_{l+1} \ldots \eta_{p+1})$. Then we see that $\varphi(\eta_l) = \varphi(\eta_{l-1})$ or $\varphi(\eta_l) = \varphi(\eta_{l+1})$. In both cases we see that for another subflag F'' of different "parity" we have $\varphi(F'') = \varphi(F')$ and for all other subflags there images are not flags. Explicitly, $F'' = (\eta_0 \ldots \eta_{l-2} \eta_l \ldots \eta_{p+1})$ or $F'' = (\eta_0 \ldots \eta_l \eta_{l+2} \ldots \eta_{p+1})$, respectively. Hence $d\varphi^*(f)_F$ is also zero.

The fact that φ^* commutes with products of K-adeles is a straightforward check. □

Gersten complex as a module over adelic complex

Let X be a regular Noetherian separable scheme of finite Krull dimension. As before, let $(\mathbf{G}_n^X)^\bullet$ be Gersten complex and $\mathbf{A}_X(\mathcal{K}_n^X)^\bullet$ be the adelic complex.

Proposition 3.16. *There is a natural morphism of complexes*

$$(\mathbf{G}_m^X)^\bullet \otimes \mathbf{A}_X(\mathcal{K}_n^X)^\bullet \to (\mathbf{G}_{m+n}^X)^\bullet,$$

which makes the sum of Gersten complexes $\oplus_{m\geq 0}(\mathbf{G}_m^X)^\bullet$ to be a left dg-module over the dg-algebra $\oplus_{n\geq 0}\mathbf{A}_X(\mathcal{K}_n^X)^\bullet$.

Proof. Let us construct the product explicitly. Consider two elements $f \in (\mathbf{G}_m^X)^p$ and $g \in \mathbf{A}_X(\mathcal{K}_n^X)^q$. We put

$$(f\cdot g)_{\eta_{p+q}} = \sum_{\eta_p\ldots\eta_{p+q}} \nu_{\eta_p\ldots\eta_{p+q}}\{f_{\eta_p}, \overline{g}_{\eta_p\ldots\eta_{p+q}}\},$$

where $\mathrm{codim}(\eta_k) = k$ and the bar denotes taking modulo the ideal of η_p, i.e. the image under the natural homomorphism $(\mathcal{K}_n^X)_{\eta_p} = K_n(\mathcal{O}_{X,\eta}) \to K_n(k(\eta_p))$. It can be easily checked that this really defines a structure of a dg-module over the dg-algebra $\oplus_{n\geq 0}\mathbf{A}_X(\mathcal{K}_n^X)^\bullet$. □

Example 3.17. Multiplication $1 \in K_0(k(X)) = \mathbb{Z}$ by the adelic complex coincides with the morphism ν_X.

Remark 3.18. As before we could consider the complexes of flasque sheaves $(\underline{\mathbf{G}}_n^X)^\bullet$ and $\underline{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet$. The analogous reasoning as in the proof of lemma 6.13 leads to the fact that the constructed above product induces the natural product and higher products (see Appendix) on K-cohomology, when X is of finite type over a field.

The main property of this module structure is the product formula. Namely, consider a proper morphism $\varphi : X \to Y$ of regular irreducible schemes of finite type over a field. Let $d = \dim(\varphi) = \dim(X) - \dim(Y)$. Then there are two natural morphisms of complexes:

$$\varphi_* : (\mathbf{G}_n^X)^\bullet[d] \to (\mathbf{G}_{n-d}^Y)^\bullet,$$

and

$$\varphi^* : \mathbf{A}_Y(\mathcal{K}_n^Y)^\bullet \to \mathbf{A}_X(\mathcal{K}_n^X)^\bullet.$$

Proposition 3.19. *The morphism φ_* is a morphism of dg-modules over adelic dg-algebras, i.e. the product formula holds:*

$$\varphi_*(f\cdot\varphi^*(g)) = \varphi_*(f)\cdot g$$

for all $f \in (\mathbf{G}_m^X)^\bullet$, $g \in \mathbf{A}_Y(\mathcal{K}_n^Y)^\bullet$.

Proof. Let $f \in (\mathbf{G}_m^X)^{p+d}$, $g \in \mathbf{A}_Y(\mathcal{K}_n^Y)^q$, and consider a point $\eta_{p+q} \in Y^{(p+q)}$. By definition the following equality holds:

$$(\varphi_*(f) \cdot g)_{\eta_{p+q}} = \sum_{\xi_p, \eta_p, \ldots, \eta_{p+q}} \nu_{\eta_p \ldots \eta_{p+q}} \{\varphi_*(f_{\xi_p}), g_{\eta_p \ldots \eta_{p+q}}\},$$

where for each $p \leq l \leq p+q$ we have $\eta_l \in Y^{(l)}$, $\xi_p \in X^{(p+d)}$, and $\varphi(\xi_p) = \eta_p$. On the other hand,

$$\varphi_*(f \cdot \varphi^*(g))_{\eta_{p+q}} = \varphi_* \left(\sum_{\xi_p \ldots \xi_{p+q}} \nu_{\xi_p \ldots \xi_{p+q}} \{f_{\xi_p}, \overline{\varphi^*(g_{\eta_p \ldots \eta_{p+q}})}\} \right),$$

where the sum is taken over all flags $\xi_p \ldots \xi_{p+q}$ such that $\varphi(\xi_p \ldots \xi_{p+q}) = (\eta_p \ldots \eta_{p+q})$ is a flag on Y, and as before $\xi_l \in X^{(l+d)}$, $\eta_l \in Y^{(l)}$ for all $p \leq l \leq p+q$. Note that

$$\overline{\varphi^*(g_{\eta_p \ldots \eta_{p+q}})} = \varphi^* \overline{g}_{\eta_p \ldots \eta_{p+q}},$$

and also projection formula for K-groups implies that

$$\varphi_*(\{f_{\xi_p}, \varphi^* \overline{g}_{\eta_p \ldots \eta_{p+q}}\}) = \{\varphi_*(f_{\xi_p}), \overline{g}_{\eta_p \ldots \eta_{p+q}}\}.$$

Now the rest of the proof is lemma 3.20 applied to all pairs (ξ_p, η_p) such that $\varphi(\xi_p) = \eta_p$, flags $\eta_p \ldots \eta_{p+q}$ and $h = \{f_{\xi_p}, \varphi^* \overline{g}_{\eta_p \ldots \eta_{p+q}}\}$. □

Lemma 3.20. *Let $\varphi : X \to Y$ be a proper morphism of irreducible schemes of finite type over a field k, such that $\dim(\varphi) = 0$. Let $h \in K_n(k(X))$ and $\eta_0 \ldots \eta_r$ be a flag on Y such that $\eta_l \in Y^{(l)}$ for all $0 \leq l \leq r$. Then*

$$\varphi_* \left(\sum_{\xi_0 \ldots \xi_r} \nu_{\xi_0 \ldots \xi_r}(h) \right) = \nu_{\eta_0 \ldots \eta_r}(\varphi_*(h)),$$

where the sum is taken over all flags $\xi_0 \ldots \xi_r$ on X such that $\varphi(\xi_0 \ldots \xi_r) = (\eta_0 \ldots \eta_r)$.

Proof. When $r = 1$ this is precisely the statement about the existence of the direct image on Gersten resolution for proper morphisms (since $\nu_{\xi_0 \xi_1}$ and $\nu_{\eta_0 \eta_1}$ correspond to the differentials in Gersten resolutions for X and Y). Next, there are only finitely many $\xi_1 \in X^{(1)}$ such that $\varphi(\xi_1) = \eta_1$, and after replacing X and Y by $\overline{\xi}_1$ and $\overline{\eta}_1$, respectively, we repeat the same procedure. Thus step by step we get the result. □

Remark 3.21. For a proper morphism $\varphi : X \to Y$ of dimension $d = \dim(\varphi)$ one can also consider a composition $\mathbf{A}_X(\mathcal{K}_n^X)^p \xrightarrow{\nu_p} (\mathbf{G}_n^X)^p \xrightarrow{\varphi_*} (\mathbf{G}_{n-d}^Y)^{p-d}$. For instance, when X is proper and regular of dimension d over a field k then we get the following formula:

$$\varphi_*([f_{\eta_0 \ldots \eta_d}]) = \sum_{\eta_0 \ldots \eta_d} \mathrm{Tr}_{k(\eta_d)/k}(\nu_{\eta_0 \ldots \eta_d}(f_{\eta_0 \ldots \eta_d})) \in K_{n-d}(k),$$

where $f_{\eta_0 \ldots \eta_d} \in \mathbf{A}_X(\mathcal{K}_n^X)^d$, $[f_{\eta_0 \ldots \eta_d}]$ denotes its class in K-cohomology and φ_* is the natural direct image in K-cohomology.

Comparison with Parshin-Beilinson adeles

Let X be a regular variety over a field k.

Proposition 3.22. *There is a well-defined morphism of complexes*

$$\mathrm{dlog} : \mathbf{A}_X(\mathcal{K}_n^X)^\bullet \to \mathbf{A}_X(\Omega_X^n)^\bullet,$$

where $\mathbf{A}_X(\Omega_X^n)^\bullet$ is the complex of rational adeles defined by Parshin and Beilinson (see [24], [14]), whose local components for a flag $\eta_0 \dots \eta_p$ are induced by natural maps $K_n(\mathcal{O}_{\eta_0}) \to \Omega^n_{\mathcal{O}_{\eta_0}/k}$ (see their definition in [4]).

Proof. Let us prove that for any subset $M \subset S(X)_p$ and for any open $U \subset X$ the natural map $\mathrm{dlog} : \mathbf{A}_M(\mathcal{K}_n^U) \to \mathbf{A}_M(\Omega_U^n)$ is well-defined. First we remark that as K-sheaves do, the differential forms depend only on the divisor D in the complement $X \backslash U$. This follows from the fact that when $\mathrm{codim}(\eta) \geq 2$ the local cohomology group $H^1_\eta(X, \Omega^n_X)$ vanishes since Ω^n_X is locally free. Hence we may suppose that $X \backslash U = D$ is the divisor.

By definition

$$\mathbf{A}_M(\Omega_U^n) = \prod_{\eta \in P(X)} \mathbf{A}_{{}_\eta M}((\Omega_U^n)_\eta) = \prod_{\eta \in P(X)} \varinjlim \mathbf{A}_{{}_\eta M}(\Omega^n_{V \cap U}),$$

where for each $\eta \in P(X)$ the limit is taken over all open subsets $V \subset X$ containing x (for the second equality we use that the Parshin-Beilinson adelic functor commutes with direct limits of quasicoherent sheaves). Since the same is true for K-sheaves of finite type we see that by induction on p we have to treat the case when $p = 0$. In this case

$$\mathbf{A}_M(\mathcal{K}_n^U) = \prod_{\eta \in M} (K_n^U)_x,$$

while

$$\mathbf{A}_M(\Omega_U^n) = \varinjlim \prod_{\eta \in M} (\Omega^n_X(lD))_x,$$

where the limit is taken over l, and $U = X \backslash D$. By lemma 3.23 we get the needed result. □

The author is grateful to C. Soulé for the explanation of the proof of the following lemma.

Lemma 3.23. *Rational differential forms from the image of natural map $K_n(k(X)) \to \Omega^n_{k(X)/k}$ have pole of order at most one along each irreducible divisor $D \subset X$.*

Proof. Recall the construction of the map $K_n(R) \to \Omega^n_{R/\mathbb{Z}}$. There are universal classes $c_n \in \varprojlim H^n(GL_m(R), \Omega^n_{R/\mathbb{Z}})$ (the limit is taken over m), which define canonical maps $K_n(R) \to H_n(GL(R), \mathbb{Z}) \xrightarrow{c_n} \Omega^n_{R/\mathbb{Z}}$. Note that the map c_n is trivial on $H_n(GL_{n-1}(R), \mathbb{Z})$. Moreover, the composition map $R^* \times \ldots \times R^* \to H_1(GL_1(R), \mathbb{Z}) \times \ldots \times H_1(GL_1(R), \mathbb{Z}) \to$

$H_n(GL_n(R),\mathbb{Z}) \to H_n(GL(R),\mathbb{Z}) \xrightarrow{c_n} \Omega^n_{R/\mathbb{Z}}$ is given by formula $(r_1,\ldots,r_n) \mapsto \frac{dr_1}{r_1} \wedge \ldots \wedge \frac{dr_n}{r_n}$.

Since one may suppose that $\dim X > 0$, the field $F = k(X)$ is infinite. By results of Suslin, see [28], there is an isomorphism $H_n(GL_n(F),\mathbb{Z}) \cong H_n(GL(F),\mathbb{Z})$, and the constructed above natural map $F^* \times \ldots \times F^* \to H_n(GL_n(F),\mathbb{Z})$ induces an isomorphism $K_n^M(F) \cong H_n(GL_n(F),\mathbb{Z})/H_n(GL_{n-1}(F),\mathbb{Z})$. Since for all non-zero rational functions $f_1,\ldots,f_n \in k(X)^*$ a differential form $\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_n}{f_n}$ has pole of order at most one along each irreducible divisor $D \subset X$, the lemma is proved. □

4 Quasiisomorphism with Gersten resolution

4.1 Main theorem

$\tilde{\mathbf{A}}$-adelic groups

In what follows it would be useful to consider another adelic groups. Namely, fix the type $(i_0 \ldots i_p)$ of depth l and consider a subgroup inside the product

$$\tilde{\mathbf{A}}_{i_0\ldots i_p}(\mathcal{K}_n^X) \subset \prod_{\eta_0\ldots\eta_p} (\mathcal{K}_n^X)_{\eta_0},$$

consisting of all elements for which, informally speaking, the map

$$\nu_{0\ldots l}\colon \tilde{\mathbf{A}}_{i_0\ldots i_p}(\mathcal{K}_n^X) \to \prod_{\eta_{l+1}\ldots\eta_p} \left(\oplus_{\eta_l}(\mathcal{K}_{n-l}^X)_{\eta_l}\right)$$

is well defined. More precisely, for each flag $\eta_l \ldots \eta_p$ of type $(i_l \ldots i_p)$ each sum inside the following expression should be finite:

$$\sum_{\eta_{l-1}\ni\overline{\eta}_l} \nu_{\eta_{l-1}\eta_l}\left(\sum_{\eta_{l-2}\ni\overline{\eta}_{l-1}} \nu_{\eta_{l-2}\eta_{l-1}}\left(\ldots\left(\sum_{\eta_1\ni\overline{\eta}_2} \nu_{\eta_0\eta_1\eta_2}(f_{\eta_0\ldots\eta_p})\right)\ldots\right)\right),$$

and the result of all summations should be zero for almost all η_l when we fix $\eta_{l+1}\ldots\eta_p$. We will call these new adelic groups $\tilde{\mathbf{A}}$-groups, while the old ones will be called $\mathbf{A}$-groups.

This adelic condition is a very "rough" one. In particular, $\tilde{\mathbf{A}}_{i_0\ldots i_p}(\mathcal{K}_n^X)$ coincides with the whole product, when $i_0 \geq 1$. In fact, the restricted product condition involves only $(0\ldots l)$-type part of flags of type $(i_0 \ldots i_p)$, where l is the depth of this type.

One may easily see that by reciprocity law in evident notations there is an inclusion $\tilde{\mathbf{A}}_{S_1}(\mathcal{K}_n^X) \hookrightarrow \tilde{\mathbf{A}}_{S_2}(\mathcal{K}_n^X)$ when $S_1 \subset S_2$. However, the analog of proposition 3.9 is not true for these new adelic groups. In addition, we may form an analogous adelic complex $\tilde{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet$. It occurs that there is no any nontrivial product structure on this complex.

Main theorem and its consequences

By proposition 3.12 there is a natural inclusion of complexes

$$i_X\colon \mathbf{A}_X(\mathcal{K}_n^X)^\bullet \to \tilde{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet.$$

Also there is a natural morphism of complexes

$$\tilde{\nu}_X\colon \tilde{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet \to (\mathbf{G}_n^X)^\bullet,$$

which is equal to the map $\nu_k = \nu_{0\dots k}$ on the $(0\dots k)$-type components and is equal to zero on all the other components of the adelic complex. We will denote by ν_X the composition $\tilde{\nu}_X \circ i_X$.

Theorem 4.1. *Suppose X is a smooth variety over an infinite ground field k or $X = \mathrm{Spec}(\mathcal{O}_{Y,y})$ for some regular variety Y over a field k and a schematic point $y \in Y$. Then the morphism $\tilde{\nu}_X$, the inclusion i_X, and hence ν_X are quasiisomorphisms.*

Remark 4.2. The additive analogue of this theorem for Parshin–Beilinson adeles was proved by Huber in [14].

Let us state several important consequences of theorem 4.1.

Corollary 4.3. *Let X be a smooth variety over an infinite field k. Then the complex of sheaves $(\underline{\mathbf{A}}_X(\mathcal{K}_n^X))^\bullet$ is a flasque resolution of the sheaf $\mathcal{K}_n^X = \mathcal{K}_n(\mathcal{O}_X)$.*

Proof. This follows immediately from theorem 4.1 and lemma 4.4 applied to the morphism $\underline{\nu}_X : \underline{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet \to (\underline{\mathbf{G}}_n^X)^\bullet$. ☐

Lemma 4.4. *Suppose $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes of sheaves of abelian groups on the topological space X such that for all open subsets $U \subset X$ the induced morphisms of complexes of abelian groups $f_U : \Gamma(U, \mathcal{F}^\bullet) \to \Gamma(U, \mathcal{G}^\bullet)$ is a quasiisomorphism. Then the map f is a quasiisomorphism of complexes of sheaves.*

Proof. Almost by definition one has

$$\mathcal{H}^i(\mathcal{F}^\bullet)_x = \varinjlim H^i(\Gamma(U, \mathcal{F}^\bullet)),$$

$$\mathcal{H}^i(\mathcal{G}^\bullet)_x = \varinjlim H^i(\Gamma(U, \mathcal{G}^\bullet)),$$

where the limit is taken over all open subsets $U \subset X$ containing x. Thus we see that the stalks of cohomology sheaves of $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are isomorphic, and we are done. ☐

Let $\varphi : X \to Y$ be a morphism of smooth varieties over an infinite field. From the general properties of resolutions of sheaves and corollary 4.4 one deduces the following result.

Corollary 4.5. *The defined above in section 3.2 contravariant morphism $\varphi^* : \underline{\mathbf{A}}_Y(\mathcal{K}_n^Y)^\bullet \to \varphi_*\underline{\mathbf{A}}_X(\mathcal{K}_n^X)^\bullet$ induces a canonical map on K-cohomology $\varphi^* : H^i(Y, \mathcal{K}_n^Y) \to H^i(X, \mathcal{K}_n^X)$ after we pass to cohomology of adelic complexes.*

Suppose in addition that φ is proper. As a consequence of theorem 4.1 and proposition 3.19 we get a product formula for Massey higher products on K-cohomology using remarks 3.18 and 6.12.

Corollary 4.6. *Let $\varphi : X \to Y$ be a proper morphism of smooth varieties over an infinite field. Let $a_1 \in H^{p_1}(X, \mathcal{K}^X_{n_1})$, $a_i \in H^{p_i}(Y, \mathcal{K}^Y_{n_i})$, $2 \le i \le k$ be classes in K-cohomology groups for which the k-th higher product $m_k(a_1, \varphi^*(a_2), \dots, \varphi^*(a_k))$ is well-defined. Then $m_k(\varphi_*(a_1), a_2, \dots, a_k)$ is also well-defined and there is an equality*

$$\varphi_*(m_k(a_1, \varphi^*(a_2), \dots, \varphi^*(a_k))) = m_k(\varphi_*(a_1), a_2, \dots, a_k)$$

Remark 4.7. A product structure described in corollary 3.11 makes $\oplus_{n\ge 0}\underline{\mathbf{A}}_X(\mathcal{K}^X_n)^\bullet$ to be a multiplicative resolution of the algebra sheaf $\oplus_{n\ge 0}\mathcal{K}^X_n$ by theorem 4.1 (see Appendix). Hence, by lemma 6.13 Massey higher products on the adelic complex induce Massey higher products on K-cohomology. This allows to compute (higher) products explicitly in some particular cases, see part **??**.

Proof of theorem 4.1. We will transform Gersten resolution into both adelic complexes in the same way using several intermediate complexes.

Consider for each number $0 \le k \le d$ a complex $C^\bullet_k$ defined by

$$C^\bullet_k\ : 0 \to \prod_{i\le k} \mathbf{A}_i(\mathcal{K}^X_n) \to \dots \to \prod_{0\le i_0<\dots i_p\le k} \mathbf{A}_{i_0\dots i_p}(\mathcal{K}^X_n) \to \dots \to \mathbf{A}_{0\dots k}(\mathcal{K}^X_n) \to$$

$$\to \oplus_{\eta\in X^{(k+1)}} K_{n-k-1}(k(\eta)) \to \dots \to \oplus_{\eta\in X^{(d)}} K_{n-d}(k(\eta)).$$

The differential in the "left part" of this complex coincides with that of the adelic complex, the differential in the "right part" — with that of Gersten resolution, and the "middle" map $\mathbf{A}_{0\dots k}(\mathcal{K}^X_n) \to \oplus_{\eta\in X^{(k+1)}} K_{n-k-1}(k(\eta))$ is the composition of the map ν_k and the differential in Gersten resolution. One may form the analogous complex $\tilde{C}^\bullet_k$ starting with $\tilde{\mathbf{A}}$-groups. Thus $C^\bullet_0 = \tilde{C}^\bullet_0 = (\mathbf{G}^X_n)^\bullet$, $C^\bullet_d = \mathbf{A}_X(\mathcal{K}^X_n)$ and $\tilde{C}^\bullet_d = \tilde{\mathbf{A}}_X(\mathcal{K}^X_n)$.

In addition, for $1 \le k \le d$ there are natural morphisms of complexes

$$\varphi_k \colon C^\bullet_k \to C^\bullet_{k-1},\ \tilde{\varphi}_k \colon \tilde{C}^\bullet_k \to \tilde{C}^\bullet_{k-1}$$

which are equal to ν_k on the k-th term of $\tilde{C}^\bullet_k$, and are identical on all the others. Obviously, these morphisms commute with the inclusion of $\mathbf{A}$-groups inside $\tilde{\mathbf{A}}$-groups. In particular, $\tilde{\varphi}_1 \circ \dots \circ \tilde{\varphi}_d = \tilde{\nu}_X$. Thus it is enough to prove by induction on k that all φ_k and $\tilde{\varphi}_k$ are quasiisomorphisms.

It is easy to show that φ_k and $\tilde{\varphi}_k$ are surjective. Indeed, since any given schematic point $\eta_k \in X^{(k)}$ is regular on X, we may choose a flag $\eta_0 \dots \eta_k$ such that η_l is regular in $\overline{\eta}_{l-1}$ for all $1 \le l \le k$. Hence the residue map

$$\nu_{\eta_0\dots\eta_k} \colon K_n(k(X)) \to K_{n-k}(k(\eta_k))$$

is surjective. So to obtain a surjectivity of φ_k (and consequently of $\tilde{\varphi}_k$) we may choose a suitable adele $f \in \mathbf{A}_{0\dots k}(\mathcal{K}^X_n)$, that will vanish on all flags except of a finite set of them.

So we need to show that the kernels of φ_k and $\tilde{\varphi}_k$ are exact complexes. Obviously, $\mathrm{Ker}(\varphi_k)$ is equal to the complex

$$0 \to \mathbf{A}_k(\mathcal{K}^X_n) \to \prod_{0\le i<k} \mathbf{A}_{ik}(\mathcal{K}^X_n) \to \dots \to \prod_{0\le i_0<\dots<i_p<k} \mathbf{A}_{i_0\dots i_p k}(\mathcal{K}^X_n) \to \dots \to \mathrm{Ker}(\nu_k) \to 0.$$

For each $\eta \in X^{(k)}$ consider the local scheme $\mathrm{Spec}(\mathcal{O}_\eta)$. By induction the complex $\tilde{C}^\bullet_{k-1}$ is qusiisomorphic to Gersten resolution, and by Quillen result for regular local rings we see that the complex

$$0 \to K_n(\mathcal{O}_\eta) \to \prod_{0\le i<k} \tilde{\mathbf{A}}_{i\eta}(\mathcal{K}_n^X) \to \ldots \to \prod_{0\le i_0<\ldots<i_p<k} \tilde{\mathbf{A}}_{i_0\ldots i_p\eta}(\mathcal{K}_n^X) \to \ldots$$

$$\ldots \to \tilde{\mathbf{A}}_{0\ldots(k-1)\eta}(\mathcal{K}_n^X) \xrightarrow{\nu_\eta} K_{n-k}(k(\eta)) \to 0$$

is exact for each point $\eta \in X^{(k)}$. Here the expression in the indices $(i_0 \ldots i_p\eta)$ denotes the set of all flags $\eta_0 \ldots \eta_p\eta$ on X of type $(i_0 \ldots i_pk)$. Now we take the product of these complexes over all points $\eta_k \in X^{(k)}$. More precisely, we consider an exact complex

$$B^\bullet_k:\ 0 \to \coprod_{\eta_k\in X^{(k)}} K_n(\mathcal{O}_{\eta_k}) \to \ldots \to \coprod_{\eta_k\in X^{(k)}} \left(\prod_{0\le i_0<\ldots<i_p<k} \tilde{\mathbf{A}}_{i_0\ldots i_p\eta_k}(\mathcal{K}_n^X) \right) \to \ldots$$

$$\ldots \to \coprod_{\eta_k\in X^{(k)}} \tilde{\mathbf{A}}_{0\ldots(k-1)\eta_k}(\mathcal{K}_n^X) \xrightarrow{\mu_k} \oplus_{\eta_k\in X^{(k)}} K_{n-k}(k(\eta)) \to 0,$$

where the symbol $\coprod$ denotes the restricted product, i.e. a subset inside the usual Cartesian product consisting of all elements (f_{η_k}) such that for almost all points $\eta_k \in X^{(k)}$ the component f_{η_k} belongs to the kernel $\mathrm{Ker}(\nu_{\eta_k})$. The point is that by definition the complex $\mathrm{Ker}(\tilde{\varphi}_k)$ is equal to

$$\tau_{\le(k-1)}(B^\bullet_k) \to \mathrm{Ker}(\nu_k)[-k],$$

where $\tau_{\le(k-1)}$ denotes the usual truncation of a complex. In other words, we consider a canonical filtration on $B^\bullet_k$. So we get that the complex $\mathrm{Ker}(\tilde{\varphi}_k)$ is exact, and by induction $\tilde{\nu}_X$ is a quasiisomorphism.

Now to show that i_X is also a quasiisomorphism note that the complex $\mathrm{Ker}(\varphi_k)$ is a subcomplex inside the exact complex $\mathrm{Ker}(\tilde{\varphi}_k)$. Thus theorem 4.1 follows from lemma 4.9. □

Remark 4.8. In the proof of theorem 4.1 for small n the group $\oplus_{x\in X^{(k)}} K_{n-k}$ is equal to zero by definition, when $k > n$. However, all the statements in the proof hold true, and it remains correct.

Several technical lemmas

Lemma 4.9. *Consider an element*

$$f \in \prod_{0\le i_0<\ldots<i_p<k} \tilde{\mathbf{A}}_{i_0\ldots i_pk}(\mathcal{K}_n^X),$$

where $p \le k-2$. Suppose that

$$df \in \prod_{0\le j_0<\ldots<j_{p+1}<k} \mathbf{A}_{j_0\ldots j_{p+1}k}(\mathcal{K}_n^X).$$

Then there exists

$$g \in \prod_{0 \le i_0 < \ldots < i_p < k} \mathbf{A}_{i_0 \ldots i_p k}(\mathcal{K}_n^X)$$

such that $df = dg$.

Proof. The adele f has several components with respect to codimensions of elements of flags:

$$f = \sum_{i_0 \ldots i_p} f_{i_0 \ldots i_p k}.$$

Let l be the maximal depth of indices of the nonzero components $f_{i_0 \ldots i_p k} \neq 0$. We will prove the assertion of lemma by induction on l.

Suppose $l = -1$. Then for any set $1 \le i_0 < \ldots < i_p < k$ the $(0 i_0 \ldots i_p k)$-component of df is equal to $f_{i_0 \ldots i_p k}$, and thus by proposition 3.9 $f_{i_0 \ldots i_p k} \in \mathbf{A}_{i_0 \ldots i_p k}(\mathcal{K}_n^X)$.

Now we prove the induction step from $l-1$ to l. Consider a maximal degree component $f_{0 \ldots l \ldots i_p k}$. More precisely, if $i_p > l$ then $i_{l+1} \ge l+2$, and if $i_p = l$, then $l = p \le k-2$. Anyway, we may consider the $(0 \ldots l, l+1 \ldots i_p k)$-component of the differential df. Let us apply the map $\nu_{0 \ldots l+1}$ to df. We also may apply it to each component of f by definition of $\tilde{\mathbf{A}}$-groups. By reciprocity law we get

$$\nu_{0 \ldots l+1}(f_{0 \ldots j, j+2 \ldots l, l+1 \ldots i_p k}) = 0$$

for $0 < j < l+1$. For $j = 0$ the vanishing is trivial and so we obtain the equality

$$\nu_{0 \ldots l+1}(f_{0 \ldots l \ldots i_p k}) = \nu_{0 \ldots l+1}((df)_{0 \ldots l, l+1 \ldots i_p k})$$

since l is the maximal possible depth of indices of the components of f. By the approximation lemma 4.11 there exists an adele $g_{0 \ldots l \ldots i_p k} \in \mathbf{A}_{0 \ldots l \ldots i_p k}(\mathcal{K}_n^X)$ of the same type as $f_{0 \ldots l \ldots i_p k}$ such that

$$\nu_{0 \ldots l+1}(f_{0 \ldots l \ldots i_p k}) = \nu_{0 \ldots l+1}(f'_{0 \ldots l \ldots i_p k}).$$

By lemma 4.10 there exists $h \in B_k^p$ such that

$$(dh)_{0 \ldots l \ldots i_p k} = f_{0 \ldots l \ldots i_p k} - g_{0 \ldots l \ldots i_p k}.$$

Moreover, the components of h are nonzero only for type $(0 \ldots j, j+2 \ldots l \ldots i_p k)$, $0 \le j \le l$. Hence $f - g - dh$ has strictly less nonzero components of depth l, and its differential still belongs to $\mathbf{A}$-groups. Thus we are done by induction on l. □

Lemma 4.10. *Suppose that for* $f \in \tilde{\mathbf{A}}_{i_0 \ldots i_p k}(\mathcal{K}_n^X)$ *of type of depth* l

$$\nu_{0 \ldots l+1}(f) = 0$$

Then there exist $h \in \prod_{0 \le j \le l} \tilde{\mathbf{A}}_{0 \ldots j, j+2 \ldots l \ldots i_p k}(\mathcal{K}_n^X)$ *such that*

$$f = \sum_{0 \le j \le l} h_{0 \ldots j, j+2 \ldots l \ldots i_p k}.$$

Proof. In fact this is the only one place where we really use the $\tilde{\mathbf{A}}$-complex. Possibly it could be overcome, but it seems that technically it would be of the same complexity.

Fix a flag $\eta_{l+1}\dots\eta_p k$ of type $(\eta_{l+1}\dots\eta_p k)$ (for $l=p$ just choose a schematic point η_k of codimension k). Varying the flag $\eta_0\dots\eta_l$ of type $(0\dots l)$ we may treat $f_{\eta_0\dots\eta_l\eta_{l+1}\dots\eta_k}$ as an element in $\tilde{\mathbf{A}}_{0\dots l}(\mathcal{K}_n^{\mathrm{Spec}(\mathcal{O}_{\eta_{l+1}})})$. By the condition of lemma $\nu_{0\dots l}(f)$ is a cocycle in the Gersten resolution for $\mathrm{Spec}(\mathcal{O}_{\eta_{l+1}})$. Hence by Quillen result this is a coboundary, and by surjectivity of $\tilde{\nu}$ there exists $h_{0\dots l-1,i_{l+1}\dots i_p k}\in\tilde{\mathbf{A}}_{0\dots l-1,i_{l+1}\dots i_p k}$ such that

$$\nu_{0\dots l}(f-h_{0\dots l-1,i_{l+1}\dots i_p k})=0.$$

In fact this is the place where we use that the restricted product condition on $\tilde{\mathbf{A}}$-groups is very rough.

We have shown in the proof of theorem 4.1 that $\mathrm{Ker}(\tilde{\varphi}_l)$ is exact for any smooth variety or its localization. Hence there are $h_{0\dots j,j+2\dots l,i_{l+1}\dots i_p k}\in\tilde{\mathbf{A}}_{0\dots j,j+2\dots l,i_{l+1}\dots i_p k}$ for $0\le j\le l-1$ such that

$$f-h_{0\dots l-1,i_{l+1}\dots i_p k}=\sum_{0\le j\le l-1}h_{0\dots j,j+2\dots l\dots i_p k},$$

and we get the desired statement. □

The essential part of the proof of the quasiisomorphism of i_X is the following approximation type lemma (there is a pun here).

Lemma 4.11. *Consider $f\in\tilde{\mathbf{A}}_{i_0\dots i_p k}(\mathcal{K}_n^X)$ of type of depth l and suppose that*

$$\nu_{0\dots l+1}(f)=\nu_{0\dots l+1}(f')$$

for certain $f'\in\mathbf{A}_{0\dots k}(\mathcal{K}_n^X)$. Then there exists an adele $g\in\mathbf{A}_{i_0\dots i_p k}(\mathcal{K}_n^X)$ of the same type as f, such that

$$\nu_{0\dots l+1}(f)=\nu_{0\dots l+1}(g).$$

The proof of this lemma is quite long and is the content of section 4.3.

4.2 Technical support for approximation

In this section we develop some technique needed for the proof of approximation lemma 4.11 in the next section.

Strongly locally effaceable pairs

First, let us recall several well-known facts. Quillen showed in [26] (see also theorem 4.15) that for a local regular scheme Y of geometric type there are exact triples

$$0\to K_n(\mathcal{M}^p(Y))\to\oplus_{\eta\in Y^{(p)}}K_n(k(\eta))\to K_{n-1}(\mathcal{M}^{p+1}(Y))\to 0,$$

where $\mathcal{M}^p(Y)$ denotes the exact category of coherent sheaves on Y whose support codimension is at least p. Recall that

$$K_n(\mathcal{M}^p(Y))) = \varinjlim K'_n(Z),$$

where the direct limit is taken over all closed subsets Z inside Y whose components have codimension at least p. The map $K_n(\mathcal{M}^p(Y)) \to \oplus_{\eta\in Y^{(p)}} K_n(k(\eta))$ associates to each element $\alpha \in K'_n(Z)$ the sum of restrictions of α on the generic points of all components of Z which have codimension p in Y. The differential in Gersten complex for Y is just the composition of corresponding arrows in the exact triples from above, thus Gersten complex for local regular rings of geometric type is exact, providing a resolution for the group $K_n(Y)$.

Remark 4.12. If $\{f_\eta\}$ is a cocycle in Gersten resolution, then the collection $\{f_\eta\}$ is defined by an certain element α from $K_n(\mathcal{M}^p(Y))$. From what we said above it follows that in fact $\alpha \in K'_n(Z)$. Suppose $Z_0 \subset Z$ is a codimension p component in X, which is not in the support of $\{f_\eta\}$. Then, since $K_n(k(Z_0)) = \varinjlim K'_n(U)$, where $U \subset Z_0$ are open subsest, which may be chosen to be open in Z, we see that in fact α belongs to $K'_n(Z')$, where $Z' \subset Z$ has the same components as Z besides Z_0, which is replaced by a proper closed subset. Hence, we may choose Z such that the union of all its codimension p components equals to the support of $\{f_\eta\}$.

Let X be a regular variety over a field k. For each (not necessary closed) point $x \in X$ there is a corresponding local Gersten resolution defined for the scheme $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$. Let us denote the differential in this complex by d_x. For any closed subset $C \subset X$, containing x, there is an equality

$$K'_n(C_x) = \varinjlim K'_n(C \cap V),$$

where the limit is taken over all open subsets $V \subset X$, containing x.

Consider an equidimensional cycle $Z \subset X$ of codimension p, and an equidimensional cycle $\widetilde{Z} \supset Z$ of codimension $p-1$. Suppose for each (not necessary closed) point $x \in Z$ and for any open subset $x \in V \subset X$ there exists a smaller open subset $x \in W \subset X$ such that the natural map

$$K'_n(V \cap Z) \to K'_n(W \cap \widetilde{Z})$$

is zero for all $n \geq 0$. Suppose, in addition, that for any $q \geq 0$ there exists an assignment $R \mapsto \Lambda(R)$, where R is an equidimensional cycle of codimension q in Z, $\Lambda(R)$ is an equidimensional cycle of codimension q in $\widetilde{Z}$, $R \subset \Lambda(R)$, and for any (not necessary closed) point $x \in Z$, for any open subset $x \in V \subset X$ there exists a smaller open subset $x \in W \subset X$ such that the composition

$$K'_n(V \cap (Z\backslash R)) \to K'_n(V \cap (Z\backslash \Lambda(R))) \to K'_n(W \cap (\widetilde{Z}\backslash \Lambda(R)))$$

is zero for all $n \geq 0$ (in fact, this condition makes sense when $x \in R$). We will say that such pair of cycles $(Z, \widetilde{Z})$ is *strongly locally effaceable* (developing the terminology from [7]). The existence of the assignment Λ is needed to show the relation between locally strongly effaceable pairs and Gersten resolution. Namely, the following holds true.

Proposition 4.13. *Let $(Z, \widetilde{Z})$ be a strongly locally effaceable pair of cycles in a smooth variety X over an infinite field, where Z is of codimension p in X. Choose an arbitrary (not necessary closed) point $x \in Z$ and consider a collection*

$$\{f_z\} \in \bigoplus_{z \in Z_x^{(0)}} K_n(k(z)),$$

which is a cocyle in local Gersten resolution, i.e. $d_x(\{f_z\}) = 0$. Then there exists a collection

$$\{g_{\widetilde{z}}\} \in \bigoplus_{\widetilde{z} \in \widetilde{Z}_x^{(0)}} K_{n+1}(k(\widetilde{z}))$$

such that

$$d_x(\{g_{\widetilde{z}}\}) = \{f_x\}.$$

Proof. First, replace X by its open affine subset containing x. It follows from remark 4.12 that the collection $\{f_z\}$ may be represented as an element α from $K'_n((Z \cup S)_x)$, where S is a certain closed subset in X whose all components are not contained in Z and have codimension at least $p+1$ in X. In fact, α is an element from $K'_n(V \cap (Z \cup S))$ for some open subset $x \in V \subset X$. Moreover, from the codimension condition we get that the intersection $Z \cap S$ is contained in certain equidimensional cycle $R \subset Z$ of codimension at least two in Z. Also, by enlarging S one can choose R and S such that S is equidimensional and $\dim R = \dim S - 1$. By the existence theorem 4.15 there is an equidimensional cycle $\widetilde{S}$ of codimension at least p in X such that the pair $(\Lambda(R) \cup S, \widetilde{S})$ is strongly locally effaceable. Consider the following commutative diagram, which is exact in the middle column in the middle term:

$$\begin{array}{ccccc}
 & & K'_n(\Lambda(R) \cup S) & \longrightarrow & K'_n(\widetilde{S}) \\
 & & \downarrow & & \downarrow \\
K'_n(Z \cup S) & \longrightarrow & K'_n(\widetilde{Z} \cup S) & \longrightarrow & K'_n(\widetilde{Z} \cup \widetilde{S}) \\
\downarrow & & \downarrow & & \\
K'_n(Z \backslash R) & \longrightarrow & K'_n(\widetilde{Z} \backslash (\Lambda(R) \cup S)) & &
\end{array}$$

Here the last map is the composition $K'_n(Z \backslash R) \to K'_n(\widetilde{Z} \backslash \Lambda(R)) \to \mathcal{K}'_n(\widetilde{Z} \backslash (\Lambda(R) \cup S))$. Since the pairs $(Z, \widetilde{Z})$ and $(\Lambda(R) \cup S, \widetilde{S})$ are strongly locally effaceable, for any point $x \in Z$ and any open subset $x \in V \subset X$ there exists a smaller open subset $x \in W \subset X$ such that the map

$$K'_n(V \cap (Z \cup S)) \to K'_n(W \cap (\widetilde{Z} \cup \widetilde{S}))$$

is zero. Therefore α is a coboundary of an element $\beta \in K'_{n+1}(W \cap ((\widetilde{Z} \cup \widetilde{S}) \backslash (Z \cup S)))$ in the excision exact sequence associated to the closed embedding $W \cap (Z \cup S) \hookrightarrow W \cap (\widetilde{Z} \cup \widetilde{S})$. In particular, β defines a collection

$$\{g_{\widetilde{z}}\} \in \bigoplus_{\widetilde{z} \in \widetilde{Z}_x^{(0)}} K_{n+1}(k(\widetilde{z})).$$

Note that all components of $\widetilde{Z} \cup \widetilde{S}$, which have codimension $p-1$ in X, are contained in $\widetilde{Z}$, and all codimension p components of $Z \cup S$ are in Z. Therefore, $d_x(\{g_{\widetilde{z}}\}) = \{f_z\}$, and we are done. □

Existence and addition of strongly locally effaceable pairs

Let $(Z, \widetilde{Z})$ be a strongly locally effaceable pair. Suppose that for any irreducible subvariety $C \subset X$ and an equidimensional cycle $R \subset Z$ of codimension $q \geq 0$ in Z, which does not contain C one can choose an equidimensional cycle $\Lambda_C(R) \subset \widetilde{Z}$ of codimension q in $\widetilde{Z}$, which does not contain C, contains R, and has the following property. For any f irreducible subvarieties $C_1, \ldots, C_f \subset \widetilde{Z}$ and f equidimensional cycles $R_1, \ldots, R_f \subset Z$ (maybe of different codimension) such that R_i does not contain C_i for all $1 \leq i \leq f$, and for any $x \in Z$ and an open subset $x \in V \subset X$ there exists a smaller open subset $x \in W \subset X$ such that the map

$$K'_n(V \cap (Z \backslash (R_1 \cup \ldots \cup R_f))) \to K'_n(W \cap (\widetilde{Z} \backslash (\Lambda_{C_1}(R_1) \cup \ldots \cup \Lambda_{C_f}(R_f))))$$

is zero for all $n \geq 0$. We call such pair of cycles $(Z, \widetilde{Z})$ *strongly locally effaceable pair with a freedom degree* f. Let us also say that a strongly locally effaceable pair has freedom degree 0.

Remark 4.14. The second condition of a strongly locally effaceable pair is a particular case of the freedom degree one condition, if one takes C outside of $\widetilde{Z}$.

Here is the existence theorem for such pairs.

Theorem 4.15. *Consider an equidimensional cycle Z of codimension $p \geq 2$ in the affine smooth variety X over an infinite ground field k and a closed subset $T \subset X$, none of whose irreducible components is contained inside Z. Then for any natural number $f \geq 0$ there exists a cycle $\widetilde{Z} \supset Z$ that does not contain any irreducible component of T, and such that the pair $(Z, \widetilde{Z})$ is strongly locally effaceable with a freedom degree f.*

Corollary 4.16. *Suppose in notations of theorem 4.15 that codimensions of all irreducible components of T are at most $p-1$, and that X is an arbitrary (not necessary affine) smooth variety over an infinite field k. Then there exists $\widetilde{Z}$ such that the pair $(Z, \widetilde{Z})$ is strongly locally effaceable.*

Proof. Let us cover X by affine open subsets: $X = \cup_\alpha U_\alpha$. For each α use theorem 4.15 for the intersection of all data with U_α, and get certain closed subsets $\widetilde{Z}_\alpha \subset U_\alpha$. Now take the union of their closures $\widetilde{Z} = \cup_\alpha \overline{\widetilde{Z}_\alpha}$. Evidently, $\widetilde{Z}$ does not contain any irreducible component of T due to the assumption on codimensions. Also, the pair $(Z, \widetilde{Z})$ is strongly locally effaceable, where $\Lambda(R)$ can be taken to be the union of closures of $\Lambda_\alpha(R \cap U_\alpha)$ for an equidimensional cycle $R \subset Z$. □

Proof of theorem 4.15. The way we prove this theorem decomposes into two steps.

Step 1.

During this step "a point" always means "a closed point". Choosing a (closed) point for each irreducible component of T outside of Z we may suppose T to be a finite set of points in X outside of Z.

Let us say that a morphism $\pi : X \to \mathbb{A}^{d-1}$ (recall that $d = \dim X$) *resolves* a point $x \in Z$ if π is smooth of relative dimension one at x, the restriction $\varphi = \pi|_Z$ is finite, $\varphi^{-1}(\varphi(x)) = \{x\}$, and $\pi(T) \cap \pi(Z) = \emptyset$.

The following geometric result is a globalization of Quillen's construction used in his proof of Gersten conjecture, see [26], Lemma 5.12 and [7], Claim on p.191.

Proposition 4.17. *Under above assumptions there is a finite set Σ of morphisms $\pi : X \to \mathbb{A}^{d-1}$ such that for any f points $y_1, \dots, y_f \in X$ and a point $x \in Z$ there exists $\pi \in \Sigma$ which resolves x and such that $\pi(y_i) \notin \pi(Z \backslash \{y_i\})$ for all $i = 1, \dots, f$.*

Proof. Let X be a complement to a hyperplane H of the projective variety $\overline{X} \subset \mathbb{P}^n$. In what follows let the bar denote a projective closure. For a moment let us take for grant the following statement.

Claim 4.18. *For any f points $y'_1 \dots, y'_f \in X \backslash Z$ and $f+1$ closed subsets $Z_0, \dots, Z_f \subset Z$ there are non-empty open subsets $U_i \subset Z_i$, $i = 0, \dots, f$ and a morphism $\pi : X \to \mathbb{A}^{d-1}$, which resolves all point x from U_0 (with respect to Z) and such that $\pi(y_i) \notin \pi(Z \backslash \{y_i\})$ for any point $y_i \in U_i$ and $\pi(y'_i) \notin \pi(Z)$ for all $i = 1, \dots, f$.*

Using claim 4.18, we prove by decreasing induction on e, $-1 \leq e \leq f$, that for any $e+1$ irreducible subsets $Z_0 \dots, Z_e$ in Z there are non-empty open subsets $U_0 \subset Z_0, \dots, U_e \subset Z_e$ and a finite set of morphisms Σ such that the statement of proposition 4.17 is true for all collections of points consisting of $x \in U_0$, $y_i \in U_i$ for $1 \leq i \leq e$, $y_i \in Z$ for $e+1 \leq i \leq f$, and $y'_i \in X \backslash Z$ for $0 \leq i \leq f$.

First, when $e = f$ we choose f points $y'_1, \dots, y'_f \in X \backslash Z$ and apply claim 4.18 for $Z_0, \dots, Z_f$ and $\{y'_j\}$. Since the condition $y'_j \notin \pi^{-1}(\pi(Z))$ is open, we may cover $(X \backslash Z)^f$ by a finite set of open subsets such that all collections $\{y'_1, \dots, y'_f\}$ from the same subset fit the same morphism π. After we take a (finite) intersection of corresponding open subsets in $Z_0, \dots, Z_f$ we get the needed open subsets $U_0, \dots, U_f$ and the needed finite set of morphisms.

Now let us show the induction step from e to $e-1$. Choose any irreducible component C_1 of Z. By induction there is a finite set of morphisms Σ_1 and open subsets $U_0, \dots, U_{e-1}, U_e$ in $Z_0 \dots, Z_{e-1}, C_1$, respectively, satisfying the discussed above properties. Let C_2 be one of the irreducible components of $Z \backslash U_e$ (we may suppose U_e to be also open in Z). Again, by induction we get finite set of morphisms Σ_2 and different open subsets in $Z_0, \dots, Z_{e-1}, C_2$. We repeat the same step until we come to the end of the obtained (finite) stratification of Z by open subsets in C_i. After this we intersect all (finitely many) obtained open subsets in each Z_j, $0 \leq j \leq e-1$, take the union of all (finitely many) Σ_i and thus we get the needed open subsets $U_0 \dots, U_{e-1}$ and a finite set of morphisms Σ.

Finally, when $e = -1$ we get the needed statement of proposition 4.17. $\square$

Proof of claim 4.18. First, let us enlarge T by $y'_1 \dots, y'_f$. Also, without loss of generality replace Z_i by their irreducible components. Choose smooth points $x'_i \in Z_i$, $0 \leq i \leq f$. We have the following dimension conditions: $\dim(H \cap \overline{Z}) \leq d-3$, $\dim(H \cap (T * \overline{Z})) \leq d-2$ (here "star" denotes a join), $\dim(H \cap \overline{T_{x'_0} X}) = d-1$, and $\dim(H \cap \overline{T_{x'_i} Z_i}) \leq d-3$ for all $0 \leq i \leq f$. Since the field k is infinite there exists a projective subspace $L' \subset H$ of codimension $d-2$ in H such that L' does not intersect $\overline{Z}$, nor any of $\overline{T_{x'_i} Z_i}$, intersects $T * \overline{Z}$ in a finite set of points and intersects all $\overline{T_{x'_0} X}$ in a line. Note that the projection $\pi_{L'}$ with the center at L' defines on $\overline{Z}$ a finite morphism $\varphi_{L'}$. Consider $Z_i' = \varphi_{L'}^{-1}(\varphi_{L'}(\overline{Z_i})) \subset \overline{Z}$ for $0 \le i \le f$. For each $0 \le i \le f$ choose a point $x_i \in Z_i \subset Z_i'$ which is smooth on Z_i', and such that $\overline{T_{x_i} Z_i}$ does not intersect L', while $\overline{T_{x_0} X}$ intersect L' in a line. We claim that all intersections $L' \cap (x_i * Z_i')$ are finite sets of points. Indeed, each join consits of a tangent space which does not intersect L', and of lines passing through x_i and other points from Z_i', whose intersection with L' corresponds to the fiber of x_i under the finite morphism $\varphi_{L'}$. Hence, there exists a hyperplane $L \subset L'$ which does not intersect the joins $T * \overline{Z}$ and $x_i * Z_i'$ for all $0 \le i \le f$ and which intersects the tangent spaces $\overline{T_{x_0} X}$ in one point. Evidently, the projection π_L with the center at L can not glue points from Z_i' with points from $\overline{Z} \backslash Z_i'$ for all $0 \le i \le f$. Therefore, by lemma 4.19 used with $Y = Z_i'$ there are non-empty open subsets $x_i \in U_i \subset Z_i$ such that π_L resolves all points x from U_0, and $\varphi_L^{-1}(\varphi(y_i)) = \{y_i\}$ for all points y_i from U_i, $1 \le i \le f$, where $\varphi_L = \pi_L|_Z$. In addition, $\pi(T)$ does not intersect with $\pi(Z)$, and $\pi(y_i') \notin \pi(Z)$ for all $1 \le i \le f$. $\square$

We have used the following fact from projective geometry.

Lemma 4.19. *Let $Y \subset \mathbb{P}^N$ be a projective variety, $x \in Y$ be a smooth point on Y. Suppose a projective subspace $M \subset \mathbb{P}^N$ does not intersect the join $y * Y$. Then there exists an open subset $x \in W \subset Y$ such that $\varphi^{-1}(\varphi(y)) = \{y\}$ for all $y \in W$, where φ is the restriction of the projection π_M to Y.*

Step 2.

We still suppose that the field k is infinite and perfect. Apply construction from proposition 4.17. We use notations from its formulation. Let $\widetilde{Z} = \cup \pi^{-1}(\pi(Z))$ where the union is taken over all $\pi \in \Sigma$. Then by construction $\widetilde{Z}$ does not contain any component of T.

Proposition 4.20. *The pair $(Z, \widetilde{Z})$ is strongly locally effaceable of a freedom degree f.*

Proof. Essentially, we repeat the proof of [26], Theorem 5.11 with minor modifications. First, we note that after we choose a suitable closed point on $\overline{x} \subset Z$ we may suppose that the given open subset $x \in V$ is in fact an open neighborhood of this closed point. Thus we may suppose x to be closed.

Choose $\pi \in \Sigma$ which resolves x, and, as before, put $\varphi = \pi|_Z$. Following the construction of Quillen, consider the Cartesian square:

$$\begin{array}{ccc} Y & \xrightarrow{\varphi'} & X \\ \downarrow \pi' & & \downarrow \pi \\ Z & \xrightarrow{\varphi} & \mathbb{A}^{d-1}. \end{array}$$

Note that φ' is finite onto its image, $\varphi'(Y) = \pi^{-1}(\pi(Z))$ is a closed subset in $\widetilde{Z}$, and $(\varphi')^{-1}(x)$ consists of one point z. Besides, the morphism π' is smooth at z and admits a canonical section $\sigma : Z \to Y$ (with $\sigma(x) = z$). During the proof of Theorem 5.11, [26] it was shown that is implies that the composition $K_n'(Z) \xrightarrow{\sigma_*} K_n'(Y) \to K_n'(Y')$ is zero for all $n \ge 0$, and for some suitable open subset $Y' \subset Y$ containing z. Hence the map $K_n'(Z) \to K_n'(\widetilde{Z} \cap U)$ is zero as well for all $n \ge 0$ for some suitable open subset

$U \subset X$ containing x such that $(\varphi')^{-1}(U) \subset Y'$ (such U exists since φ' is finite and $(\varphi')^{-1}(x) = \{z\}$).

Now take an arbitrary open subset $x \in V \subset X$. Since $\varphi^{-1}(\varphi(x)) = \{x\}$ and φ is finite, one may find an open subset $D \subset \mathbb{A}^{d-1}$ such that $x \in \varphi^{-1}(D) \subset V$. After we restrict the Cartesian diagram from $\mathbb{A}^{d-1}$ on D we get that the natural map $K'_n(V \cap Z) \to K'_n(W \cap \widetilde{Z})$ is zero for all $n \geq 0$ for some suitable open subset $W \subset V$ containing x.

Further, consider an irreducible subset $C \subset X$ and an equidimsnional cycle $R \subset Z$ of codimension q in Z, which does not contain C. We put

$$\Lambda_C(R) = \cup_{y \in C \backslash R}(\cup_{\Sigma_y} \pi^{-1}(\pi(R))),$$

where Σ_y is the set of all π such that $\pi(y) \notin \pi(Z \backslash \{y\})$. For example, if C is not contained in $\widetilde{Z}$ then $\Lambda_C(R) = \cup \pi^{-1}(\pi(R))$ where the union is taken over all $\pi \in \Sigma$. Now for irreducible subsets $C_1, \ldots, C_f$ in X and cycles $R_1, \ldots, R_f$ in Z satisfying the needed conditions one chooses closed points $y_i \in C_i \backslash R_i$. By construction, for any closed point $x \in Z$ there is a morphism $\pi \in \Sigma$ such that it resolves x and belongs to $\Sigma_{y_1} \cap \ldots \cap \Sigma_{y_f}$. Thus, the same argument with a corresponding Cartesian diagram as before leads to the needed result. ☐

Theorem 4.15 is now proven. ☐

As a consequence of theorem 4.15 and proposition 4.13 we get the following statement, which could be considered as a uniform version of Gersten conjecture for smooth varieties, and has interest in its own right.

Corollary 4.21. *Let X be a smooth variety over an infinite field k. Then for any equidimensional cycle $Z \subset X$ of codimension p in X there exists an equidimensional cycle $\widetilde{Z} \supset Z$ of codimension $p-1$ in X with the following property. For any (not necessary closed) point $x \in Z$ and a collection*

$$\{f_z\} \in \bigoplus_{z \in Z_x^{(0)}} K_n(k(z)),$$

which is a cocycle in local Gersten resolution at x, i.e. $d_x(\{f_z\}) = 0$, there exists a collection

$$\{g_{\widetilde{z}}\} \in \bigoplus_{\widetilde{z} \in \widetilde{Z}_x^{(0)}} K_{n+1}(k(\widetilde{z}))$$

such that $d_x(\{g_{\widetilde{z}}\}) = \{f_z\}$.

The following proposition allows add strongly locally effaceable pairs.

Proposition 4.22. *Consider two equidimensional cycles Z_1 and Z_2 of codimension $p \geq 2$ inside an affine smooth variety X over an infinite ground field k. Suppose we are given a cycle $\widetilde{Z}_1 \supset Z_1$ such that the pair $(Z_1, \widetilde{Z}_1)$ is strongly locally effaceable with a freedom degree $f \geq 2$. Consider a closed subset $T \subset X$, whose all irreducible components are of codimension at most $p-1$ in X and are not contained inside Z_2, and an irreducible* *subvariety $F \subset X$ which is not contained inside Z_2. Then there exists a cycle $\widetilde{Z}_2$ that does not contain any irreducible component of T and F, and such that the pair $(Z_1 \cup Z_2, \widetilde{Z}_1 \cup \widetilde{Z}_2)$ is strongly locally effaceable with a freedom degree $f-1$.*

Proof. If F is not contained inside Z_1 then the statement of proposition follows directly from theorem 4.15 after we enlarge T by F (in this case the freedom degree does not decrease).

Suppose that $C \subset Z_1$. We use the same construction as in the proof of proposition 4.13. One can choose a codimension one cycle Z_2' in Z_1 which does not contain F and contains the intersection of Z_1 with all irreducible components of Z_2, which are not in Z_1. Let $Z_3 = \Lambda_F(Z_2')$. Note that $Z_2 \cup Z_3$ does not contain any irreducible component of T due to the codimensions assumption. Hence, by theorem 4.15, there exists a cycle $\widetilde{Z}_2$ such that the pair $(Z_2 \cup Z_3, \widetilde{Z}_2)$ is strongly locally effaceable with a freedom degree $f-1$, and $\widetilde{Z}_2$ does not contain any irreducible component of T and F. We claim that the pair $(Z_1 \cup Z_2, \widetilde{Z}_1 \cup \widetilde{Z}_2)$ is strongly locally effaceable with a freedom degree $f-1$.

Indeed, the first condition of strongly locally effaceability holds by the following commutative diagram, which is exact in the middle column in the middle term:

$$\begin{array}{ccccc} & & K_n'(Z_2 \cup Z_3) & \longrightarrow & K_n'(\widetilde{Z}_2) \\ & & \downarrow & & \downarrow \\ K_n'(Z_1 \cup Z_2) & \longrightarrow & K_n'(\widetilde{Z}_1 \cup Z_2) & \longrightarrow & K_n'(\widetilde{Z}_1 \cup \widetilde{Z}_2) \\ \downarrow & & \downarrow & & \\ K_n'(Z_1 \backslash Z_2') & \longrightarrow & K_n'(\widetilde{Z}_1 \backslash (Z_2 \cup Z_3)) & & \end{array}$$

Here the last map is the composition

$$K_n'(Z_1 \backslash Z_2') \to K_n'(\widetilde{Z}_1 \backslash Z_3) \to K_n'(\widetilde{Z}_1 \backslash (Z_2 \cup Z_3)).$$

Since the pairs $(Z_1, \widetilde{Z}_1)$ and $(Z_2 \cup Z_3, \widetilde{Z}_2)$ are strongly locally effaceable, for any $x \in Z_1 \cup Z_2$ and any open subset $x \in V \subset X$ there exists a smaller open subset $x \in W \subset X$ such that the map

$$K_n'(V \cap (Z_1 \cup Z_2)) \to K_n'(W \cap (\widetilde{Z}_1 \cup \widetilde{Z}_2))$$

is zero for all $n \geq 0$.

Now consider an irreducible subvariety $C \subset X$ and an equidimensional cycle $R \subset Z_1 \cup Z_2$ of codimension q, which does not contain C. Let R' be the union of all components of R which are not contained in $Z_2' \cup Z_2$. Let $\Lambda_C'(R')$ be the union of all components of $\Lambda_C(R')$, which are not contained in $Z_2 \cup Z_3$. Then one may choose an equidimensional cycle R'' of codimension q in $Z_2 \cup Z_3$, which contains the intersection $(\Lambda_C'(R') \cup R) \cap (Z_2 \cup Z_3)$ and does not contain C. Consider $\Lambda_C(R'')$, where Λ_C is taken with respect to the strongly locally effaceable pair $(Z_2 \cup Z_3, \widetilde{Z}_2)$ of a freedom degree $f-1$. We can take $\Lambda_C'(R') \cup \Lambda_C(R'')$ to be $\Lambda_C(R)$ with respect to the pair $(Z_1 \cup Z_2, \widetilde{Z}_1 \cup \widetilde{Z}_2)$. This follows from the commutative diagram analogous to the previous one, which is composed of two diagrams:

$$\begin{array}{ccc} K_n'((Z_1 \cup Z_2) \backslash \{R_i\}) & \longrightarrow & K_n'((\widetilde{Z}_1 \cup Z_2) \backslash \{\Lambda_{C_i}'(R_i') \cup R_i\}) \\ \downarrow & & \downarrow \\ K_n'(Z_1 \backslash (Z_2' \cup \{R_i\})) & \longrightarrow & K_n'(\widetilde{Z}_1 \backslash (Z_2 \cup Z_3 \cup \{\Lambda_{C_i}'(R_i')\})), \end{array}$$

$$\begin{array}{ccc} K'_n((Z_2\cup Z_3)\backslash\{\Lambda'_{C_i}(R'_i)\cup R_i\}) & \longrightarrow & K'_n(\widetilde{Z}_2\backslash\{\Lambda_{C_i}(R_i)\}) \\ \downarrow & & \downarrow \\ K'_n((\widetilde{Z}_1\cup Z_2)\backslash\{\Lambda'_{C_i}(R'_i)\cup R_i\}) & \longrightarrow & K'_n((\widetilde{Z}_1\cup \widetilde{Z}_2)\backslash\{\Lambda_{C_i}(R_i)\}), \end{array}$$

which we glue by the following exact sequence in the middle term:

$$K'_n((Z_2\cup Z_3)\backslash\{\Lambda'_{C_i}(R'_i)\cup R_i\}) \to K'_n((\widetilde{Z}_1\cup Z_2)\backslash\{\Lambda'_{C_i}(R'_i)\cup R_i\}) \to K'_n(\widetilde{Z}_1\backslash(Z_2\cup Z_3\cup\{\Lambda'_{C_i}(R'_i)\})).$$

Here $\{O_i\}$ means the union of objects O_i over all $i = 1, \ldots, f-1$, and the horizontal maps in the diagrams are compositions of direct images under closed embeddings and restrictions on open subsets. The diagrams are used in the same way as the previous one. □

Remark 4.23. For $p = 1$ by Quillen result the only possible pair (Z, X) is strongly locally effaceable. However, there can be no analogue of theorem 4.15 and proposition 4.22 for non-empty T and F.

4.3 Approximation

Reduction to the affine case

Till now during the proof of theorem 4.1 we did not use any special properties of the scheme X at all. In fact, "geometrical" properties of X are needed only for lemma 4.11, which uses results from section 4.2. We do several reductions of lemma 4.11.

Proposition 4.24. *If lemma 4.11 is true for a scheme X, then it is true for $Y = \mathrm{Spec}(\mathcal{O}_{X,x})$ for any schematic point $x \in X$.*

Proof. Recall that the canonical morphism $j : \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is an embedding at the set-theoretical level and also preserves the codimension of points. Thus, evidently, for any type $(i_0 \ldots i_p)$ the group

$$\prod_{\eta_0\ldots\eta_p} (\mathcal{K}_n^Y)_{\eta_0}$$

is a direct summand inside the group

$$\prod_{\xi_0\ldots\xi_p} (\mathcal{K}_n^X)_{\xi_0},$$

where $\mathrm{codim}_Y(\eta_j) = i_j$ and $\mathrm{codim}_X(\xi_j) = i_j$ for $0 \le j \le p$.

Moreover, it easily follows from definitions, that $\mathbf{A}$ and $\tilde{\mathbf{A}}$ condition are preserved by the corresponding embedding and projection. Indeed, for $\mathbf{A}$-adeles remark that we may restrict any system of divisors, encountered in the definition, from X to Y, or oppositely, take its closure inside X, if it is given on Y. As for $\tilde{\mathbf{A}}$-adeles, remark that taking the residue along the flag on Y is the same as taking the residue along its closure on X.

Hence both $\mathbf{A}$ and $\tilde{\mathbf{A}}$ adelic groups associated to Y are direct summands, respectively, in $\mathbf{A}$ and $\tilde{\mathbf{A}}$ adelic groups associated to X with the same indexes, and from this one deduces proposition 4.24. □

Proposition 4.25. *If lemma 4.11 is true for any affine scheme then it is true for any scheme X which admits a finite open affine covering.*

Proof. Take a finite covering

$$X = \bigcup_{i=1}^{n} U_i$$

by affine open subsets. The same reasoning as before leads to the fact that for any open subset $U \subset X$ again both $\mathbf{A}$ and $\tilde{\mathbf{A}}$ adelic groups associated to U are direct summands, respectively, in $\mathbf{A}$ and $\tilde{\mathbf{A}}$ adelic groups associated to X.

Apply lemma 4.11 to U_1 and to restrictions $f|_{U_1}$, $f'|_{U_1}$. Thus we get an element g_{U_1} from $\mathbf{A}$ adelic group on U and we may extend it by zero on the whole X and get an element g_1 from the corresponding $\mathbf{A}$ adelic group on X such that

$$\nu_{0\ldots l+1}(g_1) = \nu_{0\ldots l+1}(f)|_{U_1}.$$

Now do the same for U_2 and get a certain element g_{U_2} from the $\mathbf{A}$ adelic group on U_2. We may truncate this element. Namely, put zero values on all its components in flags $\eta_0 \ldots \eta_k$ such that $\eta_k \notin U_2 \backslash U_1$ and preserve the component otherwise. Evidently, this will be still an element from $\mathbf{A}$ adelic group on U_2 and then we extend it by zero to an element g_2 from the corresponding $\mathbf{A}$ adelic group on X such that

$$\nu_{0\ldots l+1}(g_2) = \nu_{0\ldots l+1}(f)|_{U_2 \backslash U_1},$$

where the restriction $|_{U_2 \backslash U_1}$ is in the sense that we consider only flags which end up with a point from $U_2 \backslash U_1$. Then we repeat the same for U_3 and $U_3 \backslash (U_1 \cup U_2)$, $\ldots$, $U_n \backslash (U_1 \cup \ldots U_{n-1})$, and we get elemets $\{g_1, \ldots, g_n\}$ from $\mathbf{A}$-adelic groups such that

$$\nu_{0\ldots l+1}(\sum_i g_i) = \nu_{0\ldots l+1}(f),$$

and we are done. □

Thus we have reduced the problem to the case when X is an affine smooth variety over an infinite field.

Patching systems

Choose an equidimensional cycle Z of codimension p on a (not necessary affine) smooth variety X. Suppose we are given a system of equidimensional cycles $\{Z_m^{1,2}\}$, $1 \leq m \leq p-1$ of codimension m on X with the following properties:

(i) the cycle Z is contained inside both Z_{p-1}^1 and Z_{p-1}^2, and $Z_m^1 \cup Z_m^2$ is contained in both Z_{m-1}^1 and Z_{m-1}^2 for $2 \leq m \leq p-1$;

(ii) all the pairs (Z, Z_{p-1}^1), (Z, Z_{p-1}^2), $(Z_m^1 \cup Z_m^2, Z_{m-1}^1)$, $(Z_m^1 \cup Z_m^2, Z_{m-1}^2)$ for $2 \leq m \leq p-1$ are strongly locally effaceable with a freedom degree f;

(iii) the cycles Z_m^1 and Z_m^2 have no common irreducible components for $1 \le m \le p-1$.

We will call such system of cycles a *patching system* for the cycle Z *with a freedom degree* f.

Proposition 4.26. *For arbitrary number f and for any equidimensional cycle Z on an affine smooth variety X over an infinite field there exists a patching system for Z with freedom degree f.*

Proof. Suppose Z is of codimension p on X. Apply theorem 4.15 first with empty T and get an equidimensional cycle $\widetilde{Z}_2^1$ of codimension $p-1$ in X. Then apply theorem 4.15 with $T = \widetilde{Z}_2^1$, and get an equidimensional cycle $\widetilde{Z}_2^2$ of codimension $p-1$ in X. Note that $\widetilde{Z}_2^1$ have no common components with $\widetilde{Z}_2^2$. Put $Z_{p-1}^1 = \widetilde{Z}_2^1$ and $Z_{p-1}^2 = \widetilde{Z}_2^2$.

Now repeat the same procedure for the cycle $Z_{p-1}^1 \cup Z_{p-1}^2$ and get two equidimensional cycles Z_{p-2}^1, Z_{p-2}^2 of codimension $p-2$ in X without common components such that the pairs $(Z_{p-1}^1 \cup Z_{p-1}^2, Z_{p-2}^i)$ are strongly locally effaceable for $i = 1, 2$.

Repeat this inductively until we get two divisors Z_1^1 and Z_1^2 without common irreducible components. □

Remark 4.27. By the same method using corollary 4.16 instead of theorem 4.15 one shows that for any equidimensional cycle Z on a (not necessary affine) smooth variety X over an infinite field there exists a patching system with freedom degree 0.

Proof of approximation lemma

Now we prove lemma 4.11. We use the same notations as in its formulation and we suppose X to be an affine smooth variety over an infinite ground field k. During the proof η_p denotes a schematic point on X of codimension p. By proposition 3.12 and proposition 3.9 the residue $\nu_{0\ldots l+1}(f')$ belongs to the finite sum of $\mathbf{A}$-adelic group:

$$\nu_{0\ldots l+1}(f') = \oplus h_{\eta(i_{l+1}-l-1)\ldots(i_p-l-1)(k-l-1)} \in \oplus_{\eta \in X^{(l+1)}} \mathbf{A}_{0(i_{l+1}-l-1)\ldots(i_p-l-1)(k-l-1)}(\mathcal{K}_{n-l-1}^{\overline{\eta}}).$$

Let $Z_{l+1} = \cup \overline{\eta}$ be the union of closures of the finite set of schematic points $\eta \in X^{(l+1)}$ for which $h_{\eta(i_{l+1}-l-1)\ldots(i_p-l-1)(k-l-1)}$ is not zero.

Step 1.

Reciprocity law for f implies that for any flag $\eta_{i_{l+1}} \ldots \eta_{i_p} \eta_k$ and for any schematic point $\eta_{l+2} \supseteq \eta_{i_{l+1}}$ we have

$$\sum_{\eta_{l+1} \supset \eta_{l+2}} \nu_{\eta_{l+1},\eta_{l+2}}(h_{\eta_{l+1}\eta_{i_{l+1}}\ldots\eta_{i_p}\eta_k}) = 0.$$

On the other hand, $d^2_{\eta_k}(h_{\eta_{l+1}\eta_{i_{l+1}}\ldots\eta_{i_p}\eta_k}) = 0$ for a fixed flag $\eta_{l+1}\eta_{i_{l+1}} \ldots \eta_{i_p}\eta_k$. We deduce that after we fix a flag $\eta_{i_{l+1}} \ldots \eta_{i_p}\eta_k$ the collection

$$g_{l+2} = \sum_{\eta_{l+1}} \left(d_{\eta_k}(\{h_{\eta_{l+1}\eta_{i_{l+1}}\ldots\eta_{i_p}\eta_k}\}) - \sum_{\eta_{l+2} \supset \eta_{i_{l+1}}} \nu_{\eta_{l+1},\eta_{l+2}}(h_{\eta_{l+1}\eta_{i_{l+1}}\ldots\eta_{i_p}\eta_k}) \right) \in$$

$$\in \bigoplus_{\eta_{l+2}\in Z^{(1)}_{l+1}} K_{n-l-2}(k(\eta_{l+2}))$$

is a cocycle in Gersten resolution with respect to d_{η_k} (this collection is defined for the flag $\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k$). In other words, for each flag $\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k$ we consider the sum over all $\eta_{l+1}\in Z^{(0)}_{l+1}$ of residues of $h_{\eta_{l+1}\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k}$ on divisors in $\overline{\eta_{l+1}}$ through η_k, which are not irreducible components from the divisor $D_{\eta_{l+1}}\subset\overline{\eta}_{l+1}$ in notations from section 3.1 (here $D_{\eta_{l+1}\dots}$ denotes a system of divisors arising from the adelic conditions on $\overline{\eta}_{l+1}$).

Now we want to express this as a coboundary in local Gersten resolution at η_k of a certain element, controlling its support, namely, we want its support to be a system of divisors in the sense of section 3.1, associated with a flag $\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k$. To do this we use the technique from section 4.2.

We will use the sign "$-$" for two arbitrary equidimensional cycles C_1 and C_2 of equal dimensions in the following sense: C_1-C_2 is an equidimensional cycle of the same dimension, which consists of all components of C_1, which are not contained inside C_2.

Choose $\eta_{i_{l+1}}$ and consider the union

$$\bigcup_{\eta_{l+1}\in Z^{(0)}_{l+1}} (D_{\eta_{l+1}\eta_{i_{l+1}}}-D_{\eta_{l+1}}),$$

which is an equidimensional cycle of codimension $l+2$ in X. It does not contain $\eta_{i_{l+1}}$ by properties of the system of divisors $D_{\eta_{l+1}\dots}$. Apply theorem 4.15 for $f=p-l$, $T=\overline{\eta}_{i_{l+1}}$ and Z being the cycle from above. Thus we get a certain equidimensional cycle $z_{l+1;\eta_{i_{l+1}}}$ of codimension $l+1$ in X, such that it does not contain $\eta_{i_{l+1}}$ and verifies a certain strongly locally effaceable condition with a freedom degree $p-l$.

Now choose a flag $\eta_{i_{l+1}}\eta_{i_{l+2}}$ and consider the union

$$\bigcup_{\eta_{l+1}\in Z^{(0)}_{l+1}} D_{\eta_{l+1}\eta_{i_{l+1}}\eta_{i_{l+2}}}-D_{\eta_{l+1}\eta_{i_{l+1}}},$$

which is again an equidimensional cycle of codimension $l+2$ in X. It does not contain $\eta_{i_{l+2}}$ by properties of the system of divisors $D_{\eta_{l+1}\dots}$. Now apply proposition 4.22 for $Z_1=\cup_{\eta_{l+1}\in Z^{(0)}_{l+1}}(D_{\eta_{l+1}\eta_{i_{l+1}}}-D_{\eta_{l+1}})$, $\widetilde{Z}_1=z_{l+1;\eta_{l+1}}$, Z_2 is the cycle from above, and $C=\overline{\eta_{i_{l+2}}}$. Thus we get a certain equidimensional cycle $z_{l+1;\eta_{i_{l+1}}\eta_{i_{l+2}}}=\widetilde{Z}_2$ of codimension $l+1$ in X. As before, it does not contain $\eta_{i_{l+2}}$, and $z_{l+1;\eta_{i_{l+1}}}\cup z_{l+1;\eta_{i_{l+1}}\eta_{i_{l+2}}}$ verifies a certain strongly locally effaceable condition with a freedom degree $p-l-1$.

We continue this procedure inductively for longer flags $\eta_{i_{l+1}}\dots\eta_{i_j}$, $l+1\le j\le p$, and we get a system of equidimensional cycles $\{z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_j}}\}$ of codimension $l+1$ in X with the following properties: if we put

$$Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_j}}=Z_{l+1}\cup\bigcup_{l+1\le m\le j} z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_m}}$$

then

$$Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_j}}(\eta_{i_j})=Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_{j-1}}}(\eta_{i_j})$$

for $l+2 \leq j \leq p$, and

$$Z_{l+1;\eta_{i_{l+1}}}(\eta_{i_{l+1}}) = Z_{l+1}(\eta_{i_{l+1}}).$$

In addition, the pair of cycles

$$(\cup_{\eta_{l+1}\in Z_{l+1}^{(0)}} D_{\eta_{l+1}\eta_{i_{l+1}}\dots\eta_{i_p}} - D_{\eta_{l+1}}, Z_{l+1;\eta_{i_{l+1}}\eta_{i_{l+2}}\dots\eta_{i_p}} - Z_{l+1})$$

is strongly locally effaceable with a freedom degree one.

Now fix a flag $\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k$ and consider the constructed above collection g_{l+2}. It has support on

$$\bigcup_{\eta_{l+1}\in Z_{l+1}^{(0)}} D_{\eta_{l+1}\eta_{i_{l+1}}\dots\eta_{i_p}} - D_{\eta_{l+1}},$$

and thus by proposition 4.13 there exists a collection g'_{l+1} with support on $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_p}} - Z_{l+1}$, such that $d_{\eta_k}(g'_{l+1}) = g_{l+2}$.

Finally, we obtain a collection

$$g_{l+1} = \{h_{\eta_{l+1}\eta_{l+2}\eta_{i_{l+2}}\dots\eta_{i_p}\eta_d}\} + (-g'_{l+1})$$

with support on $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_p}}$ which is a cocycle in the local Gersten resolution associated to η_k.

Thus we have modified the collection $\{h_{\eta_{l+1}\eta_{l+2}\eta_{i_{l+2}}\dots\eta_{i_p}\eta_k}\}$ into a cocycle in local Gerten resolution at η_k, controlling the support of the difference. In particular, this support does not contain $\eta_{i_{l+1}}$.

Step 2.

By proposition 4.26 there exists a patching system $\{Z_m^{1,2}\}$, $1 \leq j \leq l$ for Z_{l+1} of a freedom degree $p-l+1$. Now we should extend this patching system to patching systems for each cycle $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_j}}$ using proposition 4.22.

Namely, first we define $Z^1_{l;\eta_{i_{l+1}}}$ as the result $\widetilde{Z}_1 \cup \widetilde{Z}_2$ of proposition 4.22 applied to $Z_1 = Z_{l+1}$, $\widetilde{Z}_1 = Z_l^1$, $T = Z_l^2$, $C = \overline{\eta}_{i_{l+1}}$, $Z_2 = z_{l+1;\eta_{i_{l+1}}}$. Then we repeat the same for $Z_1 = Z_{l+1}$, $\widetilde{Z}_1 = Z_l^2$, $T = Z^1_{l+1} \cup Z^1_{l+1;\eta_{i_{l+1}}}$, $C = \overline{\eta}_{i_{l+1}}$, $Z_2 = z_{l+1;\eta_{i_{l+1}}}$ and we obtain $Z^2_{l;\eta_{i_{l+1}}}$ as $\widetilde{Z}_1 \cup \widetilde{Z}_2$.

Then we repeat the same until we obtain two divisors $Z^1_{1;\eta_{i_{l+1}}}$ and $Z^2_{1;\eta_{i_{l+1}}}$ such that

$$(Z^1_{1;\eta_{i_{l+1}}} \cup Z^2_{1;\eta_{i_{l+1}}})(\eta_{i_{l+1}}) = (Z_1^1 \cup Z_1^2)(\eta_{i_{l+1}}),$$

and $Z^{1,2}_{m;\eta_{i_{l+1}}}$, $1 \leq m \leq l$ is a patching system for $Z_{l+1;\eta_{i_{i+1}}}$ of a freedom degree $p-l$.

Then we pass to the flag $\eta_{i_{l+1}}\eta_{i_{l+2}}$ and in the same way we define the patching system $Z^{1,2}_{m;\eta_{i_{l+1}}\eta_{i_{l+2}}}$ such that

$$(Z^1_{1;\eta_{i_{l+1}}\eta_{i_{l+2}}} \cup Z^2_{1;\eta_{i_{l+1}}\eta_{i_{l+2}}})(\eta_{i_{l+2}}) = (Z^1_{1;\eta_{i_{l+1}}} \cup Z^2_{1;\eta_{i_{l+1}}})(\eta_{i_{l+2}}).$$

Then we continue this until we get a patching system $Z^{1,2}_{m;\eta_{i_{l+1}}\dots\eta_{i_p}}$, $1 \leq m \leq l$, for $Z_{l+1;\eta_{i_{l+1}}\dots\eta_{i_p}}$ of a freedom degree one.

Now fix a flag $\eta_{i_{l+1}} \dots \eta_{i_p}\eta_k$ and consider a collection g_{l+1}, obtained at the end of Step 1. By proposition 4.13 there are two collections

$$g_l^1 \in \bigoplus_{\eta \in (Z^1_{l;\eta_{i_{l+1}}\dots\eta_{i_p}})^{(0)}} K_{n-l}(\eta)$$

and

$$g_l^2 \in \bigoplus_{\eta \in (Z^2_{l;\eta_{i_{l+1}}\dots\eta_{i_p}})^{(0)}} K_{n-l}(\eta)$$

such that their local differentials at η_k are equal to g_{l+1}. Hence for $g_l = g_l^1 \oplus (-g_l^2)$ we have $d_{\eta_d}(g_l) = 0$.

Repeat this until we get two collections g_1^1 and g_1^2 with support on $Z^1_{1;\eta_{i_{l+1}}\dots\eta_{i_p}}$ and $Z^2_{1;\eta_{i_{l+1}}\dots\eta_{i_p}}$. By induction, $d_{\eta_d}(g_1) = 0$ for $g_1 = g_1^2 \oplus (-g_1^2)$, and we get an element $g_0 \in K_n(k(X))$ such that $d_{\eta_d}(g_0) = g_1$.

Having fixed the flag $\eta_{i_{l+1}} \dots \eta_{i_p}\eta_k$, put

$$g_{\eta_0\eta_1\dots\eta_l\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k} = g_0,$$

if $\eta_j \in (Z_j^1)^{(0)}$ for all $1 \le j \le l$, and

$$g_{\eta_0\eta_1\dots\eta_l\eta_{i_{l+1}}\dots\eta_p\eta_d} = 0$$

otherwise. We see that by construction

$$\nu_{0\dots l}(g) = \nu_{0\dots l}(f),$$

therefore we are done if we show that g verifies an adelic condition in terms of certain system of divisors $E_{\eta_0\dots\eta_j}$ on X. Let us describe this system of divisors explicitly.

Note that the support of

$$d_{\eta_k}(g_{\eta_0\eta_1\dots\eta_l\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k})$$

is contained inside

$$Z^1_{1;\eta_{i_{l+1}}\dots\eta_{i_p}} \cup Z^2_{1;\eta_{i_{l+1}}\dots\eta_{i_p}}.$$

Define

$$E_F = Z_1^1 \cup Z_1^2$$

for any flag $F = (\eta_0 \dots \eta_j)$ on X such that $j \le l$ and $\eta_q \in (Z_q^1)^{(0)}$ for all $1 \le q \le j$, and

$$E_F = Z^1_{1;\eta_{i_{l+1}}\dots\eta_{i_k}} \cup Z^2_{1;\eta_{i_{l+1}}\dots\eta_{i_k}}$$

for any flag $F = (\eta_0 \dots \eta_l\eta_{i_{l+1}} \dots \eta_{i_j})$ on X such that $l+1 \le j \le p$, and $\eta_q \in (Z_q^1)^{(0)}$ for all $1 \le q \le l$. By construction this system of divisors on X satisfies the condition $(*)$ from proposition 3.6, and thus we see that $g_{\eta_0\eta_1\dots\eta_l\eta_{i_{l+1}}\dots\eta_{i_p}\eta_k}$ belongs the adelic group $\mathbf{A}_{01\dots l i_{l+1}\dots i_p k}$. So, approximation lemma 4.11 and, hence, theorem 4.1 are now proved.

5 Products in K-cohomology

5.1 Explicit classes

In this section we describe explicitly cocyles in the adelic resolution corresponding to that of Gersten resolution.

Examples

Consider a smooth variety X of dimension d over an infinite field k. Let D be a (not necessary reduced and effective) divisor on X. For each schematic point $\eta \in X$ consider the local equation $s_\eta \in k(X)^*$ of D in $\mathrm{Spec}(\mathcal{O}_\eta)$. Evidently, $s_\xi/s_\eta \in \mathcal{O}_\eta^*$ when $\xi \in \overline{\eta}$. Thus we get a 1-cocycle $[D] \in \mathbf{A}_X(\mathcal{K}_1^X)^1$, whose $(X\eta)$-component is s_η for $\eta \neq X$, and $(\eta\xi)$-component is s_ξ/s_η for $\eta \neq X$, $\xi \in \overline{\eta}$, $\xi \neq \eta$. It turns out that this cocycle satisfies the adelic condition described in the beginning of section 3.1. This is easy to check for the case of divisors, while for higher codimension cycles the situation becomes more complicated, see below. By construction, the class of $[D]$ in $H^1(\mathbf{A}_X(\mathcal{K}_1^X)^\bullet) = CH^1(X)$ coincides with the class of D in the first Chow group. In [10] and [12] it was proved that the intersection product in Chow groups coisides with the natural product in corresponding K-cohomology. Combining this with remarks 4.7 and 3.21 we get an adelic formula for the intersection index of divisors $D_1, \ldots, D_d$ when X is proper:

$$(D_1, \ldots, D_d) = \sum_{\eta_0 \ldots \eta_d} [k(\eta_d) : k] \nu_{\eta_0 \ldots \eta_d}\{s_{1,\eta_1}, s_{2,\eta_2}/s_{2,\eta_1}, \ldots, s_{d,\eta_d}/s_{d,\eta_{d-1}}\} =$$

$$= \sum_{\eta_0 \ldots \eta_d} [k(\eta_d) : k] \nu_{\eta_0 \ldots \eta_d}\{s_{1,\eta_1}, s_{2,\eta_2}, \ldots, s_{d,\eta_d}\},$$

where the last identity is obtained by use of reciprocity laws. This formula was first proved for $d = 2$ by Parshin in see [24], and then for arbitrary d by Lomadze in [15]. However, their proofs are different from the present approach, which considers adelic complex as a multiplicative resolution of sheaves of K-groups.

The next example is the intersection of a 1-cycle C and a divisor D in the three-dimensional variety X over a field k. We describe explicitly a 2-cocyle in the adelic complex $\mathbf{A}_X(\mathcal{K}_2^X)^\bullet$ corresponding to C. Let us choose an effective reduced divisor E with the following properties: for each schematic point η of codimension at least two there exists an element $t_\eta \in K_2(k(X))$ and a subdivisor $E_\eta \subset E$ such that $\mathrm{div}(t_\eta) \subset E$ and $d_\eta(\nu_{XE_\eta}(t_\eta)) = C_\eta$. Here ν_{XE_η} denotes the residue from the field $k(X)$ to the sum of fields of rational functions of all components of E_η, C_η is the cycle inside $\mathrm{Spec}(\mathcal{O}_\eta)$ defined by C, and d_η stays for the differential in the Gersten resolution for $\mathrm{Spec}(\mathcal{O}_\eta)$. Proposition 4.13 and remark 4.27 imply the existence of such divisor E.

Next, define adeles f_{012} and f_{013} such that $f_{XE_\eta\eta} = t_\eta \in K_2(k(X))$ and equals to an arbitrary element from $K_2(\mathcal{O}_\eta)$ otherwise. For each flag $\eta\xi$ of type (23) we have

$$d_\eta(\nu_{XE_\xi}(t_\xi)/\nu_{XE_\eta}(t_\eta)) = 0,$$

hence there exists an element $t_{\eta\xi} \in K_2(k(X))$ such that

$$d_\eta(t_{\eta\xi}) = \nu_{XE_\xi}(t_\xi)/\nu_{XE_\eta}(t_\eta).$$

This defines an adele f_{023}. Finally, one sees that for each flag $\mu\eta\xi$ of type (123) the product $f_{X\eta\xi}f^{-1}_{X\mu\xi}f_{X\mu\eta}$ belongs to $(\mathcal{K}_2^X)_\mu$ and is also an adele. Thus we have defined a cocycle $f \in \mathbf{A}_X(\mathcal{K}_2^X)^2$, which represents the class of C in $CH^2(X)$ by construction. From this one gets the intersection formula:

$$(D, C) = \sum_{\mu\eta\xi}[k(\xi) : k]\nu_{X\mu\eta\xi}\{s_\mu, f_{\mu\eta\xi}\} = \sum_{\mu\eta\xi}[k(\xi) : k]\nu_{X\mu\eta\xi}\{s_\mu, t_{\eta\xi}\},$$

where s_μ is the local equation of the divisor D (see above). Here again one uses reciprocity laws to obtain the last equality.

The class of a cocycle in Gersten resolution

Let $Y \subset X$ be an equidimensional cycle of codimension p in X. Consider a collection

$$\{f_y\} \in \bigoplus_{y\in Y^{(0)}} K_m(k(y)),$$

which is a cocyle in (global) Gersten resolution $(\mathbf{G}^X_{p+m})^\bullet$. Generalizing what we have done for a curve on a threefold, we construct explicitly a cocyle $f = [\{f_y\}]$ in $\mathbf{A}_X(\mathcal{K}^X_{p+m})^p$ such that $\nu_p(f) = \{f_y\}$.

Choose a patching system $Y_r^{1,2}$, $1 \le r \le p-1$ for the cycle Y of a freedom degree zero, see remark 4.27. We define the adele f inductively starting from flags $\eta_0 \dots \eta_{p-1}\eta_i$ of type $(0, 1, \dots, p-1, i)$ for some $i \ge p$. If η_i does not belong to Y then we put the corresponding component of f to be equal to zero (we speak about K-groups in the additive way). If η_i belongs to Y then we choose a “local equation” of $\{f_y\}$ with respect to the patching system $Y_r^{1,2}$, as it was done at the end of section 4.3, Step 2 (where it was denoted by g_0). Namely, we choose an element $\widetilde{f}_{\eta_i} \in K_{m+p}(k(X))$ such that $\mathrm{div}(\widetilde{f}_{\eta_i}) \subset Y_1^1 \cup Y_1^2$ and

$$d_{\eta_i}\nu_{XY_1^1Y_2^1\dots Y_{p-1}^1}(\widetilde{f}_{\eta_i}) = \{f_y\}_{\eta_i},$$

where d_{η_i} is the differential in the Gersten resolution for $X_{\eta_i} = \mathrm{Spec}(\mathcal{O}_{\eta_i})$, and ν_{ST} denotes a part of the differential in (global) Gersten resolution wich correspond to the generic points of irreducible components of S and T (provided that S and T have suitable dimensions), and the index η_i by the collection means the restriction on $X_{\eta_i} = \mathrm{Spec}(\mathcal{O}_{X,\eta_i})$. Then we put $f_{\eta_0\dots\eta_{p-1}\eta_i} = h$ if η_r is a component of Y_r^1 for all $1 \le r \le p-1$, and $f_{\eta_0\dots\eta_{p-1}\eta_i} = 0$ otherwise.

Now for each pair $i_2 > i_1 > p-1$ consider a flag $\eta_0 \dots \eta_{p-2}\eta_{i_1}\eta_{i_2}$ of type $(0, 1, \dots, p-2, i_1, i_2)$. If η_{i_1} does not belong to Y_{p-1}^1 then we put the corresponding component of f to be equal to zero. If η_{i_1} does belong to Y^1_{p-1}, then we take $\widetilde{f}_{\eta_{i_1}\eta_{i_2}} \in K_{p+m}(k(X))$ such that $\mathrm{div}(h) \subset Y_1^1 \cup Y_1^2$ and

$$d_{\eta_{i_1}}\nu_{XY_1^1Y_2^1\dots Y_{p-2}^1}(\widetilde{f}_{\eta_{i_1}\eta_{i_2}}) = \nu_{p-1}(f_{01\dots p-1\eta_{i_2}} - f_{01\dots p-1\eta_{i_1}})_{\eta_{i_1}} \in \bigoplus_{\eta\in X^{(p-1)}_{\eta_{i_1}}} K_{m+1}(k(\eta)),$$

where $f_{01\dots p-1\eta_{i_1}}$ stays for the sum over all flags of type $(01\dots p-1, i_1)$ ending by η_{i_1} (the same for η_{i_2}), and for the definition of ν_{p-1} see example 3.13. Now we put $f_{\eta_0\dots\eta_{p-2}\eta_{i_1}\eta_{i_2}}$ to be equal to $\widetilde{f}_{\eta_{i_1}\eta_{i_2}}$ if η_r is a component of Y^1_r for all $1 \le r \le p-2$, and zero otherwise. We see that for any triple $i_3 > i_2 > i_1 > p-2$ the following equality holds true:

$$d_{\eta_{i_1}}\nu_{p-2}\left(f_{01\dots p-2\eta_{i_2}\eta_{i_3}} - f_{01\dots p-2\eta_{i_1}\eta_{i_3}} + f_{01\dots p-2\eta_{i_1}\eta_{i_2}}\right) = 0.$$

Next, we repeat inductively the same procedure. More precisely, we proceed by the following formula (using the same type of notations as before):

$$d_{\eta_{i_1}}\nu_{XY^1_1\dots Y^1_r}(\widetilde{f}_{\eta_{i_1}\dots\eta_{i_{p-r}}}) = \nu_{r+1}\left(\sum_{j=0}^{r}(-1)^{j+1}f_{01\dots r+1\eta_{i_1}\dots\eta_{i_{j-1}}\eta_{i_{j+1}}\dots\eta_{i_{p-r}}}\right)_{\eta_{i_1}},$$

while defining the component of f corresponding to flags $\eta_0\dots\eta_r\eta_{i_1}\dots\eta_{i_{p-r}}$ of type $(0, 1, \dots, r, i_1, \dots, i_{p-r})$.

At the end we define the components of type $(0i_1\dots i_p)$ for $i_p > \dots > i_1 > 1$, and one sees that

$$d_{\eta_{i_0}}\left(\sum_{j=1}^{p}(-1)^j f_{X\eta_{i_0}\dots\eta_{i_{j-1}}\eta_{i_{j+1}}\dots\eta_{i_p}}\right) = 0$$

for any flag $\eta_{i_0}\dots\eta_{i_p}$, $i_0 > 0$. Therefore, the alternated sum in brackets belongs to $K_{p+m}(\mathcal{O}_{\eta_0})$ and, putting this to be a $\eta_{i_0}\dots\eta_{i_p}$-component of f, we finally get a cocycle f in $\mathbf{A}_X(\mathcal{K}^X_p)^p$. The adelic condition is satisfied since $\mathrm{div}(f_{\eta_{i_0}\dots\eta_{i_p}})$ is in $Y^1_1 \cup Y^2_1$ for all flags $\eta_{i_0}\dots\eta_{i_p}$.

Remark 5.1. From this explicit construction it can be easily deduced by induction on r that one may choose the adele $f = [\{f_y\}]$ such that the necessary condition for the non-vanishing of f on the flag $\eta_{i_0}\dots\eta_{i_p}$ is that η_{i_r} belongs to Y^1_r for all $1 \le r \le p-1$, and η_{i_p} belongs to Y.

Remark 5.2. For a flag $\eta_{i_0}\dots\eta_{i_p}$ satisfying this condition, but for which η_{i_r} does not belong to Y^1_{r+1} for $0 \le r \le p-2$, and $\eta_{i_{p-1}}$ does not belong to Y, we may choose the adele $f = [\{f_y\}]$ such that

$$f_{\eta_{i_0}\dots\eta_{i_p}} = \widetilde{f}_{\eta_{i_p}} \in K_{p+m}(k(X)),$$

where as before $\widetilde{f}_{\eta_{i_p}}$ is such that $d_{\eta_{i_p}}\nu_{XY^1_1\dots Y^1_{p-1}}(\widetilde{f}_{\eta_{i_p}}) = \{f_y\}_{\eta_{i_p}}$ and depends only on η_{i_p}. Indeed, this follows by induction from the explicit formula defining the class $[\{f_y\}]$, using the previous remark 5.1, which provides a lot of cancellations.

In the proof of theorem 5.16 we will also need the following technical integrality condition on the adele $[\{f_y\}]$.

Claim 5.3. *Suppose in addition that for certain schematic point η on Y there corresponding local cocyle $\{f_y\}_\eta$ in Gersten resolution is represented by an element $\alpha \in K'_m(Y_\eta)$. Then the element $\widetilde{f}_\eta \in K_{p+m}(k(X))$ from remark 5.2 may chosen such that the collection*

$$d_\eta\nu_{XY^1_1\dots Y^1_{r-1}}(\widetilde{f}_\eta) \in K_{p+m-r}(k((Y^1_r \cup Y^2_r)_{\eta_q}))$$

is actually represented by an element $\alpha_r \in K'_{p+m-r}((Y_r^1 \cup Y_r^2)_\eta)$ *for all* $1 \le r \le p-1$, *and is represented by* α *for* $r = p$. *Moreover, the restriction of* α_r *on* $K'_{q+m-r}((Y_r^1)_\eta \backslash Y_r^2)$ *is in fact an element from* $K'_{p+m-r}((Y_r^1)_\eta \backslash (Y_{r+1}^1 \cup Y_{r+1}^2))$. *In particular,* $\widetilde{f}_\eta = \alpha_0 \in K'_{p+m}(X_\eta \backslash (Y_1^1 \cup Y_1^2)) = K_{p+m}(X_\eta \backslash (Y_1^1 \cup Y_1^2))$.

Proof. We use a decreasing induction on r, $1 \le r \le p-1$. First, since $Y_r^{1,2}$ is a patching system of freedom degree zero, by definition we have that locally around η the maps $K'_m(Y_\eta) \to K'_m((Y_{p-1}^1)_\eta)$ and $K'_m(Y_\eta) \to K'_m((Y_{p-1}^2)_\eta)$ are both zero. Therefore, after we take elements $\alpha_{p-1}^i \in K'_{m+1}((Y_{p-1}^i)_\eta \backslash Y)$, $i = 1, 2$ whose coboundary is $\alpha \in K'_m(Y_\eta)$, they both are restrictions from an element $\alpha_{p-1} \in K'_{m+1}((Y_{p-1}^1 \cup Y_{p-1}^2)_\eta)$. The last fact follows from the excision sequence associated to the closed embedding $(Y_{p-1}^1 \cap Y_{p-1}^2 \hookrightarrow Y_{p-1}^1 \cup Y_{p-1}^2)$. Next, we proceed in the analogous way replacing Y with $Y_{p-1}^1 \cup Y_{p-1}^2$, and so on until we come to $Y_1^1 \cup Y_1^2$, and, thus, define a suitable element $\widetilde{f}_\eta = \alpha_0 \in K'_{m+p}(X_\eta \backslash (Y_1^1 \cup Y_1^2))$, which satisfies by construction all needed properties. □

Remark 5.4. In particular, the condition of claim 5.3 is satisfied for all $\eta \in X^{(p)}$. Indeed, in this case $\alpha = f_\eta \in K_m(k(\eta)) = K'_m(Y_\eta)$.

We will call classes $[\{f_y\}]$ satisfying all properties from remarks 5.1, 5.2 and claim 5.3 *good classes*. In particular, there is a good class $[Y] = [\{1_Y\}] \in \mathbf{A}_X(\mathcal{K}_p^X)^p$ for any equidimensional cycle Y of codimension p in X, where $\{1_Y\}$ denotes a collection from $\oplus_{\eta \in X^{(p)}} \mathbb{Z}$ which is a "delta-funtion" of Y.

Corollary 5.5. *Let* $Y \subset X$ *be an equidimensional cycle of codimension* p *in* X. *Then there is an explicit construction of its class in Parshin-Beilinson adelic group* $\mathbf{A}_X(\Omega_X^p)^p$, *which is the image under* dlog *of the (good) class of* $[Y]$ *in* $\mathbf{A}_X(\mathcal{K}_n^X)^p$ *(see proposition 3.22).*

5.2 K-theoretical background

The following results, needed in the proof of theorem 5.16 in the next section, are without doubt not new and follow, for example, from Waldhausen K- theory of perfect complexes, developed in [29]. However, the author did not find a reference for a simpler construction which would use only Quillen K-theory, and hence here is the description of such construction.

Let $f : (X, x_0) \to (Y, y_0)$ be a continious map of pointed topological spaces. We denote by ΩS the loop space of a pointed space (S, s_0). Consider the mapping path fibration

$$M(f) = \{(x, \varphi) | x \in X, \varphi : I \to Y, \varphi(0) = f(x)\},$$

where $I = [0, 1]$, and the homotopy fiber $F(f)$, which is the fiber over y_0 of the natural map $M(f) \to Y$, $(x, \varphi) \mapsto \varphi(1)$. Note that $F(f)$ and $M(f)$ are pointed spaces with the point (x_0, φ_0), where φ_0 is the constant map to y_0. Recall that there is a natural map $\Omega Y \to F(f)$, defined by $\gamma \mapsto (x_0, \gamma)$. Moreover, the composition $\Omega X \to \Omega Y \to F(f)$ is canonically homotopic to the constant map to $(x_0, \varphi_0) \in F(f)$. Indeed, the homotopy

$$G : \Omega X \times I \to F(f)$$

is given by

$$(\gamma, t) \mapsto (\gamma(t), \varphi_t),$$

where $\varphi_t(s) = (f \circ \gamma)(t + s(1-t))$.

Let $\mathcal{M}$ be an exact category, $\mathcal{E}_3$ be the exact category of exact triples of objects in $\mathcal{M}$. A well-known result of Quillen ([26], Theorem 2) stays that two natural maps $BQ\mathcal{E}_3 \to BQ\mathcal{M} \times BQ\mathcal{M}$, $BQ\mathcal{M} \times BQ\mathcal{M} \to BQ\mathcal{E}_3$, induced by

$$\{0 \to M' \to M \to M'' \to 0\} \mapsto (M', M'')$$

and

$$(M', M'') \mapsto \{0 \to M' \to M' \oplus M'' \to M''\},$$

correspondingly, are homotopy inverse. In fact, one may also consider two different exact subcategories $\mathcal{M}'$ and $\mathcal{M}''$ in $\mathcal{M}$ and let $\mathcal{E}_3$ be the category of exact triples in $\mathcal{M}$ such that M' is in $\mathcal{M}'$, M'' is in $\mathcal{M}''$. Then $BQ\mathcal{E}_3$ is homotopy equivalent to $BQ\mathcal{M}' \times BQ\mathcal{M}''$. The proof remains the same.

In what follows we suppose for simplicity that $\mathcal{M}$ is in fact an abelian category (which holds true in further application).

Lemma 5.6. *Let $\mathcal{C}_n$ be the exact category of complexes of length n of objects in $\mathcal{M}$, $\mathcal{E}_n$ be the subcategory in $\mathcal{C}_n$ consisting of all exact complexes. We put $B^i = \mathrm{Im}(M^{i-1} \to M^i)$ for a complex $M^\bullet$. Then the natural maps $BQ\mathcal{C}_n \to BQ\mathcal{M}^{n+1}$, $BQ\mathcal{E}_n \to BQ\mathcal{M}^n$ induced by the exact functors*

$$\{0 \to M^0 \to \ldots \to M^n \to 0\} \mapsto (M^0, \ldots, M^n)$$

and

$$\{0 \to M^0 \to \ldots \to M^n \to 0\} \mapsto (B^1, \ldots, B^n),$$

respectively, are homotopy equivalences. Moreover, the following diagram commutes up to homotopy:

$$\begin{array}{ccc} BQ\mathcal{E}_n & \longrightarrow & BQ\mathcal{M}^n \\ \downarrow & & \downarrow i \\ BQ\mathcal{C}_n & \longrightarrow & BQ\mathcal{M}^{n+1}, \end{array}$$

where the horizontal maps are as defined above, the left vertical arrow is the natural inclusion, and i is given by

$$(B^1, \ldots, B^n) \mapsto (B^1, B^1 \oplus B^2, \ldots, B^{n-1} \oplus B^n, B^n).$$

Proof. We follow a part of the proof of Theorem 1.11.7, [29]. The only difference is that we are not speaking about K-theory spectra of Waldhausen categories and do not use results about them.

Consider the natural inclusion $\mathcal{E}_{n-1} \to \mathcal{E}_n$, defined by

$$\{0 \to M^0 \to \ldots M^{n-1} \to 0\} \mapsto \{0 \to M_0 \to \ldots \to M^{n-1} \to 0 \to 0\},$$

and the natural inclusion $\mathcal{M} \to \mathcal{E}_n$, defined by

$$M \mapsto \{0 \to 0 \ldots \to 0 \to M \to M \to 0\}.$$

One may treat $\mathcal{E}_n$ as the category of exact triples in $\mathcal{C}_n$, which start with an object from $\mathcal{E}_{n-1}$, and end with an object in $\mathcal{E}_2 = \mathcal{M}$. Indeed, the explicit equivalence is given by

$$M^\bullet \mapsto \{0 \to \tau_{\leq(n-1)}(M^\bullet) \to M^\bullet \to \{0 \to B^n \to B^n \to 0\}\},$$

where $\tau_{\leq i}$ is the usual truncation functor associated to the canonical filtration on complexes. Thus, applying the modified above result of Quillen we get that $BQ\mathcal{E}_n$ is homotopy equivalent to $BQ\mathcal{E}_{n-1} \times BQ\mathcal{E}$, and from the explicit view of this homotopy one deduces the desired result for $\mathcal{E}_n$ by induction on n. The analogous reasoning leads to the needed result for $\mathcal{C}_n$. In this case we should use "bête" filtration instead of canonical one, and the inclusion $\mathcal{M} \to \mathcal{C}_n$, defined by

$$M \mapsto \{0 \to \ldots \to 0 \to M \to 0\}.$$

Finally, the exact sequences

$$0 \to B^{i-1} \to M^i \to B^i \to 0$$

for all $0 \leq i \leq n$ lead to the needed homotopy equivalence in the diagram of the lemma, see [26], §3, proof of Corollary 1. □

Let F be a homotopy fiber for the natural inclusion $BQ\mathcal{E}_n \to BQ\mathcal{C}_n$. Recall that by definition $\mathbf{K}\mathcal{N} = \Omega BQ\mathcal{N}$ for an exact category $\mathcal{N}$.

Corollary 5.7. *The inclusion of categories* $\mathcal{M} \to \mathcal{C}_n$, $M \mapsto \{0 \to M \to 0 \to \ldots \to 0\}$ *induces a map* $\mathbf{K}\mathcal{M} \to \mathbf{K}\mathcal{C}_n \to F$, *and the composite is a homotopy equivalence.*

Proof. Let us compute the induced map on homotopy groups. By lemma 5.6 for all $i \geq 0$ there is a commutative diagram

$$\begin{array}{ccc} \pi_i(\mathbf{K}\mathcal{E}_n) & \longrightarrow & \pi_{i+1}(BQ\mathcal{M})^n \\ \downarrow & & \downarrow i_* \\ \pi_i(\mathbf{K}\mathcal{C}_n) & \longrightarrow & \pi_{i+1}(BQ\mathcal{M})^{n+1} \end{array}$$

where the horizontal arrows are canonical isomorphisms, induced by maps described in lemma 5.6. Thus there is a canonical isomorphism $\pi_i(F) \cong \pi_{i+1}(BQ\mathcal{M})$ given by the alternate sum homomorphism $\pi_{i+1}(BQ\mathcal{M})^{n+1} \to \pi_{i+1}(BQ\mathcal{M})$. Moreover, the composition $\pi_{i+1}(BQ\mathcal{M}) \cong \pi_i(\mathbf{K}\mathcal{M}) \to \pi_i(\mathbf{K}\mathcal{C}_n) \to \pi_i(F) \cong \pi_{i+1}(BQ\mathcal{M})$ is the identity map and we conclude by Whitehead theorem (see [30]), since $\mathbf{K}\mathcal{M}$ has the homotopy type of a CW-complex by Milnor's result (see [19] and also [27], Appendix A). □

For a scheme S denote by $\mathcal{M}(S)$ the abelian category of coherent sheaves on it, and let $\mathcal{E}_n(S) = \mathcal{E}_n$, $\mathcal{C}_n(S) = \mathcal{C}_n$, $F(S) = F$, $\mathbf{K}(S) = \mathbf{K}\mathcal{M}$ for $\mathcal{M} = \mathcal{M}(S)$.

Proposition 5.8. *Let S be a (not necessary reduced) closed subscheme in the scheme T, $\mathcal{C}_n(T,S)$ be a full subcategory in $\mathcal{C}_n(T)$ of complexes whose cohomology sheaves are supported on S, i.e. complexes, whose restriction on $T\backslash S$ is in $\mathcal{E}_n(T\backslash S)$. Then there exists an "Euler characteristic" map $\chi : \mathbf{K}\mathcal{C}_n(T,S) \to \mathbf{K}(S)$, which is well-defined up to homotopy, with the following properties:*

(i) the induced homomorpshim $\chi_* : K_0(\mathcal{C}_n(T,S)) \to K_0'(S)$ equals to

$$[\mathcal{F}^\bullet] \mapsto \sum (-1)^i [H^i(\mathcal{F}^\bullet)],$$

where $\mathcal{F}^\bullet$ is in $\mathcal{C}(T,S)$ (here we use the isomorphism $K_0'(S) \cong K_0'(\widetilde{S})$, induced by the closed embedding, where $\widetilde{S}$ is any closed subscheme in T, containing S and with the same support);

(ii) χ commutes with closed embeddings; namely, consider a closed subset $i : T' \hookrightarrow T$, $S' = S \times_T T'$. Then the following diagram of spaces is commutative up to homotopy:

$$\begin{array}{ccc} \mathbf{K}\mathcal{C}_n(T,S) & \xrightarrow{\chi} & \mathbf{K}(S) \\ \uparrow i_* & & \uparrow i_* \\ \mathbf{K}\mathcal{C}_n(T',S') & \xrightarrow{\chi} & \mathbf{K}(S'); \end{array}$$

(iii) χ commutes with restriction on open subsets; namely, consider an open subset $j : U \hookrightarrow T$, $U' = U \times_T S$. Then the following diagram of spaces is commutative up to homotopy:

$$\begin{array}{ccc} \mathbf{K}\mathcal{C}_n(T,S) & \xrightarrow{\chi} & \mathbf{K}(S) \\ \downarrow j^* & & \downarrow j^* \\ \mathbf{K}\mathcal{C}_n(U,U') & \xrightarrow{\chi} & \mathbf{K}(U'). \end{array}$$

Proof. The natural map $\mathbf{K}\mathcal{C}_n(T,S) \to F(T\backslash S)$, induced by following diagram

$$\begin{array}{ccc} \mathbf{K}\mathcal{C}_n(T,S) & \longrightarrow & \mathbf{K}\mathcal{E}_n(T\backslash S) \\ \downarrow & & \downarrow \\ \mathbf{K}\mathcal{C}_n(T) & \longrightarrow & \mathbf{K}\mathcal{C}_n(T\backslash S) \\ \downarrow & & \downarrow \\ F(T) & \longrightarrow & F(T\backslash S), \end{array}$$

is canonically homotopy trivial. Thus there is a well-defined map

$$\mathbf{K}\mathcal{C}_n(T,S) \to F\left\{F(T) \to F(T\backslash S)\right\},$$

and this defines the needed map χ, since the diagram

$$\begin{array}{ccc} \mathbf{K}(T) & \longrightarrow & \mathbf{K}(T\backslash S) \\ \downarrow & & \downarrow \\ F(T) & \longrightarrow & F(T\backslash S) \end{array}$$

induces a map $\mathbf{K}(S) \to F\left\{F(X) \to F(T\backslash S)\right\}$, which is a homotopy equivalence.

Now we prove (i), i.e. we compute explicitly χ_* on π_0-groups. Consider a point $[\mathcal{F}^\bullet]$ in $\mathbf{K}\mathcal{C}_n(T,S)$ corresponding to a loop in $BQ\mathcal{C}_n(T,S)$ defined by a complex $\mathcal{F}^\bullet$. There exists a homotopy inside $BQ\mathcal{C}_n(T,S)$ between the loop $[\mathcal{F}^\bullet]$ and the sum of loops

$$[\tau_{\leq(n-1)}\mathcal{F}^\bullet] + [\{0 \to \ldots \to B^n \to B^n \to 0\}] + [H^n(\mathcal{F}^\bullet)[-n]].$$

In addition, $[H^n(\mathcal{F}^\bullet)[-n]]$ is homotopic inside $BQ\mathcal{C}_n(T,S)$ to the sum

$$(-1)^n[H^n(\mathcal{F}^\bullet)] + \sum_{j=0}^{n-1}(-1)^j[\{0 \to H^n(\mathcal{F}^\bullet) \to H^n(\mathcal{F}^\bullet) \to 0\}[-j]],$$

where the short complexes have support in degrees 0 and 1. Continuing, we show by induction that the initial loop may be homotoped inside $BQ\mathcal{C}_n(T,S)$ to a the sum of $\sum(-1)^i[H^i(\mathcal{F}^\bullet)]$, and a sum of loops inside $BQ\mathcal{E}_n(T)$. The points in $\mathbf{K}\mathcal{C}_n(T,S)$, corresponding to loops in $BQ\mathcal{E}_n(T)$, evidently have zero image under ψ_* on π_0-groups, and we are done.

The proof if (ii) and (iii) is a trivial check, which uses the following facts: for (ii) the natural map

$$T'\backslash S' = T' \times_T (T\backslash S) \hookrightarrow T\backslash S$$

is a closed embedding, for (iii) the natural map

$$U\backslash U' = U \times_T (T\backslash S) \hookrightarrow T\backslash S$$

is an open embedding, and if in the commutative diagram of topological spaces

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

the vertical arrows are homotopy equivalences, then after taking the homotopy inverse to them the diagram remains commutative up to homotopy. □

Now consider a finite complex $\mathcal{P}^\bullet$ from $\mathcal{C}_n(T,S)$ of flat sheaves on T. Then there is a well-defined map $*\cdot\mathcal{P}^\bullet : \mathbf{K}(T) \to \mathbf{K}(S)$, which is unique up to homotopy and is the composite of the map induced by $*\otimes_{\mathcal{O}_T} : \mathcal{M}(T) \to \mathcal{C}_n(T,S), \mathcal{F} \mapsto \mathcal{F}\otimes_{\mathcal{O}_T}\mathcal{P}^\bullet$ and $\chi : \mathbf{K}\mathcal{C}_n(T,S) \to \mathbf{K}(S)$. For a class $[\mathcal{F}]$ in $K_0'(T)$ we have by proposition 5.8, (i) the equality $[\mathcal{F}]\cdot\mathcal{P}^\bullet = \sum(-1)^i H^i(\mathcal{F}\otimes_{\mathcal{O}_T}\mathcal{P}^\bullet) \in K_0'(S)$.

Proposition 5.9. *Suppose $\mathcal{P}^\bullet$ is a finite flat resolution of $\mathcal{O}_S$ on $\mathcal{O}_T$ (here we suppose that the complex $\mathcal{P}^\bullet$ has support in non-positive terms), T admits an ample line bundle (e.g. T is quasi-projective). Then $*\cdot\mathcal{P}^\bullet$ is homotopic to the map f^*, where $f : S \hookrightarrow T$ is the closed embedding (see [26], §7, 2.5).*

Proof. By Quillen resolution theorem ([26], §4, Corollary 1) $BQ\mathcal{M}(T)$ is homotopy equivalent to its subspace $BQ\mathcal{F}(T)$, where $\mathcal{F}(T)$ is the exact category of coherent sheaves, which are Tor-independent with $\mathcal{O}_S$. For $BQ\mathcal{F}(T)$ we have an exact sequence

$$0 \to \tau^{\leq 0}(*\otimes_{\mathcal{O}_T}\mathcal{P}^\bullet) \to *\otimes_{\mathcal{O}_T}\mathcal{P}^\bullet \to *\otimes_{\mathcal{O}_T}\mathcal{O}_S \to 0$$

of functors from $\mathcal{F}(T)$ to $\mathcal{C}_n(T,S)$, where by definition

$$\tau^{\leq 0}(A^\bullet) = \{\ldots \to A^i \to \ldots \to A^{-1} \to \operatorname{Im}\left\{A^{-1} \to A^0\right\} \to 0\}$$

for a complex $A^\bullet$. Thus for the induced maps on $\mathbf{K}$-spaces the map $\mathbf{K}(\tau^{\leq 0}(*\otimes_{\mathcal{O}_T}\mathcal{P}^\bullet)) + \mathbf{K}(*\otimes_{\mathcal{O}_T}\mathcal{O}_S)$ is homotopic to $\mathbf{K}(*\otimes_{\mathcal{O}_T}\mathcal{P}^\bullet)$, where the sum is taken with respect to the natural H-structures on BQ-spaces of exact categories, see [26], §3, proof of Corollary 1. Moreover, the first map is in fact a map to $\mathbf{K}(\mathcal{E}_n(T))$, the second one equals to f^* on $\mathbf{K}(\mathcal{F}(T))$, and the proposion is proved. □

Remark 5.10. For the elements from $K_*(T)$ the map $*\cdot\mathcal{P}^\bullet$ is just the composition of the usual restriction on S with multiplication by $\chi_*([\mathcal{P}^\bullet]) \in K'_0(S)$.

Proposition 5.11. *Suppose we are given a closed embedding $i : T' \hookrightarrow T$, and put $S' = S\times_T T'$, $j : U = T\backslash T' \hookrightarrow T$, $U' = U\times_T S$. Consider arbitrary elements $x \in K_m(S)$, $y \in K'_n(U)$, $m, n \geq 0$, $m+n \geq 1$. Then in the above notations*

$$\nu(x\cdot(y\cdot\mathcal{P}^\bullet)) = x\cdot(\nu(y)\cdot i^*\mathcal{P}^\bullet) \in K'_{m+n-1}(S')$$

where $\nu : K'_*(U) \to K'_{*-1}(T')$ denotes the usual boundary map (the same for S' and U').

Proof. Both squares in the following diagram of spaces are commutative up to homotopy by proposition 5.8, (ii), (iii):

$$\begin{array}{ccc}
\mathbf{K}(T')\wedge\mathbf{K}(\mathcal{P}(S)) & \stackrel{*\cdot\mathcal{P}^\bullet\otimes *}{\longleftarrow} & \mathbf{K}(S') \\
\downarrow i_*\times\mathrm{id} & & \downarrow i_* \\
\mathbf{K}(T)\wedge\mathbf{K}(\mathcal{P}(S)) & \stackrel{*\cdot\mathcal{P}^\bullet\otimes *}{\longleftarrow} & \mathbf{K}(S) \\
\downarrow j^*\times\mathrm{id} & & \downarrow j^* \\
\mathbf{K}(U)\wedge\mathbf{K}(\mathcal{P}(S)) & \stackrel{*\cdot\mathcal{P}^\bullet\otimes *}{\longleftarrow} & \mathbf{K}(U'),
\end{array}$$

where $\mathcal{P}(S)$ denotes the exact category of locally free $\mathcal{O}_S$-modules of finite rank, and "$\wedge$" denotes the wedge product of pointed topological spaces. Writing explicitly the boundary map in the corresponding long homotopy sequences we get the result. □

Remark 5.12. The last reasoning is a particular case of a more general Theorem 2.5, [12] in which in fact we need the diagrams involved to be commutative only up to homotopy.

Let us also mention the following simple fact:

Lemma 5.13. *Let $S \hookrightarrow T$ be a closed embedding, S_{red} be the reduced scheme, $j : S_{red} \to S$ be a natural embedding. Then $j_*\circ\nu_{red} = \nu$, where $\nu : K'_*(T\backslash S) \to K'_{*-1}(S)$ is the boundary map (analogously for ν_{red}).*

Proof. This follows immediately from the commutativity of the diagram of CW-spaces

$$\begin{array}{ccc}
BQ\mathcal{M}(S_{red}) & \stackrel{j_*}{\longrightarrow} & BQ\mathcal{M}(S) \\
\downarrow & & \downarrow \\
BQ\mathcal{M}(T) & \stackrel{=}{\longrightarrow} & BQ\mathcal{M}(T) \\
\downarrow & & \downarrow \\
BQ\mathcal{M}(T\backslash S_{red}) & \stackrel{=}{\longrightarrow} & BQ\mathcal{M}(T\backslash S)
\end{array}$$

after passing to long homotopy sequences, associated to the vertical sequences, which are fibrations up to homotopy. □

Lemma 5.13 immediately implies the following statement.

Corollary 5.14. *Proposition 5.11 remains true after we change schematic intersections by their reduced parts.*

Example 5.15. Let T be a local scheme $\mathbb{A}^2_{(0,0)}$ with coordinates (x,y), $T'=\{xy=0\}$, $S=\{x+y=0\}$, $\mathcal{P}^\bullet=\{\mathcal{O}_T\xrightarrow{x+y}\mathcal{O}_T\}$. Choose two rational functions $f(x),g(y)$ on the corresponding components of S having the opposite valuations at the origin. They define an element $\alpha\in K_1'(T')$. Then $\alpha\cdot i^*\mathcal{P}^\bullet=a/b\in K_1(k)$, where a and b are the main parts of $f(x)$ and $g(y)$ in x and y, respectively, and $i:T'\hookrightarrow S$ is a closed embedding.

5.3 Formula for a product of adelic cocyles

Let X be a smooth variety over an infinite field k, Y, Z be two equidimensional cycles in X of codimensions p and q, respectively. Consider two cocycles in Gersten resolutions $(\mathbf{G}^X_{m+p})^p$ and $(\mathbf{G}^X_{n+q})^q$, respectively,

$$\{f_y\}\in\bigoplus_{y\in Y^{(0)}}K_m(k(y)),$$

and

$$\{g_z\}\in\bigoplus_{z\in Z^{(0)}}K_n(k(z)).$$

Suppose that Y and Z intersect properly. In addition, suppose that for any irreducible component w of the intersection $W=Y\cap Z$ the collection $\{f_y\}_w$ may be represented as an element from $K_p(Y_w)$, which will be denoted by α_w, while the collection $\{g_z\}_w$ may be represented as an element from $K_q'(Z_w)$, which will be denoted by β_w (here as above the index w by a collection means restriction on $X_w=\mathrm{Spec}(\mathcal{O}_{X,w})$).

Let $f\in\mathbf{A}_X(\mathcal{K}^X_{p+m})^p$ and $g\in\mathbf{A}_X(\mathcal{K}^X_{q+n})^q$ be good classes corresponding to the collections $\{f_y\}$ and $\{g_z\}$ defined by certain patching systems $Y_r^{1,2}$, $1\le r\le p-1$ and $Z_s^{1,2}$, $1\le s\le q-1$, respectively (see section 5.1). Moreover, we may choose the patching system $Z_s^{1,2}$ in such a way that no component of $Z_s^1\cap Y$ is contained inside Z_s^2 for all $1\le s\le q-1$. In fact, to do this one should use theorem 4.15 in the affine case, since the codimension condition from corollary 4.16 are no more satisfied. What we do is choosing a patching system on the whole X such that for each irreducible component w of W there exists a patching subsytem satisfying the above condition locally around w (the existence of such patching system on X can be shown by the same method as in the proof of proposition 4.26). Then, defining the parts of the adele g corresponding to flags ending up by w we will use the latter patching subsystem. This will be always implied in what follows.

Theorem 5.16. *Let $\mathcal{P}^\bullet \to \mathcal{O}_Y$ be a finite projective resolution of $\mathcal{O}_Y$ on X. Under the above assumptions the following relation holds true:*

$$\nu_{p+q}(f\cdot g) = \{\alpha_w \cdot (\beta_w \cdot i_Z^*\mathcal{P}^\bullet)\} \in \bigoplus_{w\in W^{(0)}} K_{m+n}(k(w)),$$

where $i_Z : Z \hookrightarrow X$ is a closed embedding (considered locally around w in each summand).

Proof. First, according to remark 5.1 for the computation of the $p+q$ part of the adele $f\cdot g$ one may consider only components $f_{\eta_0\dots\eta_p}$ of f for which η_r is one of the irreducible components of Y_r^1 for all $1\le r\le p-1$, and η_p is one of the irreducible components of Y. Analogously, on may consider only components $g_{\eta_p\dots\eta_{p+q}}$ of g for which η_p is one of the irreducible components of Y, η_{p+s} is one of the irreducible components of the intersection $Y\cap Z_s^1$ for all $1\le s\le q-1$, and η_{p+q} is one of the irreducible components of the intersection $Y\cap Z$. Choose one flag $\eta_p\dots\eta_{p+q}$ satishying the above condition.

Note that since Y and Z intersect properly, the intersections of $Y_r^{1,2}$ and Y with $Z_s^{1,2}$ and Z are also proper for all $1\le r\le p-1$, $1\le s\le q-1$. Consequently, since the classes f and g are good we have $f_{\eta_0\dots\eta_p} = \widetilde{f}_{\eta_p} \in K_{m+p}(X_{\eta_p}\backslash(Y_1^1\cup Y_1^2))$, and $g_{\eta_p\dots\eta_{p+q}} = \widetilde{g}_{\eta_{p+q}} \in K_{q+n}(X_{\eta_{p+q}}\backslash(Z_1^1\cup Z_1^2))$.

We claim that the residue $\nu_p((f\cdot g)_{0\dots p-1\eta_p\dots\eta_{p+q}}) \in K_{m+q+n}(k(\eta_p))$ depends only on η_p and η_{p+q}, and equals to $f_{\eta_p}\cdot i_{\eta_p}^*\widetilde{g}_{\eta_{p+q}} \in K_{m+q+n}(k(\eta_p))$, where $i_{\eta_p} : \mathrm{Spec}k(\eta_p)\hookrightarrow X_{\eta_p}$ is the closed embedding. This can be shown using proposition 5.11 first with $S=T=X_{\eta_p}$, $\mathcal{P}^\bullet = \mathcal{O}_S$, $T' = (Y_1^1\cup Y_1^2)_{\eta_p}$, and then inductively with $S=T=(Y_s^1)_{\eta_p}$, $\mathcal{P}^\bullet=\mathcal{O}_S$, $T'=(Y_{s+1}^1\cup Y_{s+1}^2)_{\eta_p}$ for $1\le s\le p-1$ (more precisely, for the induction step we use that classes are good and satisfy properties from claim 5.3).

Further, note that in fact $\nu_p((f\cdot g)_{0\dots p-1\eta_p\dots\eta_{p+q}})$ is represented by an element

$$\alpha_{\eta_{p+q}}\cdot i_Y^*\widetilde{g}_{\eta_{p+q}} \in K_{m+q+n}(Y_{\eta_{p+q}}\backslash(Z_1^1\cup Z_1^2)),$$

where $i_Y : Y\hookrightarrow X$ is the closed embedding. In what follows we consider all varieties locally around η_{p+q} denoting them by the same letter. Put $Z_0^1 = X$, $Z_q^1 = Z$ and let $i_s : Z_s^1 \hookrightarrow X$ denote a closed embedding for $1\le s\le q-1$.

By proposition 5.8 and corollary 5.14 the following diagram commutes for all $0\le s\le q-1$:

$$\begin{array}{ccc} K'_{*+m}(Y\cap(Z_s^1\backslash(Z_{s+1}^1\cup Z_{s+1}^2))) & \longleftarrow & K'_*(Z_s^1\backslash(Z_{s+1}^1\cup Z_{s+1}^2)) \\ \downarrow & & \downarrow \\ K'_{*-1}(Y\cap(Z_{s+1}^1\backslash Z_{s+1}^2)) & \longleftarrow & K'_{*+m-1}(Z_{s+1}^1\backslash Z_{s+1}^2)), \end{array}$$

where vertical arrows are compositions of coboundary maps with restrictions on open subsets, and horizontal arrows are compositions of $*\cdot i_s^*\mathcal{P}^\bullet$ (respectively, $*\cdot i_{s+1}^*\mathcal{P}^\bullet$) with multiplication by $\alpha_{\eta_{p+q}}\in K_m(Y)$. Therefore, since we required that Z_s^2 contains no irreducible components of the intersection $Z_s^1\cap Y$ for all $1\le s\le q-1$ (locally around η_{p+q}), we get by induction on s that

$$\nu_{\eta_p\dots\eta_{p+q}}(\alpha_{\eta_{p+q}}\cdot i_Y^*\widetilde{g}_{\eta_{p+q}}) = \nu_{\eta_p\dots\eta_{p+q}}(\alpha_{\eta_{p+q}}\cdot(\widetilde{g}_{\eta_{p+q}}\cdot\mathcal{P}^\bullet)) = \alpha_{\eta_{p+q}}\cdot(\nu_{\eta_p\dots\eta_{p+q}}(\widetilde{g}_{\eta_{p+q}})\cdot i_Z^*\mathcal{P}^\bullet).$$

Now take the sum of these expressions over all flags $\eta_p \dots \eta_{p+q}$ such that η_{p+s} is an irreducible component of $Y \cap Z_s^1$ for all $1 \leq s \leq q-1$, η_0 is a component of Y and η_q is a component of $Y \cap Z$, and get the desired identity. $\square$

Suppose $\{f_y\} \in (\mathbf{G}_m^X)^p$ and $\{g_z\} \in (\mathbf{G}_n^X)^q$ are two Gersten cocyles satisfying all above properties and an additional one: the collection $\{h_z\}_w$ may be represented by an element $\beta_w \in K_n(Z_w)$ for all irreducible components w of W.

Corollary 5.17. *Under the above conditions the product in K-cohomology of classes of $\{g_y\}$ and $\{h_z\}$ is represented by a cocycle*

$$\{i(Y,Z;w)\overline{\alpha}_w \cdot \overline{\beta}_w\} \in \bigoplus_{w \in W^{(0)}} K_{m+n}(k(w)),$$

where $i(Y,Z;w)$ is the local intersection index of Y and Z at w, and the bar denotes natural homomorphisms $K_m(Y_w) \to K_m(k(w))$ and $K_n(Z_w) \to K_n(k(w))$. In particular, the intersection product in Chow groups coincides with the natural product in K-cohomology (the last assertion have beed already proved in [10] and [12]).

Proof. Recall that

$$i(Y,Z;w) = \sum_{i \geq 0} (-1)^i l(\mathrm{Tor}_i^{\mathcal{O}_{X,w}}(\mathcal{O}_{Y,w}, \mathcal{O}_{Z,w})),$$

where $l(\cdot)$ is the length of an $\mathcal{O}_{X,w}$-module (i.e. a length of filtration whose adjoint quotients are one-dimensional spaces over the field $k(w)$). Thus, the lemma follows directly from theorem 5.16 and remark 5.10. $\square$

Remark 5.18. The conditions of corollary 5.17 are satisfied, in particular, if each schematic point w is regular on Y and Z.

Remark 5.19. To prove theorem 5.16 we do not need theorem 4.1. Indeed, since the composition of morphisms $\mathcal{K}_n(\mathcal{O}_X) \to \underline{\mathbf{A}}_X(\mathcal{K}_n)^\bullet \to (\underline{\mathbf{G}}_n^X)^\bullet$ is identity, what we only need is the explicit construction of good classes (which uses, nevertheless, the notion of strongly locally effaceable pairs).

6 Triple products

6.1 Zero-cycles and divisors

As it is explained in section 6.3, the existence of the multiplicative adelic resolution for the sheaves $\mathcal{K}_n^X$ gives rise to some explicit (adelic) formulas for their Massey higher products.

The case of a curve

Let X be a projective nonsingular curve over a field k. Consider elements $\mathcal{L}, \mathcal{M} \in H^1(X, \mathcal{K}_1^X)_l = \mathrm{Jac}(X)_l$ and $l \in H^0(X, \mathcal{K}_0^X) = \mathbb{Z}$. The triple $(\mathcal{L}, l, \mathcal{M})$ verifies the condition $\mathcal{L} \cdot l = l \cdot \mathcal{M} = 0$ in K-cohomology groups, hence one may define a triple product

$$m_3(\mathcal{L}, l, \mathcal{M}) \in H^1(X, \mathcal{K}_2^X)/(k^* \cdot \mathrm{Jac}(X)).$$

Note, that $k^* \cdot \mathrm{Jac}(X)$ is in the kernel of the direct image map $H^1(X, \mathcal{K}_2^X) \to k^*$, so one obtains a well-defined element $\overline{m}_3(\mathcal{L}, l, \mathcal{M}) \in \mu_l \subset k^*$. The explicit formula is as follows. Choose two adeles (in fact, ideles) $f_{01}, g_{01} \in \mathbf{A}_X(\mathcal{K}_1^X)^1$, corresponding to $\mathcal{L}$ and $\mathcal{M}$, respectively. Then there are two adeles $\widetilde{f}, \widetilde{g} \in \mathbf{A}_X(\mathcal{K}_1^X)^0$ such that $d\widetilde{f} = \widetilde{f}_1 \widetilde{f}_0^{-1} = f_{01}^l$, $d\widetilde{g} = \widetilde{g}_1 \widetilde{g}_0^{-1} = g_{01}^l$. By definition, we have

$$\overline{m}_3(\mathcal{L}, l, \mathcal{M}) = \sum_{x \in X} \nu_{Xx}(\{\widetilde{f}_X, g_{Xx}\}\{f_{Xx}, \widetilde{g}_x\}),$$

where $\{\cdot, \cdot\}$ denotes the product in K-groups. In particular, when $\widetilde{f}_x = \widetilde{g}_x = 1$ for all $x \in X$ (which can be always obtained if k is algebraically closed), this is the adelic formula for the Weil pairing of $\mathcal{L}$ and $\mathcal{M}$. Thus, by corollary 2.20 we get the following statement.

Proposition 6.1. *The direct image of the triple product $\overline{m}_3(\mathcal{L}, l, \mathcal{M})$ coincides with the Weil pairing $\psi_l(\mathcal{L}, \mathcal{M})$ of $\mathcal{L}$ and $\mathcal{M}$.*

The case of arbitrary dimension

Let X be a smooth variety of dimension d over an infinite field k. Let l be a natural number, and take elements $\alpha \in H^d(X, \mathcal{K}_d^X)_l = CH^d(X)_l$, $\mathcal{L} \in H^1(X, \mathcal{K}_1^X)_l = CH^1(X)_l$. Then there is a well-defined triple product

$$m_3(\alpha, l, \mathcal{L}) \in H^d(X, \mathcal{K}_{d+1})/(\alpha \cdot H^0(X, \mathcal{K}_1^X) + H^{d-1}(X, \mathcal{K}_d^X) \cdot \mathcal{L}).$$

Suppose that X is proper, and that α, $\mathcal{L}$ are homologically trivial, i.e. $\deg(\alpha) = 0$ (in fact, this is automatically satisfied) and $\mathcal{L} \in \mathrm{Pic}^0(X)$.

Proposition 6.2. *The subgroup $\alpha \cdot H^0(X, \mathcal{K}_1^X) + H^{d-1}(X, \mathcal{K}_d^X) \cdot \mathcal{L}$ lies in the kernel of the direct image map $H^{d+1}(X, \mathcal{K}_d^X) \to k^*$.*

Proof. We use corollary 5.17 to compute the products in K-cohomology explicitly. Thus, the pairing

$$H^0(X, \mathcal{K}_1^X) \times H^d(X, \mathcal{K}_d^X) \to H^d(X, \mathcal{K}_{d+1}^X) \to k^*$$

occurs to be $c^{\deg(\alpha)}$, where $c \in k^* = H^0(X, \mathcal{K}_1^X)$, $\alpha \in CH^d(X) = H^d(X, \mathcal{K}_d^X)$. Hence when α is a homologically trivial this is trivial as well. Further, without loss of generality we may suppose that the field k is algebraically closed. The pairing

$$\langle \cdot, \cdot \rangle : H^1(X, \mathcal{K}_1^X) \times H^{d-1}(X, \mathcal{K}_d^X) \to H^d(X, \mathcal{K}_{d+1}^X) \to k^*$$

defines a function $\langle \cdot, \beta \rangle : \mathrm{Pic}^0(X) \to k^*$ for each $\beta \in H^{d-1}(X, \mathcal{K}_d^X)$. We claim that this is a regular map from $\mathrm{Pic}^0(X)$ to k^* and, thus, is a constant unit function. To show this let us represent $\beta \in H^{d-1}(X, \mathcal{K}_d^X)$ by a Gersten cocycle $\sum_i f_i[C_i]$, where C_i are irreducible curves on X, $f_i \in k(C_i)^*$, and let the divisor $D \subset X$ be a representative for $\mathcal{L} \in H^1(X, \mathcal{K}_X^1)$. Suppose that D intersects properly $C = \cup_i C_i$ and does not pass through the finite set $\cup_i \mathrm{div}(f_i) \cup C^{sing}$. This can be always achieved by moving the divisor D in its linear system. Then by corollary 5.17 we have $\langle \mathcal{L}, \beta \rangle = \prod_{x \in D \cap C} f^{n_x}(x)$, where f denotes f_i for the unique C_i which contains the corresponding x, and n_x is the multiplicity of the intersection of D with C at x. To show the algebraicity of the map $\langle \cdot, \beta \rangle$ take a rational section of the Poincaré line bundle on the product $X \times \mathrm{Pic}^0(X)$ (after any choice of a closed point on X). This defines locally on $\mathrm{Pic}^0(X)$ a correspondence of degree r with $C_i \subset X$, where r is the locally defined number of points in the intersection of D with C_i. Hence, locally on $\mathrm{Pic}^0(X)$ we get a regular map to $\mathrm{Sym}^r(C_i)$, whose composition with $\mathrm{Sym}^r(f_i)$ defines a C_i-part of the pairing $\langle \cdot, \beta \rangle$, proving its regularity. $\square$

Thus, there is a well-defined number

$$\overline{m}_3(\alpha, l, \mathcal{L}) \in \mu_l \subset k^*,$$

or in other words we get a pairing

$$\phi_l : CH^0(X)_l^0 \times CH^1(X)_l^0 \to \mu_l.$$

Theorem 6.3. *Suppose the X is a smooth projective variety of dimension d over an algebraically closed field k, $(\mathrm{char} k, l) = 1$. Then the constructed above pairing ϕ_l coincides with the Weil pairing ψ_l on the l-torsion of Albanese and Picard varieties of X.*

Proof. Suppose X is not a curve. Let $i : C \hookrightarrow X$ be a general $(d-1)$-th hyperplane section of X. Recall that the map $CH^1(C)_l \to \mathrm{Alb}(X)_l$ is surjective. Indeed, there is an exact sequence (arising from Kummer sequence)

$$0 \to \mathrm{Pic}^0(X)_l \to H^1_{\acute{e}t}(X, \mu_l(1)) \to NS(X)_l \to 0,$$

Dualizing one gets a surjective map

$$H^{2d-1}_{\acute{e}t}(X, \mu_l(d)) \to \mathrm{Alb}(X)_l,$$

Weak Lefschetz theorem for étale cohomology (see [18], Chapter VI, §7) asserts that the composition of two natural maps (namely, inverse and direct image maps)

$$H^1_{\acute{e}t}(X, \mu_l(1)) \to H^1_{\acute{e}t}(C, \mu_l(1)) \to H^{2d-1}_{\acute{e}t}(X, \mu_l(d))$$

is an isomorphism. Hence the second (direct image) map is surjective. Finally, $H^1_{\acute{e}t}(C, \mu_l(1)) = CH^1(C)_l$, and we get the desired statement.

Remark 6.4. In fact, by a theorem of Roitman (e.g. see [5]) there is an isomorphism $CH^d(X)_l \cong \mathrm{Alb}(X)_l$, therefore the natural map $CH^1(C)_l \to CH^d(X)_l$ is surjective as well, however we need it only for Albenese torsion.

Consider two elements $\mathcal{M} \in CH^1(C)_l$, $\mathcal{L} \in CH^1(X)_l$. The product formula for Weil pairing (arising from that for étale cohomology, for example) implies that

$$\psi_l(i_*(\mathcal{M}), \mathcal{L}) = \psi_l(\mathcal{M}, i^*(\mathcal{L})).$$

On the other other hand, the product formula for higher Massey products, see corollary 4.6, implies that

$$\phi_l(i_*(\mathcal{M}), \mathcal{L}) = \phi_l(\mathcal{M}, i^*(\mathcal{L}))$$

Thus, the surjectivity of the direct image map together with proposition 6.1 give the needed result. □

Theorem 6.3 and Proposition 6.7 imply the following fact.

Corollary 6.5. *Let the 0-cycle $\sum_i n_i[x_i]$ be the representative for $\alpha \in CH^d(X)_l$ D be a representive for $\mathcal{L} \in CH^1(X)_l$ such that $x_i \notin D$ for all i. Suppose $d(\sum_j f_j[C_j]) = \sum_i n_i[x_i]$ where d is the differential in Gersten resolution, and $l[D] = \mathrm{div}(g)$ where $g \in k(X)^*$. Suppose that the proper intersection of D with the curve $C = \cup_j C_j$ does not contain the finite set $\cup_j \mathrm{div}(f_j) \cup C^{sing}$. Then the following formula for the Weil pairing of α and $\mathcal{L}$ holds true:*

$$\psi_l(\alpha, \mathcal{L}) = \prod_{y \in D \cap C} f(y) \cdot \prod_{x_i} g^{-1}(x_i).$$

Further, suppose that $H^d(X, \mathcal{K}_{d+1}) = k^*$. For example, by a classical result of Moore this holds when k is a subfield in the algebraic closure of a finite field. Then the forth product m_4 defines a map

$$m_4 : (\mathrm{Alb}(X)_l \times \mathrm{Pic}^0(X)_l)_0 \to H^{d-1}(X, \mathcal{K}_{d+1})/l,$$

where the index "$0''$" denotes that we consider only the part consisting of all $(\alpha, \mathcal{L})$ such that $\psi_l(\alpha, \mathcal{L}) = 1$. It can be checked that this pairing commutes with multiplication by l and thus we get a pairing

$$(T_l(\mathrm{Alb}(X)) \times T_l(\mathrm{Pic}^0(X)))_0 \to \varprojlim H^{d-1}(X, \mathcal{K}_{d+1})/l^r.$$

It is an interesting question to find the image of this pairing and also to compute its image in $H^{2d}_{\acute{e}t}(X, \mathbb{Z}_l(d+1))$ under the l-adic regulator map. For example, when $d = 1$ one could compare this with a result of Tate, see [2], Théorème 8.

6.2 General case

Let X be a smooth variety of dimension d over an infinite field k. Let l be a natural number, p, q be non-negative integers such that $p + q = d + 1$, and take elements $Y \in H^p(X, \mathcal{K}_p^X)_l = CH^p(X)_l$, $Z \in H^q(X, \mathcal{K}_q^X)_l = CH^q(X)_l$. Then there is a well-defined triple product

$$m_3(a, l, b) \in H^d(X, \mathcal{K}_{d+1})/(Y \cdot H^{q-1}(X, \mathcal{K}_q^X) + H^{p-1}(X, \mathcal{K}_p^X) \cdot Z).$$

Let us compute this product explicitly. For this purpose we will need the following lemma, which illustrates the freedom of choice in computations with adeles.

Lemma 6.6. *Let X be a smooth variety over an infinite field k. Let the collection $\{f_y\} \in (\mathbf{G}_m^X)^p$ be a couboundary in Gersten resolution with support in an equidimensional cycle $Y \subset X$, and let $\{\widetilde{f_{\widetilde{y}}}\} \in (\mathbf{G}_m^X)^{p-1}$ be such that $d(\{\widetilde{f_{\widetilde{y}}}\}) = \{f_y\}$. Supose we are given a cocycle $f \in \mathbf{A}_X(\mathcal{K}_m^X)^p$ such that $\nu_X(f) = \{f_y\}$, and whose restriction on $U = X\backslash Y$ is zero in the adelic group $\mathbf{A}_U(\mathcal{K}_m^U)^p$. Then there exists an adele $\widetilde{f} \in \mathbf{A}_X(\mathcal{K}_m^X)^{p-1}$ such that $d(\widetilde{f}) = f$, $\nu_X(\widetilde{f}) = \{\widetilde{f_{\widetilde{x}}}\}$ and $\widetilde{f}|_U$ is a good adelic class on U with respect to a Gersten cocyle $\{\widetilde{f_{\widetilde{x}}}\}|_U$ (see section 5.1 for the definition of a good class).*

Proof. Let $B^\bullet = \mathrm{Ker}(\nu_X : \mathbf{A}_X(\mathcal{K}_m^X)^\bullet \to (\mathbf{G}_m^X)^\bullet)$. Since ν_X is surjective, by theorem 4.1 the complex $B^\bullet$ is exact.

Choose $\widetilde{f}_1 \in \mathbf{A}_X(\mathcal{K}_m^X)^{p-1}$ such that $\nu_X(\widetilde{f}_1) = \{\widetilde{f_{\widetilde{x}}}\}$. The cocyle $d(\widetilde{f}_1) - f$ belongs to B^p, hence there exists $h \in B^{p-1}$ such that $d(h) = d(\widetilde{f}_1) - f$. The adele $\widetilde{f}_2 = \widetilde{f}_1 - h$ requires all needed properties except from being good on U.

If $p = 1$ we are done. Otherwise, choose $\widetilde{f}_U \in \mathbf{A}_U(\mathcal{K}_m^U)^{p-1}$ to be a good class with respect to the Gersten cocycle $\{\widetilde{f_{\widetilde{x}}}\}|_U$. Since $f|_U$=0, by the same reason as above there is $h_U \in B_U^{p-2}$ such that $d_U(h_U) = \widetilde{f}_U - \widetilde{f}_2|_U$. Here by definition $B_U^\bullet = \ker(\nu_U)$ and d_U denotes the differential in the adelic complex on U. Now let h'_U be the extension of h_U by zero to X. This means that for all flags in U we keep the same value as h_U has and we put zero at the other flags on X. Though this operation does not commute with differential, it commutes via ν with the analogous operation on Gersten complex. Hence, $h'_U \in B^{p-2}$. Moreover, the restriction of h'_U on U is again h_U. Finally, the adele $\widetilde{f} = \widetilde{f}_2 + d(h'_U) \in \mathbf{A}_X(\mathcal{K}_m^X)^{p-1}$ satisfies all required properties. □

Consider two non-intersecting equidimensional cycles Y, Z in X whose classes are in $CH^p(X)_l$ and $CH^q(X)_l$, respectively, $p + q = d + 1$. Let $\{f_{\widetilde{y}}\} \in (\mathbf{G}_p^X)^p$, $\{g_{\widetilde{z}}\} \in (\mathbf{G}_q^X)^q$ be two collections such that $d(\{f_{\widetilde{y}}\}) = lY$, $d(\{g_{\widetilde{z}}\}) = lZ$ and whose supports are equidimensional cycles $\widetilde{Y}$ and $\widetilde{Z}$, respectively. Suppose also that the rational functions $f_{\widetilde{y}}$ and $g_{\widetilde{z}}$ are regular at all points from the intersections $\widetilde{Y} \cap Z$ and $Y \cap \widetilde{Z}$, respectively (the latter intersections are automatically proper). Then the following formula holds true.

Proposition 6.7. *The triple Masey product $m_3(Y, l, Z)$ is represented by a Gersten cocycle*

$$\sum_{x \in \widetilde{Y} \cap Z} f(x)[x] + \sum_{x \in Y \cap \widetilde{Z}} g^{-1}(x)[x] \in (G_{d+1}^X)^d,$$

where $f(x)$, $g(x)$ mean the values of $f_{\widetilde{y}}$ and $g_{\widetilde{z}}$ whose corresponding components contain x.

Proof. Choose two good adelic classes $[Y]$ and $[Z]$ for Y and Z, respectively. By lemma 6.6 there are adele $f \in \mathbf{A}_X(\mathcal{K}_p^X)^{p-1}$, $g \in \mathbf{A}_X(\mathcal{K}_q^X)^{q-1}$ such that $df = l[Y]$, $dg = l[Z]$ and the restrictions f_U and g_V are good with respect to the corresponding Gersten cocyles on open subsets $U = X\backslash Y$ and $V = X\backslash Z$. Moreover, we may also require that the patching systems defining $[Z]$ and g satisfy the additional intersection condition from theorem 5.16 with respect to the patching systems of f and $[Y]$ (this may be achieved by the same method as in the proof of lemma 6.6). By definition, $m_3(Y, l, Z)$ is represented by a Gersten cocycle $\nu_X(f \cdot [Z] - [Y] \cdot g)$. Since $\nu_X(f \cdot [Z]) = \nu_U((f \cdot [Z])|_U)$ and $\nu_X([Y] \cdot g) = \nu_V(([Y] \cdot g)|_V)$, we are done by theorem 5.16. $\square$

Now suppose that X is proper, and that the cycles Y, Z are homologically trivial. For the case when $\mathrm{char}(k) = 0$ this means that the cycle map images of Y and Z in Betti cohomology $H^{2p}_B(X_{\mathbb{C}}, \mathbb{Z}))$ and $H^{2q}_B(X_{\mathbb{C}}, \mathbb{Z})$, respectively, are trivial (as usual, after we choose any model of X defined over $\mathbb{C}$). For the positive characterictic case one should consider the same for étale cohomology groups $H^{2p}_{ét}(X_{\overline{k}}, \mathbb{Z}_l(p))$ and $H^{2q}_{ét}(X_{\overline{k}}, \mathbb{Z}_l(q))$, respectively, for any prime $l \neq \mathrm{char}(k)$, where $\overline{k}$ is the algebraic closure of k. Recall that for all $m \geq 0$ there are canonical homomorphisms $H^m(X, \mathcal{K}^X_m) \to CH^m(X, 0)$, $H^m(X, \mathcal{K}^X_{m+1}) \to CH^{m+1}(X, 1)$ of K-cohomology groups to higher Chow groups (the first one being isomorphism while the second one is isomorphism modulo torsion), which commute with products and direct image maps (for example, this follows from [10], Proposition 1.7). Thus, the following result is proved by Bloch in [6], Lemma 1, being a generalization of proposition 6.2.

Proposition 6.8. *When Y and Z are topologically trivial the subgroup*

$$Y \cdot H^{n-1}(X, \mathcal{K}^X_n) + H^{m-1}(X, \mathcal{K}^X_m) \cdot Z \subset H^d(X, \mathcal{K}_{d+1})$$

lies in the kernel of the direct image map $H^d(X, \mathcal{K}^X_{d+1}) \to k^*$.

Corollary 6.9. *There is a well-defined pairing*

$$\psi^{p,q}_l = \overline{m}_3 : CH^p_0(X)_l \times CH^q_0(X)_l \to \mu_l$$

given in notations from proposition 6.7 by formula

$$\psi^{p,q}_l(Y, Z) = f(\widetilde{Y} \cap Z) \cdot g^{-1}(Y \cap \widetilde{Z}),$$

where $CH^*_0(X)$ denotes a subgroup of topologically trivial cycles.

Recall that higher Chow groups equal to motivic cohomology by formula $CH^i(X, m) = H^{2i-m}_M(X, \mathbb{Z}(i))$. The last groups could be defined as Zarisky hypercohomology of a complex of sheaves $\mathbb{Z}(i)$ on X, e.g. see [16]. Moreover, there are canonical morphisms of complexes $\mathbb{Z}(i) \otimes \mathbb{Z}(j) \to \mathbb{Z}(i + j)$ defining Massey higher products on motivic cohomology.

It seems that the following conjecture seems can be checked by explicit computations.

Conjecture 6.10. *The natural maps $H^m(X, \mathcal{K}^X_m) \to H^{2m}(X, \mathbb{Z}(m))$ commute with triple products and $H^m(X, \mathcal{K}^X_{m+1}) \to H^{2m+1}(X, \mathbb{Z}(m+1))$ commute with Massey higher products which preserve this type of indices.*

On the other hand for complex varieties there is a regulator map from motivic cohomology to Deligne cohomology which preserves Massey higher products. Thus, conjecture 6.10, lemma 6.11 and corollary 6.9 imply the following formula. Let Y, X be two equidimensional cycles on a smooth complex projective variety X of codimensions p and

q, respectively, $p+q=d+1$, $d=\dim X$. Suppose Y and Z are topologically trivial and their classes belong to l-torsion in Chow groups. Then in notations from proposition 6.7 we have

$$\psi_l(Y,Z) = f(\widetilde{Y}\cap Z)\cdot g^{-1}(Y\cap \widetilde{Z}),$$

where ψ_l denotes Weil pairing between l-torsion in corresponding Griffiths intermediate Jacobians. This last formula, which seems not to appear before, should have a direct explanation in classical terms.

Lemma 6.11. *Let X be a smooth complex projective variety. Consider classes $a \in J^p(X)_l \subset H_D^{2p}(X,\mathbb{Z}(p))_l$ and $b \in J^q(X)_l \subset H_D^{2q}(X,\mathbb{Z}(q))_l$ where $p+q=2d+1$, $d=\dim X$, and $J^*(X)$ denote Griffiths intermediate Jacobians. Then the image $\overline{m}_3(a,l,b)$ of the triple product $m_3(a,l,b)\in H_D^{2d+1}(X,\mathbb{Z}(d+1))$ under the direct image map $H_D^{2d+1}(X,\mathbb{Z}_D(d+1)) \to H_D^1(\mathrm{Spec}(\mathbb{C}),\mathbb{Z}(1))=\mathbb{C}^*$ equals to the Weil pairing $\psi_l(a,b)$ of a and b considered as l-torsion elements in the dual complex tori.*

Proof. We choose suitable classes representing a and b in Chech resolutions $\check{C}(X,\mathbb{Z}(\cdot)_D)$ of the corresponding Deligne complexes $\mathbb{Z}(\cdot)_D$. Namely, since a is topologically trivial its classe may be represented by a cocycle $\{\omega_i\}$ in the bicomplex $\check{C}(X,\Omega^\bullet)^\bullet$, where $\omega_i \in \check{C}(X,\Omega_X^i)^{2p-i}$, $0\le i\le p-1$. Let $l\{\omega_i\}=(d+\partial)(z+\{\widetilde{\omega}_i\})$, where d denotes de Rham differential, ∂ denotes the differential in Chech complex, and z is a cocycle in $\check{C}(X,\mathbb{Z}(p))^{2p-1}$. After we substract $d(\frac{1}{l}\{\widetilde{\omega}_j\})$ from $\{\omega_i\}$ the class a occurs to be represented by a "constant" cocycle $f\in\check{C}(X,\mathbb{C})^{2p-1}$ such that $lf=z\in\check{C}(X,\mathbb{Z}(p))^{2p-1}$. We may repeat the same for b and get, analogously, $g\in\check{C}(X,\mathbb{C})^{2q-1}$, such that $lg=u\in\check{C}(X,\mathbb{Z}(q))^{2q-1}$. It follows from the explicit formula for product in Deligne cohomology that $m_3(a,l,b)=z\cdot g-f\cdot u=\frac{1}{l}(z\cdot u)\in\check{C}(X,\mathbb{C})^{2d}$, where a dot stays for the product in Chech complex. The direct image of this class in $H_D^1(\mathrm{Spec}(\mathbb{C}),\mathbb{Z}(1))=\mathbb{C}^*$ equals to $\exp(\frac{1}{l(2\pi i)^d}(z\cdot u))$, which is by definition the Weil pairing $\psi_l(a,b)$ between l-torsion points in dual tori $J^p(X)$ and $J^q(X)$, which are quotients of suitable spaces over the dual lattices $H^{2p-1}(X,\mathbb{Z})$ and $H^{2q-1}(X,\mathbb{Z})$, respectively (after we divide the corresponding Deligne cohomology by $(2\pi i)^p$ and $(2\pi i)^q$, respectively). □

6.3 Appendix: Massey higher products

Let $A^\bullet$ be a dg-algebra. By definition this means that $A^\bullet$ is a complex (whose differential will be denoted by d) and there is a morphism of complexes

$$A^\bullet\otimes A^\bullet\to A^\bullet,$$

satisfying the associativity property. Let $M^\bullet$ be a left dg-module over $A^\bullet$, i.e. there is a morphism of complexes

$$M^\bullet\otimes A^\bullet\to A^\bullet,$$

satisfying natural properties of the module structure. Consider a bicomplex $C^{\bullet,\bullet}$ with $C^{p,q}=(M^\bullet\otimes(A^\bullet)^{\otimes(p-1)})^q$. The differential $\partial: C^{p,q}\to C^{p-1,q}$ is defined by a usual formula

$$\partial(m\otimes a_1\otimes\ldots\otimes a_{p-1}) = ma_1\otimes a_2\ldots\otimes a_{p-1}+\sum_{i=1}^{p-1}(-1)^i m\otimes a_1\ldots\otimes a_{i-1}\otimes a_i a_{i+1}\otimes a_{i+1}\ldots\otimes a_{p-1},$$

while the differential $C^{p,q}\to C^{p,q+1}$ is induced by the differentials in $A^\bullet$ and $M^\bullet$.

The bicomplex $C^{\bullet,\bullet}$ defines a spectral sequence $E_r^{p,q}$ with $E_1^{p,q} = H^q(M^\bullet\otimes(A^\bullet)^{\otimes(p-1)})$. The higher differentials in this spectral sequence define Massey higher products

$$m_k : (H^{i_1}(M^\bullet)\otimes H^{i_2}(A^\bullet)\otimes\ldots\otimes H^{i_k}(A^\bullet))^\circ \to (E_1^{k,i_1+\ldots+i_k})^\circ \to E_k^{k,i_1+\ldots+i_k}\to$$
$$\to E_k^{0,i_1+\ldots+i_k-k+2} = {}^\circ(H^{i_1+\ldots+i_k-k}(M^\bullet)),$$

where $(G)^\circ$ denotes that we take a certain subgroup inside G, and ${}^\circ(G)$ denotes that we take a quotient of the group G. More precisely, the condition on the definition domain of m_k is the following: the product m_k is well-defined for such classes (in cohomology groups) $m = a_0, a_1, \ldots, a_{k-1}$ that for all $1 < l < k$, $0\le i\le k-l$ we have $m_l(a_i, a_{i+1}\ldots, a_{i+l}) = 0$. See more details in [9].

In particular, we could take $M^\bullet = A^\bullet$ and obtain higher products on cohomology groups $H^i(A^\bullet)$. For instance, m_2 is just multiplication in cohomology groups. The operation m_3 is defined as follows: take $a\in H^i(M^\bullet)$, $b\in H^j(A^\bullet)$ and $c\in H^k(A^\bullet)$. Suppose that $a\cdot b=0$, $b\cdot c=0$ (the product is taken in cohomology groups). Let $\tilde a$, $\tilde b$ and $\tilde c$ be representatives of classes a, b and c, respectively. Then there exist elements $\widetilde{ab}\in M^{i+j-1}$, $\widetilde{bc}\in A^{j+k-1}$ such that $d(\widetilde{ab}) = \tilde a\cdot\tilde b$ and $d(\widetilde{bc}) = \tilde b\cdot\tilde c$. Then

$$m_3(a,b,c) = [\tilde a\cdot\widetilde{bc} - \widetilde{ab}\cdot\tilde c],$$

where bracket $[-]$ means the class in the group

$$H^{i+j+k-2}(M^\bullet)/(a\cdot H^{j+k-1}(A^\bullet) + H^{i+j-1}(M^\bullet)\cdot c).$$

Remark 6.12. Let $\varphi : A^\bullet\to B^\bullet$ be a homomorphism of dg-algebras. Then Massey higher products commute with the induced map on cohomology groups $\varphi : H^i(A^\bullet)\to H^i(B^\bullet)$. Let $f : M^\bullet\to N^\bullet$ be a morphism of left dg-modules over $A^\bullet$. Then Massey higher products commute with the induced map on cohomology $f : H^i(M^\bullet)\to H^i(N^\bullet)$.

Now consider a sheaf of associative algebras $\mathcal{A}$ on the topological space X. Let $C(\mathcal{A})^\bullet$ be its Godement resolution (see [11], section 4.3 and also section 6.6, where this resolution is called "canonical resolution"). It is multiplicative in the sense that there is a morphism $C(\mathcal{A})^\bullet\otimes C(\mathcal{A})^\bullet\to C(\mathcal{A})^\bullet$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \longrightarrow & C(\mathcal{A})^\bullet\otimes C(\mathcal{A})^\bullet \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & C(\mathcal{A})^\bullet. \end{array}$$

Then we obtain a dg-algebra $A^\bullet$, where $A^p = \Gamma(X, C(\mathcal{A})^p)$, whose cohomology groups are equal to $H^i(X,\mathcal{A})$. What we said before implies that in such a manner we get higher Massey products on $H^i(X,\mathcal{A})$.

Suppose now that we are given another multiplicative resolution $\mathcal{I}^\bullet$ of the sheaf $\mathcal{A}$. Then the canonical diagram

$$C(\mathcal{A})^\bullet \to C(\mathcal{I}^\bullet)^\bullet \leftarrow \mathcal{I}^\bullet$$

provides the analogous diagram of dg-algebras after taking $\Gamma(X,-)$ (where arrows are homomorphisms of dg-algebras due to functoriality of Godement resolution). Moreover, the first homomorphism is a quasiisomorphism, hence we get that the canonical map $H^i(\Gamma(X,\mathcal{I}^\bullet)) \to H^i(X,\mathcal{A})$ commutes with Massey higher products. This reasoning is just a copy of the proof of Theorem 6.6.1 from [11]. Thus we get the following result:

Lemma 6.13. *The Massey higher products structure on cohomology groups $H^i(X,\mathcal{A})$ does not depend on the choice of a flasque multiplicative resolution of $\mathcal{A}$.*

7 Remarks and open questions

There are several open questions around the adelic complex of sheaves of K-groups.

In section 3.2 we had to consider (covariant) Gersten resolution as a module over the adelic resolution, since there is no way to construct a direct image map on the adelic complex as defined here. This can be already seen in the simplest examples with closed embedding (of a point in a curve, for example) or with finite morphisms. Both situations seem to be overcome if one considers a *complete* version of K-adeles. Also, this considering should correspond to global class field theory of schemes, see [23]. Some particular cases were treated in [21]. However, the "complete" theory is still to be built.

The other possible development is the study of singular schemes. Note, that the additive version of the main theorem 4.1 is valid for any Noetherian schemes, see [14]. The adelic complex might always provide a resolution for the sheaves $\mathcal{K}_n^X$, even in the non-geometric case (the easiest examples of singular curves yield this property). We also did a strong restriction on the ground field to be infinite and on the scheme to be smooth of finite type over a ground field. In fact, the only one place where we have used the infinity of a field was a geometric proof of claim 4.18. It seems possible to prove the same result over an arbitrary field for smooth varieties, and then to extend to arbitrary regular varieties choosing their models over a field finitely generated over a prime (finite) field. Note, that Quillen's result is true in this generality. Let us note, that when $\dim X \leq 3$ one can prove theorem 4.1 over an arbitrary field, avoiding claim 4.18.

Another remark to make is that during the proof of theorem 4.1 we have treated K-groups as a black box. The only one non-trivial property we have used was in the proof of proposition 4.20. Namely, for a smooth morphism $\pi : E \to B$ having a section σ the direct image map σ_* on K'-groups is zero. Hence, the same construction of the adelic resolution should work if one replaces the sheaf $\mathcal{K}_n^X$ by any sheaf of cohomology theory with support in the sence of Bloch–Ogus, see [7], or even by any homotopy invariant sheaf with transfer in the sense of Voevodsky. Note that for such sheaves Voevodsky has established Gersten resolution, e.g. see [16], and the same construction of the adelic resolution seems to be valid in this case as well. It would be also interesting to construct the adelic complex for motivic cohomology, by doing this for the sheaves involved in Suslin–Voevodsky complexes $\mathbb{Z}(n)$. These sheaves are not homotopy invariant, but there is still some hope for the existence of the adelic resolution in this case (note, that the additive adelic resolution resolves a non-homotopy invariant sheaf of differential forms). The existence of such resolution could give rise to some explicit formulas for Massey higher products on motivic cohomology, see also [9].

Finally, there could be a higher-dimensional analogue of the biextension from section 2.2 over the product of corresponding groups of K-adeles, related to the one constructed in [6]. Maybe there is even an analogue of the central extension of idèles on a curve for the group $\mathbf{A}_X(\mathcal{K}_p^X)^p$ where $2p = d+1$, $d = \dim X$, see also [22].

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