THE MAIN COMPONENT OF THE TORIC HILBERT SCHEME

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Abstract

Let $\mathbb{X}$ be an affine toric variety under a torus $\mathbb{T}$ and let T be a subtorus. The generic T-orbit closures in $\mathbb{X}$ and their flat limits are parametrized by the main component H_0 of the toric Hilbert scheme (whose existence follows from work of Haiman and Sturmfels). Further, the quotient torus $\mathbb{T}/T$ acts on H_0 with a dense orbit. We describe the fan of this toric variety; this leads us to an integral analogue of the fiber polytope of Billera and Sturmfels. We also describe the relation of H_0 to the toric Chow quotient of Craw and Maclagan.

1 Introduction

The multigraded Hilbert scheme parametrizes, in a technical sense specified below, all homogeneous ideals in a polynomial algebra (or, more generally, in an arbitrary finitely generated algebra) having a fixed Hilbert function with respect to a grading by an abelian group. In [7] it was shown that the multigraded Hilbert scheme always exists as a quasiprojective scheme.

Let $\mathbb{X}$ be an affine algebraic variety over an algebraically closed field k with an action of an algebraic torus T, so its algebra of regular functions $S := k[\mathbb{X}]$ is graded by the group $\Lambda(T)$ of characters of T :

$$S = \bigoplus_{\lambda \in \Lambda(T)} S_\lambda,$$

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where S_λ is the subspace of T-semiinvariant functions of weight λ. Let

$$\Sigma := \{\lambda \in \Lambda(T) \ : \ S_\lambda \neq 0\}.$$

This is a finitely generated monoid. Conversely, if S is a finitely generated algebra without nilpotent elements graded by $\Lambda(T)$, then we have a T-action on the affine algebraic variety $\mathbb{X} = \operatorname{Spec} S$.

The following definition was introduced in [7].

DEFINITION 1.1. Given a function $h : \Lambda(T) \to \mathbb{N}$, the *Hilbert functor* is the covariant functor $\underline{H^h_{\mathbb{X},T}}$ from the category of k-algebras to the category of sets assigning to any k-algebra R the set of all T-invariant ideals $I \subseteq R \otimes_k S$ such that $(R \otimes_k S_\lambda)/I_\lambda$ is a locally free R-module of rank $h(\lambda)$ for any $\lambda \in \Lambda(T)$.

In [7, Th. 1.1] it was proved that there exists a quasiprojective scheme $H^h_{\mathbb{X},T}$ which represents this functor in the case when $\mathbb{X}$ is a finite-dimensional T-module V. If the grading is positive (i.e., $k[V]_0 = k$), then $H^h_{V,T}$ is projective (see [7, Cor. 1.2]). In the case of an arbitrary $\mathbb{X}$ there exists a T-equivariant closed immersion $\mathbb{X} \hookrightarrow V$, where V is a finite-dimensional T-module. Then the Hilbert functor $\underline{H^h_{\mathbb{X},T}}$ is represented by a closed subscheme of $H^h_{V,T}$ (see [1, Lemma 1.6]).

We consider the following case. Let $\mathbb{X}$ be an affine toric (not necessarily normal) variety under a torus $\mathbb{T}$. We have

$$S = k[\mathbb{X}] = \bigoplus_{\chi \in \Omega} S_\chi,$$

where $\Omega \subset \Lambda(\mathbb{T})$ is a finitely generated monoid and S_χ is the subspace of $\mathbb{T}$-semiinvariant functions of weight χ ($\dim S_\chi = 1$). Let $T \subset \mathbb{T}$ be a subtorus. We have a surjective linear map $\pi : \Lambda(\mathbb{T}) \to \Lambda(T)$ given by the restriction. The action of T on $\mathbb{X}$ arising from the action of $\mathbb{T}$ gives a grading

$$S = \bigoplus_{\lambda \in \Sigma} S_\lambda,$$

where $\Sigma = \pi(\Omega)$. In this paper we consider the following Hilbert function:

$$h(\lambda) := \begin{cases} 1 & \text{if } \ \lambda \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $H_{\mathbb{X},T}$ be the corresponding Hilbert scheme (we shall also denote it by $H_{S,T}$). It is the *toric Hilbert scheme* in the sense that it parametrizes all ideals in S with the same Hilbert function as the ideal I_X corresponding to the toric T-variety $X = \overline{T \cdot x}$, where $x \in \mathbb{X}$ is a point in the open $\mathbb{T}$-orbit (see [10]). We have a natural action of $\mathbb{T}$ on $H_{\mathbb{X},T}$.

There is a canonical irreducible component H_0 of $H_{\mathbb{X},T}$ which is the $\mathbb{T}/T$-orbit closure of I_X (see Prop. 3.1(2)). This component is called the *main component* of the toric Hilbert scheme $H_{\mathbb{X},T}$. The scheme H_0 parametrizes generic T-orbit closures in $\mathbb{X}$ and their flat limits. In fact, H_0 is a toric (not necessarily normal) $\mathbb{T}/T$-variety (see Prop. 3.1(1)). We describe its fan in terms of the fiber polyhedron for the map of monoids $\Omega \to \Sigma$ given by π (Theorem 3.8).

In the last section we consider the toric Chow morphism from the Hilbert scheme to the inverse limit $\mathbb{X}/\!\!/_C T$ of GIT quotients $\mathbb{X}/\!\!/_\lambda T$. This morphism was constructed in [7, Sect. 5] in the case when $\mathbb{X} = \mathbb{A}^n$ and $\mathbb{T} = \mathbb{G}_m^n$ acts by rescaling of coordinates. We generalize this to the case of a normal affine toric $\mathbb{T}$-variety $\mathbb{X}$ (Theorem 4.5). The toric Chow quotient is a toric $\mathbb{T}/T$-variety arising as an irreducible component of $\mathbb{X}/\!\!/_C T$. The notion of the toric Chow quotient of a quasiprojective toric variety by a subtorus was introduced in [3]; this is a generalization of the toric Chow quotient of a projective variety studied by Kapranov-Sturmfels-Zelevinsky [9]. In [3] it was shown that the fan of the toric Chow quotient is the fiber fan, that is the normal fan to the fiber polytope of Billera-Sturmfels (see [2]) generalized to the case of a linear map of polyhedra. We show that the fan of H_0 is an integral analogue of the fiber fan. If $\mathbb{X} = \mathbb{A}^n$, $\mathbb{T} = \mathbb{G}_m^n$ acts by rescaling of coordinates, and the grading of $S = k[x_1, \ldots, x_n]$ by the weights of T is positive, then the fan of H_0 coincides with the normal fan to the state polytope of Sturmfels (see [11, Th. 2.5]).

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2 Terminology and notations

We consider the category of schemes over an algebraically closed field k. An *algebraic variety* is a separated integral scheme of finite type. Any scheme Z is characterized by its *functor of points* from the category of k-algebras to the category of sets:

$$\underline{Z} : \underline{k - Alg} \to \underline{Set}, \quad \underline{Z}(R) := \mathrm{Mor}(\mathrm{Spec}\, R, Z),$$

where $\mathrm{Mor}(\mathrm{Spec}\, R, Z)$ is the set of morphisms of schemes over k from $\mathrm{Spec}\, R$ to Z (we denote the functor of points of a scheme by the corresponding underlined letter). Our main reference on schemes is [4]. If $\phi : Y \to Z$ is a morphism of schemes, then $\underline{\phi}(R)$ denotes the corresponding map of sets $\underline{Y}(R) \to \underline{Z}(R)$. We denote by $\mathcal{O}_Z$ the structure sheaf of Z, and if Z is affine, then $k[Z]$ denotes the algebra of sections of $\mathcal{O}_Z$ over Z. We denote by $\mathbb{A}^n$ the affine space $\mathrm{Spec}\, k[x_1, \ldots, x_n]$.

An n-dimensional *algebraic torus* T is an algebraic group isomorphic to the direct product of n copies of the multiplicative group $\mathbb{G}_m$ of the field k. For the lattices of characters and one-parameter subgroups of T, we use the notations $\Lambda(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $\Gamma(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between $\Lambda(T)$ and $\Gamma(T)$. For a lattice Λ, let $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. If $\Sigma \subset \Lambda$ is a monoid, then $\Sigma_{\mathbb{R}}$ denotes the subspace in $\Lambda_{\mathbb{R}}$ generated by Σ, and $\Sigma_{\mathbb{R}_+}$ denotes the cone in $\Lambda_{\mathbb{R}}$ generated by Σ. For subsets D_1, D_2 of a vector space, we denote by $D_1 + D_2$ the Minkovski sum.

By a *toric variety* under an algebraic torus T we mean an algebraic variety X such that T is embedded as an open subset into X and the action of T on itself by multiplication extends to an action on X. We do not require X to be normal.

We denote by $\mathcal{C}_X$ the associated fan of a toric variety X, so $\mathcal{C}_X \subset \Gamma(T)_{\mathbb{R}}$ (see [6, Sec. 1.4]). The T-orbits on X are in order-reversing one-to-one correspondence with the cones of $\mathcal{C}_X$. If X_σ is the T-orbit corresponding to a cone σ in $\mathcal{C}_X$, then a one-parameter subgroup $\lambda \in \Gamma(T)$ lies in σ if and only if $\lim_{s \to 0} \lambda(s)$ exists and lies in the closure of the orbit X_σ in X. A toric variety is determined by its fan up to normalization.

3 Fan of a toric Hilbert scheme

Fix a $\mathbb{T}$-equivariant closed embedding $\mathbb{X} \hookrightarrow V$, where V is a finite-dimensional $\mathbb{T}$-module such that $\mathbb{X}$ is not contained in a proper $\mathbb{T}$-submodule, and fix a basis of V consisting of $\mathbb{T}$-semiinvariant vectors. Let $x_1, \ldots, x_n$ be the coordinates in this basis and let $\mathbb{T}$ act on V by $t \cdot (x_1, \ldots, x_n) = (\chi_1(t)x_1, \ldots, \chi_n(t)x_n)$, $\chi_i \in \Lambda(\mathbb{T})$ (so T acts on V by $t \cdot (x_1, \ldots, x_n) = (\lambda_1(t)x_1, \ldots, \lambda_n(t)x_n)$, where $\lambda_i = \pi(\chi_i)$). Up to rescaling of the basis vectors we can assume that $\mathbb{X} = \overline{\mathbb{T} \cdot x}$, where $x = (1, \ldots, 1)$. We denote by X the T-orbit closure $\overline{T \cdot x}$ and $I_X \subset k[V]$ denotes the corresponding ideal. The characters $-\chi_1, \ldots, -\chi_n$ generate the monoid Ω (and the characters $-\lambda_1, \ldots, -\lambda_n$ generate the monoid Σ). We have

$$S = k[\mathbb{X}] = k[x_1, \ldots, x_n]/I_{\mathbb{X}},$$

where $I_{\mathbb{X}}$ is generated by all the binomials of the form $x_1^{c_1} \ldots x_n^{c_n} - x_1^{b_1} \ldots x_n^{b_n}$ such that $\sum c_i\chi_i = \sum b_i\chi_i$ (see [11, Lemma 4.1]). Note that the $\Lambda(T)$-grading is positive if and only if the cone $\Sigma_{\mathbb{R}_+}$ is strictly convex and $\lambda_i \neq 0$ for all $i = 1, \ldots, n$. Notice also that in the case of a positive grading there exists a unique minimal system of generators of Σ.

We consider the Hilbert scheme $H_{\mathbb{X},T}$ as a closed subscheme of $H_{V,T}$. For an algebra R the subset $\underline{H_{\mathbb{X},T}}(R) \subset \underline{H_{V,T}}(R)$ consists of those ideals $I \subset R \otimes_k k[V]$ that $I \in \underline{H_{V,T}}(R)$ and $R \otimes_k I_{\mathbb{X}} \subset I$. We can also view $\underline{H_{\mathbb{X},T}}(R)$ as a set of closed subschemes $Y \subset \mathrm{Spec}\, R \times \mathbb{X}$ such that the projection $Y \to \mathrm{Spec}\, R$ is flat. All the ideals $I \in \underline{H_{V,T}}(k)$ are binomial (see [5, Prop. 1.11]), and we have a special point $X \in \underline{H_{\mathbb{X},T}}(k)$.

Recall that the *universal family* is the closed subscheme $\mathbb{W}_{\mathbb{X},T}$ of $H_{\mathbb{X},T}\times\mathbb{X}$ corresponding to the identity map $\{\mathrm{Id}: H_{\mathbb{X},T}\to H_{\mathbb{X},T}\}\in \underline{H_{\mathbb{X},T}}(H_{\mathbb{X},T})$. For any $Y\in\underline{H_{\mathbb{X},T}}(R)$ (so Y is a closed subscheme in $\mathrm{Spec}\,(R\otimes_k S)$) we have $Y=\mathbb{W}_{\mathbb{X},T}\times_{H_{\mathbb{X},T}}\mathrm{Spec}\,R$. In fact, the k-rational points of $\mathbb{W}$ are those pairs (y,Y), where $Y\in\underline{H_{\mathbb{X},T}}(k)$ and $y\in\underline{Y}(k)$.

The group $\underline{\mathbb{T}}(R)$ acts on $\underline{H_{\mathbb{X},T}}(R)$ in the natural way. Namely, we have an action of $\underline{\mathbb{T}}(R)$ on $R\otimes_k S:$ for $f\in R\otimes_k S_\chi$, where $\chi\in\Omega$, and $t\in\underline{\mathbb{T}}(R)$ let $t\cdot f=\chi(t)f$. Hence for $I\in\underline{H_{\mathbb{X},T}}(R)$ let $t\cdot I=\{t\cdot f\ :\ f\in I\}$. These actions commute with base extensions, thus we have an action of $\mathbb{T}$ on $H_{\mathbb{X},T}$. Since T acts trivially, this yields an action of the torus $\mathbb{T}/T$. The universal family $\mathbb{W}_{\mathbb{X},T}$ is invariant under the diagonal action of $\mathbb{T}$ on $\mathbb{X}\times H_{\mathbb{X},T}$.

Let H_0 be the toric orbit closure $\overline{\mathbb{T}\cdot X}\subset H_{\mathbb{X},T}$, and denote by $\mathbb{W}_0$ its preimage under the projection

$$p:\mathbb{W}_{\mathbb{X},T}\to H_{\mathbb{X},T}$$

(we consider H_0 and $\mathbb{W}_0$ with their structure of reduced schemes).

Proposition 3.1. (1) *The stabilizer of X under the action of $\mathbb{T}$ on H_0 is T, so H_0 is a toric variety under the torus $\mathbb{T}/T$.*

(2) *The orbit $\mathbb{T}\cdot X$ is open in $H_{\mathbb{X},T}$. Consequently, H_0 is an irreducible component of $H_{\mathbb{X},T}$.*

(3) $\mathbb{W}_0$ *is a toric variety under the torus* $\mathbb{T}$ (*and, consequently,* $\mathbb{W}_0$ *is an irreducible component of* $\mathbb{W}_{\mathbb{X},T}$).

Proof. (1) If $t\cdot X=X$ for $t\in\mathbb{T}$, then $t\cdot x\in T\cdot x$ and $t\in T$.

(2) We shall prove that $\mathbb{T}\cdot X$ is open in $H_{\mathbb{X},T}$. Since the stabilizer of X in $\mathbb{T}$ is T, it suffices to prove that $\dim T_XH_{\mathbb{X},T}\le\dim\mathbb{T}\cdot X=\dim\mathbb{T}-\dim T$, where $T_XH_{\mathbb{X},T}$ denotes the tangent space to $H_{\mathbb{X},T}$ at X. By [7, Prop. 1.6], we have

$$T_XH_{\mathbb{X},T}=\mathrm{Hom}_{k[\mathbb{X}]}(I_X,k[X])_0=$$

$$\mathrm{Hom}_{k[\mathbb{T}]}(I_T,k[T])_0=\mathrm{Hom}_{k[\mathbb{T}]}(I_T/I_T^2,k[T])_0,$$

where I_T is the ideal of functions in $k[\mathbb{T}]$ vanishing on T. We can choose coordinates on $\mathbb{T}$ such that

$$k[\mathbb{T}]=k[t_1,t_1^{-1},\dots,t_m,t_m^{-1},s_1,s_1^{-1},...,s_r,s_r^{-1}],$$

where $r=\dim\mathbb{T}-\dim T$, and the ideal I_T is generated by s_i-1 for $i=1,\dots,r$. The linear space I_T is spanned by the elements $t_1^{a_1}\dots t_n^{a_n}s_1^{b_1}\dots s_m^{b_m}(s_i-1)$, where $a_i,b_j\in\mathbb{Z}$, and the projections of the elements $t_1^{a_1}\dots t_n^{a_n}(s_i-1)$ span the linear space I_T/I_T^2 (since $s_i(s_j-1)=(s_j-1)+(s_i-1)(s_j-1)$ and $s_i^{-1}(s_j-1)=(s_j-1)-s_i^{-1}(s_i-1)(s_j-1)$). Hence a homomorphism of $k[\mathbb{T}]$-modules from I_T to $k[T]$ is uniquely determined by the images of s_i-1. Thus the dimension of the vector space of such homomorphisms of degree zero is not greater then r.

(3) Consider the restriction p_0 of p to $\mathbb{W}_0$:

$$p_0 : \mathbb{W}_0 \to H_0.$$

This is a flat morphism. By Lemma 3.2 below and [8, Cor. 9.6], the dimension of any irreducible component Z of $\mathbb{W}_0$ is equal to $\dim \mathbb{T}$. This implies that $p_0(Z) = H_0$ and $Z \subset \overline{p^{-1}(\mathbb{T} \cdot X)}$. Thus $\mathbb{W}_0 = \overline{p^{-1}(\mathbb{T} \cdot X)} = \overline{\mathbb{T} \cdot (x, X)}$ is irreducible and $\mathbb{T} \cdot (x, X) \subset \mathbb{W}_0$ is dense and, consequently, open. □

LEMMA 3.2. *For any point $Y \in H_{\mathbb{X},T}$, the dimension of any irreducible component of its fibre $p^{-1}(Y)$ equals* $\dim T$.

PROOF. We denote by $k(Y)$ the residue field of $Y \in H_{\mathbb{X},T}$. Then we have

$$p^{-1}(Y) = \operatorname{Spec} k(Y) \times_{H_{\mathbb{X},T}} \mathbb{W}_{\mathbb{X},T} = \operatorname{Spec} L,$$

where L is a coherent sheaf of Σ-graded $k(Y)$-algebras:

$$L = \bigoplus_{\lambda \in \Sigma} L_\lambda,$$

and $L_\lambda := k(Y) \otimes_{\mathcal{O}_{H_{\mathbb{X},T}}} (\mathcal{O}_{\mathbb{W}_{\mathbb{X},T}})_\lambda$ is isomorphic to $k(Y)$.

Every point $Y \in H_{\mathbb{X},T}$ gives us a subdivision of $\Sigma_{\mathbb{R}_+}$ into subcones, namely two points $\lambda, \lambda' \in \Sigma$ lie in the same cone if and only if $L_\lambda L_{\lambda'} \neq 0$. The irreducible components Z of $p^{-1}(Y)$ correspond to the maximal cones C of this subdivision:

$$Z = \operatorname{Spec} \Big(\bigoplus_{\lambda \in \Sigma \cap C} L_\lambda \Big).$$

Let $\Sigma_C := \{\lambda \in \Sigma \cap C \ : \ L_\lambda \text{ is not nilpotent}\}$. Note that Σ_C is a monoid and $(\Sigma_C)_{\mathbb{R}_+} = C$. It suffices to prove that the dimension of $Z_{red} = \operatorname{Spec} (\bigoplus_{\lambda \in \Sigma_C} L_\lambda)$ is equal to $\dim T$. We can extend the action of T on Z_{red} to an action of the torus $T \times \operatorname{Spec} k(Y)$ (over the field $k(Y)$). Thus Z_{red} is a toric variety under the torus $T \times \operatorname{Spec} k(Y)$ and $\dim Z = \dim C = \dim T$. □

Our aim is to describe the fans of the toric varieties H_0 and $\mathbb{W}_0$. Let

$$\gamma : \mathbb{G}_m \to \mathbb{T}$$

be a one-parameter subgroup. Consider the closed embedding

$$\mathbb{G}_m \times X \subset \mathbb{G}_m \times \mathbb{X},$$

$$(s, x) \to (s, \gamma(s) \cdot x).$$

Let Ξ be the closure of the image of this embedding in $\mathbb{A}^1 \times \mathbb{X}$ (so Ξ is an algebraic variety). Since the projection $\Xi \to \mathbb{A}^1$ is a flat morphism, we have a morphism $\mathbb{A}^1 \to H_{\mathbb{X},T}$ such that $\Xi = \mathbb{W}_{\mathbb{X},T} \times_{H_{\mathbb{X},T}} \mathbb{A}^1$. Thus the limit of X under γ is the fiber of Ξ over 0 (we denote it by Ξ_0). This limit exists if and only if Ξ_0 is non-empty. Consider the commutative diagram:

$$\begin{array}{ccc} \Xi & \supset & \mathbb{G}_m \times X \\ \cap & & \cap \\ \mathbb{A}^1 \times \mathbb{X} & \supset & \mathbb{G}_m \times \mathbb{X}. \end{array}$$

We have the corresponding homomorphisms of algebras :

$$\begin{array}{ccc} k[\Xi] & \hookrightarrow & k[\mathbb{G}_m \times X] \\ \uparrow & & \uparrow \\ k[\mathbb{A}^1 \times \mathbb{X}] & \hookrightarrow & k[\mathbb{G}_m \times \mathbb{X}], \end{array}$$

where the vertical maps are surjective.

Denote by s the coordinate in $\mathbb{A}^1$, let $\gamma_i := \langle \gamma, \chi_i \rangle$, and let I'_X denote the ideal generated by $I_X \subset k[x_1, \ldots, x_n]$ in $k[x_1, \ldots, x_n, s, s^{-1}]$. Thus we obtain

$$\Xi = \mathrm{Spec}\ k[s^{\gamma_1} x_1, \ldots, s^{\gamma_n} x_n, s]/I_\Xi,$$

where $I_\Xi = I'_X \cap k[s^{\gamma_1} x_1, \ldots, s^{\gamma_n} x_n, s]$. Hence

$$k[\Xi_0] = k[s^{\gamma_1} x_1, \ldots, s^{\gamma_n} x_n, s]/(I_\Xi, s) \simeq k[x_1, \ldots, x_n]/I_\gamma,$$

where I_γ denotes the following ideal : for $f \in k[x_1, \ldots, x_n]$ let $in_\gamma(f)$ be the sum of all terms $cx_1^{a_1} \ldots x_n^{a_n}$ of f such that $\sum a_i \gamma_i$ is maximal, then I_γ is the ideal generated by the polynomials $in_\gamma(f)$, where $f \in I_X$.

Example 3.3. Let $\mathbb{X} = \mathbb{A}^n$, $\mathbb{T} = \mathbb{G}_m^n$ act on $\mathbb{A}^n$ by rescaling of coordinates, $T = \mathbb{G}_m$, and let the $\Lambda(T)$-grading of $k[x_1, \ldots, x_n]$ be positive.

(1) Consider the case $n = 3$. It was proved by Arnold, Korkina, Post, Roelfols (see, for example, [11, Th. 10.2]), that any ideal $I \in \underline{H_{\mathbb{A}^n,T}}(k)$ is of the form $t \cdot I_\gamma$ for some $t \in \mathbb{G}_m^n$ and $\gamma \in \Gamma(\mathbb{G}_m^n)$. This means that in this case the toric Hilbert scheme is irreducible.

(2) Let $n = 4$ and $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4, \lambda_4 = 7$. Then the toric Hilbert scheme is reducible. Moreover, in $H_{\mathbb{A}^n,T}$ there are infinitely many orbits of $\mathbb{G}_m^n$ (see [11, Th. 10.4]).

We can also write

$$k[\Xi_0] = \bigoplus_{\lambda \in \Sigma} k s^{n_\gamma(\lambda)} t^\lambda,$$

where

$$n_\gamma(\lambda) := \min_{\chi \in \pi^{-1}(\lambda) \cap \Omega} \langle \gamma, \chi \rangle.$$

Here $s^{n_\gamma(\lambda)}t^\lambda$ denotes the image in $k[\Xi_0]$ of the monomial $p(x_1, \ldots, x_n) \in k[\mathbb{A}^1 \times \mathbb{X}]$ of weight $\chi_\lambda \in \Omega$, where $\chi_\lambda \in \pi^{-1}(\lambda) \cap \Omega$ is such that $\langle \gamma, \chi_\lambda \rangle$ is minimal. This image is a T-semiinvariant function of weight λ. The product $s^{n_\gamma(\lambda_1)}t^{\lambda_1}s^{n_\gamma(\lambda_2)}t^{\lambda_2} = s^{n_\gamma(\lambda_1)+n_\gamma(\lambda_2)}t^{\lambda_1+\lambda_2}$ equals zero if and only if $n_\gamma(\lambda_1) + n_\gamma(\lambda_2) > n_\gamma(\lambda_1 + \lambda_2)$. Consider the convex hull

$$P_\lambda := \operatorname{conv}(\pi^{-1}(\lambda) \cap \Omega) \subset \Lambda(\mathbb{T})_\mathbb{R}.$$

This is a convex polyhedron.

Definition 3.4. Let P be a convex polyhedron. For any face F of P the *normal cone* $N_F(P)$ is the cone in the dual vector space consisting of those linear functions w that F is the face of P minimizing w. The *normal fan* $N(P)$ of P is the fan whose cones are normal cones to the faces of P. (This definition is taken from [11], which we shall use as a general reference on convex polyhedra.)

Let $\mathcal{C}_\lambda \subset \Gamma(\mathbb{T})_\mathbb{R}$ denote the normal fan of P_λ. Note that any cone of $\mathcal{C}_\lambda$ contains $\Gamma(T)_\mathbb{R} \subset \Gamma(\mathbb{T})_\mathbb{R}$.

Remark 3.5. We can consider fans in $\Gamma(\mathbb{T}/T)_\mathbb{R}$ as fans in $\Gamma(\mathbb{T})_\mathbb{R}$ whose cones contain $\Gamma(T)_\mathbb{R}$. In particular, we view the fan of the toric $\mathbb{T}/T$-variety H_0 as a fan in $\Gamma(\mathbb{T})_\mathbb{R}$.

Definition 3.6. A fan $\mathcal{C}_1$ is a *refinement* of a fan $\mathcal{C}_2$ if any cone of $\mathcal{C}_1$ is contained in some cone of $\mathcal{C}_2$.

Definition 3.7. We say that two polyhedra $P_1, P_2 \subset \Lambda(\mathbb{T})_\mathbb{R}$ are *equivalent* if they have the same normal fan.

Theorem 3.8. (1) *The fan $\mathcal{C}_{H_0} \subset \Gamma(\mathbb{T})_\mathbb{R}$ of the toric $\mathbb{T}/T$-variety H_0 is the maximal common refinement of the fans $\mathcal{C}_\lambda$, where $\lambda \in \Sigma$.*

(2) *The support of any $\mathcal{C}_\lambda$ is the cone generated by those one-parameter subgroups γ that $\langle \gamma, \chi \rangle \geq 0$ for any $\chi \in \pi^{-1}(0) \cap \Omega$. In particular, the grading of S by $\Lambda(T)$ is positive if and only if this support is the whole space $\Gamma(\mathbb{T})_\mathbb{R}$, i.e., any polyhedron P_λ is a polytope. This holds if and only if H_0 is projective.*

(3) *There are only finitely many non-equivalent polyhedra P_λ for $\lambda \in \Sigma$. Hence $\mathcal{C}_{H_0}$ is the normal fan of the Minkovski sum of representatives of the equivalence classes* (*we denote this sum by P_{H_0}*).

Proof. Statement (1) immediately follows from the description of limits under one-parameter subgroups given above.

(2) First note that P_0 is a cone and its normal cone $\mathcal{C}_0$ is generated by those one-parameter subgroups γ that $\langle \gamma, \chi \rangle \geq \langle \gamma, 0 \rangle = 0$ for any $\chi \in \pi^{-1}(0) \cap \Omega$. Further note that the recession cone of any P_λ is P_0 (the *recession cone* of a polyhedron P is the set

of those vectors v such that $u+v \in P$ for any $u \in P$). Indeed, S_λ is a finitely generated S_0-module. Let $\mu_1, \ldots, \mu_d \in \Lambda(\mathbb{T})$ be the weights of a set of $\mathbb{T}$-semiinvariant generators. Then

$$P_\lambda = \mathrm{conv}(\bigcup_{i=1}^{d}(\mu_i + P_0)) = \mathrm{conv}(\mu_1, \ldots, \mu_d) + P_0.$$

It follows that the support of $\mathcal{C}_\lambda$ is $\mathcal{C}_0$.

If the support of $\mathcal{C}_{H_0}$ is not $\Gamma(\mathbb{T})_\mathbb{R}$, then H_0 is not complete and, consequently, is not projective. Conversely, if the grading is positive, then the Hilbert scheme $H_{\mathbb{X},T}$ is projective, and H_0 is projective.

(3) There are only finitely many fans $\mathcal{C}$ such that $\mathcal{C}_{H_0}$ is a refinement of $\mathcal{C}$ and the supports of $\mathcal{C}$ and $\mathcal{C}_{H_0}$ coincide. □

Since $\mathbb{W}_0 \subset H_0 \times \mathbb{X}$, it follows that the fan of $\mathbb{W}_0$ is the maximal common refinement of the fans $\mathcal{C}_{H_0}$ and $N(\Omega_{\mathbb{R}_+})$. This is the normal fan of the Minkovski sum $P_{H_0} + \Omega_{\mathbb{R}_+}$.

Remark 3.9. By [11, Th. 7.15], it follows that in the case when $\mathbb{X} = \mathbb{A}^n$, $\mathbb{T} = \mathbb{G}_m^n$ acts by rescaling of coordinates, and the $\Lambda(T)$-grading of $k[\mathbb{X}]$ is positive, the polytope P_{H_0} is equivalent to the Minkovski sum of P_λ corresponding to the weights λ of the elements of the universal Gröbner basis of $I_\mathbb{X}$.

Let $\mathbb{X}$ be normal. Now we are going to give a precise description of those characters $\lambda \in \Sigma$ having equivalent polyhedra P_λ. Recall that we have a homomorphism of lattices $\pi : \Lambda(\mathbb{T}) \to \Lambda(T)$, a finitely generated monoid $\Omega \subset \Lambda(\mathbb{T})$ such that $\Omega = \Omega_{\mathbb{R}_+} \cap \Lambda(\mathbb{T})$, and we put $\Sigma = \pi(\Omega)$. To any point $\lambda \in \Sigma$ we associate the polyhedron $P_\lambda = \mathrm{conv}(\pi^{-1}(\lambda) \cap \Omega) \subset \Lambda(\mathbb{T})_\mathbb{R}$. Two points $\lambda, \lambda' \in \Sigma$ are said to be *equivalent* if the corresponding polyhedra P_λ and $P_{\lambda'}$ are equivalent. The question is to describe equivalence classes constructively.

First consider $\pi_\mathbb{R} : \Lambda(\mathbb{T})_\mathbb{R} \to \Lambda(T)_\mathbb{R}$, the linear map induced by π. Let $\mathcal{C}_\lambda^\mathbb{R}$ denote the normal fan to the polyhedron

$$P_\lambda^\mathbb{R} := \pi_\mathbb{R}^{-1}(\lambda) \cap \Omega_{\mathbb{R}_+}.$$

Consider the cell decomposition of $\Sigma_{\mathbb{R}_+}$ induced by π (see [3]). Namely, the characters λ and λ' lie in the interior of the same cone of this decomposition if and only if the set of those faces of $\Omega_{\mathbb{R}_+}$ whose images under $\pi_\mathbb{R}$ contain λ coincides with the set of such faces for λ'.

Fix a cone σ of this decomposition. Note that if λ lies in the interior of σ and $\lambda' \in \sigma$, then $\mathcal{C}_\lambda^\mathbb{R}$ refines $\mathcal{C}_{\lambda'}^\mathbb{R}$. In particular, the polyhedra $P_\lambda^\mathbb{R}$ corresponding to interior points λ of σ are equivalent. Let $P_\mathbb{R}$ denote the Minkovski sum of $P_\lambda^\mathbb{R}$ for representatives of interior points for all cones of the cell decomposition and let $\mathcal{C}_\mathbb{R}$ denote the normal fan to $P_\mathbb{R}$ (note that in the Minkovski sum it suffices to take representatives of interior points for the maximal cones of the cell decomposition).

Remark 3.10. In [3] the fan $\mathcal{C}_{\mathbb{R}}$ is called the *fiber fan* by analogy with the normal fan of the fiber polytope for a linear projection of polytopes (see [2]).

Consider a point λ lying in the interior of σ and the corresponding polyhedron $P^{\mathbb{R}}_{\lambda}$. For any vertex v of $P^{\mathbb{R}}_{\lambda}$ there exists a unique minimal face F of $\Omega_{\mathbb{R}_+}$ such that $F \cap P^{\mathbb{R}}_{\lambda} = \{v\}$ (indeed, since $P^{\mathbb{R}}_{\lambda}$ is the intersection of the cone Ω with the affine subspace $\pi^{-1}_{\mathbb{R}}(\lambda)$, it follows that any face of $P^{\mathbb{R}}_{\lambda}$ is the intersection of $\pi^{-1}_{\mathbb{R}}(\lambda)$ with some face of Ω). Let $v^{\lambda}_1, \ldots, v^{\lambda}_{l(\sigma)} \in \Lambda(\mathbb{T})_{\mathbb{R}}$ be the vertices of $P^{\mathbb{R}}_{\lambda}$ and let $F^{\sigma}_1, \ldots, F^{\sigma}_{l(\sigma)}$ be the corresponding faces (the set of such faces does not depend on a point λ in the interior of σ). For two vectors $\chi, \chi' \in \Lambda(\mathbb{T})_{\mathbb{R}}$ we say $\chi \prec \chi'$ if $\chi' - \chi \in \Omega_{\mathbb{R}_+}$.

Definition 3.11. (See [7, Def 5.4].) A character $\lambda \in \Sigma$ is *integral* if the inclusion of the convex polyhedra $P_{\lambda} \subseteq P^{\mathbb{R}}_{\lambda}$ is an equality.

We shall denote by $\Sigma^{int}_{\mathbb{X}}$ the set of integral characters. Note that $\Sigma^{int}_{V} \subset \Sigma^{int}_{\mathbb{X}}$, where V is considered as a toric variety under the torus $\mathbb{G}^n_m$ acting by rescaling of coordinates.

Lemma 3.12. *Let $\lambda_0 \in \sigma$ be integral, $\lambda \in \sigma$, and let $v^{\lambda}_i \succ (l(\sigma) - 1)v^{\lambda_0}_i$ for any $i = 1, \ldots, l(\sigma)$. Then $P_{\lambda+\lambda_0} = P_{\lambda} + P_{\lambda_0}$.*

Proof. The inclusion $P_{\lambda} + P_{\lambda_0} \subseteq P_{\lambda+\lambda_0}$ is evident. Denote by D_{μ} the convex hull of the v^{μ}_i, $i = 1, \ldots, l(\sigma)$, where μ is a point in the interior of σ. By Theorem 3.8 (2), $P_{\mu} = D_{\mu} + P_0$. Thus it suffices to prove that $D_{\lambda+\lambda_0} \subseteq P_{\lambda} + P_{\lambda_0}$. Let χ lie in $D_{\lambda+\lambda_0}$, i.e., $\chi = \sum_{i=1}^{l(\sigma)} q_i v^{\lambda+\lambda_0}_i$, where $q_i \geq 0$ and $\sum_{i=1}^{l(\sigma)} q_i = 1$. There exists i such that $q_i \geq 1/l$. Hence $\chi \succ q_i v^{\lambda+\lambda_0}_i = q_i(v^{\lambda}_i + v^{\lambda_0}_i) \succ v^{\lambda_0}_i$. Thus $\chi - v^{\lambda_0}_i \in \Omega_{\mathbb{R}_+} \cap \Lambda(\mathbb{T}) = \Omega$. □

Let $\mu_1, \ldots, \mu_r$ be generators of the monoid $\sigma \cap \Sigma$ and let $c_1, \ldots, c_r \in \mathbb{N}$ be such that $c_i\mu_i$ are integral, $i = 1, \ldots, r$. By Lemma 3.12, it follows that taking P_{λ} for $\lambda = \sum_{i=1}^{r} d_i\mu_i$, where $0 < d_i < lc_i$, we obtain representatives of all equivalence classes of points in σ up to the Minkovski sum with $P^{\mathbb{R}}_{\lambda}$. Hence P_{H_0} is the Minkovski sum of such representatives for all (maximal) cones σ of the subdivision of $\Sigma_{\mathbb{R}_+}$ induced by $\pi_{\mathbb{R}}$.

Example 3.13. Let $\mathbb{X} = \mathbb{A}^n$, $\mathbb{T} = \mathbb{G}^n_m$ act be rescaling of coordinates, and let $T = \mathbb{G}_m$ act on $\mathbb{A}^n$ with characters $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. Then $\Omega \subset \mathbb{Z}^n$ is the set of vectors with integral non-positive coordinates, and $\Sigma \subset \mathbb{Z}$ is the monoid generated by $-\lambda_i$. Moreover, $\Sigma = (\Sigma \cap \mathbb{Z}_+) \cup (\Sigma \cap \mathbb{Z}_-)$ is the subdivision of Σ induced by π. Let n_+ and n_- be the numbers of positive and negative λ_i respectively. A number $\lambda \in \mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) is integral (in the sense of Definition 3.11) if and only if λ is divisible by any $\lambda_i < 0$ (resp. > 0). Let λ_+ (resp. λ_-) be the least common (positive) multiple of all positive (resp. negative) λ_i. Then P_{H_0} is the Minkovski sum of polyhedra P_{λ} for $-n_+\lambda_+ < \lambda < n_-\lambda_-$.

4 Toric Chow morphism

We are going to describe the toric Chow morphism from the Hilbert scheme to the inverse limit of GIT quotients $\mathbb{X}/\!\!/_\lambda T$. In [7, Sect. 5] the toric Chow morphism was constructed in the case when $\mathbb{X} = \mathbb{A}^n$ is a T-module. We generalize this to the case of a normal affine toric $\mathbb{T}$-variety $\mathbb{X}$.

We use the notations of the previous sections. Let

$$S^{(\lambda)} := \bigoplus_{r=0}^{\infty} S_{r\lambda},$$

and let

$$\mathbb{X}/\!\!/_\lambda T := \text{Proj } S^{(\lambda)}$$

be the GIT quotient. In particular, $\mathbb{X}/\!\!/_0 T = \mathbb{X}/\!\!/T = \text{Spec}\,(S_0)$. Notice also that $\mathbb{X}/\!\!/_\lambda T = \mathbb{X}^{ss}_\lambda/\!\!/T$, where

$$\mathbb{X}^{ss}_\lambda := \{x \in \mathbb{X} : f(x) \neq 0 \text{ for some homogeneous } f \in S^{(\lambda)}\}.$$

If λ lies in the interior of $\Sigma_{\mathbb{R}_+}$, then $\mathbb{X}/\!\!/_\lambda T$ is a normal toric $\mathbb{T}/T$-variety whose fan is $\mathcal{C}^{\mathbb{R}}_\lambda$, the normal fan to the polyhedron $P^{\mathbb{R}}_\lambda$.

We are going to define the toric Chow quotient of a toric variety $\mathbb{X}$ by a subtorus T (see [3, Sect. 3.2]). If a fan $\mathcal{C}_1$ is a refinement of $\mathcal{C}_2$, then we have a projective morphism $Y_{\mathcal{C}_1} \to Y_{\mathcal{C}_2}$ between the corresponding normal toric varieties (see [6, Th. 2.4]), so the varieties $\mathbb{X}/\!\!/_\lambda T$, where $\lambda \in \Sigma$, form an inverse system. Consider the inverse limit

$$\mathbb{X}/\!\!/_C T := \varprojlim\{\mathbb{X}/\!\!/_\lambda T : \lambda \text{ lies in the interior of } \Sigma\}.$$

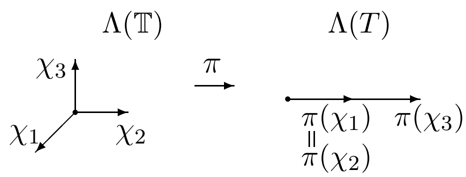
Note that $\mathbb{X}/\!\!/_C T$ is a closed subscheme in $V/\!\!/_C T$. By [3, Prop. 3.8], it follows that $\mathbb{X}/\!\!/_C T$ has an irreducible component that is a toric variety under the torus $\mathbb{T}/T$. Moreover, this component is the toric Chow quotient in the sense of the following definition.

Definition 4.1. (See [3, Def. 3.9].) The *toric Chow quotient* of a toric $\mathbb{T}$-variety $\mathbb{X}$ by a subtorus T is the irreducible component $\text{Chow}(\mathbb{X}, T)$ of $\mathbb{X}/\!\!/_C T$ such that
(1) $\text{Chow}(\mathbb{X}, T)$ is a toric $\mathbb{T}/T$-variety;
(2) given a $\mathbb{T}/T$-variety Y containing an irreducible component Y_0 such that Y_0 is a toric $\mathbb{T}/T$-variety, and given $\mathbb{T}/T$-equivariant morphisms $\phi_\lambda : Y \to \mathbb{X}/\!\!/_\lambda T$, where λ lies in the interior of Σ, such that the ϕ_λ induce birational morphisms $Y_0 \to \mathbb{X}/\!\!/_\lambda T$ and the ϕ_λ are compatible with the morphisms of the inverse system (so the ϕ_λ give a morphism $\phi : Y \to \mathbb{X}/\!\!/_C T$); then restricting the morphism ϕ to Y_0 we have a birational morphism of toric $\mathbb{T}/T$-varieties $Y_0 \to \text{Chow}(\mathbb{X}, T)$.

Remark 4.2. By [3, Prop. 3.10], it follows that the fan of $\mathrm{Chow}(\mathbb{X}, T)$ is $\mathcal{C}_{\mathbb{R}}$, the maximal common refinement of all the normal fans to the polyhedra $P_\lambda^{\mathbb{R}}$, $\lambda \in \Sigma$. Since every character $\lambda \in \Sigma$ has some integral positive multiple $c\lambda \in \Sigma_{\mathbb{X}}^{int}$ ($c \in \mathbb{N}$), the fan $\mathcal{C}_{H_0}$ is a refinement of the fan $\mathcal{C}_{\mathbb{R}}$.

The following example shows that $\mathcal{C}_{H_0}$ and $\mathcal{C}_{\mathbb{R}}$ do not always coincide.

Example 4.3. Let $\mathbb{X} = \mathbb{A}^3$, $\mathbb{T} = \mathbb{G}_m^3$ act by rescaling of coordinates, and let $T = \mathbb{G}_m$ act by $t(x_1, x_2, x_3) = (tx_1, tx_2, t^2 x_3)$.



The Hilbert scheme $H_{\mathbb{A}^3, T}$ is the closed subscheme in $\mathbb{P}^1 \times \mathbb{P}^3$ defined by the equations $z_1 w_3 - z_2 w_1 = 0$ and $z_1 w_2 - z_2 w_3 = 0$ (where z_1, z_2 and w_1, w_2, w_3, w_4 are homogeneous coordinates in $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively). The integral (in the sense of Definition 3.11) degrees are even. The fan $\mathcal{C}_{H_0}$ consists of the following cones:

$$\mathbb{R}_+(e_1 + e_2) + \mathbb{R}_+ e_2,$$

$$\mathbb{R}_+(e_1 + e_2) + \mathbb{R}_+(-e_2),$$

$$\mathbb{R}_+(e_2 - e_1) + \mathbb{R}_+ e_2,$$

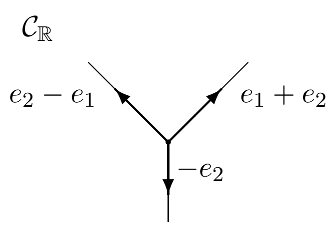
$$\mathbb{R}_+(e_2 - e_1) + \mathbb{R}_+(-e_2),$$

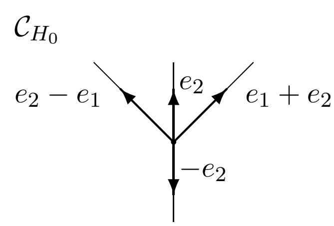
where $e_1 = \chi_1^* + \chi_3^*, e_2 = -\chi_3^*$ is a basis of $\Gamma(\mathbb{T}/T)$. The toric Chow quotient is $\mathbb{A}^3 /\!\!/_C T = \mathrm{Proj}\ k[x_1, x_2, x_3]$ (where $k[x_1, x_2, x_3]$ is graded by the weights of T), and its fan $\mathcal{C}_{\mathbb{R}}$ consists of the following cones:

$$\mathbb{R}_+(e_1 + e_2) + \mathbb{R}_+(-e_2),$$

$$\mathbb{R}_+(e_2 - e_1) + \mathbb{R}_+(-e_2),$$

$$\mathbb{R}_+(e_1 + e_2) + \mathbb{R}_+(e_2 - e_1).$$





By Lemma 3.12, it follows that if a character $\lambda \in \Sigma$ is integral, then there exists r_0 such that $S_{(r+1)\lambda} = S_\lambda S_{r\lambda}$ for all $r \geq r_0$. The statement of the following lemma was given in [7, Sect. 5] with a proof for algebras generated by elements of degree 1. For the convenience of the reader we give a complete proof here.

LEMMA 4.4. *Let P be an $\mathbb{N}$-graded algebra: $P = \bigoplus_{r\geq 0} P_r$, and*

$$(*) \quad \textit{there exists } r_0 \textit{ such that } P_{r+1} = P_1 P_r \textit{ for any } r \geq r_0.$$

Then the Hilbert scheme H_P of the graded algebra P for the Hilbert function

$$h(r) := \begin{cases} 1 & \textit{if} \ \ r \geq 0, \\ 0 & \textit{otherwise} \end{cases}$$

is isomorphic to $\operatorname{Proj} P$.

PROOF. We shall show that $\operatorname{Proj} P$ represents the Hilbert functor $\underline{H_P}$. For this we prove that the tautological bundle over $\operatorname{Proj} P$ is the universal family.

(1) Consider the open subscheme in $\operatorname{Spec} P$ that is the complement to the subscheme defined by the ideal $\bigoplus_{r>0} P_r$:

$$(\operatorname{Spec} P)_0 = \{p \in \operatorname{Spec} P \ : \ p \not\supseteq (\bigoplus_{r>0} P_r)\}$$

and the natural morphism

$$\psi : (\operatorname{Spec} P)_0 \to \operatorname{Proj} P.$$

Locally ψ is given by the embeddings of algebras $(P_f)_0 \subset P_f$, where $f \in P$ is homogeneous, $\deg f > 0$ (it is clear that the corresponding morphisms of affine schemes satisfy the compatibility conditions). Note that ψ is a locally trivial bundle with fiber $\mathbb{G}_m$. Indeed, by the condition (*), it follows that $\operatorname{Proj} P$ is covered by open affine subschemes $\operatorname{Spec}(P_h)_0$, where $h \in P_1$, and for any $h \in P_1$ we have $P_h = (P_h)_0[h, h^{-1}]$.

(2) The grading of P defines an action of $\mathbb{G}_m$ on $\operatorname{Spec} P$ and on $(\operatorname{Spec} P)_0$. Consider

$$E := (\operatorname{Spec} P)_0 \times_{\mathbb{G}_m} \mathbb{A}^1.$$

Here $(\operatorname{Spec} P)_0 \times_{\mathbb{G}_m} \mathbb{A}^1$ denotes the categorical quotient $((\operatorname{Spec} P)_0 \times \mathbb{A}^1)/\!/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ as follows: $t \cdot s = t^{-1}s$, $t \in \mathbb{G}_m, s \in \mathbb{A}^1$ (locally we have $E_f = \operatorname{Spec} \bigoplus_{r\geq 0}(P_f)_r$, where $f \in P$ is homogeneous of positive degree). Then ψ gives a morphism

$$E \to \operatorname{Proj} P$$

which is a locally trivial bundle with fiber $\mathbb{A}^1$.

(3) Now we are going to prove the universal property for E. Let $Z = \operatorname{Spec} R$ be an affine scheme and $Y = \operatorname{Spec}(R \otimes_k P/I) \in \underline{H_P}(R)$. Consider $Y_0 = Y \cap (Z \times (\operatorname{Spec} P)_0)$. We have the morphisms

$$Y_0 \xrightarrow{\rho} \operatorname{Proj}(R \otimes_k P/I) \xrightarrow{\delta} Z.$$

Since $R \otimes_k P/I$ satisfies the condition (*), by (1), it follows that ρ is a locally trivial bundle with fiber $\mathbb{G}_m$. So δ is an isomorphism. Consider the following morphism from Z to $\operatorname{Proj} P$:

$$Z \cong \operatorname{Proj}(R \otimes_k P/I) \subset Z \times \operatorname{Proj} P \xrightarrow{p} \operatorname{Proj} P,$$

where p is the projection. We shall show that $Y = E \times_{\operatorname{Proj} P} Z$.

(a) Note that $Y_0 = (\operatorname{Spec} P)_0 \times_{\operatorname{Proj} P} Z$. Indeed, locally we have

$$R \otimes_k P_f/I_f \simeq P_f \otimes_{(P_f)_0} (R \otimes_k (P_f)_0/(I_f)_0),$$

where $f \in P$ is homogeneous of positive degree.

(b) As in (2), consider $Y' = Y_0 \times_{\mathbb{G}_m} \mathbb{A}^1$ and the natural morphism $\eta : Y' \to Y$, which is locally given by the homomorphisms

$$R \otimes_k P/I \to \bigoplus_{r \geq 0} (R \otimes_k P_f/I_f)_r,$$

where $f \in P$ is homogeneous of positive degree. So we have a commutative diagram:

$$\begin{array}{ccc} Y_0 \times_{\mathbb{G}_m} \mathbb{A}^1 & \xrightarrow{\eta} & Y \\ & \searrow \quad \swarrow & \\ & Z. & \end{array}$$

Since $Y, Y' \in \underline{H_P}(R)$, likewise, η is an isomorphism.

(c) Thus we have $Y = Y_0 \times_{\mathbb{G}_m} \mathbb{A}^1 = ((\operatorname{Spec} P)_0 \times_{\operatorname{Proj} P} Z) \times_{\mathbb{G}_m} \mathbb{A}^1 = E \times_{\operatorname{Proj} P} Z.$ □

This lemma implies that

$$H_{S^{(\lambda)},T} = \operatorname{Proj} S^{(\lambda)} = \mathbb{X} /\!\!/_{\lambda} T,$$

for any $\lambda \in \Sigma_{\mathbb{X}}^{int}$.

For any subset $D \subset \Sigma$ we can consider the restriction of the Hilbert scheme $H_{\mathbb{X},T}$ on degrees D, that is, the quasiprojective scheme $H_{\mathbb{X},T}^D$ representing the covariant functor

$$\underline{H_{\mathbb{X},T}^D} : \underline{k - Alg} \to \underline{Set}$$

such that $\underline{H_{\mathbb{X},T}^D}(R)$ is the set of families $\{L_\lambda\}_{\lambda \in D}$, where $L_\lambda \subset R \otimes_k S_\lambda$ is an R-submodule, such that $(R \otimes_k S_\lambda)/L_\lambda$ is a locally free R-module of rank 1 and $f L_{\lambda_2} \subset L_{\lambda_1}$ for any $\lambda_1, \lambda_2 \in D$ and any $f \in S_{\lambda_1 - \lambda_2}$ (see [7]). In particular, $H_{\mathbb{X},T}^{\Sigma} = H_{\mathbb{X},T}$ and $H_{S^{(\lambda)},T} = H_{\mathbb{X},T}^{D^\lambda}$, where $D^\lambda := \{c\lambda \ : \ c \in \mathbb{Z}_+\}$. Note also that $H^D_{\mathbb{X},T}$ is a closed subscheme of $H^D_{V,T}$. For any $D \subset \Sigma$ we have a degree restriction morphism $H_{\mathbb{X},T} \to H^D_{\mathbb{X},T}$. In particular, we have canonical morphisms

$$\phi^\lambda_{\mathbb{X}} : H_{\mathbb{X},T} \to \mathbb{X}/_{\lambda}T.$$

The following theorem was proved in [7, Th. 5.6] for the case when $\mathbb{X} = \mathbb{A}^n$ and $\mathbb{T} = \mathbb{G}^n_m$ acts by rescaling of coordinates.

THEOREM 4.5. *Let $H^{int}_{\mathbb{X},T} := H^{\Sigma^{int}_{\mathbb{X}}}_{\mathbb{X},T}$ be the toric Hilbert scheme restricted to the set of integral degrees. Then there is a canonical morphism*

$$\phi^{int}_{\mathbb{X}} : H^{int}_{\mathbb{X},T} \to \mathbb{X}/_{C}T$$

which induces an isomorphism of the corresponding reduced schemes. In particular, composing $\phi^{int}_{\mathbb{X}}$ with the degree restriction morphism, we obtain a canonical Chow morphism from the toric Hilbert scheme to the inverse limit of the GIT quotients

$$\phi_{\mathbb{X}} : H_{\mathbb{X},T} \to \mathbb{X}/_{C}T.$$

PROOF. As in [7, Lemma 5.7], we see that the morphisms $\phi^\lambda_{\mathbb{X}}$ satisfy the compatibility conditions for $\lambda \in \Sigma^{int}_{\mathbb{X}}$ and, consequently, give a canonical morphism

$$H_{\mathbb{X},T} \to H^{int}_{\mathbb{X},T} \xrightarrow{\phi^{int}_{\mathbb{X}}} \mathbb{X}/_{C}T.$$

Futher, note that for any algebra R the morphism

$$\underline{\phi^{int}_{\mathbb{X}}}(R) : \underline{H^{int}_{\mathbb{X},T}}(R) \to \underline{\mathbb{X}/_{C}}T(R)$$

is injective (since $H_{S^{(\lambda)},T} = \mathbb{X}/_{\lambda}T$, we view any element of $\underline{\mathbb{X}/_{C}}T(R)$ as a family of ideals $\{I^{(\lambda)} \in \underline{H_{S^{(\lambda)},T}}(R)\}_{\lambda\in\Sigma^{int}_{\mathbb{X}}}$ satisfying the compatibility conditions of the direct system). Hence to prove that $\phi^{int}_{\mathbb{X}}$ induces an isomorphism of the reduced schemes, it suffices to show that $\underline{\phi^{int}_{\mathbb{X}}}(R)$ is surjective for any reduced R.

Note that $\phi^\lambda_{\mathbb{X}}$ coincides with the restriction of ϕ^λ_V to $H_{\mathbb{X},T} \subset H_{V,T}$ for any $\lambda \in \Sigma^{int}_V \subset \Sigma^{int}_{\mathbb{X}}$. By [7, Th. 5.6], the map $\underline{\phi^{int}_V}(R)$ is surjective for any reduced R, and it follows that any element $\{I^{(\lambda)} \in \underline{H_{S^{(\lambda)},T}}(R)\}_{\lambda\in\Sigma^{int}_{\mathbb{X}}}$ in $\underline{\mathbb{X}/_{C}}T(R) \subset \underline{V/_{C}}T(R)$ gives an element $\{I^{(\lambda)} \in \underline{H_{S^{(\lambda)},T}}(R)\}_{\lambda\in\Sigma^{int}_V}$ in $\underline{H^{int}_{V,T}}(R)$, i.e., $fI^{(\lambda_2)} \subset I^{(\lambda_1)}$ for any $\lambda_1, \lambda_2 \in \Sigma^{int}_V$ and any $f \in S_{\lambda_1-\lambda_2}$. We have to prove that this condition holds for any $\lambda_1, \lambda_2 \in \Sigma^{int}_{\mathbb{X}}$. There exists $c \in \mathbb{N}$ such that $c\lambda_1, c\lambda_2 \in \Sigma^{int}_V$. For any $f' \in I^{(\lambda_2)}$ we have $f^c(f')^c \in I^{(\lambda_1)}$. Applying paragraph (3) of the proof of Lemma 4.4 to the algebra $P = (R \otimes_k S^{(\lambda_1)})/I^{(\lambda_1)}$, we see that the projection of $\operatorname{Spec}((R \otimes_k S^{(\lambda_1)})/I^{(\lambda_1)})$ to $\operatorname{Spec} R$ is a locally trivial bundle with fiber $\mathbb{A}^1$. Consequently, $(R \otimes_k S^{(\lambda_1)})/I^{(\lambda_1)}$ is reduced and $ff' \in I^{(\lambda_1)}$. $\square$

Remark 4.6. Note that restricting $\phi_{\mathbb{X}}$ to the main component H_0, we obtain a birational morphism of toric $\mathbb{T}/T$-varieties from H_0 to the toric Chow quotient $\mathrm{Chow}(\mathbb{X}, T)$.

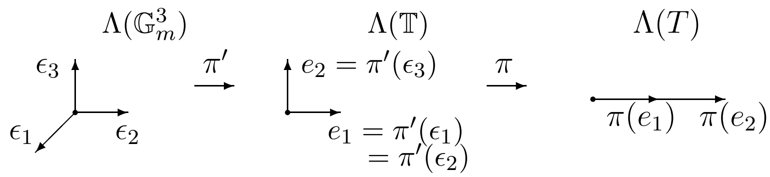
Example 4.7. Let $V = \mathbb{A}^3$ where $\mathbb{G}_m^3$ and $T = \mathbb{G}_m$ act as in Example 4.3, and let $\mathbb{T} = \mathbb{G}_m^2$ be embedded in $\mathbb{G}_m^3$ by $(t_1, t_2) \to (t_1, t_1, t_2)$. Consider the variety $\mathbb{X} = \overline{\mathbb{T} \cdot (1, 1, 1)} = \operatorname{Spec} S$, where $S = k[x_1, x_2, x_3]/I_{\mathbb{X}}$ and $I_{\mathbb{X}} = (x_1 - x_2)$. So $H_{\mathbb{X},T}$ is defined in $H_{\mathbb{A}^3,T}$ by the equation $z_1 = z_2$. We have the homomorphisms of groups of characters

$$\mathbb{Z}^3 = \Lambda(\mathbb{G}_m^3) \xrightarrow{\pi'} \mathbb{Z}^2 = \Lambda(\mathbb{T}) \xrightarrow{\pi} \mathbb{Z} = \Lambda(T)$$

and of monoids

$$\Omega_{\mathbb{A}^3} \to \Omega \to \Sigma,$$

where $\Omega_{\mathbb{A}^3}$ is the monoid in $\Lambda(\mathbb{G}_m^3)$ generated by characters with negative coordinates.



Note that $\Sigma_{\mathbb{X}}^{int} = \Sigma_{\mathbb{A}^3}^{int}$ is the set of even numbers. The scheme $H_{\mathbb{A}^3,T}^{int}$ is the closed subscheme in $\mathbb{P}^3$ defined by the equation $w_3^2 = w_1 w_2$, and $H_{\mathbb{X},T}^{int}$ is defined by the equations $w_1 = w_2 = w_3$. The isomorphism

$$\phi_{\mathbb{X}}^{int} : H_{\mathbb{X},T}^{int} \to \mathbb{X}/_{C}T = \operatorname{Proj} S$$

is the restriction of the isomorphism

$$\phi_{\mathbb{A}^3}^{int} : H_{\mathbb{A}^3,T}^{int} \to \mathbb{A}^3/_{C}T = \operatorname{Proj} k[x_1, x_2, x_3],$$

where the inverse isomorphism is given by

$$(\phi_{\mathbb{A}^3}^{int})^{-1}(x_1 : x_2 : x_3) = (x_1^2 : x_2^2 : x_1 x_2 : x_3).$$

Concerning the morphism $\phi_{\mathbb{A}^3} : H_{\mathbb{A}^3,T} \to \mathbb{A}^3/_{C}T$, note that $\phi_{\mathbb{A}^3}^{-1}(\mathbb{X}/_{C}T)$ is not contained in $H_{\mathbb{X},T}$. Indeed, consider the ideal $I = (x_1, x_2^2) \in \underline{H_{\mathbb{A}^3,T}}(k)$. We have $(I_{\mathbb{X}})_r \subset I_r$ for any even r, so $\phi_{\mathbb{A}^3}(I) \in \underline{\mathbb{X}/_{C}T}(k)$, but $I \notin \underline{H_{\mathbb{X},T}}(k)$.

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